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Constitutive relations, off shell duality rotations and the hypergeometric form of Born-Infeld theory[†]

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Abstract

We review equivalent formulations of nonlinear and higher derivatives theories of electromagnetism exhibiting electric-magnetic duality rotations symmetry. We study in particular on shell and off shell formulations of this symmetry, at the level of action functionals as well as of equations of motion. We prove the conjecture that the action functional leading to Born-Infeld nonlinear electromagnetism, that is duality rotation invariant off shell and that is known to be a root of an algebraic equation of fourth order, is a hypergeometric function.

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1 Introduction

Electric-magnetic duality is a symmetry of Maxwell electromagnetism and also, as remarked by Schrödinger [1], of the nonlinear theory of electromagnetism proposed by Born and Infeld [2]. This symmetry does not leave the Lagrangians invariant, only the equations of motion, and therefore it is not immediately detectable. This symmetry was subsequently discovered to be present in extended supergravity theories [3–5]. In [4] the first example of a noncompact duality rotation group was considered, it is due to scalar fields transforming nonlinearly under duality rotations. These results triggered further investigations in the general structure of self-dual theories. In particular the symplectic formalism for nonlinear electromagnetism coupled to scalar and fermion fields was initiated in [6], there the duality groups were shown to be subgroups of noncompact symplectic groups (compact groups being recovered in the absence of scalar fields). Also nonlinear theories admit noncompact duality symmetry, a most studied example is Born-Infeld electrodynamics coupled to axion and dilaton fields [7]. A relevant aspect of Born-Infeld theory [10] is that the spontaneous breaking of $N = 2$ rigid supersymmetry to $N = 1$ can lead to a Goldstone vector multiplet whose action is the supersymmetric and self-dual Born-Infeld action [8, 9]. Higher supersymmetric Born-Infeld type actions are also self-dual and related to spontaneous supersymmetry breakings in field theory [11–14] and in string theory [15, 16].

Another recent motivation for the renewed study of duality symmetry is due to its relevance for investigating the structure of possible counterterms in extended supergravity. After the explicit computations that showed the 3-loop UV finiteness of $N = 8$ supergravity [17], an explanation based on $E_{7(7)}$ duality symmetry was provided [18–21]. Furthermore duality symmetry arguments have also been used to suggest all loop finiteness of $N = 8$ supergravity [22]. Related to these developments, in [23] a proposal on how to implement duality rotation invariant counterterms in a corrected action $S[F]$ leading to a self-dual theory was put forward under the name of "deformed twisted self-duality conditions" . The proposal (renamed "nonlinear twisted self-duality conditions") was further elaborated in [24] and [25]; see also [26], and [27–29], for the supersymmetric extensions and examples. The proposal encompasses theories that depend nonlinearly on the field strength F and also on the partial derivative terms $\partial F, \partial\partial F, \ldots$. That is why we speak of *nonlinear and higher derivatives theories.*

The proposal is equivalent to a formulation of self-dual theories using auxiliary fields studied in [30] and [31] in case of nonlinear electromagnetism without higher derivatives of the field strength. This coincidence has been brought to light in a recent paper [32]. In [33] two of us presented a systematic and general study of the different formulations of $U(1)$ gauge theories and of self-dual ones. This lead to a closed form expression of the duality invariant action functional describing Born-Infeld theory.

Before outlining the content of the present work let us recall the notion of *constitutive* *relations*. A nonlinear and higher derivative electromagnetic theory is determined by defining, eventually implicitly, the relation between the electric field strength F (given by the electric field $\overrightarrow{E}$ and the magnetic induction $\overrightarrow{B}$) and the magnetic field strength G (given by the magnetic field $\overrightarrow{H}$ and the electric displacement $\overrightarrow{D}$). We call *constitutive relations* the relations defining G in terms of F or vice versa. Different constitutive relations determine different $U(1)$ gauge theories.

In this paper we first review and clarify the relations between constitutive relations and action functionals in nonlinear and higher derivative electromagnetism. Then we provide a pedagogical analysis of the "deformed twisted self duality conditions" and introduce the action functional $\mathcal{I}[T^-,\overline{T^-}]$ obtained via a Legendre transformation from the usual $S[F]$ action functional in the field strength F. All theories defined via an action functional $S[F]$ and having duality symmetry have a formulation via an action functional $\mathcal{I}[T^-,\overline{T^-}]$ that is off shell invariant under duality rotations.

We then further study the different formulations of the constitutive relations of nonlinear and higher derivatives electromagnetism and then of self-dual theories. These different formulations are all equivalent *on shell*. Finally we prove the conjecture formulated in [33] concerning the hypergeometric function expression of the functional $\mathcal{I}$ of Born-Infeld theory. The proof uses Cauchy residue theorem in order to show that the hypergeometric function satisfies the algebraic quartic equation characterizing the functional $\mathcal{I}$.

2 U(1) duality rotations in nonlinear and higher derivatives electromagnetism

2.1 Action functionals from equations of motion

Nonlinear and higher derivatives electromagnetism is described by the equations of motion

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0 \ , \tag{2.1}$$

$$\partial_\mu \widetilde{G}^{\mu\nu} = 0 \ , \tag{2.2}$$

$$\widetilde{G}^{\mu\nu} = h^{\mu\nu}[F,\lambda] \ . \tag{2.3}$$

The first two simply state that the 2-forms F and G are closed, $dF = dG = 0$, indeed $\widetilde{F}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, $\widetilde{G}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}$ (with $\varepsilon^{0123} = 1$). The last set $\widetilde{G}^{\mu\nu} = h^{\mu\nu}[F,\lambda]$, where λ is the dimensionful parameter typically present in a nonlinear theory*, are the constitutive relations. They specify the dynamics and determine the magnetic field

*Nonlinear and higher derivatives theories of electromagnetism admit one (or more) dimensionful coupling constant(s) λ.

strength G as a functional in terms of the electric field strength F, and, vice versa, determine F in term of G, indeed F and G should be treated on equal footing in (2.1)-(2.3). The square bracket notation $h^{\mu\nu}[F,\lambda]$ stems from the possible dependence of $h^{\mu\nu}$ on derivatives of F.

