Local $C^{1,\beta}$-regularity at the boundary of two dimensional sliding almost minimal sets in $\mathbb{R}^3$

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Abstract

In this paper, we will give a $C^{1,\beta}$-regularity result on the boundary for two dimensional sliding almost minimal sets in $\mathbb{R}^3$. This effect may lead to the existence of a solution to the Plateau problem with sliding boundary conditions proposed by Guy David in [6] in the case that the boundary is a 2-dimensional smooth manifold.

1 Introduction

Jean Taylor, in [12], proved a celebrated regularity result of Almgren almost minimal sets, that gives a complete classification of the local structure of 2-dimensional (almost) minimal sets. This result may apply to many actual surfaces, soap films are considered as typical examples. Guy David, in [4], gave a new proof of this result and generalized it to any codimension. That is, every 2-dimensional almost minimal set, in an open set $U \subseteq \mathbb{R}^n$ with gauge function $h(t) \leq Ct^\alpha$, is local $C^{1,\beta}$ equivalent to a 2-dimensional minimal cone.

In [6], Guy David proposed to consider the Plateau Problem with sliding boundary conditions, since it is very natural to soap films and Jean Taylor’s regularity also applies for sliding almost minimal sets away from the boundary, and it also has some advantages to consider the local structure at the boundary. Motivated by these, regularity at the boundary would be well worth our considering. In fact, a result similar to Jean Taylor’s will be a satisfactory conclusion, for which together with Jean Taylor’s theorem will imply the local Lipschitz retract property of sliding (almost) minimal sets, and the existence of minimizers for the sliding Plateau Problem easily follows.

One of advantages of the sliding boundary conditions is that we have chance to determine the possibility of minimal cones in the upper half space $\Omega_0$ of $\mathbb{R}^3$, where minimal cone is a cone but minimal, and minimal is understood with sliding on the boundary $\partial \Omega_0$. Indeed, there no more than seven kinds of cones which are minimal, they are $\partial \Omega_0$, cones of type $\mathcal{V}$, cones of type $\mathcal{P}_+$, cones of type $\mathcal{Y}_+$, cones of type $\mathcal{T}_+$ and cones $\partial \Omega_0 \cup Z$ where $Z$ are cones of type $\mathcal{P}_+$ or $\mathcal{Y}_+$, see Section 3 in [8] for the precise definition of cones of type $\mathcal{P}_+$, $\mathcal{Y}_+$, $\mathcal{T}_+$ and $\mathcal{V}$, and also Remark 3.11 for the claim. We ascertain that there are only one kinds of cones which are minimal and contains the boundary $\partial \Omega_0$, they are $\partial \Omega_0$ and $\partial \Omega_0 \cup Z$ where $Z$ is cone of type $\mathcal{P}_+$ or $\mathcal{Y}_+$, see Theorem 3.10 in [8] for the statement.

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Another advantages of the sliding boundary conditions is that we can easily establish a monotony density property at the boundary, see Theorem 2.3 for precise statement. In fact, the monotony density property is not enough, we have estimated the decay of the almost density, and that is also possible with sliding on the boundary, see Corollary 3.16.

In [8], we proved a Hölder regularity of two dimensional sliding almost minimal set at the boundary. That is, suppose that \( \Omega \subseteq \mathbb{R}^3 \) is a closed domain with boundary \( \partial \Omega \) a \( C^1 \) manifold of dimension 2, \( E \subseteq \Omega \) is a 2 dimensional sliding almost minimal set with sliding boundary \( \partial \Omega \), and that \( \partial \Omega \subseteq E \). Then \( E \), at the boundary, is locally biHölder equivalent to a sliding minimal cone in the upper half space \( \Omega_0 \). In this paper, we will generalized the biHölder equivalence to a \( C^{1,\beta} \) equivalence when the gauge function \( h \) satisfies that \( h(t) \leq C t^{\alpha_1} \) and \( \partial \Omega \subseteq E \). Let us refer to Theorem 1.2 for details. Where the sliding minimal cones always contain the boundary \( \partial \Omega_0 \), namely only there kinds of cones can appear: \( \partial \Omega_0 \) and \( \partial \Omega_0 \cap Z \), where \( Z \) are cones of type \( \mathbb{P}_+ \) or \( \mathbb{Y}_+ \).

Let us introduce some notation and definitions before state our main theorem. A gauge function is a nondecreasing function \( h : [0, \infty) \to [0, \infty] \) with \( \lim_{t \to 0} h(t) = 0 \). Let \( \Omega \) be a closed domain of \( \mathbb{R}^3 \), \( L \) be a closed subset in \( \mathbb{R}^3 \), \( E \subseteq \Omega \) be a given set. Let \( U \subseteq \mathbb{R}^3 \) be an open set. A family of mappings \( \{ \varphi_t \}_{0 \leq t \leq 1} \), from \( E \) into \( \Omega \), is called a sliding deformation of \( E \) in \( U \), where \( \varphi_1(E) \) is called a competitor of \( E \) in \( U \), if following properties hold:

1. \( \varphi_t(x) = x \) for \( x \in E \setminus U \), \( \varphi_t(x) \subseteq U \) for \( x \in E \cap U \), \( 0 \leq t \leq 1 \),
2. \( \varphi_t(x) \in L \) for \( x \in E \cap L \), \( 0 \leq t \leq 1 \),
3. the mapping \( [0, 1] \times E \to \Omega, (t, x) \mapsto \varphi_t(x) \) is continuous,
4. \( \varphi_1 \) is Lipschitz and \( \psi_0 = \text{id}_E \).

**Definition 1.1.** We say that an nonempty set \( E \subseteq \Omega \) is locally sliding almost minimal at \( x \in E \) with sliding boundary \( L \) and with gauge function \( h \), called \( (\Omega, L, h) \) locally sliding almost at \( x \in E \) for short, if \( \mathcal{H}^2(E \cap B(x, r)) \) is locally finite, and for any sliding deformation \( \{ \varphi_t \}_{0 \leq t \leq 1} \) of \( E \) in \( B(x, r) \), we have that

\[
\mathcal{H}^2(E \cap B(x, r)) \leq \mathcal{H}^2(\varphi_1(E) \cap B(x, r)) + h(r)r^2.
\]

We say that \( E \) is sliding almost minimal with sliding boundary \( L \) and gauge function \( h \), denote by \( \text{SAM}(\Omega, L, h) \) the collection of all such sets, if \( E \) is locally sliding almost minimal at all points \( x \in E \).

For any \( x \in \mathbb{R}^3 \), let \( \tau_x : \mathbb{R}^3 \to \mathbb{R}^3 \) be the translation defined by \( \tau_x(y) = y + x \), and let \( \mu_r : \mathbb{R}^3 \to \mathbb{R}^3 \) be the mapping defined by \( \mu_r(y) = ry \) for any \( r > 0 \). For any \( S \subseteq \mathbb{R}^3 \) and \( x \in S \), a blow-up limit of \( S \) at \( x \) is any closed set in \( \mathbb{R}^3 \) that can be obtained as the Hausdorff limit of a sequence \( \mu_{1/r_k} \circ \tau_{-x}(S) \) with \( \lim_{k \to \infty} r_k = 0 \). A set \( X \) in \( \mathbb{R}^3 \) is called a cone centered at the origin 0 if for any \( \mu_r(X) = X \) for any \( t \geq 0 \); in general, we call a cone \( X \) centered at \( x \) if \( \tau_{-x}(X) \) is a cone centered at 0. We denote by \( \text{Tan}(S, x) \) the tangent cone of \( S \) at \( x \), see Section 2.1 in [1]. We see that if there is unique blow-up limit of \( S \) at \( x \), then it coincide with the tangent cone \( \text{Tan}(S, x) \). Our main theorem is the following.

**Theorem 1.2.** Let \( \Omega \subseteq \mathbb{R}^3 \) be a closed set such that the boundary \( \partial \Omega \) is a 2-dimensional manifold of class \( C^{1,\alpha} \) for some \( \alpha > 0 \) and \( \text{Tan}(\Omega, z) \) is a half space for any \( z \in \partial \Omega \). Let \( E \subseteq \Omega \) be a closed set such that \( E \supseteq \partial \Omega \) and \( E \) is a sliding almost minimal set with sliding boundary \( \partial \Omega \) and with gauge function \( h \) satisfying that

\[
h(t) \leq C h t^{\alpha_1}, \quad 0 < t \leq t_0, \quad \text{for some } C_h > 0, \alpha_1 > 0 \text{ and } t_0 > 0.
\]
Then for any $x_0 \in \partial \Omega$, there is unique blow-up limit of $E$ at $x_0$; moreover, there exist a radius $r > 0$, a sliding minimal cone $Z$ in $\Omega$ with sliding boundary $\partial \Omega$, and a mapping $\Phi : \Omega \cap B(0,1) \to \Omega$ of class $C^{1,\beta}$, which is a diffeomorphism between its domain and image, such that $\Phi(0) = x_0$, $|\Phi(x) - x_0 - x| \leq 10^{-2}r$ for $x \in B(0,2r)$, and

$E \cap B(x_0,r) = \Phi(Z) \cap B(x_0,r)$.

Theorem 1.2 and Jean Taylor’s theorem imply that any set $E$ as in above theorem is Lipschitz neighborhood retract. This effect gives the existence of a solution to the Plateau problem with sliding boundary conditions in a special case, see Theorem 8.1.

2 Lower bound of the decay for the density

In this section, we will consider a simple case that $\Omega$ is a half space and $L$ is its boundary; without loss of generality, we assume that $\Omega$ is the upper half space, and change the notation to be $\Omega_0$ for convenience, i.e.

$$\Omega_0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 \geq 0\}, L_0 = \partial \Omega_0.$$

It is well known that for any 2-rectifiable set $E$, there exists an approximate tangent plane $\text{Tan}(E, y)$ of $E$ at $y$ for $\mathcal{H}^2$-a.e. $y \in E$. We will denote by $\theta(y) \in [0, \pi/2]$ the angle between the segment $[0, y]$ and the plane $\text{Tan}(E, y)$, by $\theta_x(y) \in [0, \pi/2]$ the angle between the segment $[x, y]$ and the plane $\text{Tan}(E, y)$, for $x \in \mathbb{R}^3$.

In this section, we assume that there is a number $r_h > 0$ such that

$$\int_0^{r_h} \frac{h(2t)}{t}dt < \infty,$$

and put

$$h_1(t) = \int_0^t \frac{h(2s)}{s}ds, \text{ for } 0 \leq t \leq r_h.$$

Lemma 2.1. Let $E \subseteq \Omega_0$ be any 2-rectifiable set. Then, by putting $u(r) = \mathcal{H}^2(E \cap B(x,r))$, we have that $u$ is differentiable almost every $r > 0$, and for such $r$,

$$\mathcal{H}^1(E \cap \partial B(x,r)) \leq u'(r).$$

Proof. Considering the function $\psi : \mathbb{R}^3 \to \mathbb{R}$ defined by $\psi(y) = |y - x|$, we have that, for any $y \neq x$ and $v \in \mathbb{R}^3$,

$$D\psi(y)v = \left\langle \frac{y - x}{|y - x|}, v \right\rangle,$$

thus

$$\text{ap } J_1(\psi|_E)(y) = \sup\{|D\psi(y)v| : v \in \text{Tan}(E, x), |v| = 1\} = \cos \theta_x(y).$$

Employing Theorem 3.2.22 in [9], we have that, for any $0 < r < R < \infty$,

$$\int_r^R \mathcal{H}^1(E \cap \partial B(x,t))dt = \int_{E \cap B(x,R) \setminus B(x,r)} \cos \theta_x(y)d\mathcal{H}^2(y) \leq u(R) - u(r),$$
we get so that, for almost every \( r \in (0, \infty) \),
\[
\mathcal{H}^1(E \cap \partial B(x, t)) \leq u'(r).
\]

Lemma 2.2. Let \( E \) be a 2-rectifiable \((\Omega_0, L_0, h)\) locally sliding almost minimal at \( x \in E \).

- If \( x \in E \cap L_0 \), then for \( \mathcal{H}^1\)-a.e. \( r \in (0, \infty) \),
  \[
  \mathcal{H}^2(E \cap B(x, r)) \leq \frac{r}{2} \mathcal{H}^1(E \cap \partial B(x, r)) + h(2r)(2r)^2.
  \] (2.3)

- If \( x \in E \setminus L_0 \), then inequality (2.2) holds for \( \mathcal{H}^1\)-a.e. \( r \in (0, \text{dist}(x, L_0)) \).

Proof. If \( \mathcal{H}^2(E \cap \partial B(x, r)) > 0 \), then \( \mathcal{H}^1(E \cap \partial B(x, r)) = \infty \), and nothing need to do. We assume so that \( \mathcal{H}^2(E \cap \partial B(x, r)) = 0 \).

Let \( f : [0, \infty) \to [0, \infty) \) be any Lipschitz function, we let \( \phi : \Omega_0 \to \Omega_0 \) be defined by
\[
\phi(y) = f(|y - x|) \frac{y - x}{|y - x|}.
\]
Then, for any \( y \neq x \) and any \( v \in \mathbb{R}^3 \), by putting \( \tilde{y} = y - x \), we have that
\[
D\phi(y)v = \frac{f(|\tilde{y}|)}{|\tilde{y}|} v + \frac{|\tilde{y}| f'(|\tilde{y}|) - f(|\tilde{y}|)}{|\tilde{y}|^2} \left< \frac{\tilde{y}}{|\tilde{y}|}, v \right> \tilde{y}
\]
If the tangent plane \( \text{Tan}^2(E, y) \) of \( E \) at \( y \) exists, we take \( v_1, v_2 \in \text{Tan}^2(E, y) \) such that \( |v_1| = |v_2| = 1 \), \( v_1 \) is perpendicular to \( y = x \), and that \( v_2 \) is perpendicular to \( v_1 \), let \( v_3 \) be a vector in \( \mathbb{R}^3 \) which is perpendicular to \( \text{Tan}^2(E, y) \) and \( |v_3| = 1 \), then
\[
\tilde{y} = \langle \tilde{y}, v_2 \rangle v_2 + \langle \tilde{y}, v_3 \rangle v_3 = |\tilde{y}| \cos \theta_x(y)v_2 + |\tilde{y}| \sin \theta_x(y)v_3,
\]
and
\[
D\phi(y)v_1 \wedge D\phi(y)v_2 = \frac{f(|\tilde{y}|)^2}{|\tilde{y}|^2} v_1 \wedge v_2 + \frac{|\tilde{y}| f'(|\tilde{y}|) f(|\tilde{y}|) - f(|\tilde{y}|)^2}{|\tilde{y}|^2} \cos \theta_x(y)v_1 \wedge \tilde{y},
\]
thus
\[
\text{ap} J_2(\phi_E)(y) = \left| D\phi(y)v_1 \wedge D\phi(y)v_2 \right|
\]
\[
= \frac{f(|\tilde{y}|)}{|\tilde{y}|} \left( f'(|\tilde{y}|)^2 \cos^2 \theta_x(y) + \frac{f(|\tilde{y}|)^2}{|\tilde{y}|^2} \sin^2 \theta_x(y) \right)^{1/2}.
\]
We consider the function \( \psi : \mathbb{R}^3 \to \mathbb{R} \) defined by \( \psi(y) = |y - x| \). Then, by (2), we have that
\[
\text{ap} J_1(\psi_E)(y) = \cos \theta_x(y).
\]
For any \( \xi \in (0, r/2) \), we consider the function \( f \) defined by
\[
f(t) = \begin{cases} 
0, & 0 \leq t \leq r - \xi \\
\frac{r}{\xi}(t - r + \xi), & r - \xi < t \leq r \\
t, & t > r.
\end{cases}
\]
Then we have that
\[
\text{ap} \int_2 (\phi|_E)(y) \leq \frac{f(|\tilde{g}|)f'(|\tilde{g}|)}{|\tilde{g}|} \cos \theta_x(y) + \frac{f(|\tilde{g}|)^2}{|\tilde{g}|^2} \sin \theta_x(y).
\]
Applying Theorem 3.2.22 in [9], by putting \( A_\xi = E \cap B(0, r) \setminus B(0, r - \xi) \), we get that
\[
\mathcal{H}^2(\phi(E \cap B(0, r))) \leq \int_{A_\xi} \frac{r^2}{\xi^2} \frac{|\tilde{g}| - r + \xi}{|\tilde{g}|} \cos \theta_x(y) d\mathcal{H}^2(y) + \frac{r^2}{(r - \xi)^2} \mathcal{H}^2(A_\xi)
\]
thus
\[
\mathcal{H}^2(E \cap B(0, r)) \leq (2r)^2 h(2r) + \lim_{\xi \to 0^+} r^2 \int_{r-\xi}^r \frac{t - r + \xi}{t\xi^2} \mathcal{H}^1(E \cap \partial B(x, t)) dt.
\]
Since the function \( g(t) = \mathcal{H}^1(E \cap B(x, t))/t \) is a measurable function, we have that, for almost every \( r \),
\[
\lim_{\xi \to 0^+} \int_0^\xi \frac{t g(t) - r + \xi}{\xi^2} dt = \frac{1}{2} g(r),
\]
thus for such \( r \),
\[
\mathcal{H}^2(E \cap B(x, r)) \leq (2r)^2 h(2r) + \frac{r}{2} \mathcal{H}^1(E \cap B(x, r)).
\]
\[]

For any set \( E \subseteq \mathbb{R}^3 \), we set
\[
\Theta_E(x, r) = r^{-2} \mathcal{H}^2(E \cap B(x, r)), \text{ for any } r > 0,
\]
and denote by \( \Theta_E(x) = \lim_{r \to 0^+} \Theta_E(x, r) \) if the limit exist, we may drop the script \( E \) if there is no danger of confusion.

**Theorem 2.3.** Let \( E \) be a 2-rectifiable \((\Omega_0, L_0, h)\) locally sliding almost minimal at \( x \in E \).

- If \( x \in L_0 \), then \( \Theta(x, r) + 8h_1(r) \) is nondecreasing as \( r \in (0, r_h) \).
- If \( x \notin L_0 \), then \( \Theta(x, r) + 8h_1(r) \) is nondecreasing as \( r \in (0, \min\{r_h, \text{dist}(x, L)\}) \).

**Proof.** From Lemma 2.2 and Lemma 2.1, by putting \( u(r) = \mathcal{H}^2(E \cap B(x, r)) \), we get that, if \( x \in L \),
\[
u(r) \leq \frac{r}{2} u'(r) + h(2r)(2r)^2,
\]
for almost every \( r \in (0, \infty); \) if \( x \notin L \), then (2) holds for almost every \( r \in (0, \min\{r_h, \text{dist}(x, L)\}) \).

We put \( v(r) = r^{-2} u(r) \), then \( v'(r) \geq -8r^{-2} h(2r) \), we get that \( \Theta(x, r) + 8h_1(r) \) is nondecreasing. \[\]

**Remark 2.4.** Let \( E \) be a 2-rectifiable \((\Omega_0, L_0, h)\) locally sliding almost minimal at some point \( x \in E \). Then by Theorem 2.3, we get that \( \Theta_E(x) \) exists.
3 Estimation of upper bound

Let \( Z \) be a collection of cones. We say that a set \( E \subseteq \mathbb{R}^3 \) is locally \( C^{k,\alpha} \)-equivalent (resp. \( C^k \)-equivalent) to a cone in \( Z \) at \( x \in E \) for some nonnegative integer \( k \) and some number \( \alpha \in (0,1] \), if there exist \( q_0 > 0 \) and \( \tau_0 > 0 \) such that for any \( \tau \in (0,\tau_0) \) there is \( \varrho \in (0,q_0) \), a cone \( Z \in Z \) and a mapping \( \Phi : B(0,2\varrho) \to \mathbb{R}^3 \), which is a homeomorphism of class \( C^{k,\alpha} \) (resp. \( C^k \)) between \( B(0,2\varrho) \) and its image \( \Phi(B(0,2\varrho)) \) with \( \Phi(0) = x \), satisfying that

\[
\| \Phi - \text{id} - \Phi(0) \|_\infty \leq \varrho \tau
\]  
(3.1)

and

\[
E \cap B(x, \varrho) \subseteq \Phi (Z \cap B (0,2\varrho)) \subseteq E \cap B(x, 3\varrho).
\]  
(3.2)

Similarly, if \( \Omega \subseteq \mathbb{R}^3 \) is a closed set with the boundary \( \partial \Omega \) is a 2-dimensional manifold, a set \( E \subseteq \Omega \) is called locally \( C^{k,\alpha} \)-equivalent to a minimal cone \( Z \) in \( \Omega_0 \) at \( x \in E \cap \partial \Omega \), if there exist \( q_0 > 0 \) and \( \tau_0 > 0 \) such that for any \( \tau \in (0,\tau_0) \) there is \( \varrho \in (0,q_0) \) and a mapping \( \Phi : B(0,2\varrho) \cap \Omega_0 \to \Omega \), which is a diffeomorphism of class \( C^{k,\alpha} \) between its domain and image with \( \Phi(0) = x \) satisfying that \( \Phi(L_0 \cap B(0,2\varrho)) \subseteq \partial \Omega \) and (3) and (3).

Suppose that \( \Omega \subseteq \mathbb{R}^3 \) is closed set with the boundary \( \partial \Omega \) is a 2-dimensional \( C^1 \) manifold. Suppose that \( E \subseteq \Omega \) is sliding almost minimal with sliding boundary \( \partial \Omega \) and gauge function \( h \). Then, by putting \( U = \Omega \setminus \partial \Omega \), we see that \( E \cap U \) is almost minimal in \( U \), applying Jean Taylor’s theorem, \( E \) is locally \( C^{1,\beta} \)-equivalent to a minimal cone at each point \( x \in E \cap U \) for some \( \beta > 0 \) in case \( h(r) \leq c r^\alpha \) for some \( c > 0 \), \( \alpha > 0 \), \( r_0 > 0 \) and \( 0 < r < r_0 \). We see from [8, Theorem 6.1] that, at \( x \in E \cap \partial \Omega \), \( E \) is locally \( C^{0,\beta} \)-equivalent to a sliding minimal cone in \( \Omega_0 \) in case the gauge function \( h \) satisfying (2).

3.1 Approximation of \( E \cap \partial B(0, r) \) by rectifiable curves

For any sets \( X,Y \subseteq \mathbb{R}^3 \), any \( z \in \mathbb{R}^3 \) and any \( r > 0 \), we denote by \( d_{z,r} \) the normalized local Hausdorff distance defined by

\[
d_{z,r}(X,Y) = \frac{1}{r} \sup \{ \text{dist}(x,Y) : x \in X \cap B(z,r) \} + \frac{1}{r} \sup \{ \text{dist}(y,X) : y \in Y \cap B(z,r) \}.\]

A cone in \( \mathbb{R}^3 \) is called of type \( \mathbb{Y} \) if it is the union of three half planes with common boundary line and that make \( 120^\circ \) angles along the boundary line. A cone \( Z \subseteq \Omega_0 \) is called of type \( \mathbb{P}_+ \) if it is a half plane perpendicular to \( L_0 \); a cone \( Z \subseteq \Omega_0 \) is called of type \( \mathbb{Y}_+ \) if \( Z = \Omega_0 \cap Y \), where \( Y \) is a cone of type \( \mathbb{Y} \) perpendicular to \( L_0 \); for convenient, we will also use the notation \( \mathbb{P}_+ \), to denote the collection of all of cones of type \( \mathbb{P}_+ \), and \( \mathbb{Y}_+ \) to denote the collection of all of cones of type \( \mathbb{Y}_+ \).

For any set \( E \subseteq \Omega_0 \) with \( 0 \in E \), and any \( r > 0 \), we set

\[
\varepsilon_P(r) = \inf \{ d_{0,r}(E,Z) : Z \in \mathbb{P}_+ \},
\]

\[
\varepsilon_Y(r) = \inf \{ d_{0,r}(E,Z) : Z \in \mathbb{Y}_+ \}.\]

If \( E \) is 2-rectifiable and \( \mathcal{H}^2(E) < \infty \), then \( E \cap \partial B(0, r) \) is 1-rectifiable and \( \mathcal{H}^1(E \cap \partial B(0, r)) < \infty \) for \( \mathcal{H}^1 \)-a.e. \( r \in (0,\infty) \); we consider the function \( u : (0,\infty) \to \mathbb{R} \) which is defined by \( u(r) = \mathcal{H}^2(E \cap B(0, r)) \), it is quite easy to see that \( u \) is nondecreasing, thus \( u \) is differentiable for \( \mathcal{H}^1 \)-a.e.; we will denote \( \mathcal{B} \) the set \( r \in (0,\infty) \) such that

\[
\mathcal{H}^1(E \cap \partial B(0, r)) < \infty, \ u \text{ is differentiable at } r,
\]
Lemma 3.1. Let $E \subseteq \mathbb{R}^3$ be a connected set. If $\mathcal{H}^1(E) < \infty$, then $E$ is path connected.

For a proof, see for example Lemma 3.12 in [7], so we omit it here.

Lemma 3.2. Let $X$ be a locally connected and simply connected compact metric space. Let $A$ and $B$ be two connected subsets of $X$. If $F$ is a closed subset of $X$ such that $A$ and $B$ are contained in two different connected components of $X \setminus F$, then there exists a connected closed set $F_0 \subseteq F$ such that $A$ and $B$ still lie in two different connected components of $X \setminus F_0$.

Proof. See for example 52.III.1 on page 335 in [11], so we omit the proof here. \hfill \Box

For any $r > 0$, we put $3_r = (0, 0, r) \in \mathbb{R}^3$.

Lemma 3.3. Let $E \subseteq \Omega_0$ be a 2-rectifiable set with $\mathcal{H}^2(E) < \infty$. Suppose that $0 \in E$, and that $E$ is locally $C^0$-equivalent to a sliding minimal cone of type $\mathbb{P}_+$ at 0. Then for any $\tau \in (0, \tau_0)$ there exist $r = r(\tau) > 0$ such that, for any $r \in (0, \tau)$ and $\epsilon > \epsilon(r)$, we can find $y_r \in E \cap \partial B(0, r) \setminus L$, $x_{r, 1}, x_{r, 2} \in E \cap L \cap \partial B(0, r)$ and two simple curves $\gamma_{r, 1}, \gamma_{r, 2} \subseteq E \cap \partial B(0, r)$ satisfying that

1. $|y_r - 3_r| \leq \epsilon r$ and $|z_{r, 1} - z_{r, 2}| \geq (2 - 2\epsilon)r$;
2. $\gamma_{r, i}$ joins $y_r$ and $x_{r, i}$, $i = 1, 2$;
3. $\gamma_{r, 1}$ and $\gamma_{r, 2}$ are disjoint except for point $y_r$.

Proof. Since $E$ is locally $C^0$-equivalent to a sliding minimal cone of type $\mathbb{P}_+$ at 0, for any $\tau \in (0, \tau_0)$, there exist $\varrho > 0$, sliding minimal cone $Z$ of type $\mathbb{P}_+$, and a mapping $\Phi : \Omega_0 \cap B(0, 2\varrho) \to \Omega_0$ which is a homeomorphism between $\Omega_0 \cap B(0, 2\varrho)$ and $\Phi(\Omega_0 \cap B(0, 2\varrho))$ with $\Phi(0) = 0$ and $\Phi(\partial \Omega_0 \cap B(0, 2\varrho)) \subseteq \partial \Omega_0$ such that (3) and (3) hold. We new take $r = \varrho$. Then for any $r \in (0, \tau)$,

$$\Phi^{-1}[E \cap \partial B(0, r)] \subseteq Z \cap B(0, 3\varrho).$$

Without loss of generality, we assume that $Z = \{(x_1, 0, x_3) \mid x_1 \in \mathbb{R}, x_3 \geq 0\}$. Applying Lemma 3.2 with $X = Z \cap B(0, 3\varrho)$, $F = \Phi^{-1}[E \cap \partial B(0, r)]$, $A = \{0\}$ and $B = Z \cap \partial B(0, 3\varrho)$, we get that there is a connected closed set $F_0 \subseteq F$ such that $A$ and $B$ lie in two different connected components of $A \setminus F_0$, thus $\phi(F_0) \subseteq E \cap \partial B(0, r)$ is connected. We put $a_1 = \{(x_1, 0, 0) \mid x_1 < 0\}$ and $a_2 = \{(x_1, 0, 0) \mid x_1 > 0\}$. Then $F_0 \cap a_i \neq \emptyset$, $i = 1, 2$; otherwise $A$ and $B$ are contained in a same connected component of $X \setminus F_0$. We take $z_{r, i} \in F_0 \cap a_i$, and let $x_{r, i} = \phi(z_{r, i}) \in E \cap \partial B(0, r)$. Then $|x_{r, 1} - x_{r, 2}| \geq (2 - 2\epsilon)r$.

Since $F_0$ is connected and $\mathcal{H}^1(F_0) < \infty$, by Lemma 3.1, $F_0$ is path connected. Let $\gamma$ be a simple curve which joins $z_{r, 1}$ and $z_{r, 2}$. We see that $B(3_r, \varepsilon \gamma) \cap \gamma \neq \emptyset$, because $\varepsilon \gamma(r) < \varepsilon$ and $3_r \in Z$ for sliding minimal cone $Z$ of type $\mathbb{P}_+$. We take $y_r \in B(3_r, \varepsilon \gamma) \cap \gamma$. \hfill \Box

Lemma 3.4. Let $E \subseteq \Omega_0$ be a 2-rectifiable set with $\mathcal{H}^2(E) < \infty$. Suppose that $0 \in E$, and that $E$ is locally $C^0$-equivalent to a sliding minimal cone of type $\mathbb{P}_+$ at 0. Then for any $\tau \in (0, \tau_0)$ there exist $r = r(\tau) > 0$ such that, for any $r \in (0, \tau)$ and $\epsilon > \epsilon(\tau)$, we have

$$\lim_{\xi \to 0^+} \frac{1}{\xi} \int_{t \in (\xi, r)} \int_{E \cap \partial B(0, t)} f(z) \mathcal{H}^1(z) dt = \int_{E \cap \partial B(0, r)} f(z) \mathcal{H}^1(z),$$

and

$$\sup_{\xi > 0} \frac{1}{\xi} \int_{t \in (\xi, r)} \mathcal{H}^1(E \cap \partial B(0, t)) dt < +\infty.$$
can find $y_r \in E \cap \partial B(0, r) \setminus L$, $\mathcal{X}_{r,1}, \mathcal{X}_{r,2}, \mathcal{X}_{r,3} \in E \cap L \cap \partial B(0, r)$ and three simple curves $\gamma_{r,1}, \gamma_{r,2}, \gamma_{r,3} \subseteq E \cap \partial B(0, r)$ satisfying that

1. $|y_r - y_r| \leq \pi r/6$, and there exists $Z \in Y_+$ with $\text{dist}(x, Z) \leq \epsilon r$ for $x \in \gamma$;
2. $\gamma_{r,i} \cup \mathcal{X}_{r,i}$;
3. $\gamma_{r,i}$ and $\gamma_{r,j}$ are disjoint except for point $y_r$.

