

Holographic two-point functions for Janus interfaces in the $D1/D5$ CFT

Marco Chiodaroli,^a John Estes^b and Yegor Korovin^c

^a*Department of Physics and Astronomy, Uppsala University,
SE-75108 Uppsala, Sweden*

^b*Department of Physics, Long Island University,
1 University Plaza, Brooklyn, NY 11201, U.S.A.*

^c*Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut,
Am Mühlenberg 1, 14476 Golm, Germany*

E-mail: marco.chiodaroli@physics.uu.se, John.Estes@liu.edu,
jegor.korovins@aei.mpg.de

ABSTRACT: This paper investigates scalar perturbations in the top-down supersymmetric Janus solutions dual to conformal interfaces in the $D1/D5$ CFT, finding analytic closed-form solutions. We obtain an explicit representation of the bulk-to-bulk propagator and extract the two-point correlation function of the dual operator with itself, whose form is not fixed by symmetry alone. We give an expression involving the sum of conformal blocks associated with the bulk-defect operator product expansion and briefly discuss finite-temperature extensions. To our knowledge, this is the first computation of a two-point function which is not completely determined by symmetry for a fully-backreacted, top-down holographic defect.

KEYWORDS: AdS-CFT Correspondence, Conformal Field Models in String Theory

ARXIV EPRINT: [1612.08916](https://arxiv.org/abs/1612.08916)

Contents

1	Introduction	1
2	Basic facts about defect CFTs	3
3	Supergravity solution	5
4	Solving the massless Klein-Gordon equation	10
5	Bulk-to-bulk propagator and two-point correlation function	13
5.1	Bulk-to-bulk propagator	13
5.2	Bulk-to-boundary propagator	15
5.3	Correlation functions	16
6	Discussion	19
A	Bulk-to-boundary propagator in pure anti-de Sitter	21

1 Introduction

The structure of conformal field theories (CFTs) can be enriched by introducing boundary conditions which break part of the conformal symmetry. A simple way to do so is to add a codimension-one planar boundary or interface which preserves a $SO(d, 1)$ subgroup of the $SO(d + 1, 1)$ conformal group. Such extended objects are ubiquitous in string theory and condensed matter physics and are referred to as conformal defects.

In the presence of defects, conformal symmetry poses less stringent constraints on the structure of the correlation functions. The two-point function is no longer completely fixed by symmetry. Rather, it is an arbitrary function of an invariant cross-ratio. The implications of conformal invariance in the presence of defects have been studied extensively [1–4], most recently in the context of quantum information theory, where a proof of the g-theorem for boundary CFTs in $1 + 1$ dimensions has been obtained together with its higher-dimensional analogs [5–7].

In the context of the AdS_{d+1}/CFT_d correspondence, the simplest holographic realizations of defects are obtained by adding AdS_d probe branes into the AdS_{d+1} bulk geometry [8, 9] or by introducing an additional boundary in the bulk supergravity solution [10–12]. In such setups, one can successfully model many features of condensed matter systems such as the Kondo model [13–17].

Obtaining backreacted analytic solutions dual to defects is a considerably more involved enterprise. A non-supersymmetric “Janus” solution dual to an interface in a four-dimensional bulk CFT was first constructed in ref. [18]. This solution has a geometry

obtained by foliating the spacetime with AdS_4 slices and gives a non-singular dilatonic deformation of the $\text{AdS}_5 \times S^5$ vacuum of type IIB supergravity in which the dilaton smoothly interpolates between two different asymptotic values (the two “faces” of Janus). It is interpreted as the holographic dual of two four-dimensional CFTs with different coupling constants which are glued together along a codimension-one interface [19, 20].

Over the years, the Janus solution has produced a variegated offspring. Supersymmetric extensions that possess asymptotic regions where the geometry is locally $\text{AdS}_5 \times S^5$ were first obtained in refs. [21–23] and further studied in [24–26]. The corresponding solutions in $\mathcal{N} = 2$ gauged supergravity in five dimensions and their embedding in $\mathcal{N} = 8$, five-dimensional gauged supergravity were discussed in [27, 28]. Gaiotto and Witten subsequently classified the intersecting-brane configurations corresponding to the supersymmetric Janus solutions and extended the solutions to nonzero theta angle [29, 30]. Later work also uncovered Janus solutions in M-theory [31–33] and attained a classification relying on superconformal algebras [34]. Non-supersymmetric solutions with asymptotic locally- $\text{AdS}_3 \times S^3$ geometries were constructed in ref. [35]. The supersymmetric $\text{AdS}_3 \times S^3$ Janus solutions which we will utilize in this paper were first obtained in ref. [36] and further studied in refs. [37–39] together with large classes of string-junction solutions which possess more than two asymptotic regions. The solutions obtained in [36] have geometries given by a $\text{AdS}_2 \times S^2 \times M_4$ fibration over a two-dimensional base space, where M_4 is either the four-torus or a $K3$ manifold, and, in the simplest cases, are holographic duals to interfaces in the $D1/D5$ CFT. Analogous solutions in Roman’s type 4b six-dimensional supergravity were constructed in refs. [40–42]. Finally, the literature has discussed various finite-temperature versions of Janus solutions either numerically [43] or analytically [44].

The advantage of top-down constructions is that often the dual (deformed) field theory is known explicitly. The drawback is that these supergravity solutions are considerably more complicated than their artificial bottom-up relatives. In particular, most holographic computations conducted to date with top-down solutions have focused on boundary or entanglement entropy [35, 37, 45–48], for which there exist a simple holographic prescription due to Ryu and Takayanagi [49] (see also [50–52] for non-supersymmetric examples). It seems particularly difficult to extract any correlation function whose form is not completely fixed by the residual defect conformal symmetry.

In this paper, we take a concrete step towards the computation of more general holographic observables in Janus backgrounds, and study one particular family of top-down defects for which we are able to perform explicitly a number of analytic computations. We identify a particular field which decouples from other fields, solve the linearized field equations, construct the bulk-to-bulk propagator and obtain the two-point correlator, which a priori is an arbitrary function of the conformal crossratio. To our knowledge, this is the first computation of this kind for a fully-backreacted, top-down holographic defect.¹

One interesting feature of our holographic calculation is that the final result for the two-point function is explicitly expressed in terms of the conformal blocks associated with

¹Two-point functions of the stress-energy tensor with scalar operators were computed in [19], taking an AdS background and considering linear fluctuations in the presence of a source.

the boundary operator product expansion (BOPE). As we review in the next section, there are two kinds of operator product expansions (OPEs) in the presence of a defect. One is the usual OPE, which can be used when two operators are close to each other but stay away from the defect. The second kind of OPE, the BOPE, is relevant when an operator is brought close to the defect but is separated from the other insertions. It is the latter BOPE which is made explicit in our holographic computation. Of course, writing a two-point function as a sum of conformal blocks is not automatically consistent. As a non-trivial check of our final result, we have partially resummed the BOPE conformal blocks and reproduced the identity contribution in the crossed channel, i.e. the identity contribution to the ordinary OPE.

This paper is organized as follows. In the next section, we collect some relevant facts about defect CFTs. In section 3, we discuss the background supergravity solution and scalar perturbations. In section 4, we solve the linearized field equations for the decoupled scalar field. Finally, in section 5, we construct the bulk-to-bulk propagator from the linear modes and compute the two-point correlation function of the dual operator.

2 Basic facts about defect CFTs

Here we review the basic facts about codimension-one defects. For a more detailed field-theory discussion we refer to [2, 3].

For definiteness, we consider an Euclidean CFT on \mathbb{R}^d . A conformal defect of codimension one preserves $\text{SO}(d, 1)$ defect conformal symmetry. The coordinates along the defect are labelled as x^a with $a = 1, \dots, d-1$, whereas those in the bulk are labelled as $x^\mu = (x^a, x_\perp)$.

In the presence of a defect, one distinguishes between bulk and defect excitations. The defect does not affect the fusion of primary operators away from the defect (the so-called bulk operators). Thus we have the usual bulk OPE channel

$$\mathcal{O}(x_1)\mathcal{O}(x_2) = \sum_k \frac{c_{12k}}{(x_{12}^2)^{(\Delta_1+\Delta_2-\Delta_k+J)/2}} C^{(J)}(x_{12}, \partial_2)_{\mu_1 \dots \mu_J} \mathcal{O}_k^{\mu_1 \dots \mu_J}(x_2), \quad (2.1)$$

where $C^{(J)}$ are some known differential operators.

Any bulk operator \mathcal{O} can be restricted to a defect. In general, the defect possesses its own local excitations as well. We will refer to both kinds of operators as the defect operators $\hat{\mathcal{O}}_i$. When a bulk operator is brought close to the defect, it becomes indistinguishable from a defect operator. This is captured by the bulk-to-defect OPE (also referred to as BOPE),

$$\mathcal{O}(x) = \sum_{\hat{\mathcal{O}}} \frac{b_{\mathcal{O}\hat{\mathcal{O}}}}{|x_\perp|^{\Delta-\hat{\Delta}}} C_{\hat{\mathcal{O}}}(|x_\perp|^2 \partial_\parallel^2) \hat{\mathcal{O}}(x^a), \quad (2.2)$$

with some differential operators $C_{\hat{\mathcal{O}}}$. Here ∂_\parallel is the derivative operator along the defect (in the x^a directions).

