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Conformal anomalies and gravitational waves

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ABSTRACT

We argue that the presence of conformal anomalies in gravitational theories can lead to observable modifications to Einstein's equations via the induced anomalous effective actions, whose non-localities can overwhelm the smallness of the Planck scale. The fact that no such effects have been seen in recent cosmological or gravitational wave observations therefore imposes strong restrictions on the field content of possible extensions of Einstein's theory: all viable theories should have vanishing conformal anomalies. We then show that a complete cancellation of conformal anomalies in $D = 4$ for both the C^2 invariant and the Euler (Gauss–Bonnet) invariant E_4 can only be achieved for N-extended supergravity multiplets with $N \geqslant 5$, as well as for M theory compactified to four dimensions. Although there remain open questions, in particular concerning the true significance of conformal anomalies in non-conformal theories, as well as their possible gauge dependence for spin $s \geqslant \frac{3}{2}$, these cancellations suggest a hidden conformal structure of unknown type in these theories.

1. Introduction

In this Letter, we study the effect of conformal anomalies on low energy (sub-Planckian) gravitational physics. While the effect of the non-anomalous part of the effective action on Einstein's equations for any matter coupling is completely negligible away from the Planck regime, due to the powers of ℓ_{Pl} (the Planck length) that come with higher order curvature and other quantum corrections, we will here argue that the (non-unique) *anomalous part* Γ_{anom} of the non-local effective action, which is responsible for the conformal anomaly via formula (3) below, is not only relevant for the quantised theory, but could also affect classical low energy processes (such as the propagation of gravitational waves) in observable ways, for instance by exciting longitudinal gravitational degrees of freedom. This effect is made possible through the compensation of the smallness of the Planck scale ℓ_{Pl} by a large factor coming from the non-locality, analogous to the standard axial anomaly which arises by the competition of a UV divergence and a vanishing expression in $D = 4$ (*e.g.* in dimensional regularisation). It can thus be viewed as a low energy (IR) manifestation of trans-Planckian physics, in analogy with other anomaly induced low energy effects, such as axion couplings. The absence of such modifications in recent cosmological or gravitational wave observations leads us to conclude that all conformal anomalies must cancel. In principle, our considerations apply to any context where Einstein's equations play a central role (*e.g.* black holes, dispersion relations for gravitational waves, amplification of primordial quantum fluctuations, *etc.*).

2. Conformal anomaly

We refer to [1–10] for a derivation and basic properties of the conformal anomaly, as well as for further references. The conformal anomaly has two sources, namely the fact that the UV regulator for any conformal matter system coupled to gravity necessarily breaks conformal invariance, and secondly the fact that even for classically conformal fields the functional measure depends on the metric in a non-local manner. In four dimensions the anomaly takes the form

$$T^{\mu}{}_{\mu} = \frac{1}{180(4\pi)^2}\left(c_s C^2 + a_s \mathrm{E}_4\right) + \cdots \equiv \mathcal{A} + \cdots \tag{1}$$

where

$$C^2 \equiv C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$$
$$\mathrm{E}_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \tag{2}$$

and the coefficients c_s and a_s depend on the spin s of the field that couples to gravity, and on whether these fields are massless

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or massive, see Table 1. E_4 is the Gauss-Bonnet density, a total derivative that gives a topological invariant when integrated over a 4-dimensional manifold. A further possible term $\propto \Box_g R$ to (1) can be dropped as it is obtainable by variation of a local functional ($\propto R^2$). Because we will also be dealing with *non-conformal* theories we have put dots on the r.h.s. of (1) to allow for extra terms originating from explicit or spontaneous breaking of conformal invariance, such as mass terms $\propto m^2\varphi^2$; in general, there might even be non-local contributions, although we are not aware of any example where such a non-locality has been exhibited explicitly. The r.h.s. of (1), being expressible as the variational derivative of the (total) effective action w.r.t. the conformal factor, must satisfy the Wess–Zumino consistency condition, *independently* of whether the theory is classically conformal or not.[1]