Proof. Since $E$ is locally $C^0$-equivalent to a sliding minimal cone of type $Y_+$ at 0, for any $\tau \in (0, \tau_0)$, there exist $\tau > 0$, $\varrho > 0$, and a mapping $\Phi : \Omega_0 \cap B(0, 2\varrho) \to \Omega_0$ which is a homeomorphism between $\Omega_0 \cap B(0, 2\varrho)$ and $\Phi(\Omega_0 \cap B(0, 2\varrho))$ with $\Phi(0) = 0$ and $\Phi(\partial \Omega_0 \cap B(0, 2\varrho)) \subseteq \partial \Omega_0$ such that (3) and (3) hold. We now take $\tau = \varrho$. Then for any $r \in (0, \tau)$,

$$\Phi^{-1}[E \cap \partial B(0, r)] \subseteq Z \cap B(0, 2\varrho).$$

Applying Lemma 3.2 with $Z = Z \cap B(0, 3\varrho), F = \Phi^{-1}[E \cap \partial B(0, r)], A = \{0\}$ and $B = Z \cap \partial B(0, 3\varrho)$, we get that there is a connected closed set $F_0 \subseteq F$ such that $A$ and $B$ lie in two different connected components of $A \setminus F_0$, thus $\phi(F_0) \subseteq Z \cap \partial B(0, r)$ is connected. We let $a_i, i = 1, 2, 3$, be the three connected components of $Z \cap L_0 \setminus A$. Then $F_0 \cap a_i \neq \emptyset$, $i = 1, 2, 3$; otherwise $A$ and $B$ are contained in a same connected component of $X \setminus F_0$. We take $z_{r,i} \in F_0 \cap a_i$, and let $\mathcal{X}_{r,i} = \phi(z_{r,i}) \in E \cap \partial B(0, r)$. Then $|\mathcal{X}_{r,1} - \mathcal{X}_{r,2}| \geq (\sqrt{3} - 2\epsilon)r$.

Since $F_0$ is connected and $H^1(F_0) < \infty$, by Lemma 3.1, $F_0$ is path connected.

3.2 Approximation of rectifiable curves in $S^2$ by Lipschitz graph

We denote by $S^2$ the unit sphere in $\mathbb{R}^3$. We say that a simple rectifiable curve $\gamma \subseteq S^2$ is a Lipschitz graph with constant at most $\eta$, if it can be parametrized by

$$z(t) = \left(\sqrt{1 - v(t)^2} \cos \theta(t), \sqrt{1 - v(t)^2} \sin \theta(t), v(t)\right),$$

where $v$ is Lipschitz with $\text{Lip}(v) \leq \eta$.

Lemma 3.5. Let $T \in [\pi/3, 2\pi/3]$ be a number, and $\gamma : [0, T] \to S^2$ a simple rectifiable curve given by

$$\gamma(t) = \left(\sqrt{1 - v(t)^2} \cos \theta(t), \sqrt{1 - v(t)^2} \sin \theta(t), v(t)\right),$$

where $v$ is a continuous function with $v(0) = v(T) = 0$, $\theta$ is a continuous function with $\theta(0) = 0$ and $\theta(T) = T$. Then there is a small number $\tau_0 \in (0, 1)$ such that whenever $|v(t)| \leq \tau_0$, we have that

$$|v(t)| \leq 10\sqrt{H^1(\gamma) - T}.$$
We see that $\mathfrak{c}_1 \cup \mathfrak{c}_2$ is a simple Lipschitz curve joining $A$ and $B$, and let $\gamma_3 : [0, \ell] \to S^2$ giving by

$$\gamma_3(t) = \left( \sqrt{1 - w(t)^2} \cos \theta(t), \sqrt{1 - w(t)^2} \sin \theta(t), w(t) \right)$$

be its parametrization by length. We assume that $\gamma_3(t_1) = D$, then $w'(t) > 0$ on $(0, t_1)$, or $w'(t) < 0$ on $(0, t_1)$, thus $|w(t)| = \int_0^{t_1} |w'(t)|dt$.

We let the number $\tau_0 \in (0, 1)$ to be the small number $\tau_0$ in Lemma 7.8 in [4]. If $\mathcal{H}^1(\gamma) - T \leq \tau_0$, then we have that

$$\int_0^\ell |w'(t)|^2dt \leq 14(\ell - T),$$

thus

$$|w(t_1)| = \int_0^{t_1} |w'(t)|dt \leq \left( t_1 \int_0^{t_1} |w'(t)|^2dt \right)^{1/2} \leq \sqrt{14(\ell - T)}.$$

We get so that

$$|v(t_0)| \leq \mathcal{H}^1(\mathfrak{c}_3) + |w(t_1)| \leq (\mathcal{H}^1(\gamma) - \ell) + 14\sqrt{\ell - T} \leq \sqrt{14\mathcal{H}^1(\gamma)(\mathcal{H}^1(\gamma) - T)} \leq 10\sqrt{\mathcal{H}^1(\gamma) - T}.$$

If $\mathcal{H}^1(\gamma) - T > \tau_0$, then $v(t) \leq \tau_0 \leq 10\sqrt{\tau_0} \leq 10\sqrt{\mathcal{H}^1(\gamma) - T}$.

\begin{lemma}
Let $a$ and $b$ be two points in $\Omega_0 \cap \partial B(0, 1)$ satisfying

$$\frac{\pi}{3} \leq \text{dist}_{S^2}(a, b) \leq \frac{2\pi}{3}.$$ 

Let $\gamma$ be a simple rectifiable curve in $\Omega_0 \cap \partial B(0, 1)$ which joins $a$ and $b$, and satisfies

$$\text{length}(\gamma) \leq \text{dist}_{S^2}(a, b) + \tau_0,$$

where $\tau_0 > 0$ is as in Lemma 3.5. Then there is a constant $C > 0$ such that, for any $\eta > 0$, we can find a simple curve $\gamma_\ast$ in $\Omega_0 \cap \partial B(0, 1)$ which is a Lipschitz graph with constant at most $\eta$ joining $a$ and $b$, and satisfies that

$$\mathcal{H}^1(\gamma_\ast \setminus \gamma) \leq \mathcal{H}^1(\gamma \setminus \gamma_\ast) \leq C\eta^{-2}(\text{length}(\gamma) - \text{dist}_{S^2}(a, b)).$$

The proof will be the same as in [4, p.875-p.878], so we omit it.

\subsection{Compare surfaces}
Let $\Gamma$ be a Lipschitz curve in $S^2$. We assume for simplicity that its extremities $a$ and $b$ lie in the horizontal plane. Let us assume that $a = (1, 0, 0)$ and $b = (\cos T, \sin T, 0)$ for some $T \in [\pi/3, 2\pi/3]$. We also assume that $\Gamma$ is a Lipschitz graph with constant at most $\eta$, i.e. there is a Lipschitz function $s : [0, T] \to \mathbb{R}$ with $s(0) = s(T) = 0$ and $\text{Lip}(s) \leq \eta$, such that $\Gamma$ is parametrized by

$$z(t) = (w(t) \cos t, w(t) \sin t, s(t)) \text{ for } t \in [0, T],$$

where $w(t) = (1 - |s(t)|^2)^{1/2}$.

We set

$$D_T = \{(r \cos t, r \sin t) \mid 0 < r < 1, 0 < t < T\},$$
and consider the function \( v : T \to \mathbb{R} \) defined by

\[
v(r \cos t, r \sin t) = \frac{r s(t)}{w(t)} \quad \text{for } 0 \leq r \leq 1 \text{ and } 0 \leq t \leq T.
\]

For any function \( f : T \to \mathbb{R} \), we denote by \( \Sigma_f \) the graphs of \( f \) over \( T \).

**Lemma 3.7.** There is a universal constant \( \kappa > 0 \) such that we can find a Lipschitz function \( u \) on \( T \) satisfying that

\[
\text{Lip}(u) \leq C \eta,
\]

\[
\begin{align*}
u(r, 0) &= u(r \cos T, r \sin T) = 0, \quad \text{for } 0 \leq r \leq 1, 0 \leq t \leq T, \\
u(r \cos t, r \sin t) &= v(r \cos t, r \sin t) \quad \text{for } 0 \leq r \leq 1, 0 \leq t \leq T, \\
u(r \cos t, r \sin t) &= 0, \quad \text{for } 0 \leq r \leq 2\kappa, 0 \leq t \leq T
\end{align*}
\]

and

\[
\mathcal{H}^2(\Sigma_v) - \mathcal{H}^2(\Sigma_u) \geq 10^{-4}(\mathcal{H}^1(\Gamma) - T).
\]

**Proof.** The proof is the same as Lemma 8.8 in [4], we omit it. \( \square \)

### 3.4 Retractions

We let \( \Pi : \mathbb{R}^3 \setminus \{0\} \to S^2 \) be the projection defined by \( \Pi(x) = x/|x| \). In this subsection, we assume that \( E \subseteq \Omega_0 \) is a 2-rectifiable set satisfying that

- \( \mathcal{H}^2(E) < \infty, \ 0 \in E \),
- \( E \) is locally \((\Omega_0, L_0, h)\) sliding almost minimal at \( 0 \),
- \( E \) is locally \( C^0\)-equivalent to a sliding minimal cone of type \( \mathbb{P}_+ \) or \( \mathbb{Y}_+ \).

For convenient, we put

\[
j(r) = \frac{1}{r} \mathcal{H}^1(E \cap \partial B(0, r)) - \mathcal{H}^1(X \cap \partial B(0, 1)),
\]

and denote by \( \mathcal{R}_1 \) the set \( \{r \in \mathcal{R} : j(r) \leq \tau_1\} \), where \( \tau_1 \) is the small number considered as in Lemma 3.5.

For any \( r \in (0, t) \cap \mathcal{R}_1 \), we take \( X_r \subseteq E \cap B(0, r) \cap L_0 \) as following: if \( E \) is locally \( C^0\)-equivalent to a sliding minimal cone of type \( \mathbb{P}_+ \), we let \( X_{r,1}, X_{r,2} \) be the same as in Lemma 3.3, and let \( X_r = \{X_{r,1}, X_{r,2}\} \); if \( E \) is locally \( C^0\)-equivalent to a sliding minimal cone of type \( \mathbb{Y}_+ \), we let \( X_{r,1}, X_{r,2}, X_{r,3} \) be the same as in Lemma 3.4, and let \( X_r = \{X_{r,1}, X_{r,2}, X_{r,3}\} \).

We take \( y_r \) as in Lemma 3.3 or Lemma 3.4. For any \( x \in X_r \), let \( \gamma^x \) be the curve which joins \( x \) and \( y_r \) as in Lemma 3.3 and Lemma 3.4, let \( D_{x,y_r} \) be the sector determined by points \( 0, y_r, x \). We denote by \( P_{x,y_r} \), the plane that contains \( 0, x \) and \( y_r \), \( R_{x,y_r} \) be a rotation such that \( R_{x,y_r}(y_r) = (r, 0, 0) \) and \( R_{x}(y_r) = (r \cos T_x, r \sin T_x, 0) \), where \( T_x \in [\pi/3, 2\pi/3] \).

For any \( x \in X_r \), \( \gamma^x \) is a simple rectifiable curve in \( \Omega_0 \cap \partial B(0, r) \), thus the curve \( \Gamma^x = \Pi(\gamma^x) \) is a simple rectifiable curve in \( \Omega_0 \cap \partial B(0, 1) \), let \( \Gamma^x_\ast \) be the corresponding curve with respect to \( \Gamma^x \) as in Lemma 3.6. Let \( z(t) = (w(t) \cos t, w(t) \sin t, s(t)) \) be a parametrization of \( R_{x,y_r}(\Gamma^x_\ast) \), where \( w(t) = \sqrt{1 - s(t)^2} \). Let \( \Sigma^x_\ast \) and \( \Sigma^x_\circ \) be the same as in Lemma 3.7. We put

\[
T = \sum_{x \in X_r} T_x,
\]

and put

\[
X = \bigcup_{x \in X_r} D_{x,y_r}, \quad \Gamma_\ast = \bigcup_{x \in X_r} \Gamma^x_\ast, \quad M = \bigcup_{x \in X_r} \Sigma^x_\ast, \quad \text{and} \quad \Sigma = \bigcup_{x \in X_r} \Sigma^x_\circ.
\]
By Lemma 3.7, we have that
\[ \mathcal{H}^2(M) - \mathcal{H}^2(\Sigma) \geq 10^{-4} (\mathcal{H}^1(\Gamma_r) - T), \tag{3.4} \]
and by Lemma 3.5, we have that
\[ d_{0,1}(X, M) \leq 10j(r)^{1/2}. \tag{3.5} \]

**Lemma 3.8.** If \( \varepsilon(r) < 1/2 \), then for any \( \varepsilon(r) < \varepsilon < 1/2 \), there is a sliding minimal cone \( Z = Z_r \) such that
\[ d_{0,1}(X, Z) \leq 4\varepsilon. \]
Moreover
\[ d_{0,r}(E, X) \leq 5\varepsilon(r). \]

**Proof.** There exists sliding minimal cone \( Z \) such that \( d_{0,r}(E, Z) \leq \varepsilon \), thus for any \( x \in \mathcal{X}_r \), there is \( x_z \in Z \cap (L_0 \cap \partial B_r) \) satisfying that \( |x - x_z| \leq 2\varepsilon r \). We get so that
\[ d_H([x, y_r], [x_z, 3_r]) \leq 2\varepsilon r. \]
Since \( \text{dist}(0, [x, y_r]) > r/2 \) for any \( x \in \mathcal{X}_r \), we have that
\[ d_H(X \cap B(0, r/2), Z \cap B(0, r/2)) \leq 2\varepsilon r. \]
Thus
\[ d_{0,1}(X, Z) = d_{0,r/2}(X, Z) \leq 4\varepsilon, \]
and
\[ d_{0,r}(E, X) \leq d_{0,r}(E, Z) + d_{0,r}(Z, X) \leq 5\varepsilon. \]
\[ \square \]

**Lemma 3.9.** Let \( 0 < \delta, \varepsilon < 1/2 \) be positive numbers. Let \( v_1, v_2, v_3 \in \mathbb{R}^3 \) be three unit vectors.

- If \( |\langle v_2, v_i \rangle| \leq \delta \) for \( i = 1, 3 \), then for any \( v \in \mathbb{R}^3 \) with \( \langle v, v_2 \rangle = 0 \) and \( \text{dist}(v, \text{span}\{v_1, v_2\}) \leq \varepsilon |v| \), we have that
\[ |\langle v, v_3 \rangle - \langle v_1, v_3 \rangle \langle v, v_2 \rangle| \leq (\varepsilon + \delta) |v|, \quad \text{and} \quad |\langle v, v_1 \rangle| \geq (1 - \varepsilon - \delta) |v|. \]

- If \( \langle v_1, v_3 \rangle < 1 \) and \( 0 < \delta < 10^{-2}(1 - \langle v_1, v_3 \rangle)^2 \), then for any \( w_1, w_3 \in \mathbb{R}^3 \) with \( \langle v_1, w_i \rangle \geq (1 - \delta) |w_i|, i = 1, 3 \), we have that
\[ |w_1| + |w_3| \leq \sqrt{2} \cdot \left( 1 - \langle v_1, v_3 \rangle - 4\sqrt{2}\delta \right)^{-1/2} |w_1 - w_3|. \tag{3.6} \]

**Proof.** We write \( v = v^\perp + \lambda_1 v_1 + \lambda_2 v_2 \), \( \lambda_i \in \mathbb{R}, \langle v^\perp, v_i \rangle = 0 \). Since \( \langle v, v_2 \rangle = 0 \), we have that \( \lambda_2 = -\lambda_1 \langle v_1, v_2 \rangle \), thus
\[ \lambda_1 = \frac{\langle v, v_1 \rangle}{1 - \langle v_1, v_2 \rangle^2}, \quad \lambda_2 = -\frac{\langle v, v_1 \rangle \langle v_1, v_2 \rangle}{1 - \langle v_1, v_2 \rangle^2}. \]
Thus and Lemma 3.10. For any \( H \) we get so that
\[
v = v^\perp + \frac{(v, v_1)v_1 - (v, v_1)(v_1, v_2)v_2}{1 - (v_1, v_2)^2},
\]
and then
\[
\langle v, v_3 \rangle = \langle v^\perp, v_3 \rangle + \frac{(v_1, v_3) - (v_2, v_3)(v_1, v_2)}{1 - (v_1, v_2)^2} \langle v, v_1 \rangle,
\]
thus
\[
|\langle v, v_3 \rangle - \langle v, v_1 \rangle| \langle v, v_1 \rangle| \leq \varepsilon|v| + \frac{\delta^2 + \delta}{1 - \delta^2}|v| \leq (\varepsilon + 2\delta)|v|.
\]
We get also, from (3.4), that
\[
|\langle v, v_1 \rangle| \geq (1 - \varepsilon)(1 - \delta)|v| \geq (1 - \varepsilon - \delta)|v|.
\]
We can certainly assume \( w_i \neq 0 \), otherwise the inequality (3.9) will be trivial true. Since
\( \langle v_i, w_i \rangle \geq (1 - \delta)|w_i| \), we have that \( \langle v_i, w_i/|w_i| \rangle \geq 1 - \delta \), and
\[|v_i - w_i/|w_i||^2 = 2 - 2\langle v_i, w_i/|w_i| \rangle \leq 2\delta.\]
Thus
\[
|\frac{w_1}{|w_1|} - \frac{w_2}{|w_2|}|^2 = \left| \left( \frac{w_1}{|w_1|} - v_1 \right) - \left( \frac{w_2}{|w_2|} - v_2 \right) + (v_1 - v_2) \right|^2 \\
\geq |v_1 - v_2|^2 - 2|v_1 - v_2| \left( \frac{|w_1|}{|w_1|} - v_1 \right) + \left( \frac{|w_2|}{|w_2|} - v_2 \right) \\
\geq 2 - 2\langle v_1, v_2 \rangle - 8\sqrt{\delta},
\]
and
\[
\langle w_1, w_2 \rangle = |w_1||w_2| \left( \frac{w_1}{|w_1|} \cdot \frac{w_2}{|w_2|} \right) \leq |w_1||w_2| \left( \langle v_1, v_2 \rangle + 4\sqrt{\delta} \right).
\]
Hence
\[
|w_1 - w_2|^2 \geq |w_1|^2 + |w_2|^2 - 2|w_1||w_2| \left( \langle v_1, v_2 \rangle + 4\sqrt{\delta} \right) \geq (1 - s)(|w_1| + |w_2|)^2,
\]
where \( s = (1 + \langle v_1, v_2 \rangle + 4\sqrt{\delta})/2 \in (0, 1). \)

\[\square\]

**Lemma 3.10.** For any \( r \in (0, r) \cap \mathcal{R}_1 \), we let \( \Sigma \) be as in (3.4). Then there is a Lipschitz mapping \( p : \Omega_0 \to \Sigma \) with \( \text{Lip}(p) \leq 50 \), such that \( p(z) \in L \) for \( z \in L \), and that \( p(z) = z \) for \( z \in \Sigma \).

**Proof.** By definition, we have that
\[\Sigma \setminus B(0, 9/10) = \mathcal{M} \setminus B(0, 9/10),\]
and that
\[\Sigma \cap B(0, 2\kappa) = X \cap B(0, 2\kappa).\]
For any \( z \in \Omega_0 \setminus \{0\} \), we denote by \( \ell(z) \) the line which is through 0 and \( z \). Then \( \partial D_{x,y_r} = \ell(x) \cup \ell(y_r) \). We fix any \( \sigma \in (0, 10^{-2}) \), put
\[
R^x = \{ z \in \Omega_0 \mid \text{dist}(z, D_{x,y_r}) \leq \sigma \text{dist}(z, \partial D_{x,y_r}) \}, \\
R^1 = \{ z \in \Omega_0 \mid \text{dist}(z, D_{x,y_r}) \leq \sigma \text{dist}(z, \ell(y_r)) \},
\]
and
\[
R = \bigcup_{x \in \mathcal{X}_r} R^x, R_1 = \bigcup_{x \in \mathcal{X}_r} R^1.
\]
Then we see that \( R^x \subseteq R^1 \), and that both of them are cones,
\[
R^x_i \cap R^x_j = R^{x_i}_1 \cap R^{x_j}_1 = \ell(y_r) \text{ for } x_i, x_j \in \mathcal{X}_r, x_i \neq x_j.
\]

Since \( \Sigma^x_u \) is a small Lipschitz graph over \( D_{x,y_r} \), bounded by two half lines of \( \partial D_{x,y_r} \) with constant at most \( \eta \), there is a constant \( \bar{\eta} \) such that
\[
\Sigma^x_u \subseteq R^x,
\]
when \( 0 < \eta < \bar{\eta} \).

We will construct a Lipschitz retraction \( p_0 : \Omega_0 \to R_1 \) such that \( p_0(z) = z \) for \( z \in R_1 \), \( p_0(z) \in L \) for \( z \in L \), and \( \text{Lip}(p_0) \leq 3 \). We now distinguish two cases, depending on cardinality of \( \mathcal{X}_r \).

Case 1: \( \text{card}(\mathcal{X}_r) = 2 \). We assume that \( \mathcal{X}_r = \{x_1, x_2\} \). Then \( |y_r| = |x_1| = |x_2| = r \), and
\[
0 \leq \langle x_1, x_2 \rangle + r^2 \leq 2 \varepsilon^2 r^2.
\]
Since \( |y_r - z| \leq \varepsilon r \), we have that \( |\langle y_r, x \rangle| \leq \varepsilon r^2 \) for any \( x \in L \cap \partial B(0, r) \).

We now let \( e_1 \) and \( e_2 \) be two unit vectors in \( L \) such that \( \langle x_1, e_1 \rangle = \langle x_2, e_1 \rangle \geq 0 \) and \( e_2 = -e_1 \). Then
\[
0 \leq \langle x_i, e_1 \rangle \leq \varepsilon r.
\]
We let \( \Omega'_1 \) and \( \Omega'_2 \) be the two connected components of \( \Omega_0 \setminus (\cup_i D_{x_i,y_r}) \) such that \( e_i \in \Omega'_i \). We put \( \Omega_i = \Omega'_i \setminus R_1 \). We claim that
\[
|\langle z_1 - z_2, e_i \rangle| \leq 5(\sigma + \varepsilon)|z_1 - z_2|
\]
whenever \( z_1, z_2 \in \partial \Omega_i, z_1 \neq z_2, i \in \{1, 2\} \).

Without loss of generality, we assume \( z_1, z_2 \in \partial \Omega_1 \), because for another case we will use the same treatment. We see that
\[
\text{dist}(z_i, D_{x_j,y_r}) = \sigma \text{dist}(z_i, \ell(y_r)).
\]

(1) In case \( z_1, z_2 \in \partial R^x_1 \cap \Omega_1 \), without loss of generality, we assume that \( z_1, z_2 \in \partial R^{x_1}_1 \cap \Omega_1 \).

We let \( \tilde{z}_i \in D_{x_i,y_r} \) be such that \( z_i - \tilde{z}_i = \text{dist}(z_i, D_{x_i,y_r}), i = 1, 2, \)

and let \( z'_i \in \ell(y_r) \) be such that \( |z_i - z'_i| = \text{dist}(z_i, \ell(y_r)) \).
and put
\[ w_1 = z_1 - \tilde{z}_1 + \tilde{z}_2, \quad w_2 = z_1 - z_1' + z_2', \]
then we get that \( z_1 - z_2 = (z_1 - w_2) + (w_2 - z_2) \). Moreover, we have that \( z_1 - w_2 \) is perpendicular to \( w_2 - z_2 \) and parallel to \( y_r \). Thus \( |w_2 - z_2| \leq |z_1 - z_2|, \quad |z_1 - w_2| \leq |z_1 - z_2| \) and
\[ \text{dist}(w_2 - z_2, \text{span}\{x_1, y_r\}) = \sigma|w_2 - z_2|. \]
We apply Lemma 3.9 to get that
\[ |\langle z_1 - w_2, e_1 \rangle| \leq \varepsilon|z_1 - w_2| \]
and
\[ |\langle w_2 - z_2, e_1 \rangle| \leq (\sigma + 3\varepsilon)|w_2 - z_2|, \]
thus
\[ |\langle z_1 - z_2, e_1 \rangle| \leq |\langle z_1 - w_2, e_1 \rangle| + |\langle w_2 - z_2, e_1 \rangle| \leq (\sigma + 4\varepsilon)|z_1 - z_2|. \]
(2) In case \( z_1 \in \partial R^{x_1} \cap \Omega_1, \ z_2 \in \partial R^{x_2} \cap \Omega_1 \). We let \( \tilde{z}_i \in D_{x_i, y_r} \) be such that
\[ |z_i - \tilde{z}_i| = \text{dist}(z_i, D_{x_i, y_r}), \quad i = 1, 2, \]
and let \( z_i' \in \ell(y_r) \) be such that
\[ |z_i - z_i'| = \text{dist}(z_i, \ell(y_r)), \quad i = 1, 2. \]
Then by Lemma 3.9, we have that
\[ \left\langle z_i - z_i', \frac{x_i}{|x_i|} \right\rangle \geq (1 - \sigma - \varepsilon)|z_i - z_i'|, \quad i = 1, 2. \]
Since \( z_1 - z_2 = (z_1 - z_1') + (z_2' - z_2) + (z_1' - z_2'), \)
\[ |\langle z_1' - z_2', e_1 \rangle| \leq \varepsilon|z_1' - z_2'| \leq \varepsilon|z_1 - z_2| \]
and
\[ |\langle z_1 - z_1', e_1 \rangle| \leq (\sigma + \varepsilon)|z_1 - z_1'|, \]
we get that
\[ |\langle z_1 - z_2, e_1 \rangle| \leq |\langle z_1 - z_1', e_1 \rangle| + |\langle z_2' - z_2, e_1 \rangle| + |\langle z_1' - z_2', e \rangle| \]
\[ \leq 2 \cdot (\sigma + \varepsilon)(|z_1 - z_1'| + |z_2 - z_2'|) + \varepsilon|z_1 - z_2|. \]
Since $z_1' - z_2'$ is perpendicular to $z_1 - z_1'$ and $z_2 - z_2'$, and
\[
\left\langle z_i - z_i', \frac{x_i}{|x_i|} \right\rangle \geq (1 - \sigma - \varepsilon)|z_i - z_i'|, \quad i = 1, 2,
\]
and
\[
\left\langle \frac{x_1}{|x_1|}, \frac{x_2}{|x_2|} \right\rangle \leq -1 + 2\varepsilon^2,
\]
we get, by Lemma 3.9, that
\[
|z_1 - z_1'| + |z_2 - z_2'| \leq (1 - \varepsilon^2 - 5\sqrt{\sigma + \varepsilon})^{-1/2} |(z_1 - z_1') - (z_2 - z_2')| \leq 2|z_1 - z_2|.
\]
Thus
\[
\langle z_1 - z_2, e \rangle \leq (4\sigma + 5\varepsilon)|z_1 - z_2|.
\]
We now define $p_0 : \Omega \to R_1$ as follows: for any $z \in \Omega$, we let $p_0(z)$ be the unique point in $\partial \Omega$ such that $p_0(z) - z$ parallels $e$; and for any $z \in R_1$, we let $p_0(z) = z$. Since $p_0(z) - z$ parallels $e$, we see that $p_0(L) \subseteq L$. We will check that
\[
p_0 \text{ is Lipschitz with } \text{Lip}(p_0) \leq \frac{2}{1 - 5(\sigma + \varepsilon)}.
\]
Indeed, for any $z_1, z_2 \in \Omega$, we put
\[
p_0(z_i) = z_i + t_i e, \quad t_i \in \mathbb{R},
\]
then
\[
|t_1 - t_2| = |\langle (t_1 - t_2)e, e \rangle| \leq |\langle p_0(z_1) - p_0(z_2), e \rangle| + |\langle z_1 - z_2, e \rangle| \leq 5(\sigma + \varepsilon)|p_0(z_1) - p_0(z_2)| + |z_1 - z_2|,
\]
and
\[
|p_0(z_1) - p_0(z_2)| \leq |z_1 - z_2| + |t_1 - t_2| \leq 5(\sigma + \varepsilon)|p_0(z_1) - p_0(z_2)| + 2|z_1 - z_2|,
\]
thus
\[
|p_0(z_1) - p_0(z_2)| \leq \frac{2}{1 - 5(\sigma + \varepsilon)}|z_1 - z_2|.
\]
Case 2: $\text{card}(\mathcal{X}) = 3$. We assume that $\mathcal{X} = \{x_1, x_2, x_3\}$, then
\[
|\langle x_i, y_r \rangle| \leq \varepsilon r^2 \left( -\sqrt{3\varepsilon - \frac{1}{2}} \right) r^2 \leq \langle x_i, x_j \rangle \leq \left( -\frac{1}{2} + 2\varepsilon \right) r^2.
\]
We put
\[
e_1 = \frac{x_2 + x_3}{|x_2 + x_3|}, \quad e_2 = \frac{x_1 + x_3}{|x_1 + x_3|}, \quad e_3 = \frac{x_2 + x_1}{|x_2 + x_1|},
\]
and let $\Omega_1', \Omega_2'$ and $\Omega_3'$ be the three connected components of $\Omega \setminus (\cup_i D_{x_i, y_r})$ such that $e_i \in \Omega_i'$. By putting $\Omega_i = \Omega_i' \setminus R_1$, we claim that
\[
\left( \frac{1}{2} - 5(\sigma + \varepsilon) \right) |z_1 - z_2| \leq |\langle z_1 - z_2, e_i \rangle| \leq \left( \frac{1}{2} + 5(\sigma + \varepsilon) \right) |z_1 - z_2|.
\]
whenever \( z_1, z_2 \in \partial \Omega_1, z_1 \neq z_2, i \in \{1, 2, 3\} \).

Indeed, we only need to check the case \( z_1, z_2 \in \partial \Omega_1 \), and the other two cases will be the same. Since \(-\sqrt{3}\varepsilon - 1/2 \leq \langle x_i, x_j \rangle \leq 1/2 + 2\varepsilon \), we have that \((1/2 - \varepsilon)r \leq \langle x_i, e_i \rangle \leq (1/2 + \varepsilon)r \) for \( i = 2, 3 \).