A trivial example of a BOPE is obtained in the case in which there is no actual defect. In this case, (2.2) reduces essentially to a Taylor expansion.²

²With a caveat that the operators $\hat{\mathcal{O}}_n = \partial_{x_\perp}^n \mathcal{O}(x)|_{x_\perp=0}$ have to be diagonalized by adding terms of the form $c_i \partial_{x_\perp}^{n-2i} (\partial_a \partial^a)^i \mathcal{O}(x)|_{x_\perp=0}$. See [53] for a detailed discussion.

In the presence of a conformal defect, the one-point correlation function of a bulk operator and the two-point correlator of a bulk and a defect operator are fixed by the unbroken defect conformal symmetry to be

$$\langle \mathcal{O}(x) \rangle = \frac{a_{\mathcal{O}}}{|x_{\perp}|^{\Delta}}, \quad \langle \mathcal{O}(x) \hat{\mathcal{O}}(0) \rangle = \frac{b_{\mathcal{O}\hat{\mathcal{O}}}|x_{\perp}|^{\hat{\Delta}-\Delta}}{|x^{\mu}|^{2\hat{\Delta}}}, \quad (2.3)$$

with some constants $a_{\mathcal{O}}$ and $b_{\mathcal{O}\hat{\mathcal{O}}}$, which in principle can be different on the two sides of the defect.

The two-point function of bulk operators is not fixed by symmetry alone and in general depends on one invariant crossratio ξ ,

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle = \frac{1}{|x_{1\perp}|^{\Delta_1} |x_{2\perp}|^{\Delta_2}} f(\xi), \quad \xi = \frac{(x_1^a - x_2^a)^2 + (x_{1\perp} - x_{2\perp})^2}{4x_{1\perp}x_{2\perp}}. \quad (2.4)$$

There are two interesting limits to consider. First, when $\xi \rightarrow 0$, the two insertions approach each other but stay away from the defect. This limit is governed by the bulk OPE of the CFT without the defect. We will focus on another limit, $\xi \rightarrow \infty$, i.e. when both insertions are close to the defect but remain apart from each other. This limit is governed by the BOPE:

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle \sim \sum_{\hat{\mathcal{O}}} \frac{b_{\mathcal{O}_1\hat{\mathcal{O}}} b_{\mathcal{O}_2\hat{\mathcal{O}}}}{|x_{1\perp}|^{\Delta_1-\hat{\Delta}} |x_{2\perp}|^{\Delta_2-\hat{\Delta}}} C_{\hat{\mathcal{O}}}(|x_{1\perp}|^2 \partial_{1\parallel}^2) C_{\hat{\mathcal{O}}}(|x_{2\perp}|^2 \partial_{2\parallel}^2) \langle \hat{\mathcal{O}}(x_1^a) \hat{\mathcal{O}}(x_2^a) \rangle. \quad (2.5)$$

Thus, we see that every $\hat{\mathcal{O}}$ (primary or descendant) which appears in the BOPE of both \mathcal{O}_1 and \mathcal{O}_2 with the defect contributes to $f(\xi)$ as

$$f(\xi) \sim \sum_{\hat{\mathcal{O}}} b_{\mathcal{O}_1\hat{\mathcal{O}}} b_{\mathcal{O}_2\hat{\mathcal{O}}} \left(\frac{4}{\xi}\right)^{\hat{\Delta}}. \quad (2.6)$$

Therefore, from the $\xi \rightarrow \infty$ limit, we can read off the dimensions of operators $\hat{\mathcal{O}}$ appearing in the BOPE of both \mathcal{O}_1 and \mathcal{O}_2 . This provides very nontrivial information about the defect.

The contribution of descendants to (2.5) can be resummed by solving the Casimir equation (as for bulk conformal blocks). The result was obtained in [2] (see also [3, 54]) and reads

$$f(\hat{\Delta}, \xi) = b_{\mathcal{O}\hat{\mathcal{O}}}^2 \frac{1}{\xi^{\hat{\Delta}}} {}_2F_1\left(\hat{\Delta}, \hat{\Delta} + 1 - d/2; 2\hat{\Delta} + 2 - d; -\frac{1}{\xi}\right) \quad (2.7)$$

for identical scalar operators. For later reference, we note that in two dimensions the hypergeometric function reduces to the Legendre function of the second kind [55]

$${}_2F_1\left(\hat{\Delta}, \hat{\Delta} + 1 - d/2; 2\hat{\Delta} + 2 - d; -\frac{1}{\xi}\right) \stackrel{d=2}{=} \frac{2^{2\hat{\Delta}}}{\sqrt{\pi}} \frac{\Gamma[\hat{\Delta} + 1/2]}{\Gamma[\hat{\Delta}]} \xi^{\hat{\Delta}} Q_{\hat{\Delta}-1}(2\xi + 1). \quad (2.8)$$

3 Supergravity solution

We now review the supersymmetric Janus solutions constructed in [36]. The starting point is type IIB supergravity compactified on the four-torus T^4 or on the $K3$ manifold. In this paper, we will focus on the simpler case in which the compactification manifold is a four torus. We consider solutions with asymptotic geometries which are locally $\text{AdS}_3 \times S^3 \times T^4$. Their holographic duals are marginal deformations of the two-dimensional $\mathcal{N} = (4, 4)$ SCFT which arises as the IR fixed-point of the worldvolume theory of the $D1/D5$ system [56–58] (see ref. [59] for a review). This CFT can be described more explicitly as the two-dimensional sigma-model with the symmetric-orbifold target-space $(T^4)^{Q_{D1}Q_{D5}}/S_{Q_{D1}Q_{D5}}$. The sigma-model is associated to the free Lagrangian

$$S = \frac{1}{2} \int \partial x_A^i \bar{\partial} x_A^i - \psi_A^i(z) \bar{\partial} \psi_A^i(z) - \tilde{\psi}_A^i(\bar{z}) \partial \tilde{\psi}_A^i(\bar{z}), \tag{3.1}$$

where the index $i = 1, \dots, 4$ runs over the coordinates of the four-torus and the symmetric group acts by permuting the indices A of the $Q_{D1}Q_{D5}$ copies of the four-torus.

The particular Janus solutions under consideration are supported by the three- and five-form fluxes and are dual to conformal interfaces obtained by marginally deforming the above CFT. In the supergravity solutions, the six-dimensional dilaton and a linear combination of the axion with the four-form potential interpolate smoothly between two different asymptotic values on opposite sides of the interface. The six-dimensional dilaton is dual to the volume of the four-torus of the CFT, while the axion/four-form potential linear combination corresponds to the source for the orbifold \mathbb{Z}_2 twist operator. In the CFT marginal deformation dual to the Janus solution, these operators have a step-function profile jumping across the interface.

The ten-dimensional metric for the supersymmetric Janus solution is written as

$$ds_{10}^2 = f_{1,10}^2 ds_{\text{AdS}_2}^2 + f_{2,10}^2 ds_{S^2}^2 + \rho_{10}^2 dw d\bar{w} + f_{3,10}^2 ds_{T^4}^2, \tag{3.2}$$

where the metric factors $f_{1,10}, f_{2,10}, f_{3,10}$ and ρ_{10} depend on the coordinates w, \bar{w} of a two-dimensional Riemann surface Σ with boundary. The fields and metric factors have expressions in terms of two harmonic functions, H and K , and two meromorphic functions, A and B . The harmonic functions $H, K, A + \bar{A}$ and $B + \bar{B}$ all obey vanishing Dirichlet boundary conditions on the boundary $\partial\Sigma$.

Global regularity requires that the harmonic functions have isolated singular points on the boundary of Σ and imposes conditions on zeros and singularities of the relevant functions. In particular, the singular points of H correspond to asymptotic regions in which the geometry is locally $\text{AdS}_3 \times S^3$. Aside from the Janus solutions that are the main focus of the present paper, string-junction solutions with more than two asymptotic regions have been studied in refs. [36, 37] and found to be dual to CFTs defined on star graphs.

The ten dimensional solutions solve the field equations

$$\nabla^\mu P_\mu - 2iQ^\mu P_\mu + \frac{1}{24}G_{\mu\nu\rho}G^{\mu\nu\rho} = 0, \quad (3.3)$$

$$\nabla^\rho G_{\mu\nu\rho} - iQ^\rho G_{\mu\nu\rho} - P^\rho \bar{G}_{\mu\nu\rho} + \frac{2}{3}iF_{(5)\mu\nu\rho\sigma\lambda}G^{\rho\sigma\lambda} = 0, \quad (3.4)$$

$$R_\mu^\nu - P_\mu \bar{P}^\nu - \bar{P}_\mu P^\nu - \frac{1}{6}(F_{(5)}^2)_\mu^\nu - \frac{1}{8}(G_{\mu\rho\sigma}\bar{G}^{\nu\rho\sigma} + \bar{G}_{\mu\rho\sigma}G^{\nu\rho\sigma}) + \frac{1}{48}\delta_\mu^\nu G_{\rho\sigma\lambda}\bar{G}^{\rho\sigma\lambda} = 0, \quad (3.5)$$

together with the Bianchi identities

$$dP - 2iQ \wedge P = 0, \quad (3.6)$$

$$dQ + iP \wedge \bar{P} = 0, \quad (3.7)$$

$$dG - iQ \wedge G + P \wedge \bar{G} = 0, \quad (3.8)$$

$$dF_{(5)} - \frac{i}{8}G \wedge \bar{G} = 0, \quad (3.9)$$

and the self-duality condition for the five-form field strength $F_{(5)}$. In the above expressions, P and Q are a composite one-form field strength and connection which can be expressed in terms of the axion χ and dilaton ϕ as

$$P = \frac{1}{2}(d\phi + ie^\phi d\chi), \quad Q = -\frac{1}{2}e^\phi d\chi. \quad (3.10)$$