While the anomaly $\mathcal{A}$ in (1) itself is a local expression, it is well known that it cannot be obtained by variation of a local action functional (otherwise the anomaly could be cancelled by a local counterterm). Accordingly, we are here concerned with the *anomalous* part $\Gamma_{\rm anom}[g,\phi]$ of the non-local effective action that gives rise to (1) via variation of the conformal factor,

$$\mathcal{A}(x) = -\frac{2}{\sqrt{-g(x)}}\, g_{\mu\nu}(x)\frac{\delta\Gamma_{\rm anom}[g,\phi]}{\delta g_{\mu\nu}(x)} \tag{3}$$

where ϕ stands for any kind of matter field. In four dimensions such actions – apart from not being unique – are not known in completely explicit form, although there are partial results. This is in contrast to the case $D=2$ where there *is* a unique expression satisfying all consistency requirements, namely the famous Polyakov formula $\Gamma^{D=2}_{\rm anom}[g] = -\frac{1}{2}\int\sqrt{-g}R\,\Box_g^{-1}R$, where $\Box_g^{-1}$ is the Green's function [14]. As is very well known [14,15], the presence of this anomalous term in the effective action changes the behaviour of the quantised theory in dramatic ways – altering the critical dimension of the effective string theory and adding a new propagating (Liouville) degree of freedom to restore quantum conformal invariance. For higher dimensional gravity the *non-local* effects of $\Gamma_{\rm anom}[g,\phi]$ have not been much discussed so far (but see [16–18]), not only because of the absence of sufficiently explicit expressions, but also for the obvious reason that gravity is non-renormalizable. By contrast, for a finite or renormalizable theory of quantum gravity the extra term does make a difference, like for Yang–Mills theories where anomalies are well known to spoil the renormalizability, hence the consistency, of the quantised theory. Similarly, the presence of conformal anomalies can spoil the finiteness or unitarity (or both) of Weyl invariant theories of gravity and supergravity [7].

3. Classical considerations

We now show that the presence of an anomalous contribution $\Gamma_{\rm anom}$ to the effective action *may affect the classical theory in observable ways*. The addition of $\Gamma_{\rm anom}$ to the effective action leads to an effective modification of Einstein's equations, *viz.*

$$\ell_{\rm Pl}^{-2}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)(x) = -\frac{2}{\sqrt{-g}}\frac{\delta\Gamma_{\rm anom}[g,\phi]}{\delta g^{\mu\nu}(x)} + \cdots \tag{4}$$

where the dots stand for non-anomalous matter contributions as well as contributions from non-anomalous higher order curvature corrections (which are negligible). The above equation entails the consistency condition

$$\nabla^\mu\left(\frac{2}{\sqrt{-g(x)}}\frac{\delta\Gamma_{\rm anom}[g,\phi]}{\delta g^{\mu\nu}(x)}\right) = 0 \tag{5}$$

which is automatically satisfied also for non-local $\Gamma_{\rm anom}[g]$ if the latter is diffeomeorphism invariant. The vanishing divergence of the contribution from the non-anomalous part of the effective action follows from the standard Ward–Takahashi identities.

Our main interest here is in the variation of $\Gamma_{\rm anom}$ w.r.t. metric deformations which are *neither* trace *nor* diffeomorphisms; we may generically refer to such deformations as 'gravitational waves' (in the broadest sense of the word, that these must only satisfy the conditions at infinity first identified by Trautman [19]). Unlike for (3), such variations cannot be expected to be local, as is already evident for the Polyakov action for which

$$\begin{aligned}\frac{2}{\sqrt{-g}}\frac{\delta\Gamma^{D=2}_{\rm anom}[g]}{\delta g_{\mu\nu}} &= 2g^{\mu\nu}R - 2\nabla^\mu\nabla^\nu\left(\frac{1}{\Box_g}R\right) \\ &+ \left(g^{\mu\alpha}g^{\nu\beta} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\right)\nabla_\alpha\left(\frac{1}{\Box_g}R\right)\nabla_\beta\left(\frac{1}{\Box_g}R\right)\end{aligned} \tag{6}$$

Tracing with $g_{\mu\nu}$ we recover the standard conformal anomaly; also, this expression does satisfy (5) identically. This result illustrates our claim that variations of $\Gamma_{\rm anom}$ other than trace variations remain non-local. Equally important, it exhibits potential singularities associated with the Green's functions $\Box_g^{-1}(x,y)$. However, in this particular case the singularities are harmless because of a well known peculiarity of $D=2$ gravity: there are no deformations of the metric other than conformal or diffeomorphism variations.[2]