Since in general we consider curved background metrics $g_{\mu\nu}$, it is convenient to introduce the $*$-Hodge operator; on an arbitrary antisymmetric tensor $F_{\mu\nu}$ it is defined by

$$
{}^*F_{\mu\nu} = \frac{1}{2\sqrt{g}} g_{\mu\alpha} g_{\nu\beta}\, \varepsilon^{\alpha\beta\rho\sigma} F_{\rho\sigma} = \frac{1}{\sqrt{g}} \widetilde{F}_{\mu\nu} \ , \tag{2.4}
$$

where $g = -\det(g_{\mu\nu})$, and it squares to minus the identity. The constitutive relations (2.3) implicitly include also a dependence on the background metric $g_{\mu\nu}$ and for example in case of usual electromagnetism they read $G_{\mu\nu} = {}^*F_{\mu\nu} = \frac{1}{\sqrt{g}}\widetilde{F}_{\mu\nu}$, while for Born-Infeld theory,

$$
S_{BI} = \frac{1}{\lambda} \int d^4x\, \sqrt{g} \Big(1 - \sqrt{1 + \frac{1}{2}\lambda F^2 - \frac{1}{16}\lambda^2 (F\,{}^*F)^2}\, \Big) \ , \tag{2.5}
$$

where $F^2 = FF = F_{\mu\nu}F^{\mu\nu}$ and $F\,{}^*F = F_{\mu\nu}\,{}^*F^{\mu\nu}$, they read

$$
G_{\mu\nu} = \frac{{}^*F_{\mu\nu} + \frac{1}{4}\lambda (F\,{}^*F)\, F_{\mu\nu}}{\sqrt{1 + \frac{1}{2}\lambda F^2 - \frac{1}{16}\lambda^2 (F\,{}^*F)^2}} \ . \tag{2.6}
$$

The constitutive relations (2.3) define a nonlinear and higher derivatives extension of electromagnetism because we require that setting $\lambda = 0$ in (2.3) we recover usual electromagnetism: $G_{\mu\nu} = {}^*F_{\mu\nu}$.

We now recall [33] that in the general nonlinear case (where the constitutive relations do not involve derivatives of F) the equations of motion (2.1)-(2.3) can always be obtained from a variational principle provided they satisfy the integrability conditions

$$
\frac{\partial h^{\mu\nu}}{\partial F_{\rho\sigma}} = \frac{\partial h^{\rho\sigma}}{\partial F_{\mu\nu}} \ . \tag{2.7}
$$

These conditions are necessary in order to obtain (2.3) from an action $S[F] = \int d^4x \mathcal{L}(F)$. Indeed if[†] $h^{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}$ then (2.7) trivially holds.

In order to show that (2.7) is also a sufficient condition we recall that the field strength $F_{\mu\nu}(x)$ locally is a map from spacetime to $\mathbb{R}^6$ (with coordinates $F_{\mu\nu}$, $\mu < \nu$). We assume $h^{\mu\nu}(F,\lambda)$ to be well defined functions on $\mathbb{R}^6$ or more generally on an open submanifold $M \subset \mathbb{R}^6$ that includes the origin ($F_{\mu\nu} = 0$) and that is a star shaped region w.r.t. the origin (e.g. a 6-dimensional ball or cube centered in the origin).

[†]The factor 2 is due to the convention $\frac{\partial F_{\rho\sigma}}{\partial F_{\mu\nu}} = \delta^\mu_\rho \delta^\nu_\sigma$ adopted in [6] and in the review [35]. It will be used throughout the paper.

Then condition (2.7) states that the 1-form $\hbar = h^{\mu\nu} dF_{\mu\nu}$ is closed, and hence, by Poincaré lemma, exact on M; we write $\hbar = d\mathcal{L}$. We have $\mathcal{L}(F) - \mathcal{L}(0) = \int_\gamma \hbar$ for any curve $\gamma(c)$ of coordinates $\gamma_{\mu\nu}(c)$ such that $\gamma_{\mu\nu}(0) = 0$ and $\gamma_{\mu\nu}(1) = F_{\mu\nu}$. In particular, choosing the straight line from the origin to the point with coordinates $F_{\mu\nu}$, and setting $S = \int d^4x\, \mathcal{L}(F)$, we immediately conclude:

Under the integrability conditions (2.7) locally the equations of motion of nonlinear electromagnetism (2.1)-(2.3) can be obtained from the action

$$S = \frac{1}{2} \int d^4x \int_0^1 dc\, cF\widetilde{G}_c \ , \tag{2.8}$$

where $\widetilde{G}_c = \frac{1}{c} h(cF, \lambda)$.

One can also consider the more general case of nonlinear and higher derivatives electromagnetism. Here too if the theory is obtained from an action functional $S[F]$ then we have

$$S[F] = \frac{1}{2} \int d^4x \int_0^1 dc\, F h[cF, \lambda] \ , \tag{2.9}$$

that we simply rewrite $S = \frac{1}{2} \int d^4x \int_0^1 dc\, cF\widetilde{G}_c$.

Proof. Consider the one parameter family of actions $S_c[F] = \frac{1}{c^2} S[cF]$. Deriving with respect to c we obtain

$$-c\frac{\partial S_c}{\partial c} = 2S_c - \int d^4x\ F\frac{\delta S_c[F]}{\delta F} \ , \tag{2.10}$$

i.e. $-c\frac{\partial S_c}{\partial c} = 2S_c - \frac{1}{2}\int d^4x\ F\widetilde{G}_c$. It is easy to see that $S_c = \frac{1}{2c^2}\int d^4x \int_0^c dc'\, c'F\widetilde{G}_{c'}$ is the primitive with the correct behaviour under rescaling of c and F. We conclude that $\frac{1}{c^2}S[cF] = \frac{1}{2c^2}\int d^4x \int_0^c dc'\, c'F\widetilde{G}_{c'}$, and setting $c = 1$ we complete the proof.

An equivalent form of the expression $S = \frac{1}{2} \int d^4x \int_0^1 dc\, cF\widetilde{G}_c$ has been considered, for self-dual theories, in [25] and called reconstruction identity. It has been used to reconstruct the action S from equations of motion with duality rotation symmetry in examples with higher derivatives of F.

