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One-loop four-point amplitudes in pure and matter-coupled $\mathcal{N} \leq 4$ supergravity

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Abstract

We construct *all* supergravity theories that can be obtained through factorized orbifold projections of $\mathcal{N} = 8$ supergravity, exposing their double-copy structure, and calculate their one-loop four-point scattering amplitudes. We observe a unified structure in both matter and gravity amplitudes, and demonstrate that the four-graviton amplitudes are insensitive to the precise nature of the matter couplings. We show that these amplitudes are identical for the two different realizations of $\mathcal{N} = 4$ supergravity with two vector multiplets, and argue that this feature extends to all multiplicities and loop orders as well as to higher dimensions. We also construct a selected set of supergravities obtained through a non-factorized orbifold action. Furthermore we calculate one-loop four-point amplitudes for all pure super-Yang-Mills theories with less-than-maximal supersymmetry using the duality between color and kinematics, finding here a unified expression that holds for all four gluon amplitudes in these theories. We recover the related amplitudes of factorized $\mathcal{N} \leq 4$ supergravities employing the double-copy construction. We observe a requirement that the four-point loop-level amplitudes have non-local integrand representations, exhibiting a mild non-locality in the form of inverse powers of the three external Mandelstam invariants. These are the first loop-level color-kinematic-satisfying representations in reduced supersymmetry theories.

1 Introduction

This last decade has marked tremendous progress in multi-loop scattering amplitude calculations, especially for maximally supersymmetric gauge and gravitational theories. This has been accompanied by the discovery of novel mathematical structures with potentially broad applicability such as integrability in the planar $\mathcal{N} = 4$ super Yang-Mills (sYM) theory [1], on-shell recursion [2], dual superconformal and Yangian symmetries [3], a Grassmannian description of the S-matrix [4], and the duality between color and kinematics [5]. Most notably, recent calculations have revealed a better-than-expected ultraviolet (UV) behavior for amplitudes in $\mathcal{N} = 8$ supergravity up to at least four loops [6, 7, 8, 9] which has in turn lead to the conjecture that $\mathcal{N} = 8$ supergravity may be perturbatively finite [10][1].

As new techniques to study amplitudes in maximally supersymmetric theories are developed, it is natural to ask whether they can lead to new insight and to the discovery of novel structures in theories with reduced supersymmetry. At the same time, the conjectured finiteness of $\mathcal{N} = 8$ supergravity raises the question of whether more phenomenologically viable theories of gravity could share similar UV properties.

Compactification of closed string theories on orbifolds [19] – smooth spaces modded out by some discrete group Γ – is a classic strategy for constructing four-dimensional models with reduced supersymmetry. The spectrum of light (massless) states relevant to the low-energy limit is composed of *untwisted* sector and *twisted* sector states. The former are the states of ten-dimensional string theory in flat space that are invariant under the action of Γ; the latter are the states of strings which would be open in a flat ten-dimensional space but are closed due to the action of the orbifold group. Since the mass of a string state is proportional to the length of the string, if the action of Γ has fixed points (and therefore the orbifold space is singular) the twisted sector has massless states localized at the singularities; otherwise twisted sector states are massive and not directly relevant to the low-energy limit. However, in all cases twisted sectors are crucial for the consistency of the string construction, in particular for its modular invariance.

Twisted sector states carry charges under the quantum symmetry[2] of the orbifold. Thus, in the field theory limit, they may be consistently truncated away[3]. This leads to a field theory of the Γ-invariant fields of ten-dimensional string theory in flat ten-dimensional

[1] Subsequent duality arguments have also been used to show that candidate counterterms for $\mathcal{N} = 8$ supergravity cannot appear until seven loops [11, 12, 13, 14], as well as to argue in favor of the conjectured perturbative finiteness of the theory [15, 16, 17], although subtleties with the decoupling of massive string states may alter this conclusion [18].

[2] If the orbifold group is Abelian, the quantum symmetry group of the orbifold string theory is a group isomorphic to Γ.

[3] The quantum symmetry guarantees that they appear at least bilinearly in the low-energy effective action and thus setting them to zero gives a consistent solution to the nonlinear classical equations of motion.

Minkowski space, and to an action that is the truncation of the effective action of the appropriate ten-dimensional supergravity[4].

The same action may be constructed by starting directly with the ten-dimensional supergravity theory and projecting out all states which are not invariant under a discrete Abelian subgroup Γ of the $SU(8)$ R-symmetry group[5]. Invoking a mild indulgence of nomenclature, these constructions will be referred to as supergravity orbifolds throughout the paper. Similar constructions have been discussed extensively for open string theories [20] where they lead to vast classes of quiver gauge theories whose planar limits have special properties.

The UV properties of gravity amplitudes in theories with $\mathcal{N} \geq 4$ supersymmetry have come under recent scrutiny [21, 22, 23, 24]. In particular, the authors of ref. [23] have identified startling[6] cancellations rendering four-graviton amplitudes in pure $\mathcal{N} = 4$ supergravity finite through three loops. One-loop four-graviton amplitudes in two specific string theory orbifold constructions – referred to as the $(2, 2)$ and $(4, 0)$ models – have been discussed in [26]; very recently it was shown there that, in the field theory limit, both the $(2, 2)$ and $(4, 0)$ orbifold string models lead to the same one-loop four-graviton amplitude. It was nevertheless suggested that amplitudes with external vector and scalar fields are different in the two constructions. We will find this not to be the case and argue that all one-loop four-point amplitudes (and quite likely all-loop all-multiplicity amplitudes) in these theories are the same.

Generically, an orbifold supergravity theory has reduced (or no) supersymmetry and has matter supermultiplets in addition to the gravity multiplet. Matter-coupled supergravity theories, in particular in the presence of vector multiplets, are expected to be divergent already at one loop due to the existence of a candidate counterterm of the form

$$T_M^{\mu\nu} T_{M\mu\nu} \ , \tag{1.1}$$

where T_M is the matter energy-momentum tensor. However, so far explicit computations have been carried out only in theories with $\mathcal{N} = 4$ supersymmetry and an arbitrary number of vector multiplets [27, 28, 24].

It has been known since the late 1980s that the semi-classical (tree-level) S-matrix of Yang-Mills theory encodes all necessary information to generate the semi-classical S-matrix of gravity. This was first demonstrated in the tree-level relations of Kawai, Lewellen and Tye (KLT) [34] which express gravity tree-level amplitudes as the sum over permutations of products between two color-ordered (or color-stripped) gauge theory amplitudes. This double-copy structure was clarified recently by Bern, Johansson, and one

[4]Alternatively, if a geometric resolution of the (generically asymmetric) orbifold singularities is possible, it will generate a nonzero mass for the twisted sector states and thus remove them from the low-energy effective action.

[5]This construction guarantees that the truncation of ten-dimensional supergravity is consistent.

[6]See refs. [25] for recent analysis and proposed explanations.

of the current authors in [5], where it was realized that gauge theory amplitudes can be arranged in a graph-organized representation[7] so as to make manifest a duality between their color and kinematic factors (modulo scalar propagators). Related gravity theories are then trivially given in the same graph organization but with the color factors replaced by another copy of gauge theory kinematic factors. This duality and its associated double-copy construction are expected to hold at loop level [29] for all gauge theories and related gravity theories that can be organized around scalar graphs.

The duality between color and kinematics, as well as the associated double-copy structure of the related gravity theories, such as $\mathcal{N}=8$ supergravity, provide powerful tools for the study of orbifold supergravity theories. A particularly relevant case is when the orbifold group Γ is the product of two groups each acting on a different $SU(4)$ factor of the $SU(4)_R \times SU(4)_L \subset SU(8)$ symmetry which is manifest in the KLT relations; such *factorizable* supergravity orbifolds of $\mathcal{N}=8$ supergravity can be expressed as double-copies of suitable orbifolds of $\mathcal{N}=4$ sYM theory.

With this paper, we present a systematic study of one-loop amplitudes in orbifold supergravity theories as a concrete step in the search for supergravity theories with reduced supersymmetry that are finite at one loop and beyond. In particular, we focus on orbifolds of $\mathcal{N}=8$ supergravity which preserve $1 \leq \mathcal{N} \leq 4$ supersymmetry and have various matter multiplets. We study both gravity and (when applicable) matter amplitudes, and find that all the factorizable orbifold theories with matter have divergent one-loop amplitudes when the external states are taken in the same matter supermultiplet. In contrast, pure (matter-free) orbifold supergravity theories with $\mathcal{N} \leq 4$ are finite at one loop, as expected on general grounds[8].

The structure of the paper is as follows. In section 2, we review orbifolds of $\mathcal{N}=8$ supergravity and present an exhaustive list of factorizable orbifolds together with some non-factorizable examples. We work out the maps between gauge theory and supergravity theory states for the factorizable orbifolds, extending some results already in the literature and obtained by other means – see [31] and references therein. These maps can be used to construct all loop-level amplitudes using unitarity methods [32, 33] and the KLT tree-level relations [34], or directly through the double-copy construction when gauge theory representations that make manifest the duality between color and kinematics are available [5, 29]. For all theories, we identify the moduli space. We give particular attention to two specific theories, namely $\mathcal{N}=4$ Maxwell-Einstein supergravity (MESGT)[9] with two vector multiplets and $\mathcal{N}=2$ Maxwell-Einstein supergravity with seven vector

[7] Cubic graphs for D-dimensional Yang-Mills theories [5, 29], but quartic graph organizations have made manifest such a duality [30] for (superconformal) Chern-Simons-matter theories in three dimensions.

[8] As we will see, with the exception of $\mathcal{N}=4$ supergravity, such pure reduced supersymmetry theories can be obtained only as non-factorizable orbifolds.

[9] Maxwell-Einstein supergravity theories (MESGT) describe the coupling of supergravity to vector multiplets.

multiplets. They form a twin pair of supergravity theories that have the same bosonic field content but different number of supersymmetries and hence different fermionic field contents. They can be obtained by consistent truncations of the twin pair of pure $\mathcal{N}=6$ supergravity and $\mathcal{N}=2$ magical MESGT with 15 vector multiplets [35]. Both the pure $\mathcal{N}=6$ supergravity and the magical MESGT with 15 vector multiplets were obtained as four-dimensional low-energy effective theories of type II superstrings on orbifolds by Sen and Vafa in their study of dual pairs of compactifications [36, 37].

Magical supergravity theories are a very special class of $\mathcal{N}=2$ MESGT whose global symmetry groups in five, four and three dimensions coincide with the symmetry groups of the famous Magic Square of Freudenthal-Rozenfeld and Tits. In five dimensions they are the only *unified* MESGTs with symmetric scalar manifolds and are uniquely defined by the four simple Euclidean Jordan algebras $J_3^{\mathbb{A}}$ of 3×3 Hermitian matrices over the four division algebras $\mathbb{A}$ (real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$ and octonions $\mathbb{O}$). They describe the coupling of 5 (6), 8 (9), 14 (15) and 26 (27) vector multiplets to $\mathcal{N}=2$ supergravity in five (four) dimensions, respectively [35, 39]. Of these magical supergravities, the quaternionic theory defined by $J_3^{\mathbb{H}}$ has the same bosonic field content as the pure $\mathcal{N}=6$ supergravity [35].

Using four-dimensional generalized unitarity [32, 33], in section 3 we construct the one-loop four-point amplitudes of the various orbifold theories. In particular, we consider four-graviton amplitudes and matter amplitudes of the form $F\bar{F}\to F\bar{F}$ and $\phi\bar{\phi}\to\phi\bar{\phi}$. We find that, after the integral reduction is performed, all amplitudes differ from the ones of $\mathcal{N}=8$ supergravity by additional terms proportional to four-dimensional bubble integrals and one particular six-dimensional box integral, all multiplied by rational functions in the Mandelstam variables. For theories that may be obtained by dimensional reduction from six dimensions, we comment on their potential additional rational terms from the perspective of the six-dimensional spinor-helicity formalism.

In section 4 we discuss the color/kinematics duality and find representations of one-loop four-point amplitudes for gauge theory orbifolds with $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetry where the duality is manifest. We demonstrate that the factorized supergravity amplitudes listed in section 2 can be obtained as double-copies of suitable gauge amplitudes. We conclude the paper with a discussion of open questions and directions for further research.

2 Supergravity orbifolds and truncations of $\mathcal{N}=8$ supergravity

Eleven-dimensional supergravity and type IIB supergravity reduce to ungauged maximal supergravity theories in lower dimensions, $D\leq 9$, under dimensional reduction, which corresponds simply to compactification on tori followed by consistent truncation to the massless sector. The resulting maximal supergravity theory in D dimensions has a global symmetry group G_D and its scalar fields parametrize the symmetric space $M_D=\frac{G_D}{K_D}$ where

K_D is the maximal compact subgroup of G_D. Eleven-dimensional supergravity reduces to type IIA supergravity in $D = 10$ under dimensional reduction. The product manifolds of the D-dimensional Minkowski spaces Min_D with the tori $T^{(10-D)}$ provide consistent backgrounds for type II string theories whose low-energy effective theories are simply the maximal supergravity theories in D-dimensions. Type IIA and IIB superstring theories compactified over tori are related via T-duality, which in the low-energy limit is only a field redefinition. The eleven-dimensional supergravity theory arises as the low-energy effective theory of a strongly-coupled phase of superstring theory [38].

The fields of maximal supergravity theories in various dimensions and their symmetry groups are listed in Table 1.

Orbifolds, obtained from modding out smooth manifolds by a discrete symmetry group Γ, can also provide consistent backgrounds for string theories [19]. Low-energy effective theories of type II superstrings on orbifolds of tori $T^{(10-D)}$ can further be consistently truncated – as field theories – to the massless fields coming from the *untwisted* sector. The resulting theories are equivalent to consistent truncations of the D-dimensional maximal supergravity to theories with fewer supersymmetries and fields that are invariant under the action of Γ. We will refer to them as *supergravity orbifolds*, in analogy with the similar gauge theory constructions [20].

Our main goal is to analyze – from the perspective of their scattering amplitudes – a large set of supergravity theories that are orbifold projections of $\mathcal{N} = 8$ supergravity. We will use the KLT relations to describe the tree-level amplitudes of these theories as sums of products of color-stripped tree-level amplitudes of sYM theories with less-than-maximal supersymmetry. The map of states relevant to maximal supergravity [34] uses the $SU(4)_L \times SU(4)_R$ basis of the R-symmetry group $SU(8)$ of four-dimensional maximal supergravity. Each $SU(4)$ factor is identified with the R-symmetry group of an $\mathcal{N} = 4$ sYM theory. The orbifold projection may be chosen to act either independently on the two factors or in a correlated fashion. As usual, we will refer to them as *factorizable* and *non-factorizable* orbifolds, respectively, and focus mainly on the factorizable ones. There exists a limited set of relevant gauge theories which can be obtained as orbifold projections of $\mathcal{N} = 4$ sYM theory by some discrete group Γ and have only fields in the adjoint representation of $SU(N)$[10]:

1. A unique $\Gamma = \mathbf{Z}_2$ orbifold preserving $\mathcal{N} = 2$ supersymmetry where Γ is generated by $\mathrm{diag}(1, 1, -1, -1)$. The theory contains one vector, two fermions and two real scalars or, equivalently, one $\mathcal{N} = 2$ vector multiplet.

2. A unique $\Gamma = \mathbf{Z}_3$ orbifold preserving $\mathcal{N} = 1$ supersymmetry where Γ is generated by $\mathrm{diag}(1, e^{\frac{2}{3}\pi i}, e^{\frac{2}{3}\pi i}, e^{\frac{2}{3}\pi i})$. The theory contains one vector, one fermion and no scalars,

[10] Of course, there exists a wide choice of orbifold theories which have fields in bifundamental representation [20]; the main difference between those orbifolds and the ones discussed here is that *here* the action of the orbifold group on the gauge degrees of freedom is taken to be trivial.

