

Spherically symmetric potential in noncommutative spacetime with a compactified extra dimensions

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The Schrödinger equation of the spherical symmetry quantum models such as the hydrogen atom problem seems to be analytically non-solvable in higher dimensions. When we try to compactifying one or several dimensions this question can maybe solved. This paper stands for the study of the spherical symmetry quantum models on noncommutative spacetime with compactified extra dimensions. We provide analytically the resulting spectrum of the hydrogen atom and Yukawa problem in $4 + 1$ dimensional noncommutative spacetime in the first order approximation of noncommutative parameter. The case of higher dimensions $D \geq 4$ is also discussed.

Key words: Compactified extra dimensions, hydrogen atom, noncommutative quantum mechanics, eigenvalue problem.

I. INTRODUCTION

One of the recently discovered concepts that has impacted the theoretical physics community in a major way is most likely the idea of a noncommutative (NC) spacetime which led to a NC generalization of quantum mechanics and field theory. The idea of noncommutativity of spacetime was first discussed in the work by Snyder [1] and Connes [2]-[3]. The above concept (NC) spacetime allows to find possible solution to ultraviolet divergencies in quantum field theory [4]-[5]. The NC physics also arises as a possible scenario for the short-distance behaviour of physical theories (the Planck scale). At this scale, the universal constants c, \hbar and G appear naturally equivalent. Under the Planck length, the distance loses its meaning [4]-[6] and the physical phenomena are believed to be nonlocal. NC geometry could be realized by introducing the noncommutativity through the coordinates which satisfy the commutation relations $[x^\mu, x^\nu] = i\theta^{\mu\nu}$, where $\theta^{\mu\nu}$ is a skew-symmetric matrix characterizing the deformation of the spacetime. This leads to a new Heisenberg uncertainty relation, given on the spacetime coordinates by $\Delta x^\mu \Delta x^\nu \geq \theta^{\mu\nu}$, and makes this spacetime a quantum space [6]-[7]. The important implications of noncommutativity is the loss of Lorentz

invariant in the dispersion relations and the loss of causality [8]-[13]. Intuitive arguments involving quantum mechanics in NC space is realized by imposing the commutation relations, now between coordinates and momentums as

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \quad [p^\mu, p^\nu] = i\gamma^{\mu\nu}, \quad [x^\mu, p^\nu] = i\hbar\kappa^{\mu\nu} \quad (1)$$

where $\gamma^{\mu\nu}$ is also skew symmetric matrices. In this paper we restrict ourself to the case where $\gamma^{\mu\nu} = 0$, this implies that $\kappa^{\mu\nu} = \delta^{\mu\nu}$, the Kronecker symbole. We also assume that the tensor $\theta^{\mu\nu}$ is chosen to have the dimension of *length · time* i.e. $\theta^{0j} = \theta^j \in \mathbb{R}$, $\theta^{ij} = 0$, $i, j = 1, 2, \dots, D$. The noncommutative variables can be expressed in terms of commutative coordinates as $x^j = x_c^j - i\theta^j \partial_0 = (x)$, and $p^j = p_c^j$, $p^0 = i\hbar \partial_0 = E$, where the index “c” is used to specify the commutative variables and where E is the energy of the system. The Hamiltonian of quantum system on NC space can be expressed with the commutative coordinates $H(x, p) \equiv H_c(x_c, p_c, \theta)$, where the parameter $\theta = \theta^j$ is showed to have the fundamental limit $\theta \preceq 1.6 \cdot 10^{-27} m \cdot s \approx 0.3(keV)^{-2}$ which is smaller than the one obtained by the theory of quantum gravity [14]-[15].

The compactified extra dimension is motivated by string theory, which predicts the existence of extra dimensions and noncommutativity between coordinates. Our idea is to understand how the eigenvalue problem changes if we periodically identify one of the NC coordinates $x^j = (x^1, x^2, x^3, x^4)$ in the target space, say $[-\pi R, \pi R] \ni w$, such that $x^4 = w - 2\pi Rk$, $k \in \mathbb{Z}$

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and R is the radius of the circle. The wave function $\psi(x^0, x^{\bar{\ell}}, x^4)$, $\bar{\ell} = 1, 2, 3$, can be expanded in the Fourier mode as [20]

$$\psi(x^0, x^{\bar{\ell}}, x^4) = \frac{1}{\sqrt{2\pi R}} \sum_{n=-\infty}^{\infty} \psi_n(x^0, x^{\bar{\ell}}) \exp\left(\frac{in x^4}{R}\right). \quad (2)$$

