The Harper-Hofstadter Hamiltonian and conical diffraction in photonic lattices with grating assisted tunneling

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We introduce a grating assisted tunneling scheme for tunable synthetic magnetic fields in photonic lattices, which can be implemented at optical frequencies in optically induced one- and two-dimensional dielectric photonic lattices. We demonstrate a conical diffraction pattern in particular realization of these lattices which possess Dirac points in k-space, as a signature of the synthetic magnetic fields. The two-dimensional photonic lattice with grating assisted tunneling constitutes the realization of the Harper-Hofstadter Hamiltonian.

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Synthetic magnetism for photons is a unique tool for the manipulation and control of light, and for the design of novel topological phases and states in photonic systems.1–12 Topological photonics is a rapidly growing field, advancing in parallel to analogous efforts in ultracold atomic gases,16,17 inspired by the development of topological insulators in condensed matter physics.18 One motivating aspect of topological photonic systems is the existence of unidirectional backscattering immune states,1–4, which are robust to imperfections, and thus may serve as novel waveguides and for building integrated photonic devices. The first experimental observations of such edge states were in the microwave domain, in magneto-optical photonic crystals,1 theoretically proposed in Refs. 1–3. In the optical domain, imaging of topological edge states was reported in Floquet topological insulators, implemented in modulated honeycomb photonic lattices,5 and in the two-dimensional array of coupled optical-ring resonators.6 The strategies for obtaining synthetic magnetic/gauge fields and topological phases for optical photons are closely related to the system at hand. In systems of coupled optical resonators, the strategy is to tune the phase of the tunneling between coupled cavities; for example, by using link resonators of different length or time-modulation of the coupling. Photonic topological insulators were also proposed in superlattices of metamaterials with strong magneto-electric coupling.13 In 2D photonic lattices (waveguide arrays), pseudomagnetic fields have been demonstrated by inducing ‘strain’ in optical graphene.10 By modulating 1D photonic lattices along the propagation axis, one can choose the sign of the hopping parameter between neighboring sites,11 whereas topological states were achieved using 1D photonic quasicrystals.12 However, even though photonic lattices possess a great potential for exploring synthetic magnetism and topological effects, a viable scheme for arbitrary design of the phases of the complex tunneling matrix elements still needs to be developed. Here we introduce one such scheme termed grating assisted tunneling, and propose its implementation in optically induced photonic lattices.19,20 We demonstrate the conical diffraction pattern in particular realizations of one- (1D) and two-dimensional (2D) square photonic lattices with grating assisted tunneling. The latter constitutes demonstration of the the (‘single particle’) Harper-Hofstadter Hamiltonian (HHH)26,27 in these systems.

The proposed method is inspired by the so called laser assisted tunneling scheme, which was implemented in optical lattices with ultracold atoms.28–30 The scheme is based on theoretical proposals in Refs. 31,32, subsequently developed to experimentally realize the HHH,24,30, and staggered magnetic fields in optical superlattices.28 Many of the results obtained with ultracold atoms are viable in photonic lattices, and perhaps even more of them are feasible, because heating by spontaneous emission that is present in ultracold atomic systems, is absent in photonic systems.

We consider the paraxial propagation of light in a photonic lattice defined by the index of refraction n = n0 + δn(x, y, z) (δn ≪ n0), where n0 is the constant background index of refraction, and δn(x, y, z) describes small spatial variations, which are slow along the propagation z-axis. The slowly varying amplitude of the electric field ψ(x, y, z) follows the (continuous) Schrödinger equation:

$$i \frac{\partial \psi}{\partial z} = - \frac{1}{2k} \nabla^2 \psi - \frac{k \delta n}{n_0} \psi;$$  \hspace{1cm} (1)

here, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, and $k = 2\pi n_0/\lambda$, where $\lambda$ is the wavelength in vacuum. From now on we will refer to $\delta n(x, y, z)$ as the potential or index of refraction. In the

\[ \frac{\partial \psi}{\partial z} = - \frac{1}{2k} \nabla^2 \psi - \frac{k \delta n}{n_0} \psi; \]
In order to interpret it, we point out that, for resonant tunneling, we can use grating assisted tunneling to tune the system. For this 1D lattice, we seek a scheme which effectively renormalizes the tunneling parameters of the discrete Schrödinger equation (e.g., see 29 for ultracold atoms):