2.2 Conditions for $U(1)$ duality rotation symmetry of the equations of motion

Nonlinear and higher derivatives electromagnetism admits $U(1)$ duality rotation symmetry if given a field configuration F, G that satisfies (2.1)-(2.3) then the rotated configuration

$$\begin{pmatrix} F' \\ G' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} \ , \tag{2.11}$$

that is trivially a solution of $\partial_\mu \widetilde{F}^{\mu\nu} = 0\,,\ \partial_\mu \widetilde{G}^{\mu\nu} = 0\,$, satisfies also $\widetilde{G}'_{\mu\nu} = h_{\mu\nu}[F', \lambda]$, so that F', G' is again a solution of the equations of motion. If we consider an infinitesimal duality rotation, $F \to F + \Delta F$, $G \to G + \Delta G$ then condition $\widetilde{G}'_{\mu\nu} = h_{\mu\nu}[F', \lambda]$ reads $\Delta \widetilde{G}_{\mu\nu} = \int d^4x \, \frac{\delta h_{\mu\nu}}{\delta F_{\rho\sigma}} \, \Delta F^{\rho\sigma}$, i.e., $\widetilde{F}_{\mu\nu} = -\int d^4x \, \frac{\delta h_{\mu\nu}}{\delta F_{\rho\sigma}} \, G^{\rho\sigma}$, that we simply rewrite

$$\widetilde{F}_{\mu\nu} = -\int d^4x \, \frac{\delta \widetilde{G}_{\mu\nu}}{\delta F_{\rho\sigma}} \, G^{\rho\sigma} \ . \tag{2.12}$$

It is straightforward to check that electromagnetism and Born-Infeld theory satisfy (2.12).

If the theory is obtained from an action functional $S[F]$ (in the field strength F and its derivatives) then (2.3) is given by

$$\widetilde{G}^{\mu\nu} = 2 \frac{\delta S[F]}{\delta F_{\mu\nu}} \ . \tag{2.13}$$

In particular it follows that

$$\frac{\delta \widetilde{G}^{\mu\nu}}{\delta F_{\rho\sigma}} = \frac{\delta \widetilde{G}^{\rho\sigma}}{\delta F_{\mu\nu}} \ , \tag{2.14}$$

hence the duality symmetry condition (or self-duality condition) (2.12) equivalently reads $\widetilde{F}_{\mu\nu} = -\int d^4x \, \frac{\delta \widetilde{G}_{\rho\sigma}}{\delta F_{\mu\nu}} \, G^{\rho\sigma}$. Now writing $\widetilde{F}_{\mu\nu} = \frac{\delta}{\delta F_{\mu\nu}} \, \frac{1}{2} \int d^4x \; F_{\rho\sigma} \widetilde{F}^{\rho\sigma}$ we equivalently have

$$\frac{\delta}{\delta F_{\mu\nu}} \int d^4x \; (F\widetilde{F} + G\widetilde{G}) = 0 \ , \tag{2.15}$$

where $F\widetilde{F} = F_{\rho\sigma}\widetilde{F}^{\rho\sigma}$ and similarly for $G\widetilde{G}$. We require this condition to hold for any field configuration F (i.e. off shell of (2.1), (2.2)) and hence we obtain the Noether-Gaillard-Zumino (NGZ) self-duality condition[‡]

$$\int d^4x \; (F\widetilde{F} + G\widetilde{G}) = 0 \ . \tag{2.16}$$

The vanishing of the integration constant is determined for example by the condition $G = {}^*F$ for weak and slowly varying fields, i.e. by the condition that in this regime the theory is approximated by usual electromagnetism.

We also observe that the NGZ self-duality condition (2.16) is equivalent to the invariance of $S^{inv} = S - \frac{1}{4} \int d^4x \, F\widetilde{G}$, indeed under a rotation (2.11) with infinitesimal parameter α we have $S^{inv}[F'] - S^{inv}[F] = -\frac{\alpha}{4} \int d^4x \; (F\widetilde{F} + G\widetilde{G}) = 0$.

[‡]Note that (2.16) (the integrated form of the more restrictive self-duality condition $F\widetilde{F} + G\widetilde{G}$) also follows in a straightforward manner by repeating the passages in [6] but with G the functional derivatives of the action rather than the partial derivatives of the lagrangian [12, 35]. This makes a difference for nonlinear theories which also contain terms with derivatives of F.

From this relation it follows that the action $S[F]$ is not invariant under duality rotations and that under a finite transformation (2.11) we have

$$S[F'] = S[F] + \frac{1}{8}\int d^4x \left(\sin(2\alpha)(F\widetilde{F} - G\widetilde{G}) - 4\sin^2(\alpha)F\widetilde{G} \right) . \qquad (2.17)$$

Thus the action changes by the integral of the four-forms $F \wedge F - G \wedge G$ and $F \wedge G$, that, on the equations of motion $dF = dG = 0$ (cf. (2.1), (2.2)), are locally total derivatives. This is a sufficient condition for the transformation (2.11) with $\widetilde{G}^{\mu\nu} = 2\frac{\delta S[F]}{\delta F_{\mu\nu}}$ to be a symmetry.

We summarize the results thus far obtained: The self-duality condition (2.16) is off shell of (2.1) and (2.2) but on shell of (2.3). The action functional $S[F]$ provides a variational principle for the equation (2.3) and under duality rotations changes by a term that on shell of (2.1) and (2.2) is a total derivative

2.3 Off shell formulation of duality symmetry

We here provide an off shell formulation of duality symmetry by considering a Legendre transformation to new variables. The new action functional, off shell of the equations of motion (2.1), (2.2) and (2.3), is invariant under duality rotations. This formulation allows for a classification of duality rotation symmetric theories (an ackward task using the action functional $S[F]$).

An example of functional invariant under duality rotations is provided by the Hamiltonian action functional. Indeed the Hamiltonian itself (and more generally the energy-momentum tensor) of duality symmetric theories is invariant under duality rotations [6][§]. The problem with the Hamiltonian formulation is however the lack of explicit Lorentz covariance.