Note that the orthonormalized functions $(2\pi R)^{-1/2} \exp(ix^4 n/R)$ are eigenfunctions of the operator $\nabla_{x^4}^2$ with eigenvalues $E_{n0} = -n^2/R^2$. This means that the spectrum of quantum system defined with one dimension compactify is in the form $E_{nl} = E_{n0} + E'_{nl}$, where l is a positive integer, E'_{nl} depends on the potential associated to the system which requires to be computed. In the following paper we investigate the spectrum of the Coulomb and Yukawa Hamiltonian on $(4 + 1)$ -dimensional NC spacetime. Using the first order approximation of the deformation parameter θ and by compactifying one extra dimension x^4 resulting topology $\mathbb{R}^{3+1} \times S^1$ (see [18] and [19] for the essential reviews), the spectrum may be given exactly. We prove that in the case of “space · time” noncommutativity, the correction of the energy spectrum does not depend on the NC deformation parameter θ but rather on the dimension compactified parameter.

Our paper is organized as follows. In section 2, we focus on the hydrogen atom in $(D + 1)$ -dimensional noncommutative space with non-compactified extra dimension. We discuss the particular case where $D = 4$ in which the solution of the spectral problem can be solved. The Yukawa potential is also discussed in this section. In section 3 the same problem is solved with now compactified extra dimensions. The discussion and conclusion are given in section 4.

II. HYDROGEN ATOM IN NONCOMMUTATIVE SPACE WITH NON-COMPACTIFIED EXTRA DIMENSION

In this section we focus on the hydrogen atom problem defined in $(D+1)$ dimensional NC spacetime (we consider the particular case where $D = 4$). To be specific, the model is given with the spherical potential of the form

$$V(\vec{r}_{nc}) = -\frac{q_e^2}{|\vec{r}_{nc}|^{D-2}}, \quad (3)$$

where q_e is related to the atomic charge and where we use the following notation $\vec{r}_{nc} = \vec{r} - i\theta\partial_0$, i.e., (the NC coordinates are $\vec{r}_{nc} = (x)$) and (the commutative coordinates are $\vec{r} = (x_c)$). It would be advisable to work in spherical coordinates system $\vec{r} = (r, \alpha_1, \alpha_2, \alpha_3)$ such that $r \in \mathbb{R}_+$, $0 < \alpha_{\bar{\ell}} < \pi$, $\bar{\ell} = 1, 2$ and $0 < \alpha_3 < 2\pi$. It thus follows that the Hamiltonian of the system is

$$H = -\frac{\hbar^2}{2m} \left[\frac{\partial^2}{\partial r^2} + \frac{D-1}{r} \frac{\partial}{\partial r} - \frac{\mathcal{L}^2(D-1)}{r^2} \right] + V(\vec{r}_{nc}), \quad (4)$$

where $\mathcal{L}^2(D-1)$ is the Laplace-Beltrami operator on the $(D-1)$ -sphere. Hence the potential (3), using the first

order Taylor expansion on $\vec{\theta}$ is

$$\begin{aligned} V(\vec{r}_{nc}) &= -\frac{q_e^2}{|\vec{r} - i\vec{\theta}\partial_0|^{D-2}} \\ &\approx -\frac{q_e^2}{r^{D-2}} \left(1 + i(D-2) \frac{\vec{r} \cdot \vec{\theta}}{r^2} \partial_0 \right). \end{aligned} \quad (5)$$

We consider the adequate choice, such that the vector $\vec{\theta}$ is transform in the spherical coordinates as $\vec{r} \cdot \vec{\theta} \equiv r\theta$ [14]. Furthermore, the spherical function $\mathcal{Y}_\ell^{(D-1)}(\alpha_1, \alpha_2, \dots, \alpha_{D-1})$, which are the eigenfunctions of the operator $\mathcal{L}(D-1)$ is considered:

$$\mathcal{L}^2(D-1) \mathcal{Y}_\ell^{(D-1)} = \ell(\ell + D - 2) \mathcal{Y}_\ell^{(D-1)} = \lambda_D \mathcal{Y}_\ell^{(D-1)}, \quad (6)$$

where ℓ is the orbital angular momentum quantum number.