Dimension	Field Content	$M_D = G_D/K_D$
$D=6$	$g_{\mu\nu}, 4\psi_\mu, 5B_{\mu\nu}, 16A_\mu, 20\lambda, 25\phi$	$SO(5,5)/\big(SO(5)\times SO(5)\big)$
$D=5$	$g_{\mu\nu}, 8\psi_\mu, 27A_\mu, 48\lambda, 42\phi$	$E_{6(6)}/USp(8)$
$D=4$	$g_{\mu\nu}, 8\psi_\mu, 28A_\mu, 56\lambda, 70\phi$	$E_{7(7)}/SU(8)$
$D=3$	$128\lambda, 128\phi$	$E_{8(8)}/SO(16)$

Table 1: Fields of maximal Poincaré supergravity in various dimensions and their scalar manifolds.

that is a single $\mathcal{N}=1$ vector multiplet.

3. A non-supersymmetric $\Gamma = \mathbf{Z}_2$ orbifold generated by $-I_4$. The theory contains one vector, six real scalars and no fermions.

4. A non-supersymmetric $\Gamma = \mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold generated by $\mathrm{diag}(1,1,-1,-1)$ and $\mathrm{diag}(-1,-1,1,1)$. This theory has one vector, two real scalars and no fermions.

5. A non-supersymmetric $\Gamma = \mathbf{Z}_4$ orbifold generated by $e^{\frac{\pi}{2}i} I_4$. This theory contains one vector and no other fields.

6. A non-supersymmetric $\Gamma = \mathbf{Z}_4$, which we shall denote as $\tilde{\mathbf{Z}}_4$ orbifold, generated by $\mathrm{diag}(i,-i,-i,i)$. This theory contains one vector, four real scalars and no fermions.

In Table 2 we list factorizable orbifold groups of the form $\mathbf{Z_L} \times \mathbf{Z_R}$, where $\mathbf{Z_L}(\mathbf{Z_R})$ is a discrete subgroup of $SU(4)_L(SU(4)_R)$, together with the massless field content of the corresponding supergravity orbifolds. [11]

Other choices of $\mathbf{Z_L}$ and $\mathbf{Z_R}$ do not lead to additional new theories as long as their action on the gauge indices of the fields of $\mathcal{N}=4$ sYM is trivial; therefore, the orbifold supergravities in Table 2 are all the factorizable ones that preserve at least minimal amount of supersymmetry.

As an example, we work out the massless field content of the $\mathcal{N}=4$ theory with two vector multiplets on the fifth row of Table 2. For this theory, the $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold group is generated by the elements[12] $\mathrm{diag}(1,1,-1,-1,1,1,1,1)$ and $\mathrm{diag}(1,1,1,1,1,1,-1,-1)$. Aside from the graviton, which is unaffected by the projection, four gravitini, namely ψ^a_μ

[11] By taking non-supersymmetric orbifold groups for both $\mathbf{Z_L}$ and $\mathbf{Z_R}$ one could also construct non-supersymmetric gravity theories with various amounts of matter. We will however restrict ourselves to supergravity theories with at least $\mathcal{N}=1$ supersymmetry.

[12] Let us note that, from the perspective of the $SU(8)$ R-symmetry group of $\mathcal{N}=8$ supergravity, the position of the negative entries in the generator of $\mathbf{Z}_2 \times \mathbf{Z}_2$ is given by a particular choice of labels of the $SU(8)$ Cartan generators, which is completely arbitrary.

with $a = 1, 2, 5, 6$, are left in the orbifold theory. The theory also has six vector fields which are part of the gravity multiplet, A_μ^{ab} with $a, b = 1, 2, 5, 6$, and two additional matter vectors, A_μ^{34} and A_μ^{78}. Moreover, a total of twelve spin-$\frac{1}{2}$ fields survive the projection, namely l^{abc}, l^{a34} and λ^{a78} with $a, b, c = 1, 2, 5, 6$. The theory is completed by a total of fourteen real scalars of the form $\phi^{1256}, \phi^{ab34}, \phi^{ab78}$ and ϕ^{3478} with $a, b = 1, 2, 5, 6$. It is immediate to verify that all other fields are not invariant under at least one of the two generators of the orbifold group.

$\mathcal{N}$	**Orbifold**	**Factors**	**Massless Fields**	**Matter**
6	$\mathbf{Z}_2$	$\mathcal{N}=4 \times \mathcal{N}=2\vert_{\mathbf{Z}_2}$	$g_{\mu\nu}, 6\psi_\mu, 16A_\mu, 26\lambda, 30\phi$	pure
5	$\mathbf{Z}_3$	$\mathcal{N}=4 \times \mathcal{N}=1\vert_{\mathbf{Z}_3}$	$g_{\mu\nu}, 5\psi_\mu, 10A_\mu, 11\lambda, 10\phi$	pure
4	$\mathbf{Z}_2$	$\mathcal{N}=4 \times \mathcal{N}=0\vert_{\mathbf{Z}_2}$	$g_{\mu\nu}, 4\psi_\mu, 12A_\mu, 28\lambda, 38\phi$	6 vectors
4	$\tilde{\mathbf{Z}}_4$	$\mathcal{N}=4 \times \mathcal{N}=0\vert_{\tilde{\mathbf{Z}}_4}$	$g_{\mu\nu}, 4\psi_\mu, 10A_\mu, 20\lambda, 26\phi$	4 vectors
4	$\mathbf{Z}_2 \times \mathbf{Z}_2$	$\mathcal{N}=4 \times \mathcal{N}=0\vert_{\mathbf{Z}_2 \times \mathbf{Z}_2}$	$g_{\mu\nu}, 4\psi_\mu, 8A_\mu, 12\lambda, 14\phi$	2 vectors
4	$\mathbf{Z}_2 \times \mathbf{Z}_2$	$\mathcal{N}=2\vert_{\mathbf{Z}_2} \times \mathcal{N}=2\vert_{\mathbf{Z}_2}$	$g_{\mu\nu}, 4\psi_\mu, 8A_\mu, 12\lambda, 14\phi$	2 vectors
4	$\mathbf{Z}_4$	$\mathcal{N}=4 \times \mathcal{N}=0\vert_{\mathbf{Z}_4}$	$g_{\mu\nu}, 4\psi_\mu, 6A_\mu, 4\lambda, 2\phi$	pure
3	$\mathbf{Z}_2 \times \mathbf{Z}_3$	$\mathcal{N}=2\vert_{\mathbf{Z}_2} \times \mathcal{N}=1\vert_{\mathbf{Z}_3}$	$g_{\mu\nu}, 3\psi_\mu, 4A_\mu, 5\lambda, 6\phi$	1 vector
2	$\mathbf{Z}_2 \times \mathbf{Z}_2$	$\mathcal{N}=2\vert_{\mathbf{Z}_2} \times \mathcal{N}=0\vert_{\mathbf{Z}_2}$	$g_{\mu\nu}, 2\psi_\mu, 8A_\mu, 14\lambda, 14\phi$	7 vectors
2	$\mathbf{Z}_2 \times \tilde{\mathbf{Z}}_4$	$\mathcal{N}=2\vert_{\mathbf{Z}_2} \times \mathcal{N}=0\vert_{\tilde{\mathbf{Z}}_4}$	$g_{\mu\nu}, 2\psi_\mu, 6A_\mu, 10\lambda, 10\phi$	5 vectors
2	$\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$	$\mathcal{N}=2\vert_{\mathbf{Z}_2} \times \mathcal{N}=0\vert_{\mathbf{Z}_2 \times \mathbf{Z}_2}$	$g_{\mu\nu}, 2\psi_\mu, 4A_\mu, 6\lambda, 6\phi$	3 vectors
2	$\mathbf{Z}_2 \times \mathbf{Z}_4$	$\mathcal{N}=2\vert_{\mathbf{Z}_2} \times \mathcal{N}=0\vert_{\mathbf{Z}_4}$	$g_{\mu\nu}, 2\psi_\mu, 2A_\mu, 2\lambda, 2\phi$	1 vector
2	$\mathbf{Z}_3 \times \mathbf{Z}_3$	$\mathcal{N}=1\vert_{\mathbf{Z}_3} \times \mathcal{N}=1\vert_{\mathbf{Z}_3}$	$g_{\mu\nu}, 2\psi_\mu, 1A_\mu, 2\lambda, 4\phi$	1 hyper
1	$\mathbf{Z}_3 \times \mathbf{Z}_2$	$\mathcal{N}=1\vert_{\mathbf{Z}_3} \times \mathcal{N}=0\vert_{\mathbf{Z}_2}$	$g_{\mu\nu}, 1\psi_\mu, 6A_\mu, 7\lambda, 2\phi$	6 vecs, 1 chiral
1	$\mathbf{Z}_3 \times \tilde{\mathbf{Z}}_4$	$\mathcal{N}=1\vert_{\mathbf{Z}_3} \times \mathcal{N}=0\vert_{\tilde{\mathbf{Z}}_4}$	$g_{\mu\nu}, \psi_\mu, 4A_\mu, 5\lambda, 2\phi$	4 vecs, 1 chiral
1	$\mathbf{Z}_3 \times \mathbf{Z}_2 \times \mathbf{Z}_2$	$\mathcal{N}=1\vert_{\mathbf{Z}_3} \times \mathcal{N}=0\vert_{\mathbf{Z}_2 \times \mathbf{Z}_2}$	$g_{\mu\nu}, 1\psi_\mu, 2A_\mu, 3\lambda, 2\phi$	2 vecs, 1 chiral
1	$\mathbf{Z}_3 \times \mathbf{Z}_4$	$\mathcal{N}=1\vert_{\mathbf{Z}_3} \times \mathcal{N}=0\vert_{\mathbf{Z}_4}$	$g_{\mu\nu}, 1\psi_\mu, 1\lambda, 2\phi$	1 chiral

Table 2: Factorizable orbifold groups and massless field content of the low-energy four-dimensional supergravities coming from the untwisted sectors of the corresponding string theory orbifolds. The list includes all the supergravity theories that can be obtained though a factorized orbifold group action. The scalar fields in the fourth column are real.

In Table 3 we list the number of supersymmetries coming from left and right sectors of four-dimensional orbifold supergravity theories and the corresponding moduli spaces, which are all submanifolds of $E_{7(7)}/SU(8)$.

We should note that there also exist orbifold theories which are non-factorizable. These theories are largely outside of the scope of this paper. A partial list can be found in Table 4.

#	$\mathcal{N}$	**Factors**	**Scalar Manifold**	**Supergravity**
1	6	$\mathcal{N}=4\times\mathcal{N}=2\vert_{\mathbf{Z}_2}$	$\frac{SO^*(12)}{U(6)}$	pure $\mathcal{N}=6$ sugra
2	5	$\mathcal{N}=4\times\mathcal{N}=1\vert_{\mathbf{Z}_3}$	$\frac{SU(5,1)}{U(5)}$	pure $\mathcal{N}=5$ sugra
3	4	$\mathcal{N}=4\times\mathcal{N}=0\vert_{\mathbf{Z}_2}$	$\frac{SO(6,6)\times SU(1,1)}{SO(6)\times SO(6)\times U(1)}$	$\mathcal{N}=4$ sugra, 6 vector multiplets
4	4	$\mathcal{N}=4\times\mathcal{N}=0\vert_{\tilde{\mathbf{Z}}_4}$	$\frac{SO(6,4)\times SU(1,1)}{U(4)\times SO(4)}$	$\mathcal{N}=4$ sugra, 4 vector multiplets
5	4	$\mathcal{N}=4\times\mathcal{N}=0\vert_{\mathbf{Z}_2\times\mathbf{Z}_2}$	$\frac{SO(6,2)\times SU(1,1)}{SO(6)\times SO(2)\times U(1)}$	$\mathcal{N}=4$ sugra, 2 vector multiplets
6	4	$\mathcal{N}=2\vert_{\mathbf{Z}_2}\times\mathcal{N}=2\vert_{\mathbf{Z}_2}$	$\frac{SO(6,2)\times SU(1,1)}{SO(6)\times SO(2)\times U(1)}$	$\mathcal{N}=4$ sugra, 2 vector multiplets
7	4	$\mathcal{N}=4\times\mathcal{N}=0\vert_{\mathbf{Z}_4}$	$\frac{SU(1,1)}{U(1)}$	pure $\mathcal{N}=4$ sugra
8	3	$\mathcal{N}=2\vert_{\mathbf{Z}_2}\times\mathcal{N}=1\vert_{\mathbf{Z}_3}$	$\frac{SU(3,1)}{U(3)}$	$\mathcal{N}=3$ sugra, 1 vector multiplet
9	2	$\mathcal{N}=2\vert_{\mathbf{Z}_2}\times\mathcal{N}=0\vert_{\mathbf{Z}_2}$	$\frac{SO(6,2)\times SU(1,1)}{SO(6)\times SO(2)\times U(1)}$	$\mathcal{N}=2$ sugra, 7 vector multiplets
10	2	$\mathcal{N}=2\vert_{\mathbf{Z}_2}\times\mathcal{N}=0\vert_{\tilde{\mathbf{Z}}_4}$	$\frac{SO(4,2)\times SU(1,1)}{SO(4)\times SO(2)\times U(1)}$	$\mathcal{N}=2$ sugra, 5 vector multiplets
11	2	$\mathcal{N}=2\vert_{\mathbf{Z}_2}\times\mathcal{N}=0\vert_{\mathbf{Z}_2\times\mathbf{Z}_2}$	$\frac{SU(1,1)\times SU(1,1)\times SU(1,1)}{U(1)\times U(1)\times U(1)}$	$\mathcal{N}=2$ sugra, 3 vector multiplets
12	2	$\mathcal{N}=2\vert_{\mathbf{Z}_2}\times\mathcal{N}=0\vert_{\mathbf{Z}_4}$	$\frac{SU(1,1)}{U(1)}$	$\mathcal{N}=2$ sugra, 1 vector multiplet
13	2	$\mathcal{N}=1\vert_{\mathbf{Z}_3}\times\mathcal{N}=1\vert_{\mathbf{Z}_3}$	$\frac{SU(2,1)}{SU(2)\times U(1)}$	$\mathcal{N}=2$ sugra, 1 hypermultiplet
14	1	$\mathcal{N}=1\vert_{\mathbf{Z}_3}\times\mathcal{N}=0\vert_{\mathbf{Z}_2}$	$\frac{SU(1,1)}{U(1)}$	$\mathcal{N}=1$ sugra, 6 vector and 1 chiral multiplets
15	1	$\mathcal{N}=1\vert_{\mathbf{Z}_3}\times\mathcal{N}=0\vert_{\tilde{\mathbf{Z}}_4}$	$\frac{SU(1,1)}{U(1)}$	$\mathcal{N}=1$ sugra, 4 vector and 1 chiral multiplets
16	1	$\mathcal{N}=1\vert_{\mathbf{Z}_3}\times\mathcal{N}=0\vert_{\mathbf{Z}_2\times\mathbf{Z}_2}$	$\frac{SU(1,1)}{U(1)}$	$\mathcal{N}=1$ sugra, 2 vector and 1 chiral multiplets
17	1	$\mathcal{N}=1\vert_{\mathbf{Z}_3}\times\mathcal{N}=0\vert_{\mathbf{Z}_4}$	$\frac{SU(1,1)}{U(1)}$	$\mathcal{N}=1$ sugra, 1 chiral multiplets

Table 3: Factorizable $\mathcal{N}\geq 1$ supergravity orbifolds and their corresponding scalar manifolds. The second column lists the total number of preserved $D=4$ supersymmetry. The third column lists the number of supersymmetries coming from the different gauge factors. The fifth column lists the supergravity theories and the fourth column their scalar manifolds. Note that the theories with $\mathcal{N}=2,3,4$ are ungauged Maxwell-Einstein Supergravity Theories (MESGTs).