These observations lead to consider a Legendre transformation of $S[F]$ to an action functional in new variables that transform linearly under duality rotations and that are Lorentz tensors.

The action $S[F]$ determines the submanifold of equations $\widetilde{G} = 2\frac{\partial S[F]}{\partial F}$ in the plane of coordinates F and G. Equivalently, defining the complex self-dual combinations

$$F^- = \frac{1}{2}(F - i^*F) , \qquad (2.18)$$

$$G^- = \frac{1}{2}(G - i^*G) , \qquad (2.19)$$

[§]In a general nonlinear theory the Hamiltonian depends on the magnetic field $\overrightarrow{B}$ and on the electric displacement $\overrightarrow{D} = \frac{\delta S[F]}{\delta \overrightarrow{E}}$, that rotate into each other under the duality (2.11), $\begin{pmatrix} \overrightarrow{B}' \\ -\overrightarrow{D}' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \overrightarrow{B} \\ -D \end{pmatrix}$. Since the composite fields $\overrightarrow{B}^2 + \overrightarrow{D}^2$ and $(\overrightarrow{B} \times \overrightarrow{D})^2$ are duality invariant, Hamiltonians that depend upon these combinations and their derivatives are trivially duality invariant and lead to duality symmetric theories.

and their complex conjugates $\overline{F^-} = F^+ = \frac{1}{2}(\overline{F} + i\,{}^*F), \overline{G^-} = G^+ = \frac{1}{2}(G + i^*G)$, the action $S[F^-, \overline{F^-}] = S[F]$ determines the submanifold of equations $G^- = -2i\frac{\partial S}{\partial F^-}$ in the plane of coordinates F^-, G^-.

We want to retrieve this submanifold using the new variables

$$T^- = F^- - iG^- \ , \tag{2.20}$$
$$\overline{T^+} = F^- + iG^- = 2F^- - T^- \ , \tag{2.21}$$

and their complex conjugates $\overline{T^-} = F^+ + iG^+$, $T^+ = F^+ - iG^+ = 2F^+ - \overline{T^-}$. These variables transform simply with a phase under duality rotations, ${T^-}' = e^{i\alpha}T^-$, $\overline{T^+}' = e^{-i\alpha}\overline{T^+}$; hence the formulation of a theory symmetric under duality rotations should be facilitated in these variables. The change of variables $(F^-, G^-) \to (T^-, \overline{T^+})$ is achieved by first changing from G^- to T^-, then by a Legendre transformation so that T^- become the independent variables and F^- the dependent ones, and finally changing further the dependent variables from F^- to $\overline{T^+} = 2F^- - iT^-$. Schematically we undergo the following chain of change of variables

$$(F^-, G^-) \longrightarrow (F^-, T^-) \longrightarrow (T^-, F^-) \longrightarrow (T^-, \overline{T^+}) \ . \tag{2.22}$$

More explicitly the equation in the (F^-, G^-)-plane

$$G^- = -2i\frac{\partial S}{\partial F^-} \tag{2.23}$$

is equivalent to the equation in the (F^-, T^-)-plane

$$T^- = \frac{\partial U}{\partial F^-} \tag{2.24}$$

where $U[F^-, F^+] = -2S[F^-, F^+] + \frac{1}{2}\int d^4x\sqrt{g}\big({F^-}^2 + {F^+}^2\big)$. Furthermore, via Legendre transform, this last equation is equivalent to the equation in the (T^-, F^-)-plane

$$F^- = \frac{\delta V}{\delta T^-} \tag{2.25}$$

where $V[T^-, \overline{T^-}] = -U[F^-, F^+] + \int d^4x\sqrt{g}\ \ (T^-F^- + \overline{T^-}F^+)$. Finally we rewite this equation in the $(T^-, \overline{T^+})$-plane as

$$\overline{T^+} = \frac{\delta \mathcal{I}}{\delta T^-} \tag{2.26}$$

where

$$\mathcal{I}[T^-, \overline{T^-}] = 2V[T^-, \overline{T^-}] - \frac{1}{2}\int d^4x\sqrt{g}\ \left({T^-}^2 + \overline{T^-}^2\right) \ . \tag{2.27}$$

In conclusion, as pioneered in [31] (in the case of no derivatives of F in the action), we have that $\mathcal{I}[T^-,\overline{T^-}]$ and $S[F]$ are related by

$$
\begin{aligned}
\frac{1}{4}\mathcal{I}[T^-,\overline{T^-}] &= S[F] \\
&+\int d^4x\sqrt{g}\left(\frac{1}{2}T^-F^- - \frac{1}{8}T^{-2} - \frac{1}{4}F^{-2} + \frac{1}{2}\overline{T^-}F^+ - \frac{1}{8}\overline{T^-}^2 - \frac{1}{4}F^{+2}\right) .
\end{aligned}
\tag{2.28}
$$

The equations of motion (2.26) were studied in [23], where a nontrivial example of a self-dual action with an infinite number of derivatives of the field strength F is considered (see also the generalizations in the appendix of [33]).

Let's now study duality rotations. We consider F to be the elementary fields and let $S[F]$ be the action functional of a self-dual theory. Under infinitesimal duality rotations (2.11), $F \to F+\Delta F = F-\alpha G$, $G \to G+\Delta G = G+\alpha F$ we have (since $T^- = F^- - \frac{2}{\sqrt{g}}\frac{\delta S}{\delta F^-}$) that $T^- \to T^- + \Delta T^- = T^- - i\alpha T^-$. We calculate the variation of (2.28) under duality rotations. After a little algebra we see that

$$
\begin{aligned}
\Delta\mathcal{I} &= \mathcal{I}[T^- + \Delta T^-,\overline{T^-} + \Delta\overline{T^-}] - \mathcal{I}[T^-,\overline{T^-}] \\
&= S[F+\Delta F] - S[F] + \frac{\alpha}{4}\int d^4x\sqrt{g}\,\left(G\widetilde{G} - F\widetilde{F}\right) \\
&= -\frac{\alpha}{4}\int d^4x\sqrt{g}\,\left(G\widetilde{G} + F\widetilde{F}\right) = 0
\end{aligned}
\tag{2.29}
$$

where we used that $S[F+\Delta F]-S[F] = \int d^4x\,\frac{\delta S}{\delta F}\Delta F = -\frac{\alpha}{2}\int d^4x\,\widetilde{G}G$, and the self-duality conditions (2.16). Hence $\mathcal{I}$ is invariant under duality rotations.