Note that the Hamiltonian (4) depend on the partial derivative with respect to the time t , due to the relation (5). But one can show that, the wave function $\psi(\vec{r}, t)$, namely the solution of the Schrödinger equation is expressed as $\psi(\vec{r}, t) = \mathcal{Y}_\ell^{(D-1)} \psi(r) f(t)$, where the time dependent function is $f(t) = \exp(-\frac{i}{\hbar} E t)$ and where $\psi(r)$ satisfied the radial equation

$$\left[\frac{d^2}{dr^2} + \frac{D-1}{r} \frac{d}{dr} - \left(\frac{\lambda_D}{r^2} - \frac{\nu^2}{r^{D-2}} - \frac{\mu_D}{r^{D-1}} - \alpha^2 \right) \right] \psi(r) = 0, \quad (7)$$

with $\alpha^2 = \frac{2mE}{\hbar^2}$, $\nu^2 = \frac{2mq_e^2}{\hbar^2}$, $\mu_D = (D-2)\nu^2\theta E/\hbar$. Now for $D = 4$, this equation turns to be:

$$\left[\frac{d^2}{dr^2} + \frac{3}{r} \frac{d}{dr} - \left(\frac{\lambda_4 - \nu^2}{r^2} - \frac{\mu_4}{r^3} - \alpha^2 \right) \right] \psi(r) = 0. \quad (8)$$

However in this case, (unlike for the commutative case discussed in [18]-[19]), such equation seems to be non-solvable. We provide an algebraic method, which will allow us to derive the solution of this equation. For this, let us reparameterized the function $\psi(r)$ as $\psi(r) := \psi(r, \theta)$. Then $\psi(r, 0)$ corresponds to the solution of the equation (8) in the case where $\theta = 0$. The first order Taylor expansion on θ of the function $\psi(r, \theta)$ takes the form

$$\psi(r, \theta) = \psi(r, 0) + \theta \frac{d\psi(r, \theta)}{d\theta} \Big|_{\theta=0} + \mathcal{O}(\theta^2). \quad (9)$$

We get simply

$$\psi(r, 0) = \frac{c}{r} J_\nu(\alpha r) + \frac{c'}{r} Y_\nu(\alpha r), \quad c, c' \in \mathbb{R}, \quad (10)$$

where $J_\nu(\alpha r)$ and $Y_\nu(\alpha r)$ are respectively the first and second kind Bessel functions (see [19] for more detail). By replacing the solution (9) in the partial differential equation (8), we get

$$\left[\frac{d^2}{dr^2} + \frac{3}{r} \frac{d}{dr} + \left(\alpha^2 - \frac{\lambda_4 - \nu^2}{r^2} \right) \right] \tilde{\chi}(r) = -\frac{2\nu^2 E}{\hbar r^3} \psi(r, 0), \quad (11)$$

where $\tilde{\chi}(r) = \frac{d\psi(r, \theta)}{d\theta} \Big|_{\theta=0}$. This equation corresponds to a nonhomogeneous differential equation, which can

be solved easily. For $\epsilon = (1 + \lambda_4 - \nu^2)^{1/2}$ and $g(r) = -2\nu^2 E\psi(r, 0)/\hbar r^3$, by using the Wronskian method, the solution of the equation (11) takes the form

$$\begin{aligned} \tilde{\chi}(r) &= cJ_\epsilon(\alpha r) + c'Y_\epsilon(\alpha r) \\ &- \frac{\pi}{2r} J_\epsilon(\alpha r) \int_1^r x^2 Y_\epsilon(\alpha x) g(x) dx \\ &+ \frac{\pi}{2r} Y_\epsilon(\alpha r) \int_1^r x^2 J_\epsilon(\alpha x) g(x) dx, \quad c, c' \in \mathbb{R}. \end{aligned} \quad (12)$$

Remark that there are some difficulties however. One defect of this method (in the commutative and NC case) is that the energy spectrum can only be determined numerically, and we do not deal here with a numerical method to provide this spectrum. In more than $(4 + 1)$ dimensions, the differential equations (7) are much more complicated to be solve.

Now let us discuss the case of Yukawa potential :

$$V(\vec{r}_{nc}) = -V_0 \frac{e^{-\eta r_{nc}}}{|\vec{r}_{nc}|^{D-2}}, \quad (13)$$

where V_0 and η depend on the constant of the neutral atom. In order to probe this potential, we write the expression (13) at the first order on θ as

$$V(\vec{r}_{nc}, t) = -\frac{V_0 e^{-\eta r}}{r^{D-2}} \left[1 + i(\eta r + D - 2) \frac{\theta}{r} \partial_0 \right]. \quad (14)$$