Similarly, the complex three-form field strength G is given in terms of the R-R and NS-NS field strengths $F_{(3)}$ and $H_{(3)}$ as

$$G = e^{-\phi/2}H_{(3)} + ie^{\phi/2}(F_{(3)} - \chi H_{(3)}). \quad (3.11)$$

A key feature of the Janus solutions is that the three-form and five-form fluxes are related to the volume forms of the unit S^2 , AdS_2 and T^4 fibres $\hat{\omega}_{S^2}$, $\hat{\omega}_{\text{AdS}_2}$ and $\hat{\omega}_{T^4}$ as

$$F_{(3)} = dc^{(1)} \wedge \hat{\omega}_{\text{AdS}_2} + dc^{(2)} \wedge \hat{\omega}_{S^2}, \quad (3.12)$$

$$H_{(3)} = db^{(1)} \wedge \hat{\omega}_{\text{AdS}_2} + db^{(2)} \wedge \hat{\omega}_{S^2}, \quad (3.13)$$

$$F_{(5)} = dC_K \wedge \hat{\omega}_{\text{AdS}_2} \wedge \hat{\omega}_{S^2} + d\tilde{C}_K \wedge \hat{\omega}_{T^4}, \quad (3.14)$$

where expressions for the scalar potentials $c^{(1,2)}$, $b^{(1,2)}$, C_K and \tilde{C}_K in terms of the relevant harmonic functions are given in ref. [36]. Note that this structure is dictated by the $\text{SO}(2,1) \times \text{SO}(3)$ symmetry imposed in constructing the solutions. For the reader's convenience, non-zero components of the various tensor fields are listed in table 1.

It is convenient to reduce the solutions to six dimensions and write the metric in the Einstein frame as

$$ds_6^2 = f_1^2 ds_{\text{AdS}_2}^2 + f_2^2 ds_{S^2}^2 + \rho^2 dz d\bar{z}, \quad (3.15)$$

where $f_1^2 = f_{1,10}^2 f_{3,10}^2$, $f_2^2 = f_{2,10}^2 f_{3,10}^2$ and $\rho^2 = \rho_{10}^2 f_{3,10}^2$. Note that large classes of Janus and string-junction solutions have also been obtained directly in six-dimensional supergravity in refs. [40–42]. Here we started with simple ten-dimensional solutions and reduced them

	AdS ₂		S ²		Σ		T ⁴			
	<i>t</i>	<i>z</i>	<i>θ</i>	<i>φ</i>	<i>x</i>	<i>y</i>	<i>u</i> ₁	<i>u</i> ₂	<i>u</i> ₃	<i>u</i> ₄
<i>P, Q</i>					X					
<i>H</i> _{(3), <i>F</i>} (3)	X	X			X					
	X	X	X	X	X	X				
<i>F</i> ₍₅₎	X	X	X	X	X					
	X	X	X	X		X	X	X	X	X

Table 1. List of nonzero components of the five- and three-form fluxes and of the axion-dilaton composite field strength and connection for general ten-dimensional Janus and string-junction solutions.

to six dimensions to better illustrate the top-down origin of our scalar deformations. A useful identity involving the metric factors,

$$f_1^2 f_2^2 = H^2, \tag{3.16}$$

permits us to write the square root of the determinant of the metric as

$$\sqrt{-g} = \rho^2 H^2. \tag{3.17}$$

We shall now focus on the simplest solutions which have two asymptotic AdS₃ × S³ regions, a simply-connected Riemann surface, and are supported only by R-R charges (see ref. [37] for a discussion of the Page charges associated to our solutions). We take the infinite strip with coordinates *x, y* as the Riemann surface Σ. Figure 1 displays the pole configuration for the solutions of interest in the *x-y* plane, as well as a sketch of the *x-z* plane which illustrates the position of the defect in our coordinates. The two asymptotically AdS₃ regions are approached as *x* → ±∞. Nontrivial Janus solutions are parameterized by two parameters, *ψ* and *θ*, associated to the jump in the dilaton and axion respectively. The undeformed case is obtained by *θ* = 0 = *ψ*. A plot of the ten-dimensional dilaton for the solution with nonzero *ψ* can be found in figure 2.

The following simple expressions for the harmonic function *H* and the ratios of metric factors [36, 39],

$$H = 2\hat{L} \cosh x \sin y, \quad \frac{\rho^2}{f_1^2} = \frac{\kappa^2}{\cosh^2 x}, \quad \frac{\rho^2}{f_2^2} = \frac{1}{\sin^2 y} + \frac{\kappa^2 - 1}{\cosh^2 x}, \tag{3.18}$$

will be useful in the remainder of the paper. We have introduced the interface parameter *κ* = cosh *ψ* cosh *θ* and denoted with *x, y* real and imaginary part of the complex coordinate of Σ.

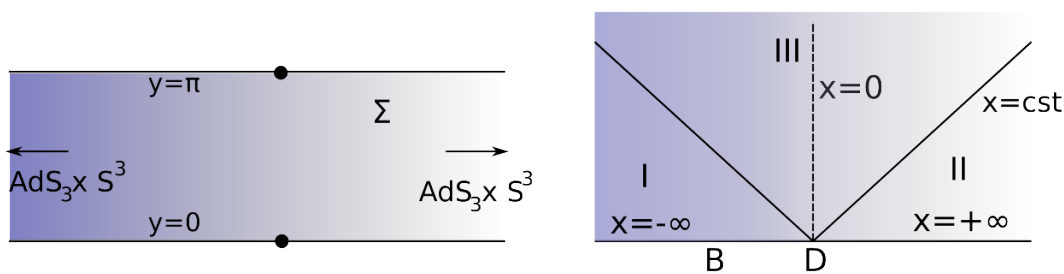


Figure 1. Structure of the Janus solutions on the x - y plane and x - z plane. The Riemann surface Σ is the infinite strip. For a solution supported only by RR fluxes, the functions A, B and K have singularities on both boundaries at $x = 0$. The spacetime boundary B is reached for $x \rightarrow \pm\infty$ and for $z \rightarrow 0$ on the AdS slice. The latter limit corresponds to the location of the defect D .

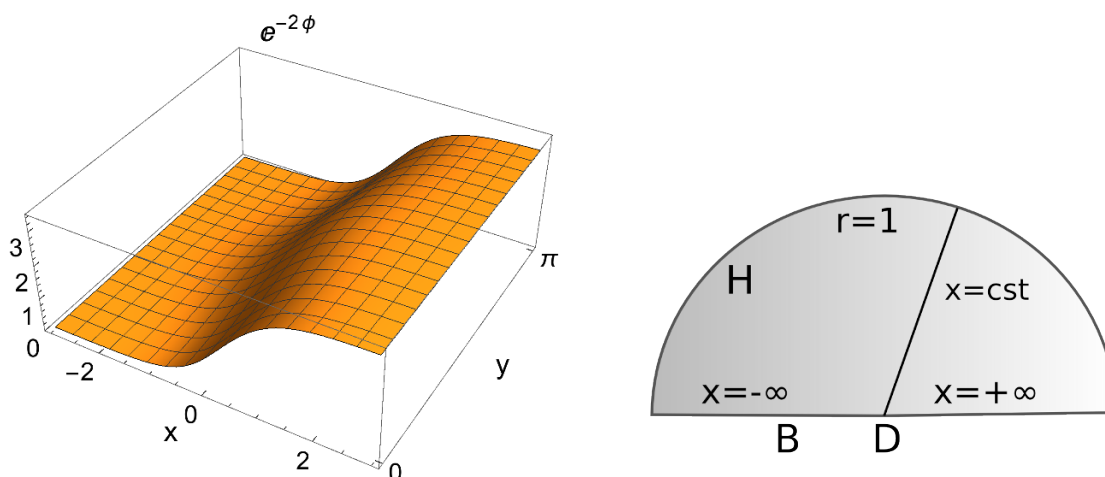


Figure 2. Dilaton profile for the supersymmetric Janus solution with $\psi = 1, \theta = 0$ in the x - y plane (left) and sketch of the finite-temperature solution (right). The black-hole horizon H is located at $r = 1$, the defect D is located at $r = 0$.

So far, we have discussed zero-temperature solutions. However, as explained in ref. [44], a black-hole solution can be obtained by replacing the AdS_2 fibre with the “black-hole slice” metric

$$ds_{AdS_2}^2 \mapsto ds_{BH}^2 = -\frac{1-r^2}{r^2}(2\pi T)^2 dt^2 + \frac{dr^2}{r^2(1-r^2)}, \quad 0 \leq r \leq 1, \quad (3.19)$$

where we have introduced the explicit temperature dependence as T and we have employed the coordinate transformation $\rho = 1/r$ as compared to ref. [44]. For these solutions, the interface is located at $r = 0$ and the horizon at $r = 1$. In the absence of an interface, the Fefferman-Graham coordinate expansion close to the boundary reproduces the one of the BTZ black hole order by order in the radial parameter. Black-hole solutions obtained with this technique are dual to finite-temperature CFTs defined on the real line times the thermal circle.³ The structure of the solution on the r - x plane can be found in figure 2.

