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Conformal Anomalies and Gravitational Waves

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We argue that the presence of conformal anomalies in gravitational theories can lead to observable modifications to Einstein's equations via the induced anomalous effective actions, whose non-localities can overwhelm the smallness of the Planck scale. The fact that no such effects have been seen in recent cosmological or gravitational wave observations therefore imposes strong restrictions on the field content of possible extensions of Einstein's theory: all viable theories should have vanishing conformal anomalies. We then show that a complete cancellation of conformal anomalies in $D = 4$ for both the C^2 invariant and the Euler (Gauss-Bonnet) invariant E_4 can only be achieved for N-extended supergravities with $N \geqslant 5$, as well as for M theory compactified to four dimensions.

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1. Introduction. In this Letter, we study the effect of conformal anomalies on low energy (sub-Planckian) gravitational physics. While the effect of the non-anomalous part of the non-local effective action on Einstein's equations for any matter coupling is completely negligible away from the Planck regime, due to the powers of $\ell_{\rm Pl}$ (the Planck length) that come with higher order curvature and other quantum corrections, we will here argue that the *anomalous part* $\Gamma_{\rm anom}[g,\phi]$ of the non-local effective action, which is responsible for the conformal anomaly via formula (3) below, is not only relevant for the quantised theory, but could also affect classical low energy processes (such as the propagation of gravitational waves) in observable ways, for instance by exciting longitudinal gravitational degrees of freedom. This effect is made possible through the compensation of the smallness of the Planck scale $\ell_{\rm Pl}$ by a large factor coming from the non-locality, analogous to the standard axial anomaly which arises by the competition of a UV divergence and a vanishing expression in $D = 4$ (*e.g.* in dimensional regularisation). It can thus be viewed as a low energy (IR) manifestation of trans-Planckian physics. The absence of such modifications in recent cosmological or gravitational wave observations leads us to conclude that all conformal anomalies must cancel – in analogy with the cancellation of chiral gauge anomalies in the Standard Model of particle physics, required for the latter's consistency and renormalizability. Our considerations apply to any context where Einstein's equations play a central role (*e.g.* black holes, dispersion relations for gravitational waves, amplification of primordial quantum fluctuations, *etc.*).

2. Conformal Anomaly. We refer to [1–7] for a derivation and basic properties of the conformal anomaly, as well as for further references. The conformal anomaly has two sources, namely the fact that the UV regulator for any conformal matter system coupled to gravity necessarily breaks conformal invariance, and secondly the fact that even for classically conformal fields the functional measure depends on the metric in a non-local manner. In four dimensions the anomaly takes the form [1–3]

$$\mathcal{A} = T^{\mu}{}_{\mu} = \frac{1}{180(4\pi)^2}\left(c_s C^2 + a_s \mathrm{E}_4\right) \qquad (1)$$

where

$$\begin{aligned} C^2 &\equiv C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2 \\ \mathrm{E}_4 &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \end{aligned} \qquad (2)$$

and the coefficients c_s and a_s depend on the spin s of the field that couples to gravity, and on whether these fields are massless or massive, see Table below. E_4 is the Gauss-Bonnet density, a total derivative that gives a topological invariant when integrated over a 4-dimensional manifold.

While the anomaly (1) itself is a local expression, it is well known that it cannot be obtained by variation of a local action functional (otherwise the anomaly could be cancelled by a local counterterm). Accordingly, we are here concerned with the *anomalous* part $\Gamma_{\rm anom}[g,\phi]$ of the non-local effective action that gives rise to (1) via variation of the conformal factor,

$$\mathcal{A}(x) = -\frac{2}{\sqrt{-g(x)}}g_{\mu\nu}(x)\frac{\delta\Gamma_{\rm anom}[g,\phi]}{\delta g_{\mu\nu}(x)} \qquad (3)$$