Vice versa, we can consider T^-, $\overline{T^-}$ to be the elementary fields and assume $\mathcal{I}[T^-,\overline{T^-}]$ to be duality invariant. Then from $2F^- - T^- = \frac{1}{\sqrt{g}}\frac{\delta\mathcal{I}[T^-,\overline{T^-}]}{\delta T^-_{\mu\nu}}$, and $F^- - iG^- = T^-$, it follows that under the infinitesimal rotation $T^- \to T^- + \Delta T^- = T^- - i\alpha T^-$ we have $F \to F + \Delta F = F - \alpha G$, $G \to G + \Delta G = G + \alpha F$, and from (2.29) we recover the self-duality conditions (2.16) for the action $S[F]$.

This shows the equivalence betweeen the $S[F]$ and the $\mathcal{I}[T^-,\overline{T^-}]$ formulations of self-dual constitutive relations. Hence the deformed twisted self-duality condition proposal originated in the context of supergravity counterterms is actually the general framework needed to discuss self-dual theories starting from a variational principle.

We stress that while we needed to use the equations of motion in order to verify that the action $S[F]$ leads to a duality rotation symmetric theory, we do not need to use the equations of motion in order to verify that the action $\mathcal{I}[T^-,\overline{T^-}]$ is duality invariant. In the formulation with the $\mathcal{I}[T^-,\overline{T^-}]$ action functional duality rotations are an off shell symmetry provided that $\mathcal{I}[T^-,\overline{T^-}]$ is invariant under $T^- \to e^{i\alpha}T^-$ and $\overline{T^-} \to e^{-i\alpha}\overline{T^-}$.

3 Constitutive relations without self-duality

3.1 The $\mathcal{N}$ and $\mathcal{M}$ matrices

More insights in the constitutive relations (2.3) can be obtained if we restrict our study to the wide subclass that can be written as

$$^*G_{\mu\nu} = \mathcal{N}_2\, F_{\mu\nu} + \mathcal{N}_1\, {}^*F_{\mu\nu} \ , \tag{3.1}$$

where $\mathcal{N}_2$ is a real scalar field, while $\mathcal{N}_1$ is a real pseudo-scalar field (i.e., it is not invariant under parity, or, if we are in curved spacetime, it is not invariant under an orientation reversing coordinate transformation). As usual in the literature we set

$$\mathcal{N} = \mathcal{N}_1 + i\mathcal{N}_2 \ . \tag{3.2}$$

In nonlinear theories $\mathcal{N}$ depends on the field strength F, and in higher derivatives theories also on derivatives of F, we have therefore in general a functional dependence $\mathcal{N} = \mathcal{N}[F, \lambda]$. Furthermore $\mathcal{N}$ is required to satisfy $\mathcal{N} \to -i$ in the limit $\lambda \to 0$ so that we recover classical electromagnetism when the coupling constant(s) $\lambda \to 0$, or otherwise stated, in the weak and slowly varying field limit, i.e., when we discard higher powers of F and derivatives of F. Since $\mathcal{N}_2 \to -1$ for $\lambda \to 0$, $\mathcal{N}_2$, at least for sufficiently weak and slowly varying fields, is invertible. It follows that the constitutive relation (3.1) is equivalent to the more duality symmetric one

$$\begin{pmatrix} {}^*F \\ {}^*G \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathcal{M} \begin{pmatrix} F \\ G \end{pmatrix} \tag{3.3}$$

where the matrix $\mathcal{M}$ is given by

$$\mathcal{M}(\mathcal{N}) = \begin{pmatrix} 1 & -\mathcal{N}_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{N}_2 & 0 \\ 0 & \mathcal{N}_2^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\mathcal{N}_1 & 1 \end{pmatrix} = \begin{pmatrix} \mathcal{N}_2 + \mathcal{N}_1 \mathcal{N}_2^{-1} \mathcal{N}_1 & -\mathcal{N}_1 \mathcal{N}_2^{-1} \\ -\mathcal{N}_2^{-1} \mathcal{N}_1 & \mathcal{N}_2^{-1} \end{pmatrix} . \tag{3.4}$$

The matrix $\mathcal{M}$ is symmetric and sympletic and $\mathcal{M} \to -1$ for $\lambda \to 0$. Actually any such matrix is of the kind (3.4) with $\mathcal{N}_1$ real and $\mathcal{N}_2$ real and negative.

Finally, in order to really treat on equal footing the electric and magnetic field strengths F and G, we should consider functionals $N_1[F, G, \lambda]$ and $N_2[F, G, \lambda]$ such that the constitutive relations ${}^*G = N_2[F, G, \lambda]\, F + N_1[F, G, \lambda]\, {}^*F$ are equivalent to (3.1), i.e., such that on shell of these relations, $N_1[F, G, \lambda] = \mathcal{N}_1[F, \lambda]$ and $N_2[F, G, \lambda] = \mathcal{N}_2[F, \lambda]$. Henceforth, with slight abuse of notation, from now on the $\mathcal{N}, \mathcal{N}_1, \mathcal{N}_2$ fields in (3.1)-(3.4) will in general be functionals of both F and G.

We now reverse the argument that led from (3.1) to (3.3). We consider constitutive relations of the form

$$\begin{pmatrix} {}^*F \\ {}^*G \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathcal{M}[F, G, \lambda] \begin{pmatrix} F \\ G \end{pmatrix} \tag{3.5}$$

that treat on equal footing F and G, and where $\mathcal{M} = \mathcal{M}[F, G, \lambda]$ is now an *arbitrary* real 2×2 matrix (with scalar entries $\mathcal{M}_{ij}$). We require $\mathcal{M} \to -1$ for $\lambda \to 0$, so that we recover classical electromagnetism when the coupling constant $\lambda \to 0$. A priory (3.5) is a set of 12 real equations, twice as much as those present in the constitutive relations (3.1). We want only 6 of these 12 relations to be independent in order to be able to determine G in terms of independent fields F (or equivalently F in terms of independent fields G). Only in this case the constitutive relations are well given. In [33] we show

Proposition 1. The constitutive relations (3.5) with $\mathcal{M}[F, G, \lambda]|_{\lambda=0} = -1$ are well given if and only if on shell of (3.5) the matrix $\mathcal{M}[F, G, \lambda]$ is symmetric and symplectic. They are equivalent to the constitutive relations (3.1) provided that on shell the relation between the $\mathcal{M}$ and $\mathcal{N}$ matrices is as in (3.4).