³We would like to emphasize that additional global identifications, analogous to the ones arising for the BTZ black-hole solution in Poincaré coordinates, are required to reproduce the global structure of a solution dual to a finite-temperature CFT defined on a two-torus.

We are now ready to discuss the ten-dimensional origin of the probe scalars obeying the massless Klein-Gordon equation which we will solve in the next section. Given the general structure of our interface solutions, we consider simple deformations of the supersymmetric Janus geometry which leave invariant the volume forms of the fibres.

The simplest of such deformations affects only the components of the metric along the T^4 fibre and has the form

$$g_{ij} = f_{3,10}^2 \exp(t_{ij}^a \Phi^a) = f_{3,10}^2 (\delta_{ij} + \Phi^a t_{ij}^a + \dots), \quad (t^a)_{ii} = 0, \quad i, j = 6, 7, 8, 9, \quad (3.20)$$

where t_{ij}^a ($a = 1, \dots, 9$) are constant symmetric-traceless tensors along the compact T^4 directions.

A deformation of this form leaves the three-form and five-form fields invariant, as well as the axion and dilaton. Inspecting the equations of motion (3.3)–(3.5), it is immediate to verify that the only non-trivial equation is (3.5), which reduces to the equation for a free scalar,⁴

$$R_\mu^\nu = 0 \quad \Rightarrow \quad \nabla^2 \Phi^a = 0. \quad (3.21)$$

The same equation can be obtained from the reduced action

$$S[\Phi] = \int d^4x \sqrt{g} \left(-\partial_\mu \Phi^a \partial^\mu \Phi^a \right). \quad (3.22)$$

In the remainder of the paper, we will study the equation of motion for these free scalars in the Janus geometry and, in particular, find that it is separable, allowing for an explicit analytic solution. As mentioned before, one of the advantages of the top-down approach is that the dual $\mathcal{N} = (4, 4)$ SCFT is known explicitly. In particular, its operators are classified using the $\mathcal{N} = (4, 4)$ superconformal algebra and are labelled by their representation of the $SO(4)_I$ symmetry acting on the coordinates of the four-torus. Hence, the operator dual to our scalar deformation is identified as

$$\mathcal{O}^{\{ij\}} = \partial x_A^i \bar{\partial} x_A^j = \partial x_A^i \bar{\partial} x_A^j + \partial x_A^j \bar{\partial} x_A^i - \frac{1}{4} \delta^{ij} \partial x_A^k \bar{\partial} x_A^k.$$

The reader might wonder if considering deformations leaving the fibres' volume forms invariant might lead to other simple perturbations. In particular, a potentially interesting deformation involves the metric on the AdS_2 slice,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad h_{\mu\nu} \bar{g}^{\mu\nu} = 0, \quad \mu, \nu = 0, 1, \quad (3.23)$$

and leaves all other fields invariant. Repeating the argument above, we obtain

$$R_\mu^\nu \Big|_{\mathcal{O}(h)} = 0 \quad \Rightarrow \quad \nabla^2 h_{\mu\nu} - \nabla_\mu \nabla^\rho h_{\nu\rho} - \nabla^\nu \nabla^\rho h_{\mu\rho} + 2\bar{R}^\rho_{\mu\ \nu} h_{\rho\sigma} = 0, \quad (3.24)$$

where $\bar{R}^\rho_{\mu\ \nu}$ is the Riemann tensor of the background geometry. These are three differential equations in two functions h_{tt} and h_{tz} and force a trivial dependence on the t, z coordinates for h_{tt} and h_{tz} . Hence, to have non-trivial solutions, one needs to turn on the trace part of the AdS_2 metric perturbation. While deformations of this sort are very interesting, their treatment is considerably more involved and will be deferred to future work.

⁴To obtain the equation for the free scalar (3.21), it is simpler to consider the field equations with mixed indices.

4 Solving the massless Klein-Gordon equation

From now on, we drop the index parametrizing different deformations on the torus and work with the single scalar Φ . The massless Klein-Gordon equation (3.21) for the Janus geometry with two AdS₃ asymptotic regions can then be written as

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) = \frac{1}{\rho^2}\left\{\frac{\partial_a(H^2\partial_a)}{H^2} + \frac{\rho^2}{f_1^2}\nabla_{\text{AdS}_2}^2 + \frac{\rho^2}{f_2^2}\nabla_{S^2}^2\right\}\Phi = 0, \quad a = x, y \quad (4.1)$$

where $\nabla_{S^2}^2$ is the Laplacian on the unit two-sphere and $\nabla_{\text{AdS}_2}^2$ is the Laplacian on the unit AdS₂ slice,

$$\nabla_{\text{AdS}_2}^2 = z^2(\partial_z^2 - \partial_t^2). \quad (4.2)$$

In case of the black-hole solution, we replace

$$\nabla_{\text{AdS}_2}^2 \mapsto \nabla_{\text{BH}}^2 = r^2\left(\partial_r(1-r^2)\partial_r - \frac{1}{(2\pi T)^2}\frac{\partial_t^2}{1-r^2}\right). \quad (4.3)$$

For the massless scalar field, the Klein-Gordon equation is separable. In this section we will present the solution.

First, we note that the differential equations on the two-dimensional slices,

$$\nabla_{\text{AdS}_2}^2 f = \nu(\nu+1)f, \quad \nabla_{\text{BH}}^2 f = \nu(\nu+1)f, \quad (4.4)$$

admit solutions in terms of Bessel and associated Legendre functions,

$$\text{AdS: } f(z, t) = C_1\sqrt{z}J_{\nu+\frac{1}{2}}(\omega z)e^{\pm i\omega t} + C_2\sqrt{z}Y_{\nu+\frac{1}{2}}(\omega z)e^{\pm i\omega t}, \quad (4.5)$$

$$\text{BH: } f(r, t) = C_1P_\nu^{\frac{i\omega}{2\pi T}}\left(\frac{1}{r}\right)e^{\pm i\omega t} + C_2Q_\nu^{\frac{i\omega}{2\pi T}}\left(\frac{1}{r}\right). \quad (4.6)$$

In the Euclidean signature, the Bessel functions $J_{\nu+1/2}$ and $Y_{\nu+1/2}$ should be replaced by the modified Bessel functions $K_{\nu+1/2}$ and $I_{\nu+1/2}$.

The effective mass on the AdS₂ slice is related to the dimension of the defect operator simply by $\nu = \hat{\Delta} - 1$. We will later fix the possible values of ν by imposing the absence of sources on the boundary. This will result in a discrete tower of possible masses on the AdS₂ slice and determine the dimensions of the operators appearing in the BOPE of the dual operator with the defect. To extract the BOPE coefficients themselves, we need to compute the two-point correlation function of bulk operators.

The Laplace operator on the two-sphere has the usual spherical harmonics as its eigenfunctions. Thus the probe scalar can be expanded as

$$\text{AdS: } \Phi = \sum\left(\Phi_{\nu\omega lm}^{(1)}(x, y)\sqrt{z}J_{\nu+\frac{1}{2}}(\omega z) + \Phi_{\nu\omega lm}^{(2)}(x, y)\sqrt{z}Y_{\nu+\frac{1}{2}}(\omega z)\right)e^{\pm i\omega t}Y_{lm}(\theta, \phi), \quad (4.7)$$

$$\text{BH: } \Phi = \sum\left(\Phi_{\nu\omega lm}^{(1)}(x, y)P_\nu^{\frac{i\omega}{2\pi T}}\left(\frac{1}{r}\right) + \Phi_{\nu\omega lm}^{(2)}(x, y)Q_\nu^{\frac{i\omega}{2\pi T}}\left(\frac{1}{r}\right)\right)e^{\pm i\omega t}Y_{lm}(\theta, \phi), \quad (4.8)$$

where the spherical harmonics satisfy $\nabla_{S^2}^2 Y_{lm}(\theta, \phi) = -l(l+1)Y_{lm}(\theta, \phi)$. In both cases, the Klein-Gordon equation becomes

$$\left\{\frac{\partial_a(H^2\partial_a)}{H^2} + \nu(\nu+1)\frac{\rho^2}{f_1^2} - l(l+1)\frac{\rho^2}{f_2^2}\right\}\Phi_{\nu\omega lm}^{(i)} = 0, \quad (4.9)$$

or using (3.18)

$$\left\{ \frac{\partial_x \cosh^2 x \partial_x + \alpha(\alpha + 1)}{\cosh^2 x} + \frac{\partial_y \sin^2(y) \partial_y - l(l + 1)}{\sin^2(y)} \right\} \Phi_{\nu\omega lm}^{(i)} = 0, \quad i = 1, 2, \quad (4.10)$$

where the constant α is determined in terms of l and ν by

$$\alpha(\alpha + 1) = l(l + 1) + \kappa^2(\nu(\nu + 1) - l(l + 1)). \quad (4.11)$$

Equation (4.10) is separable. Moreover, the y -dependent part combines with the Laplacian on the S^2 slice to give the Laplacian on the three-sphere, with eigenvalues $-k(k + 2)$.⁵

To get a better intuition, it is useful to represent the field equation in the form

$$\square_{3d} \Phi_{l,k} = \left(\frac{l(l + 1)}{\cosh^2 x} (\kappa^2 - 1) + k(k + 2) \right) \Phi_{l,k} = M_{3d}^2(x) \Phi_{l,k}, \quad (4.12)$$

where \square_{3d} is the Laplacian on the three-dimensional asymptotically locally-AdS₃ space with the metric

$$ds_{3d}^2 = dx^2 + \frac{\cosh^2 x}{\kappa^2} ds_{\text{AdS}_2}^2 = dx^2 + \frac{\cosh^2 x}{\kappa^2} \frac{dz^2 - dt^2}{z^2}. \quad (4.13)$$

Thus the presence of the defect does not just modify the AdS₃ metric, but also introduces a position-dependent mass, $M_{3d}^2(x)$. Without the defect, the dimension of the dual operator is simply $\Delta = k + 2$. In the presence of the defect, this operator gets decomposed into a tower of operators with dimensions $\hat{\Delta}_n = \nu(n) + 1$, the precise form of which we will deduce later. Importantly, the dimensions of the members of this tower are not related in a simple way to the dimension of the original operator, in contrast to the case in which the defect is absent. This is a characteristic feature of the BOPE. In the process of bringing an operator close to the defect, divergences can appear and regularization is required as in the case of composite operators.