Dual pairs of type II string compactifications on orbifolds were studied by Sen and Vafa [36], who constructed examples of dual pairs with $\mathcal{N}=2$, $\mathcal{N}=4$ and $\mathcal{N}=6$ supersymmetry. Among the four-dimensional $\mathcal{N}=2$ theories obtained in this way, there is one with fifteen vector multiplets. As it was pointed out in [37], this $\mathcal{N}=2$ MESGT

$\mathcal{N}$	**Orbifold**	**Generators**	**Field Content**	**Matter**
3	$\mathbf{Z}_5$	$I_3 \times e^{\frac{2}{5}\pi i} I_5$	$g_{\mu\nu}, 3\psi_\mu, 3A_\mu, 1\lambda$	pure
2	$\mathbf{Z}_6$	$I_2 \times e^{\frac{1}{3}\pi i} I_6$	$g_{\mu\nu}, 2\psi_\mu, 1A_\mu$	pure
2	$\mathbf{Z}_3$	$I_2 \times e^{\frac{2}{3}\pi i} I_6$	$g_{\mu\nu}, 2\psi_\mu, 1A_\mu, 20\lambda, 40\phi$	10 hypers
2	$\mathbf{Z}_2$	$I_2 \times e^{\pi i} I_6$	$g_{\mu\nu}, 2\psi_\mu, 16A_\mu, 30\lambda, 30\phi$	15 vectors
1	$\mathbf{Z}_7$	$I_1 \times e^{\frac{2}{7}\pi i} I_7$	$g_{\mu\nu}, 1\psi_\mu$	pure
1	$\mathbf{Z}_5 \times \mathbf{Z}_2$	$I_1 \times (-I_2) \times I_5;\ I_3 \times e^{\frac{2}{5}\pi i} I_5$	$g_{\mu\nu}, 1\psi_\mu, 1A_\mu, 1\lambda$	1 vector

Table 4: Examples of non-factorizable supergravity orbifolds preserving some supersymmetry. The third column lists the orbifold group generators and the $SU(8)$ indices they act on, up to conjugation; for example, $I_3 \times e^{\frac{2}{5}\pi i} I_5$ generates a $\mathbf{Z}_5$ group that acts on five of the eight values of an $SU(8)$ fundamental representation index. The $\mathcal{N} = 2$ theory with 10 hypermultiplets has the moduli space $\frac{E_{6(2)}}{SU(6)\times SU(2)}$ and the $\mathcal{N} = 2$ theory with 15 vector multiplets has the same moduli space as the $\mathcal{N} = 6$ supergravity, namely $\frac{SO^*(12)}{U(6)}$.

is simply the $\mathcal{N} = 2$ magical supergravity theory defined by the Jordan algebra $J_3^{\mathbb{H}}$ of 3×3 Hermitian matrices over the quaternions [35, 39]. This quaternionic $\mathcal{N} = 2$ magical supergravity and $\mathcal{N} = 6$ supergravity have the same bosonic field content with the scalar manifold $SO^*(12)/U(6)$, and hence form a twin pair of supergravities [35][13]. $\mathcal{N} = 6$ supergravity is obtained as a factorized orbifold (the first row in Table 3) while the quaternionic magical supergravity is obtained as a non-factorizable orbifold (fourth row in Table 4). We should also note that under dimensional reduction $\mathcal{N} = 6$ supergravity and the magical quaternionic MESGT yield twin supergravities in three dimensions with $\mathcal{N} = 12$ and $\mathcal{N} = 4$ supersymmetries and the same scalar manifold, namely $\frac{E_{7(-5)}}{SO(12)\times SU(2)}$.

The $\mathcal{N} = 6$ supergravity and the magical supergravity defined by $J_3^{\mathbb{H}}$ can be truncated to $\mathcal{N} = 4$ and $\mathcal{N} = 2$ MESGTs that also form a twin pair with the same bosonic field content. The resulting twin theories are $\mathcal{N} = 4$ supergravity coupled to two vector multiplets and $\mathcal{N} = 2$ supergravity coupled to seven vector multiplets and have the same scalar manifold, $\frac{SO(6,2)\times SU(1,1)}{U(4)\times U(1)}$. The $\mathcal{N} = 4$ MESGT with two vector multiplets can be obtained as a factorized orbifold theory in two different ways, which are listed in rows five and six of Table 3. The corresponding twin $\mathcal{N} = 2$ MESGT with seven vector multiplets is listed in row nine of the same table. When reduced to three dimensions they yield a twin pair of $\mathcal{N} = 8$ and $\mathcal{N} = 4$ supergravity theories, respectively, with the same scalar manifold $\frac{SO(8,4)}{SO(8)\times SO(4)}$.

We should note that the magical $\mathcal{N} = 2$ supergravity defined by the quaternionic Jordan algebra $J_3^{\mathbb{H}}$ can be further truncated to the magical supergravity theory defined by complex Jordan algebra $J_3^{\mathbb{C}}$ with nine vector multiplets and scalar manifold $\frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}$

[13] Both the quaternionic magical MESGT and the $\mathcal{N} = 6$ supergravity are self-dual as type II compactifications in the sense of S and T duality exchange [36].

and to the magical supergravity defined by real Jordan algebra $J_3^{\mathbb{R}}$ with six vector multiplets and scalar manifold $\frac{Sp(6,\mathbb{R})}{U(3)}$. None of the magical supergravity theories involves any hypermultiplets.

One can go further and truncate the smallest magical supergravity theory to the STU model which describes the coupling of three $\mathcal{N}=2$ vector multiplets to supergravity in four dimensions. The STU model has the global symmetry group $SU(1,1)^3$ under which the vector field strengths and their magnetic duals transform in the eight-dimensional representation $(1/2,1/2,1/2)$. The STU model coupled to four hypermultiplets is one of the theories obtained by Sen and Vafa by orbifolding [36].

In light of the above comments, we should stress that every toroidal orbifold compactification of type II superstring theory leads to a supergravity orbifold of maximal supergravity involving the massless fields of the untwisted sector. At the supergravity level one can consistently truncate these theories further by discarding certain fields. However, the truncated theory may generally not be the low-energy effective theory of the massless untwisted sector of some orbifold of type II superstring. For example, ref. [40] lists pure supergravity theories in various dimensions that cannot be obtained by toroidal orbifolds from M-theory. They can however all be obtained by consistent truncation of the maximal supergravity in the respective dimension. Among the supergravity orbifolds we have listed in Table 3 there is a $\mathcal{N}=2$ MESGT with only one $\mathcal{N}=2$ vector multiplet. This theory can be obtained by dimensional reduction from the pure $\mathcal{N}=2$ supergravity in five dimensions which has no moduli. Elaborating further on the results of [40], it was argued in [41] that the moduli-free pure $\mathcal{N}=2$ supergravity in five dimensions cannot be obtained from type II superstring by orbifolding and that any such orbifolding must involve additional multiplets containing some moduli. This implies that there must be additional massless fields coming from the twisted sector of the corresponding orbifold.

3 One-loop amplitudes at four points

We now turn to the calculation of one-loop amplitudes in the factorizable orbifold theories listed in Table 3 and in the non-factorizable ones listed in Tables 4; we will use the generalized unitarity method, originally introduced in [32], [33] and further developed for one-loop amplitudes in [42]. We will evaluate unitarity cuts in four dimensions and thus our results could potentially miss[14] rational functions of momentum invariants. In cases where D-dimensional unitarity cuts are readily accessible (e.g. $\mathcal{N}=4$ supergravity coupled with two vector multiplets) we will nevertheless see that our results are complete through $\mathcal{O}(\epsilon^0)$.

[14]This is to be contrasted with constructing supersymmetric gauge theory amplitudes at one loop, where improved power counting guarantees [33] that four-dimensional unitarity methods yield a complete result through $\mathcal{O}(\epsilon^0)$ in the integrated expression.

3.1 On tree-level amplitudes and unitarity cuts

In analogy to gauge theory orbifolds [20], the fields of orbifold supergravity theories can be obtained by suitably projecting the fields of the $\mathcal{N}=8$ supergravity parent theory. Assuming that the orbifold group is Abelian and denoting it by Γ, the relevant projection operators are

$$\mathcal{P}_\Gamma\ \Phi_{a_1\ldots a_n} = \frac{1}{|\Gamma|}\sum_{r\in\Gamma} r_{a_1}^{a_1}\ldots r_{a_n}^{a_n}\Phi_{a_1\ldots a_n}\ , \tag{3.1}$$

where $\Phi_{a_1\ldots a_n}$ is a generic field of $\mathcal{N}=8$ supergravity labeled by the fundamental $SU(8)$ indices $a_1\ldots a_n = 1,2,\ldots,8$ and the diagonal matrices r form a representation of Γ and act on the fundamental representation of $SU(8)$. This projection operator may also be interpreted as acting on the Grassmann parameters labeling the $\mathcal{N}=8$ on-shell superfield; it may therefore be used to extract from the $\mathcal{N}=8$ tree-level superamplitudes the component amplitudes of the orbifolded theory. An n-point superamplitude in the orbifold supergravity theory is thus given as

$$\mathcal{M}_{n,\Gamma}^{\text{tree}} = \frac{1}{|\Gamma|}\sum_{r_1\in\Gamma}\cdots\frac{1}{|\Gamma|}\sum_{r_n\in\Gamma}\mathcal{M}_n^{\text{tree}}\Big((r_1)_a^a\eta_{(1)}^a,\ldots,(r_n)_a^a\eta_{(n)}^a\Big)\ . \tag{3.2}$$

This identification of tree superamplitudes implies that we can carry out the sum over states that cross (generalized) unitarity cuts (or supersums) as a Grassmann integral using familiar techniques developed for $\mathcal{N}=4$ sYM and $\mathcal{N}=8$ supergravity [43, 44, 45, 46, 47, 48]. For example, the s-channel two-particle cut of the four-point one-loop superamplitude is

$$\mathcal{M}_{4,\Gamma}^{(1)}(1,2,3,4)\Big|_{s-\text{cut}} = \Big(\frac{\kappa}{2}\Big)^4\sum_{r\in\Gamma}\frac{s^2\ \delta^8\Big(\sum_{k=1}^2\langle kl_1\rangle\eta_k^a + r_a^a\sum_{k=3}^4\langle kl_1\rangle\eta_k^a\Big)\delta^8\Big(\sum_{k=1}^4\langle kl_2\rangle\eta_k^a\Big)}{\langle 12\rangle^2\langle 34\rangle^2\langle l_1l_2\rangle^4\prod_{k=1}^4\langle kl_1\rangle\langle kl_2\rangle}\ . \tag{3.3}$$

The s-channel cuts of the component amplitudes can be obtained by acting with an appropriate set of Grassmann derivatives on the expression (3.3). Similarly, the t- and u-channel cuts at one loop can be obtained by relabeling the external legs.

We proceed to construct the one-loop amplitudes as follows:

1. We start with an ansatz involving box and triangle scalar integrals multiplied by rational functions of the Mandelstam variables and of the scalar products $\tau_i = k_i\cdot l$, where l is the loop momentum. The degrees of the polynomials appearing in the ansatz depend on the number of supersymmetries of the particular orbifold theory we are considering.

2. We fix most of the coefficients in the ansatz by requiring that its s-, t- and u-channel two-particle cuts match the direct evaluation of these cuts in terms of tree-level (super)amplitudes (cf. eq. (3.3)). At this stage, some of the coefficients in the

ansatz are still free; some of them correspond to tadpoles and bubble-on-external-line (snail) integrals[15].

3. We perform a Passarino-Veltman reduction using the expressions collected in appendix A. Due to the presence of supersymmetry, tadpole and bubble-on-external-line integrals should not be present at this loop level; while they may emerge from the integral reduction due to the features of the original ansatz, they are discarded.

After the reduction is carried out, all the remaining free coefficients disappear and the results assume remarkably simple expressions, which are listed in the following subsections. Our results are given up to $\mathcal{O}(\epsilon)$ terms and up to rational terms. Using the six-dimensional spinor helicity formalism [49] we comment in sec. 3.4 on the properties of the rational terms in some of these theories (which, in cases that we have checked, turn out to be complete).

3.2 Four-graviton amplitudes

In this subsection we focus on four-graviton amplitudes $\mathcal{M}(--++)$ for the various orbifold theories listed in Table 2. These amplitudes are expected to be finite regardless of the theory, since the first candidate on-shell counterterm involving four gravitons arises only at three loops. The amplitudes are constructed using an ansatz of the form[16],

$$\mathcal{M}_4^{(1)} = \left(\frac{\kappa}{2}\right)^4 \frac{M_{\text{tree}}}{s^{7-\mathcal{N}}} \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \left(\sum_{n=1}^{3} P^{(n)}_{10-\mathcal{N},0}(s_{ij}) \mathcal{I}^{D=4-2\epsilon}_{4,n} + \sum_{m=1}^{6} Q^{(m)}_{9-\mathcal{N},N_L}(s_{ij},\tau_k) \mathcal{I}^{D=4-2\epsilon}_{3,m} \right) , \tag{3.4}$$

where $\mathcal{N}$ is the number of supersymmetries and

$$M_{\text{tree}} = i\frac{st}{u} A^2_{\text{tree}} = -is \frac{\langle 12\rangle^6}{\langle 34\rangle^2 \langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle} , \tag{3.5}$$

with A_{tree} being the tree-level four-gluon color-ordered amplitude. $P^{(n)}_{p,q}$ and $Q^{(m)}_{p,q}$ are generic polynomials of degree p in the Mandelstam variables $\{s_{12}, s_{23}, s_{13}\} = \{s,t,u\}$, with $s_{ij} \equiv (k_i + k_j)^2$, and in the scalar products $\tau_i = k_i \cdot l$ with terms up to degree q in the loop momentum l. $\mathcal{I}^{D=4-2\epsilon}_{4,n}$ and $\mathcal{I}^{D=4-2\epsilon}_{3,m}$, with $n = 1,2,3$ and $m = 1,\ldots,6$, are the three independent scalar box integrands and the six independent (before integration) scalar triangle integrands, respectively. The four-point amplitudes in these theories do not require that explicit bubble integrands be present in the ansatz. Nevertheless, bubble integrals can (and in some cases do) appear after the reduction to an integral basis is

[15] Such contributions should be absent from supersymmetric scattering amplitudes. They appear however because we made no effort to remove then at the previous step.

[16] Parity-odd terms, which integrate to zero, are ignored.