For $D=4$, matters are much more involved, because the metric now carries propagating degrees of freedom; in this case $\Gamma_{\rm anom}[g]$ will be a rather more complicated functional. Various candidate actions have been discussed in the literature, either of non-local type [20,8,21,22,9], or, for spontaneously broken conformal symmetry, of local type with a dilaton [18,23–25]. In particular, for E_4 there is a closed form action ('Riegert action') analogous to the Polyakov action, and involving the inverse of a 4th order differential operator [7,20] which however does not produce conformally covariant correlators in the flat space limit [22,9,26] (in fact, no closed form action with this property seems to be known). Even if (5) is satisfied, hence diffeomorphism invariance maintained, the example of Liouville theory teaches us that $\Gamma_{\rm anom}$ must be expected to excite *new* (longitudinal) gravitational degrees of freedom. In quantum gauge theory, this effect would entail the breaking of gauge invariance via the insertion of anomalous triangles into vector boson self-energy diagrams. In addition, the presence of $\Gamma_{\rm anom}$ can lead to observable consequences even in the classical approximation, provided we can show that the smallness of $\ell_{\rm Pl}$ can be overcome by the non-localities on the r.h.s. of (4).

Because a detailed analysis of the resulting non-local effects and closed formulas for the Riegert and the dilaton actions are presented in [27], we will content ourselves here with a more general argument. To this aim we observe that, in lowest order, $\Gamma^{D=4}_{\rm anom}$ must contain a term $\propto E_4\,(\Box_g + \frac{1}{6}R)^{-1}R$, which is present in all actions that have been proposed so far when suitably expanded. Varying this term it is easy to see that the Einstein equation (4) will be modified by a term proportional to

[1] For *conformal* theories, there is a direct link between the one-loop divergence and the trace anomaly, as explained in [11,6,7]. This link is broken for non-conformal theories in the presence of a (gauge-dependent) divergent counterterm $\propto R^2$. The issue of gauge dependence and the status of the conformal anomaly for non-conformal theories is not completely settled [12], but see [13] for a proposal how to fix the potential ambiguities in accord with the Wess–Zumino consistency condition.

[2] In Liouville theory with *Euclidean* signature on compact Riemann surfaces all singularities are actually integrable because the scalar propagators only exhibit logarithmic singularities at coincident points and the worldsheets are compact.

$$\propto \nabla_\mu (G \star \mathrm{E}_4) \nabla_\nu (G \star R) + (\mu \leftrightarrow \nu) \tag{7}$$

where G is the Green's function associated with the conformal d'Alembertian, for which $\big(-\Box_g + \frac{1}{6}R(x)\big)G(x) = \delta^{(4)}(x)$, and the symbol $\star$ stands for the convolution integral (the locality after tracing with $g^{\mu\nu}$ is ensured by further contributions that we have not written out). The above term represents a physical effect because, unlike (6), it cannot be rendered local by an appropriate gauge choice. Unlike for the conventional treatment of Liouville theory, we now have to deal squarely with the Lorentzian signature, as there is no equivalence theorem relating pseudo-Riemannian and Riemannian geometry.

To arrive at a quantitative estimate we consider a cosmological solution of the classical Einstein equations with metric $ds^2 = a^2(\eta)\big(-d\eta^2 + d\mathbf{x}^2\big)$, where η is conformal time, and with the retarded Green's function [28]

$$G\big(\eta, \mathbf{x}; \eta', \mathbf{y}\big) = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} \cdot \frac{\delta(\eta - \eta' - |\mathbf{x}-\mathbf{y}|)}{a(\eta)a(\eta')} \tag{8}$$

valid for any profile of the scale factor $a(\eta)$. As an example let us consider the radiation era ending at $t = t_{\rm rad}$ and starting at $t_0 = n_* l_P$ where n_* can be large (say, 10^8, *i.e.* long after the Planck era). Then $\eta = 2\sqrt{t\,t_{\rm rad}}$ and $a(\eta) = \eta/(2t_{\rm rad})$; furthermore $R = 0$ for radiation, so the last convolution in (7) receives contributions only from primordial perturbations, therefore we can very crudely estimate this contribution as $\nabla_0 (G \star R) \approx 10^{-5} t_{\rm rad}^{-1}$ [29]. However, the first convolution integral gives a dramatically bigger contribution since in the radiation era $\mathrm{E}_4 = -3/(2t^4)$. With $\mathbf{x} = 0$ and $t = t_{\rm rad}$, and neglecting subleading contributions we get (with $t' = (\eta')^2/t_{\rm rad}$)