After solving the AdS₂ and S^3 parts of the problem, we are left with a single ordinary differential equation,

$$\frac{\partial_x \cosh^2 x \partial_x + \alpha(\alpha + 1)}{\cosh^2 x} \chi_{\omega\nu klm}^{(i)}(x) = k(k + 2) \chi_{\omega\nu klm}^{(i)}(x), \quad i = 1, 2, \quad (4.14)$$

which is solved by

$$\chi_{\omega\nu klm}^{(i)}(x) = \frac{c_3 P_\alpha^{k+1}(\tanh x) + c_4 Q_\alpha^{k+1}(\tanh x)}{\cosh x}. \quad (4.15)$$

⁵We introduce $\mathcal{Y}_{klm}(y, \theta, \phi)$, which satisfies $\nabla_{S^2}^2 \mathcal{Y}_{klm} = -l(l+1)\mathcal{Y}_{klm}$ and $[\partial_y(\sin^2(y) \partial_y) - l(l+1)]\mathcal{Y}_{klm} = -k(k+2)\sin^2(y)\mathcal{Y}_{klm}$.

To summarize, there are eight classes of solutions for the probe scalar in the Janus background. For the AdS slicing we have:

$$\begin{aligned}
 \text{AdS : } \quad \Phi_{\omega\nu klm}^{(1)} &= \frac{\sqrt{z} \omega^{-(\nu+\frac{1}{2})}}{\cosh x} P_{\alpha}^{k+1}(\tanh x) J_{\nu+\frac{1}{2}}(\omega z) e^{\pm i\omega t} \mathcal{Y}_{klm}(y, \theta, \phi), \\
 \Phi_{\omega\nu klm}^{(2)} &= \frac{\sqrt{z} \omega^{(\nu+\frac{3}{2})}}{\cosh x} P_{\alpha}^{k+1}(\tanh x) Y_{\nu+\frac{1}{2}}(\omega z) e^{\pm i\omega t} \mathcal{Y}_{klm}(y, \theta, \phi), \\
 \Phi_{\omega\nu klm}^{(3)} &= \frac{\sqrt{z} \omega^{-(\nu+\frac{1}{2})}}{\cosh x} Q_{\alpha}^{k+1}(\tanh x) J_{\nu+\frac{1}{2}}(\omega z) e^{\pm i\omega t} \mathcal{Y}_{klm}(y, \theta, \phi), \\
 \Phi_{\omega\nu klm}^{(4)} &= \frac{\sqrt{z} \omega^{(\nu+\frac{3}{2})}}{\cosh x} Q_{\alpha}^{k+1}(\tanh x) Y_{\nu+\frac{1}{2}}(\omega z) e^{\pm i\omega t} \mathcal{Y}_{klm}(y, \theta, \phi), \tag{4.16}
 \end{aligned}$$

where \mathcal{Y}_{klm} are the three-sphere spherical harmonics and $\omega \geq 0$. The factors of ω have been included so that the limit $\omega \rightarrow 0$ can be taken smoothly. For the black-hole solutions we have

$$\begin{aligned}
 \text{BH : } \quad \Phi_{\omega\nu klm}^{(1)} &= \frac{1}{\cosh x} P_{\alpha}^{k+1}(\tanh x) P_{\nu}^{\frac{i\omega}{2\pi T}}\left(\frac{1}{r}\right) e^{\pm i\omega t} \mathcal{Y}_{klm}(y, \theta, \phi), \\
 \Phi_{\omega\nu klm}^{(2)} &= \frac{1}{\cosh x} P_{\alpha}^{k+1}(\tanh x) Q_{\nu}^{\frac{i\omega}{2\pi T}}\left(\frac{1}{r}\right) e^{\pm i\omega t} \mathcal{Y}_{klm}(y, \theta, \phi), \\
 \Phi_{\omega\nu klm}^{(3)} &= \frac{1}{\cosh x} Q_{\alpha}^{k+1}(\tanh x) P_{\nu}^{\frac{i\omega}{2\pi T}}\left(\frac{1}{r}\right) e^{\pm i\omega t} \mathcal{Y}_{klm}(y, \theta, \phi), \\
 \Phi_{\omega\nu klm}^{(4)} &= \frac{1}{\cosh x} Q_{\alpha}^{k+1}(\tanh x) Q_{\nu}^{\frac{i\omega}{2\pi T}}\left(\frac{1}{r}\right) e^{\pm i\omega t} \mathcal{Y}_{klm}(y, \theta, \phi). \tag{4.17}
 \end{aligned}$$

Note that at this stage the parameters k, l, m are quantized. The general solution is obtained by superimposing these modes.

The defect modes correspond to those solutions which vanish at the boundary away from the defect, i.e. when $x \rightarrow \pm\infty$. Using simple properties of Legendre functions, we find that this requirement is satisfied only if $\alpha = k + 1 + n$ for a non-negative integer n . Note that (4.11) tells us what discrete values the ν parameter can take,

$$\nu(n) = \left\{ \frac{1}{4} + \frac{(k+n+1)(k+n+2)}{\kappa^2} + l(l+1) \left(1 - \frac{1}{\kappa^2} \right) \right\}^{\frac{1}{2}} - \frac{1}{2}, \tag{4.18}$$

where we have taken the positive solution to (4.11).

From (4.4), the effective mass on the AdS₂ slice is $m^2 = \nu(\nu+1) = \hat{\Delta}_n(\hat{\Delta}_n-1)$. Thus, for the defect modes, we obtain the conformal dimension of the corresponding fields:

$$m_{l,n}^2 = \hat{\Delta}_n(\hat{\Delta}_n-1) = l(l+1) + \frac{1}{\kappa^2} \left[(k+n+2)(k+n+1) - l(l+1) \right], \tag{4.19}$$

where $n = 0, 1, 2, 3, \dots$ is an integer and we have set $\alpha = k + 1 + n$. Note that the massless AdS₂ scalar cannot appear in the spectrum.

As a simple check, consider the pure AdS₃ scenario, i.e. assume that the dilaton and axion do not jump across the defect ($\kappa = 1$). Then for $k = 0$, we have

$$m_n^2 = (n+2)(n+1), \quad \Rightarrow \quad \hat{\Delta}_n = n+2 \tag{4.20}$$

The higher values of n then give the dimension of the higher-order operators appearing in the expansion.⁶ For higher values of k , we have $\Delta_n = k + 2 + n$, in agreement with the fact that the AdS₃ mass is given by $m_{\text{AdS}_3}^2 = k(k + 2)$.

5 Bulk-to-bulk propagator and two-point correlation function

In this section, we use the linearized modes to construct the bulk-to-bulk propagator satisfying vanishing boundary conditions at the boundary of the spacetime and infalling boundary conditions at the Poincaré horizon. The latter choice is appropriate for the retarded holographic propagator. In Euclidean signature, it is sufficient to impose regularity in the spacetime's interior. The propagator can be constructed as a summation over modes which satisfy prescribed boundary conditions. The modes which vanish at the boundary (away from the defect) must necessarily have $\alpha \in \mathbb{N}$. This gives the quantization condition for the masses on the AdS₂ slice and hence the dimensions of operators appearing in the BOPE of the dual operator with the defect.

In what follows, we will effectively ignore the S^3 part of the geometry and construct propagators directly for the equation (4.12).