Notice that off shell of (3.5) the matrix $\mathcal{M}$ does not need to be symmetric and symplectic. This is what happens with Schrödinger's formulation of Born-Infeld theory (see (4.11) and comments thereafter).

3.2 Schrödinger's variables

Following Schrödinger [1, 34] it is fruitful to consider the complex variables

$$T = F - iG \ , \ \ \overline{T} = F + iG \ . \tag{3.6}$$

The transition from the real to the complex variables is given by the symplectic and unitary matrix $\mathcal{A}^t$ where

$$\mathcal{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \quad , \qquad \mathcal{A}^{-1} = \mathcal{A}^{\dagger} \ . \tag{3.7}$$

The equation of motions in these variables read $dT = 0$, with constitutive relations obtained applying the matrix $\mathcal{A}^t$ to (3.5):

$$\begin{pmatrix} {}^*T \\ {}^*\overline{T} \end{pmatrix} = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathcal{A}^t \mathcal{M} \overline{\mathcal{A}} \begin{pmatrix} T \\ \overline{T} \end{pmatrix} \ , \tag{3.8}$$

where $\mathcal{A}^t \mathcal{M} \overline{\mathcal{A}}$, on shell of (3.8), is complex symplectic and pseudounitary w.r.t the metric $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, i.e. it belongs to $Sp(2, \mathbb{C}) \cap U(1,1) = SU(1,1)$. It is also Hermitian and negative definite. These properties uniquely characterize the matrices $\mathcal{A}^t \mathcal{M} \overline{\mathcal{A}}$ as the matrices

$$\begin{pmatrix} -\sqrt{1 + \tau\overline{\tau}} & -i\tau \\ i\overline{\tau} & -\sqrt{1 + \tau\overline{\tau}} \end{pmatrix} \tag{3.9}$$

where $\tau = \tau[T, \overline{T}]$ is a complex field that depends on T, $\overline{T}$ and possibly also their derivatives. We then see that the constitutive relations (3.8) are equivalent to the equations

$$ {}^*T_{\mu\nu} = i\sqrt{1+\tau\overline{\tau}}\,T_{\mu\nu} - \tau\,\overline{T}_{\mu\nu}\ . \tag{3.10} $$

In conclusion the most general set of equations in the T variables that is well defined in the sense that it allows to express $G = \frac{i}{2}(T + \overline{T})$ in terms of $F = \frac{1}{2}(T + \overline{T})$ as in (3.1) (equivalently F in terms of G) is equivalent, on shell, to the equations (3.10) for a given $\tau = \tau[T, \overline{T}]$. In this sense equations (3.10) are the most general way of defining constitutive relations of electromagnetism. The constitutive relations (3.1) are determined by the complex function $\mathcal{N}$ (depending on F, G and their derivatives $\mathcal{N} = \mathcal{N}[F, G]$) the equivalent constitutive relations (3.10) are determined by the complex function τ (depending on $T, \overline{T}$ and their derivatives $\tau = \tau[T, \overline{T}]$).

4 Schrödinger's approach to self-duality conditions

In the previous section we have clarified the structure of the constitutive relations for an arbitrary nonlinear theory of electromagnetism. The theory can also be with higher derivatives of the field strength because the complex field $\mathcal{N}$, or equivalently the matrix $\mathcal{M}$ in (3.5) of (pseudo)scalar entries, can depend also on derivatives of the electric and magnetic field strengths F and G.

We now further examine the constitutive relations for theories that satisfy the NGZ self-duality condition

$$ F\widetilde{F} + G\widetilde{G} = 0\ , \tag{4.1} $$

i.e., $\overline{T}\widetilde{T} = 0$, or equivalently,

$$ \overline{T}\,{}^*T = 0\ . \tag{4.2} $$

We multiply (3.10) by ${}^*T$ and obtain

$$ -\,T^2 = i\sqrt{1+\tau\overline{\tau}}\,T\,{}^*T \tag{4.3} $$

It is convenient to consider modulus and argument of these complex scalar expressions. Setting

$$ T^2 = |T^2|e^{i\alpha} \tag{4.4} $$

from (4.3) we have

$$ T\,{}^*T = |T\,{}^*T|ie^{i\varphi} \tag{4.5} $$

We also contract (3.10) with ${}^*\overline{T}^{\mu\nu}$ and obtain $-T\overline{T} = -\tau\overline{T\,{}^*T}$ that implies

$$ |\tau| = \frac{T\overline{T}}{|\overline{T\,{}^*T}|}\ . \tag{4.6} $$

Use of (4.3) then gives the moduli relations

$$|T^2|^2 = |T\,^*T|^2 + (T\overline{T})^2 \ . \tag{4.7}$$

The constitutive relations (3.10) can also be rewritten using the chiral variables $T^{\pm} = T \pm i\,^*T$, they read

$$T^+_{\mu\nu} = te^{i\varphi}\overline{T^-}_{\mu\nu} \tag{4.8}$$

where $t = \frac{T\overline{T}}{|T^2|+|T^*T|}$. In order to obtain the explicit relation between the ratio $|\tau| = T\overline{T}/|T\,^*T|$ and t we calculate

$$|T^{-2}|(1-t^2) = \frac{1}{2}(|T^2| + |T\,^*T|)(1-t^2) = |T\,^*T| \ , \tag{4.9}$$

multiply this last equality by $|\tau|$ and obtain

$$(1-t^2)|\tau| = 2t \ . \tag{4.10}$$

Examples 2. *Linear electromagnetism* ($G = {}^*F$) corresponds to $|\tau| = 0$. *Born-Infeld nonlinear theory* satisfies the relations