$$\frac{\partial}{\partial t}\int d^3y \int_{\eta_0}^{\eta_{\rm rad}} d\eta' \, \frac{a(\eta')^4}{4\pi|\mathbf{y}|}\frac{\delta(\eta - \eta' - |\mathbf{y}|)}{a(\eta)a(\eta')}\frac{3}{2t'^4} = \frac{8}{(n_*\ell_{\rm Pl})^3} \tag{9}$$

We therefore see that the pre-factor $\ell_{\rm Pl}^{-2}$ in (4) that would normally suppress the corrections on the r.h.s. of (4) can be 'beaten' by the anomalous action because the convolution integral picks up the contributions along the past lightcone back to $t_0 = n_*\ell_{\rm Pl}$, yielding an anomalous $T_{00}(t_{\rm rad}) \sim 10^{-5} t_{\rm rad}^{-1}(n_*\ell_{\rm Pl})^{-3}$ that with our assumptions on n_* is much bigger than the l.h.s. $\sim t_{\rm rad}^{-2}\ell_{\rm Pl}^{-2}$. Note that we do not even need to go back all the way to the Planck era $t \sim \ell_{\rm Pl}$ for this argument! In this way a perfectly smooth solution to Einstein's equation could receive a very large 'jolt' from the non-local contributions. Because the enhancement effect exhibited above persists also for the closed form Riegert action, where exact computations can be done, as well as for actions with a dilaton we expect that the precise form of $\Gamma_{\rm anom}$ will not alter this main conclusion [27].

Let us emphasise that no such effects are expected to arise from the non-anomalous part of the non-local effective action which is obtained by integrating out massive modes. To be sure, the split of the effective action into an anomalous and a non-anomalous part is not unique, but we here suppose that the ambiguities (which are Weyl invariant functionals) are not relevant to the enhancement effects discussed above. At low energies the contributions of the standard effective action reduce to vertices which are effectively local (similar to the 4-Fermi interaction in electroweak theory), and hence would remain suppressed. By contrast, the conformal anomaly should be viewed as an IR manifestation of physics in the deep UV region (similar to the axion gluon vertex which is thought to arise from anomalous triangles with superheavy quarks). This feature is reflected in the anomalous part of the effective action, whose non-localities are independent of the masses of the particles contributing to the anomaly, as a consequence of which the resulting corrections to Einstein's equations need not remain suppressed. Consequently, the terms indicated by ellipses in (1) are not relevant to the effects discussed above, which are present likewise for conformal *and* for non-conformal theories. To clarify this point it is helpful to recall the well known example of the axial anomaly. In the presence of a mass term (which breaks axial symmetry explicitly), the divergence of the axial current is

$$\partial_\mu J^{5\mu} = m\bar{\chi}\gamma^5\chi + \frac{\alpha}{4\pi}F^{\mu\nu}\widetilde{F}_{\mu\nu} \tag{10}$$

The important point here is that *even in the presence of explicit breaking, the anomaly is always present* and may actually dominate over the classical contribution, as put in evidence by (10) and the example of π^0 decay. Likewise the physical effects induced by $\Gamma_{\rm anom}$ can dominate over other (local) contributions coming from the explicit breaking of conformal invariance, as we have shown above.