\[ i\frac{d\psi_m}{dz} = -(K_x e^{i\phi_m-\psi_{m-1} + K_x e^{-i\phi_m}\psi_{m+1}}), \]

where \( \phi_m = q \cdot R_m = q_x a_m + q_y a_n \). In Fig. 1(g) we used \( q_x = \pi/a \), i.e., \( \phi_m = m \pi \). Such a discrete lattice is illustrated in Fig. 1(c); its dispersion having a 1D Dirac cone at \( k_x = 0 \). For a wavepacket that initially excites modes close to \( k_x = 0 \), the diffraction in the discrete model yields the so-called (1D) conical diffraction pattern, as illustrated in Fig. 1(h). The initial conditions for propagation in the discrete model can be set to initial conditions in the continuous system, \( \psi_m(0) = \sqrt{T}e^{-(m/\pi)^2} \), and \( k_x = 0.053 \text{ mm}^{-1} \). Thus, we interpret the diffraction pattern in Fig. 1(g) as 1D conical diffraction, a signature of the discrete model depicted in Fig. 1(c). A comparison of the discrete and the realistic continuous model, clearly shows that we can use grating assisted tunneling to tune the phases of the tunneling parameters in the discrete Schrödinger equation, thereby realizing synthetic magnetic fields.

Before proceeding to 2D systems, we discuss the amplitude of the tunneling matrix elements \( K_x \) as a function of \( \delta n_{G0} \). Figure 1(d) shows \( K_x \) (green squares) and \( J_x \) (blue circles) versus \( \delta n_{G0} \), where \( J_x \) corresponds to the potential which includes the lattice and the grating, but no tilt. The amplitudes \( J_x \) and \( K_x \) are obtained by comparing the diffraction pattern of the discrete with the continuum model, and adjusting \( J_x \) and \( K_x \) until the two patterns coincide, as in Figs. 1(g) and (h). Our results in Fig. 1(d) are in agreement with those in ultracold atoms [e.g., see Fig. 3(a) in Ref. 29].

The extension of the scheme to 2D lattices is straightforward. We consider a square lattice, \( \delta n_{L} = \delta n_{G0} \cos^2(\pi x/a) + \cos^2(\pi y/a) \), the tilt in the \( x \) direction, \( \delta n_{T}(x) = \eta x \), and the grating which has the form \( \delta n_{C}(x, y, z) = \delta n_{G0} \cos^2((q_x x + q_y y - \kappa z)/2) \). Propagation of light in the total potential \( \delta n(x, y, z) = \delta n_{L}(x, y) + \delta n_{T}(x) + \delta n_{C}(x, y, z) \) can be modeled by the discrete Schrödinger equation (the derivation is equivalent to that in Ref. 29 for ultracold atoms):

\[ i\frac{d\psi_{m,n}}{dz} = -(K_x e^{i\phi_{m-1,n}}\psi_{m-1,n} + K_x e^{-i\phi_{m,n}}\psi_{m+1,n} + J_y \psi_{m,n-1} + J_y \psi_{m,n+1}), \]

where \( \phi_{m,n} = q \cdot R_{m,n} = q_x a_m + q_y a_n \). Note that the tunneling along \( y \) does not yield a phase because there is no tilt in the \( y \) direction; the tunneling amplitude along \( y \) depends on the depth of the grating, as illustrated in Fig. 1(d) with blue circles.

In order to demonstrate that the propagation of light in the continuous 2D potential \( \delta n(x, y, z) \) is indeed equivalent to the dynamics of Eq. 1, we compare propagation in the discrete model [3], with that of the
Fig. 2(b). Suppose that the incoming beam at \(z = 0\) excites the modes which are in the vicinity of these two Dirac points. Around these points, for a given \(m, n\), the group velocity, \(\nabla_{k}(k_x, k_y)\), is constant. Thus, the beam will undergo conical diffraction, which has been thoroughly addressed with Dirac points in honeycomb optical lattices [24]. To demonstrate this effect, we consider the propagation of a beam with the initial profile given by \(\psi(x, y, 0) = \sqrt{J} \cos(y\pi/2a) \exp(-x^2/3a^2 - (y/3a)^2)\) in the 2D potential \(\delta n(x, y, z)\) (the cosine term ensures that we excite modes close to the two Dirac points). In Fig. 2(c) we show this beam after propagation for \(z = 162\ \text{mm}\). The two concentric rings are a clear evidence of conical diffraction [24, 25]. We also compare this with the propagation in the discrete model [14], with tunneling phases plotted in Fig. 2(a); \(K_x = 0.11\ \text{mm}^{-1}, J_y = 0.14\ \text{mm}^{-1}\). The results of the propagation in the discrete model are shown in Fig. 2(d). It is evident that for our choice of the tilt and the grating, we have effectively realized the lattice plotted in Fig. 2(a). This is in fact a realization of the HHF for \(\alpha = 1/2\), where \(\alpha\) is the flux per plaquette in units of the flux quantum [24, 25].