If \( z_1, z_2 \in \partial R^{x_2} \cap \Omega_1 \) or \( z_1, z_2 \in \partial R^{x_3} \cap \Omega_1 \), we assume that \( z_1, z_2 \in \partial R^{x_2} \cap \Omega_1 \), and let \( \tilde{z}_i \in D_{x_2, y_r} \) be such that

\[
\tilde{z}_i - z_i = \text{dist}(z_i, D_{x_2, y_r}), \quad i = 1, 2,
\]

and let \( z_i' \in \ell(y_r) \) be such that

\[
|z_i - z_i'| = \text{dist}(z_i, \ell(y_r)),
\]

and put

\[
w_1 = z_1 - \tilde{z}_1 + \tilde{z}_2, \quad w_2 = z_1 - z_1' + z_2',
\]

then we get that \( z_1 - w_2 \) is perpendicular to \( w_2 - z_2 \) and parallel to \( y_r \). Since \( z_1 - z_2 = (z_1 - w_2) + (w_2 - z_2) \), we have that \(|w_2 - z_2| \leq |z_1 - z_2|, |z_1 - w_2| \leq |z_1 - z_2| \) and

\[
\text{dist}(w_2 - z_2, \text{span}\{x_1, y_r\}) = \sigma |w_2 - z_2|.
\]

We apply Lemma 3.9 to get that

\[
|\langle z_1 - w_2, e_1 \rangle| \leq \varepsilon |z_1 - w_2|
\]

and

\[
|\langle w_2 - z_2, e_1 \rangle| \leq \left(\frac{1}{2} + \varepsilon + \sigma + \varepsilon\right) |w_2 - z_2|,
\]

thus

\[
|\langle z_1 - z_2, e_1 \rangle| \leq |\langle z_1 - w_2, e_1 \rangle| + |\langle w_2 - z_2, e_1 \rangle| \leq \left(\frac{1}{2} + \sigma + 3\varepsilon\right) |z_1 - z_2|.
\]

If \( z_1 \in \partial R^{x_2} \cap \Omega_1, z_2 \in \partial R^{x_3} \cap \Omega_1 \), we let \( \tilde{z}_i \in D_{x_2, y_r} \) be such that

\[
|z_1 - \tilde{z}_1| = \text{dist}(z_1, D_{x_2, y_r}), \quad |z_2 - \tilde{z}_2| = \text{dist}(z_2, D_{x_3, y_r})
\]

and let \( z_i' \in \ell(y_r) \) be such that

\[
|z_i - z_i'| = \text{dist}(z_i, \ell(y_r)), \quad i = 1, 2.
\]

Since \( z_1 - z_2 = (z_1 - z_1') + (z_2' - z_2) + (z_1' - z_2') \),

\[
|\langle z_1' - z_2', e_1 \rangle| \leq \varepsilon |z_1' - z_2'| \leq \varepsilon |z_1 - z_2|
\]

and

\[
|\langle z_1 - z_1', e_1 \rangle| \leq \left(\frac{1}{2} + \varepsilon + \sigma + \varepsilon\right) |z_1 - z_1'|,
\]

we get that

\[
|\langle z_1 - z_2, e_1 \rangle| \leq |\langle z_1 - z_1', e_1 \rangle| + |\langle z_2' - z_2, e_1 \rangle| + |\langle z_1' - z_2', e \rangle| \leq \left(\frac{1}{2} + \sigma + 2\varepsilon\right) \left( |z_1 - z_1'| + |z_2 - z_2'| \right) + \varepsilon |z_1 - z_2|.
\]

(3.8)
By Lemma 3.9, we have that
\[
\langle z_1 - z_1', \frac{x_2}{|x_2|} \rangle \geq (1 - \sigma - \varepsilon)|z_1 - z_1'|
\]
and
\[
\langle z_2 - z_2', \frac{x_3}{|x_3|} \rangle \geq (1 - \sigma - \varepsilon)|z_2 - z_2'|.
\]
Applying Lemma 3.9 with \(\langle x_2/|x_2|, x_3/|x_3| \rangle \leq -1/2 + 2\varepsilon\), we get that
\[
|z_1 - z_1'| + |z_2 - z_2'| \leq \left( \frac{2}{1 + 1/2 - 2\varepsilon - 4\sqrt{2\sigma + 2\varepsilon}} \right)^{1/2} \left| \langle z_1 - z_1' \rangle - (z_2 - z_2') \right|
\]
\[
\leq \frac{2}{\sqrt{3}} \left( 1 - \frac{2\varepsilon + 4\sqrt{2\sigma + 2\varepsilon}}{3} \right) |z_1 - z_2|.
\]
We get, from (3.4), that
\[
|\langle z_1 - z_2, e_1 \rangle| \leq \frac{2}{3} |z_1 - z_2|.
\]
For any \(z \in \Omega_i\), we now let \(p_0(z)\) be the unique point in \(\partial \Omega_i\) such that \(p_0(z) - z\) parallels \(e_i\); and for \(z \in R_i\), we let \(p_0(z) = z\). Then \(p_0(L) \subseteq L\). We will check that
\[
p_0 \text{ is Lipschitz with } \text{Lip}(p_0) \leq 6.
\]
For any \(z_1, z_2 \in \Omega_i\), we put
\[
p_0(z_j) = z_j + t_je_i, \ t_i \in \mathbb{R}, \ j = 1, 2,
\]
then
\[
|t_1 - t_2| = |\langle (t_1 - t_2)e_i, e_i \rangle| \\
\leq |\langle p_0(z_1) - p_0(z_2), e_i \rangle| + |\langle z_1 - z_2, e_i \rangle| \\
\leq \frac{2}{3} |p_0(z_1) - p_0(z_2)| + |z_1 - z_2|,
\]
and
\[
|p_0(z_1) - p_0(z_2)| \leq |z_1 - z_2| + |t_1 - t_2| \leq \frac{2}{3} |p_0(z_1) - p_0(z_2)| + 2|z_1 - z_2|,
\]
thus
\[
|p_0(z_1) - p_0(z_2)| \leq 6|z_1 - z_2|.
\]
By the definition of \(R^x\) and \(R^z\), we have that
\[
R_c = \{ z \in R^c_1 \ | \ \text{dist}(z, D_{x,y}) \leq \sigma \text{ dist}(z, \ell(x)) \}.
\]
Similar as above, we can that, for any \(z_1, z_2 \in R^c_1 \cap \partial R^x\) with \([z_1, z_2] \cap D_{x,y} = \emptyset\), if \(\text{card}(X_c) = 2\) then
\[
|\langle z_1 - z_2, e_i \rangle| \leq 5(\sigma + \varepsilon)|z_1 - z_2|;
\]
if \(\text{card}(X_c) = 3\) then
\[
|\langle z_1 - z_2, e_i \rangle| \leq \left( \frac{1}{2} + \sigma + 3\varepsilon \right) |z_1 - z_2|,
\]
where \(e_i\) is the vector in (3.4) such that \(z_1, z_2 \in \Omega_i\).

We now consider the mapping \(p_1 : R_1 \to R\) defined by

\[
p_1(z) = \begin{cases} 
  z, & \text{for } z \in R, \\
  z - te_i \in \partial R \cap \Omega_i, & \text{for } z \in \Omega_i.
\end{cases}
\]

By the same reason as above, we get that

\[
\text{Lip}(p_1) \leq \frac{2}{1 - 1/2 - \sigma - 3\varepsilon} \leq 5.
\]

We define a mapping \(p_2 : R \cap B(0, 1) \to \Sigma\) as follows: we know \(\Sigma^\varepsilon_u\) is the graph of \(u\) over \(D_{x,y},\) thus for any \(z \in R^x,\) there is only one point in the intersection of \(\Sigma^\varepsilon_u\) and the line which is perpendicular to \(D_{x,y}\) and through \(z,\) we define \(p_2(z)\) to be the unique intersection point. That is, \(p_2(z)\) is the unique point in \(\Sigma^\varepsilon_u\) such that \(p_2(z) - z\) is perpendicular to \(D_{x,y}\). We will show that \(p_2\) is Lipschitz and \(\text{Lip}(p_2) \leq 1 + 10^4\eta.\) Indeed, for any points \(z_1, z_2 \in R^x,\) we let \(\tilde{z}_i, i = 1, 2,\) be the points in \(D_{x,y}\) such that \(z_i - \tilde{z}_i\) is perpendicular to \(D_{x,y},\) then

\[
|p_2(z_1) - z_1)\rangle - (p_2(z_2) - z_2)| = |u(\tilde{z}_1) - u(\tilde{z}_2)| \leq \text{Lip}(u)|\tilde{z}_1 - \tilde{z}_2| \leq \text{Lip}(u)|z_1 - z_2|,
\]

thus

\[
|p_2(z_1) - p_2(z_2)| \leq (1 + \text{Lip}(u))|z_1 - z_2| \leq (1 + 10^4\eta)|z_1 - z_2|.
\]

Let \(p_3 : \mathbb{R}^3 \to \mathbb{R}^3\) be the mapping defined by

\[
p_3(x) = \begin{cases} 
  x, & |x| \leq 1 \\
  \frac{x}{|x|}, & |x| > 1.
\end{cases}
\]

Then \(p = p_3 \circ p_2 \circ p_3 \circ p_1 \circ p_0\) is our desire mapping.

\[\square\]

**Lemma 3.11.** For any \(r \in (0, \varepsilon) \cap R_1,\) we let \(\Sigma\) be as in (3.4), and let \(\Sigma_r\) be given by \(\mu_r(\Sigma)\). Then we have that

\[
\mathcal{H}^2(E \cap B(0, r)) \leq \mathcal{H}^2(\Sigma_r) + 2550 \int_{E \cap \partial B(0, r)} \text{dist}(z, \Sigma_r) d\mathcal{H}^1(z) + (2r)^2 h(2r).
\]

**Proof.** For any \(\xi > 0,\) we consider the function \(\psi_\xi : [0, \infty) \to \mathbb{R}\) defined by

\[
\psi_\xi(t) = \begin{cases} 
  1, & 0 \leq t \leq 1 - \xi \\
  -\frac{t - 1}{\xi}, & 1 - \xi < t \leq 1 \\
  0, & t > 1,
\end{cases}
\]

and the mapping \(\phi_\xi : \Omega_0 \to \Omega_0\) defined by

\[
\phi_\xi(z) = \psi_\xi(|z|) p(z) + (1 - \psi_\xi(|z|)) z.
\]

Then we get that \(\phi_\xi(L) \subseteq L.\) For any \(t \in [0, 1],\) we put

\[
\varphi_t(z) = tr\phi_\xi(z/r) + (1 - t)z, \text{ for } z \in \Omega_0.
\]
Then \( \{ \varphi_t \}_{0 \leq t \leq 1} \) is a sliding deformation, and we get that
\[
\mathcal{H}^2(E \cap B(0, r)) \leq \mathcal{H}^2(\varphi_1(E) \cap \overline{B(0, r)}) + (2r)^2 h(2r).
\]
Since \( \psi_\xi(t) = 1 \) for \( t \in [0, 1 - \xi] \), we get that
\[
\varphi_1(E \cap B(0, (1 - \xi)r)) = p(E \cap B(0, (1 - \xi)r)) \subseteq \Sigma_r.
\]
We set \( A_\xi = B(0, r) \setminus B(0, (1 - \xi)r) \). By Theorem 3.2.22 in [9], we get that
\[
\mathcal{H}^2(\varphi_1(E \cap A_\xi)) \leq \int_{E \cap A_\xi} \mathrm{ap} J_2(\varphi_1|E)(z) d\mathcal{H}^2(z). \tag{3.9}
\]
For any \( z \in A_\xi \) and \( v \in \mathbb{R}^3 \), we have, by setting \( z' = z/r \), that
\[
D\varphi_1(z)v = \psi_\xi(|z'|)Dp(z')v + (1 - \psi_\xi(|z'|))v + \psi_\xi(|z'|)(z/|z|, v)(rp(z') - z).
\]
For any \( z \in A_\xi \cap E \), we let \( v_1, v_2 \in T_z E \) be such that
\[
|v_1| = |v_2| = 1, \ v_1 \perp z \text{ and } v_2 \perp v_1,
\]
then we have that \( \langle z/|z|, v \rangle = \cos \theta(z) \), and that
\[
|\psi_\xi(|z'|)Dp(z')v_1 + (1 - \psi_\xi(|z'|))v_1| \leq |Dp(z')v_1| \leq \text{Lip}(p),
\]
thus
\[
\text{ap} J_2(\varphi_1|E)(z) = |D\varphi_1(z)v_1 \wedge D\varphi_1(z)v_2| \
\leq \text{Lip}(p)^2 + \frac{1}{\xi} \text{Lip}(p) \cos \theta(z) |rp(z') - z|. \tag{3.10}
\]
Since \( p(\tilde{z}) = \tilde{z} \) for any \( \tilde{z} \in \Sigma \), we have that
\[
|p(z') - z'| = |p(z') - p(\tilde{z}) + \tilde{z} - z'| \leq (\text{Lip}(p) + 1)|\tilde{z} - z'|,
\]
then we get that
\[
|p(z') - z'| \leq (\text{Lip}(p) + 1) \text{dist}(z, \Sigma).
\]
We now get, from (3.4), that
\[
\text{ap} J_2(\varphi_1|E)(z) \leq \text{Lip}(p)^2 + \frac{1}{\xi} \text{Lip}(p)(\text{Lip}(p) + 1) \text{dist}(z, \Sigma_r) \cos \theta(z),
\]
plug that into (3.4) to get that
\[
\mathcal{H}^2(\varphi_1(E \cap A_\xi)) \leq 2500 \mathcal{H}^2(E \cap A_\xi) + \frac{2550}{\xi} \int_{E \cap A_\xi} \text{dist}(z, \Sigma_r) \cos \theta(z) d\mathcal{H}^2(z)
\leq 2500 \mathcal{H}^2(E \cap A_\xi) + \frac{2550}{\xi} \int_{(1-\xi)r} \int_{E \cap \partial B(0, t)} \text{dist}(z, \Sigma_r) d\mathcal{H}^1(z) dt,
\]
we let \( \xi \to 0^+ \), then we get that, for such \( r \),
\[
\lim_{\xi \to 0^+} \mathcal{H}^2(\varphi_1(E \cap A_\xi)) \leq 2550r \int_{E \cap \partial B(0, r)} \text{dist}(z, \Sigma_r) d\mathcal{H}^1(z),
\]
thus
\[
\mathcal{H}^2(E \cap B(0, r)) \leq \mathcal{H}^2(\Sigma_r) + 2550r \int_{E \cap \partial B(0, r)} \text{dist}(z, \Sigma_r) d\mathcal{H}^1(z) + (2r)^2 h(2r).
\]
\( \square \)
3.5 The main comparison statement

For any $x, y \in \Omega \cap \partial B(0,1)$, if $|x - y| < 2$, we denote by $g_{x,y}$ the unique geodesic on $\Omega \cap \partial B(0,1)$ which join $x$ and $y$.

We will denote by $B_t$ the open ball $B(0,t)$ sometimes for short.

**Lemma 3.12.** Let $\tau \in (0, 10^{-4})$ be a given. Then there is a constant $\vartheta > 0$ such that the following hold. Let $a \in \partial B(0,1)$ and $b, c \in L \cap \partial B(0,1)$ be such that $\text{dist}(a, (0,0,1)) \leq \tau$, $\text{dist}(b, (1,0,0)) \leq \tau$ and $\text{dist}(c, (-1,0,0)) \leq \tau$. Let $X$ be the cone over $g_{a,b} \cup g_{a,c}$. Then there is a Lipschitz mapping $\varphi : \Omega \to \Omega$ with $\varphi(L) \subseteq L_0$, $|\varphi(z)| \leq 1$ when $|z| \leq 1$, and $\varphi(z) = z$ when $|z| > 1$, such that

$$
\mathcal{H}^2(\varphi(X) \cap \overline{B(0,1)}) \leq (1 - \vartheta)\mathcal{H}^2(X \cap B(0,1)) + \frac{\vartheta \pi}{2}.
$$

**Proof.** We let $b_0$ a unit vector in $L_0$ which is perpendicular to $b$, and let $c_0$ be a unit vector in $L_0$ which is perpendicular to $c$, such that $b_0 + c_0$ is parallel to $b + c$, and take

$$
u_a = \lambda_a(e_b + e_c), \quad v_b = \lambda_b b_0 \quad \text{and} \quad v_c = \lambda_c c_0,$
$$

where $\lambda_j \in \mathbb{R}$, $j \in \{a,b,c\}$, will be chosen later. We let $\psi_1 : \mathbb{R} \to \mathbb{R}$ be a function of class $C^1$ such that $0 \leq \psi_1 \leq 1$, $\psi_1(x) = 0$ for $x \in (-\infty, 1/4) \cup (3/4, +\infty)$, $\psi_1(x) = 1$ for $x \in [2/5, 3/5]$, and $|\psi_1'| \leq 10$. We let $\psi_2 : \mathbb{R} \to \mathbb{R}$ be a non increasing function of class $C^1$ such that $0 \leq \psi_2 \leq 1$, $\psi_2(x) = 1$ for $x \in (-\infty, 0]$, $\psi_2(x) = 0$ for $x \in [1/5, +\infty)$, and $|\psi_2'| \leq 10$. We let $\psi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be a function defined by

$$
\psi(z,v) = \psi_1(\langle z, v \rangle)\psi_2(|z - \langle z, v \rangle|). \quad (3.11)
$$

We now consider the mapping $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$
\varphi(z) = z + \psi(z,a)v_a + \psi(z,b)v_b + \psi(z,c)v_c.
$$

We see that $\text{supp}(\varphi(\cdot, a))$, $\text{supp}(\varphi(\cdot, b))$ and $\text{supp}(\varphi(\cdot, c))$ are mutually disjoint, and that

$$
\{ z \in \mathbb{R}^3 : \varphi(z) \neq z \} \subseteq B(0,1), \quad \varphi(\Omega_0) \subseteq \Omega_0, \quad \varphi(L_0) \subseteq L_0.
$$

We have that

$$
D\varphi(z)w = w + (D\psi(\cdot, a), w)v_a + (D\psi(\cdot, b), w)v_b + (D\psi(\cdot, c), w)v_c.
$$

By setting $z_v^+ = z - \langle z, v \rangle v$ for convenient, if $w \neq 0$ and $z_v^+ \neq 0$, we have that

$$
D\psi(\cdot, v)w = \psi'_1(\langle z, v \rangle)\psi_2(|z_v^+|)\langle w/|w|, v \rangle + \psi_1(\langle z, v \rangle)\psi'_2(|z_v^+|)\langle w_v^+, z_v^+/|z_v^+| \rangle.
$$

If $w$ is perpendicular to $v$, then $w_v^+ = w$; if $w$ is parallel to $v$ and $|v| = 1$, then $w_v^+ = 0$. We denote by $W_j = \text{supp}(\varphi(\cdot, j))$ for $j \in \{a, b, c\}$. Then

$$
D\psi(\cdot, v)w = \begin{cases} 
w, & z \notin W_a \cup W_b \cup W_c, \\
w + \langle D\psi(\cdot, v), w \rangle v_j, & z \in W_a \cup W_b \cup W_c.
\end{cases}
$$

But

$$
\langle D\psi(\cdot, j), j \rangle = \psi'_1(\langle z, j \rangle)\psi_2(|z_j^+|), \quad j \in \{a, b, c\},
$$

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\[ \langle D\psi(\cdot, i), u_i \rangle = \psi_1(\langle z, i \rangle)\psi'_2(|z^+_i|)\langle u_i, z^+_i/|z^+_i| \rangle, \ i \in \{b, c\}, \]

and

\[ \langle D\psi(\cdot, a), e_i \rangle = \psi_1(\langle z, a \rangle)\psi'_2(|z^+_a|)\langle e_i, z^+_a/|z^+_a| \rangle, \ i \in \{b, c\}, \]

by putting

\[ g_j(z) = \psi'_1(\langle z, j \rangle)\psi_2(|z^+_j|), \ j \in \{a, b, c\}, \]

\[ g_{a,i}(z) = \psi_1(\langle z, a \rangle)\psi'_2(|z^+_a|)\langle e_i, z^+_a/|z^+_a| \rangle, \ i \in \{b, c\} \]

and

\[ g_{i,i}(z) = \psi_1(\langle z, i \rangle)\psi'_2(|z^+_i|)\langle v_i, z^+_i/|z^+_i| \rangle, \ i \in \{b, c\}, \]

and denote by \( X_i \) the cone over \( g_{a,i}, i \in \{b, c\} \), we have that

\[ D\varphi(z) a \land D\varphi(z) e_i = a \land e_i + g_a(z)v_a \land e_i + g_{a,i}(z) a \land v_a, \ z \in X_i \cap W_a \]

and

\[ D\varphi(z) i \land D\varphi(z) u_i = i \land u_i + g_i(z)v_i \land u_i + g_{i,i}(z) i \land v_i, \ z \in X_i \cap W_i. \]

If \( z \in X_i \cap W_a, i \in \{b, c\} \), we have that

\[ J_{2\varphi}|X(z) = \|D\varphi(z)a \land D\varphi(z)e_i\| \]

\[ \leq 1 + (a \land e_i, g_a(z)v_a \land e_i + g_{a,i}(z) a \land v_a) + \frac{1}{2}\|g_a(z)v_a \land e_i + g_{a,i}(z) a \land v_a\|^2 \]

\[ = 1 + g_a(z)\langle a, v_a \rangle + g_{a,i}(z)\langle e_i, v_a \rangle + \frac{1}{2} (g_a(z)^2\|v_a \land e_i\|^2 + g_{a,i}(z)^2|v_a|^2) \]

\[ \leq 1 + g_{a,i}(z)\langle e_i, v_a \rangle + 100|v_a|^2. \]

Similarly, we have that, for \( z \in X_i \cap W_i, \)

\[ J_{2\varphi}|X(z) = \|D\varphi(z)i \land D\varphi(z)u_i\| \leq 1 + g_{i,i}(z)\langle u_i, v_i \rangle + 100|v_i|^2. \]

We see that \( z^+_a/|z^+_a| = e_i \) when \( z \in X_i \setminus \text{span}\{a\} \), and \( z^+_i/|z^+_i| = u_i \) in case \( z \in X_i \setminus \text{span}\{i\} \), thus

\[ g_{a,i}(z) = \psi_1(\langle z, a \rangle)\psi'_2(|z^+_a|) \]

and

\[ g_{i,i}(z) = \psi_1(\langle z, i \rangle)\psi'_2(|z^+_i|). \]

Hence, for \( j = a \) or \( i \), we have that

\[ \int_{z \in X_i \cap W_j} g_{j,i}(z) \mathcal{H}^2(z) = \int_{z \in X_i \cap W_j} \psi_1(\langle z, j \rangle)\psi'_2(|z^+_j|) \mathcal{H}^2(z) \]

\[ = \int_0^{+\infty} \int_0^{+\infty} \psi_1(t)\psi'_2(s) dtds \]

\[ = -\int_0^{+\infty} \psi_1(t) dt < \frac{1}{5}. \]

Thus

\[ \mathcal{H}^2(\varphi(X \cap B_1)) = \int_{z \in X \cap B(0,1)} J_{2\varphi}|X(z) d\mathcal{H}^2(z) \]

\[ \leq (1 + 100 \sum_j |v_j|^2) \mathcal{H}^2(X \cap B_1) - \frac{1}{5} \langle v_a, e_b + e_c \rangle + \sum_i \langle u_i, v_i \rangle \]

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If we take $\lambda_a = 10^{-3}H^2(X \cap B_1)^{-1}$ and $\lambda_i = 10^{-3}H^2(X \cap B_1)^{-1}(u_i, i_0)$, $i \in \{b, c\}$, then
\[
H^2(\varphi(X \cap B_1)) \leq H^2(X \cap B_1) - 10^{-4}(|e_b + e_c|^2 + \langle u_b, b_0 \rangle^2 + \langle u_c, c_0 \rangle^2).
\]
Since $|(a, w)| \leq \tau |w|$ for $w \in L_0$, and $-1 \leq \langle b, c \rangle \leq -1 + 2\tau^2$, we get that
\[
|e_b + e_c|^2 = 2(1 + \langle e_b, e_c \rangle) \geq \frac{2}{1 - \langle e_b, e_c \rangle}(1 - \langle e_b, e_c \rangle)
\]
\[
\geq 1 - \frac{(\langle b, c \rangle - \langle a, b \rangle \langle a, c \rangle)^2}{(1 - \langle a, b \rangle^2)(1 - \langle a, c \rangle^2)}
\]
\[
\geq 1 - \langle a, b \rangle^2 - \langle a, c \rangle^2 - \langle b, c \rangle^2 + 2\langle a, b \rangle \langle b, c \rangle \langle c, a \rangle
\]
\[
= (1 - \langle b, c \rangle + 2\langle a, b \rangle \langle c, a \rangle)(1 + \langle b, c \rangle) - \langle a, b + c \rangle^2
\]
\[
\geq (1 - 3\tau^2)|b + c|^2.
\]
Since $\arcsin x = x + \sum_{n \geq 1} C_n x^{2n+1}$ for $|x| \leq 1$, where $C_n = \frac{(2n)!}{4^n n! (2n+1)}$, we have that
\[
H^2(X \cap B_1) - \frac{\pi}{2} = \frac{1}{2}(\arccos(a, b) + \arccos(a, c)) - \frac{\pi}{2}
\]
\[
= -\frac{1}{2}(\arcsin(a, b) + \arcsin(a, c)) \leq \frac{1}{2}(1 + \tau)|\langle a, b + c \rangle|.
\]
If $b + c \neq 0$, then $|b_0 + c_0| \geq 1$, and we have that
\[
\langle a, \frac{b + c}{|b + c|} \rangle^2 = \langle a, \frac{b_0 + c_0}{|b_0 + c_0|} \rangle^2 \leq 2(\langle a, b_0 \rangle^2 + \langle a, c_0 \rangle^2).
\]
We get so that in any case
\[
|\langle a, b + c \rangle| \leq \frac{1}{2}(|b + c|^2 + 2\langle a, b_0 \rangle^2 + 2\langle a, c_0 \rangle^2).
\]
Since
\[
\langle u_b, b_0 \rangle^2 + \langle u_c, c_0 \rangle^2 = \frac{\langle a, b_0 \rangle^2}{1 - \langle a, b \rangle^2} + \frac{\langle a, c_0 \rangle^2}{1 - \langle a, c \rangle^2} \geq \langle a, b_0 \rangle^2 + \langle a, c_0 \rangle^2,
\]
we get that
\[
H^2(\varphi(X \cap B_1)) \leq H^2(X \cap B_1) - 10^{-4}\left(\frac{1}{2}|b + c|^2 + \langle a, b_0 \rangle^2 + \langle a, c_0 \rangle^2\right)
\]
\[
\leq H^2(X \cap B_1) - 10^{-4}\left(H^2(X \cap B_1) - \frac{\pi}{2}\right).
\]

Lemma 3.13. Let $\tau \in (0, 10^{-4})$ be a given. Then there is a constant $\vartheta > 0$ such that the following hold. Let $a \in \partial B(0, 1)$ and $b, c, d \in L_0 \cap \partial B(0, 1)$ be such that $\text{dist}(a, (0, 0, 1)) \leq \tau$, $\text{dist}(b, (-1/2, \sqrt{3}/2, 0)) \leq \tau$, $\text{dist}(c, (-1/2, -\sqrt{3}/2, 0)) \leq \tau$ and $\text{dist}(d, (1, 0, 0)) \leq \tau$. Let $X$ be the cone over $g_{a,b} \cup g_{a,c} \cup g_{a,d}$. Then there is a Lipschitz mapping $\varphi : \Omega_0 \to \Omega_0$ with $\varphi(E \cap L) \subseteq L$, $|\varphi(z)| \leq 1$ when $|z| \leq 1$, and $\varphi(z) = z$ when $|z| > 1$, such that
\[
H^2(\varphi(X \cap B(0, 1))) \leq (1 - \vartheta)H^2(X \cap B(0, 1)) + \vartheta \frac{3\pi}{4}.
\]
Proof. We let $b_0$, $c_0$ and $d_0$ be unit vectors in $L_0$ such that

$$b_0 \perp b, c_0 \perp c, d_0 \perp d.$$ 

For $i \in \{b, c, d\}$, we put

$$u_i = \frac{a - \langle a, i \rangle i}{|a - \langle a, i \rangle i|}, \quad e_i = \frac{i - \langle i, a \rangle a}{|i - \langle i, a \rangle a|}.$$ 

We take $v_a = \lambda_a(e_b + e_c + e_d)$ and $v_i = \lambda_i b_0$, where $\lambda_i > 0$, $i \in \{b, c, d\}$, will be chosen later. We let $\psi$ be the same as in (3.5), and consider the mapping $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\varphi(z) = z + \psi(z) v_a + \psi(z, b) v_b + \psi(z, c) v_c + \psi(z, d) v_d.$$ 

We see that $\text{supp}(\psi(\cdot, a))$, $\text{supp}(\psi(\cdot, b))$, $\text{supp}(\psi(\cdot, c))$ and $\text{supp}(\psi(\cdot, d))$ are mutually disjoint, and that

$$\{z \in \mathbb{R}^3 : \varphi(z) \neq z\} \subseteq B(0, 1), \quad \varphi(\Omega_0) \subseteq \Omega_0, \quad \varphi(L_0) \subseteq L_0.$$ 

By putting $W_j = \text{supp}(\psi(\cdot, j))$ for $j \in \{a, b, c, d\}$, we have that

$$D\psi(\cdot, v) w = \begin{cases} w, & z \notin W_a \cup W_b \cup W_c \cup W_d, \\ w + \langle D\psi(\cdot, v), w \rangle v_j, & z \in W_a \cup W_b \cup W_c \cup W_d, \end{cases}$$ 

and

$$\langle D\psi(\cdot, j), j \rangle = \psi_1((z, j)) \psi_2(|z_j^+|), \quad j \in \{a, b, c, d\},$$ 

$$\langle D\psi(\cdot, i), u_i \rangle = \psi_1((z, i)) \psi_2(|z_i^+|) \langle u_i, z^+_i / z^+_i \rangle,$$

$$\langle D\psi(\cdot, a), e_i \rangle = \psi_1((z, a)) \psi_2(|z_a^+|) \langle e_i, z^+_a / z^+_a \rangle, \quad i \in \{b, c, d\},$$

where $z_w = z - \langle z, w \rangle w$. By putting

$$g_j(z) = \psi_1((z, j)) \psi_2(|z_j^+|), \quad j \in \{a, b, c, d\},$$

$$g_{a,i}(z) = \psi_1((z, a)) \psi_2(|z_a^+|) \langle e_i, z^+_a / z^+_a \rangle,$$

$$g_{i,i}(z) = \psi_1((z, i)) \psi_2(|z_i^+|) \langle e_i, z^+_i / z^+_i \rangle, \quad i \in \{b, c, d\},$$

and denote by $X_i$ the cone over $g_{a,i}$, $i \in \{b, c, d\}$, we have that

$$D\varphi(z) a \wedge D\varphi(z) e_i = a \wedge e_i + g_{a,i}(z) v_a \wedge e_i + g_{a,i}(z)a \wedge v_a, \quad z \in X_i \cap W_a,$$

$$D\varphi(z) i \wedge D\varphi(z) u_i = i \wedge u_i + g_{i,i}(z) v_i \wedge u_i + g_{i,i}(z)i \wedge v_i, \quad z \in X_i \cap W_i.$$ 