4. Cancelling the conformal anomaly

Let us now survey the theories for which the contributions from the different spins to the conformal anomaly cancel. For *conformal* theories it was already shown in [7] that a cancellation is only possible for $N = 4$ conformal supergravity coupled to $N = 4$ Yang–Mills with any gauge group of dimension $r = 4$; see [30] for a recent update on the status of these theories. For non-conformal (Poincaré or AdS) supergravities, the situation is less clear because there remain open issues, *e.g.* related to gauge dependence. Even though these theories do not respect conformal invariance (this is not immediately obvious for spins $s \le 1$ as the associated kinetic terms *are* conformally invariant, but the conformal invariance is lost because the scalar fields couple via a non-linear σ-model which is not conformal), we will nevertheless use the coefficients as determined with the harmonic gauge from the conformal actions in [31] by the procedure explained in the appendix. In Table 1 we list the coefficients c_s and a_s for spins up to $s = 2$ (first two columns). For spins $s \le 1$, these coefficients have been known for a long time [32,3,4,6,33–36]; the entries labelled 0^* give the results for two-form fields (which have no massive analog). For $s = \frac{3}{2}$ and $s = 2$ there is the further complication [4,6] that only the sums $(a_s + c_s)$ (multiplying the square of the Riemann tensor) are gauge-invariant, but to extract the a_s and c_s coefficients separately, one has to go off-shell and fix a gauge (but see [13] for a proposal how to fix the gauge ambiguity by invoking the Wess–Zumino consistency condition). We note that these numbers have also been used more recently in connection with the AdS/CFT correspondence [37]. The numbers for the massive fields (last two columns) are obtained in the usual way by adding the lower helicities, *e.g.* $\bar{c}_2 = c_2 + c_1 + c_0$. This is in accord with the fact that spontaneous symmetry breaking does not affect the conformal anomaly, and that the anomalies must match between the phases with broken and unbroken conformal symmetry [23].

We first consider the *sum* of the anomaly coefficients. As shown long ago [38] the sum $\sum(c_s + a_s)$ can be made to vanish for all

Table 1
Anomaly coefficients for the harmonic gauge.

	Massless		Massive	
	c_s	a_s	$\bar{c}_s$	$\bar{a}_s$
$0(0^*)$	$\frac{3}{2}(\frac{3}{2})$	$-\frac{1}{2}(\frac{179}{2})$	$\frac{3}{2}(\varnothing)$	$-\frac{1}{2}(\varnothing)$
$\frac{1}{2}$	$\frac{9}{2}$	$-\frac{11}{4}$	$\frac{9}{2}$	$-\frac{11}{4}$
1	18	-31	$\frac{39}{2}$	$-\frac{63}{2}$
$\frac{3}{2}$	$-\frac{411}{2}$	$\frac{589}{4}$	-201	$\frac{289}{2}$
2	783	-571	$\frac{1605}{2}$	$-\frac{1205}{2}$

$N=3$ supermultiplets with maximum spin $s \leqslant 2$ by replacing one scalar by one two-form field in the spin-$\frac{3}{2}$ multiplet; the resulting multiplets can then be used as building blocks to arrange for all $N \geqslant 4$ supergravities to have vanishing $\sum(c_s + a_s)$. This implies in particular that the combined contribution vanishes for the maximal $N=4$ Yang–Mills multiplet.

Next, turning to the C^2 anomaly, it is a remarkable fact that the presence of gravitinos is mandatory for any cancellation involving the coefficients c_s because the gravitinos are the only fields that contribute with a negative sign. It is easy to check that no $N \leqslant 4$ supergravity can achieve this because the summed contribution from the pure supergravity multiplets for $N \leqslant 4$ is positive, hence of the same sign as the one from any of the associated matter multiplets, so no cancellation of the C^2 anomaly is possible for these theories.

One then checks that $\sum c_s$ vanishes for the $N \geqslant 5$ supergravity multiplets,

$$\begin{aligned} &c_2 + 5c_{\frac{3}{2}} + 10c_1 + 11c_{\frac{1}{2}} + 10c_0 = 0 \\ &c_2 + 6c_{\frac{3}{2}} + 16c_1 + 26c_{\frac{1}{2}} + 30c_0 = 0\,, \\ &c_2 + 8c_{\frac{3}{2}} + 28c_1 + 56c_{\frac{1}{2}} + 70c_0 = 0\,, \end{aligned} \tag{11}$$

but for no other theories. This cancellation requires the harmonic gauge adopted in [31] (an analogous cancellation for the $N=8$ multiplet in a different gauge was already noticed in [6]). These cancellations are analogous to the ones for the β-function in gauged extended supergravities [39], and due to similar spin sum rules [40]. Unlike the cancellation of gauge anomalies which can in principle be arranged for arbitrary chiral gauge groups by suitable choice of matter multiplets, the above cancellation thus singles out three very special cases.