$\mathcal{N}$	**Orbifold**	**Matter**	$\mathcal{P}_1(\alpha)$	$\mathcal{P}_2(\alpha)$	$\mathcal{Q}(\alpha)$
4	$\mathcal{N}=4\times\mathcal{N}=0\vert_{\mathbf{Z}_2}$	6 vecs	4	$4-8\alpha$	4
4	$\mathcal{N}=4\times\mathcal{N}=0\vert_{\tilde{\mathbf{Z}}_4}$	4 vecs	3	$4-6\alpha$	3
4	$\mathcal{N}=4\times\mathcal{N}=0\vert_{\mathbf{Z}_2\times\mathbf{Z}_2}$	2 vecs	2	$4-4\alpha$	2
4	$\mathcal{N}=2\vert_{\mathbf{Z}_2}\times\mathcal{N}=2\vert_{\mathbf{Z}_2}$	2 vecs	2	$4-4\alpha$	2
4	$\mathcal{N}=4\times\mathcal{N}=0\vert_{\mathbf{Z}_4}$	pure	1	$4-2\alpha$	1
3	$\mathcal{N}=2\vert_{\mathbf{Z}_2}\times\mathcal{N}=1\vert_{\mathbf{Z}_3}$	1 vec	3	$5-6\alpha$	3
2	$\mathcal{N}=2\vert_{\mathbf{Z}_2}\times\mathcal{N}=0\vert_{\mathbf{Z}_2}$	7 vecs	$\frac{20}{3}-8\alpha$	$6-16\alpha+16\alpha^2$	$\frac{20}{3}-4\alpha$
2	$\mathcal{N}=2\vert_{\mathbf{Z}_2}\times\mathcal{N}=0\vert_{\tilde{\mathbf{Z}}_4}$	5 vecs	$6-6\alpha$	$6-14\alpha+12\alpha^2$	$6-3\alpha$
2	$\mathcal{N}=2\vert_{\mathbf{Z}_2}\times\mathcal{N}=0\vert_{\mathbf{Z}_2\times\mathbf{Z}_2}$	3 vecs	$\frac{16}{3}-4\alpha$	$6-12\alpha+8\alpha^2$	$\frac{16}{3}-2\alpha$
2	$\mathcal{N}=2\vert_{\mathbf{Z}_2}\times\mathcal{N}=0\vert_{\mathbf{Z}_4}$	1 vec	$\frac{14}{3}-2\alpha$	$6-10\alpha+4\alpha^2$	$\frac{14}{3}-\alpha$
2	$\mathcal{N}=1\vert_{\mathbf{Z}_3}\times\mathcal{N}=1\vert_{\mathbf{Z}_3}$	1 hyper	$\frac{9}{2}$	$6-9\alpha$	$\frac{9}{2}$
1	$\mathcal{N}=1\vert_{\mathbf{Z}_3}\times\mathcal{N}=0\vert_{\mathbf{Z}_2}$	6 vecs, 1 chiral	$8-12\alpha$	$7-20\alpha+24\alpha^2$	$8-6\alpha$
1	$\mathcal{N}=1\vert_{\mathbf{Z}_3}\times\mathcal{N}=0\vert_{\tilde{\mathbf{Z}}_4}$	4 vecs, 1 chiral	$\frac{15}{2}-9\alpha$	$7-18\alpha+18\alpha^2$	$\frac{15}{2}-\frac{9}{2}\alpha$
1	$\mathcal{N}=1\vert_{\mathbf{Z}_3}\times\mathcal{N}=0\vert_{\mathbf{Z}_2\times\mathbf{Z}_2}$	2 vecs, 1 chiral	$7-6\alpha$	$7-16\alpha+12\alpha^2$	$7-3\alpha$
1	$\mathcal{N}=1\vert_{\mathbf{Z}_3}\times\mathcal{N}=0\vert_{\mathbf{Z}_4}$	1 chiral	$\frac{13}{2}-3\alpha$	$7-14\alpha+6\alpha^2$	$\frac{13}{2}-\frac{3}{2}\alpha$

Table 5: Four-gravitons amplitudes at one loop for factorizable orbifolds. The polynomials $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{Q}$ depend on the ratio of Mandelstam variables $\alpha = tu/s^2$.

carried out. All bubbles-on-external-legs are discarded. The maximum number of loop momenta in the numerators of the triangle integrals, N_L, is chosen to be

$$N_L = 6-\mathcal{N}, \qquad \mathcal{N}=2,3,4\ . \tag{3.6}$$

Interestingly, this pattern is improved for $\mathcal{N}=1$, in which case it is sufficient to have four loop momenta in the numerators.

The one-loop amplitudes for all the orbifold theories in Tables 2 and 3 and with $\mathcal{N}=1,2,3,4$ supersymmetry have the following general form,

$$\begin{aligned}\mathcal{M}_4^{(1)} &= \mathcal{M}_{4,\mathcal{N}=8}^{(1)} + i\Big(\frac{\kappa}{2}\Big)^4 M_{\rm tree}\frac{tu}{s}\Bigg(\frac{u-t}{s}\mathcal{P}_1\Big(\frac{tu}{s^2}\Big)\big(I_2(t)-I_2(u)\big)\\ &+\epsilon\ \mathcal{Q}\Big(\frac{tu}{s^2}\Big)\big(I_2(t)+I_2(u)\big)+\ \mathcal{P}_2\Big(\frac{tu}{s^2}\Big)\ \Big(tI_3(t)+uI_3(u)-\frac{tu}{2}I_4(t,u)\Big)\Bigg)+\mathcal{O}(\epsilon),\end{aligned} \tag{3.7}$$

where $\mathcal{P}_1,\mathcal{P}_2$ and $\mathcal{Q}$ are polynomials which depend on the particular orbifold theory and $\mathcal{M}_{4,\mathcal{N}=8}^{(1)} = i\left(\frac{\kappa}{2}\right)^4 stuM_{\rm tree}\big(I_4(s,t)+I_4(s,u)+I_4(t,u)\big)$ is the known [50] one-loop four-graviton amplitude in the $\mathcal{N}=8$ theory. All scalar integrals in eqn. 3.7 are to be evaluated in $D=4-2\epsilon$ dimensions. The combination of triangle and box integrals in the parenthesis on its second line may be recognized as a multiple of the six-dimensional box integral $tI_3(t)+uI_3(u)-\frac{tu}{2}I_4(t,u)=sI_4^{D=6-2\epsilon}(t,u)$.

The polynomials $\mathcal{P}_1,\mathcal{P}_2$ and $\mathcal{Q}$ for the factorizable orbifold supergravity theories from Table 3 are listed in Table 5. Amplitudes for $\mathcal{N}=4$ theories have already been studied extensively in [22, 23, 24, 51, 52], and are included here only for completeness. We note here that the $\mathcal{N}=4$, $\mathbf{Z}_2\times\mathbf{Z}_2$ amplitude on row 3 differs from the $\mathcal{N}=2$, $\mathbf{Z}_2\times\mathbf{Z}_2$ amplitude on row 7, despite the theories forming a twin pair (i.e. having the same bosonic content).

Since all matter fields couple canonically to the gravity multiplet through their energy-momentum tensors, all one-loop amplitudes depend only on the number of supersymmetries and matter multiplets, and are thus insensitive to the precise nature of the matter couplings. We can decompose the field content of the theories in terms of $\mathcal{N}=1$ vector and chiral multiplets, and present unified expressions for all of the amplitudes described in Table 5

$$\mathcal{P}_1(\alpha) = \Big(\frac{17}{2}-\frac{25}{12}\mathcal{N}+\frac{1}{4}n_V+\frac{1}{12}n_C\Big)-\Big(6-\frac{5}{2}\mathcal{N}+\frac{3}{2}n_V-\frac{1}{2}n_C\Big)\alpha\ , \tag{3.8}$$

$$\mathcal{P}_2(\alpha) = \Big(8-\mathcal{N}\Big)-\Big(19-5\mathcal{N}+n_V\Big)\alpha+2\Big(6-\frac{5}{2}\mathcal{N}+\frac{3}{2}n_V-\frac{1}{2}n_C\Big)\alpha^2\ , \tag{3.9}$$

$$\mathcal{Q}(\alpha) = \Big(\frac{17}{2}-\frac{25}{12}\mathcal{N}+\frac{1}{4}n_V+\frac{1}{12}n_C\Big)-\frac{1}{2}\Big(6-\frac{5}{2}\mathcal{N}+\frac{3}{2}n_V-\frac{1}{2}n_C\Big)\alpha\ , \tag{3.10}$$

where n_V and n_C are the numbers of vector and chiral multiplets. We will see in the next subsection that matter amplitudes do not in general share this structure and depend on the specific nature of the matter couplings.

It is interesting to note that the constant term in $\mathcal{P}_2$ does not depend on the matter content of the theory and is always equal to $8-\mathcal{N}$. The various multiplets of the existing supersymmetry algebra contribute independently to one-loop amplitudes. Thus, the terms depending on the polynomials $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{Q}$ can be read as removing from the $\mathcal{N}=8$ amplitude the contribution of the multiplets that have been orbifolded away. See for example the constant term in $\mathcal{P}_2$. The form $8-\mathcal{N}$ represents the (ultimately removed) contribution of the $(8-\mathcal{N})$ gravitino multiplets that vary under the action of the orbifold group Γ. Moreover, one can note that the highest-degree terms in the three polynomials are proportional to each other and the constant terms in $\mathcal{P}_1$ and $\mathcal{Q}$ are identical.

An interesting example is provided by the two apparently different realizations of $\mathcal{N}=4$ supergravity coupled to two vector multiplets, realized from the double-copy perspective as different $\mathbf{Z_2}\times\mathbf{Z_2}$ orbifolds, which appear on rows five and six of Tables 2 and 3. Their

$\mathcal{N}$	**Orbifold**	**Matter**	$\mathcal{P}_1(\alpha)$	$\mathcal{P}_2(\alpha)$	$\mathcal{Q}(\alpha)$
3	$\mathbf{Z}_5$	pure	$\frac{5}{2}$	$5-5\alpha$	$\frac{5}{2}$
2	$\mathbf{Z}_6$	pure	$\frac{13}{3}-\alpha$	$6-9\alpha+2\alpha^2$	$\frac{13}{3}-\frac{1}{2}\alpha$
1	$\mathbf{Z}_7$	pure	$\frac{77}{12}-\frac{7}{2}\alpha$	$7-14\alpha+7\alpha^2$	$\frac{77}{12}-\frac{7}{4}\alpha$

Table 6: Four-gravitons amplitudes at one loop for some non-factorizable orbifolds.

amplitudes turn out to be identical –

$$\begin{aligned}\mathcal{M}^{(1)}_{4,\mathbf{Z_2}\times\mathbf{Z_2}} &= \mathcal{M}^{(1)}_{4,\mathcal{N}=8} + i\Big(\frac{\kappa}{2}\Big)^4 M_{\text{tree}}\bigg(2tu\frac{u-t}{s^2}\big(I_2(t)-I_2(u)\big) + 4\epsilon\frac{tu}{s}\big(I_2(u)+I_2(t)\big) \\ &\quad +4\frac{tu}{s}\Big(1-\frac{tu}{s^2}\Big)\Big(tI_3(t)+uI_3(u)-\frac{tu}{2}I_4(t,u)\Big)\bigg) \ . \end{aligned} \tag{3.11}$$

One may convince oneself that this is indeed so by noticing that the supersums that govern the four-dimensional cuts of all four-point one-loop amplitudes are identical:

$$\begin{aligned}&\Big(\langle 1l_1\rangle^4\langle 2l_2\rangle^4 - 2\langle 1l_1\rangle^3\langle 1l_2\rangle\langle 2l_1\rangle\langle 2l_2\rangle^3 + 2\langle 1l_1\rangle^2\langle 1l_2\rangle^2\langle 2l_1\rangle^2\langle 2l_2\rangle^2 - 2\langle 1l_1\rangle\langle 1l_2\rangle^3\langle 2l_1\rangle^3\langle 2l_2\rangle \\ &\quad +\langle 1l_2\rangle^4\langle 2l_1\rangle^4\Big)^2 = \Big(\langle 1l_1\rangle\langle 2l_2\rangle - \langle 1l_2\rangle\langle 2l_1\rangle\Big)^4\Big(\langle 1l_1\rangle^2\langle 2l_2\rangle^2 + \langle 1l_2\rangle^2\langle 2l_1\rangle^2\Big)^2 \ . \end{aligned} \tag{3.12}$$

The expression on the left-hand side is the supersum for the $\mathcal{N}=2|_{\mathbf{Z_2}} \times \mathcal{N}=2|_{\mathbf{Z_2}}$ theory, while the expression on the right-hand side is the supersum for the $\mathcal{N}=4\times\mathcal{N}=0|_{\mathbf{Z_2}\times\mathbf{Z_2}}$ theory. One may also understand this equivalence as a consequence of the fact that, up to conjugation by $SU(8)$ elements, there is a unique embedding of the orbifold group $\Gamma = \mathbf{Z_2}\times\mathbf{Z_2}$ inside the $SU(8)$ R-symmetry group of $\mathcal{N}=8$ supergravity. As we will see in the next subsections, matter amplitudes exhibit the same feature. We will comment in sections 3.4 and 4.4 on the equivalence of the two theories at the level of rational terms and for higher-loop and higher-point amplitudes.

A thorough analysis of the non-factorizable orbifolds such as those shown in Table 4 is beyond the scope of this paper. However, as examples, we present in Table 6 the four-graviton amplitudes for the pure $\mathcal{N}=1,2,3$ supergravities. We notice the close similarity of the graviton amplitude in the pure $\mathcal{N}=3$ and $\mathcal{N}=4$ supergravities (cf. first row of Table 6 and fifth row of Table 5). It would be interesting to check whether this similarity continues at higher loops.

$\mathcal{N}$	**Orbifold**	**Matter**	c_1	c_2	c_3	$\mathcal{P}(\alpha)$
4	$\mathcal{N}=4\times\mathcal{N}=0\vert_{\mathbf{Z}_2}$	6 vecs	4	0	0	4
4	$\mathcal{N}=4\times\mathcal{N}=0\vert_{\tilde{\mathbf{Z}}_4}$	4 vecs	3	0	0	4
4	$\mathcal{N}=4\times\mathcal{N}=0\vert_{\mathbf{Z}_2\times\mathbf{Z}_2}$	2 vecs	2	0	0	4
4	$\mathcal{N}=2\vert_{\mathbf{Z}_2}\times\mathcal{N}=2\vert_{\mathbf{Z}_2}$	2 vecs	2	0	0	4
2	$\mathcal{N}=2\vert_{\mathbf{Z}_2}\times\mathcal{N}=0\vert_{\mathbf{Z}_2}$	7 vecs	$\frac{14}{3}$	2	$\frac{16}{9}$	$6-4\alpha$
2	$\mathcal{N}=2\vert_{\mathbf{Z}_2}\times\mathcal{N}=0\vert_{\tilde{\mathbf{Z}}_4}$	5 vecs	4	2	$\frac{11}{6}$	$6-4\alpha$
2	$\mathcal{N}=2\vert_{\mathbf{Z}_2}\times\mathcal{N}=0\vert_{\mathbf{Z}_2\times\mathbf{Z}_2}$	3 vecs	$\frac{10}{3}$	2	$\frac{17}{9}$	$6-4\alpha$
1	$\mathcal{N}=1\vert_{\mathbf{Z}_3}\times\mathcal{N}=0\vert_{\mathbf{Z}_2}$	6 vecs, 1 chiral	5	3	$\frac{8}{3}$	$7-6\alpha$
1	$\mathcal{N}=1\vert_{\mathbf{Z}_3}\times\mathcal{N}=0\vert_{\tilde{\mathbf{Z}}_4}$	4 vecs, 1 chiral	$\frac{9}{2}$	3	$\frac{11}{4}$	$7-6\alpha$
1	$\mathcal{N}=1\vert_{\mathbf{Z}_3}\times\mathcal{N}=0\vert_{\mathbf{Z}_2\times\mathbf{Z}_2}$	2 vecs, 1 chiral	4	3	$\frac{17}{6}$	$7-6\alpha$

Table 7: One-loop vector amplitudes $F^{123456}F^{78}\to F^{123456}F^{78}$ for factorizable orbifolds. $\mathcal{P}$ is a function of the ratio of Mandelstam variables $\alpha=\frac{tu}{s^2}$

3.3 Matter amplitudes

Let us now study vector and scalar amplitudes of the form $F\bar{F}\to F\bar{F}$ or $\phi\bar{\phi}\to\phi\bar{\phi}$ with all fields in the same matter supermultiplet. Specifically, there are three relevant amplitudes,

$$\mathcal{M}_V\left(F^{123456}F^{123456}F^{78}F^{78}\right)\ , \tag{3.13}$$
$$\mathcal{M}_{V'}\left(F^{125678}F^{125678}F^{34}F^{34}\right)\ , \tag{3.14}$$
$$\mathcal{M}_H\left(\phi^{1234}\phi^{1234}\phi^{5678}\phi^{5678}\right)\ . \tag{3.15}$$

In some particular cases – namely all the $\mathcal{N}=4$ orbifolds and the $\mathcal{N}=2$, $\mathbf{Z}_2\times\mathbf{Z}_2\times\mathbf{Z}_2$ theory – the two vector amplitudes (3.13) and (3.14) are equivalent. In some other cases, one of the two amplitudes does not exist as the corresponding fields are projected out by the orbifold projection.