After separation variables in the Schrödinger equation, it become easy to show that the radial equation is given by the following:

$$\left[\frac{\partial^2}{\partial r^2} + \frac{D-1}{r} \frac{\partial}{\partial r} + s_D(r) \right] \psi(r) = 0. \quad (15)$$

where

$$\begin{aligned} s_D(r) &= \frac{2mV_0 e^{-\eta r}}{\hbar^3 r^{D-2}} \left(D - 2 + \eta r \right) \frac{E\theta}{r} \\ &+ \frac{2mE}{\hbar^2} - \frac{\lambda_D}{r^2} + \frac{2mV_0 e^{-\eta r}}{\hbar^2 r^{D-2}}, \end{aligned}$$

In the particular case where $D = 4$, this equation is reduced to

$$\frac{d^2 \psi(r)}{dr^2} + \frac{3}{r} \frac{d\psi(r)}{dr} + s_4(r) \psi(r) = 0. \quad (16)$$

This equation, (including now the occurrence of the exponential factor $e^{-\eta r}$), has the same shape as (8), and therefore the same conclusion with (12) will be made.

III. HYDROGEN ATOM IN NONCOMMUTATIVE SPACE WITH COMPACTIFIED EXTRA DIMENSION

In this section we consider $(D + 1)$ NC spacetime , where one dimension x^D is compactified on a circle of radius R . This means that \mathbb{R}^{D+1} is reduced to $\mathbb{R}^{D-1+1} \times [-\pi R, \pi R]$ and $x^D = \omega - 2\pi nR$, $n \in \mathbb{Z}$. The interaction potential (5) written now with the required coordinates

$\vec{r} = (r, \alpha_1, \dots, \alpha_{D-2})$ and the compactify coordinate w is

$$\begin{aligned} V(\vec{r}, w) &= -q_e^2 \sum_{n=-\infty}^{\infty} \left\{ \frac{1}{(r^2 + (w - 2\pi nR)^2)^{\frac{D-2}{2}}} \right. \\ &\left. + \frac{i(D-2)r\theta}{(r^2 + (w - 2\pi nR)^2)^{\frac{D}{2}}} \partial_0 \right\}, \end{aligned} \quad (17)$$

where r is now the radial coordinates in $(D - 1)$ -dimensional space, and the extra dimension x^D satisfy the condition $|x^D - 2\pi nR| \leq \pi R$. For $D = 4$, we get

$$\begin{aligned} V(\vec{r}, w) &= -q_e^2 \sum_{n \in \mathbb{Z}} \left\{ \frac{1}{r^2 + (w - 2\pi nR)^2} + \right. \\ &\left. \frac{2ir\theta}{(r^2 + (w - 2\pi nR)^2)^2} \partial_0 \right\}. \end{aligned} \quad (18)$$

Then we can compute the followings identities:

$$\sum_{n \in \mathbb{Z}} \frac{1}{r^2 + (w - 2\pi nR)^2} = \frac{1}{2Rr} \frac{\sinh(r/R)}{\cosh(r/R) - \cos(w/R)} \quad (19)$$

and

$$\sum_{n \in \mathbb{Z}} \frac{2r}{(r^2 + (w - 2\pi nR)^2)^2} = G(r) + F(r) \quad (20)$$

where

$$G(r) = -\frac{1}{2Rr^2} \frac{\sinh(r/R)}{\cosh(r/R) - \cos(w/R)} \quad (21)$$

$$F(r) = \frac{1}{2R^2 r} \frac{1 - \cosh(r/R) \cos(w/R)}{(\cosh(r/R) - \cos(w/R))^2}. \quad (22)$$

The potential $V(\vec{r}, w)$ is periodic with respect to the w -direction, and it can be expanded to a Fourier series as

$$V(\vec{r}, w) = \sum_{n \in \mathbb{Z}} a_n(r) e^{inw/R} + i\theta \sum_{n \in \mathbb{Z}} b_n(r) e^{inw/R} \partial_0, \quad (23)$$

where

$$\begin{aligned} a_n(r) &= -\frac{q_e^2}{2rR} e^{-|n|r/R}, \\ b_n(r) &= -\frac{q_e^2}{2Rr^2} [1 + |n|] e^{-|n|r/R}, \end{aligned} \quad (24)$$

and such that

$$\begin{aligned} V(\vec{r}, w) &= \\ &= -\frac{q_e^2}{2rR} \sum_{n \in \mathbb{Z}} \left[1 + i\theta \left(\frac{1}{r} + \frac{|n|}{R} \right) \partial_0 \right] e^{-(|n|r - inw)/R}. \end{aligned} \quad (25)$$

The separation of the variables in the Schrödinger equation shows that the radial function $\psi(r)$ satisfies

$$\left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{n^2}{R^2} + \alpha^2 + \frac{\zeta}{r} - \frac{\nu_\theta^2}{r^2} \right) \psi_n(r) = 0, \quad (26)$$

with $\alpha^2 = \frac{2mE}{\hbar^2}$, $\nu_\theta^2 = \lambda_4 - \zeta E\theta/\hbar$, $\zeta = \nu^2/(2R)$, $\nu^2 = \frac{2mq_e^2}{\hbar^2}$.