We have that, for $i \in \{b, c, d\}$,

$$J_2\varphi|_X(z) = ||D\varphi(z) a \wedge D\varphi(z) e_i|| \leq 1 + g_{a,i}(z) \langle e_i, v_a \rangle + 100|v_a|^2, \quad z \in X_i \cap W_a,$$

$$J_2\varphi|_X(z) = ||D\varphi(z) i \wedge D\varphi(z) u_i|| \leq 1 + g_{i,i}(z) \langle u_i, v_i \rangle + 100|v_i|^2, \quad z \in X_i \cap W_i.$$ 

Since $z^+_a / |z^+_a| = e_i$ when $z \in X_i \setminus \text{span}\{a\}$, and $z^+_i / |z^+_i| = u_i$ in case $z \in X_i \setminus \text{span}\{i\}$, we have that

$$g_{a,i}(z) = \psi_1((z, a)) \psi_2(|z_a^+|) \quad \text{and} \quad g_{i,i}(z) = \psi_1((z, i)) \psi_2(|z_i^+|).$$

Thus, for $j = a$ or $i$,

$$\int_{z \in X_i \cap W_j} g_{j,j}(z) dH^2(z) = - \int_0^\infty \psi_1(t) dt < -\frac{1}{5}.$$
Hence
\[
\mathcal{H}^2(\varphi(X \cap B_1)) = \int_{z \in X \cap B_1} J_2 \varphi(z) d\mathcal{H}^2(z) \\
\leq (1 + 100(|v_a|^2 + |v_b|^2 + |v_c|^2 + |v_d|^2)) \mathcal{H}^2(X \cap B_1) \\
- \frac{1}{5} ((v_a, e_b + e_c + e_d) + (v_b, v_b) + (v_c, v_c) + (v_d, v_d)) .
\]

If we take \( \lambda_a = 10^{-3}\mathcal{H}^2(X \cap B_1)^{-1} \) and \( \lambda_i = 10^{-3}\mathcal{H}^2(X \cap B_1)^{-1} \langle w_i, i \rangle, \ i \in \{b, c, d\} \), then
\[
\mathcal{H}^2(\varphi(X \cap B_1)) \leq \mathcal{H}^2(X \cap B_1) - 10^{-4} \left( |e_b + e_c + e_d|^2 + \sum_i (\langle w_i, i \rangle)^2 \right) .
\]

Since \( |\langle a, w \rangle| \leq \tau |w| \), for \( w \in L_0 \), and \(-1/2 - \sqrt{3} \tau \leq \langle i_1, i_2 \rangle \leq -1/2 + \sqrt{3} \tau \), \( i_1, i_2 \in \{b, c, d\} \), \( i_1 \neq i_2 \), we get that \( \langle i, j \rangle - \langle a, i \rangle \langle a, j \rangle < 0 \). By putting \( e = (0, 0, 1) \), it is evident that
\[
\langle a, w \rangle^2 \leq 1 - \langle a, e \rangle^2, \text{ for any } w \in L_0 \text{ with } |w| = 1 .
\]

We put \( N = \langle a, b \rangle^2 + \langle a, c \rangle^2 + \langle a, d \rangle^2 \), and we claim that
\[
N \leq (3/2 + 25\tau) (1 - \langle a, e \rangle^2) . \tag{3.12}
\]

Indeed, for any \( w = \lambda b + \mu c \) with \( \lambda, \mu \geq 0 \), we have that
\[
|w|^2 = \lambda^2 + \mu^2 + 2\lambda\mu \langle b, c \rangle \geq \lambda^2 + \mu^2 - (1 + 4\tau)\lambda\mu ,
\]
\[
\langle w, d \rangle^2 \leq (1/2 + \sqrt{3} \tau)^2 (\lambda + \mu)^2 \leq (1/4 + 2\tau)(\lambda + \mu)^2
\]
and
\[
\langle w, b \rangle^2 + \langle w, b \rangle^2 + \langle w, b \rangle^2 = (\lambda^2 + \mu^2)(1 + \langle b, c \rangle^2) + 4\lambda\mu \langle b, c \rangle + \langle w, d \rangle^2
\leq (3/2 + 4\tau) (\lambda^2 + \mu^2) - (3/2 - 10\tau)\lambda\mu
\leq (3/2 + 25\tau)|w|^2 .
\]

Hence, for any \( w \in L_0 \), we have that
\[
\langle w, b \rangle^2 + \langle w, b \rangle^2 + \langle w, b \rangle^2 \leq (3/2 + 25\tau)|w|^2 ,
\]
we now take \( w = a - \langle a, e \rangle e \), then
\[
N \leq (3/2 + 25\tau)|a - \langle a, e \rangle e|^2 = (3/2 + 25\tau)(1 - \langle a, e \rangle^2) ,
\]
the claim \( (3.5) \) follows.

Since \( (1 - x)^{1/2} \leq 1 - x/2 - x^2/8 \) for any \( x \in (0, 1) \), and
\[
(1 - \langle a, b \rangle^2)(1 - \langle a, c \rangle^2)(1 - \langle a, d \rangle^2) \geq 1 - N ,
\]
we have that, for \( \{i, j, k\} = \{b, c, d\} \),
\[
\langle e_i, e_j \rangle = \frac{\langle i, j \rangle - \langle a, i \rangle \langle a, j \rangle}{(1 - \langle a, i \rangle^2)^{1/2}(1 - \langle a, j \rangle^2)^{1/2}} \\
\geq \frac{(\langle i, j \rangle - \langle a, i \rangle \langle a, j \rangle)(1 - \langle a, k \rangle^2/2 - \langle a, k \rangle^4/8)}{(1 - N)^{1/2}} .
\]

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Note that
\[ \langle a, b \rangle^4 + \langle a, c \rangle^4 + \langle a, d \rangle^4 \geq N^2/3, \]
and
\[ |\langle a, b + c + d \rangle| \leq \frac{1}{2} (|b + c + d|^2 + 1 - \langle a, e \rangle^2), \]
we get so that
\[
|e_b + e_c + e_d|^2 \geq 3 + (1 - N)^{-1/2} \left( -3 + (3/2 - \sqrt{3} \tau) N + \frac{1}{12} (1/2 - \sqrt{3} \tau) N^2 \right.
\[
+ |b + c + d|^2 - \langle a, b + c + d \rangle^2 + \langle a, b \rangle \langle a, c \rangle \langle a, d \rangle \langle a, b + c + d \rangle
\[
+ \frac{1}{4} \langle a, b \rangle \langle a, c \rangle \langle a, d \rangle \left( \langle a, b \rangle^3 + \langle a, c \rangle^3 + \langle a, d \rangle^3 \right)
\[
\geq (1 - N)^{-1/2} \left( (1 - \tau^2) |b + c + d|^2 - 2\tau N - 2\tau^3 |\langle a, b + c + d \rangle| \right)
\[
\geq (1 - \tau) |b + c + d|^2 - 6\tau (1 - \langle a, e \rangle^2) .
\]
Since \( 1/(1 - x) = 1 + x + x^2/(1 - x) \) for \( x \in [0, 1] \), and \( \langle a, i \rangle^2 \leq 1 - \langle a, e \rangle^2 \) for \( i \in \{ b, c, d \} \), we have that
\[
\frac{\langle a, e \rangle^2}{1 - \langle a, i \rangle^2} = \frac{\langle a, e \rangle^2}{1 - \langle a, i \rangle^2} \leq \frac{(1 - \langle a, e \rangle^2 - \langle a, i \rangle^2}{1 - \langle a, i \rangle^2}
\]
and
\[
\langle u_b, b_0 \rangle^2 + \langle u_c, c_0 \rangle^2 + \langle u_d, d_0 \rangle^2 = \sum_{i \in \{ b, c, d \}} \frac{1 - \langle a, e \rangle^2 - \langle a, i \rangle^2}{1 - \langle a, i \rangle^2}
\]
\[= 3(1 - \langle a, e \rangle^2) - N \]
\[\geq (1 - \tau) (1 - \langle a, e \rangle^2). \]
We get so that
\[
\mathcal{H}^2(\varphi(X \cap B_1)) \leq \mathcal{H}^2(X \cap B_1) - 10^{-4} (1 - 10\tau) \left( |b + c + d|^2 + 1 - \langle a, e \rangle^2 \right)
\]
Since \( \arcsin x = x + \sum_{n \geq 1} C_n x^{2n+1} \) for \( |x| \leq 1 \), where \( C_n = \frac{(2n)!}{4^n n! (2n+1)!} \), we have that \( \arcsin \langle a, i \rangle \geq \langle a, i \rangle - \tau \langle a, i \rangle^2 \), thus
\[
\mathcal{H}^2(X \cap B_1) - \frac{3\pi}{4} = \frac{1}{2} (\arcsin \langle a, b \rangle + \arcsin \langle a, c \rangle + \arcsin \langle a, c \rangle)
\[
\leq \frac{1}{2} (\langle a, b + c + d \rangle + \tau N
\]
\[\leq \frac{1}{2} \left( |b + c + d|^2 + 1 - \langle a, e \rangle^2 \right) + \tau \left( 1 - \langle a, e \rangle^2 \right). \]
Thus
\[
\mathcal{H}^2(\varphi(X \cap B_1)) \leq (1 - 10^{-4}) \mathcal{H}^2(X \cap B_1) - 10^{-4} \cdot \frac{3\pi}{4}.
\]
\[\square\]

Let \( E \subseteq \Omega_0 \) be a 2-rectifiable set satisfying (a), (b) and (c). We will denote by \( \mathcal{R}_2 \) the set
\[
\left\{ r \in \mathcal{R}_1 : \varepsilon(r) + j(r)^{1/2} \leq 10^{-6} (1 - 2 \cdot 10^{-4}) \right\}.
\]
Lemma 3.14. For any $r \in (0, r) \cap \mathcal{R}_2$, we have that
\[
\mathcal{H}^2(E \cap B_r) \leq (1 - 2 \cdot 10^{-4}) \frac{r^2}{2} \mathcal{H}^1(E \cap \partial B_r) + (2 \cdot 10^{-4} - \vartheta \kappa^2) \frac{r^2}{2} \mathcal{H}^1(X \cap \partial B_1)
+ \vartheta \kappa^2 r^2 \Theta(0) + (2r)^2 h(2r).
\]

Proof. Let $\Sigma$, $\Sigma_r$, $\xi$, $\psi$, $\phi_\Sigma$ and $\{\varphi_r\}_{0 \leq t \leq 1}$ be the same as in the proof of Lemma 3.11. We see that
\[
\varphi_1(E \cap B(0,(1-\xi)r)) = \varphi(E \cap B(0,(1-\xi)r)) \subseteq \Sigma_r,
\]
and that $\Sigma \cap B(0,2\kappa) = X \cap B(0,2\kappa)$, where $X$ is a cone defined in (3.4). We see that if $\Theta(0) = \pi/2$, then $X$ satisfies the conditions in Lemma 3.12; if $\Theta(0) = 3\pi/4$, then $X$ satisfies the conditions in Lemma 3.13. Thus we can find a Lipschitz mapping $\Omega_0 \to \Omega_0$ with $\varphi(E \cap L) \subseteq L$, $|\varphi(z)| \leq 1$ when $|z| \leq 1$, and $\varphi(z) = z$ when $|z| > 1$, such that
\[
\mathcal{H}^2(\varphi(X) \cap B(0,1)) \leq (1 - \vartheta)\mathcal{H}^2(X \cap B(0,1)) + \vartheta \Theta(x).
\]

Let $\tilde{\varphi} : \Omega_0 \to \Omega_0$ be the mapping defined by $\tilde{\varphi}(x) = r\varphi(x/r)$, then
\[
\mathcal{H}^2(E \cap B(0,r)) \leq \mathcal{H}^2(\tilde{\varphi} \circ \varphi_1(E) \cap B(0,r)) + (2r)^2 h(2r)
\leq \mathcal{H}^2(\tilde{\varphi} \circ \varphi_1(E \cap B(0,(1-\xi)r))) + \mathcal{H}^2(\varphi_1(E \cap A_\xi))
\leq \mathcal{H}^2(\Sigma_r \setminus B(0,\kappa r)) + (1 - \vartheta)(\kappa r)^2 \mathcal{H}^2(X \cap B(0,1))
+ \vartheta \cdot (\kappa r)^2 \Theta(0) + \mathcal{H}^2(\varphi_1(E \cap A_\xi)).
\]

But we see that $\Sigma_r = \{rx : x \in \Sigma\}$, $\Sigma \cap B(0,2\kappa) = X \cap B(0,2\kappa)$, and
\[
\lim_{\xi \to 0^+} \mathcal{H}^2(\varphi_1(E \cap A_\xi)) \leq 2550 \int_{E \cap \partial B(0,r)} \text{dist}(z, \Sigma_r) d\mathcal{H}^1(z),
\]
we get so that
\[
\mathcal{H}^2(\Sigma_r \setminus B(0,\kappa r)) = r^2 \left( \mathcal{H}^2(\Sigma) - \mathcal{H}^2(X \cap B(0,\kappa)) \right),
\]
and
\[
\mathcal{H}^2(E \cap B(0,r)) \leq r^2 \mathcal{H}^2(\Sigma) - (\kappa r)^2 \mathcal{H}^2(X \cap B(0,1))
+ (1 - \vartheta)(\kappa r)^2 \mathcal{H}^2(X \cap B(0,1)) + (\kappa r)^2 \vartheta \cdot \Theta(0)
+ 2550 \int_{E \cap \partial B(0,r)} \text{dist}(z, \Sigma_r) d\mathcal{H}^1(z) + (2r)^2 h(2r).
\]

By (3.4), we get that
\[
\mathcal{H}^2(\Sigma) \leq \mathcal{H}^2(M) - 10^{-4}(\mathcal{H}^1(\Gamma_\star) - T)
= (1/2 - 10^{-4})\mathcal{H}^1(\Gamma_\star) + 10^{-4} \mathcal{H}^1(X \cap \partial B(0,1)),
\]
and then
\[
\mathcal{H}^2(E \cap B_r) \leq (1/2 - 10^{-4}) r^2 \mathcal{H}^1(\Gamma_\star) + (10^{-4} - \vartheta \kappa^2/2) r^2 \mathcal{H}^1(X \cap \partial B_1)
+ \vartheta \kappa^2 r^2 \Theta(0) + 2550 \int_{E \cap \partial B_r} \text{dist}(z, \Sigma_r) d\mathcal{H}^1(z) + (2r)^2 h(2r).
\]
By (3.4) and Lemma 3.8, we have that
\[ d_{0,r}(E, \mathcal{M}) \leq 5\varepsilon(r) + 10j(r)^{1/2}. \]
We get that for any \( z \in E \cap \partial B(0, r) \),
\[ \text{dist}(\mu_{1/r}(z), \mathcal{M}) \leq 5\varepsilon(r) + 10j(r)^{1/2}. \]
Since \( \Sigma \setminus B(0, 9/10) = \mathcal{M} \setminus B(0, 9/10) \), we have that
\[ \text{dist}(z, \Sigma_r) = r \text{dist}(\mu_{1/r}(z), \Sigma) = r \text{dist}(\mu_{1/r}(z), \mathcal{M}) \leq 5r\varepsilon(r) + 10rj(r)^{1/2}. \]
We get so that
\[ \int_{E \cap \partial B(0, r)} \text{dist}(z, \Sigma_r) dH^1(z) \leq 5r(\varepsilon(r) + 10j(r)^{1/2})H^1(E \cap \partial B, \mathcal{M} \setminus \mathcal{M}_{\Sigma_r}) \leq 10r(\varepsilon(r) + j(r)^{1/2})(H^1(E \cap \partial B_r) - rH^1(\Gamma_*)). \]
By Lemma 3.6, we have that
\[ H^1(\Gamma_* \setminus \Gamma) \leq H^1(\Gamma \setminus \Gamma) \leq C\gamma^2(H^1(\Gamma) - H^1(X \cap \partial B(0, 1))), \]
so that
\[ H^1(X \cap \partial B(0, 1)) \leq H^1(\Gamma_*) \leq H^1(\mu_{1/r}(E \cap \partial B_r)), \]
thus
\[ H^2(E \cap B_r) \leq (1/2 - 10^{-4})r^2H^1(\Gamma_*) + (10^{-4} - \vartheta \kappa^2/2)r^2H^1(X \cap \partial B_1) + 10^5(\varepsilon(r) + j(r)^{1/2})r(H^1(E \cap \partial B_r) - rH^1(\Gamma_*)) + \vartheta \kappa^2 r^2 \Theta(0) + (2r)^2 h(2r). \]
Since \( r \in (0, \tau) \cap \mathcal{R}_2 \), we have that
\[ 10^5(\varepsilon(r) + 10j(r)^{1/2}) \leq \frac{1}{10}(1 - 2 \cdot 10^{-4}) \]
thus
\[ H^2(E \cap B_r) \leq (1 - 2 \cdot 10^{-4})\frac{r}{2}H^1(E \cap \partial B_r) + (2 \cdot 10^{-4} - \vartheta \kappa^2)\frac{r^2}{2}H^1(X \cap \partial B_1) + \vartheta \kappa^2 r^2 \Theta(0) + (2r)^2 h(2r). \]
\[ \square \]

**Theorem 3.15.** There exist \( \lambda, \mu \in (0, 10^{-3}) \) and \( r_1 > 0 \) such that, for any \( 0 < r < r_1 \),
\[ H^2(E \cap B_r) \leq (1 - \mu - \lambda)\frac{r}{2}H^1(E \cap \partial B_r) + \mu \frac{r^2}{2}H^1(X \cap \partial B_1) + \lambda \Theta(0)r^2 + 4r^2 h(2r). \]
Proof. We put $\tau_1 = \min\{\tau_0, 10^{-12}(1 - \vartheta^2)^2\}$, and take $\delta$ such that
\[ \kappa < \delta < \kappa + (8\vartheta)^{-1}(1 - 2 \cdot 10^{-4})\Theta(0)\tau_1. \]  
(3.13)
We see that $\varepsilon(r) \to 0$ as $r \to 0+$, there exist $r_1 \in (0, r)$ such that, for any $r \in (0, r_1)$,
\[ \varepsilon(r) \leq 10^{-1}\min\{\tau_1, \vartheta(\delta^2 - \kappa^2)\}. \]  
(3.14)

If $r \in (0, r_1)$ and $j(r) \leq \tau_1$, then $r \in \mathcal{R}_2$, then by Lemma 3.14, we have that
\[ \mathcal{H}^2(E \cap B_r) \leq (1 - 2 \cdot 10^{-4})\frac{r^2}{2}\mathcal{H}^1(E \cap \partial B_r) + (2 \cdot 10^{-4} - \vartheta^2)\frac{r^2}{2}\mathcal{H}^1(X \cap \partial B_1) + \vartheta\kappa^2r^2\Theta(0) + (2\vartheta)^2h(2r). \]
We only need to consider the case $r \in (0, r_1)$, $j(r) > \tau_1$ and $\mathcal{H}^1(E \cap \partial B_r) < +\infty$, thus
\[ \mathcal{H}^1(X \cap \partial B_1) + \tau_1 \leq \frac{1}{r}\mathcal{H}^1(E \cap B(0, r)). \]  
(3.15)

By the construction of $X$, we see that $X \cap B(0, 1)$ is Lipschitz neighborhood retract, let $U$ be a neighborhood of $X \cap B(0, 1)$ and $\varphi_0 : U \to X \cap B(0, 1)$ be a retraction such that $|\varphi_0(x) - x| \leq r/2$. We put $U_1 = \mu_{8r/9}(U)$, $\varphi_1 = \mu_{8r/9} \circ \varphi_0 \circ \mu_{9/(8r)}$, and let $s : [0, \infty) \to [0, 1]$ be a function given by
\[ s(t) = \begin{cases} 1, & 0 \leq t \leq 3r/4, \\ -(8/r)(t - 7r/8), & 3r/4 < t \leq 7r/8, \\ 0, & t > 7r/8. \end{cases} \]
We see that there exist sliding minimal cone $Z$ such that $d_{0,1}(X, Z) \leq \varepsilon(r)$, thus $d_{0,r}(E, X) \leq 2\varepsilon(r)$, then for any $x \in E \cap B(0, r) \setminus B(0, 3r/4),$
\[ \text{dist}(x, X) \leq 2\varepsilon(r)r \leq \frac{8\varepsilon(r)}{3}|x|. \]
We consider the mapping $\psi : \Omega_0 \to \Omega_0$ defined by
\[ \psi(x) = s(|x|)\varphi_1(x) + (1 - s(|x|))x, \]
then $\psi(L) = L$ and $\psi(x) = x$ for $|x| \geq 8r/9$.
We take $\tau_1 > 0$ such that, for any $r \in (0, \tau_1),$
\[ \{x \in \Omega_0 \cap B(0, 1) : \text{dist}(x, X) \leq 3\varepsilon(r)\} \subseteq U. \]
Then we get that $\psi(x) \in X$ for any $x \in E \cap B(0, 3r/4);$
\[ \text{dist}(\psi(x), X) \leq 3\varepsilon(r)|x| \] for any $x \in E \cap B(0, r) \setminus B(0, 3r/4);$
and $\Psi(E \cap B_r) \cap B(0, r/4) = X \cap B(0, r/4).$
We now consider the mapping $\Pi_1 : \Omega_0 \to \Omega_0$ defined by
\[ \Pi_1(x) = s(4|x|)x + (1 - s(4|x|))\Pi(x), \]
and the mapping $\psi_1 : \Omega_0 \to \Omega_0$ defined by
\[ \psi_1(x) = \begin{cases} \Pi_1 \circ \psi(x), & |x| \leq r, \\ x, & |x| \geq r. \end{cases} \]

We have that $\psi_1$ is Lipschitz, $\psi_1(L_0) = L_0$ and $\psi_1(B(0, r)) \subseteq \overline{B(0, r)}$, $\psi_1(E \cap B(0, r)) \subseteq X \cap B(0, r) \cup \{ x \in \partial B_r : \text{dist}(x, X) \leq 3r \varepsilon(r) \}$.

Let $\varphi$ be the same as in Lemma 3.12 and Lemma 3.13, and let $\psi_2 = \mu_\delta \circ \varphi \circ \mu_{1/\delta} \circ \psi_1$. Then we have that
\[
\mathcal{H}^2(E \cap B(0, r)) \leq \mathcal{H}^2(\varphi_2(E \cap B(0, r))) + (2r)^2 h(2r)
\leq (1 - \vartheta \delta^2)\mathcal{H}^2(X \cap B(0, r)) + \vartheta \delta^2 \Theta(0) r^2 + 4r^2 h(2r)
+ \mathcal{H}^2(\{ x \in \partial B_r : \text{dist}(x, X) \leq 3r \varepsilon(r) \})
\leq (1 - \vartheta \delta^2)\mathcal{H}^2(X \cap B(0, r)) + \vartheta \delta^2 \Theta(0) r^2 + 4r^2 h(2r)
\leq (1 - \vartheta \delta^2 + 8 \varepsilon(r)) \frac{r^2}{2} \mathcal{H}^1(X \cap \partial B_1) + \vartheta \delta^2 \Theta(0) r^2 + 4r^2 h(2r)
\]

We take $\mu = 2 \cdot 10^{-4} - \vartheta \kappa^2$ and $\lambda = \vartheta \kappa^2$, then by (3.5) and (3.5), we have that
\[ 8 \varepsilon(r) < \vartheta (\delta^2 - \kappa^2) \]
and
\[ \vartheta (\delta^2 - \kappa^2) \Theta(0) \leq (1 - 2 \cdot 10^{-4}) \frac{\tau_1}{2}. \]

We get from (3.5) and (3.5) that
\[
\mathcal{H}^2(E \cap B_r) \leq (1 - 2 \cdot 10^{-4}) \frac{r^2}{2} \left( \mathcal{H}^1(X \cap \partial B_1) + \tau_1 \right) - (1 - 2 \cdot 10^{-4}) \frac{r^2}{2} 
+ \frac{\mu r^2}{2} \mathcal{H}^1(X \cap \partial B_1) + \vartheta \kappa^2 \Theta(0) r^2 + 4r^2 h(2r)
+ (8 \varepsilon(r) - \vartheta \delta^2 + \vartheta \kappa^2) \frac{r^2}{2} \mathcal{H}^1(X \cap \partial B_1) + (\vartheta \delta^2 - \vartheta \kappa^2) \Theta(0) r^2
\leq (1 - \lambda - \mu) \frac{r^2}{2} \mathcal{H}^1(E \cap \partial B_r) + \mu \frac{r^2}{2} \mathcal{H}^1(X \cap \partial B_1) + \lambda \Theta(0) r^2 + 4r^2 h(2r).
\]

For convenient, we put $\lambda_0 = \lambda/(1 - \lambda)$, $f(r) = \Theta(0, r) - \Theta(0)$ and $u(r) = \mathcal{H}^1(E \cap B(0, r))$ for $r > 0$. Since $f(r) = r^{2\lambda} u(r) - \Theta(0)$ and $u$ is a nondecreasing function, we have that, for any $\lambda_1 \in \mathbb{R}$ and $0 < r \leq R < +\infty$,
\[
R^{\lambda_1} f(R) - r^{\lambda_1} f(r) \geq \int_r^R \left( t^{\lambda_1} f(t) \right)' \, dt,
\]
thus
\[
f(r) \leq r^{-\lambda_1} R^{\lambda_1} f(R) + r^{-\lambda_1} \int_r^R \left( t^{\lambda_1} f(t) \right)' \, dt. \tag{3.17}
\]

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Corollary 3.16. If the gauge function $h$ satisfy

$$h(t) \leq C h^\alpha, \ 0 < t \leq \tau_1 \text{ for some } C_h > 0, \ \alpha > 0,$$

then for any $0 < \beta < \min\{\alpha, 2\lambda_0\}$, there is a constant $C = C(\lambda_0, \alpha, \beta, \tau_1, C_h) > 0$ such that

$$|\Theta(0, \rho) - \Theta(0)| \leq C \rho^\beta$$

(3.18)

for any $0 < \rho \leq \tau_1$.

Proof. For any $r > 0$, we put $u(r) = \mathcal{H}^2(E \cap B(0,\tau_1))$. Then $u$ is differentiable for $\mathcal{H}^1$-a.e. $r \in (0, \infty)$.

By Theorem 3.15 and Lemma 2.1, we have that for any $r \in (0, \tau_1) \cap R$,

$$u(r) \leq (1 - \lambda)\frac{r}{2} \mathcal{H}^1(E \cap \partial B(0, r)) + \lambda \Theta(0)r^2 + 4r^2 h(2r)$$

$$\leq (1 - \lambda)\frac{r}{2} u'(r) + \lambda \Theta(0)r^2 + 4r^2 h(2r),$$

thus

$$rf'(r) \geq \frac{2\lambda}{1 - \lambda} f(r) - \frac{8}{1 - \lambda} h(2r) = 2\lambda_0 f(r) - 8(1 + \lambda_0) h(2r),$$

and

$$\left(r^{-2\lambda_0} f(r)\right)' = r^{-1-2\lambda_0} \left(r f'(r) - 2\lambda_0\right) \geq -8(1 + \lambda_0) r^{-1-2\lambda_0} h(2r).$$

Recall that $\mathcal{H}^1((0, \infty) \setminus R) = 0$. We get, from (3.5), so that, for any $0 < r < R \leq \tau_1$,

$$f(r) \leq r^{2\lambda_0} R^{-2\lambda_0} f(R) + 8(1 + \lambda_0) r^{2\lambda_0} \int_r^R t^{-1-2\lambda_0} h(2t) dt.$$  

(3.19)

Since $h(t) \leq C h^\alpha$, we have that

$$f(r) \leq (r/R)^{-2\lambda_0} f(R) + 2^{3+\alpha} (1 + \lambda_0) C_h r^{2\lambda_0} \int_r^R t^{\alpha-2\lambda_0-1} dt.$$  

If $\alpha > 2\lambda_0$, then

$$f(r) \leq \left(f(R) + 2^{3+\alpha} (1 + \lambda_0)(1 + \lambda_0)(\alpha - 2\lambda_0)^{-1} C_h R^\alpha\right) \left(r/R\right)^{2\lambda_0};$$

(3.20)

if $\alpha = 2\lambda_0$, then

$$f(r) \leq f(R)(r/R)^{\alpha} + 2^{\alpha+3}(1 + \lambda_0) C_h r^\alpha \ln(R/r),$$

thus, for any $\beta \in (0, \alpha)$,

$$f(r) \leq f(R)r^{\alpha} + 2^{\alpha+3}(1 + \lambda_0) C_h r^\beta R^{\alpha-\beta} \frac{\ln(R/r)}{(r/R)^{\alpha-\beta}}$$

$$\leq \left(f(R) + 2^{\alpha+3}(1 + \lambda_0) C_h (\alpha - \beta)^{-1} e^{-1} R^\alpha\right) \left(r/R\right)^{\beta};$$

(3.21)

if $\alpha < 2\lambda_0$, then

$$f(r) \leq f(R)(r/R)^{2\lambda_0} + 2^{\alpha+3}(1 - \lambda_0) C_h r^{2\lambda_0} \cdot (2\lambda_0 - \alpha)^{-1} \left(r^{\alpha-2\lambda_0} - R^{\alpha-2\lambda_0}\right)$$

$$\leq \left((r/R)^{2\lambda_0 - \alpha} f(R) + 2^{\alpha+3}(1 - \lambda_0) C_h (2\lambda_0 - \alpha)^{-1} R^\alpha\right) \left(r/R\right)^{\alpha}.$$  

(3.22)
Hence (3.16) follows from (3.5), (3.5), (3.5) and Theorem 2.3. Indeed, there is a constant 
\(C_1(\alpha, \beta, \lambda_0) > 0\) such that
\[
\int_{r}^{R} r^{2\lambda_0} \int_{r}^{t} r^{\alpha - 2\lambda_0 - 1} dt \leq C_1(\alpha, \beta, \lambda_0) R^\alpha \cdot (r/R)^\beta,
\]
(3.23)
and there is a constant \(C_2(\alpha, \beta, \lambda_0) > 0\) such that
\[
f(r) \leq (f(R) + C_2(\alpha, \beta, \lambda_0) C_h \cdot R^\alpha) (r/R)^\alpha.
\]

\[\square\]

Remark 3.17. If the gauge function \(h\) satisfy that
\[
h(t) \leq C \left( \ln \left( \frac{A}{t} \right) \right)^{-b}
\]
for some \(A, b, C > 0\), then (3.5) implies that there exist \(R > 0\) and constant \(C(R, \lambda, b)\) such that
\[
f(r) \leq C(R, \lambda, b) \left( \ln \left( \frac{A}{r} \right) \right)^{-b} \text{ for } 0 < r \leq R.
\]

4 Approximation of \(E\) by cones at the boundary

In this section, we also assume that \(E \subseteq \Omega_0\) is a 2-rectifiable set satisfying (a), (b) and (c). We let \(\varepsilon(r) = \varepsilon_P(r)\) if \(E\) is locally \(C^0\)-equivalent to a sliding minimal cone of type \(\mathbb{P}_+\); and let \(\varepsilon(r) = \varepsilon_Y(r)\) if \(E\) is locally \(C^0\)-equivalent to a sliding minimal cone of type \(\mathbb{Y}_+\).