All amplitudes are constructed using the ansatz,

$$\mathcal{M}_4^{(1)}=\left(\frac{\kappa}{2}\right)^4\frac{M_{\text{tree}}}{s^{5-\mathcal{N}}tu}\int\frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}}\left(\sum_{n=1}^{3}P^{(n)}_{10-\mathcal{N},0}\left(s_{ij}\right)\mathcal{I}^{D=4-2\epsilon}_{4,n}+\sum_{m=1}^{6}Q^{(m)}_{9-\mathcal{N},N_L}\left(s_{ij},\tau_k\right)\mathcal{I}^{D=4-2\epsilon}_{3,m}\right) \tag{3.16}$$

where as usual all integrals are in $D=4-2\epsilon$ dimensions. As for the four-graviton one-loop amplitudes, the maximum number of loop momenta in the numerators is $N_L=6-\mathcal{N}$ for $\mathcal{N}=2,3,4$ and $N_L=4$ for $\mathcal{N}=1$. After integral reduction all these amplitudes take the

$\mathcal{N}$	**Orbifold**	**Matter**	c_1	c_2	c_3	$\mathcal{P}(\alpha)$
4	$\mathcal{N}=4\times\mathcal{N}=0\vert_{\mathbf{Z}_2}$	6 vecs	4	0	0	4
4	$\mathcal{N}=4\times\mathcal{N}=0\vert_{\tilde{\mathbf{Z}}_4}$	4 vecs	3	0	0	4
4	$\mathcal{N}=4\times\mathcal{N}=0\vert_{\mathbf{Z}_2\times\mathbf{Z}_2}$	2 vecs	2	0	0	4
4	$\mathcal{N}=2\vert_{\mathbf{Z}_2}\times\mathcal{N}=2\vert_{\mathbf{Z}_2}$	2 vecs	2	0	0	4
3	$\mathcal{N}=2\vert_{\mathbf{Z}_2}\times\mathcal{N}=1\vert_{\mathbf{Z}_3}$	1 vec	3	0	0	5
2	$\mathcal{N}=2\vert_{\mathbf{Z}_2}\times\mathcal{N}=0\vert_{\mathbf{Z}_2}$	7 vecs	$\frac{8}{3}$	4	$\frac{34}{9}$	$6-8\alpha$
2	$\mathcal{N}=2\vert_{\mathbf{Z}_2}\times\mathcal{N}=0\vert_{\tilde{\mathbf{Z}}_4}$	5 vecs	$\frac{9}{3}$	3	$\frac{17}{6}$	$6-6\alpha$
2	$\mathcal{N}=2\vert_{\mathbf{Z}_2}\times\mathcal{N}=0\vert_{\mathbf{Z}_2\times\mathbf{Z}_2}$	3 vecs	$\frac{10}{3}$	2	$\frac{17}{9}$	$6-4\alpha$
2	$\mathcal{N}=2\vert_{\mathbf{Z}_2}\times\mathcal{N}=0\vert_{\mathbf{Z}_4}$	1 vec	$\frac{11}{3}$	1	$\frac{17}{18}$	$6-2\alpha$

Table 8: One-loop vector amplitudes $F^{125678}F^{34}\to F^{125678}F^{34}$ for factorizable orbifolds.

$\mathcal{N}$	**Orbifold**	**Matter**	c_1	c_2	c_3	$\mathcal{P}(\alpha)$
2	$\mathcal{N}=1\vert_{\mathbf{Z}_3}\times\mathcal{N}=1\vert_{\mathbf{Z}_3}$	1 hyper	$\frac{9}{2}$	0	0	6
1	$\mathcal{N}=1\vert_{\mathbf{Z}_3}\times\mathcal{N}=0\vert_{\mathbf{Z}_2}$	6 vecs, 1 chiral	4	4	$\frac{11}{3}$	$7-8\alpha$
1	$\mathcal{N}=1\vert_{\mathbf{Z}_3}\times\mathcal{N}=0\vert_{\tilde{\mathbf{Z}}_4}$	4 vecs, 1 chiral	$\frac{9}{2}$	3	$\frac{11}{4}$	$7-6\alpha$
1	$\mathcal{N}=1\vert_{\mathbf{Z}_3}\times\mathcal{N}=0\vert_{\mathbf{Z}_2\times\mathbf{Z}_2}$	2 vecs, 1 chiral	5	2	$\frac{11}{6}$	$7-4\alpha$
1	$\mathcal{N}=1\vert_{\mathbf{Z}_3}\times\mathcal{N}=0\vert_{\mathbf{Z}_4}$	1 chiral	$\frac{11}{2}$	1	$\frac{11}{12}$	$7-2\alpha$

Table 9: One-loop scalar amplitudes $\phi^{1234}\phi^{5678}\to\phi^{1234}\phi^{5678}$ for factorizable orbifolds.

following form,

$$\begin{aligned}\mathcal{M}_4^{(1)} &= \mathcal{M}_{4,\mathcal{N}=8}^{(1)} + i\Big(\frac{\kappa}{2}\Big)^4 M_{\text{tree}}\frac{tu}{s}\bigg(\big(c_1+\epsilon\, c_3\big)\big(I_2(t)+I_2(u)\big) + c_2\frac{u-t}{s}\big(I_2(t)-I_2(u)\big)\\ &\quad +\mathcal{P}\Big(\frac{tu}{s^2}\Big)\Big(tI_3(t)+uI_3(u)-\frac{tu}{2}I_4(t,u)\Big)\bigg) + \mathcal{O}(\epsilon)\ , \qquad (3.17)\end{aligned}$$

which depends on the polynomial $\mathcal{P}$ and on the constants c_1, c_2, c_3.

We start by listing amplitudes $\mathcal{M}_V$ of the form (3.13) in Table 7. As expected, the amplitudes with fields in the same matter multiplet are all divergent, i.e. all constants c_1 are different from zero. The divergence of the $\mathcal{N}=4$ amplitudes is proportional to $(2+n_v)$, where n_v is the number of vector multiplets, in agreement with the results of Fischler, Fradkin and Tseytlin [27, 28] (see also [24] for a recent discussion of UV properties of these theories from the perspective of the double-copy property of half-maximal supergravities).

The $\mathcal{N}=3$ and $\mathcal{N}=2$ orbifolds with one vector multiplet do not have any non-vanishing amplitude of this class because the external fields do not survive the $\mathbf{Z}_2\times\mathbf{Z}_3$ and $\mathbf{Z}_2\times\mathbf{Z}_4$ projections.

In Table 8 we list the numerical coefficients c_1, c_2, c_3 and the polynomial $\mathcal{P}$ for amplitudes $\mathcal{M}_V{}'$ of the form (3.14). The $\mathcal{N}=1$ orbifold supergravities do not have amplitudes of this class, because the external fields in (3.14) do not survive the projection. For the $\mathcal{N}=4$ orbifolds, amplitudes in this class are equal to the ones already listed.

The amplitudes of $\mathcal{N}=2$ MESGT with three vector multiplets quoted in Tables 7 and 8 coincide. For this theory the U-duality group is $SO(2,2)\times SU(1,1)$ which is isomorphic to $SU(1,1)^3$ under which the field strengths of vector fields and their magnetic duals transform in the eight-dimensional representation $(1/2,1/2,1/2)$. Because of the triality symmetry permuting the three factors of $SU(1,1)$ all vectors are on equal footing with respect to the full U-duality group.

In the case of the $\mathcal{N}=2$ orbifolds with seven and five vector multiplets we find that $\mathcal{M}_V{}'$ is different from $\mathcal{M}_V$. To explain this, we note that the $\mathcal{N}=2$ MESGTs with $(n+1)$ vector multiplets and symmetric scalar manifolds

$$\frac{SO(n,2)\times SU(1,1)}{SO(n)\times SO(2)\times U(1)} \tag{3.18}$$

descend from generic Jordan family of five-dimensional MESGTs with target manifolds [39]

$$\frac{SO(n-1,1)\times SO(1,1)}{SO(n-1)}\ . \tag{3.19}$$

Under $SO(n,2)\times SU(1,1)$ symmetry group $n+2$ vector field strengths and their magnetic duals transform in the representation $(n+2,2)$, i.e. they form $(n+2)$ doublets of $SU(1,1)$. Pure five-dimensional $\mathcal{N}=2$ supergravity reduces to $\mathcal{N}=2$ supergravity coupled to one vector multiplet with scalar manifold $\frac{SU(1,1)_0}{U(1)}$. Under the isometry group $SU(1,1)_0$ the field strengths of the graviphoton and the single vector field and their magnetic duals transform in the four dimensional "spin"=3/2 representation. This $SU(1,1)_0$ symmetry group can be identified with a diagonal subgroup of $SO(1,2)\times SU(1,1)$ subgroup of the isometry group $SO(n,2)\times SU(1,1)$ of generic Jordan family of MESGTs

$$SU(1,1)_0\times SO(n-1)\subset SU(1,1)\times SO(1,2)\times SO(n-1)\subset SU(1,1)\times SO(n,2)\ . \tag{3.20}$$

Under the $SU(1,1)_0$ subgroup vector field strengths and their magnetic duals transform as n doublets ("spin"=1/2) and one quartet ("spin"=3/2). This shows that a certain linear combination of the vector fields couples differently than the rest. This gives a physical interpretation to the result that $\mathcal{M}_V$ and $\mathcal{M}_V{}'$ are different in the $\mathcal{N}=2$ orbifold supergravities with seven and five vector multiplets on rows nine and ten of Table 3.

Finally, scalar amplitudes of the form (3.15) in theories with hypermultiplets or chiral multiplets are shown in Table 9.

As in the case of the one-loop four-graviton amplitudes, the numerical coefficients of the expressions in Tables 7-9 are not all independent. In particular, the constant term in $\mathcal{P}$ only depends on the amount of supersymmetry and is equal to $(8-\mathcal{N})$. Moreover, the linear term in $\mathcal{P}$ is equal to $2c_2$. Finally, $c_1 + c_2$ is always equal to the constant term in $\mathcal{P}_1$ shown in Table 5.

3.4 Two presentations of $\mathcal{N} = 4$ supergravity with two vector multiplets

In previous sections, using unitarity, we calculated many four-point one-loop amplitudes in various matter-coupled supergravity theories that can be obtained by orbifolding $\mathcal{N} = 8$ supergravity. We have also observed that such amplitudes in the theories on rows 5 and 6 of Table 3, $\mathcal{N} = 2|_{\mathbf{Z}_2} \times \mathcal{N} = 2|_{\mathbf{Z}_2}$ and $\mathcal{N} = 4 \times \mathcal{N} = 0|_{\mathbf{Z}_2\times\mathbf{Z}_2}$, are the same in four dimensions. We will demonstrate here that this equality extends to D-dimensions.

Unitarity carried out strictly in four-dimensions may miss non-trivial contributions known as *rational terms*. These arise from tree-level amplitudes that vanish identically in four dimensions but are non-vanishing in $D \neq 4$ dimensions, such as the all-plus amplitudes in pure Yang-Mills (YM) theory. So, strictly speaking, in order to verify that one has not missed any contributions, it is necessary to use unitarity with D-dimensional constituent amplitudes. For theories that can be consistently expressed as a dimensional-reduction of a higher-dimensional theory one can simply use a higher-dimensional formalism, such as the six-dimensional helicity formalism introduced in [53, 49] and used in [54, 55] for $\mathcal{N} = 4$ sYM as well as pure YM calculations.

The minimal amount of supersymmetry for a six-dimensional sYM theory is $\mathcal{N} = (1, 0)$ or $\mathcal{N} = (0, 1)$; it reduces to $\mathcal{N} = 2$ sYM theory in four dimensions. Moreover, six-dimensional non-supersymmetric YM theory may be realized as an orbifold projection of this theory and reduces upon dimensional reduction to pure YM theory coupled to two real scalar fields. Thus, from the supergravity theories listed in Table 3, only those that have such factors should be treated in this formalism when allowing *all* states to run across all loops. We evaluate here the various supersums appearing in the corresponding supergravity calculations and discuss their implications. In our discussion we will also label the external states using a six-dimensional notation; thus, all possible four-point amplitudes allowed by supersymmetry are covered simultaneously. One should note that it is certainly possible to consistently handle the other theories using these same methods, but one must constrain the state-sums to only allow states corresponding to the appropriate theory. In principle this is just a matter of book-keeping. For example, to consider pure YM in four-dimensions, using higher-dimensional methods, one simply restricts [54] to state-sums that only allow two gluonic states[17] to run around the loop.

The tree-level color-ordered four-point superamplitude of the $(1, 1)$ sYM theory in six

[17] Or $2 - \epsilon$ gluonic states depending on the regularization scheme.

dimensions is [49]:

$$A_4^{\rm tree}(1,2,3,4) = \frac{1}{st}\delta^6(\sum_{i=1}^4 p_i)\delta^4(\sum_{i=1}^4 q_i^A)\delta^4(\sum_{i=1}^4 \tilde{q}_i^A)\ , \tag{3.21}$$

where

$$p^{AB} = \epsilon^{ab}\lambda_a^A\lambda_b^B\ ,\quad p_{AB} = \frac{1}{2}\epsilon_{ABCD}p^{CD} = \tilde{\lambda}_A^{\dot a}\tilde{\lambda}_B^{\dot b}\epsilon_{\dot a\dot b}\ ,\quad q_i^A = \lambda_{i,a}^A\eta_{+,i}^a\ ,\quad \tilde{q}_{iA} = \tilde{\lambda}_{i,A}^{\dot a}\tilde{\eta}_{+,i,\dot a}\ . \tag{3.22}$$

Similarly to the four-dimensional on-shell superspace, component amplitudes are obtained by extracting from (3.21) the appropriate combinations of Grassmann variables η. The six-dimensional on-shell superfield that contains the various component fields is [56]:

$$\begin{aligned}\Phi^{D=6}(\eta,\tilde{\eta}) \quad = \quad & \phi + \chi^a\eta_a + \tilde{\chi}_{\dot a}\tilde{\eta}^{\dot a} + \phi'(\eta)^2 + g^a{}_{\dot a}\eta_a\tilde{\eta}^{\dot a} \\ & +\phi''(\tilde{\eta})^2 + \tilde{\lambda}_{\dot a}\tilde{\eta}^{\dot a}(\eta)^2 + \lambda^a\eta_a(\tilde{\eta})^2 + \phi'''(\eta)^2(\tilde{\eta})^2\ .\end{aligned} \tag{3.23}$$

To extract component amplitudes we simply multiply (3.21) by the desired superfields (each of which contains a single nonzero component field) and integrate over all Grassmann variables.

In the same way as pure $\mathcal{N}=2$ sYM theory in four dimensions may be realized as an orbifold projection of $\mathcal{N}=4$ sYM theory, the $\mathcal{N}=(1,0)$ sYM theory may be realized as an orbifold projection of the $\mathcal{N}=(1,1)$ sYM theory: one projects out the fields $\phi, \phi', \phi'', \phi'''$, $\tilde{\chi}_{\dot a}$ and $\tilde{\lambda}_{\dot a}$. All surviving states have exactly one η Grassmann variable and up to two $\tilde{\eta}$ Grassmann variables. The $\mathcal{N}=(0,1)$ sYM theory is obtained by conjugation.