$$^*T_{\mu\nu} = -\frac{T^2}{T\,^*T}T_{\mu\nu} - \frac{\lambda}{8}(T\,^*T)\,\overline{T}_{\mu\nu} \tag{4.11}$$

as remarked by Schrödinger [1], see [34] for a clear account in nowadays notations. Comparison with (3.10) shows that, on shell of (4.11) and (4.2), i.e. using (4.3) and (4.6), $\frac{T^2}{T^*T} = i\sqrt{1+\tau\overline{\tau}}$ and $\tau = \frac{\lambda}{8}T\,^*T$. Hence Born-Infeld theory is determined by

$$|\tau| = \frac{\lambda}{8}|T\,^*T| \ . \tag{4.12}$$

Schrödinger's formulation of Born-Infeld theory uses the freedom, dicussed in Proposition 1, of considering a matrix $\mathcal{M}$ that off shell of (3.5) is not symmetric and symplectic. Indeed the term $\frac{T^2}{T^*T}$ is not pure imaginary off shell. Schrödinger's elegant variational principle formulation of Born-Infeld constitutive relations is also due to this freedom. Defining the "Lagrangian" $\Upsilon(T) = \frac{4T^2}{T^*T}$ we have that (4.11) is equivalent to

$$\lambda^*\overline{T}^{\mu\nu} = \frac{\partial}{\partial T_{\mu\nu}}\Upsilon(T) \ . \tag{4.13}$$

5 Nonlinear theories without higher derivatives

We now consider theories (possibly in curved spacetime) that depend only on the (pseudo)scalars F^2 and $F\,^*F$, or T^{-2} and $\overline{T^-}^2$. Since the action functional $\mathcal{I}[T^-, \overline{T^-}]$

studied in Section 2.3 and the scalar field t defined in (4.8) are duality invariant, and under a duality of angle α we have the phase rotation $T^{-2} \to e^{2i\alpha}T^{-2}$, we conclude that $\mathcal{I}$ and t depend only on the modulus of T^{-2}, hence $\mathcal{I} = \mathcal{I}[T^-, \overline{T^-}]$ and $t = t[T^-, \overline{T^-}]$ simplify to

$$\mathcal{I} = \frac{1}{\lambda}\int d^4x\sqrt{g}\, I(u) \ , \quad t = t(u) \ , \tag{5.1}$$

where $I(u)$ is an adimensional scalar function, and the variable u is defined by

$$u \,\equiv\, 2\lambda|T^{-2}| \,=\, \lambda(|T^2| + |T^*T|) \ . \tag{5.2}$$

Similarly, the constitutive relations (2.26) simplify to

$$T^{+\mu\nu} = \frac{1}{\lambda}\frac{\partial I}{\partial \overline{T^-}_{\mu\nu}} = \frac{1}{\lambda}\frac{dI}{du}\,\frac{\partial u}{\partial \overline{T^-}_{\mu\nu}} \ , \tag{5.3}$$

and comparison with (4.8) leads to

$$t = 2\frac{dI}{du} \ . \tag{5.4}$$

[Hint: calculate $\frac{\partial u^2}{\partial \overline{T^-}_{\mu\nu}}$ and use $T^{-2} = |T^{-2}|e^{i\varphi}$].

5.1 Born-Infeld nonlinear theory

We determine the scalar field $t = t(u) = 2\frac{dI}{du}$ in case of Born-Infeld theory. This is doable thanks to Schrödinger's formulation (4.11) of Born-Infeld theory, that explicitly gives $|\tau| = \frac{\lambda}{8}|T^*T|$, see (4.12). Then from (4.9) we have

$$|\tau| = \frac{1}{16}u(1 - t^2) \ , \tag{5.5}$$

and recalling (4.10) we obtain [32, 33]

$$(1 - t^2)^2 u = 32t \ . \tag{5.6}$$

Now in the limit $u \to 0$, i.e., $\lambda \to 0$, we see from the definition of t that $t \to 0$. The function $t = t(u)$ defining Born-Infeld theory is then given by the unique positive root of the fourth order polynomial equation (5.6) that has the limit $t \to 0$ for $\lambda \to 0$. Explicitly,

$$t = \frac{1}{\sqrt{3}}\Big(\sqrt{1 + s + s^{-1}} - \sqrt{2 - s - s^{-1} + \frac{24\sqrt{3}}{u\sqrt{1 + s + s^{-1}}}}\ \Big) \ , \tag{5.7}$$

where

$$s = \frac{1}{u}\Big(216\,u + 12\sqrt{3}\sqrt{108 + u^2}\,u + u^3\Big)^{\frac{1}{3}} \ . \tag{5.8}$$

5.2 The hypergeometric function and its hidden identity

In [24] the action functional $\mathcal{I}$ and the function $t(u)$ corresponding to the Born-Infeld action were found via an iterative procedure order by order in λ (or equivalently in u). The first coefficients of the power series expansion of $t(u)$ were recognized to be those of a generalized hypergeometric function, leading to the conclusion

$$
\begin{aligned}
t(u) &= \frac{u}{32}\,{}_3F_2\Big(\frac{1}{2},\frac{3}{4},\frac{5}{4};\,\frac{4}{3},\frac{5}{3};\,-\frac{u^2}{3^3\cdot 2^2}\Big)\ , \\
&= \frac{2u}{32}\sum_{k=0}^{\infty}\frac{(4k+1)!}{(3k+2)!k!}\Big(-\frac{u^2}{4^5}\Big)^k
\end{aligned}
\tag{5.9}
$$

and, integrating (5.4),

$$
I(u) = 6\left(1-\,{}_3F_2\Big(-\frac{1}{2},-\frac{1}{4},\frac{1}{4};\,\frac{1}{3},\frac{2}{3};\,-\frac{u^2}{3^3\cdot 2^2}\Big)\right)\ . \tag{5.10}
$$

In [33] we conjectured, and checked up to order $O(u^{1000})$, that the expansion in power series of u of the closed form expression of $t(u)$ derived in (5.7), (5.8) coincides with the power series expansion in (5.9).