The solution of the above equation is expressed as

$$\psi_n(r) = \left[\frac{(2\zeta)^{a+2} l!}{(2l+1+a)^{a+3} \Gamma(l+1+a)} \right]^{\frac{1}{2}} r^{\frac{1}{2}(a-1)} e^{-\frac{\zeta r}{2l+a+1}} \times L_l^a \left(\frac{2\zeta r}{2l+a+1} \right), \quad a = \sqrt{4\nu_\theta^2 + 1}, \quad (27)$$

where we have used the normalisation condition $\int_0^\infty e^{-z} z^{a+1} [L_l^a(z)]^2 dz = \frac{(2l+1+a)\Gamma(l+1+a)}{l!}$, and where L_l^a stands for the generalized Laguerre polynomial. The quantum number l is a positive integer, which correspond to the physical situation. This integer is given by

$$l = -\frac{1}{2} - \frac{1}{2} \sqrt{4\nu_\theta^2 + 1} + \frac{\zeta R}{2\sqrt{n^2 - \alpha^2 R^2}}. \quad (28)$$

Two energies contribution appear from the relation (28).

$$E_{nl}^{(1)} = \frac{\hbar^2}{2m} \left(\frac{n^2}{R^2} - \frac{\zeta^2}{(2l+1+\sqrt{1+4\lambda_4})^2} \right) \quad (29)$$

and

$$E_{nl}^{(2)} = \frac{\hbar^2}{2m} \frac{\zeta^2}{(2l+1+\sqrt{1+4\lambda_4})^2} + \frac{\hbar\sqrt{1+4\lambda_4}}{4\theta\zeta} (2l+1+\sqrt{1+4\lambda_4}). \quad (30)$$

Let us discuss the energy spectrum (30). In the limit where $\theta \rightarrow 0$, $E_{nl}^{(2)}$ is not well defined. Also as expected in our introduction the eigenfunctions of the operator $\nabla_{x^4}^2$ with eigenvalues $E_{n0} = -n^2/R^2$ is not recovered. Finally this expression can not be taking into account as solution of eigenvalues problem. Then the energy spectrum becomes

$$E_{nl} = \frac{\hbar^2}{2m} \left(\frac{n^2}{R^2} - \frac{\zeta^2}{(2l+1+\sqrt{1+4\lambda_4})^2} \right). \quad (31)$$

Remark 1. • Our result shows that the energy spectrum (31) does not depend on the NC parameter θ if we consider the first order approximation of this parameter. The solution of eigenvalue problem of the hydrogen atom with compactified one dimension is solved numerically in [18]-[19] (see also [16]-[17] in the case where no dimensions are compactified). Due to the fact that $\lim_{\theta \rightarrow 0} E_{nl} = E_{nl}$, the expression (31) can be condered as the solution of Hydrogen atom in 4 + 1-dimensional spacetime for both NC[21] and commutative case, where one dimension is compactified.

- The quantity

$$E_{nl}' = -\frac{\hbar^2 \zeta^2}{2m(2l+1+\sqrt{1+4\lambda_4})^2}. \quad (32)$$

correspond to the reduce dimension energy spectrum and is discussed in the introduction of our paper.

We consider now the case of Yukawa potential (13) for $D = 4$. On shell, and compactified the x^4 direction on the circle we get the reduce potential

$$V(\vec{r}, w) = iV_0\theta \left[-\frac{1}{2Rr^2} \frac{\sinh(r/R)}{\cosh(r/R) - \cos(w/R)} + \frac{1}{2R^2 r} \frac{1 - \cosh(r/R) \cos(w/R)}{(\cosh(r/R) - \cos(w/R))^2} \right] \partial_0 - \frac{V_0}{2Rr} \frac{\sinh(r/R)}{\cosh(r/R) - \cos(w/R)}. \quad (33)$$