Each of the supergravity tree-level amplitudes can be expressed as a sum over products of tree-level color-ordered (s)YM amplitudes using the KLT relations [34]. As discussed at length in [57], cuts of such supergravity theories factorize over these relations to sums over products of color-ordered cuts of the corresponding (s)YM amplitudes. The four (s)YM cuts which determine the cuts of all one-loop four-point amplitudes of supergravity theories we analyze here can be easily computed though the appropriate restriction on the dependence of the Grassmann δ-functions in (3.21) on η and $\tilde{\eta}$. They are:

$$\mathcal{S}_4^{(1,1)} \quad = \quad \sum_s A_{4;(1,1)}^{\rm tree}(1,2,l_1^s,l_2^s)A_{4;(1,1)}^{\rm tree}(l_2^s,l_1^s,3,4) = \frac{\delta^4(q_{\rm ext})\delta^4(\tilde{q}_{\rm ext})}{(k_2+l_1)^2(k_3-l_1)^2}\ , \tag{3.24}$$

$$\begin{aligned}\mathcal{S}_4^{(0,0)} \quad = \quad & \sum_s A_{4;(0,0)}^{\rm tree}(1,2,l_1^s,l_2^s)A_{4;(0,0)}^{\rm tree}(l_2^s,l_1^s,3,4) = \frac{1}{s_{12}^2\,(k_2+l_1)^2(k_3-l_1)^2} \\ & \times\epsilon_{ABCD}{q_1}^A{q_2}^B\ \epsilon_{EFGH}{q_3}^E{q_4}^F(l_1)^{CG}(l_2)^{DH} \\ & \epsilon^{A'B'C'D'}\tilde{q}_{1,A}\tilde{q}_{2,B'}\ \epsilon^{E'F'G'H'}\tilde{q}_{3,E}\tilde{q}_{4,F'}(l_1)_{C'G'}(l_2)_{D'H'}\ ,\end{aligned} \tag{3.25}$$

$$\begin{aligned}\mathcal{S}_4^{(1,0)} \quad = \quad & \sum_s A_{4;(1,0)}^{\rm tree}(1,2,l_1^s,l_2^s)A_{4;(1,0)}^{\rm tree}(l_2^s,l_1^s,3,4) = \frac{\delta^4(q_{\rm ext})(2l_1\cdot l_2)}{s_{12}^2(k_2+l_1)^2(k_3-l_1)^2} \\ & \times\epsilon^{A'B'C'D'}\tilde{q}_{1,A}\tilde{q}_{2,B'}\ \epsilon^{E'F'G'H'}\tilde{q}_{3,E}\tilde{q}_{4,F'}(l_1)_{C'G'}(l_2)_{D'H'}\ ,\end{aligned} \tag{3.26}$$

$$\mathcal{S}_4^{(0,1)} \quad = \quad \sum_s A_{4;(0,1)}^{\rm tree}(1,2,l_1^s,l_2^s)A_{4;(0,1)}^{\rm tree}(l_2^s,l_1^s,3,4) = \frac{\delta^4(\tilde{q}_{\rm ext})(2l_1\cdot l_2)}{s_{12}^2(k_2+l_1)^2(k_3-l_1)^2}$$

$$\times \epsilon_{ABCD} {q_1}^A {q_2}^B \, \epsilon_{EFGH} {q_3}^E {q_4}^F (l_1)^{CG} (l_2)^{DH} \ . \tag{3.27}$$

These unitarity cuts are all that is required to construct the D-dimensional version of all the one-loop four-point amplitudes of the supergravities listed on rows one, five, six and eleven of Table 3.

While the resulting expressions are slightly more involved, the construction described above can be modified to also accommodate the pure YM theory coupled to other even numbers of scalar fields. For example, for a vector field coupled with six real scalars in four dimensions (the $\mathcal{N} = 0|_{\mathbf{Z}_2}$ theory) it is necessary to project onto the invariant part of the $\mathbf{Z}_2$ transformation

$$\mathbf{Z}_2 : (\eta, \tilde{\eta}) \mapsto (-\eta, -\tilde{\eta}) \ ; \tag{3.28}$$

This transformation projects out all fermions in eq. (3.23) while keeping all bosonic fields.

Using eqs. (3.24)-(3.27) we can see that the observed equality of the four-point one-loop amplitudes of the $\mathcal{N} = 2|_{\mathbf{Z}_2} \times \mathcal{N} = 2|_{\mathbf{Z}_2}$ and $\mathcal{N} = 4 \times \mathcal{N} = 0|_{\mathbf{Z}_2 \times \mathbf{Z}_2}$ theories holds in higher dimensions. Indeed, we note that the four supersums (3.24)-(3.27) are related by

$$\mathcal{S}_4^{(1,1)} \mathcal{S}_4^{(0,0)} = \mathcal{S}_4^{(0,1)} \mathcal{S}_4^{(1,0)} \ ; \tag{3.29}$$

This implies that all cuts of the one-loop four-point amplitudes of the theories on rows five and six of Table 3 – and therefore also their complete one-loop four-point amplitudes – are the same in any $D \leq 6$. This includes the "rational" one-loop amplitudes in the $\mathcal{N} = 2|_{\mathbf{Z}_2} \times \mathcal{N} = 2|_{\mathbf{Z}_2}$ supergravity theory whose integrand vanishes identically on four dimensional cuts. This might come as a surprise since no such amplitude contributes to either of the sYM factors. How could this be reconciled with the double-copy property of these theories? They are generated by terms that integrate to zero in four dimensions in the two gauge theory factors, but whose double-copy no longer vanishes under integration.

4 $\mathcal{N} = 1$ and $\mathcal{N} = 2$ sYM one-loop amplitudes from color/kinematics duality

Explicit calculations in the $\mathcal{N} = 4$ sYM theory revealed that the integrands of scattering amplitudes in this theory may be organized such that they exhibit a certain duality between color and kinematics. Such a duality, which we will review below, is believed to hold in all theories whose amplitudes may be organized in terms of cubic graphs.

4.1 On the color/kinematics duality and the double-copy structure of supergravity amplitudes

The general expression of the dimensionally-regularized L-loop m-point scattering amplitude in such a theory is

$$\mathcal{A}_m^{L-\text{loop}} \;=\; i^L\, g^{m-2+2L} \sum_{i\in\Gamma} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i C_i}{\prod_{\alpha_i} p_{\alpha_i}^2}\,, \tag{4.1}$$

where g is the gauge coupling constant. The sum runs over the complete set Γ of m-point L-loop graphs with only cubic (trivalent) vertices, including all permutations of external legs, the integration is over the L independent loop momenta p_l and the denominator is determined by the product of all propagators of the corresponding graph. The coefficients C_i are the color factors obtained by assigning to every three-vertex in the graph a factor of the structure constant

$$\tilde{f}^{abc} = i\sqrt{2} f^{abc} = \text{Tr}([T^a, T^b]T^c)\,, \tag{4.2}$$

while respecting the cyclic ordering of edges at the vertex. The hermitian generators T^a of the gauge group are normalized via $\text{Tr}(T^aT^b) = \delta^{ab}$. The coefficients n_i are kinematic numerator factors depending on momenta, polarization vectors and spinors. For supersymmetric amplitudes in an on-shell superspace, they will also contain Grassmann parameters. The symmetry factors S_i of each graph remove any overcount introduced by summing over all permutations of external legs (included by definition in the set Γ), as well as any internal automorphisms of the graph – i.e. symmetries of the graph with fixed external legs.

The color/kinematics duality [5] is manifest when the kinematic numerators of a graph representation of the amplitude satisfy antisymmetry and (generalized) Jacobi relations around each propagator – in one-to-one correspondence with the color-factors. That is, schematically for cubic-graph representations, it requires that

$$C_i + C_j + C_k = 0 \qquad \Rightarrow \qquad n_i + n_j + n_k = 0\,. \tag{4.3}$$

The existence of such representations are conjectured [29] to hold all loop orders and to all multiplicities.

There is by now substantial evidence for the color/kinematics duality, especially at tree level [58, 59, 60, 61], where explicit representations of the numerators in terms of partial amplitudes are known for any number of external legs [62]. A partial Lagrangian understanding of the duality has also been given [60]. An alternative trace-inspired presentation of the duality relation (4.3), which emphasizes its group-theoretic structure, was described in [63].

Once amplitudes of a gauge theory are organized to exhibit color/kinematics duality (4.3), numerator factors of L-loop m-point amplitudes in several supergravity or matter-coupled supergravity theories may be obtained in a straightforward way by simply multiplying together kinematic numerator factors of this and other gauge theories [5, 29],

$$\mathcal{M}_m^{L-\text{loop}} = i^{L+1} \left(\frac{\kappa}{2}\right)^{m-2+2L} \sum_{i\in\Gamma} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2} \,, \tag{4.4}$$

where κ is the gravitational coupling. The factors n_i are the numerator factors of the amplitude satisfying color/kinematics duality (4.3) and $\tilde{n}_i$ the numerator factors of the amplitude of a second, potentially different gauge-theory (at the same loop order and with the same number of external legs) and the sum runs over the same set of graphs as in eq. (4.1); moreover, they are not required to satisfy the duality. The construction (4.4) is expected to hold in a large class of (super)gravity theories in which the field content may be written as the tensor product of the field content of two gauge theories, such as the factorized orbifold supergravity theories discussed in previous sections. For $\mathcal{N}=8$ supergravity both n_i and $\tilde{n}_i$ are numerators of amplitudes of $\mathcal{N}=4$ sYM theory.

The KLT relations [34] described in earlier sections, emerge from the double-copy construction at tree-level when the Jacobi satisfying cubic tree-graph numerators are expressed in terms of color-ordered partial amplitudes [5] – the so-called amplitude-encoded representations.

The double-copy formula (4.4) has been proven [60] for both pure gravity and for $\mathcal{N}=8$ supergravity tree amplitudes, under the assumption that the duality (4.3) holds in the corresponding gauge theories, pure YM and $\mathcal{N}=4$ sYM theory, respectively. The nontrivial part of the loop-level conjecture is the existence of a representation of gauge-theory amplitudes that satisfies the duality constraints. The multi-loop double-copy property was explicitly exposed for the three-loop four-point [29] and the four-loop four-point [64] amplitudes of $\mathcal{N}=8$ supergravity. The color/kinematics duality and double-copy property have also been confirmed in one- and two-loop five-point amplitudes in $\mathcal{N}=8$ supergravity [65]. For less-than-maximal supergravities, the double-copy property has been checked explicitly for the one-loop four- and five-graviton amplitudes of $\mathcal{N}=4,5,6$ supergravity [66] by showing it reproduces known results [67]. The infrared divergences and other properties of the four-graviton amplitudes in these theories have also been shown to be consistent with it at two loops [68] and to all orders [69].

The color/kinematics duality and the double-copy property were also discussed and shown to hold under certain conditions in the presence of deformations by higher-dimension operators [70]. Additionally the duality and double-copy have been shown to lead to some suggestive relations between certain $\mathcal{N}\geq 4$ supergravity and subleading-color sYM amplitudes [71].

In the explicit multi-loop calculations carried out in the maximally supersymmetric sYM theory it was possible to choose Jacobi-satisfying kinematic numerator factors n_i which respect the symmetries of graphs; that is, if two graphs are obtained from each other by some map of their edges labels, then so are their kinematic numerator factors. Sign factors in such maps are taken to be the same as the sign appearing in the corresponding map of color factors. Intriguingly the symmetric double-copy representations of the maximally supersymmetric gravity theories through four-loops at four-point and two-loops at five-point make manifest the ultraviolet-behavior of the theory [29, 64, 65]. The double-copy property has also been recently shown to directly lead to certain UV cancellations in half-maximal supergravities [24].

Through the calculation of the four-gluon amplitudes in pure $\mathcal{N} = 1$ and $\mathcal{N} = 2$ sYM theories such external-state symmetry restrictions should not be expected, and we are naturally led to presentations of amplitudes asymmetric in external state labels. We find that in this case it is quite possible to find representations that do not make the UV properties manifest and it is such representations we present below.

4.2 Color/kinematics duality in the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ sYM theories

In general, the one-loop four-point amplitude in any gauge theory with fields in the adjoint representation is constructed in terms of the graphs shown in figs. 1 and 2 as well as graphs containing bubble and tadpole subgraphs whose topologies are shown in fig. 3. To construct the relevant Jacobi relations we start with the box graphs, iteratively select one of its internal lines and carry out the Jacobi transformation (4.3) of the numerator factors while paying close attention to the sign factors arising from reordering the lines at each vertex. For example, the numerator Jacobi relations on the edge carrying momentum l in the first graph in fig. 1 and the first graph in fig. 2 are

$$\begin{aligned} n_1(k_1, k_2, k_3, q + k_2 + k_3) - n_3(k_1, k_2, k_3, q + k_3) - n_4(k_1, k_2, k_3, -q) &= 0 \\ n_4(k_1, k_2, k_3, k_3 + q) + n_4(k_1, k_2, k_3, k_1 + k_2 + k_3 - q) - n_{10}(k_1, k_2, k_3, -q) &= 0 \ , \end{aligned} \tag{4.5}$$

where n_{10} is the kinematic numerator of the first graph in fig. 3 and q is a generic momentum. In general, the Jacobi relations single out (non-unique) subsets of color graphs – master graphs [72, 64] – whose numerators determine all the other ones through Jacobi transformations. In our case it is not difficult to see that eqs. (4.5), (B.1) allow us to express the numerator factors of graphs with triangle subintegrals in terms of the numerator factors of the box integrals, the numerator factors of graphs with bubble sub-integrals in terms of those of graphs with triangle sub-integrals and so on. In other words, the three box graphs can be chosen as master graphs.

This construction is valid in any gauge theory with adjoint matter at one loop level. We found that in the cases of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ sYM theory it is also possible to impose a further requirement. While generic one-loop Jacobi-satisfying representations have bubble contributions, for these theories, the bubble and tadpole graphs can be chosen to have vanishing numerator factors; in particular, we can require that $n_{10,11,12}(k_1, k_2, k_3, q) = 0$.

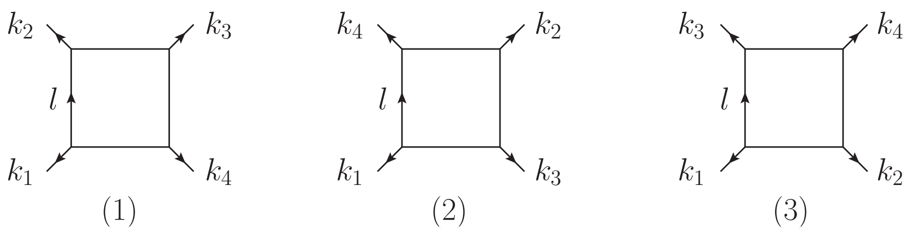

Figure 1: Box graphs contributing to the four-point one-loop amplitudes in $\mathcal{N} \leq 4$ sYM theories.

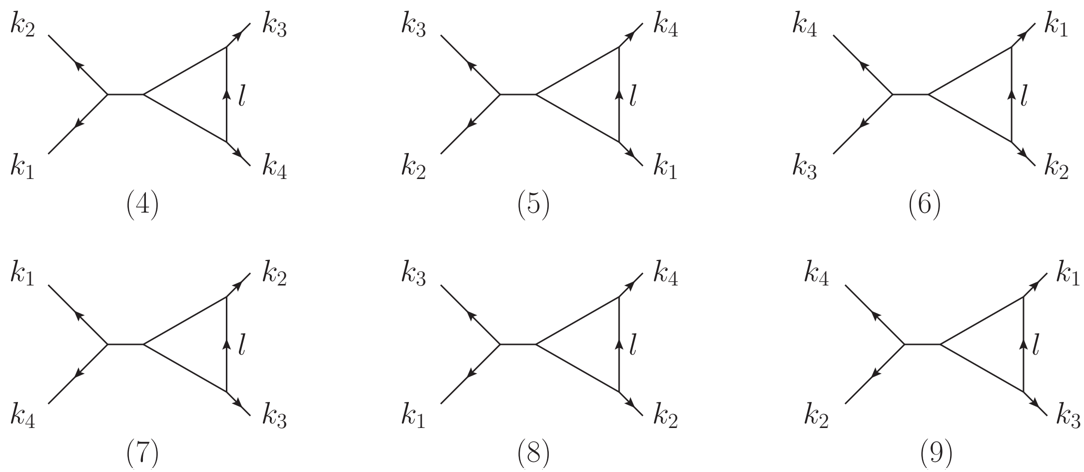

Figure 2: Graphs with triangle subgraphs contributing to the four-point one-loop amplitudes in $\mathcal{N} < 4$ sYM theories.

Such a requirement could have been inconsistent with the cut conditions, but is not for these theories.

While in $\mathcal{N} = 4$ sYM the numerator factors of the three box graphs in fig. 1 can be related by the relabeling of the graphs' external legs, for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ theories such a symmetric choice of numerator factors does not appear to be possible. Without loss of generality, we will choose k_1 and k_2 to be associated with negative helicity gluons, and k_3 and k_4 to be associated with positive helicity gluons.