For the experimental implementation of the scheme, we propose the so-called optical induction technique in photosensitive materials, which can be implemented in photorefractives [19–23]. In these systems, both the lattice \(\delta n_L(x, y)\) and the grating potential \(\delta n_G(x, y, z)\) can be obtained in a straightforward fashion by using interference of plane waves in the medium [23]. The lattice constant \(a\), the grating parameter \(q\), and hence the hopping phase \(\phi_{m,n} = q \cdot R_{m,n}\), are tunable by changing the angle between the interfering beams. This could enable manipulation of the phases of the tunneling matrix elements in real time [20].
The propagation for \( z \) temporarily excites modes in the vicinity of the Dirac points, after the propagation constant. (c) Intensity of a beam, which initially excites modes in the vicinity of the Dirac points, after propagation for \( z = 162 \text{ mm} \). The intensity has two concentric rings corresponding to the conical diffraction pattern. (d) Simulation of the diffraction pattern in the discrete model corresponding to the lattice in (a), which also exhibits conical diffraction (see text for details).

The most challenging part of the implementation appears to be creation of the linear tilt potential. To observe the conical diffraction effects, one needs a linear gradient over the \( \sim 20 \) lattice sites, which implies a total index tilt \( \Delta n = 20\eta a \sim 20 \times 0.1\delta n_{LO} \sim 8 \times 10^{-4} \). In principle, the linear tilt could be achieved by using a spatial light modulator. Another possibility which could create a linear tilt is to use crystals with linear dependence of the index of refraction on temperature, and a temperature gradient across the crystal.

However, it should be emphasized that if one uses a superlattice for \( \delta n_L \), rather than the linear tilt potential, as in Ref. [28] for ultracold atomic gases, the grating assisted tunneling scheme proposed here would straightforwardly yield staggered synthetic magnetic fields for photons. The achievement of such staggered fields is straightforward with optically induced lattices proposed here.

Before closing, we emphasize that the spatial phase between the periodic lattice potential \( \delta n_L \) and the grating \( \delta n_G \) is a relevant parameter. This point has recently been examined in detail in the context of ultracold atoms [29].

To translate it to our system, suppose that in our 1D system, we shift the grating along the \( x \)-direction, such that \( \delta n_G = \delta n_{G0} \cos^2((q_G x - \kappa z + \xi)/2) \). This simply shifts the phases in the discrete lattice [shown in Fig. 1(c)] such that \( \phi_m \) is replaced by \( \phi_m + \xi \), which moves the 1D Dirac point in \( k_z \)-space by \( \Delta k_z = \xi/a \). We have numerically verified that this indeed happens.

In conclusion, we have introduced a grating assisted tunneling scheme for tunable synthetic magnetic fields in photonic lattices, and proposed its implementation at optical frequencies in optically induced one- and two-dimensional dielectric photonic lattices. As a signature of the synthetic magnetic fields, we have demonstrated the conical diffraction pattern in particular realization of these lattices [shown in Fig. 1(c) and Fig. 2(a)]. The two-dimensional photonic lattice with grating assisted tunneling constitutes the realization of the HHH scheme is well suited for realization of staggered synthetic magnetic fields in photorefractics. We envision that this proposal will open the way to other studies of light propagation in photonic lattices with complex tunneling matrix elements, including nonlinear propagation, solitons and instabilities in synthetic magnetic fields, and creation of synthetic dimensions in photonic lattices [30] by employing synthetic fields.

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Note added. A few days before this paper was submitted to arXiv, a paper by a different group [37] appeared on the arXiv. It is also inspired by laser assisted tunneling in cold gases, and experimentally demonstrated modulation-assisted tunneling in laser-fabricated photonic Wannier-Stark ladders.