For any \(r > 0\), we put
\[
f(r) = \Theta(0, r) - \Theta(0), \quad F(r) = f(r) + 8h_1(r), \quad F_1(r) = F(r) + 8h_1(r),
\]
and for \(r \in \mathcal{R}\), we put
\[
\Xi(r) = rf'(r) + 2f(r) + 16h(2r) + 32h_1(r).
\]

We denote by \(X(r)\) and \(\Gamma(r)\), respectively, the cone \(X\) and the set \(\Gamma\) which are defined in (3.4), and by \(\gamma(r)\) the set \(\mu_r(\Gamma(r))\). For any \(r_2 > r_1 > 0\), we put
\[
A(r_1, r_2) = \{ x \in \mathbb{R}^3 : r_1 \leq |x| \leq r_2 \}.
\]

Lemma 4.1. For any \(0 < r < R < \infty\) with \(\mathcal{H}^2(E \cap \partial B_r) = \mathcal{H}^2(E \cap \partial B_R) = 0\), we have that
\[
\int_{E \cap A(r, R)} \frac{1 - \cos \theta(x)}{|x|^2} d\mathcal{H}^2(x) \leq F(R) - F(r),
\]
(4.1)
and
\[
\mathcal{H}^2(\Pi(E \cap A(r, R))) \leq \int_{E \cap A(r, R)} \frac{\sin \theta(x)}{|x|^2} d\mathcal{H}^2(x).
\]
(4.2)
Proof. We see that for $\mathcal{H}^2$-a.e. $x \in E$, the tangent plane $\text{Tan}(E, x)$ exists, we will denote by $\theta(x)$, the angle between the line $[0, x]$ and the plane $\text{Tan}(E, x)$. For any $t > 0$, we put $u(t) = \mathcal{H}^2(E \cap B(0, t))$, then $u : (0, \infty) \to [0, \infty]$ is a nondecreasing function. By Lemma 2.2, we have that

$$u(t) \leq \frac{t}{2} \mathcal{H}^1(E \cap \partial B(0, t)) + 4t^2 h(2t),$$

for $\mathcal{H}^1$-a.e. $t \in (0, \infty)$. Considering the mapping $\phi : \mathbb{R}^3 \to [0, \infty)$ given by $\phi(x) = |x|$, we have, by (2), that

$$\text{ap} J_1(\phi|_E)(x) = \cos \theta(x)$$

for $\mathcal{H}^2$-a.e. $x \in E$.

Apply Theorem 3.2.22 in [9], we get that

$$\int_{E \cap A(r, R)} \frac{1}{|x|^2} \cos \theta(x) d\mathcal{H}^2(x) = \int_{0}^{R} \frac{1}{t^2} \mathcal{H}^1(E \cap \partial B(0, t)) dt$$

$$\geq 2 \int_{0}^{R} \frac{u(t)}{t^3} dt - 8 \int_{0}^{R} \frac{h(2t)}{t} dt$$

$$= 2 \int_{0}^{R} \frac{1}{t^3} \int_{E \cap B(0, t)} d\mathcal{H}^2(x) dt = 8(h_1(R) - h_1(r))$$

$$= 2 \int_{E \cap B(0, R)} \int_{0}^{R} \frac{1}{t^3} dt d\mathcal{H}^2(x) - 8(h_1(R) - h_1(r))$$

$$= \int_{E \cap A(r, R)} \frac{1}{|x|^2} d\mathcal{H}^2(x) + r^{-2}u(r) - R^{-2}u(R) - 8(h_1(R) - h_1(r)),$$

thus (4.1) holds.

By a simple computation, we get that

$$\text{ap} J_2 \Pi(x) = \frac{\sin \theta(x)}{|x|^2},$$

we now apply Theorem 3.2.22 in [9] to get (4.1). \hfill \square

We get from above Lemma that

$$\mathcal{H}^2(\Pi(E \cap A(r, R))) \leq \frac{r^2}{r_1}\ (2\Theta(0, R))^{1/2} (F(R) - F(r))^{1/2}$$

Lemma 4.2. For any $r \in (0, r_1) \cap \mathcal{D}$, if $\Xi(r) \leq \mu r_0$, then

$$d_H(\Gamma(r), X(r) \cap \partial B(0, 1)) \leq 10\mu^{-1/2}\Xi(r)^{1/2}.$$

Proof. By lemma 2.1, we get that

$$\frac{1}{r} \mathcal{H}^1(E \cap \partial B(0, r)) \leq 2\Theta(0) + rf'(r) + 2f(r),$$

By Theorem 3.15, we get that

$$r^2\Theta(0, r) \leq (1 - \lambda - \mu) \frac{r}{2} \mathcal{H}^1(E \cap \partial B_r) + \mu \frac{r^2}{2} \mathcal{H}^1(X \cap \partial B_1) + \lambda \Theta(0)r^2 + 4r^2 h(2r)$$

$$\leq \frac{1}{2} (1 - \lambda - \mu) r^2 (2\Theta(0) + rf'(r) + 2f(r)) + \mu \frac{r^2}{2} \mathcal{H}^1(X \cap \partial B_1) + \lambda \Theta(0)r^2 + 4r^2 h(2r),$$

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such that
\[ \mathcal{H}^1(X \cap \partial B_1) \geq 2\Theta(0) + \frac{2(\lambda + \mu)}{\mu} f(r) - \frac{1 - \lambda - \mu}{\mu} r f'(r) - \frac{\mu}{8} h(2r). \]

Hence
\begin{align*}
    j(r) &= \frac{1}{r} \mathcal{H}^1(E \cap B_r) - \mathcal{H}^1(X \cap \partial B_1) \\
    &\leq \frac{1 - \lambda}{\mu} r f'(r) - \frac{2\lambda}{\mu} f(r) + \frac{8}{\mu} h(2r) \\
    &\leq \frac{1}{\mu} (rf'(r) + 16h_1(r) + 16h(2r)).
\end{align*}

Since
\[ \mathcal{H}^1(X \cap \partial B_1) \leq \mathcal{H}^1(\Gamma_*(r)) \leq \mathcal{H}^1(\Gamma(r)) \leq \mathcal{H}^1(\mu_1.,(E \cap \partial B_r)), \]
we have that
\[ 0 \leq \mathcal{H}^1(\Gamma(r)) - \mathcal{H}^1(X \cap B_1) \leq j(r) \leq \frac{1}{\mu} \Xi(r), \]
by Lemma 3.5, we get that for any \( z \in \Gamma(r), \)
\[ \text{dist} \ (z, X \cap B(0, 1)) \leq 10 \left( \frac{\Xi(r)}{\mu} \right)^{1/2}. \]

\[ \square \]

**Lemma 4.3.** For any \( 0 < r_1 < r_2 < (1 - \tau)r, \) if \( P \) is a plane such that \( \mathcal{H}^1(E \cap P \cap B_\tau) < \infty \) and \( P \cap X_r = \emptyset \) for any \( r \in [r_1, r_2], \) then there is a compact path connected set
\[ C_{P,r_1,r_2} \subseteq E \cap P \cap A(r_2, r_1) \]
such that
\[ C_{P,r_1,r_2} \cap \gamma(t) \neq \emptyset \text{ for } r_1 \leq t \leq r_2. \]

**Proof.** We let \( q \) be the same as in 3. Since \( \|\Phi - \text{id}\|_\infty \leq \tau q, \) we get that
\[ \Phi^{-1} \left( E \cap B(0, r_2) \right) \subseteq Z_{0,q} \cap B(0, r_2 + \tau q). \]

We put
\begin{align*}
    X &= Z_{0,q} \cap B(0, r_2 + \tau q), \\
    F &= X \cap \Phi^{-1}(E \cap P). 
\end{align*}

We take \( x_1, x_2 \in X_r, \) \( x_2 \neq x_1, \) such that \( \Phi^{-1}(x_1) \) and \( \Phi^{-1}(x_2) \) are contained in two different connected components of \( X \setminus F. \) By Lemma 3.2, there is a connected closed subset \( F_0 \) of \( F \) such that \( \Phi^{-1}(x) \) and \( \Phi^{-1}(x_2) \) are still contained in two different connected components of \( X \setminus F_0. \) Then \( F_0 \cap \phi^{-1}(\gamma(t)) \neq \emptyset \) for \( 0 < t \leq r_2; \) otherwise, if \( F_0 \cap \phi^{-1}(\gamma(t_0)) = \emptyset, \) then \( x_1 \) and \( x_2 \) are in the same connected component of \( \Phi(X) \setminus \Phi(F_0), \) thus \( \Phi^{-1}(x_1) \) and \( \Phi^{-1}(x_2) \) are in the same connected component of \( X \setminus F_0, \) absurd!

Since \( \mathcal{H}^1(\Phi(F_0)) \leq \mathcal{H}^1(E \cap P \cap B_\tau) < \infty, \) we get that \( \Phi(F_0) \) is path connected. We take \( z_1 \in \Phi(F_0) \cap \gamma(r_1) \) and \( z_2 \in \Phi(F_0) \cap \gamma(r_2), \) and let \( g : [0, 1] \to \Phi(F_0) \) be a path such that \( g(0) = z_1 \) and \( g(1) = z_2. \) We take \( t_1 = \sup \{ t \in [0, 1] : |g(t)| \leq r_1 \} \) and \( t_2 = \inf \{ t \in [t_1, 1] : |g(t)| \geq r_2 \}. \) Then \( C_{z,r_1,r_2} = g([t_1, t_2]) \) is our desire set. \[ \square \]
Lemma 4.4. Let $T \in [\pi/4, 3\pi/4]$ and $\varepsilon \in (0, 1/2)$ be given. Suppose that $F$ a 2-rectifiable set satisfying

$$F \subseteq \partial B(0,1) \cap \{(t \cos \theta, t \sin \theta, x_3) \in \mathbb{R}^3 \mid t \geq 0, |\theta| \leq T/2, |x_3| \leq \varepsilon\}.$$  

Then we have, by putting $P_\theta = \{(t \cos \theta, t \sin \theta, x_3) \mid t \geq 0, x_3 \in \mathbb{R} \}$, that

$$\int_{-T/2}^{T/2} \mathcal{H}^1(F \cap P_\theta) d\theta \leq (1 + \varepsilon) \mathcal{H}^2(F).$$

Proof. For any $x = (x_1, x_2, x_3) \in F$, we have that $x_1^2 + x_2^2 + x_3^2 = 1$ and $|x_3| \leq \varepsilon$, thus $x_1^2 + x_2^2 \geq 1 - \varepsilon^2$. Since $|\theta| \leq T/2 \leq 3\pi/8$, we get that the mapping $\phi : F \to \mathbb{R}$ given by

$$\phi(x_1, x_2, x_3) = \arctan \frac{x_2}{x_1}$$

is well defined and Lipschitz. Moreover, we have that

$$\text{ap} J_1 \phi(x) = (x_1^2 + x_2^2)^{-1/2} \leq (1 - \varepsilon^2)^{-1/2} \leq 1 + \varepsilon.$$  

Hence

$$\int_{-T/2}^{T/2} \mathcal{H}^1(F \cap P_\theta) d\theta = \int_F \text{ap} J_1 \phi(x) d\mathcal{H}^2(x) \leq (1 + \varepsilon) \mathcal{H}^2(F).$$

For any $0 < t_1 \leq t_2$, we put

$$E_{t_1,t_2} = \Pi \{x \in E : t_1 \leq |x| \leq t_2\}.$$  

For any $t > 0$, we put

$$\varepsilon(t) = \sup \{\varepsilon(r) : r \leq t\}.$$  

Lemma 4.5. If $r_2 > r_1 > 0$ satisfy that $10(1 + r_2/r_1)\varepsilon(r_2) < 1/2$, then we have that

$$\int_{X(t) \cap \partial B(0,1)} \mathcal{H}^1(P_z \cap E_{r_1,r_2}) d\mathcal{H}^1(z) \leq 2 \mathcal{H}^2(E_{r_1,r_2}), \ \forall r_1 \leq t \leq r_2.$$  

Proof. By Lemma 3.8, we have that, for any $r > 0$, if $\varepsilon(r) < 1/2$, then

$$d_{0,r}(E, X(r)) \leq 5\varepsilon(r).$$  

We get so that

$$d_{0,1}(X(t), X(r_2)) = d_{0,t}(X(t), X(r_2)) \leq d_{0,t}(E, X(t)) + d_{0,t}(E, X(r_2)) \leq 5\varepsilon(r_2) + 5\varepsilon(r_2).$$  

Since

$$\text{dist}(x, X(r_2)) \leq 5r_2\varepsilon(r_2), \ \text{for any } x \in E \cap B(0,r_2),$$
we have that
\[ \text{dist}(\Pi(x), X(r_2)) \leq \frac{5r_2\varepsilon(r_2)}{|x|}, \text{ for any } x \in E \cap A(r_1, r_2), \]
we get so that
\[ \text{dist}(\Pi(x), X(t)) \leq \frac{5r_2\varepsilon(r_2)}{|x|} + 5\varepsilon(r_2) + 5\frac{r_2}{t}\varepsilon(r_2) \leq 10(r_2/r_1 + 1)\varepsilon(r_2) < 1/2. \]
We now apply Lemma 4.4 to get the result. \hfill \Box

**Lemma 4.6.** Let \( \varepsilon \in (0, 1/2) \) be given. Let \( A \subseteq \partial B(0, 1) \) be an arc of a great circle such that \( 0 < H^1(A) \leq \pi \) and
\[ \text{dist}(x, L_0) \leq \varepsilon, \forall x \in A. \]
Then
\[ \text{dist}(x, L_0) \leq \frac{\pi^2}{2H^1(A)^2} \int_A \text{dist}(x, L_0)dH^1(x), \forall x \in A. \]

**Proof.** We let \( P \) be the plane such that \( A \subseteq P \), let \( v_0 \in P \cap L_0 \cap \partial B(0, 1) \) and \( v_2 \in P \cap \partial B(0, 1) \) be two vectors such that \( v_0 \) is perpendicular to \( v_1 \). Then \( A \) can be parametrized as \( \gamma : [\theta_1, \theta_2] \to A \) given by
\[ \gamma(t) = v_0 \cos t + v_1 \sin t, \]
where \( \theta_2 - \theta_1 = H^1(A) \). We write \( v_1 = w + w^\perp \) with \( w \in L_0 \) and \( w^\perp \) perpendicular to \( L_0 \).

Since \( \text{ap} J_1 \gamma(t) = 1 \) for any \( t \in [\theta_1, \theta_2] \), by Theorem 3.2.22 in [9], we have that
\[ \int_A \text{dist}(x, L_0)H^1(x) = \int_{\theta_1}^{\theta_2} \text{dist}(\gamma(t), L_0)dt = \int_{\theta_1}^{\theta_2} |w^\perp \sin t|dt \]
\[ \geq 2|w^\perp| \left( 1 - \cos \frac{\theta_2 - \theta_1}{2} \right) \geq \frac{2(\theta_2 - \theta_1)^2}{\pi^2} |w^\perp|, \]
and that
\[ \text{dist}(x, L_0) \leq |w^\perp| \leq \frac{\pi^2}{2H^1(A)^2} \int_A \text{dist}(x, L_0)dH^1(x). \]
\hfill \Box

**Lemma 4.7.** Let \( r_1 \) and \( r_2 \) be the same as in Lemma 4.3. If \( \Xi(r_i) \leq \mu r_0, 10(1 + r_2/r_1)\varepsilon(r_2) \leq 1 \), then we have that
\[ d_{0,1}(X(r_1), X(r_2)) \leq \frac{30r_2}{r_1} \Theta(0, r_2)^{1/2} \cdot F(r_2)^{1/2} + 20\pi \mu^{-1/2} \cdot \left( \Xi(r_1)^{1/2} + \Xi(r_2)^{1/2} \right). \]

**Proof.** For \( z \in X(r_2) \cap \partial B_1 \), if \( z \notin \{ y_r \} \cup \mathcal{X}_r \), we will denote by \( P_z \) the plane which is through \( 0 \) and \( z \) and perpendicular to \( \text{Tan}(X(r_2) \cap \partial B_1, z) \). By Lemma 4.2, we have that
\[ |z - a| \leq 10\mu^{-1/2}\Xi(r_1)^{1/2}, \forall a \in \Gamma(r_2) \cap P_z. \]

Since \( \mathcal{C}_{P_z, r_1, r_2} \cap \gamma(r_i) \neq \emptyset, i = 1, 2 \), we take \( b_i \in \mathcal{C}_{P_z, r_1, r_2} \cap \gamma(r_i) \), then
\[ |\Pi(b_1) - \Pi(b_2)| \leq H^1(\Pi(\mathcal{C}_{P_z, r_1, r_2})) \leq H^1(P_z \cap E_{r_1, r_2}), \]
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thus
\[
\text{dist}(z, X(r_1) \cap \partial B_1) \leq |z - \Pi(b_2)| + |\Pi(b_1) - \Pi(b_2)| + \text{dist}(\Pi(b_1), X(r_1) \cap \partial B_1)
\]
\[
\leq H^1(P_2 \cap E_{r_1,r_2}) + 10\mu^{-1/2}\left(\Xi(r_1)^{1/2} + \Xi(r_2)^{1/2}\right).
\]

For any \(x \in X_r\), we let \(A_x\) be the arc in \(\partial B(0,1)\) which join \(\Pi(x)\) and \(\Pi(y_r)\). We see that \(X(r_2) \cap \partial B(0,1) = \cup_{x \in X_r} A_x\), and \(H^1(A_x) \geq (1/2 - \varepsilon(r_2))\pi \geq \pi/4\). Suppose \(z \in A_x\), then
\[
\text{dist}(z, X(r_1)) \leq \frac{\pi^2}{2H^1(A_x)^2} \int_{A_x} \text{dist}(z, X(r_1))dH^1(x)
\]
\[
\leq \frac{2\pi}{H^1(A_x)} \int_{A_x} H^1(P_2 \cap E_{r_1,r_2})dH^1(x) + 20\pi\mu^{-1/2}\left(\Xi(r_1)^{1/2} + \Xi(r_2)^{1/2}\right)
\]
\[
\leq 16H^2(E_{r_1,r_2}) + 20\pi\mu^{-1/2}\left(\Xi(r_1)^{1/2} + \Xi(r_2)^{1/2}\right)
\]
\[
\leq \frac{16r_2}{r_1} (2\Theta(0,r_2))^{1/2} F(r_2)^{1/2} + 20\pi\mu^{-1/2}\left(\Xi(r_1)^{1/2} + \Xi(r_2)^{1/2}\right)
\]

\(\square\)

**Remark 4.8.** For any cones \(X_1\) and \(X_2\), we see that
\[d_H(X_1 \cap \partial B(0,1), X_2 \cap \partial B(0,1)) \leq 2d_{0,1}(X_1, X_2).\]

Since \(\Xi(r) = [rF_1(r)]'\) for any \(r \in \mathcal{R}\), we get that
\[\int_{r_1}^{r_2} \Xi(t)dt \leq r_2F_1(r_2) - r_1F_1(r_1),\]

For any \(\zeta > 2\), if \(r_1 \leq r_2 \leq r\), then by Chebyshev’s inequality, we get that,
\[H^1\left(\left\{t \in [r_1, r_2] \mid \Xi(t) \leq \zeta F_1(r)^{2/3}\right\}\right) \geq r_2 - r_1 - \frac{1}{\zeta}rF_1(r)^{1/3},\]
thus \(\left\{t \in [r_1, r_2] \mid \Xi(t) \leq \zeta F_1(r)^{2/3}\right\} \neq \emptyset\) when \(r_2 - r_1 > (1/\zeta)rF_1(r)^{1/3}\).

**Lemma 4.9.** Let \(R_0 < (1 - \tau)r\) be a positive number such that \(F(R_0) \leq \mu\tau_0/4\) and \(\varepsilon(R_0) \leq 10^{-4}\). For any \(r \in \mathcal{R} \cap (0, R_0)\), if \(\Xi(r) \leq \mu\tau_0\), then there is a constant \(C = C(\mu, \Theta(0))\) such that
\[\text{dist}(x, E) \leq Cr\left(F_1(r)^{1/3} + \Xi(r)^{1/2}\right), \quad x \in X(r) \cap B_r.\]

**Proof.** For any \(k \geq 0\), we take \(r_k = 2^{-k}r\). Then there exists \(t_k \in [r_k, r_{k-1}]\) such that
\[\Xi(t_k) \leq \frac{\int_{r_k}^{r_{k-1}} \Xi(t)dt}{r_{k-1} - r_k} \leq \frac{r_{k-1}F_1(r_{k-1})}{r_{k-1}/2} = 2F_1(r_{k-1}).\]
We let $X_k = X(t_k)$, then for any $j > i \geq 1$, we have that

$$d_{0,1}(X_i, X_j) \leq \sum_{k=i+1}^{j} d_{0,1}(X_k, X_{k+1})$$

$$\leq 60 (\Theta(0) + \mu \tau_0/4)^{1/2} \frac{1}{2} \sum_{k=i}^{j-1} F_1(t_k)^{1/2} + 20 \pi \mu^{-1/2} \sum_{k=i}^{j-1} (\Xi(t_k)^{1/2} + \Xi(t_{k+1})^{1/2})$$

$$\leq \left( 60 (\Theta(0) + \mu \tau_0/4)^{1/2} + 40 \pi \mu^{-1/2} \right) \sum_{k=i}^{j-1} 2F_1(t_k)^{1/2} + F_1(t_{k-1})^{1/2}$$

$$\leq C_1(\mu, \Theta(0))(j - i) F_1(r_{i-1})^{1/2} = C_1(\mu, \Theta(0)) F_1(r_{i-1})^{1/2} \log_2(r_i/r_j),$$

where $C_1(\mu, \Theta(0)) = 3 \left( 60 (\Theta(0) + \mu \tau_0/4)^{1/2} + 40 \pi \mu^{-1/2} \right)$.

For any $x \in X(r) \cap B_r$ with $\Xi(|x|) \leq \mu \tau_0$, we assume that $t_{k+1} \leq |x| < t_k$, then

$$\text{dist}(x, E) \leq d_H(X(r) \cap B_{|x|}, X(|x|) \cap B_{|x|}) + d_H(X(|x|) \cap B_{|x|}, \gamma(|x|))$$

$$\leq 2|x|d_{0,1}(X(|x|), X(|x|)) + 10 \mu^{-1/2} |x| \Xi(|x|)^{1/2}$$

$$\leq 2|x|d_{0,1}(X(|x|), X_k) + d_{0,1}(X_k, X_1) + d_{0,1}(X_1, X(r)) + 10 \mu^{-1/2} |x| \Xi(|x|)^{1/2}$$

$$\leq (40 \pi + 10) \mu^{-1/2} |x| \left( \Xi(|x|)^{1/2} + \Xi(r)^{1/2} \right) + C_2(\mu, \Theta(0)) |x| F_1(r)^{1/2} \log_2(r/|x|)$$

$$\leq (40 \pi + 10) \mu^{-1/2} |x| \Xi(|x|)^{1/2} + C_3(\mu, \Theta(0)) r \left( \Xi(r)^{1/2} + F_1(r)^{1/2} \right)$$

For any $0 \leq a \leq b \leq r$, we put

$$I(a, b) = \left\{ t \in [a, b] \mid \Xi(t) \leq F_1(r)^{2/3} \right\},$$

then $I(a, b) \neq \emptyset$ when $b - a > r F_1(r)^{1/3}$. If $|x| \in I(0, r)$, then

$$\text{dist}(x, E) \leq C_4(\mu, \Theta(0)) r \left( F_1(r)^{1/3} + \Xi(r)^{1/2} \right).$$

We let $\{s_i\}_{i=0}^{m+1} \subseteq [0, r]$ be a sequence such that

$$0 = s_0 < s_1 < \cdots < s_m < s_{m+1} = r, \ s_i \in I(0, r),$$

and

$$s_{i+1} - s_i \leq 2r F_1(r)^{1/3}.$$ 

For any $x \in X(r) \cap B_r$, if $s_i \leq |x| < s_{i+1}$ for some $0 \leq i \leq m$, we have that

$$\text{dist}(x, E) \leq \left| x - \frac{s_i}{|x|} x \right| + \text{dist}(\frac{s_i}{|x|} x, E)$$

$$\leq (s_{i+1} - s_i) + C_4(\mu, \Theta(0)) r \left( F_1(r)^{1/3} + \Xi(r)^{1/2} \right)$$

$$\leq (C_4(\mu, \Theta(0)) + 2) r \left( F_1(r)^{1/3} + \Xi(r)^{1/2} \right).$$
Definition 4.10. Let \( U \subseteq \mathbb{R}^3 \) be an open set, \( E \subseteq \mathbb{R}^3 \) be a set of Hausdorff dimension 2. \( E \) is called Ahlfors-regular in \( U \) if there is a \( \delta > 0 \) and \( \xi_0 \geq 1 \) such that, for any \( x \in E \cap U \), if \( 0 < r < \delta \) and \( B(x, r) \subseteq U \), we have that
\[
\xi_0^{-1}r^2 \leq \mathcal{H}^2(E \cap B(x, r)) \leq \xi_0 r^2.
\]

Lemma 4.11. Let \( R_0 \) be the same as in Lemma 4.9. If \( E \) is Ahlfors-regular, and \( r \in \mathcal{R} \cap (0, R_0) \) satisfies \( \Xi(r) \leq \mu \tau_0 \), then there is a constant \( C = C(\mu, \xi_0, \Theta(0)) \) such that
\[
\text{dist}(x, X(r)) \leq Cr \left( F_1(r)^{1/4} + \Xi(r)^{1/2} \right), \quad x \in E \cap B(0, 9r/10).
\]

Proof. Let \( \{X_k\}_{k \geq 1} \) be the same as in (4). For any \( t \in \mathcal{R} \) with \( t_{k+1} \leq t < t_k \), \( \Xi(t) \leq \mu \tau_0 \) and \( x \in \gamma(t) \), we have that
\[
\text{dist}(x, X(r)) \leq d_H(\gamma(t), X(|x|) \cap B_{|x|}) + d_H(X(|x|) \cap B_{|x|}, X(r)) \leq (40 \pi + 10)\mu^{-1/2}|x|\Xi(|x|)^{1/2} + C_3(\mu, \Theta(0))r \left( \Xi(r)^{1/2} + F_1(r)^{1/2} \right)
\]
We put
\[
J(0, r) = \{ t \in [0, r] : \Xi(t) > F_1(r)^{1/2} \}.
\]
For any \( x \in \gamma(t) \) with \( t \in (0, r) \setminus J(0, r) \), we have that
\[
\text{dist}(x, X(r)) \leq C_5(\mu, \Theta(0))r \left( \Xi(r)^{1/2} + F_1(r)^{1/4} \right).
\]
We put
\[
E_1 = \bigcup_{t \in J(0, r)} (\ E \cap \partial B_t), \quad E_2 = \bigcup_{t \in (0, r) \setminus J(0, r)} (E \cap B_t \setminus \gamma(t)),
\]
and
\[
E_3 = E \cap B_r \setminus (E_1 \cup E_2) = \bigcup_{t \in (0, r) \setminus J(0, r)} \gamma(t).
\]
Then
\[
\mathcal{H}^2(E_1 \cup E_2) = \int_{E \cap B_r} d\mathcal{H}^2(x) - \int_{E_3} d\mathcal{H}^2(x)
\]
\[
\leq \int_{E \cap B_r} d\mathcal{H}^2(x) - \int_{E_3} \cos \theta(x) d\mathcal{H}^2(x)
\]
\[
= \int_{E \cap B_r} (1 - \cos \theta(x)) d\mathcal{H}^2(x) + \int_{E_1 \cup E_2} \cos \theta(x) d\mathcal{H}^2(x)
\]
\[
\leq r^2 F(r) + \int_{J(0, r)} \mathcal{H}^1(E_1 \cap \partial B_t)dt + \int_{0}^{r} \mathcal{H}^1(E_2 \cap \partial B_t)dt
\]
\[
\leq r^2 F(r) + \int_{J(0, r)} (2\Theta(0) + tf'(t) + 2f(t))tdt + \mu^{-1} \int_{0}^{r} t\Xi(t)dt
\]
\[
\leq (2 + \mu^{-1})r^2 F_1(r) + 2\Theta(0) \int_{\{t \in [0, r] : \Xi(t) > F_1(r)^{1/2} \}} tdt
\]
\[
\leq (2 + \mu^{-1})r^2 F_1(r) + \frac{2\Theta(0)}{F_1(r)^{1/2}} \int_{0}^{r} t\Xi(t)dt
\]
\[
\leq C_6(\mu, \Theta(0)) r^2 F_1(r)^{1/2},
\]
where \( C_6(\mu, \Theta(0)) = (2 + \mu^{-1})(\mu r_0/4)^{1/2} + 2\Theta(0) \).