To determine the kinematic numerators associated with these three master graphs we proceed to construct an ansatz for each of them, subject to the constraints mentioned above which are also sufficiently large to solve all the unitarity constraints. The known results on the structure of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ amplitudes [73] suggests that this ansatz should contain at least two powers of the loop momentum. Moreover, the presence of inverse factors of Mandelstam invariants [73] suggests that this ansatz should also contain such factors (and thus is mildly nonlocal). Last, factors of spinor products are necessary to account for the helicity of the scattered particles. We will choose this factor

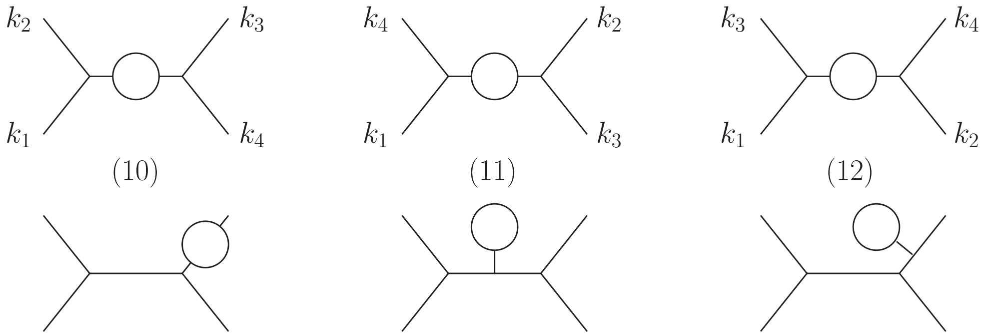

Figure 3: Cubic graphs containing bubble and tadpole subgraphs; they need not contribute to the four-gluon amplitudes in the $\mathcal{N}=1$ and $\mathcal{N}=2$ sYM theory.

to be the completely symmetric combination

$$istA(1^-\,2^-\,3^+\,4^+) = \langle 12\rangle^2[34]^2\ . \tag{4.6}$$

such a choice may in principle introduce additional s^{-1} and t^{-1} factors compared to the expected dependence of amplitudes on Mandelstam invariants. The inclusion of inverse factors of Mandelstam invariants in the kinematic numerator factors, organized in particular as inverse powers of Gram determinants, was shown [74] to be necessary already at tree-level for color-kinematic satisfying representations if one requires that the external state information is encoded symmetrically by a tree-level amplitude factor.

A further departure form the structure of the $\mathcal{N}=4$ sYM integrand is the presence of parity-odd terms. While such terms necessarily integrate to zero, they are required for the cut conditions to be satisfied. They are also crucial for the double-copy construction of gravity amplitudes[18]. Collecting everything, the ansatz for the kinematic numerator factors of the master graphs is

$$\begin{aligned} n_i &= stA(k_1,k_2,k_3,k_4)\,N_i \\ N_i &= \frac{1}{(stu)^2}\Big(P_{i;(6,2)}(\tau_{l,k_1},\tau_{l,k_2},\tau_{l,k_3};s,t) + 4i\varepsilon(k_1,k_2,k_3,l)\,P_{i;(4,0)}(\tau_{l,k_1},\tau_{l,k_2},\tau_{l,k_3};s,t)\Big)\ , \end{aligned} \tag{4.7}$$

where $i=1,2,3$ labels the master graph, $P_{i;(j,k)}(\tau_{l,k_1},\tau_{l,k_2},\tau_{l,k_3};s,t)$ are degree-j polynomials which also have degree-k in their first three arguments, and $\varepsilon(k_1,k_2,k_3,l) = \varepsilon_{\mu\nu\rho\sigma}k_1^\mu k_2^\nu k_3^\rho l^\sigma$. We recognize the overall factor as the square of the inverse of the Gram determinant of the three independent momenta. Depending on the helicity configuration it is moreover possible to choose two of the three odd terms to vanish identically. Recall that we chose for particles 1 and 2 to have negative helicity, so in this case graphs 1 and 3 will have vanishing parity-odd terms. Using the above ansatz, Jacobi constraints and

[18]Nontrivial contribution to supergravity amplitudes from terms that vanish upon integration in gauge theory was noticed in $\mathcal{N}=8$ supergravity at four loops in [64].

imposing the absence of bubbles and tadpoles leaves us with 45 free parameters which are theory dependent and to be constrained by cuts. Due to the Jacobi relations, which determine six numerator factors in terms of those of the three master graphs, this number of free parameters is far smaller than the number of free parameters of an ansatz covering all nine contributing integrals.

4.3 Four-gluon amplitudes in $\mathcal{N} = 1$ and $\mathcal{N} = 2$ sYM theories

To determine the remaining 45 coefficients we require that the cuts of the ansatz are the same as the unitarity cuts of the one-loop four-gluon amplitudes in these theories evaluated directly in terms of state sums over products of tree-level amplitudes. We could use either color-dressed cuts or, alternatively, project the ansatz onto its color-ordered and color-stripped components and use color-ordered cuts. In either case the supersums (i.e. the sum over intermediate states) are evaluated through the general strategy [44] that relates them to the corresponding $\mathcal{N} = 4$ supersums:

$$\mathcal{S}_{\mathcal{N}=4} = (A + B + C + \dots)^4 \;\longrightarrow \\ \mathcal{S}_{\mathcal{N}<4} = (A + B + C + \dots)^{\mathcal{N}}(A^{4-\mathcal{N}} + B^{4-\mathcal{N}} + C^{4-\mathcal{N}} + \dots)\;, \quad (4.8)$$

where A^4, B^4, etc. are the contribution of gluon intermediate states to a given MHV amplitude. The MHV-vertex expansion straightforwardly carries this organization of amplitudes in terms of $\mathcal{N} = 4$ gluonic contributions to any $\mathrm{N}^{k\geq 0}$MHV amplitude as a sum over such expressions [44]. The presence of supersymmetry implies [33] that one-loop $\mathrm{N}^{k\geq 0}$MHV amplitudes in these theories are four-dimensional cut-constructible, at least at the integrated level, and therefore, as long as the amplitudes are used in four dimensions, a four-dimensional cut calculation is sufficient. Here we will require that the ansatz satisfies the six distinct four-dimensional color-ordered two-particle cuts. The only guarantee on the supergravity side is that the double-copy amplitude will satisfy all four-dimensional cuts. Verification on higher-dimensional cuts would be sufficient to demonstrate that there are no missing contributions to the gravity amplitude. Additional work is necessary to construct $\mathcal{N} < 4$ sYM amplitudes in D-dimensions. While these (unrenormalized) amplitudes are divergent already in four dimensions [73], their D-dimensional expressions are necessary to identify the critical dimension of the supergravity theories that have them as one of the double-copy factors.

As is well-known, one-loop amplitudes in gauge and supergravity theories may be decomposed into the independent contributions from the various fields running in the loop. These contributions may also be organized in multiplets of the unbroken symmetries; in particular, for $\mathcal{N}$-extended pure sYM theories, one-loop n-point gluon amplitudes may be organized as

$$A^{(1)}_{n,\mathcal{N}=4} = A^{(1)}_{n,\mathcal{N}=4} - (4-\mathcal{N})A^{(1)}_{n,\text{chiral}}\;, \quad (4.9)$$

where $A^{(1)}_{n,\text{chiral}}$ is the contribution of a single $\mathcal{N}=1$ chiral multiplet in the loop. Since the Jacobi relations are linear one should expect that numerator factors obeying color/kinematics duality should have similar properties; that is

$$N_i^{\mathcal{N}} = N_i^{\mathcal{N}=4} - (4-\mathcal{N})N_i^{\text{chiral}} \quad , \qquad N_{1,2,3}^{\mathcal{N}=4} = 1 \ , \quad N_{4,5,6,7,8,9}^{\mathcal{N}=4} = 0 \ . \tag{4.10}$$

This is indeed what we find.

The six two-particle cut conditions fix uniquely the parity-odd part of the chiral multiplet contribution as well as the parity-even part of two of the nine graphs – graphs 4 and 6 in fig 2. The remaining factors depend on 10 free parameters; this is a reflection of the generalized-gauge symmetry [5, 29] that allows the addition of zero to an amplitude in the form of any arbitrary function multiplied by the color-space Jacobi identity. This representation freedom at the integrand level does not affect the content of the amplitude – any value of these parameters is consistent with the cuts of the theory, and therefore cannot affect the resulting integrated expressions. Indeed, upon integral reduction[19] to basis integrals all free parameters disappear and we recover the classic results of ref. [73]. To present our results we make a particular choice for the free parameters which simplifies the numerator factors; with this choice the numerator factors corresponding to the chiral multiplet contribution is given by:

$$\begin{aligned}
N_1^{\text{chiral}} &= \frac{1}{s}(\tau_{l,k_1}-\tau_{l,k_2}) + \frac{2}{stu}(s\tau_{l,k_1}\tau_{l,k_2} - u\tau_{l,k_1}\tau_{l,k_3} - t\tau_{l,k_2}\tau_{l,k_3})\\
N_2^{\text{chiral}} &= \frac{1}{s^2}(t\tau_{l,k_2}+s\tau_{l,k_3}-u\tau_{l,k_1}) + \frac{2}{stu}(s\tau_{l,k_1}\tau_{l,k_2} - u\tau_{l,k_1}\tau_{l,k_3} - t\tau_{l,k_2}\tau_{l,k_3}) - \frac{2i}{s^2}\varepsilon_{k_1,k_2,k_3,l}\\
N_3^{\text{chiral}} &= -\frac{1}{u}(\tau_{l,k_3}+\tau_{l,k_1}) + \frac{2}{stu}(s\tau_{l,k_1}\tau_{l,k_2} - u\tau_{l,k_1}\tau_{l,k_3} - t\tau_{l,k_2}\tau_{l,k_3})\\
N_4^{\text{chiral}} &= -\frac{1}{u}(\tau_{l,k_1}+\tau_{l,k_3}) + \frac{1}{t}(\tau_{l,k_2}+\tau_{l,k_3})\\
N_5^{\text{chiral}} &= \frac{t}{s^2u}(s(\tau_{l,k_1}+\tau_{l,k_3}) - u(\tau_{l,k_1}+\tau_{l,k_2})) + \frac{2i}{s^2}\varepsilon_{k_1,k_2,k_3,l}\\
N_6^{\text{chiral}} &= 0\\
N_7^{\text{chiral}} &= \frac{1}{s^2}(t\tau_{l,k_1}-u\tau_{l,k_2}-s\tau_{l,k_3}) - \frac{2i}{s^2}\varepsilon_{k_1,k_2,k_3,l}\\
N_8^{\text{chiral}} &= \frac{u}{s^2t}(s(\tau_{l,k_2}+\tau_{l,k_3}) - t(\tau_{l,k_1}+\tau_{l,k_2})) - \frac{2i}{s^2}\varepsilon_{k_1,k_2,k_3,l}\\
N_9^{\text{chiral}} &= \frac{1}{s^2}(t\tau_{l,k_1}-u\tau_{l,k_2}+s\tau_{l,k_3}) - \frac{2i}{s^2}\varepsilon_{k_1,k_2,k_3,l}
\end{aligned} \tag{4.11}$$

Note that, unlike the $\mathcal{N}=4$ sYM theory, the chiral multiplet numerator factors have both a parity-even and a parity-odd component already for the four-point amplitudes. On symmetry ground these terms integrate to zero; they however make nontrivial contributions to the corresponding supergravity amplitudes though the double-copy construction

[19] In the process of reducing to an integral basis we must drop all bubble-on-external-line integrals as such integrals should not appear at this loop order in supersymmetric theories. Here they may not cancel automatically and we did not explicitly exclude them in order to preserve the simplicity of the solution.

(4.4) which we discuss in the next section. Carrying out the reduction to an integral basis we reproduce the classic expressions of [73] through $\mathcal{O}(\epsilon^0)$.

4.4 Orbifold supergravities as double-copies

Together with the known expressions for the one-loop $\mathcal{N} = 4$ sYM amplitudes which automatically exhibit color/kinematics duality, the amplitudes constructed in the previous subsection allow us to recover through the double-copy construction

$$\mathcal{A}_4^{(1)} = i\, g^4 \sum_{i=1}^{9} \int \frac{d^D l}{(2\pi)^D} \frac{n_i C_i}{\prod_{\alpha_i} p_{\alpha_i}^2}\,, \tag{4.12}$$

$$\mathcal{M}_4^{(1)} = -\left(\frac{\kappa}{2}\right)^4 \sum_{i=1}^{9} \int \frac{d^D l}{(2\pi)^D} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2}\,, \tag{4.13}$$

some of the four-point amplitudes found though direct computation in sec. 3 for the factorized orbifold supergravity theories listed in Table 3. In eqs. (4.12) and (4.13) the numerator factors n_i and $\tilde{n}_i$ are given by eqs. (4.7), (4.10) and (4.11). We will emphasize here the main differences from the $\mathcal{N} \geq 4$ supergravity theories in which one of the two copies is $\mathcal{N} = 4$ sYM theory and discuss the theories in which both factors are the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ sYM theories studied in the previous section. While we will focus on graviton amplitudes, it is important to stress that, due to the reduced supersymmetry, there exist additional four-point one-loop scattering amplitudes not related to these by supersymmetric Ward identities; some of them are evaluated in sec. 3.

The essential feature of the one-loop four-point amplitude of $\mathcal{N} = 4$ sYM theory is that its kinematic numerator factors are independent of loop momenta. This property (which also persists at two loop level), extensively exploited in [66, 24], implies that it is possible to construct four-point amplitudes in $\mathcal{N} \geq 4$ supergravities directly in terms of integrated gauge theory amplitudes. In contrast, the kinematic numerator amplitudes found in the previous section contain nontrivial loop momentum dependence. Thus, they may be used to construct supergravity amplitudes at the integrand level. We have checked that indeed the four-graviton amplitudes in the theories listed on rows six, eight and thirteen of Table 3 are correctly reproduced. To reproduce amplitudes in the other theories though the double-copy construction one only requires an integrand representation of the numerator factors in pure YM theory coupled with some number of scalar fields.

While the parity-odd terms present in the numerator factors (4.11) trivially integrate to zero in the gauge theory amplitudes they make nontrivial contributions to the corresponding supergravity amplitude; in particular, they are required for the UV finiteness of the four-graviton amplitude (they also produce parity-odd terms which integrate to zero). Introducing a flag f to highlight the importance these contributions (which should be taken to 1 for the actual amplitude), we present the result of the integral reduction of the four-graviton amplitude in the $\mathcal{N}=2$ supergravity coupled to one hypermultiplet realized as a double-copy of the $\mathcal{N}=1$ sYM theory given in the color-kinematics-satisfying representation above:

$$\begin{aligned}\mathcal{M}^{(1)}_{4,\mathcal{N}=1\times\mathcal{N}=1} = \mathcal{M}^{(1)}_{4,\mathcal{N}=8} + i\left(\frac{\kappa}{2}\right)^4 M_{\text{tree}}\Big[\, 6\,\frac{tu}{s^2}(s^2-\frac{3(1+f)}{4}tu)\, I_4^{D=6-2\epsilon}(t,u) \\ +\frac{9}{4}\,(1+f)\frac{tu}{s^2}(t-u)(I_2(u)-I_2(t)) + \frac{9}{2}\frac{1+(3-D)f}{D-2}(I_2(u)+I_2(t))\Big]\Big|_{f\to 1}\,.\end{aligned} \tag{4.14}$$

This expressions reproduces eq. (3.7) with coefficients given by row eleven of Table 5 which was derived though the direct evaluation of supergravity cuts. Through a similar strategy we can easily construct the one-loop four-graviton amplitude for the supergravity theory on row thirteen of Table 3. Keeping the flag tracking the contribution of the terms that vanish upon integration in the sYM amplitude, it reads

$$\begin{aligned}\mathcal{M}^{(1)}_{4,\mathcal{N}=2\times\mathcal{N}=1} = \mathcal{M}^{(1)}_{4,\mathcal{N}=8} + i\left(\frac{\kappa}{2}\right)^4 M_{\text{tree}}\Big[\,\frac{tu}{s^2}(5s^2-3(1+f)tu)\, I_4^{D=6-2\epsilon}(t,u) \\ +\frac{3}{2}\,(1+f)\frac{tu}{s^2}(t-u)(I_2(u)-I_2(t)) + 3\frac{1+(3-D)f}{D-2}(I_2(u)+I_2(t))\Big]\Big|_{f\to 1}\,,\end{aligned} \tag{4.15}$$

and, as expected, reproduces eq. (3.7) with coefficients given by row six of Table 5.