With the same assumptions as above one can furthermore show that, in addition to the cancellation for the massless supergravity multiplets, the contributions to c_s also cancel for the whole tower of Kaluza–Klein states of maximal $D=11$ supergravity compactified on S^7, and they do so 'floor by floor' – exactly as for the one-loop β-function in [41,42]. For the reader's convenience we reproduce below from [41] the full table of massive Kaluza–Klein multiplets with the Dynkin labels of the associated SO(8) representations (note that some towers start 'higher up', because no entry in a Dynkin label can be negative).

Using the Weyl formula for $SO(8)$, as quoted in [41],

$$\begin{aligned} D(a_1, b, a_2, a_3) = \frac{1}{4320}&\tilde{b}\tilde{a}_1\tilde{a}_2\tilde{a}_3(\tilde{b}+\tilde{a}_1)(\tilde{b}+\tilde{a}_2)(\tilde{b}+\tilde{a}_3)\times \\ &\times(\tilde{b}+\tilde{a}_1+\tilde{a}_2)(\tilde{b}+\tilde{a}_1+\tilde{a}_3)(\tilde{b}+\tilde{a}_2+\tilde{a}_3)\times \\ &\times(\tilde{b}+\tilde{a}_1+\tilde{a}_2+\tilde{a}_3)(2\tilde{b}+\tilde{a}_1+\tilde{a}_2+\tilde{a}_3) \end{aligned} \tag{12}$$

where $\tilde{b} = b+1$, $\tilde{a}_i = a_i + 1$, the sum $N_s(n)$ over the dimensions of the representations for spin s at level n comes out to be

$$N_s(n) = d_s\left(\frac{1}{20}n^5 + \frac{13}{36}n^4 + \frac{4}{3}n^3 + \frac{949}{360}n^2 + \frac{157}{60}n + 1\right)\,, \tag{13}$$

the same for all spins s! The coefficients d_s are given in the last column of Table 2; they are equal to the multiplicities of the $N=8$ supermultiplet in $D=5$. Using the $\bar{c}_s$ coefficients for the massive representations from Table 1 it is easy to check that

$$\sum_s d_s \bar{c}_s = 0 \tag{14}$$

Consequently, the sum over spins vanishes separately for each n. Furthermore, since the anomaly is independent of the chosen background solution it follows immediately that this result remains true *for any compactification of M theory to four dimensions*. Of course, for all we presently know, this is *not* a conformal theory, and therefore the significance of these cancellations remains to be understood, see also remarks below.

Table 2
Massive states in the S^7 compactification [41].

	$SO(8)$ representations	d_s
0	[n+2 0 0 0], [n 0 2 0], [n−2 2 0 0], [n−2 0 0 2], [n−2 0 0 0]	42
$\frac{1}{2}$	[n+1 0 1 0], n−1 1 1 0], [n−2 1 0 1], [n−2 0 0 1]	48
1	[n 1 0 0], [n−1 0 1 1], [n−2 1 0 0]	27
$\frac{3}{2}$	[n 0 0 1], [n−1 0 1 0]	8
2	[n 0 0 0]	1

From the Table 1 we see that the contribution from the spin-0 fields to the Gauss–Bonnet invariant E_4 depends on the field representation, unlike c_0. From the results of [38] it follows directly that one can arrange for the sum $\sum a_s$ to vanish for all $N \geqslant 5$ supergravities with appropriate choice of field representation for the spin-0 degrees of freedom, but for no other theories. To be sure, the contributions to a_s fail to cancel if one chooses to represent all spin-0 degrees of freedom by scalar fields, and they also do not cancel for the massive Kaluza–Klein supermultiplets. Nevertheless, as shown in [41], even in that case the total contribution to a_s does cancel provided one adds up the contribution from *all* Kaluza–Klein supermultiplets. The argument involves a ζ-function regularisation and is basically due to the fact that one compactifies on an odd-dimensional manifold. Again, this result is background independent and thus remains valid for any compactification of M theory to four dimensions.