5.1 Bulk-to-bulk propagator

As we will show shortly, the bulk-to-bulk propagator is given explicitly by

$$G_\Delta = \frac{1}{2\pi} \int_0^\infty d\omega e^{-i\omega(t-t')} \sum_{n=0}^\infty \frac{(n + \Delta - 1/2)n!}{(n + 2\Delta - 2)!} \tilde{G}_{\nu+1/2}[z, z'] \times \frac{P_{n+\Delta-1}^{\Delta-1}(\tanh x)}{\cosh x} \frac{P_{n+\Delta-1}^{\Delta-1}(\tanh x')}{\cosh x'} + \text{c.c.}, \quad (5.1)$$

where

$$\tilde{G}_\nu[z, z'] = \frac{1}{W[\sqrt{z}H_\nu^{(1)}(|\omega|z), \sqrt{z}J_\nu(|\omega|z)]\Big|_{z=z'}} \left[\theta(z - z')\sqrt{z}H_\nu^{(1)}(|\omega|z)\sqrt{z'}J_\nu(|\omega|z') + \theta(z' - z)\sqrt{z}J_\nu(|\omega|z)\sqrt{z'}H_\nu^{(1)}(|\omega|z') \right] \quad (5.2)$$

is the propagator in the z direction which satisfies the equation

$$\left(z^2 \partial_z^2 + \omega^2 z^2 - \nu(\nu + 1) \right) \tilde{G}_{\nu+1/2}[z, z'] = z^2 \delta(z - z'), \quad (5.3)$$

vanishes as z approaches zero and satisfies infalling boundary conditions at the horizon of the AdS₂ slice, i.e. as z goes to infinity. $W[\sqrt{z}H_\nu^{(1)}(|\omega|z), \sqrt{z}J_\nu(|\omega|z)]$ denotes the Wronskian of the two functions, which, for the case at hand, is equal to $-2i/\pi$. The Green's function is a continuous function with discontinuous first derivative at $z = z'$.⁷ Note that

⁶Consider the expansion $\mathcal{O}(x_\perp) = \sum_{n=1}^\infty \mathcal{O}^{(n)} x_\perp^{n-1}$. The dimension of $\mathcal{O}^{(0)}$ would be the same dimension as \mathcal{O} , which is two for a marginal operator. The dimension of the higher order terms is then $[\mathcal{O}^{(n)}] = \Delta_n = n + 1$.

⁷To obtain the result in Euclidean signature we continue $\omega^2 \rightarrow -\omega^2$. Hankel function H_ν gets replaced by K_ν (up to a simple prefactor) which is regular as $z \rightarrow \infty$ and J_ν gets replaced by I_ν . The Wronskian is then $W[\sqrt{z}K_\nu(|\omega|z), \sqrt{z}I_\nu(|\omega|z)] = 1$.

the choice of the Hankel function of the first kind is determined by requiring the Green's function to satisfy ingoing boundary conditions at the horizon, while the choice of the Bessel function of the first kind $J_\nu(|\omega|z)$ is determined by requiring the bulk-to-bulk propagator to vanish at the boundary of the AdS₂ slice. The quantization of $\alpha = n + \Delta - 1 = n + k + 1$ is determined by requiring the bulk-to-bulk propagator to vanish as $x \rightarrow \pm\infty$. This means that the bulk-to-bulk propagator is decomposable in terms of the defect modes.

Let us show by explicit computation that (5.1) indeed gives the propagator. Acting with the three-dimensional Laplacian on G_Δ , we note that the non-vanishing contribution comes from the step-function in the z direction. Explicitly,

$$(\square_{3d} - M_{3d}^2) G_\Delta = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} \left[\frac{\kappa^2}{\cosh^2 x} z^2 \delta(z - z') \right] \sum_{n=0}^{\infty} \frac{(n + \Delta - 1/2)n!}{(n + 2\Delta - 2)!} \frac{P_{n+\Delta-1}^{\Delta-1}(\tanh x)}{\cosh x} \frac{P_{n+\Delta-1}^{\Delta-1}(\tanh x')}{\cosh x'}. \quad (5.4)$$

Next, we use the completeness relation for the associated Legendre polynomials

$$\cosh x \cosh x' \delta(x - x') = \sum_{n=0}^{\infty} \frac{(n + \Delta - 1/2)n!}{(n + 2\Delta - 2)!} P_{n+\Delta-1}^{\Delta-1}(\tanh x) P_{n+\Delta-1}^{\Delta-1}(\tanh x') \quad (5.5)$$

and obtain

$$(\square_{3d} - M_{3d}^2) G_\Delta = \frac{1}{\sqrt{|g_{3d}|}} \delta(t - t') \delta(x - x') \delta(z - z'), \quad (5.6)$$

with $\sqrt{|g_{3d}|} = \cosh^2 x / (\kappa^2 z^2)$. Thus, we see that the bulk-to-bulk propagator can be indeed represented as a sum of the modes derived before.

The Euclidean version is

$$G_{E,\Delta} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} \sum_{n=0}^{\infty} \frac{(n + \Delta - 1/2)n!}{(n + 2\Delta - 2)!} \tilde{G}_{E,\nu+1/2}[z, z'] \frac{P_{n+\Delta-1}^{\Delta-1}(\tanh x)}{\cosh x} \frac{P_{n+\Delta-1}^{\Delta-1}(\tanh x')}{\cosh x'}, \quad (5.7)$$

with

$$\tilde{G}_{E,\nu}[z, z'] = \frac{1}{W[\sqrt{z}K_\nu(|\omega|z), \sqrt{z}I_\nu(|\omega|z)]_{z=z'}} \left[\theta(z - z') \sqrt{z}K_\nu(|\omega|z) \sqrt{z'}I_\nu(|\omega|z') + \theta(z' - z) \sqrt{z}I_\nu(|\omega|z) \sqrt{z'}K_\nu(|\omega|z') \right]. \quad (5.8)$$

In this case, we have $W[\sqrt{z}K_\nu(|\omega|z), \sqrt{z}I_\nu(|\omega|z)] = 1$. The continuation back to Minkowski signature is achieved by replacing $J_\nu \rightarrow I_\nu$ and $K_\nu \rightarrow H_\nu^{(2)}$ for $\omega < 0$ and $K_\nu \rightarrow H_\nu^{(1)}$ for $\omega > 0$.

Finally, for the finite-temperature case, we can proceed along the same lines and obtain the expression

$$\tilde{G}_{E,T,\nu}[z, z'] = \frac{1}{W\left[Q_{\nu}^{\frac{\omega}{2\pi T}}\left(\frac{1}{r}\right), P_{\nu}^{\frac{\omega}{2\pi T}}\left(\frac{1}{r}\right)\right]\Big|_{r=r'}} \left[\theta(r-r') Q_{\nu}^{\frac{\omega}{2\pi T}}\left(\frac{1}{r}\right) P_{\nu}^{\frac{\omega}{2\pi T}}\left(\frac{1}{r'}\right) + \theta(r'-r) P_{\nu}^{\frac{\omega}{2\pi T}}\left(\frac{1}{r}\right) Q_{\nu}^{\frac{\omega}{2\pi T}}\left(\frac{1}{r'}\right) \right]. \quad (5.9)$$

This propagator is regular at the black-hole horizon at $r = 1$ and is appropriate for the holographic dual of a CFT defined on the real line times the thermal circle. Note that the boundary conditions need to be modified if we introduce additional global identifications in the black-hole geometry.

5.2 Bulk-to-boundary propagator

The bulk-to-boundary propagator $K_{\Delta}(x_0, \vec{x}, \vec{x}')$ can be obtained by taking the limit [19, 60]

$$K(z, t, x; t', x'_{\perp}) = \lim_{u' \rightarrow 0} \frac{(2\Delta - 2)}{(u')^{\Delta}} G_{\Delta}(z, t, x; u', t', x'_{\perp}), \quad (5.10)$$

where the u' coordinate denotes the radial coordinate in the Poincaré patch of AdS_3 . For the AdS-slicing coordinates we are employing, the Fefferman-Graham expansion needs to be computed separately on both sides of the defect (regions I and II in figure 1). The leading-order terms are:

$$\text{region I:} \quad x' \sim -\ln(2u'/\kappa|x'_{\perp}|), \quad z' \sim -x'_{\perp}, \quad (5.11)$$

$$\text{region II:} \quad x' \sim +\ln(2u'/\kappa|x'_{\perp}|), \quad z' \sim +x'_{\perp}. \quad (5.12)$$

To compute the limit, we use that [55]

$$P_{n+\Delta-1}^{\Delta-1}(\tanh x') \sim \frac{(-1)^{\Delta-1} 2^{-\frac{\Delta-1}{2}} \Gamma(n+2\Delta-1) (1-\tanh x')^{\frac{\Delta-1}{2}}}{(\Delta-1)! \Gamma(n+1)}, \quad \text{as } x' \rightarrow \infty. \quad (5.13)$$

The other side of the defect is approached as $x' \rightarrow -\infty$, this would introduce a $(-1)^n$ prefactor. For now we postpone this issue until later. A short computation gives

$$G_{\Delta} \sim \frac{1}{(\Delta-2)!} \left(-\frac{1}{2}\right)^{\Delta-1} \frac{1}{2(\Delta-1)\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} \times \sum_{n=0}^{\infty} (n+\Delta-1/2) \tilde{G}_{\nu+1/2}[z, z'] \frac{1}{(\cosh x')^{\Delta}} \frac{P_{n+\Delta-1}^{\Delta-1}(\tanh x)}{\cosh x} \quad \text{as } x' \rightarrow \infty, \quad (5.14)$$

so that the bulk-to-boundary propagator can be read off:

$$K_{\Delta}(z, t, x; t', x'_{\perp}) = \frac{1}{(\Delta-2)!} \left(-\frac{1}{2}\right)^{\Delta-1} \frac{1}{\pi} \frac{1}{\kappa^{\Delta}} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} \times \sum_{n=0}^{\infty} (n+\Delta-1/2) \tilde{G}_{\nu+1/2}[z, |x'_{\perp}|] \frac{1}{|x'_{\perp}|^{\Delta}} \frac{P_{n+\Delta-1}^{\Delta-1}(\tanh x)}{\cosh x}, \quad (5.15)$$

In appendix A, we show that this reduces to the well-known pure-AdS bulk-to-boundary propagator when the defect is switched off (in which case $\kappa = 1$ and $\nu = n + \Delta - 1$).