where ϕ stands for any kind of matter field. In four dimensions such actions – apart from not being unique – are not known in completely explicit form, although there are partial results [5, 6, 8, 9]. This is in contrast to the case $D = 2$ where there *is* a unique expression satisfying all consistency requirements, namely the famous Polyakov formula $\Gamma^{D=2}_{\rm anom}[g] = -\frac{1}{2}\int\sqrt{-g}R\,\Box_g^{-1}R$, where $\Box_g^{-1}$ is the Green's function [10]. As is very well known [10, 11], the presence of this anomalous term in the effective action changes the behaviour of the quantised theory in dramatic ways – altering the critical dimension of the effective string theory and adding a new propagating (Liouville) degree of freedom to restore quantum conformal invariance. For higher dimensional gravity the *non-local* effects of $\Gamma_{\rm anom}[g,\phi]$ have not been much discussed so far (but see [8, 12, 13]), not only because of the absence of sufficiently explicit expressions, but also for the obvious reason that gravity is non-renormalizable, whence the further addition of anomalous vertices would only make bad things worse. By contrast, for a finite or renormalizable theory of quantum gravity the extra term does make a difference, like for Yang-Mills theories where anomalies are well known to spoil the renormalizability, hence the consistency, of the quantised theory.

3. Classical considerations. We now show that the presence of an anomalous contribution $\Gamma_{\rm anom}$ to the effective action *may affect the classical theory in observable ways*. The addition of $\Gamma_{\rm anom}$ to the effective action leads to an effective modification of Einstein's equations, *viz.*

$$\ell_{\rm Pl}^{-2}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)(x) = -\frac{2}{\sqrt{-g}}\frac{\delta\Gamma_{\rm anom}[g,\phi]}{\delta g^{\mu\nu}(x)}+\cdots \tag{4}$$

where the dots stand for non-anomalous matter contributions as well as contributions from non-anomalous higher order curvature corrections (which are negligible). The above equation entails the consistency condition

$$\nabla^{\mu}\left(\frac{2}{\sqrt{-g(x)}}\frac{\delta\Gamma_{\rm anom}[g,\phi]}{\delta g^{\mu\nu}(x)}\right) = 0 \tag{5}$$

where the vanishing divergence of the contribution from the non-anomalous part of the effective action follows from the standard Ward-Takahashi identities.

Our main interest here is in the variation of $\Gamma_{\rm anom}$ w.r.t. metric deformations which are *neither* trace *nor* diffeomorphisms; we may generically refer to such deformations as 'gravitational waves' (in the broad sense; the proper gravitational waves must satisfy the conditions at infinity first identified by Trautman [14]). Unlike for (3), such variations cannot be expected to be local, as is already evident for the Polyakov action for which

$$\frac{2}{\sqrt{-g}}\frac{\delta\Gamma^{D=2}_{\rm anom}[g]}{\delta g_{\mu\nu}} = 2g^{\mu\nu}R-2\nabla^{\mu}\nabla^{\nu}\left(\frac{1}{\Box_g}R\right) \tag{6}$$
$$+\,(g^{\mu\alpha}g^{\nu\beta}-\frac{1}{2}g^{\mu\nu}g^{\alpha\beta})\nabla_{\alpha}\left(\frac{1}{\Box_g}R\right)\nabla_{\beta}\left(\frac{1}{\Box_g}R\right)$$

Tracing with $g_{\mu\nu}$ we recover the standard conformal anomaly; also, this expression does satisfy (5) identically. This result illustrates our claim that variations of $\Gamma_{\rm anom}$ other than trace variations remain non-local. Equally important, it exhibits potential singularities associated with the Green's functions $\Box_g^{-1}(x,y)$. However, in this particular case the singularities are harmless because of a well known peculiarity of $D=2$ gravity: there are no deformations of the metric other than conformal or diffeomorphism variations [26].

For $D=4$, matters are much more involved, because the metric now carries propagating degrees of freedom. In this case $\Gamma_{\rm anom}$ will be a very complicated functional, in general involving an infinite series of products of Riemann tensors interspersed with massless propagators in some gravitational background. Even if (5) is satisfied, hence diffeomorphism invariance maintained, the example of Liouville theory teaches us that $\Gamma_{\rm anom}$ must be expected to excite *new* (longitudinal) gravitational degrees of freedom. In quantum gauge theory, this effect would entail the breaking of gauge invariance via the insertion of anomalous triangles into vector boson self-energy diagrams. In addition, the presence of $\Gamma_{\rm anom}$ can lead to observable consequences even in the classical approximation, provided we can show that the smallness of $\ell_{\rm Pl}$ can be overcome by the non-localities on the r.h.s. of (4).