5. A hidden conformal structure?

A puzzling aspect of the present work is that the most interesting cancellations occur precisely for those combinations of fields for which no (super-)conformal actions are known. While there is no issue with conformal covariance for the $s = 0, \frac{1}{2}, 1$ couplings in lowest order, this is not the case for the $s = \frac{3}{2}$ and $s = 2$ couplings. The fact that the coefficients cancel nevertheless for the non-conformal $N \geq 5$ supergravities (albeit with a fixed choice of gauge) is, to us, indicative of a hidden underlying conformal structure that is spontaneously broken, and that cannot be of any known type because conformal supergravities stop at $N=4$ [43,30]. Likewise, the cancellations for the massive modes can possibly be understood in terms of spontaneously broken conformal symmetry. Hints of such a hidden conformal structure have appeared in various different places [44–47], but perhaps most significant here is the fact that the massless $N=8$ states constitute a (short) multiplet of the super-conformal group $SU(2,2|8)$ [44]. Let us also mention recent work [48] which, arguing from a very different perspective, reaches the same conclusion that conformal anomalies must cancel for a theory to be viable.

Finally, the complete cancellation of conformal anomalies takes place for the very same multiplets for which the composite R symmetry anomaly cancels [49]. As argued in [50,51] it is precisely the presence or absence of this anomaly that is responsible for the finiteness properties of the $N \geqslant 5$ supergravities, and that explains the appearance of non-renormalizable divergences in $N=4$ supergravity at four loops. This remarkable coincidence leads us to conjecture that the conformal anomaly may play a similar role for the finiteness properties of extended supergravities, and is suggestive of a possible link between these cancellations via supersymmetry.

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Appendix A

In this appendix we briefly explain how to extract the numbers quoted in Table 1 for the massless fields from the results derived in [31]. We restrict attention to spins $\frac{3}{2}$ and 2 since the results for lowers spins were established already long ago. The results for a_s and $(a_s + c_s)$ are obtained by specializing the relevant formulas of [31] to, respectively, conformally flat and Ricci flat backgrounds. All equation numbers below refer to [31]. Note that we use the opposite sign for a and a uniform normalization (which differs from the one in [31] by factors of 180).

- The results for bosonic a_s were given in (3.39)

$$a_s^b = -\frac{1}{2}(2 - 15s^2 + 75s^4) \tag{15}$$

whence $a_2 = -571$ (the value $a_0 = -1$ corresponds to a *complex* scalar).

- To derive the results for fermionic a_s^f we start from (4.29)

$$a_f(s,m) = 5\left(s+\tfrac{1}{2}\right)\left(s+\tfrac{3}{2}\right)\left[\left(s+\tfrac{1}{2}\right)\left(s+\tfrac{3}{2}\right) - 3m^2 + 12m - \frac{121}{10}\right] \tag{16}$$

Then from (4.10) we get

$$a_{\frac{3}{2}} = -\frac{1}{4}a_f\left(\tfrac{1}{2},-1\right) + \frac{1}{4}\left[a_f\left(\tfrac{3}{2},3\right) - a_f\left(\tfrac{1}{2},-1\right)\right] = \frac{589}{4} \tag{17}$$

- For the bosonic $a_s + c_s$ (Ricci flat background) we have (3.11)

$$\beta_1(s) = -\frac{1}{2}(s+1)^2\left[21 - 20(s+1)^2 + 3(s+1)^4\right] \tag{18}$$

Then from (3.3)

$$a_2 + c_2 = -\frac{1}{2}\left[\beta_1(2) + \beta_1(0)\right] + \beta_1(1) = 212 \tag{19}$$

Using (15) above we infer

$$c_s = 3 - 45s^2 + 60s^4 \tag{20}$$

whence $c_1 = 18$ and $c_2 = 783$ ($c_0 = 3$ again corresponds to a *complex* scalar).

- For the fermionic $a_s + c_s$ we start from the expression in (4.6):

$$\beta_1(s) = \frac{2}{8}\left(s+\tfrac{1}{2}\right)\left(s+\tfrac{3}{2}\right)\left[50 - 28\left(s+\tfrac{1}{2}\right)\left(s+\tfrac{3}{2}\right) + 3\cdot 4\left(s+\tfrac{1}{2}\right)^2\left(s+\tfrac{3}{2}\right)^2\right] \tag{21}$$

Then from (4.3) we get

$$c_{\frac{3}{2}} + a_{\frac{3}{2}} = \frac{1}{2}\beta_1\left(\tfrac{1}{2}\right) - \frac{1}{4}\beta_1\left(\tfrac{3}{2}\right) = -\frac{233}{4} \tag{22}$$

which gives $c_{\frac{3}{2}} = -\frac{411}{2}$.

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