We see that, for any \( x \in E_3 \),
\[
\text{dist}(x, X(r)) \leq C_5(\mu, \Theta(0))r \left( \Xi(r)^{1/2} + F_1(r)^{1/4} \right).
\]

If \( x \in E \cap B(0, 9r/10) \) with
\[
\text{dist}(x, X(r)) > C_5(\mu, \Theta(0))r \left( \Xi(r)^{1/2} + F_1(r)^{1/4} \right) + s
\]
for some \( s \in (0, r/10) \), then \( E \cap B(x, s) \subseteq E_1 \cup E_2 \), thus
\[
\mathcal{H}^2(E \cap B(x, s)) \leq C_6(\mu, \Theta(0))r^2 F_1(r)^{1/2}.
\]

But on the other hand, by Ahlfors-regular property of \( E \), we have that
\[
\mathcal{H}^2(E \cap B(x, s)) \geq \xi_0^{-1} s^2.
\]

We get so that
\[
s \leq C_6(\mu, \Theta(0))^{1/2} \cdot \xi_0^{1/2} \cdot r F_1(r)^{1/4}.
\]

Therefore, for \( x \in E \cap B(0, 9r/10) \),
\[
\text{dist}(x, X(r)) \leq \left( C_6(\mu, \Theta(0))^{1/2} \cdot \xi_0^{1/2} + C_5(\mu, \Theta(0)) \right) \left( \Xi(r)^{1/2} + F_1(r)^{1/4} \right).
\]

For any \( k \geq 0 \), we take \( R_k = 2^{-k} R_0 \) and \( s_k \in [R_{k+1}, R_k] \) such that
\[
\Xi(s_k) \leq \frac{\int_{R_{k+1}}^{R_k} \Xi(t) dt}{R_k - R_{k+1}} \leq 2 F_1(R_k).
\]

We put \( X_k = X(s_k) \). Then for any \( j \geq i \geq 2 \), we have that
\[
d_{0,1}(X_i, X_j) \leq \frac{C_1(\mu, \Theta(0))}{3} \sum_{k=i}^{j-1} \left( 2 F_1(s_k)^{1/2} + F_1(s_{k-1})^{1/2} \right)
\]
\[
\leq C_1(\mu, \Theta(0)) \sum_{k=i-1}^{j-1} F_1(R_k)^{1/2}
\]
\[
\leq \frac{C_1(\mu, \Theta(0))}{\ln 2} \sum_{k=i-1}^{j-1} \int_{R_k}^{R_{k-1}} \frac{F_1(t)^{1/2}}{t} dt
\]
\[
= \frac{C_1(\mu, \Theta(0))}{\ln 2} \int_{R_{i-2}}^{R_{j-1}} \frac{F_1(t)^{1/2}}{t} dt.
\]

If the gauge function \( h \) satisfy that
\[
\int_0^{R_0} \frac{F_1(t)^{1/2}}{t} dt < +\infty, \tag{4.4}
\]
then \( X_k \) converges to a cone \( X(0) \), and
\[
d_{0,1}(X(0), X_k) \leq \frac{C_1(\mu, \Theta(0))}{\ln 2} \int_0^{R_{k-2}} \frac{F_1(t)^{1/2}}{t} dt.
\]
Remark 4.12. If \( h(r) \leq C(\ln(A/r))^{-b} \), \( 0 < r \leq R_0 \), for some \( A > R_0 \), \( C > 0 \) and \( b > 3 \), then (4) holds.

Indeed,
\[
h_1(r) = \int_0^r \frac{h(2t)}{t} \, dt \leq \frac{C}{b-1} \left( \ln \left( \frac{A}{r} \right) \right)^{-b+1},
\]
and then Remark 3.17 implies that
\[
F(r) \leq C_1 \left( \ln \left( \frac{A}{r} \right) \right)^{-b} + \frac{C}{b-1} \left( \ln \left( \frac{A}{r} \right) \right)^{-b+1} \leq C_2 \left( \ln \left( \frac{A}{r} \right) \right)^{-b+1},
\]
thus (4) holds.

Lemma 4.13. If (4) holds, then \( X(0) \) is a minimal cone.

Proof. By Lemma 3.8, for any \( r \in (0, r) \cap \mathcal{R} \), there exist sliding minimal cone \( Z(r) \) such that 
\[
d_{0,1}(X(r), Z(r)) \leq 4 \varepsilon(r).
\]
But \( \varepsilon(r) \to 0 \) as \( r \to 0^+ \), we get that 
\[
d_{0,1}(Z(s_k), X(0)) \to 0.
\]
Since \( Z(s_k) \) is sliding minimal for any \( k \), we get that \( X(0) \) is also sliding minimal. \( \Box \)

For any \( r \in \mathcal{R} \cap (0, R_0) \) with \( \Xi(r) \leq \mu_0 \), we assume \( R_{k+1} \leq r < R_k \), by Lemma 4.7, we have that
\[
d_{0,1}(X(0), X(r)) \leq d_{0,1}(X(0), X_{k+3}) + d_{0,1}(X_{k+3}, X(r)) \leq C_1(\mu, \Theta(0)) \int_0^{R_{k+1}} F_1(t)^{1/2} \frac{t}{\ln 2} dt \]
\[
+ \frac{30r}{s_{k+3}} \Theta(0, r)^{1/2} F_1(r)^{1/2} + 20 \pi \mu^{-1/2} \left( \Xi(s_{k+3})^{1/2} + \Xi(r)^{1/2} \right)
\]
\[
\leq 10 C_1(\mu, \Theta(0)) \left( \Xi(r)^{1/2} + F_1(r)^{1/2} + \int_0^r F_1(t)^{1/2} \frac{t}{t} dt \right).
\]

Theorem 4.14. If (4) holds, and \( E \) is Ahlfors-regular, then \( E \) has unique blow-up limit \( X(0) \) at 0, and there is a constant \( C = C_{10}(\mu, \Theta, \xi_0) \) such that
\[
d_{0,9r/10}(E, X(0)) \leq C \left( F_1(r)^{1/4} + \int_0^r \frac{F(t)^{1/2}}{t} dt \right), \quad 0 < r < r.
\]

In particular,
- if \( h(r) \leq C_h(\ln(A/r))^{-b} \) for some \( A, C_h > 0 \), \( b > 3 \) and \( 0 < r \leq R_0 < A \), then 
\[
d_{0,r}(E, X(0)) \leq C'(\ln(A_1/r))^{-(b-3)/4}, \quad 0 < r \leq 9 R_0/10, \quad A_1 \leq 10 A/9;
\]
- if \( h(r) \leq C_h r^{\alpha_1} \), for some \( C_h, \alpha_1 > 0 \), and \( 0 < r \leq r_0, 0 < r_0 \leq \min\{1, R_0\} \), then 
\[
d_{0,r}(E, X(0)) \leq C(r/r_0)^{\beta}, \quad 0 < r \leq 9 r_0/10, \quad 0 < \beta < \alpha_1,
\]

where
\[
C \leq C_{11}(\mu, \alpha_0, \alpha_1, \beta, C_h, \xi_0, \Theta(0)) \left( F(r_0)^{1/4} + r_0^{\alpha_1/4} \right).
\]
Proof. From (4) and Lemma 4.9, we get that, for any $x \in X(0) \cap B_r$ where $r \in R \cap (0, R_0)$ such that $\Xi(r) \leq \mu \tau_0$,

$$\text{dist}(x, E) \leq C_7(\mu, \xi_0, \Theta(0)) r \left( \Xi(r)^{1/2} + F_1(r)^{1/4} + \int_0^r \frac{F_1(t)^{1/2}}{t} dt \right).$$

Similarly to the proof of Lemma 4.9, we still consider

$$I(a, b) = \left\{ t \in [a, b] \mid \Xi(t) \leq F_1(r)^{2/3} \right\}, \quad 0 \leq a \leq b \leq r,$$

we have that $I(a, b) \neq \emptyset$ whenever $b - a > r F_1(r)^{1/3}$. We let $\{s_i\}_{0}^{m+1} \subseteq [0, r]$ be a sequence such that

$$0 = s_0 < s_1 < \cdots < s_m < s_{m+1} = r, \quad s_i \in I(0, r),$$

and

$$s_{i+1} - s_i \leq 2r F_1(r)^{1/3}.$$

For any $r \in (0, R_0)$, we assume that $s_i \leq r < s_{i+1}$, $x \in X(0) \cap \partial B_r$.

$$\text{dist}(x, E) \leq \left| x - \frac{s_i}{|x|} x \right| + \text{dist} \left( \frac{s_i}{|x|} x, E \right) \leq C_8(\mu, \xi_0, \Theta(0)) r \left( F_1(r)^{1/4} + \int_0^r \frac{F_1(t)^{1/2}}{t} dt \right).$$

(4.7)

From (4) and Lemma 4.11, we have that, for any $x \in X(0) \cap B(0, 9r/10)$ where $r \in R \cap (0, R_0)$ such that $\Xi(r) \leq \mu \tau_0$,

$$\text{dist}(x, X(0)) \leq C_9(\mu, \xi_0, \Theta(0)) \left( \Xi(r)^{1/2} + F_1(r)^{1/4} + \int_0^r \frac{F_1(t)^{1/2}}{t} dt \right).$$

(4.8)

Similarly to the proof of Lemma 4.11, we can get that

$$\text{dist}(x, X(0)) \leq C_{10}(\mu, \xi_0, \Theta(0)) \left( F_1(r)^{1/4} + \int_0^r \frac{F_1(t)^{1/2}}{t} dt \right).$$

We get, from (4) and (4), that (4.14) holds.

If $h(r) \leq C_h(\ln(A/r))^{-b}$ for some $A, C_h > 0$ and $b > 3$ and $0 < r \leq R_0 < A$, then

$$h_1(r) = \int_0^r \frac{h(2t)}{t} dt \leq \frac{C_h}{b-1} \left( \ln \left( \frac{A}{r} \right) \right)^{-b+1},$$

and by Remark 3.17 we have that

$$F(r) \leq C'' \left( \ln \frac{A}{r} \right)^{-b+1}$$

where

$$C'' \leq C(R_0, \lambda, b) \left( \ln \frac{A}{r} \right)^{-1} + \frac{C_1}{b-1} \leq C(R_0, \lambda, b) \left( \ln \frac{A}{R_0} \right)^{-1} + \frac{C_1}{b-1}.$$
is bounded, thus
\[ \int_0^r \frac{F_1(t)^{1/2}}{t} dt \leq C'' \left( \ln \frac{A}{r} \right)^{-(b+3)/2}. \]
Hence we get that
\[ d_{0,9r/10}(E, X(0)) \leq C_{10}(\mu, \xi_0, \Theta(0)) \left( F_1(r)^{1/4} + \int_0^r \frac{F_1(t)^{1/2}}{t} dt \right) \]
\[ \leq C' \left( \ln \frac{A}{r} \right)^{-(b-3)/4}. \]

If \( h(r) \leq C_h r^{\alpha_1} \) for some \( C_h, \alpha_1 > 0 \) and \( 0 < r \leq r_0 \), then
\[ h_1(r) = \int_0^r \frac{h(2t)}{t} dt \leq \frac{C_h}{\alpha_1} (2r)^{\alpha_1}. \]
We see, from the proof of Corollary 3.16, that
\[ f(r) \leq (f(r_0) + C_2(\alpha_1, \beta, \lambda_0) C_h r_0^{\alpha_1}) (r/r_0)^\beta, \quad \forall 0 < \beta < \alpha_1, \]
thus
\[ F_1(r) = f(r) + 16h_1(r) \leq (f(r_0) + C_2'(\alpha_1, \beta, \lambda_0) C_h r_0^{\alpha_1}) (r/r_0)^\beta. \]
Then
\[ d_{0,9r/10}(E, X(0)) \leq C_{10}(\mu, \xi_0, \Theta(0)) \left( F_1(r)^{1/4} + \int_0^r \frac{F_1(t)^{1/2}}{t} dt \right) \]
\[ \leq C(r/r_0)^{\beta/4}, \]
where
\[ C \leq C_{10}'(\mu, \xi_0, \Theta(0))(F(r_0)^{1/4} + C_2'\left(\alpha_1, \beta, \lambda_0, C_h\right)r_0^{1/4}). \]

\[ \square \]

5 Parameterization of well approximate sets

Recall that a cone in \( \mathbb{R}^3 \) is called of type \( \mathbb{P} \) if it is a plane; a cone is called of type \( \mathbb{Y} \) if it is the union of three half planes with common boundary line and that make 120° angles along the boundary line; a cone of type \( \mathbb{T} \) if it is the cone over the union of the edges of a regular tetrahedron.

**Theorem 5.1.** Let \( E \subseteq \Omega_0 \) be a set with \( 0 \in E \). Suppose that there exist \( C > 0, \ r_0 > 0, \ \beta > 0 \) and \( 0 < \eta \leq 1 \) such that, for any \( x \in E \cap B(0, r_0) \) and \( 0 < r \leq 2r_0 \), we can find cone \( Z_{x,r} \) through \( x \) such that
\[ d_{x,r}(E, Z_{x,r}) \leq C r^\beta, \]
where \( Z_{x,r} \) is a minimal cone in \( \mathbb{R}^3 \) of type \( \mathbb{P} \) or \( \mathbb{Y} \) when \( x \notin \partial \Omega_0 \) and \( 0 < r < \eta \text{dist}(x, \partial \Omega_0) \), and otherwise, \( Z_{x,r} \) is a sliding minimal cone of type \( \mathbb{P}_+ \) or \( \mathbb{Y}_+ \) in \( \Omega_0 \) with sliding boundary \( \partial \Omega_0 \) centered at some point in \( \partial \Omega_0 \). Then there exist a radius \( r_1 \in (0, r_0/2) \), a sliding minimal cone \( Z \) centered at 0 and a mapping \( \Phi : \Omega_0 \cap B(0, r_1) \to \Omega_0 \), which is a \( C^{1,\beta} \)-diffeomorphism between its domain and image, such that \( \Phi(0) = 0, \ \Phi(\partial \Omega_0 \cap B(0, 2r_1)) \subseteq \partial \Omega_0, \ |\Phi - \text{id}|_\infty \leq 10^{-2}r_1 \) and
\[ E \cap B(0, r_1) = \Phi(Z) \cap B(0, r_1). \]
Proof. Let $\sigma : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $\sigma(x_1, x_2, x_3) = (x_1, x_2, -x_3)$. By setting $E_1 = E \cup \sigma(E)$, we have that, for any $x \in E_1 \cap B(0, r_0)$ and $0 < r < 2r_0$, there exist minimal cone $Z(x, r)$ in $\mathbb{R}^3$ centered at $x$ of type $\mathbb{P}$ or $\mathbb{Y}$ such that $Z(\sigma(x), r) = \sigma(Z(x, r))$ and

$$d_{x,r}(E, Z(x, r)) \leq C r^\beta.$$ 

By Theorem 4.1 in [8], there exist $r_1 \in (0, r_0)$, $\tau \in (0, 1)$, a cone $Z$ centered at $0$ of type $\mathbb{P}$ or $\mathbb{Y}$, and a mapping $\Phi_1 : B(0, 3r_1/2) \to B(0, 2r_1)$ such that

$$\sigma(Z) = Z, \quad \sigma \circ \Phi_1 = \Phi_1 \circ \sigma, \quad \Phi_1 - \text{id} \leq r_0 \tau,$$

$$C_1 |x - y|^{1+\tau} \leq |\Phi(x) - \Phi(y)| \leq C_1^{-1} |x - y|^{1/(1+\tau)},$$

$$E_1 \cap B(0, r_1) \subseteq \Phi_1(Z \cap B(0, 3r_1/2)) \subseteq E_1 \cap B(0, 2r_1).$$

Using the same argument as in Section 10 in [2], we get that $\Phi_1$ is of class $C^{1,\beta}$.

6 Approximation of $E$ by cones away from the boundary

In this section, we let $\Omega \subseteq \mathbb{R}^3$ be a closed set. Let $E \in SAM(\Omega, \partial \Omega, h)$ be a sliding almost minimal set, $x_0 \in E \setminus \partial \Omega$. Then $E \cap B(x, r)$ is almost minimal with gauge function $h$ for any $0 < r < \text{dist}(x_0, \partial \Omega)$. We put

$$F(x, r) = \Theta(x, r) - \Theta(x) + 8h_1(r).$$

We see from Theorem 2.3 that $F(x, r) \geq 0$ and $F(x, \cdot)$ is nondecreasing for $0 < r < \text{dist}(x_0, L)$.

Theorem 6.1. If $\int_0^{R_0} r^{-1} F(x, r)^{1/3} dr < \infty$ for some $R_0 > 0$, then $E$ has unique blow-up limit $T$ at $x$. Moreover there is a constant $C > 0$ and a radius $\rho_0 = \rho_0(x) > 0$ such that

$$d_{x,r}(E, T) \leq C \int_0^{200r} \frac{F(x, t)^{1/3}}{t} dt, \quad 0 < r \leq \rho_0.$$ 

In particular, if the gauge function $h$ satisfies that

$$h(t) \leq C h t^{\alpha_1} \text{ for some } \alpha_1 > 0 \text{ and } 0 < t \leq R_0,$$

then there is a $\beta_0 > 0$ such that, for any $0 < \beta < \beta_0$,

$$d_{x,r}(E, T) \leq C(\alpha_1, \beta) (F(x, \rho_0) + C h \rho_0^{\alpha_1})^{1/3} (r/\rho_0)^{\beta/3}.$$ 

Proof. Let $\varnothing$ be the radius defines as in (3). We take $\rho_0 = 10^{-3} \min\{R_0, \text{dist}(x_0, \partial \Omega), \varnothing\}$. By Theorem 11.4 in [4], there is a constant $C > 0$ and cone $Z_r$ for each $0 < r < \rho_0$ such that

$$d_{x,r}(E, Z_r) + \alpha_+ (Z_r) \leq C F(x, 110r)^{1/3}.$$ 

We put $\rho_k = 2^{-k} \rho_0$, and $Z_k = Z_{\rho_k}$. Then

$$d_{x,1}(Z_k, Z_{k+1}) = d_{x,\rho_{k+1}}(Z_k, Z_{k+1}) \leq d_{x,\rho_k+1}(Z_k, E) + d_{x,\rho_{k+1}}(E, Z_{k+1}) \leq C F(x, 110\rho_{k+1})^{1/3} + 2 C F(x, 110\rho_k)^{1/3}.$$ 


For any $1 \leq i < j$, we have that
\[
d_{x,1}(Z_i, Z_j) \leq 2C \sum_{k=i}^{j-1} F(x, 110 \rho_k)^{1/3} + C \sum_{k=i+1}^{j} F(x, 110 \rho_k)^{1/3} \leq 3C \sum_{k=i}^{j} F(x, 110 \rho_k)^{1/3}
\]
\[
\leq \frac{3C}{\ln 2} \int_{\rho_j}^{\rho_i} \frac{F(x, 110t)^{1/3}}{t} dt.
\]
Let $Z_0$ be the limit of $\{Z_k\}_{k=1}^{\infty}$. Then we have that
\[
d_{x,1}(Z_0, Z_i) \leq \frac{3C}{\ln 2} \int_{0}^{\rho_i} \frac{F(x, 110t)^{1/3}}{t} dt.
\]
For any $0 < r < \rho_0$, we assume that $\rho_{k+1} \leq r < \rho_k$, then
\[
d_{x,1}(Z_r, Z_0) \leq d_{x,\rho_k}(Z_r, Z_{k+1}) + d_{x,1}(Z_{k+1}, Z_0)
\]
\[
\leq d_{x,1}(Z_{k+1}, Z_0) + d_{x,\rho_k}(Z_r, E) + d_{x,\rho_k}(E, Z_{k+1})
\]
\[
\leq \frac{r}{\rho_{k+1}} d_{x,r}(Z_r, E) + d_{x,\rho_k}(E, Z_{k+1})
\]
\[
\leq 3CF(x, 110r)^{1/3} + \frac{3C}{\ln 2} \int_{0}^{\rho_k} \frac{F(x, 110t)^{1/3}}{t} dt.
\]
Hence
\[
d_{x,r}(E, Z_0) \leq d_{x,r}(E, Z_r) + d_{x,r}(Z_r, Z_0) \leq \frac{10C}{\ln 2} \int_{0}^{200r} \frac{F(x, t)^{1/3}}{t} dt (6.1)
\]
and $T = \tau_x(Z_0)$ is the only blow up limit of $E$ at $x$, which is a minimal cone.

By Theorem 4.5 in [4], we have that
\[
\Theta_E(x, r) \leq \left( \frac{1}{2} - \alpha_0 \right) \frac{\mathcal{H}^1(E \cap B(x, r))}{r} + 2\alpha_0 \Theta_E(x) + 4h(r),
\]
where we take $\alpha_0$ the constant $\alpha$ in Theorem 4.5 in [4]. For our convenient, we denote $u(r) = \mathcal{H}^2(E \cap B(x, r))$ and $f(r) = \Theta_E(x, r) - \Theta_E(x)$, then we have $\mathcal{H}^1(E \cap \partial B(x, r)) \leq u'(r)$ and
\[
f(r) + \Theta_E(x) \leq \left( \frac{1}{2} - \alpha_0 \right) \frac{u'(r)}{r} + 2\alpha_0 \Theta_E(x) + 4h(r)
\]
\[
= \left( \frac{1}{2} - \alpha_0 \right) (2f(r) + rf'(r) + 2\Theta_E(x)) + 2\alpha_0 \Theta_E(x) + 4h(r),
\]
thus
\[
r f'(r) \geq \frac{4\alpha_0}{1 - 2\alpha_0} f(r) - \frac{8}{1 - 2\alpha_0} h(r),
\]
and
\[
\left( r^{-\frac{4\alpha_0}{1 - 2\alpha_0}} f(r) \right)' \geq - \frac{8}{1 - 2\alpha_0} r^{-\frac{1 + 2\alpha_0}{1 - 2\alpha_0}} h(r).
\]
We take $\beta_0 = \min\{4\alpha_0/(1-2\alpha_0), \alpha_1\}$. Then for any $0 < \beta < \beta_0$, we have that
\[
f(r) \leq (r/\rho_0)^{4\alpha_0/(1-2\alpha_0)} f(\rho_0) + \frac{8}{1 - 2\alpha_0} r^{4\alpha_0/(1-2\alpha_0)} \int_{r}^{\rho_0} t^{-\frac{1 + 2\alpha_0}{1 - 2\alpha_0}} h(t) dt
\]
\[
\leq (r/\rho_0)^{4\alpha_0/(1-2\alpha_0)} f(\rho_0) + C_1'(\alpha_1, \beta, \alpha_0) \rho_0^{\alpha_1} \cdot (r/\rho_0)^{\beta}.
\]
We get so that
\[ F(x, r) \leq C(\alpha_1, \beta, \alpha_0)(F(x, \rho_0) + C_\lambda \rho_0^\alpha)(r/\rho_0)^\beta, \]
combine this with (6), we get the conclusion. \(\Box\)

7 Parameterization of sliding almost minimal sets

Let \(n, d \leq n\) and \(k\) be nonnegative integers, \(\alpha \in (0, 1)\). By a \(d\)-dimensional submanifold of class \(C^{k,\alpha}\) of \(\mathbb{R}^n\) we mean a subset \(M\) of \(\mathbb{R}^n\) satisfying that for each \(x \in M\) there exist neighborhood \(U\) of \(x\) in \(\mathbb{R}^n\), a mapping \(\Phi : U \to \mathbb{R}^n\) which is a diffeomorphism of class \(C^{k,\alpha}\) between its domain and image, and a \(d\) dimensional vector subspace \(Z\) of \(\mathbb{R}^n\) such that
\[
\Phi(M \cap U) = Z \cap \Phi(U).
\]

In this section, we assume that \(\Omega \subseteq \mathbb{R}^3\) is a closed set whose boundary \(\partial \Omega\) is a 2-dimensional submanifold of class \(C^{1,\alpha}\) for some \(\alpha \in (0, 1)\), and suppose that \(\partial \Omega\) has tangent cone a half space at any point in \(\partial \Omega\). Let \(E \subseteq \Omega\) be a closed set such that \(E \in \text{SAM}(\Omega, \partial \Omega, h)\) and \(\partial \Omega \subseteq E\), \(x_0 \in \partial \Omega\). We always assume that the gauge function \(h\) satisfies that
\[
\int_0^{R_0} \frac{1}{r} \left( \int_0^r \frac{h(2t)}{t} \, dt \right)^{1/2} \, dr < +\infty \quad (7.1)
\]
and
\[
\int_0^{R_0} r^{1-\frac{\lambda}{1-\alpha}} \left( \int_0^r t^{-1-2\lambda h(2t)} \, dt \right)^{1/2} \, dr < +\infty, \quad (7.2)
\]
for some \(R_0 > 0\). It is easy to see that if \(h(t) \leq Ct^{\alpha_1}\) for some \(\alpha_1 > 0\), \(C > 0\) and \(0 < t \leq R_0\), then (7) and (7) hold. For our convenience, we put \(\lambda_0 = \lambda/(1-\lambda)\),
\[
h_2(\rho) = \int_0^\rho \frac{1}{r} \left( \int_0^r \frac{h(2t)}{t} \, dt \right)^{1/2} \, dr
\]
and
\[
h_3(\rho) = \int_0^\rho r^{-1+\lambda_0} \left( \int_0^r t^{-1-2\lambda h(2t)} \, dt \right)^{1/2} \, dr.
\]

We see, from Proposition 4.1 in [5], that \(E\) is Ahlfors-regular in \(B(x_0, R_0)\), i.e. there exist \(\delta_1 > 0\) and \(\xi_1 \geq 1\) such that for any \(x \in E \cap B(x_0, R_0)\), if \(0 < r < \delta_1\) and \(B(x, r) \subseteq B(x_0, R_0)\), we have that
\[
\xi_1^{-1} r^2 \leq \mathcal{H}^2(E \cap B(x, r)) \leq \xi_1 r^2.
\]
We see from Theorem 3.10 in [8] that there only there kinds of possibility for the blow-up limits of \(E\) at \(x_0\), they are the plane \(\text{Tan}(\partial \Omega, x_0)\), cones of type \(\mathbb{P}_+\) union \(\text{Tan}(\partial \Omega, x_0)\), and cones of type \(\mathbb{Y}_+\) union \(\text{Tan}(\partial \Omega, x_0)\). By Proposition 29.53 in [5], we get so that
\[
\Theta_E(x_0) = \pi, \quad \frac{3\pi}{2}, \quad \text{or} \quad \frac{7\pi}{4}.
\]
If \(\Theta_E(x_0) = \pi\), then there is a neighborhood \(U_0\) of \(x_0\) in \(\mathbb{R}^3\) such that \(E \cap U_0 = \partial \Omega \cap U_0\). In the next content of this section, we put ourself in the case \(\Theta_E(x_0) = \pi/2\) or \(7\pi/4\).
**Lemma 7.1.** There exist \( r_0 = r_0(x_0) > 0 \) and a mapping \( \Psi = \Psi(x_0) : B(0, r_0) \rightarrow \mathbb{R}^3 \), which is a diffeomorphism of class \( C^{1,\alpha} \) from \( B(0, r_0) \) to \( \Psi(B(0, r_0)) \), such that

\[
\Psi(0) = x_0, \quad \Psi(\Omega_0 \cap B_{r_0}) \subseteq \Omega \cap B(x_0, R_0), \quad \Psi(L_0 \cap B_{r_0}) \subseteq \partial \Omega \cap B(x_0, R_0),
\]

and that \( D\Psi(0) \) is a rotation satisfying that

\[
D\Psi(0)(\Omega_0) = \Tan(\Omega, x_0) \quad \text{and} \quad D\Psi(0)(L_0) = \Tan(\partial \Omega, x_0).
\]

**Proof.** By definition, there are an open set \( U, V \subseteq \mathbb{R}^3 \) and a diffeomorphism \( \Phi : U \rightarrow V \) of class \( C^{1,\alpha} \) such that \( x_0 \in U, \quad 0 = \Phi(x_0) \in V \) and

\[
\Phi(U \cap \partial \Omega) = Z \cap V,
\]

where \( Z \) is a plane through \( 0 \). Indeed, we have that

\[
Z = D\Phi(x_0) \Tan(\partial \Omega, x_0)
\]

and

\[
\Phi(U \cap \Omega) = V \cap D\Phi(x_0) \Tan(\Omega, x_0).
\]

We will denote by \( A \) the linear mapping given by \( A(v) = D\Phi(x_0)^{-1} v \), and assume that \( A(V) = B(0, r) \) is a ball. Let \( \Phi_1 \) be a rotation such that \( \Phi_1(\Tan(\partial \Omega, x_0)) = L_0 \) and \( \Phi_1(\Tan(\Omega, x_0)) = \Omega_0 \). Then we get that \( \Phi_1 \circ A \circ \Phi \) is also \( C^{1,\alpha} \) mapping which is a diffeomorphism between \( U \) and \( B(0, r) \),

\[
D(\Phi_1 \circ A \circ \Phi)(x_0) \Tan(\Omega, x_0) = \Phi_1(\Tan(\Omega, x_0)) = \Omega_0,
\]

\[
D(\Phi_1 \circ A \circ \Phi)(x_0) \Tan(\partial \Omega, x_0) = \Phi_1(\Tan(\partial \Omega, x_0)) = L_0,
\]

and

\[
\Phi_1 \circ A \circ \Phi(U \cap \partial \Omega) = \Phi_1 \circ A(Z \cap V) = L_0 \cap B(0, r),
\]

\[
\Phi_1 \circ A \circ \Phi(U \cap \partial \Omega) = \Phi_1 \circ A(V \cap D\Phi(x_0) \Tan(\Omega, x_0)) = \Omega_0 \cap B(0, r).
\]

We now take \( r_0 = r \) and \( \Psi = (\Phi_1 \circ A \circ \Phi)^{-1}|_{B(0,r)} \) to get the result. \( \square \)

Let \( U \subseteq \mathbb{R}^n \) be an open set. For any mapping \( \Psi : U \rightarrow \mathbb{R}^n \) of class \( C^{1,\alpha} \), we will denote by \( C_\Psi \) the constant \( C_\Psi = \sup \{ ||D\Psi(x) - D\Psi(y)||/|x - y|^\alpha : x, y \in U, x \neq y \} \). Then we have that

\[
\Psi(x) - \Psi(y) = \left< x - y, \int_0^1 D\Psi(y + t(x - y)) dt \right>,
\]

and thus

\[
|\Psi(x) - \Psi(y) - D\Psi(y)(x - y)| \leq |x - y| \int_0^1 C_\Psi(t|x - y|)^{\alpha} dt \leq \frac{C_\Psi}{\alpha + 1}|x - y|^{1+\alpha}.
\]

For any \( 0 < \rho \leq r_0 \), we set \( U_\rho = \Psi(B_\rho), \quad M_\rho = \Psi^{-1}(E \cap U_\rho) \) and

\[
\Lambda(\rho) = \max \left\{ \text{Lip} \left( \Psi_{B_\rho} \right), \text{Lip} \left( \Psi_{U_\rho}^{-1} \right) \right\}.
\]  

(7.3)
Then
\[ \|D\Psi(0)\| - \|D\Psi(x) - D\Psi(0)\| \leq \|D\Psi(x)\| \leq \|D\Psi(0)\| + \|D\Psi(x) - D\Psi(0)\|, \]
thus \(1 - C_\Psi \rho^\alpha \leq \|D\Psi(x)\| \leq 1 + C_\Psi \rho^\alpha\) for \(x \in B_\rho\), and we have that
\[ \Lambda(\rho) \leq 1/(1 - C_\Psi \rho^\alpha) \]
whenever \(C_\Psi \rho^\alpha < 1. \tag{7.4} \]

**Lemma 7.2.** For any \(1 < \rho \leq \min\{\rho_0, C_\Psi^{-1/\alpha}\}\), \(M_\rho\) is local almost minimal in \(B_\rho\) at \(0\) with gauge function \(H\) satisfying that
\[ H(2r) \leq 4\Lambda(r)^2 h(2\Lambda(r)r) + 4\xi_1 C_\Psi \Lambda(\rho) r^\alpha \]
for \(0 < r < (1 - C_\Psi \rho^\alpha)\delta_1\).