We here present a proof by showing that the power series in (5.9) satisfies the quartic equation (5.6). We consider the generic power series

$$
t = \sum_{m=1}^{\infty} a_m u^m \tag{5.11}
$$

with the initial condition $t = \mathcal{O}(u)$ for $u \to 0$, and determine the coefficients a_m so as to satisfy the quartic equation (5.6). The initial condition $t = \mathcal{O}(u)$ for $u \to 0$ is compatible with (5.6), indeed from (5.6) we see that for $u \to 0$ we have $t = \frac{u}{32}$.

We extend the variables t and u to the complex plane so that use of Cauchy's residue theorem gives

$$
a_m = \frac{1}{2\pi i}\oint_{C_0} t u^{-m-1} du \tag{5.12}
$$

We next calculate from (5.6) the differential

$$
du = 32 d\frac{t}{(1-t^2)^2} = 32\frac{1+3t^2}{(1-t^2)^3}dt\ , \tag{5.13}
$$

and observe that, since for $u \to 0$, $t = \mathcal{O}(u)$, infinitesimal closed paths surrounding the origin of the complex u-plane are mapped to infinitesimal ones surrounding the origin of the complex t-plane (that we still denote C_0). We hence obtain

$$
\begin{aligned}
a_m &= \frac{32}{2\pi i}\oint_{C_0}\frac{t+3t^3}{(1-t^2)^3}\frac{(1-t^2)^{2m+2}}{(32t)^{m+1}}dt\\
&= \frac{1}{32^m\,2\pi i}\oint_{C_0}(t^{-m}+3t^{2-m})(1-t^2)^{2m-1}dt\\
&= \frac{1}{32^m\,2\pi i}\oint_{C_0}(t^{-m}+3t^{2-m})\sum_{n=0}^{2m-1}(-1)^n t^{2n}\binom{2m-1}{n}dt\\
&= \frac{1}{32^m}\sum_{n=0}^{2m-1}(-1)^n\binom{2m-1}{n}(\delta_{2n-m+1,0}+3\delta_{2n-m+3,0})\ .
\end{aligned}
\tag{5.14}
$$

We see that only the coefficients a_m with m odd are nonvanishing, setting $m=2k+1$ we have

$$
\begin{aligned}
a_{2k+1} &= \frac{(-1)^k}{32^{2k+1}}\Big[\binom{4k+1}{k}-3\binom{4k+1}{k-1}\Big]\\
&= (-1)^k\frac{2}{32^{2k+1}}\frac{(4k+1)!}{(3k+2)!k!}
\end{aligned}
\tag{5.15}
$$

that proves the conjecture.

As a corollary we have that the hypergeometric function in (5.9)

$$
\mathfrak{F}(u^2)\equiv {}_3F_2\Big(\frac{1}{2},\frac{3}{4},\frac{5}{4};\frac{4}{3},\frac{5}{3};-\frac{u^2}{3^3\cdot 2^2}\Big)=2\sum_{k=0}^{\infty}\frac{(4k+1)!}{(3k+2)!k!}\Big(-\frac{u^2}{4^5}\Big)^k
\tag{5.16}
$$

has the closed form expression $\mathfrak{F}(u^2)=\frac{32}{u}t(u)$ where $t(u)$ is given in (5.7), (5.8), and, because of (5.6), that it satisfies the "hidden" identity

$$
\mathfrak{F}(u^2)=\Big(1-\frac{u^2}{4^5}\mathfrak{F}(u^2)^2\Big)^2\ .
\tag{5.17}
$$

5.3 General nonlinear theory

Since Born-Infeld theory is singled out by setting $|\tau|=\frac{\lambda}{8}|T\,{}^*T|$, and Maxwell theory by setting $|\tau|=0$ (cf. Example 2), it is convenient to describe a general nonlinear theory without higher derivatives by setting

$$
|\tau|=\frac{\lambda}{8}|T\,{}^*T|f(u)/u
\tag{5.18}
$$

where $f(u)$ is a positive function of u. We require the theory to reduce to electromagnetism in the weak field limit, i.e., ${}^*G_{\mu\nu}=-F+o(F)$ for $F\to 0$. Then we have

$T^- = \mathcal{O}(F)$, $T^+ = o(F)$, $u = \mathcal{O}(F^2)$. Hence from (4.8) we obtain $\lim_{u\to 0} t = 0$. Moreover from (4.10), $r = \mathcal{O}(t)$ and from $r = \frac{1}{16} f(u)(1-t^2)$ (that follows from (5.18) and (4.9)) $f = \mathcal{O}(t)$. Hence the theory reduces to electromagnetism in the weak field limit if and only if $\lim_{u\to 0} f(u) = 0$.

From $r = \frac{1}{16} f(u)(1-t^2)$ (that follows from (5.18) and (4.9)) and (4.10) we obtain that the composite function $t(f(u))$ satisfies the fourth order polynomial equation

$$(1-t^2)^2 f(u) = 32t \ , \tag{5.19}$$

so that $t(f(u))$ is obtained with the substitution $u \to f(u)$ in (5.7) and (5.8), or in (5.9).

More explicitly, generalizing the results of Example 2, we conclude, as in [33], that the constitutive relations à la Schrödinger

$$^*T_{\mu\nu} = -\frac{T^2}{T\,^*T} T_{\mu\nu} - \frac{\lambda}{8}\frac{f(u)}{u}\,(T\,^*T)\,\overline{T}_{\mu\nu} \ , \tag{5.20}$$

are (on shell) equivalent to the constitutive relations (deformed twisted self-duality conditions)

$$T^{+\mu\nu} = \frac{1}{2\lambda} t(f(u))\,\frac{\partial u}{\partial \overline{T^-}_{\mu\nu}} \ , \tag{5.21}$$

where $t(f(u))$ satisfies the quartic equation (5.19), and we recall that $u = 2\lambda |T^{-2}| = \lambda(|T^2| + |T\,^*T|)$.

In other words the appearence of the quartic equation (5.19) is a general feature of the relation between the constitutive relations (5.20) and (5.21), it appears for any self-dual theory and it is not only a feature of the Born-Infeld theory.

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