To make all this a little more explicit we observe that, in lowest order, $\Gamma^{D=4}_{\rm anom}$ must contain a term $\propto \mathcal{K}\,\Box_g^{-1}R$ (also present in the anomalous effective action for E_4 proposed in [5]), where $\mathcal{K}\equiv R_{\rho\sigma\lambda\tau}R^{\rho\sigma\lambda\tau}$ is the Kretschmann scalar [27]. In analogy with (6), the Einstein equation (4) will thus be modified by a term proportional to

$$\propto\,\nabla_{\mu}\left(G\star\mathcal{K}\right)\nabla_{\nu}\left(G\star R\right)\;+\;(\mu\leftrightarrow\nu) \tag{7}$$

where G is the Green's function associated with the conformal d'Alembertian, *i.e.* $\big(-\Box_g+\frac{1}{6}R(x)\big)G(x)=\delta^{(4)}(x)$ (the symbol $\star$ stands for the convolution integral). The above term represents a physical effect because, unlike (6), it cannot be rendered local by an appropriate gauge choice. Unlike for the conventional treatment of Liouville theory, we now have to deal squarely with the Lorentzian signature, as there is no equivalence theorem relating pseudo-Riemannian and Riemannian geometry.

To arrive at a quantitative estimate we consider a cosmological solution of the classical Einstein equations with metric $ds^2=a^2(\eta)\big(-d\eta^2+d\mathbf{x}^2\big)$, where η is conformal time, and with the retarded Green's function [15]

$$G(\eta,\mathbf{x};\eta',\mathbf{y})=\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\cdot\frac{\delta(\eta-\eta'-|\mathbf{x}-\mathbf{y}|)}{a(\eta)a(\eta')} \tag{8}$$

valid for any profile of the scale factor $a(\eta)$. As an example let us consider the radiation era ending at $t=t_{\rm rad}$ and starting at $t_0=n_*l_P$ where n_* can be large (say, 10^8, *i.e.* long after the Planck era). Then $\eta=2\sqrt{t\,t_{\rm rad}}$ and $a(\eta)=\eta/(2t_{\rm rad})$; furthermore $R=0$ for radiation, so the last convolution in (7) receives contributions only from primordial perturbations, therefore we can very crudely estimate this contribution as $\nabla_0\left(G\star R\right)\approx 10^{-5}t_{\rm rad}^{-1}$ [16]. However, the first convolution integral gives a dramatically bigger contribution since in the radiation era $\mathcal{K}=3/(2t^4)$. With $\mathbf{x}=0$ and $t=t_{\rm rad}$, and neglecting subleading contributions we get

$$\frac{\partial}{\partial t}\int {\rm d}^3y\int_{\eta_0}^{\eta_{\rm rad}}{\rm d}\eta'\,\frac{a(\eta')^4}{4\pi|\mathbf{y}|}\frac{\delta(\eta-\eta'-|\mathbf{y}|)}{a(\eta)a(\eta')}\frac{3}{2t'^4}=\frac{8}{(n_*\ell_{\rm Pl})^3} \tag{9}$$

We therefore see that the pre-factor $\ell_{\rm Pl}^{-2}$ in (4) that would normally suppress the corrections on the r.h.s. of (4) can be 'beaten' by the anomalous action because the convolution integral picks up the contributions along the past lightcone back to $t_0 = n_* \ell_{\rm Pl}$, yielding an anomalous $T_{00} \sim 10^{-5} t_{\rm rad}^{-1} (n_* \ell_{\rm Pl})^{-3}$ that with our assumptions on n_* is much bigger than the l.h.s. $\sim t_{\rm rad}^{-2} \ell_{\rm Pl}^{-2}$. Note that we do not even need to go back all the way to the singularity $t \sim \ell_{\rm Pl}$ for this argument! In this way a perfectly smooth solution to Einstein's equation could receive a very large 'jolt' from the non-local contributions; similar 'jolts' could be expected for wave-like solutions.