The correlation functions in (defect) CFTs are typically presented in position space. So we would like to perform the inverse Fourier transform of (5.15). To this end, we note the Gegenbauer's formula [61]

$$\frac{H_{\nu+m}^{(1)}(\omega z)}{(\omega z)^\nu} \frac{J_{\nu+m}(\omega x'_\perp)}{(\omega x'_\perp)^\nu} = \frac{2^{\nu-1} m! \Gamma(\nu)}{\pi \Gamma(2\nu+m)} \int_0^\pi \frac{H_{\nu+m}^{(1)}(|\omega| \sqrt{z^2 + x'_\perp{}^2 - 2zx'_\perp \cos \theta})}{\omega^\nu (z^2 + x'_\perp{}^2 - 2zx'_\perp \cos \theta)^{\nu/2}} C_m^\nu(\cos \theta) \sin^{2\nu} \theta d\theta, \quad (5.16)$$

where $C_m^\nu(\cos \theta)$ denotes the Gegenbauer function. For the computation in Euclidean signature the same formula holds if we exchange $J_{\nu+m}$ by $I_{\nu+m}$ and $H_{\nu+m}$ by $K_{\nu+m}$ on the both sides of the equation. Using this relation (with $m = 0$) in (5.15) and Fourier transforming back, we are left with an integral over θ which can be expressed in terms of the hypergeometric function. Using [55]

$${}_2F_1\left(\nu+1, \nu+1, 2\nu+2, \frac{2}{1-z}\right) = 2^{\nu+1} \frac{\Gamma[\nu+3/2]}{\sqrt{\pi} \Gamma[\nu+1]} (z-1)^{\nu+1} Q_\nu(z), \quad (5.17)$$

we are left with a sum

$$K_\Delta = \left(-\frac{1}{2}\right)^{\Delta-1} \frac{1}{\pi \kappa^\Delta \Gamma(\Delta-1)} \frac{1}{|x'_\perp|^\Delta} \sum_{n=0}^{\infty} (n+\Delta-1/2) Q_\nu\left(\frac{z^2 + x'_\perp{}^2 + (t-t')^2}{2zx'_\perp}\right) \frac{P_{n+\Delta-1}^{\Delta-1}(\tanh x)}{\cosh x}. \quad (5.18)$$

For the undeformed case, $\nu = n + \Delta - 1$, and we use the identity ([62], see also [63])

$$(z - \cos(\phi))^{-k-1/2} = \frac{2^{k+1/2}}{\pi^{1/2}} \frac{\Gamma(k)}{\Gamma(k+1/2)} \sum_{n=0}^{\infty} (k+n) C_n^k(\cos(\phi)) Q_{n+k-1/2}(z), \quad \text{Re } k > -\frac{1}{2} \quad (5.19)$$

to check that the position-space propagator (A.3) is indeed reproduced. For the deformed case, the sum cannot be evaluated in closed form. As we will see later, each term in the sum corresponds to the expected contribution of a descendant of the dual operator \mathcal{O} .

5.3 Correlation functions

The bulk field Φ is dual to an operator \mathcal{O} in the dual CFT. Solving the field equation on an asymptotically locally-AdS background near the boundary, one obtains the asymptotic solution

$$\Phi = e^{(\Delta-d)r} \left(\phi_{(0)} + \dots + e^{(d-2\Delta)r} \phi_{(2\Delta-d)} + \dots \right). \quad (5.20)$$

The expectation value $\langle \mathcal{O} \rangle$ in the presence of the source $\phi_{(0)}$ is given by

$$\langle \mathcal{O} \rangle_{\phi_{(0)}} = (d-2\Delta) \phi_{(2\Delta-d)}[\phi_{(0)}] + \text{local terms}. \quad (5.21)$$

The following comments are in order. Among the local terms in (5.21) there are some which are scheme-independent. In $d = 2$, the source $\phi_{(0)}$ generically contributes to the conformal anomaly for even integer Δ . Alternatively, for certain operator dimensions, the two-point function is not a well-defined distribution unless a dependence on the renormalisation scale is introduced. This fact is reflected in the local terms appearing in (5.21). For a CFT, their relation to the conformal anomaly is well understood, see e.g. [64] for a corresponding discussion in the holographic context. In particular, these local terms may lead to delta-function contributions in the two-point function $\langle \mathcal{O} \mathcal{O} \rangle$ (see [64] for explicit expressions).

In the presence of defects, there can be additional terms in the conformal anomaly which are localised on the defect. Such terms can be constructed from the sources of operators with integer dimension and derivative normal to the defect. E.g. for $\Delta = 2$ we can construct a combination $\delta(x_{\perp})\phi_{(0)}\partial_{\perp}\phi_{(0)}$, which can contribute to the conformal anomaly. In general the local terms in (5.21) include scheme-independent terms which contribute to the conformal anomaly and may appear as a delta-function contribution in the two-point function we are going to compute. In the following we will neglect such contributions and focus on non-coincident insertion points.⁸

The bulk solution with prescribed source $\phi_{(0)}$ can be obtained with the help of the bulk-to-boundary propagator as

$$\Phi(x) = \int d^d y K_{\Delta}(x, y) \phi_{(0)}(y). \tag{5.22}$$

Since we know the bulk-to-boundary propagator for the defect geometries, we can directly use the last formula to extract the two-point function by extracting the corresponding term in the near-boundary expansion and differentiating the one-point function with respect to the source. After performing these steps, we obtain the final result:

$$\begin{aligned} \langle \mathcal{O}(x_{\perp}, t) \mathcal{O}(x'_{\perp}, t') \rangle &= \frac{2^{3-2\Delta}}{\pi \kappa^{2\Delta} [\Gamma(\Delta - 1)]^2} \frac{1}{|x_{\perp} x'_{\perp}|^{\Delta}} \times \\ &\times \sum_{n=0}^{\infty} \text{sign}(\xi)^n (n + \Delta - 1/2) \frac{\Gamma(n + 2\Delta - 1)}{\Gamma(n + 1)} Q_{\nu}(|2\xi + 1|), \end{aligned} \tag{5.23}$$

where ξ was defined in (2.4) and the above formula applies regardless of whether the insertions are on the same side of the defect ($\xi > 0$) or not ($\xi < 0$). (5.23) has exactly the functional dependence (2.7) obtained in [2]. We recognize each term in the sum as a contribution from a particular descendant $\hat{\mathcal{O}}_n = \partial_{x_{\perp}}^n \hat{\mathcal{O}}$. Plots can be found in figure 3. We see in particular that the singularity when the two insertions are brought close to each other is not sensitive to the deformation parameter. In the presence of the defect, the correlator is also singular as one of the operators approaches the defect. Finally, there is a “screening” effect when the operators are placed on opposite sides of the defect, i.e. the correlator decays to zero quickly behind the defect.

⁸When computing correlation functions on pure AdS (without defects), the anomalous part of the correlation function can be read off from the logarithmic terms in the bulk-to-boundary propagator. However our propagator in the presence of the defect (5.21) is written as an infinite sum of terms, so to detect logarithmic terms it would be necessary to resum it.

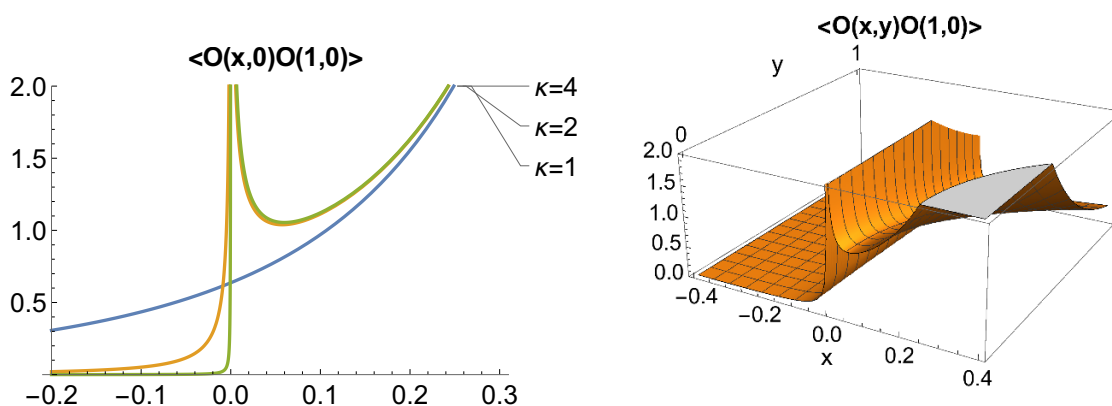


Figure 3. Plot of $\langle \mathcal{O}(x,0)\mathcal{O}(1,0) \rangle$ for $\kappa = 1, 2, 4$ and $\Delta = 2$ (left) and plot of $\langle \mathcal{O}(x,y)\mathcal{O}(1,0) \rangle$ for $\kappa = 2$ and $\Delta = 2$ (right).