**Proof.** For any open set \(U \subseteq \mathbb{R}^3\), \(M \geq 1\), \(\delta > 0\) and \(\epsilon > 0\), we let \(GSAQ(U, M, \delta, \epsilon)\) be the collection of generalized sliding Almgren quasiminimal sets which is defined in Definition 2.3 in [5]. We see that
\[ \text{diam}(U_\rho) \leq 2\rho \text{Lip}(\Psi|_{B_\rho}) \leq 2\rho \Lambda(\rho) \]
and
\[ E \cap U_\rho \in \text{GSAQ}(U_\rho, 1, \text{diam}(U_\rho), h(2 \text{diam}(U_\rho))), \]
By Proposition 2.8 in [5], we have that
\[ M_\rho \in \text{GSAQ}(B_\rho, \Lambda(\rho)^4, 2\rho, \Lambda(\rho)^4 h(2 \Lambda(\rho))) \]
By Proposition 4.1 in [5], we get that \(M_\rho\) is Ahlfors-regular in \(B_\rho\). Indeed, we can get a little more, that is, for any \(x \in M_\rho\) with \(0 < r\Lambda(\rho) < \delta_1\) and \(B(x, r) \subseteq B(0, \rho)\), we have that
\[ (\xi_1 \Lambda(\rho))^{-1} r^2 \leq \mathcal{H}^2(M_\rho \cap B(x, r)) \leq (\xi_1 \Lambda(\rho)) r^2. \tag{7.5} \]
Let \(\{\varphi_t\}_{0 \leq t \leq 1}\) be any sliding deformation of \(M_\rho\) in \(B_r\). Then
\[ \{\Psi \circ \varphi_t \circ \Psi^{-1}\}_{0 \leq t \leq 1} \]
is a sliding deformation of \(E\) in \(U_r\). Hence we get that
\[ \mathcal{H}^2(E \cap U_r) \leq \mathcal{H}^2(\Psi \circ \varphi_t \circ \Psi^{-1}(E \cap U_r)) + h(2 \text{diam}(U_r))^2 \text{diam}(U_r)^2 \tag{7.6} \]
For any 2-rectifiable set \(A \subseteq B_\rho\), by Theorem 3.2.22 in [9], we have that
\[ \text{ap J}_2^1(\Psi|_A)(x) = \|\wedge_2(D\Psi(x)|_{\text{Tan}(A, x)})\| \]
and
\[ \mathcal{H}^2(\Psi(A \cap B_r)) = \int_{A \cap B_r} \text{ap J}_2^1(\Psi|_A)(x) d\mathcal{H}^2(x) \]
By (7), we get that
\[ \int_{A \cap B_r} (1 - C_\Psi |x|^{\alpha})^2 d\mathcal{H}^2 \leq \mathcal{H}^2(\Psi(A \cap B_r)) \leq \int_{A \cap B_r} (1 + C_\Psi |x|^{\alpha})^2 d\mathcal{H}^2. \]

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Thus, by taking $A = M_r$, we have that $M_r = M_r \cap B_r$, $\Psi(M_r) = E \cup U_r$ and
\[\mathcal{H}^2(\Psi(M_r)) \geq (1 - C_\Psi r^\alpha)^2 \mathcal{H}^2(M_r);\]
by taking $A = \varphi_1(M_r)$, we have that
\[\mathcal{H}^2(\Psi(\varphi_1(M_r) \cap B_r)) \leq (1 + C_\Psi r^\alpha)^2 \mathcal{H}^2(\varphi_1(M_r) \cap B_r).
\]
Combine these two equations with (7) and (7), we get that
\[
\begin{align*}
\mathcal{H}^2(\varphi_1(M_r) \cap B_r) &\geq (1 + C_\Psi r^\alpha)^{-2} \mathcal{H}^2(\Psi(\varphi_1(M_r) \cap B_r)) \\
&\geq (1 + C_\Psi r^\alpha)^{-2} (\mathcal{H}^2(E \cap U_r) - h(4r\Lambda(r))(2r\Lambda(r))^2) \\
&\geq \left(\frac{1 - C_\Psi r^\alpha}{1 + C_\Psi r^\alpha}\right)^2 \mathcal{H}^2(M_r) - \left(\frac{2r\Lambda(r)}{1 + C_\Psi r^\alpha}\right)^2 h(4r\Lambda(r)) \\
&\geq \mathcal{H}^2(M_r) - H(2r)^2.
\end{align*}
\]

\[\square\]

**Lemma 7.3.** Let $E_1 \subseteq \Omega_0$ be a 2-rectifiable set, $x \in E_1$, $X$ a cone centered at 0, $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ a diffeomorphism of class $C^{1,\alpha}$. Then there exist $C > 0$ such that, for any $r > 0$ and $\rho > 0$ with $B(\Phi(x), \rho) \subseteq \Phi(B(x, r))$,
\[
d_{\Phi(x),\rho}(\Phi(E_1), \Phi(x) + D\Phi(x)X) \leq (C r^\alpha + \|D\Phi(x)\|d_{x,r}(E_1, x + X)) \frac{r}{\rho}.
\]

**Proof.** Since $\Phi$ is of class $C^{1,\alpha}$, we have that
\[
|\Phi(y) - \Phi(x) - D\Phi(x)(y - x)| \leq \frac{C_\Phi}{\alpha + 1} |x - y|^{1+\alpha},
\]
by putting $C_1 = C_\Phi/(\alpha + 1)$, we get that
dist$(\Phi(y), \Phi(x) + D\Phi(x)X) \leq C_1 |y - x|^{1+\alpha}$ for $y \in x + X$.

For any $z \in E_1 \cap B_r$ and $y \in x + X$, we have that
\[
|\Phi(z) - \Phi(y)| \leq |\Phi(z) - \Phi(y) - D\Phi(x)(z - y)| + \|D\Phi(x)\| \cdot |z - y| \\
\leq \|D\Phi(x)\| \cdot |z - y| + C_1 |z - x|^{1+\alpha} + C_1 |y - x|^{1+\alpha},
\]
thus
\[
\text{dist}(\Phi(z), \Phi(x + X)) \leq \|D\Phi(x)\|rd_{x,r}(E_1, x + X) + 2C_1 r^{1+\alpha},
\]
hence
\[
\text{dist}(\Phi(z), \Phi(x) + D\Phi(x)X) \leq \|D\Phi(x)\|rd_{x,r}(E_1, x + X) + 3C_1 r^{1+\alpha}.
\] (7.7)

For any $z \in X \cap B_r$, $\Phi(x) + D\Phi(x)z \in \Phi(x) + D\Phi(x)X$, and
\[
\text{dist}(\Phi(x) + D\Phi(x)z, \Phi(E_1)) = \inf\{|\Phi(y) - \Phi(x) - D\Phi(x)z| : y \in E_1\} \\
\leq \inf\{C_1 r^{1+\alpha} + \|D\Phi(x)\| \cdot |y - x - z| : y \in E_1\} \\
\leq \|D\Phi(x)\|rd_{x,r}(x + X, E_1) + C_1 r^{1+\alpha}.
\] (7.8)

We get from (7) and (7) that
\[
d_{\Phi(x),\rho}(\Phi(E_1), \Phi(x) + D\Phi(x)X) \leq \frac{r}{\rho} \left(3C_1 r^{\alpha} + \|D\Phi(x)\| \cdot d_{x,r}(E_1, x + X)\)
\]
\[\square\]
Theorem 7.4. Let $\Omega$, $E \subseteq \Omega$, $x_0 \in \partial \Omega$ and $h$ be the same as in the beginning of this section. Then there is a unique blow-up limit $X$ of $E$ at $x_0$; moreover, if the gauge function $h$ satisfy that

$$h(t) \leq C h t^{\alpha_1} \text{ for some } C_h > 0, \alpha_1 > 0 \text{ and } 0 < t < t_0,$$

(7.9)

then there exists $\rho_0 > 0$ such that, for any $0 < \beta < \min\{\alpha, \alpha_1, 2\lambda_0\}$,

$$d_{x_0, \rho}(E, x_0 + X) \leq C(\rho/\rho_0)^{\beta/4}, \ 0 < \rho \leq 9\rho_0/20,$$

where $C$ is a constant satisfying that

$$C \leq C_{20}(\mu, \lambda_0, \alpha, \alpha_1, \beta, \xi_1)(F_{E}(x_0, 2\rho_0) + C_{\Psi}\rho_0^\alpha + C_r\rho_0^{\alpha_1})^{1/4},$$

and $F_E(x_0, r) = r^{-2}H^2(E \cap B(x_0, r)) - \Theta_E(x_0) + 16h_1(r)$.

Proof. Let $r \in (0, r_0)$ be such that $C_{\Psi} r^\alpha \leq 1/2$ and $2r \leq R_0$. Then $\Lambda(r) \leq 2$. By Lemma 7.2, we have that $M_r$ is local almost minimal at 0 with gauge function $H$ satisfying that

$$H(t) \leq 16h(2t) + C_r t^\alpha, \ 0 < t < r,$$

(7.10)

where $C_r \in (0, 2^{3-\alpha}\xi_1 C_{\Psi})$ is a constant.

We put $f_{M_r}(\rho) = \Theta_{M_r}(0, \rho) - \Theta_{M_r}(0)$. Then we get, from (3.5) and (3.5), that

$$f_{M_r}(\rho) \leq \left( r^{-2\lambda_0} f_{M_r}(r) \right) \rho^{2\lambda_0} + 8(1 + \lambda_0)\rho^{2\lambda_0} \int_0^r t^{-1-2\lambda_0} H(2t) dt$$

$$\leq \left( r^{-2\lambda_0} f_{M_r}(r) \right) \rho^{2\lambda_0} + 2^7 + 2^{2+2\lambda_0} (1 + \lambda_0)\rho^{2\lambda_0} \int_{2\rho}^{2r} h(2t) \frac{1}{t^{1+2\lambda_0}} dt$$

$$+ 2^{\alpha + 3}(1 + \lambda_0) C_r \cdot C_1(\alpha, \beta, \lambda_0) r^\alpha \cdot (\rho/r)^\beta,$$

where $C_1(\alpha, \beta, \lambda_0)$ is the constant in (3.5).

We get from (7) that

$$H_1(\rho) = \int_0^\rho \frac{H(s)}{s} ds \leq 16h_1(2\rho) + C_r(2\rho)^\alpha,$$

by setting $F_1(\rho) = f_{M_r}(\rho) + 16H_1(\rho)$, we have that

$$F_1(\rho) \leq C_{12}(\lambda_0, \alpha, \beta, r)(\rho/r)^\beta + 2^8h_1(2\rho) + 2^{4+\alpha} C_r \alpha^{-1}\rho^\alpha$$

$$+ 2^{7+2\lambda_0}(1 + \lambda_0)\rho^{2\lambda_0} \int_{2\rho}^{2r} h(2t) \frac{1}{t^{1+2\lambda_0}} dt,$$

where

$$C_{12}(\lambda_0, \alpha, \beta, r) \leq f_{M_r}(r) + 2^{\alpha + 3}(1 + \lambda_0) C_r C_1(\alpha, \beta, \lambda_0) r^\alpha.$$
where $C_{13}(\alpha, r) \leq 2^{3+\alpha/2}\alpha^{-3/2}C_r^{1/2}$, thus

$$
\int_0^t \frac{F_1(\rho)^{1/2}}{\rho} d\rho < +\infty, \text{ for } 0 < t \leq r.
$$

We now apply Theorem 4.14, there is a unique tangent cone $T$ of $M_r$ at $0$, thus there is a unique tangent cone $X$ of $E$ at $x_0$.

For any $R \in (0, R_1)$, we put

$$f_E(x_0, R) = R^{-2}H^2(E \cap B(x_0, R)) - \Theta_E(x_0)$$

and

$$F_E(x_0, R) = f_E(x_0, R) + 16h_1(R).$$

We see, from (7) and $B(x_0, \rho/\Lambda(\rho)) \subseteq U_\rho \subseteq B(x_0, \rho\Lambda(\rho))$, that

$$(1 - C_\psi \rho^\alpha)^2(f_M(\rho) + \Theta_E(x_0)) \leq \rho^{-2}H^2(E \cap U_\rho) \leq (1 + C_\psi \rho^\alpha)^2(f_M(\rho) + \Theta_E(x_0)),$$

so that

$$f_M(\rho) \leq (1 - C_\psi \rho^\alpha)^{-4}f_E(x_0, \rho\Lambda(\rho)) + 4\Theta_E(x_0)C_\psi \rho^\alpha,$$

and

$$f_M(\rho) \geq (1 - C_\psi^2 \rho^{2\alpha})^2f_E(x_0, \rho/\Lambda(\rho)) + 2\Theta_E(x_0)C_\psi^2 \rho^{2\alpha}.$$
where
\[ G_E(x_0, \rho_0) = (F_E(x_0, 2\rho_0) + C_\Psi \rho_0^\alpha + C_h \rho_0^{\alpha_1})^{1/4}. \]
Apply Lemma 7.3, and by setting \( X = D\Psi(0)T \), we get that, for any \( \rho \in (0, 9\rho_0/10) \),
\[
d_{x_0, \rho/2}(E, x_0 + X) \leq d_{x_0, \rho/\Lambda(\rho)}(E, x_0 + D\Psi(0)T) \leq 6C_\Psi \rho^\alpha + 2d_{x, \rho}(M, T) \leq 6C_\Psi \rho^\alpha + C_{18}(\mu, \lambda_0, \alpha, \alpha_1, \beta, \xi_1)G_E(x_0, \rho_0)(\rho/\rho_0)^{\beta/4} \leq C_{19}(\mu, \lambda_0, \alpha, \alpha_1, \beta, \xi_1)G_E(x_0, \rho_0)(\rho/\rho_0)^{\beta/4}.
\]
The radius \( \rho_0 \) is chosen to be such that
\[
0 < \rho_0 \leq \min\left\{ 1, t_0, r_0(x_0), R_0/2, (2C_\Psi)^{-1/\alpha} \right\}
\]
and \( R_0 > 0 \) is chosen to be such that
\[
F_M(0) \leq \mu r_0/4, \quad \bar{\varepsilon}(R_0) \leq 10^{-4}, \quad R_0 < (1 - \tau)r.
\]

**Lemma 7.5.** For any \( \tau > 0 \) small enough, there exists \( \varepsilon_2 = \varepsilon_2(\tau) > 0 \) such that the following hold: \( E \) is an sliding almost minimal set in \( \Omega \) with sliding boundary \( \partial\Omega \) and gauge function \( h, x_0 \in E \cap \partial\Omega, \Psi \) is a mapping as in Lemma 7.1 and \( C_\Psi \) is the constant as in (7), if \( r_1 > 0 \) satisfy that \( C_\Psi r_1^\alpha \leq \varepsilon_2 \), \( h(2r_1) \leq \varepsilon_2 \) and \( F_E(x_0, r_1) \leq \varepsilon_2 \), then for any \( \rho \in (0, 9r_1/10) \), we can find sliding minimal cone \( Z_{x_0, \rho} \) in \( \Tan(\Omega, x_0) \) with sliding boundary \( \Tan(\partial\Omega, x_0) \) such that
\[
dist(x, Z_{x_0, \rho}) \leq \tau r, \quad x \in E \cap B(x_0, (1 - \tau)r) \leq \tau r, \quad x \in Z_{x_0, \rho} \cap B(x_0, (1 - \tau)r),
\]
and for any ball \( B(x, t) \subseteq B(x_0, (1 - \tau)r) \),
\[
|\mathcal{H}^2(Z_{x_0, \rho} \cap B(x, t)) - \mathcal{H}^2(E \cap B(x, t))| \leq \tau r^2.
\]
Moreover, if \( E \supseteq \partial\Omega \), then \( Z_{x_0, \rho} \supseteq \Tan(\partial\Omega, x_0) \).

**Proof.** It is a consequence of Proposition 30.19 in [5].

**Corollary 7.6.** Let \( \Omega, E \subseteq \Omega, x_0 \in \partial\Omega, h \) and \( F_E \) be the same as in Theorem 7.4. Suppose that the gauge function \( h \) satisfying
\[
h(t) \leq C_\Psi t^\alpha \quad \text{for some } C_\Psi > 0, \quad \alpha_1 > 0 \quad \text{and } 0 < t < t_0.
\]
Then there exists \( \delta > 0 \) and constant \( C = C_{20}(\mu, \lambda_0, \alpha, \alpha_1, \beta, \xi_1) > 0 \) for \( 0 < \beta < \min\{\alpha, \alpha_1, 2\lambda_0\} \) such that, whenever \( 0 < \rho_0 \leq \min\{1, t_0, r_0(x_0), \bar{\varepsilon}\} \) satisfying
\[
F_E(x_0, 2\rho_0) + C_\Psi \rho_0^\alpha + C_h \rho_0^{\alpha_1} \leq \delta,
\]
we have that, for \( 0 < \rho \leq 9\rho_0/20 \),
\[
d_{x_0, \rho}(E, x_0 + \Tan(E, x_0)) \leq C(F_E(x_0, 2\rho_0) + C_\Psi \rho_0^\alpha + C_h \rho_0^{\alpha_1})^{1/4}(\rho/\rho_0)^{\beta/4}.
\]
Proof. By Theorem 7.4, there exist $\rho > 0$ such that

$$d_{x_0,\rho}(E, x_0 + \text{Tan}(E, x_0)) \leq C(\rho/\rho_0)^{\delta/4}, \ 0 < \rho \leq 9\rho_0/20,$$

where $\rho_0 > 0$ is chosen to be such that

$$0 < \rho_0 \leq \min\left\{ 1, t_0, r_0(x_0), R_0/2, (2C\psi)^{-1/\alpha} \right\} \quad (7.13)$$

and $R_0 > 0$ is chosen to be such that

$$F_{M_i}(R_0) \leq \mu r_0/4, \ \bar{\varepsilon}(R_0) \leq 10^{-4}, \ R_0 < (1 - \tau)r.$$

By Lemma 7.5, there exists $\delta > 0$ such that if $F_E(x_0, 2\rho_0) + C\psi\rho_0^\alpha + Ch\rho_0^{\alpha_1} \leq \delta$, then (7) holds, and we get the result.

Lemma 7.7. Let $\Omega, E$ and $h$ be the same as in Theorem 7.4. We have that

$$E \setminus \partial \Omega \in \text{SAM}(\Omega, \partial \Omega, h).$$

Proof. We will put $E_1 = E \setminus \partial \Omega$ for convenient. We first show that $\mathcal{H}^2(E_1 \cap \partial \Omega) = 0$. Indeed, for any $x \in E_1 \cap \partial \Omega$, $\Theta_E(x) \geq 3\pi/2$. It follows from the fact that for $\mathcal{H}^2$-a.e. $x \in E$, $\Theta_E(x) = \pi$ that $\mathcal{H}^2(E_1 \cap \partial \Omega) = 0$.

Let $\{\varphi_t\}_{0 \leq t \leq 1}$ be any sliding deformation in some ball $B = B(y, r)$. Since $E \supset \partial \Omega$ and $E \in \text{SAM}(\Omega, \partial \Omega, h)$, we have that

$$\mathcal{H}^2(E_1) = \mathcal{H}^2(E \setminus \partial \Omega) \leq \mathcal{H}^2(\varphi_1(E) \setminus \partial \Omega) + 4h(2r)r^2$$

$$= \mathcal{H}^2(\varphi_1(E_1) \setminus \partial \Omega) + 4h(2r)r^2$$

$$\leq \mathcal{H}^2(\varphi_1(E)) + 4h(2r)r^2.$$ 

Thus $E_1 \in \text{SAM}(\Omega, \partial \Omega, h)$. \hfill \Box

Lemma 7.8. Let $\Omega, E, x_0$ and $h$ be the same as in Theorem 7.4. For any $\varepsilon > 0$ small enough, there exists a $\rho_0 > 0$ such that for any $0 < \rho < \rho_0$ and $x \in E \cap B(x_0, \rho)$, there exists $x_1 \in B(x_0, 5\rho) \cap \partial \Omega$ with $x_1 \in E \setminus \Omega$ such that

$$|x - x_1| \leq (1 + \varepsilon) \text{dist}(x, \partial \Omega).$$

Proof. If $\Theta_E(x_0) = \pi$, then there is an open ball $B = B(x_0, r)$ such that $E \cap B = \partial \Omega \cap B$, and we have nothing to prove.

We assume that $\Theta_E(x_0) = 3\pi/2$ or $7\pi/4$. We put $E_1 = E \setminus \partial \Omega$. Then $x_0 \in E_1$ and $\Theta_E(x_0) = \pi/2$ or $3\pi/4$, and by Lemma 7.7, we have that $E_1 \in \text{SAM}(\Omega, \partial \Omega, h)$. By Lemma 7.5, for any $\varepsilon \in (0, 10^{-3})$, there exists $\rho_0 \in (0, r_0)$ such that, for any $0 < \rho < \rho_0$, we can find sliding minimal cone $Z_\rho$ centered at $x_0$ of type $\mathbb{P}_+$ or $\mathbb{Y}_+$ satisfying that

$$d_{x_0,\rho}(E_1, Z_\rho) \leq \varepsilon.$$ 

Let $\Psi : B(0, r_0) \to \mathbb{R}^3$ be the mapping defined in Lemma 7.1, and let $A$ be the same as in (7). We put $U_\rho = \Psi(B_\rho)$, $A_1 = \Psi^{-1}(E_1 \cap U_\rho)$. By Lemma 7.3, for any $0 < r \leq \rho/\rho(\rho)$, there exist sliding minimal cone $X_r$ in $\Omega_0$ such that

$$d_{0,r}(A_1, X_r) \leq (C\rho^\alpha + \varepsilon)^{\rho}_{r}.$$ 

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Thus there exists $\rho_1 > 0$ such that for any $0 < r \leq \rho_1$, we can find sliding minimal cone $X_r$ of type $\mathbb{P}_+$ or $\mathbb{Y}_+$ such that

$$d_{0,r}(A_1, X_r) \leq 2\varepsilon.$$  

Using the same argument as in the proof Lemma 5.4 in [8], we get that there exists $\rho_2 > 0$ such that for any $x \in A_1 \cap B(0, \rho)$ with $0 < \rho \leq \rho_2$, we can find $a \in A_1 \cap L_0 \cap B(0, 3\rho)$ such that

$$|P_{L_0}(x) - a| \leq 8\varepsilon|x - a|,$$

where we denote by $P_{L_0}$ the orthogonal projection from $\mathbb{R}^3$ to $L_0$. Thus

$$|x - a| \leq |x - P_{L_0}(x)| + |P_{L_0}(x) - a| \leq \text{dist}(x, L_0) + 8\varepsilon|x - a|,$$

and we get that

$$\text{dist}(x, A_1 \cap L_0 \cap B(0, 3\rho)) \leq \frac{1}{1 - 8\varepsilon} \text{dist}(x, L_0 \cap B(0, 3\rho)).$$

We take $\rho_3 = \text{dist}(x_0, \mathbb{R}^3 \setminus U_{\rho_2}) / 10$. Then, for any $0 < \rho \leq \rho_3$ and $z \in E_1 \cap B(x_0, \rho)$,

$$\text{dist}(z, E_1 \cap \partial \Omega \cap B(x_0, 5\rho)) \leq \text{Lip}(\Psi|_{B(0,3\rho_2)}) \text{dist}(\Psi^{-1}(z), A_1 \cap L_0 \cap B(0, 3\rho))$$

$$\leq (1 - 8\varepsilon)^{-1} \Lambda(3\rho) \text{dist}(\Psi^{-1}(z), A_1 \cap L_0 \cap B(0, 3\rho))$$

$$\leq (1 - 8\varepsilon)^{-1} \Lambda(3\rho)^2 \text{dist}(z, \partial \Omega \cap B(x_0, 5\rho)).$$

We assume $\rho_2$ to be small enough such that $(1 - 8\varepsilon)^{-1} \Lambda(3\rho_2)^2 < 1 + 10\varepsilon$, then

$$\text{dist}(z, E_1 \cap \partial \Omega \cap B(x_0, 5\rho)) \leq (1 + 10\varepsilon) \text{dist}(z, \partial \Omega \cap B(x_0, 5\rho)).$$

\[\Box\]

**Lemma 7.9.** Let $\Omega, E, x_0$ and $h$ be the same as in Theorem 7.4. Suppose that $\Theta_E(x_0) = 3\pi/2$. Then, by putting $E_1 = E \setminus \partial \Omega$, there exist a radius $r > 0$, a number $\beta > 0$ and a constant $C > 0$ such that, for any $x \in B(x_0, r) \cap E_1$ and $0 < \rho < 2r$, we can find cone $Z_{x, \rho}$ such that

$$d_{x, \rho}(E_1, Z_{x, \rho}) \leq C\rho^\beta,$$

where $Z_{x, \rho} = y + \text{Tan}(E_1, y)$, $y \in E_1 \cap B(x, C\rho)$, and $y \in E_1 \cap \partial \Omega \cap B(x, C\rho)$ in case $\rho \geq \text{dist}(x, \partial \Omega)/10$.  

**Proof.** We see that $E = E_1 \cup \partial \Omega$, and $F_E(x_0, \rho) = F_{E_1}(x, \rho) + F_{\partial \Omega}(x_0, r)$. By Corollary 7.6, there exist $\delta > 0$ and $C > 0$ such that whenever $0 < \rho_0 \leq \min\{1, t_0, r_0(x_0)\}$ satisfying

$$F_{E_1}(x_0, 2\rho_0) + C_{\Psi_{x_0}}\rho_0^\alpha + C_{h}\rho_0^{\alpha_1} \leq \delta,$$

we have that, for $0 < \rho \leq 9\rho_0 / 20$,

$$d_{x_0, \rho}(E_1, x_0 + \text{Tan}(E_1, x_0)) \leq C\delta^{1/4} (\rho/\rho_0)^\beta,$$

where $0 < \beta < \min\{\alpha, \alpha_1, 2\lambda_0, \beta_0\} / 4$. We take $\rho_1 \in (0, \rho_0)$ such that

$$F_{E_1}(x_0, 2\rho) + C_{\Psi_{x_0}}\rho^\alpha + C_{h}\rho^{\alpha_1} \leq \min\{\delta/2, \varepsilon_2(\tau)\}, \forall 0 < \rho \leq \rho_1.$$
If \( x \in \partial \Omega \cap B(x_0, \rho_1/10) \), we take \( t = \rho_1/2 \), then apply Lemma 7.5 with \( r = |x - x_0| + t \) to get that
\[
\mathcal{H}^2(E_1 \cap B(x,t)) \leq \mathcal{H}^2(Z_{x,r} \cap B(x,t)) + \tau r^2,
\]
thus
\[
\Theta_{E_1}(x,t) \leq \frac{1}{r^2} \mathcal{H}^2(Z_{x,r} \cap B(x,t)) + 4 \tau \leq \frac{\pi}{2} + C\psi_{\alpha_r} r^\alpha + 4 \tau,
\]
and
\[
F_{E_1}(x,t) \leq C\psi_{\alpha_r} r^\alpha + 4 \tau + 16 h_1(t).
\]
We get that \( F_{E_1}(x,2\rho) + C\psi_{\alpha_r} r^\alpha + C h_\alpha r^\alpha \leq \delta \) for \( 0 < \rho \leq t/2 \). Thus
\[
d_{x,r}(E_1, x + \tan(E_1,x)) \leq C_2^{1/4} (r/t)^\beta, \quad 0 < r < 9 t/20.
\]

By Lemma 7.8, we assume that for any \( x \in E_1 \cap B(x_0, \rho_1/10) \), there exists \( x_1 \in E_1 \cap B(x_0, \rho_1/2) \cap \partial \Omega \) such that
\[
|x - x_1| \leq 2 \text{dist}(x, \partial \Omega).
\]
If \( x \in E_1 \cap B(x_0, \rho_1/10) \\setminus \partial \Omega \), we take \( t = t(x) = 10^{-3} \text{dist}(x, \partial \Omega) \), then apply Lemma 7.5 with \( r = |x - x_1| + t \) to get that
\[
\mathcal{H}^2(E_1 \cap B(x,t)) \leq \mathcal{H}^2(Z_{x_1,r} \cap B(x,t)) + \tau r^2,
\]
thus
\[
\Theta_{E_1}(x,t) \leq \frac{1}{r^2} \mathcal{H}^2(Z_{x_1,r} \cap B(x,t)) + (1 + 2 \cdot 10^3)^2 \tau \leq \frac{\pi}{2} + (1 + 2 \cdot 10^3)^2 \tau,
\]
and
\[
F(x,t) \leq (1 + 2 \cdot 10^3)^2 \tau + 8 h_1(t).
\]
By Theorem 6.1, there is a constant \( C_1 > 0 \) such that
\[
d_{x,r}(E_1, x + \tan(E_1,x)) \leq C_1 (r/t)^\beta, \quad 0 < r < t.
\]
Hence we get that
\[
d_{x,r}(E_1, x + \tan(E_1,x)) \leq C_2 (r/t_0)^\beta, \forall x \in E_1 \cap B(x_0, \rho_1/10), 0 < r < t_0,
\]
where
\[
t_0 = \begin{cases} 
\rho_1/10, & x \in \partial \Omega, \\
10^{-3} \text{dist}(x, \partial \Omega), & x \notin \partial \Omega.
\end{cases}
\]
We take \( 0 < a < \beta/(1 + \beta) \). For any \( x \in B(x_0, \rho_1/10) \setminus \partial \Omega \), if \( r \leq C_3 t_0^{1/(1-a)} \), then we get from (7) that
\[
d_{x,r}(E_1, x + \tan(E_1,x)) \leq C_2 C_3^{(a-1)} r^a;
\]
if \( C_3 t_0^{1/(1-a)} < r < \rho_1/5 \), then by (7), we have that
\[
d_{x,r}(E_1, x_1 + \tan(E_1,x_1)) \leq \frac{|x - x_1| + r}{r} d_{x_1,|x-x_1|+r}(E_1, x_1 + \tan(E_1,x_1))
\]
\[
\leq C_4 \left( 1 + \frac{2 \cdot 10^3 t_0}{r} \right) \left( \frac{r + 2 \cdot 10^3 t_0}{\rho_1/2} \right)^\beta
\]
\[
\leq C_5 (1 + C_6 r^{-a})^{\beta+1} r^\beta \leq C_7 r^\beta - a \beta - a.
\]

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Lemma 7.10. Let \( \Omega, E, x_0 \) and \( h \) be the same as in Theorem 7.4. Suppose that \( \Theta_E(x_0) = 7\pi/4 \). Then, by putting \( E_1 = E \setminus \partial \Omega \), there exist a radius \( r > 0 \), a number \( \beta > 0 \) and a constant \( C > 0 \) such that, for any \( x \in B(x_0, r) \cap E_1 \) and \( 0 < \rho < 2r \), we can find a cone \( Z_{x,\rho} \) such that

\[
d_{x,\rho}(E_1, Z_{x,\rho}) \leq C \rho^\beta,
\]

where \( Z_{x,\rho} = y + \tan(E_1, y), \ y \in E_1 \cap B(x, C \rho) \), and \( y \in E_1 \cap \partial \Omega \cap B(x, C \rho) \) in case \( \rho \geq \text{dist}(x, \partial \Omega)/10 \).