Let us emphasise that no such effects are expected to arise from the non-anomalous part of the non-local effective action which is obtained by summing 1PI Feynman diagrams in the usual way, or equivalently by 'integrating out' massive modes. At low energies their contribution would reduce to vertices which are effectively local (similar to the 4-Fermi interaction in electroweak theory), and hence would remain suppressed. By contrast, the conformal anomaly is independent of the mass of the state that is integrated out, hence can be viewed as an IR manifestation of physics in the deep UV region. This feature is reflected in the anomalous part of the effective action, whose non-localities are independent of the masses of the particles contributing to the anomaly, as a consequence of which the resulting corrections to Einstein's equations need not remain suppressed (similar non-localities would be present in the Standard Model if the gauge anomalies did not cancel).

Extending this argument to the full theory requires knowledge of the complete expression for $\Gamma_{\rm anom}$ which is not available, and whose construction remains a formidable theoretical challenge. Although we expect that the higher order terms in $\Gamma_{\rm anom}$ will not affect our main conclusions, let us mention two possible caveats. One is that the non-localities in $\Gamma_{\rm anom}$ and (4) could cancel between different contributions; this appears unlikely in view of the fact that the cancellation would have to involve infinitely many terms. The other caveat is that a resummation in the series expansion of $\Gamma_{\rm anom}$ might soften or even remove the singularity of the inverse differential operators (conversely, it might also introduce *new* singularities, further strengthening the arguments presented above).

4. Cancelling the conformal anomaly. We now survey the theories for which the contribution from the different spins to the coefficients c_s and a_s cancel. Below we tabulate the coefficients c_s and a_s coming from the integration over massless and massive fields (first and last two columns, respectively), with spins up to $s = 2$. These results can be excerpted from the literature, although they are often not presented in precisely this form; for a derivation for the massless fields, see refs. [2–4, 7, 17–19]. The entries labeled 0^* give the results for two-form fields (which have no massive analog); these give the same contribution to c_0 as the scalars, but their contribution to the a_0 coefficient is different. The numbers for the massive fields are obtained in the usual way by adding the lower helicities – implying that spontaneous symmetry breaking does not affect the conformal anomaly.

We first consider the *sum* of the anomaly coefficients. As shown long ago [20] the sum $\sum(c_s + a_s)$ can be made to vanish for all $N = 3$ supermultiplets with maximum spin $s \leqslant 2$ by replacing one scalar by one two-form field in the spin-$\frac{3}{2}$ multiplet; the resulting multiplets can then be used as building blocks to arrange for all $N \geqslant 4$ supergravities to have vanishing $\sum(c_s + a_s)$. This implies in particular that the combined contribution vanishes for the maximal $N = 4$ Yang-Mills multiplet.

	massless		massive	
	c_s	a_s	c_s	a_s
$0(0^*)$	$\frac{3}{2}(\frac{3}{2})$	$-\frac{1}{2}(\frac{179}{2})$	$\frac{3}{2}(\varnothing)$	$-\frac{1}{2}(\varnothing)$
$\frac{1}{2}$	$\frac{9}{2}$	$-\frac{11}{4}$	$\frac{9}{2}$	$-\frac{11}{4}$
1	18	-31	$\frac{39}{2}$	$-\frac{63}{2}$
$\frac{3}{2}$	$-\frac{411}{2}$	$\frac{589}{4}$	-201	$\frac{289}{2}$
2	783	-571	$\frac{1605}{2}$	$-\frac{1205}{2}$

Table 1. Coefficients of the conformal anomaly in (1)