Let us briefly give the proof of this relation. The goal of this prove is to compute the integral $\int_\Gamma f(z) dz$, where Γ is a closed contour on the complex plane and $f(z)$ is a holomorphic function given by

$$f(z) = \frac{\cot(\pi z) \exp[-c((a-z)^2 + b^2)^{1/2}]}{(a-z)^2 + b^2}, \quad (34)$$

where a, b, c are three real numbers. The pole of $f(z)$ are $z_n = n, n \in \mathbb{Z}$, $z_I = a + ib$ and $z_{II} = a - ib$. Using the residue theorem:

$$\int_\Gamma f(z) dz = 2i\pi \sum \text{Res}[f(z)] = 0, \quad (35)$$

with

$$\begin{aligned} \text{Res}_{z_n}[f(z)] &= \frac{\exp[-c((a-n)^2 + b^2)^{1/2}]}{\pi[(a-n)^2 + b^2]}, \\ \text{Res}_{z_I}[f(z)] &= -\frac{i}{2b} \cot[\pi(a+ib)], \quad \text{and} \\ \text{Res}_{z_{II}}[f(z)] &= \frac{i}{2b} \cot[\pi(a-ib)] \end{aligned} \quad (36)$$

Hence,

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \frac{\exp[-c((a-n)^2 + b^2)^{1/2}]}{(a-n)^2 + b^2} \\ = \frac{\pi}{b} \frac{\sinh(2\pi b)}{\cosh(2\pi b) - \cos(2\pi a)}. \end{aligned} \quad (37)$$

The first term on the right hand side of (33) is

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \frac{\exp[-\zeta(r^2 + (w - 2\pi n R)^2)^{1/2}]}{r^2 + (w - 2\pi n R)^2} \\ = \frac{1}{2Rr} \frac{\sinh(r/R)}{\cosh(r/R) - \cos(w/R)}. \end{aligned} \quad (38)$$

The second term on the right hand side of expression (33) is the first order derivative of equation (38) respect to r . Then the relation (33) is straightforward obtained.

Now using the fact that the fonction $V(\vec{r}, w)$ is periodic, the Fourier serie can be given by

$$V(\vec{r}, w) = \sum_{n \in \mathbb{Z}} \left(a_n(r) + i\theta b_n(r) \partial_0 \right) e^{inw/R}, \quad (39)$$

where the Fourier coefficients are:

$$\begin{aligned} a_n(r) &= -\frac{V_0}{2rR} e^{-|n|r/R}, \\ b_n(r) &= -V_0 \left[\frac{1}{2Rr^2} + \frac{|n|}{2R^2r} \right] e^{-|n|r/R} \end{aligned} \quad (40)$$

Finally we come to

$$\begin{aligned} V(\vec{r}, w) &= -\frac{V_0}{2rR} \sum_{n \in \mathbb{Z}} \left[1 + i\theta \left(\frac{1}{r} + \frac{|n|}{R} \right) \partial_0 \right] \\ &\times e^{-(|n|r - inw)/R}. \end{aligned} \quad (41)$$

the relation (33) can also be expand using the Fourier serie as

$$\begin{aligned} V(\vec{r}, w) &= -\frac{V_0}{2rR} \sum_{n \in \mathbb{Z}} \left[1 + i\theta \left(\frac{1}{r} + \frac{|n|}{R} \right) \partial_0 \right] \\ &\times e^{-(|n|r - inw)/R} \end{aligned} \quad (42)$$

and the radial equation takes the form

$$\left[\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{n^2}{R^2} + \alpha^2 + \frac{u}{r} - \frac{v}{r^2} \right] \psi_n(r) = 0, \quad (43)$$

where $\alpha^2 = \frac{2mE}{\hbar^2}$; $v = \lambda_4 - \frac{mV_0 E \theta}{\hbar^3 R}$, $u = \frac{mV_0}{\hbar^2 R}$. The solution of this equation leads to the same results given in (27) and (31).

IV. DISCUSSION AND CONCLUSION

In this paper we have found that the noncommutativity of spacetime can help to compute the exact expression of the energy spectrum of the hydrogen atom in $(4+1)$ dimensions with compactified one extra dimension. Unfortunately, it's clear that this method can not be used in higher dimensions. To be more precise, let us consider the particular case where $D = 6$, the compactified one dimension x^6 gives the potential

$$\begin{aligned} V(\vec{r}, w) &= -\frac{q_e^2}{(2Rr)^2} \left(\frac{R}{r} \frac{\sinh(r/R)}{\cosh(r/R) \cos(w/R) - 1} \right. \\ &+ \left. \frac{\cosh(r/R) \cos(w/R) - 1}{[\cosh(r/R) - \cos(w/R)]^2} \right) \\ &+ \frac{i\theta q_e^2}{8r^4 R^3 (\cos(w/R) - \cosh(r/R))^3} \\ &\times \left[(r^2 + 3R^2) \sinh(r/R) \cos(2w/R) \right. \\ &+ \left. \cos(w/R) \left((r^2 - 6R^2) \sinh(2r/R) \right. \right. \\ &+ \left. \left. 3rR(\cosh(2r/R) + 3) \right) \right. \\ &+ \left. 3 \sinh(r/R) \left(R^2 \cosh(2r/R) + 2R^2 - r^2 \right) \right. \\ &- \left. \left. 3rR \cosh(r/R) \left(\cos(2w/R) + 3 \right) \right] \partial_0. \end{aligned} \quad (44)$$