Proof. By Corollary 7.6, there exist \( \delta > 0 \) and \( C > 0 \) such that whenever \( 0 < \rho_0 \leq \min\{1, t_0, r_0(x_0)\} \) satisfying

\[
F_{E_1}(x_0, 2\rho_0) + C_{\Psi_\alpha} \rho_0^\alpha + C_{h} \rho_0 \alpha_1 \leq \delta,
\]

we have that, for \( 0 < \rho \leq 9\rho_0/20 \),

\[
d_{x_0,\rho}(E_1, x_0 + \tan(E_1, x_0)) \leq C \delta^{1/4}(\rho/\rho_0)^\beta,
\]

where \( 0 < \beta < \min\{\alpha, \alpha_1, 2\lambda_1\}/4 \). We take \( \rho_1 \in (0, \rho_0) \) such that

\[
F_{E_1}(x_0, 2\rho) + C_{\Psi_\alpha} \rho^\alpha + C_{h} \rho \alpha_1 \leq \min\{\delta/2, \varepsilon_2(\tau)\}, \forall 0 < \rho \leq \rho_1.
\]

If \( x \in \partial \Omega \cap B(x_0, \rho_1/10) \), we take \( t = |x - x_0|/2 \), then apply Lemma 7.5 with \( r = |x - x_0| + t \) to get that

\[
H^2(E_1 \cap B(x, t)) \leq H^2(Z_{x,\rho} \cap B(x, t)) + \tau r^2,
\]

thus

\[
\Theta_{E_1}(x, t) \leq \frac{1}{f^2} H^2(Z_{x,\rho} \cap B(x, t)) + 9\tau \leq \frac{\pi}{2} + C_{\Psi_\alpha} \rho^\alpha + 9\tau,
\]

and

\[
F_{E_1}(x, t) \leq C_{\Psi_\alpha} \rho^\alpha + 9\tau + 16\lambda_1(t).
\]

We get that \( F_{E_1}(x, 2\rho) + C_{\Psi_\alpha} \rho^\alpha + C_{h} \rho \alpha_1 \leq \delta \) for \( 0 < \rho \leq t/2 \). Thus

\[
d_{x,\rho}(E_1, x + \tan(E_1, x)) \leq C \delta^{1/4}(r/t)^\beta, \ 0 < r < 9t/20.
\]

(7.15)

By Lemma 7.8, we assume that for any \( x \in E_1 \cap B(x_0, \rho_1/10) \), there exists \( x_1 \in E_1 \cap B(x_0, \rho_1/5) \cap \partial \Omega \) such that

\[
|x - x_1| \leq 2 \text{dist}(x, \partial \Omega).
\]

If \( x \in E_1 \cap B(x_0, \rho_1/10) \setminus \partial \Omega \), then \( \Theta_{E_1}(x) = \pi \) or \( 3\pi/2 \). We put \( t(x) = \text{dist}(x, \partial \Omega) \). If \( \Theta_{E_1}(x) = 3\pi/2 \), we take \( t = 10^{-3} t(x) \), then apply Lemma 7.5 with \( r = |x - x_1| + t \) to get that

\[
H^2(E_1 \cap B(x, t)) \leq H^2(Z_{x,\rho} \cap B(x, t)) + \tau r^2,
\]

and
thus
\[ \Theta_{E_1}(x, t) \leq \frac{1}{t^2} \mathcal{H}^2(Z_{x_1, r} \cap B(x, t)) + (1 + 2 \cdot 10^3)^2 \tau \leq \frac{3\pi}{2} + (1 + 2 \cdot 10^3)^2 \tau. \]
and
\[ F_{E_1}(x, t) \leq (1 + 2 \cdot 10^3)^2 \tau + 8h_1(t). \]

By Theorem 6.1, we have that
\[ d_{x, \rho}(E_1, x + \tan(E_1, x)) \leq C_1(\rho/t)^3, \ 0 < \rho < t. \] (7.16)

We put \( E_Y = \{x_0\} \cup \{x \in E \setminus \partial \Omega : \Theta_{E_1}(x) = \pi\}. \) If \( \Theta_{E_1}(x) = \pi \) and \( \text{dist}(x, E_Y) \leq 10^{-2} \text{dist}(x, \partial \Omega), \) we take \( x_2 \in E_Y \) such that \( |x - x_2| \leq 2 \text{dist}(x, E_Y) \) and \( t = 10^{-1} \text{dist}(x, E_Y), \) then apply Lemma 7.24 in [3] with \( r = |x - x_2| + t \) to get that
\[ \mathcal{H}^2(E_1 \cap B(x, t)) \leq \mathcal{H}^2(Z_{x_2, r} \cap B(x, t)) + \tau r^2, \]

thus
\[ \Theta_{E_1}(x, t) \leq \frac{1}{t^2} \mathcal{H}^2(Z_{x_2, r} \cap B(x, t)) + 400 \tau \leq \pi + 400 \tau, \]
and
\[ F_{E_1}(x, t) \leq 4\tau + 8h_1(t). \]

By Theorem 6.1, we have that
\[ d_{x, \rho}(E_1, x + \tan(E_1, x)) \leq C_2(\rho/t)^3, \ 0 < \rho < t. \] (7.17)

If \( \Theta_{E_1}(x) = \pi \) and \( \text{dist}(x, E_Y) > 10^{-2} \text{dist}(x, \partial \Omega), \) we take \( t = 10^{-3} \text{dist}(x, \partial \Omega), \) then apply Lemma 7.5 with \( r = |x - x_1| + t \) to get that
\[ \mathcal{H}^2(E_1 \cap B(x, t)) \leq \mathcal{H}^2(Z_{x_1, r} \cap B(x, t)) + \tau r^2, \]

thus
\[ \Theta_{E_1}(x, t) \leq \frac{1}{t^2} \mathcal{H}^2(Z_{x_1, r} \cap B(x, t)) + (1 + 2 \cdot 10^3)^2 \tau \leq \pi + (1 + 2 \cdot 10^3)^2 \tau. \]
and
\[ F_{E_1}(x, t) \leq (1 + 2 \cdot 10^3)^2 \tau + 8h_1(t). \]

By Theorem 6.1, we have that
\[ d_{x, \rho}(E_1, x + \tan(E_1, x)) \leq C_3(\rho/t)^3, \ 0 < \rho < t. \] (7.18)

We get, from (7), (7), (7) and (7), so that
\[ d_{x, \rho}(E_1, x + \tan(E_1, x)) \leq C_4(\rho/t_0)^3, \ x \in E_1 \cap B(x_0, \rho_1/10), \ 0 < \rho < t_0, \] (7.19)

where
\[ t_0 = \begin{cases} \rho_1/2, & x = x_0, \\ |x - x_0|/10, & x \in \partial \Omega \setminus \{x_0\}, \\ 10^{-3} \text{dist}(x, \partial \Omega), & x \not\in \partial \Omega, \Theta_{E_1}(x) = 3\pi/2 \\ 10^{-1} \min\{10^{-2} \text{dist}(x, \partial \Omega), \text{dist}(x, E_Y)\}, & x \not\in \partial \Omega, \Theta_{E_1}(x) = \pi. \end{cases} \]

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Thus we get that, for any $x$, cone $\mathcal{Z}$.

By (7), we have that, for any $x \in \partial \Omega \cap B(x_0, \rho_1/10) \setminus \{x_0\}$, and any $0 < \rho < |x - x_0|/10,

d_{x,\rho}(E_1, x + \text{Tan}(E_1, x)) \leq C_4(\rho/t_0)^{\beta}.

If $0 < \rho \leq C_5|x - x_0|^{1/(1-\beta_1)}$, then

d_{x,\rho}(E_1, x + \text{Tan}(E_1, x)) \leq C_4(10\rho/|x - x_0|)^{\beta} = C_6\rho^{\beta_1}\beta;

if $C_5|x - x_0|^{1/(1-\beta_1)} < \rho \leq \rho_1/5$, then

d_{x,\rho}(E_1, x + \text{Tan}(E_1, x)) \leq \frac{|x - x_0| + \rho}{\rho}d_{x,\rho}[x - x_0 + \rho](E_1, x + \text{Tan}(E_1, x))

\leq (1 + C_5^{-1+\beta_1}\rho^{-\beta_1})C_4\left(\frac{C_5^{-1+\beta_1}\rho^{1-\beta_1} + \rho}{\rho_1^{1/2}}\right)^{\beta}

\leq C_7\rho^{\beta_2-\beta_1-\beta_3}.\]

Thus we get that, for any $0 < \beta_3 \leq \min\{\beta\beta_1, \beta_2, \beta_1 - \beta_1 - \beta_3\}$, there is a constant $C_8$ such that for any $x \in \partial \Omega \cap B(x_0, \rho_1/10)$ and $0 < \rho \leq \rho_1/5$ we can find cone $\mathcal{Z}_{x,\rho}$ satisfying that

\[d_{x,\rho}(E_1, Z_{x,\rho}) \leq C_8\rho^{\beta_3}.\]  

(7.21)

If $x \in E_1 \cap B(x_0, \rho_1/10) \setminus \partial \Omega$ and $\Theta_{E_1}(x) = 3\pi/2$, then for $0 < \rho \leq C_5|x - x_0|^{1/(1-\beta_1)}$, we get, from (7), that

\[d_{x,\rho}(E_1, x + \text{Tan}(E_1, x)) \leq C_4(10\rho/\text{dist}(x, \partial \Omega))^{\beta} = C_9\rho^{\beta_3}\beta;

and for $C_5|x - x_0|^{1/(1-\beta_1)} < \rho \leq \rho_1/5$, we have that

\[d_{x,\rho}(E_1, x + \text{Tan}(E_1, x)) \leq \frac{|x - x_0| + \rho}{\rho}d_{x,\rho}[x - x_0 + \rho](E_1, x + \text{Tan}(E_1, x))

\leq (1 + C_5^{-1+\beta_1}\rho^{-\beta_1})C_4\left(\frac{C_5^{-1+\beta_1}\rho^{1-\beta_1} + \rho}{\rho_1^{1/2}}\right)^{\beta}

\leq C_{10}\rho^{\beta_2-\beta_1-\beta_3}.\]

Thus we get that, for any $0 < \beta_4 \leq \min\{\beta\beta_1, \beta_1 - \beta_1 - \beta_1\}$, there is a constant $C_{11}$ such that for any $x \in E_1 \cap B(x_0, \rho_1/10) \setminus \partial \Omega$ with $\Theta_{E_1}(x) = 3\pi/2$, and $0 < \rho \leq \rho_1/5$ we can find cone $\mathcal{Z}_{x,\rho}$ satisfying that

\[d_{x,\rho}(E_1, Z_{x,\rho}) \leq C_{11}\rho^{\beta_4}.\]  

(7.22)

If $x \in E_1 \cap B(x_0, \rho_1/10) \setminus \partial \Omega$, $\Theta_{E_1}(x) = \pi$ and dist$(x, \partial \Omega) < 100\text{dist}(x, E_Y)$, then for any $0 < \rho < C_9\text{dist}(x, \partial \Omega)^{1/(1-\beta_1)}$, we get, from (7), that

\[d_{x,\rho}(E_1, x + \text{Tan}(E_1, x)) \leq C_4(10\rho/\text{dist}(x, \partial \Omega))^{\beta} = C_{12}\rho^{\beta_1}\beta;\]  

(7.23)

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and for $C_9 \text{dist}(x, \partial \Omega)^{1/(1-\beta_1)} \leq \rho \leq \rho_1/5$, in case $\rho \leq C_{13}|x - x_0|^{1/(1-\beta_2)}$, we get, from (7), that
\[
d_{x,\rho}(E_1, x_1 + \Tan(E_1, x_1)) \leq \frac{|x - x_1| + \rho}{\rho} d_{x_1,|x-x_1|+\rho}(E_1, x_1 + \Tan(E_1, x_1))
\leq (1 + 2C_9^{1+\beta_1} \rho^{-\beta_1})C_4 \left( \frac{2C_9^{1+\beta_1} \rho^{1-\beta_1} + \rho}{|x_0 - x_1|/10} \right)^\beta \tag{7.24}
\leq C_{14}\rho^{\beta_2-\beta_1-\beta_1};
\]
in case $\rho > C_{13}|x - x_0|^{1/(1-\beta_2)}$, we have that
\[
d_{x,\rho}(E_1, x_0 + \Tan(E_1, x_0)) \leq \frac{|x - x_0| + \rho}{\rho} d_{x_0,|x-x_0|+\rho}(E_1, x_0 + \Tan(E_1, x_0))
\leq (1 + C_{13}^{1+\beta_2} \rho^{-\beta_2})C_4 \left( \frac{C_{13}^{1+\beta_2} \rho^{1-\beta_2} + \rho}{\rho_1/2} \right)^\beta \tag{7.25}
\leq C_{15}\rho^{\beta_2-\beta_1-\beta_2}.
\]
If $x \in E_1 \cap B(x_0, \rho_1/10) \setminus \partial \Omega$, $\Theta_{E_1}(x) = \pi$ and $\text{dist}(x, \partial \Omega) \geq 100 \text{dist}(x, E_Y)$, then for any $0 < \rho < C_{16} \text{dist}(x, E_Y)^{1/(1-\beta_1)}$, we get, from (7), that
\[
d_{x,\rho}(E_1, x + \Tan(E_1, x)) \leq C_4(10\rho/ \text{dist}(x, E_Y))^\beta = C_{17}\rho^{\beta_1}, \tag{7.26}
\]
for $C_{16} \text{dist}(x, E_Y)^{1/(1-\beta_1)} \leq \rho \leq \rho_1/5$, we can find $y \in E_Y$ such that $|x - y| \leq 2 \text{dist}(x, E_Y)$, in case $\rho \leq C_{18} \text{dist}(y, \partial \Omega)^{1/(1-\beta_2)}$, we get, from (7), that
\[
d_{x,\rho}(E_1, y + \Tan(E_1, y)) \leq \frac{|x - y| + \rho}{\rho} d_{y,|x-y|+\rho}(E_1, y + \Tan(E_1, y))
\leq (1 + 2C_{16}^{1+\beta_1} \rho^{-\beta_1})C_4 \left( \frac{2C_{16}^{1+\beta_1} \rho^{1-\beta_1} + \rho}{10^{-\beta_1} \text{dist}(y, \partial \Omega)} \right)^\beta \tag{7.27}
\leq C_{19}\rho^{\beta_2-\beta_1-\beta_1};
\]
and in case $\rho > C_{18} \text{dist}(y, \partial \Omega)^{1/(1-\beta_2)}$, we have that
\[
|x - x_0| \geq \text{dist}(x, \partial \Omega) \geq 100 \text{dist}(x, E_Y) \geq 50|x - y|,
\]
and by (7),
\[
\text{dist}(y, \partial \Omega) \geq \eta_3|y - x_0| \geq \eta_3(|x - x_0| - |x - y|) \geq \eta_3 \cdot \frac{49}{50}|x - x_0|,
\]
thus by (7),
\[
d_{x,\rho}(E_1, x_0 + \Tan(E_1, x_0)) \leq \frac{|x - x_0| + \rho}{\rho} d_{x_0,|x-x_0|+\rho}(E_1, x_0 + \Tan(E_1, x_0))
\leq (1 + C_{20}^{1+\beta_2} \rho^{-\beta_2})C_4 \left( \frac{C_{20}^{1+\beta_2} \rho^{1-\beta_2} + \rho}{\rho_1/2} \right)^\beta \tag{7.28}
\leq C_{21}\rho^{\beta_2-\beta_2-\beta_2}.
\]
We get, from (7), (7), (7), (7), and (7), that for any $0 < \beta_5 \leq \min\{\beta_1, \beta_2 - \beta_1 - \beta_2 - \beta_2\}$, there is a constant $C_{22}$ such that for any $x \in E_1 \cap B(x_0, \rho_1/10) \setminus \partial \Omega$ with $\Theta_{E_1}(x) = \pi$, and $0 < \rho \leq \rho_1/5$ we can find cone $Z_{x,\rho}$ such that

$$d_{x,\rho}(E_1, Z_{x,\rho}) \leq C_{22} \rho^{\beta_5}. \quad (7.29)$$

Hence we get, from (7), (7), and (7), that for any $0 < \beta_6 \leq \min\{\beta_1, \beta_2 - \beta_1 - \beta_2 - \beta_2\}$, there is a constant $C_{23} > 0$ and $C_{24} > 0$ such that for any $x \in E_1 \cap B(x_0, \rho_1/10)$ and $0 < \rho \leq \rho_1/5$ we can find cone $Z_{x,\rho}$ such that

$$d_{x,\rho}(E_1, Z_{x,\rho}) \leq C_{23} \rho^{\beta_6},$$

where $Z_{x,\rho} = z + \tan(E_1, z)$ for some $z \in E_1 \cap B(x, C_{24}\rho)$, and $z \in E_1 \cap \partial \Omega \cap B(x, C_{24}\rho)$ in case $\rho \geq \max\{C_5 |x - x_0|^{1/(1-\beta_1)} + C_9 \text{dist}(x, \partial \Omega)^{1/(1-\beta_1)} + C_{18}) \text{dist}(y, \partial \Omega)^{1/(1-\beta_2)} \text{dist}(x, \partial \Omega)\}$. 

**Corollary 7.11.** Let $\Omega$, $E$ and $h$ be the same as in Theorem 7.4. Let $E_1 = E \setminus \partial \Omega$ and $x_0 \in E_1 \cap \partial \Omega$. Then there exist a radius $r > 0$, a number $\beta > 0$ and a constant $C > 0$ such that, for any $x \in E_1 \cap B(x_0, r)$ and $0 < \rho < 2r$, we can find cone $Z_{x,\rho}$ such that

$$d_{x,\rho}(E_1, Z_{x,\rho}) \leq C \rho^\beta,$$

where $Z_{x,\rho} = y + \tan(E_1, y)$, $y \in E_1 \cap B(x, C \rho)$, and $y \in E_1 \cap \partial \Omega \cap B(x, C \rho)$ in case $\rho \geq \text{dist}(x, \partial \Omega)/10$.

**Proof.** It is follow from Lemma 7.9 and Lemma 7.10.

**Lemma 7.12.** Let $\Omega$, $E$, $x_0$ and $h$ be the same as in Corollary 7.11. Let $\Psi : B(0, r_0) \to \mathbb{R}^3$ be the mapping defined in Lemma 7.1. Let $R > 0$ be such that $\Psi(B(0, R)) \subseteq B(x_0, r)$, where $B(x_0, r)$ is the ball considered as in Corollary 7.11. By putting $U_{\Psi} = \psi(B(0, R))$, $M_1 = \Psi^{-1}(E_1 \cap U)$, we have that there exist $\rho_1 > 0$, $\beta > 0$, and constant $C > 0$ such that for any $z \in M_1 \cap B(0, \rho_3)$ and $0 < t < 2\rho_3$, we can find cone $Z(z, t)$ through $z$ such that

$$d_{z,t}(M_1, Z(z, t)) \leq Ct^\beta,$$

where $Z(z, t)$ is a minimal cone of type $\mathbb{P}$ or $\mathbb{Y}$ in case $z \in M_1 \setminus L_0$ and $0 < t < \text{dist}(z, L_0)$; and in case $t \geq \text{dist}(z, L_0)$ or $z \in L_0$, $Z(z, t)$ is a sliding minimal cone in $\Omega$ with sliding boundary $L_0$, if $Z(z, t) \setminus L_0 \neq \emptyset$, we can be written as $Z(z, t) = L_0 \cup Z$, $Z$ is a sliding minimal cone of type $\mathbb{P}_+$ or $\mathbb{Y}_+$.

**Proof.** For any $x \in B(x_0, r) \cap E_1$ and $0 < \rho < 2r$, we let $Z_{x,\rho}$ be the same cone considered as in Corollary 7.11. We put $\Phi = \Psi^{-1}|_{B(x_0, r)}$ and $X = \tan(E_1, y)$ for convenient.

For any $x \in E_1 \cap B(x_0, r)$, and any $z \in E_1 \cap B(x, \rho)$, we have that

$$\text{dist}(\Phi(z), \Phi(y + X)) \leq \text{lip}(\Phi) \text{dist}(z, y + X) \leq C \text{lip}(\Phi) \rho^{1+\beta}.$$

Since

$$|\Phi(z_1) - \Phi(z_2) - D\Phi(z_2)(z_1 - z_2)| \leq C_1 |z_1 - z_2|^{1+\alpha},$$

we have that, for any $z_1 \in y + X$,

$$\text{dist}(\Phi(z_1), \Phi(y) + D\Phi(y)X) \leq C_1 |z_1 - y|^{1+\alpha}.$$
Hence
\[ \text{dist}(\Phi(z), \Phi(y) + D\Phi(y)X) \leq C \text{Lip}(\Phi) \rho^{1+\beta} + C_1(\rho + C \rho \rho^{1+\beta})^{1+\alpha} \leq C_2 \rho^{1+\beta}. \quad (7.30) \]

For any \( v \in X \), we see that \( \Phi(y) + D\Phi(y)v \in \Phi(y) + D\Phi(y)X \), and we have that
\[
\text{dist}(\Phi(y) + D\Phi(y)v, M_1) \leq \text{dist}(\Phi(y) + D\Phi(y)v, \Phi(E_1 \cap B(x, \rho))) \\
= \inf\{|\Phi(z) - \Phi(y) - D\Phi(y)v| : z \in E_1 \cap B(x, \rho)\} \\
\leq \inf\{C_1|z - y|^{1+\alpha} + \text{Lip}(\Phi)|z - y| : z \in E_1 \cap B(x, \rho)\} \\
\leq C_1(\rho + C C_0^{1+\alpha}) + \text{Lip}(\Phi) \text{dist}(y + v, E_1).
\]

Thus there exist \( C_3 > 0 \) such that, for any \( v \in X \) with \(|y + v - x| \leq \rho\),
\[
\text{dist}(\Phi(y) + D\Phi(y)v, M_1) \leq C_3 \rho^{1+\beta}. \quad (7.31)
\]

We take \( 0 < \rho_5 < C_4 < 1 \) small enough, for example \( C_4 < (10 \text{Lip}(\Phi))^{-1} \), then for any \( C_5 \rho \leq t \leq C_4 \rho \leq \rho/\text{Lip}(\Phi) - C_1(C_0)^{1+\alpha} \), we have that \( M_1 \cap B(\Phi(x), t) \subseteq \Phi(E_1 \cap B(x, \rho)) \) and
\[
[\Phi(y) + D\Phi(y)X] \cap B(\Phi(x), t) \subseteq \{\Phi(y) + D\Phi(y)v : v \in X, y + v \in B(x, \rho)\}.
\]

We get, from (7) and (7), so that
\[
d_{\Phi(x), t}(M_1, \Phi(y) + D\Phi(y)X) \leq C_6 \rho^3 \leq C_7 t^3,
\]
and
\[
|\Phi(x) - \Phi(y)| \leq \text{Lip}(\Phi) |x - y| \leq (\text{Lip}(\Phi) CC_5^{-1})t.
\]

Hence
\[
d_{\Phi(x), t}(M_1, \Phi(y) + D\Phi(y)X) \leq C_7 t^3, \quad \text{for any } 0 < t < C_4 \rho_1,
\]

where \( \rho_1 \in (0, 2\rho) \) satisfy that \( C_1 C_0^{1+\alpha} \rho_1 \leq \text{Lip}(\Phi)^{-1} - C_4 \).

We take \( \rho_2 > 0 \) such that, for any \( x \in E_1 \cap \Phi(B(x_0, \rho_2)) \) and \( 0 < \rho < 2\rho_2 \), \( Z_{x, \rho} \) can be expressed as \( Z_{x, \rho} = y + \text{Tan}(E_1, y) \) with \( y \in E_1 \cap U \). Since \( D\Phi(y)X = D\Phi(y)\text{Tan}(E_1, y) = \text{Tan}(M_1, \Phi(y)) \) in case \( y \in E_1 \cap U \), by putting \( \rho_3 = \min\{\rho_2, C_1 \rho_1/2, R\} \), we have that, for any \( z \in M_1 \cap B(0, \rho_3) \) and \( 0 < t < 2\rho_3 \), there exist cone \( Z'(z, t) \) in \( \Omega_0 \) with sliding boundary \( L_0 = \partial \Omega_0 \), such that
\[
d_{x, t}(M_1, Z'(z, t)) \leq C_7 t^3.
\]

For such cone \( Z'(z, t) \), we have that \( Z'(z, t) = w + \text{Tan}(M_1, w), w \in M_1, |w - z| \leq C_8 t, \) and \( w \in L_0 \cap B(z, C_8 t) \) in case \( t \geq \text{dist}(z, L_0)/2 \). \( Z'(z, t) \) may not pass through \( z \), but the cone \( Z(z, t) = Z'(z, t) - w + z \) pass through \( z \), and
\[
d_{x, t}(M_1, Z(z, t)) \leq C_7 t^3 + C_8 t \leq C_9 t^3.
\]

\[ \square \]

Proof of Theorem 1.2. Let \( M_1 \) be the same as in Lemma 7.12, and let \( M = \Psi^{-1}(E \cap U) \). Then by Lemma 7.12, we have that for any \( x \in M_1 \cap B(0, \rho_3) \) and \( 0 < r < 2\rho_3 \), there exist cone \( Z(x, r) \) such that
\[
d_{x, r}(M_1, Z(x, r)) \leq C r^3,
\]

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where \( Z(x, r) \) is a minimal cone in \( \mathbb{R}^3 \) of type \( \mathbb{P} \) or \( \mathbb{Y} \) in case \( x \notin L_0 \) and \( t \leq \text{dist}(x, L_0) \); and \( Z(x, r) \) is a sliding minimal cone in \( \Omega_0 \) with sliding boundary \( L_0 \) of type \( \mathbb{P}_+ \) or \( \mathbb{Y}_+ \) in other case. We apply Theorem 5.1 to get that there exist \( \rho_4 > 0 \), a sliding minimal cone \( Z' \) centered at 0, and a mapping \( \Phi_1 : \Omega_0 \cap B(0, \rho_4) \to \Omega_0 \), which is a \( C^{1,\beta} \)-differential, such that \( \Phi_1(0) = 0 \), \( \Phi_1(\partial \Omega_0 \cap B(0, \rho_4)) \subseteq L_0 \), \( \| \Phi - \text{id} \| \leq 10^{-1} \rho_4 \) and

\[
M_1 \cap B(0, \rho_4) = \Phi(Z') \cap B(0, \rho_4).
\]

We take \( Z = Z' \cup L_0 \), then we get that

\[
M \cap B(0, \rho_4) = \Phi(Z) \cap B(0, \rho_4).
\]

\[\square\]

8 Existence of the Plateau problem with sliding boundary conditions

The Plateau Problem with sliding boundary conditions arise in [6], due to Guy David. That is, given an initial set \( E_0 \), and boundary \( \Gamma \), to find the minimizers among all competitors. The author of the paper [6] also gives some hint to the existence in Section 6, and later on in [5], he pave the way. We will give an existence result in case the boundary is nice enough.

Let \( \Omega \subseteq \mathbb{R}^3 \) be a closed domain such that the boundary \( \partial \Omega \) is a 2-dimensional manifold of class \( C^{1,\alpha} \) for some \( \alpha > 0 \). Let \( E_0 \subseteq \Omega \) be a closed set with \( E_0 \supseteq \partial \Omega \). We denote by \( \mathcal{C}(E_0) \) the collection of all competitors of \( E_0 \).

**Theorem 8.1.** If there is a bounded minimizing sequence of competitors. Then there exists \( E \in \mathcal{C}(E_0) \) such that

\[
\mathcal{H}^2(E \setminus \partial \Omega) = \inf \{ \mathcal{H}^2(S \setminus \partial \Omega) : S \in \mathcal{C}(E_0) \}
\]

**Proof.** We put

\[
m_0 = \inf \{ \mathcal{H}^2(S \setminus \partial \Omega) : S \in \mathcal{C}(E_0) \}.
\]

If \( m_0 = +\infty \), we have nothing to do. We now assume that \( 0 \leq m_0 < +\infty \).

Let \( \{ S_i \} \subseteq \mathcal{C}_0 \) be a sequence of competitors bounded by \( B(0, R) \) such that

\[
\lim_{i \to \infty} \mathcal{H}^2(S_i \setminus \partial \Omega) = m_0.
\]

Apply Lemme 5.2.6 in [10], we can fined a sequence of open sets \( \{ U_i \} \) and a sequence of competitors \( \{ E_i \} \subseteq \mathcal{C}(E_0) \) of \( E_0 \) bounded by \( B(0, R + 1) \) such that

\[
\begin{align*}
&\bullet \quad U_i \subseteq U_{i+1}, \quad \cup_{i \geq 1} U_i = B(0, R + 2) \setminus \partial \Omega; \\
&\bullet \quad E_i \cap U_i \in \mathcal{C}(E_i) \subseteq \mathcal{C}(E_0) \end{align*}
\]

\[
\begin{align*}
&\bullet \quad E_i \cap U_i \in \mathcal{C}(E_i) \subseteq \mathcal{C}(E_0) \end{align*}
\]

for constant \( M > 0 \);

\[
\mathcal{H}^2(E_i) \leq \mathcal{H}^2(S_i) + 2^{-i}.
\]

We assume that \( E_i \) converge locally to \( E \) in \( B(0, R + 2) \), pass to subsequence if necessary, then by Corollary 21.15 in [5], we get that \( E \) is sliding minimal.

We get, from Theorem 1.2 and Theorem 1.15 in [4], that \( E \) is a Lipschitz neighborhood retract. But we see that \( E_i \) converges to \( E \), we get so that \( E \) contains a competitor. \[\square\]
References


[4] _____, *C^{1+\alpha}-regularity for two-dimensional almost-minimal sets in $\mathbb{R}^n$*, J. Geom. Anal. 20 (2010), no. 4, 837–954. MR2683770


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