Next, turning to the C^2 anomaly, it is a remarkable fact that the presence of gravitinos, hence supergravity, is mandatory for any cancellation involving the coefficients c_s because the gravitinos are the only fields that contribute with a negative sign. It is easy to check that no $N \leqslant 4$ supergravity can achieve this because the summed contribution from the pure supergravity multiplets for $N \leqslant 4$ is positive, hence of the same sign as the one from any of the associated matter multiplets, so no cancellation of the C^2 anomaly is possible for these theories at all. One then checks that the total contribution $\sum c_s$ vanishes for the $N \geqslant 5$ supergravities,

$$\begin{aligned} c_2 + 5c_{\frac{3}{2}} + 10c_1 + 11c_{\frac{1}{2}} + 10c_0 &= 0 \\ c_2 + 6c_{\frac{3}{2}} + 16c_1 + 26c_{\frac{1}{2}} + 30c_0 &= 0\,, \\ c_2 + 8c_{\frac{3}{2}} + 28c_1 + 56c_{\frac{1}{2}} + 70c_0 &= 0\,, \end{aligned} \tag{10}$$

but for no other theories! Unlike the cancellation of gauge anomalies which can in principle be arranged for arbitrary chiral gauge groups by suitable choice of matter multiplets, the above cancellation thus singles out three very special theories.

Using the results from [21] one can furthermore show that, in addition to the cancellation for the massless supergravity multiplets, the contributions to c_s also cancel for the whole tower of Kaluza-Klein states of maximal $D = 11$ supergravity compactified on S^7, and they do so 'floor by floor' – exactly as for the one-loop β-function in [21, 22]! For the reader's convenience we reproduce below from [21] the full table of massive Kaluza-Klein multiplets with the Dynkin labels of the associated SO(8) representations (note that some towers start 'higher up', because no entry in a Dynkin label can be negative).

	$SO(8)$ representations
0	$[n{+}2\ 0\ 0\ 0]$, $[n\ 0\ 2\ 0]$, $[n{-}2\ 2\ 0\ 0]$, $[n{-}2\ 0\ 0\ 2]$, $[n{-}2\ 0\ 0\ 0]$
$\frac{1}{2}$	$[n{+}1\ 0\ 1\ 0]$, $n{-}1\ 1\ 1\ 0]$, $[n{-}2\ 1\ 0\ 1]$, $[n{-}2\ 0\ 0\ 1]$
1	$[n\ 1\ 0\ 0]$, $[n{-}1\ 0\ 1\ 1]$, $[n{-}2\ 1\ 0\ 0]$
$\frac{3}{2}$	$[n\ 0\ 0\ 1]$, $[n{-}1\ 0\ 1\ 0]$
2	$[n\ 0\ 0\ 0]$

Table 2. Massive states in the S^7 compactification [21]

Using the c_s coefficients for the massive representations from Table 1 and the dimension formula from [21] it is straightforward to prove that the sum over spins vanishes separately for each n. Furthermore, since the anomaly is independent of the chosen background solution it follows immediately that this result remains true *for any compactification of M theory to four dimensions*. However, for other compactifications, the cancellations in general are not expected to work 'floor by floor', but rather involve different 'floors' of the Kaluza Klein tower.

From the table we see that the contribution from the spin-0 fields to the Gauss-Bonnet invariant E_4 depends on the field representation, unlike c_0. From the results of [20] it follows directly that one can arrange for the sum $\sum a_s$ to vanish for all $N \geqslant 5$ supergravities with appropriate choice of field representation for the spin-0 degrees of freedom, but for no other theories. To be sure, the contributions to a_s fail to cancel if one chooses to represent all spin-0 degrees of freedom by scalar fields, and they also do not cancel for the massive Kaluza-Klein supermultiplets. Nevertheless, as shown in [21], even in that case the total contribution to a_s does cancel provided one adds up the contribution from *all* Kaluza-Klein supermultiplets. The argument involves a ζ-function regularisation and is basically due to the fact that one compactifies on an odd-dimensional manifold. Again, this result is background independent and thus remains valid for any compactification of M theory to four dimensions.

Finally, it has not escaped our attention that the complete cancellation of conformal anomalies takes place for the very same theories for which the composite R symmetry anomaly cancels [23]. As argued in [24, 25] it is precisely the presence or absence of this anomaly that is responsible for the finiteness properties of the $N \geqslant 5$ supergravities, and that explains the appearance of nonrenormalizable divergences in $N = 4$ supergravity at four loops. This remarkable coincidence leads us to conjecture that the conformal anomaly may play a similar role for the finiteness properties of extended supergravities. Indeed, it is easy to see that the insertion of $\Gamma_{\rm anom}$ will induce new contributions from four loops onward.

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