This relation can be expanded in a Fourier series as

$$V(\vec{r}, w) = -\frac{q_e^2}{(2Rr)^2} \sum_{n \in \mathbb{Z}} \left[\left(\frac{R}{r} + |n| \right) \right.$$

$$\left. + i\theta \left(\frac{3R}{r^2} + \frac{3|n|}{r} + \frac{n^2}{R} \right) \partial_0 \right] e^{-|n|r/R} e^{inw/R}, \quad (45)$$

in which the radial part of the Schrödinger equation become

$$\begin{aligned} \left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{\lambda_4}{r^2} - \frac{n^2}{R^2} + \alpha^2 + \frac{2mq_e^2}{4\hbar^2 r^3 R} + \frac{3mq_e^2 E \theta}{2\hbar^3 r^4 R} \right) \psi_n(r) \\ = 0. \end{aligned} \quad (46)$$

The solution of this equation is not yet understood. Surprisingly, we have also show that, despite from this non-commutativity, the energy spectrum do not depend on the deformation parameter θ and therefore might be considered as the energy solution of commutative space, with compactified one extra dimension, solved in [18] and [19]. Finally let us mention that in the case of higher dimensions more than $4+1$ the compactified several extra dimensions may be considered.

Let us examined the case of the Klein-Gordon (KG) equation:

$$\begin{aligned} \left(i\hbar \partial_0 - V(\vec{r}_{nc}) \right)^2 \psi(\vec{r}_{nc}, t) \\ = m^2 c^4 \psi(\vec{r}_{nc}, t) - \hbar^2 c^2 \Delta \psi(\vec{r}_{nc}, t), \end{aligned} \quad (47)$$

where the potential $V(\vec{r}_{nc})$ is

$$V(\vec{r}_{nc}) = -\wp \frac{\hbar c}{|\vec{r}_{nc}|^{D-2}}. \quad (48)$$

\wp is related to the fine structure constant and c is the light speed. For $D = 4$, by taking into account the fact that $\vec{r}_{nc} = \vec{r} - i\theta \partial_0$, and writing the extra-dimension $x^4 = w - 2\pi Rk$, $k \in \mathbb{Z}$, we get

$$\begin{aligned} V(r, w) &= -\wp \hbar c \sum_{n=-\infty}^{\infty} \left\{ \frac{1}{r^2 + (w - 2\pi nR)^2} \right. \\ &+ \left. \frac{2ir\theta}{(r^2 + (w - 2\pi nR)^2)^2} \partial_0 \right\}. \end{aligned} \quad (49)$$

The left hand side of the equation (47), using (49) gives

$$\begin{aligned} \left(i\hbar \frac{\partial}{\partial t} - V(r, w) \right)^2 \psi(\vec{r}, t) = \\ \left[-\hbar^2 \partial_0^2 + \wp \hbar^2 c \left(2iA(r, w) \partial_0 + 3\theta \frac{\partial A(r, w)}{\partial r} \partial_0^2 \right) \right. \\ \left. + \wp^2 \hbar^2 c^2 \left(A^2(r, w) - i\theta \frac{\partial A^2(r, w)}{\partial r} \partial_0 \right) \right] \psi(\vec{r}, t) \end{aligned} \quad (50)$$

where $A(r, w)$ is written as

$$A(r, w) = \frac{1}{2Rr} \frac{\sinh(r/R)}{\cosh(r/R) - \cos(w/R)}, \quad (51)$$

with the Fourier serie

$$A(r, w) = \frac{1}{2rR} \sum_{n \in \mathbb{Z}} e^{inw/R} e^{inw/R}. \quad (52)$$

After some technical handling, we can show that the quantity $A^2(r, w)$ is expanded as

$$A^2(r, w) = \frac{1}{(2Rr)^2} \sum_{n \in \mathbb{Z}} \left[|n| + \coth \left(\frac{r}{R} \right) \right] e^{-\frac{|n|r}{R}} e^{\frac{inw}{R}}. \quad (53)$$