The third four-graviton amplitude that can be constructed from the color/kinematics-satisfying sYM amplitudes we have constructed is in the $\mathcal{N}=4$ supergravity coupled to two vector multiplets realized as a double-copy of the $\mathcal{N}=2$ sYM theory. Reducing the result of the double-copy construction to the standard integral basis yields

$$\begin{aligned}\mathcal{M}^{(1)}_{4,\mathcal{N}=2\times\mathcal{N}=2} = \mathcal{M}^{(1)}_{4,\mathcal{N}=8} + i\left(\frac{\kappa}{2}\right)^4 M_{\text{tree}}\Big[\, 4\,\frac{tu}{s^2}(s^2-\frac{1+f}{2}tu)\, I_4^{D=6-2\epsilon}(t,u) \\ +(1+f)\frac{tu}{s^2}(t-u)(I_2(u)-I_2(t)) + 2\frac{1+(3-D)f}{D-2}(I_2(u)+I_2(t))\Big]\Big|_{f\to 1}\,;\end{aligned} \tag{4.16}$$

it indeed reproduces eq. (3.11) which was derived though the direct evaluation of supergravity cuts. It also reproduces the result of the double-copy construction applied to $\mathcal{N}=4$ sYM theory and pure YM theory coupled to two scalar fields [66, 24].

We have also evaluated a number of four-dimensional cuts of higher-loop and higher-point amplitudes in the two presentations of $\mathcal{N}=4$ supergravity coupled with two vector multiplets and found that they are the same. This suggests that the scattering amplitudes of $\mathcal{N}=2|_{\mathbf{Z}_2}\times\mathcal{N}=2|_{\mathbf{Z}_2}$ and $\mathcal{N}=4\times\mathcal{N}=0|_{\mathbf{Z}_2\times\mathbf{Z}_2}$ theories are the same to all loop orders and to all multiplicities, at least at the integrand level.

5 Discussion

In this paper we discussed orbifold supergravity theories with $\mathcal{N}=8$ supergravity as parent that preserve at least minimal amount of supersymmetry; similarly to non-planar orbifold gauge theories (and contrary to their planar limit), loop amplitudes are not inherited from the parent and an explicit calculation is necessary. Realizing $\mathcal{N}=8$ supergravity tree-level amplitudes through the KLT relations, we identified *all* different supergravity theories that can be obtained with an orbifold group that acts independently on the two $\mathcal{N}=4$ sYM amplitude factors. Using generalized unitarity in four dimensions we have computed large classes (up to relabeling and use of supersymmetry Ward identities) of four-point one-loop amplitudes of all such supergravity theories. Before reduction to the usual basis of box, triangle and bubble integrals the amplitudes are sums of tensor integrals whose rank depends on the amount of supersymmetry. After reduction to the integral basis all surviving triangle integrals can be organized in terms of the six-dimensional box integral. Both before and after integral reduction the amplitudes exhibit a mild non-locality involving inverse powers of Mandelstam invariants; similar features have been previously observed in sYM theories with less-than-maximal supersymmetry.

Our results show that the twin theories with $\mathcal{N}=2$ and $\mathcal{N}=4$, sharing the same bosonic field content and scalar manifold, have different four-point one-loop scattering amplitudes. We have also observed that all four-point one-loop amplitudes of the two different realizations of the $\mathcal{N}=4$ supergravity with two vector multiplets are identical and argued that this equality persists to all loop orders and to all multiplicities as well as away from four dimensions. Thus, the two orbifolds, $\mathcal{N}=2|_{\mathbf{Z}_2}\times\mathcal{N}=2|_{\mathbf{Z}_2}$ and $\mathcal{N}=4\times\mathcal{N}=0|_{\mathbf{Z}_2\times\mathbf{Z}_2}$, appear to lead to the same theory. This D-dimensional equivalence is reminiscent of, if perhaps not quite as dramatic as, the equivalence of the double-copy construction of $\mathcal{N}=16$ supergravity in three dimensions using either two copies of $\mathcal{N}=8$ sYM theory arranged on color-kinematic satisfying cubic graphs or two copies of the BLG theory arranged on color-kinematic satisfying quartic graphs [30].

We also analyzed examples in which the orbifold group action is not factorized. Among them are pure supergravity theories with $\mathcal{N}=3$, $\mathcal{N}=2$ and $\mathcal{N}=1$ supersymmetry. It would be interesting to analyze them at higher loops and thus probe their UV properties, in particular at three loops, where the first divergence is expected to appear. More general, it would also be interesting to classify all supergravity theories that may be obtained from $\mathcal{N}=8$ supergravity though non-factorized orbifold group action and explore their properties.

The calculation of amplitudes in all theories with a factorized orbifold group action can also be approached though the expected double-copy property of supergravity amplitudes. While for one of the two gauge theories only an integrand representation is sufficient, for the other one a representation satisfying color/kinematics duality is required. We have constructed a representation (valid in four dimensions) of the one-loop four-gluon amplitude of pure $\mathcal{N} = 2$ and $\mathcal{N} = 1$ sYM theories. Due to the special relations between numerator factors imposed by the kinematic Jacobi relations, the size of the ansatz needed for such a construction is substantially smaller than that used for the direct unitarity-based construction. We have found that it is not possible to satisfy both the color/kinematics duality constraints and the cut conditions with a local choice of numerator factors for all contributing graphs and a certain amount of nonlocality in the form of inverse powers of Mandelstam invariants is required. This suggests that, in some sense, generic amplitudes will only have nonlocal representations. This is exactly the case for multi-loop maximally supersymmetric five-point color-kinematic representations [65] as well as certain tree-level representations [59]. Using these representations we have recovered through the double-copy construction the results of the direct unitarity-based calculation. It should be stressed that D-dimensional representations of the same sYM amplitudes are required to construct supergravity amplitudes whose integrand vanishes identically in four dimensions.

It should be possible and interesting to construct multi-loop and higher-multiplicity amplitudes obeying color/kinematics duality for $\mathcal{N} = 2$ and $\mathcal{N} = 1$ sYM theories. The structure of the duality satisfying representations presented here is very suggestive that the gluonic γ and β functions of ref. [65] should encode the gluonic supersymmetric color-kinematic satisfying representations, albeit with potentially higher powers of the Gram determinant in the denominator to allow for the required additional loop-momentum dependence. Such constructions would further probe the need for nonlocal kinematic numerator factors as well as the properties of nonlocal integrands for scattering amplitudes.

In any case, it is clear that powerful methods and structures have now brought the ability to directly probe the S-matrix of intriguingly more intricate (and potentially phenomenologically relevant) supersymmetric gauge and gravity theories – not only integrated, but at the integrand level as well. Indeed, integrand level considerations are an incredibly powerful perspective for exposing invariant unifying structure that remains true independent of regularization schemes. This type of data will undoubtedly prove useful in the addressing of many open theoretic questions such as the ultimate role of non-linear dualities, shift symmetries, and unifications.

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A Integral Reduction

In this section we list the integral reductions used to obtain the final expressions for the amplitudes listed in the main body of the paper. The external momenta q_1, q_2 are assumed to be strictly massless and four-dimensional. We have the following identities involving scalar triangles with numerator factors,

$$
\begin{aligned}
I_3\big(l^\mu; q_1, q_1+q_2\big) &= \frac{q_2^\mu - q_1^\mu}{2q_1\cdot q_2} I_2(q_1+q_2) - q_1^\mu I_3(q_1, q_1+q_2)\ , \\
I_3\big(l^\mu l^\nu; q_1, q_1+q_2\big) &= \frac{\eta^{\mu\nu}}{2d-4} I_2(q_1+q_2) + q_1^\mu q_1^\nu \Big(\frac{3}{4}\frac{I_2(q_1+q_2)}{q_1\cdot q_2} + I_3(q_1, q_1+q_2)\Big) - \\
&\quad \frac{q_1^{(\mu} q_2^{\nu)} d}{4(d-2)} \frac{I_2(q_1+q_2)}{q_1\cdot q_2} - \frac{q_2^\mu q_2^\nu}{4 q_1\cdot q_2} I_2(q_1+q_2)\ , \\
I_3\big(l^\mu l^\nu l^\rho; q_1, q_1+q_2\big) &= -\frac{\eta^{(\mu\nu} q_1^{\rho)} d}{4(d-1)(d-2)} I_2(q_1+q_2) - \frac{\eta^{(\mu\nu} q_2^{\rho)}}{4(d-1)} I_2(q_1+q_2) - \\
&\quad q_1^\mu q_1^\nu q_1^\rho \Big(\frac{7d-6}{8(d-1)} \frac{I_2(q_1+q_2)}{q_1\cdot q_2} + I_3(q_1, q_1+q_2)\Big) + \\
&\quad \frac{q_1^{(\mu} q_1^\nu q_2^{\rho)} d(d+2)}{8(d-1)(d-2)} \frac{I_2(q_1+q_2)}{q_1\cdot q_2} + \\
&\quad \frac{q_1^{(\mu} q_2^\nu q_2^{\rho)} (d+2)}{8(d-1)} \frac{I_2(q_1+q_2)}{q_1\cdot q_2} + \frac{q_2^\mu q_2^\nu q_2^\rho d}{8(d-1)} \frac{I_2(q_1+q_2)}{q_1\cdot q_2}\ , \\
I_3\big(l^\mu l^\nu l^\rho l^\sigma; q_1, q_1+q_2\big) &= -\frac{\eta^{(\mu\nu}\eta^{\rho\sigma)}}{4d(d-1)} (q_1\cdot q_2) I_2(q_1+q_2) + \frac{\eta^{(\mu\nu} q_1^\rho q_1^{\sigma)} (d+2)}{8(d-1)(d-2)} I_2(q_1+q_2) + \\
&\quad \frac{\eta^{(\mu\nu} q_1^\rho q_2^{\sigma)} (d+2)}{8d(d-1)} I_2(q_1+q_2) + \frac{\eta^{(\mu\nu} q_2^\rho q_2^{\sigma)}}{8(d-1)} I_2(q_1+q_2) + \\
&\quad q_1^\mu q_1^\nu q_1^\rho q_1^\sigma \Big(\frac{5(3d-2)}{16(d-1)} \frac{I_2(q_1+q_2)}{q_1\cdot q_2} + I_3(q_1, q_1+q_2)\Big) - \\
&\quad \frac{q_1^{(\mu} q_1^\nu q_1^\rho q_2^{\sigma)} (d+2)(d+4)}{16(d-1)(d-2)} \frac{I_2(q_1+q_2)}{q_1\cdot q_2} - \\
&\quad \frac{q_1^{(\mu} q_1^\nu q_2^\rho q_2^{\sigma)} (d+2)(d+4)}{16d(d-1)} \frac{I_2(q_1+q_2)}{q_1\cdot q_2} - \\
&\quad \frac{q_1^{(\mu} q_2^\nu q_2^\rho q_2^{\sigma)} (d+4)}{16(d-1)} \frac{I_2(q_1+q_2)}{q_1\cdot q_2} - \frac{q_2^\mu q_2^\nu q_2^\rho q_2^\sigma (d+2)}{16(d-1)} \frac{I_2(q_1+q_2)}{q_1\cdot q_2}\ .
\end{aligned}
\tag{A.1}
$$

Note that to obtain the expressions listed in the main body of the paper it is sufficient to apply the Passarino-Veltman reduction on the triangle integrals with up to four loop momenta in the numerator, as boxes do not have numerator factors and bubbles do not explicitly appear in the ansatz.

B Jacobi relations

We include here the Jacobi relations between box, triangle and bubble graph kinematic numerator factors.

$$
\begin{aligned}
N_1(k_1,k_2,k_3,k_2+k_3+q) - N_3(k_1,k_2,k_3,k_3+q) - N_4(k_1,k_2,k_3,-q) &= 0 \\
N_1(k_1,k_2,k_3,-k_1-q) - N_2(k_1,k_2,k_3,q) - N_5(k_1,k_2,k_3,q) &= 0 \\
N_1(k_1,k_2,k_3,-q) - N_3(k_1,k_2,k_3,q-k_1) - N_6(k_1,k_2,k_3,q) &= 0 \\
N_1(k_1,k_2,k_3,k_2+k_3+q) - N_2(k_1,k_2,k_3,-k_1+q) - N_7(k_1,k_2,k_3,-q) &= 0 \\
N_3(k_1,k_2,k_3,-q) - N_2(k_1,k_2,k_3,q-k_1) + N_9(k_1,k_2,k_3,q-k_1) &= 0 \\
N_3(k_1,k_2,k_3,k_3+q) - N_2(k_1,k_2,k_3,q) - N_8(k_1,k_2,k_3,-k_1-k_2-k_3-q) &= 0 \\
N_4(k_1,k_2,k_3,k_3+q) + N_4(k_1,k_2,k_3,k_1+k_2+k_3-q) - N_{10}(k_1,k_2,k_3,-q) &= 0 \\
N_5(k_1,k_2,k_3,-k_1-k_2-k_3+q) + N_5(k_1,k_2,k_3,-k_1-q) + N_{11}(k_1,k_2,k_3,-q) &= 0 \\
N_6(k_1,k_2,k_3,k_1+q) + N_6(k_1,k_2,k_3,-k_2-q) - N_{10}(k_1,k_2,k_3,-k_1-k_2-q) &= 0 \\
-N_8(k_1,k_2,k_3,k_4+q) - N_8(k_1,k_2,k_3,-k_2-q) - N_{12}(k_1,k_2,k_3,-k_1-k_3-q) &= 0 \\
-N_7(k_1,k_2,k_3,-q) + N_7(k_1,k_2,k_3,-k_3-q) - N_{11}(k_1,k_2,k_3,q) &= 0 \\
N_9(k_1,k_2,k_3,k_1+q) + N_9(k_1,k_2,k_3,-k_3-q) + N_{12}(k_1,k_2,k_3,q) &= 0
\end{aligned}
\tag{B.1}
$$

These relations imply that the three box integrals may be chosen as master graphs at one-loop in any gauge theory.

Further Jacobi relations link triangle and bubble graphs to tadpole graphs. Requiring that the latter vanish leads to:

$$
\begin{aligned}
N_4(k_1,k_2,k_3,q) + N_4(k_1,k_2,k_3,k_1+k_2+k_3-q) &= 0 \\
N_4(k_1,k_2,k_3,q) + N_4(k_1,k_2,k_3,k_3-q) &= 0 \\
N_5(k_1,k_2,k_3,q) + N_5(k_1,k_2,k_3,-k_1-q) &= 0 \\
N_5(k_1,k_2,k_3,q) + N_5(k_1,k_2,k_3,-k_1-k_2-k_3-q) &= 0 \\
N_6(k_1,k_2,k_3,q) + N_6(k_1,k_2,k_3,-k_2-q) &= 0 \\
N_6(k_1,k_2,k_3,q) + N_6(k_1,k_2,k_3,k_1-q) &= 0 \\
N_7(k_1,k_2,k_3,q) + N_7(k_1,k_2,k_3,-k_3-q) &= 0 \\
N_7(k_1,k_2,k_3,q) + N_7(k_1,k_2,k_3,k_2-q) &= 0 \\
N_8(k_1,k_2,k_3,q) + N_8(k_1,k_2,k_3,-k_2-q) &= 0 \\
N_8(k_1,k_2,k_3,q) + N_8(k_1,k_2,k_3,-k_1-k_2-k_3-q) &= 0 \\
N_9(k_1,k_2,k_3,q) + N_9(k_1,k_2,k_3,-k_3-q) &= 0 \\
N_9(k_1,k_2,k_3,q) + N_9(k_1,k_2,k_3,k_1-q) &= 0
\end{aligned}
$$

These relations constrain the parameters of the ansatz for the kinematic numerators for the master graphs though their relation to the triangle numerator factors implied by the Jacobi relations.

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