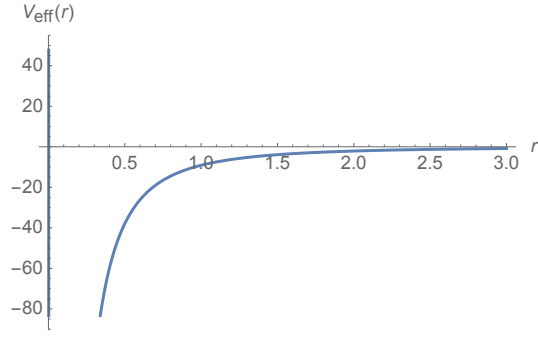


FIG. 1: Plot of the potential $V_{\text{eff}}(r)$, with $R = 0.01$, $\theta = 0.01$, $\hbar = c = 1$, $\wp = 1/137$, $E = 1$, $\ell = 1$.

By replacing (52) and (53) in (50) and separating variables as $\psi(\vec{r}, t) = \Psi(\vec{r})e^{-iEt/\hbar}$ and $\Psi(\vec{r}) = \Psi(r)\mathcal{Y}_\ell^{(3)}$ such that $\hat{L}^2\mathcal{Y}_\ell^{(3)} = \ell(\ell + 2)\mathcal{Y}_\ell^{(3)} = \lambda_4\mathcal{Y}_\ell^{(3)}$ we find the radial equation:

$$\left[\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + V_{\text{eff}}(r) + \frac{E^2}{\hbar^2 c^2} - \frac{m^2 c^2}{\hbar^2} - \frac{n^2}{R^2} \right] \Psi_n(r) = 0, \quad (54)$$

where the effective potential $V_{\text{eff}}(r)$ is

$$V_{\text{eff}}(r) = -\frac{\lambda_4}{r^2} + \frac{3\theta\wp E^2}{2r^2 R \hbar^2 c} + \frac{\wp E}{r R \hbar c} + \frac{\wp^2}{(2Rr)^2} \coth(r/R) + \frac{\theta\wp^2 E}{(2Rr)^2 \hbar} \left(\frac{2}{r} \coth(r/R) + \frac{r}{\sinh^2(r/R)} \right) \quad (55)$$

The equation (54) can be solved numerically using the approximation method. Consider the Taylor expansion of V_{eff} by using the fact that:

$$\begin{aligned} \coth(r/R) &= \frac{R}{r} + \frac{r}{3R} + \dots \\ \sinh(r/R) &= \frac{r}{R} + \dots \end{aligned} \quad (56)$$

Then (54) becomes

$$\frac{d^2 \Psi_n(r)}{dr^2} + \frac{2}{r} \frac{d\Psi_n(r)}{dr} + \sum_{j=0}^4 \frac{b_j}{r^j} \Psi_n(r) = 0. \quad (57)$$

with $b_4 = \frac{\theta\wp^2 E}{2R\hbar}$, $b_3 = \frac{\theta\wp^2 E}{4\hbar} + \frac{\wp^2}{4R}$, $b_2 = \frac{\theta\wp^2 E}{6R^3\hbar} + \frac{3\theta\wp E^2}{2R\hbar^2 c} - \lambda_4$, $b_1 = \frac{\wp^2}{12R^3} + \frac{\wp E}{R\hbar c}$, $b_0 = \frac{E^2}{\hbar^2 c^2} - \frac{m^2 c^2}{\hbar^2} - \frac{n^2}{R^2}$.

We first examined the wave functions $\Psi_n(r)$ in the asymptotic range $r \rightarrow \infty$. The potential $V_{\text{eff}}(r)$ vanish, in this limit, i.e.

$$V_{\text{eff}}(r_\infty) \rightarrow 0. \quad (58)$$

In the region r_∞ the equation (57) gives the solution of the form

$$\Psi_n^\infty(r) = U \frac{e^{-\sqrt{-b_0}r}}{r}, \quad U \in \mathbb{R}. \quad (59)$$

The general solution of the equation (57) takes the form

$$\Psi_n(r) = U(r) \frac{e^{-\sqrt{-b_0}r}}{r}, \quad (60)$$

where $U(r)$ satisfy the differential equation

$$U''(r) - 2\sqrt{-b_0}U'(r) + \sum_{j=1}^4 \frac{b_j}{r^j} U(r) = 0 \quad (61)$$

The investigation of the numerical solution of this equation can be made in forthcoming work.

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