Moreover, now we can read off the BOPE coefficient as

$$b_{\mathcal{O}\hat{\mathcal{O}}_n}^2 = 2^{2(\Delta-\hat{\Delta}_n)} \frac{\Gamma(\hat{\Delta}_n)}{\Gamma(\hat{\Delta}_n + 1/2)} \frac{2^{3-2\Delta}}{\sqrt{\pi}[\Gamma(\Delta - 1)]^2} \frac{1}{\kappa^{2\Delta}} (n + \Delta - 1/2) \frac{\Gamma(n + 2\Delta - 1)}{\Gamma(n + 1)}. \quad (5.24)$$

The formula (5.23) is the main result of this section. We already know that it does reduce to the usual CFT result when the defect goes away. Moreover the final result appears in a form which makes the BOPE block decomposition explicit. There is one more check we can perform. Let's consider the case in which the operator insertions are situated far away from the defect but close to each other. This corresponds to the limit $\xi \rightarrow 0$. In this limit we expect to find the decomposition of the two-point correlation functions in terms of the usual OPE, i.e. we should find the singularities of the form $1/\xi^\Delta$. Notice that each term in (5.23) diverges only logarithmically as $\xi \rightarrow 0$. Thus, to obtain the scaling behavior, we need to resum infinitely many terms in (5.23). To achieve this, we can use asymptotic expressions for large n .

At large n ($n \gg \kappa$) we can approximate $\nu \sim n/\kappa$. The asymptotic expansion of the Legendre function at large n is directly obtainable from the hypergeometric series representation

$$Q_\nu(z) = \sqrt{\frac{\pi}{2}} \frac{\Gamma[\nu + 1]}{\Gamma[\nu + 3/2]} \frac{[z - (z^2 - 1)^{1/2}]^{\nu+1/2}}{(z^2 - 1)^{1/4}} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \nu + \frac{3}{2}; \frac{\sqrt{z^2 - 1} - z}{2\sqrt{z^2 - 1}}\right). \quad (5.25)$$

Using this and Stirling's approximation for Γ -functions we can perform a sum in (5.23) from some large N to infinity. In the following, we will ignore numerical prefactors (except for the dependence on the deformation parameter κ) since we are after scaling behavior

only. The computation goes as follows:

$$\begin{aligned}
 & \sum_{n=N}^{\infty} (n + \Delta - 1/2) \frac{\Gamma(n + 2\Delta - 1)}{\Gamma(n + 1)} Q_{\nu}(2\xi + 1) \\
 & \sim \kappa^{1/2} \left(\frac{1}{\xi^{1/4}} + \mathcal{O}(\xi^{1/4}) \right) \sum_{n=N}^{\infty} n^{2\Delta-5/2} \left(1 + \mathcal{O}\left(\frac{1}{n}\right) \right) \zeta^{n/\kappa} \\
 & \quad \times \left(1 - \frac{\kappa}{4n} \frac{\zeta}{4\sqrt{\xi(\xi+1)}} + \mathcal{O}\left(\frac{1}{n^2}\right) \right), \tag{5.26}
 \end{aligned}$$

where we defined $\zeta = 2\xi + 1 - 2\sqrt{\xi(\xi+1)} \sim 1 - 2\sqrt{\xi}$. The leading contribution can be resummed to give⁹

$$\frac{\kappa^{1/2}}{\xi^{1/4}} \sum_{n=N}^{\infty} n^{2\Delta-5/2} \zeta^{n/\kappa} = \frac{\kappa^{1/2}}{\xi^{1/4}} \zeta^{N/\kappa} \Phi[\zeta^{1/\kappa}, \frac{3}{2} - 2\Delta, N], \tag{5.27}$$

where

$$\Phi[z, s, v] = \sum_{n=0}^{\infty} \frac{z^n}{(v+n)^s} \tag{5.28}$$

is the so-called Lerch transcendent [55]. The asymptotic behavior of the Lerch transcendent as $\zeta \rightarrow 1$ can be deduced from [55]

$$\Phi[z, s, v] \sim \frac{\Gamma[1-s]}{(1-z)^{1-s}}. \tag{5.29}$$

Plugging this into the (5.27) gives

$$\frac{\kappa^{1/2}}{\xi^{1/4}} \frac{1}{(1 - \zeta^{1/\kappa})^{2\Delta-1/2}} \sim \frac{\kappa^{2\Delta}}{2^{2\Delta-1/2}} \frac{1}{\xi^{\Delta}}, \tag{5.30}$$

which is exactly the identity contribution to the OPE in the bulk CFT!

It is easy to see from (5.23) that there are no more contributions at the leading order. In principle, one can now proceed systematically and discover additional contributions with subleading scaling behavior. These should be there since the one-point functions are in general non-zero in the presence of the defect. However, this procedure becomes prohibitively difficult even at the next order ($1/\xi^{\Delta-1/2}$). One reason is that in the expansion of the hypergeometric function there are infinitely-many terms contributing at each order.

6 Discussion

In the present paper, we have studied a particular family of top-down supersymmetric Janus solutions which asymptote to $\text{AdS}_3 \times S^3$ and are dual to two-dimensional defect conformal field theories. We have identified a scalar perturbation in the bulk which decouples at

⁹Alternatively one could approximate the sum by an integral, which in turn can be evaluated in terms of the exponential integral function. However the sum under consideration does reduce to a well-studied special function.

the leading order and solved the linearized field equation. Taking advantage of the simple structure of the background solution, we managed to construct explicitly the bulk-to-bulk propagator and to extract the two-point correlation function. The functional form of this correlator is not fixed by the symmetry alone. Rather, it involves an arbitrary function of an invariant cross-ratio. The final expression for the two-point correlation function is represented as a sum over boundary conformal blocks and makes manifest the data present in the OPE of the operator with the defect (dimensions and OPE coefficients). Finally, we have partially checked that our final result is crossing symmetric.

There are several directions one can now pursue. The theory which is holographically dual to the Janus solution under consideration is known in detail. Moreover, we have identified the microscopic description of the dual operator in the dual field theory. Hence, it is in principle possible to compute the same correlation function in the field theory at weak coupling.

Our analysis of the linearized fluctuations around the supergravity solution was not completely systematic. In particular, there is certainly room for other fluctuations to decouple at the leading order. One possibility would be to consider fluctuations of the B-field with the indices along the four-torus. Additionally, it would be natural to consider perturbations around the background solution that involve more than one field. Indeed, treating perturbations of this sort is required for computing holographically correlators involving the energy-momentum tensor or other conserved currents.

Furthermore, it would be natural to attempt extending our results to Janus solutions in higher dimension and with more asymptotic regions. Perturbations around the supersymmetric $\text{AdS}_5 \times S^5$ Janus solutions have been studied in [65], where particular modes that obey separable differential equations were identified. However, these equations are not of the hypergeometric type (they are known in the literature as Heun's equations) and in general their solutions can be studied only numerically. In case of solutions with more than two asymptotic regions, the differential equations would not be separable even in the asymptotically $\text{AdS}_3 \times S^3$ case.

Finally, further investigating finite-temperature solutions would be a particularly promising direction. Since we managed to solve for the linearized modes at finite temperature, one can proceed to compute quasi-normal modes and/or correlation functions, which are hard to compute even at weak coupling. It is not clear to us if this can be done as explicitly as for the zero-temperature case, but it is certainly possible to find solutions at least numerically. However, in order to proceed in this direction, a better understanding of the global properties of the finite-temperature solutions is required.

Acknowledgments

We would like to thank Marco Meineri and Michael Gutperle for useful discussions and Andy O'Bannon for helpful comments on the draft. The work of MC is supported in part by the Knut and Alice Wallenberg Foundation under grant KAW 2013.0235r.

A Bulk-to-boundary propagator in pure anti-de Sitter

It might be useful to demonstrate how the well-known bulk-to-boundary propagator in pure AdS can be represented using our defect modes. To do that we start with the expression

$$\phi(x_0, \vec{x}) = \int d^d \vec{x}' K_\Delta(x_0, \vec{x}, \vec{x}') \phi_0(\vec{x}'), \quad (\text{A.1})$$

where the bulk-to-boundary propagator in Poincaré coordinates is¹⁰

$$K_\Delta(x_0, \vec{x}, \vec{x}') = C \frac{x_0^\Delta}{(x_0^2 + (\vec{x} - \vec{x}')^2)^\Delta}, \quad C = \pi^{-d/2} \frac{\Gamma(\Delta)}{\Gamma(\Delta - d/2)}. \quad (\text{A.3})$$

Here $\phi_0(\vec{x}')$ represents the source on the boundary of AdS. Next we decompose boundary directions as $\vec{x} = (x_\perp, \vec{y})$ and similarly for the location of the source \vec{x}' and go to hyperbolic slicing by defining new coordinates z and x by

$$x_0 = \frac{z}{\cosh x}, \quad x_\perp = z \tanh x. \quad (\text{A.4})$$

Now we Fourier-transform along the directions of the defect to obtain (in Euclidean signature)¹¹

$$\begin{aligned} K_\Delta(x_0, \vec{x}, \vec{x}') &= \frac{2}{\sqrt{\pi} \Gamma(\Delta - d/2)} \frac{z^\Delta}{\cosh^\Delta x} \frac{1}{(2\pi)^{d-1}} \\ &\times \int d^{d-1} k e^{-i\vec{k} \cdot (\vec{y} - \vec{y}')} \left(\frac{k^2}{2}\right)^{\Delta + \frac{1-d}{2}} \frac{K_{\Delta + \frac{1-d}{2}}(kw)}{(k\omega)^{\Delta + \frac{1-d}{2}}}, \end{aligned} \quad (\text{A.6})$$

where

$$w = \sqrt{z^2 - 2z \tanh x x'_\perp + x'^2_\perp}, \quad (\text{A.7})$$

and K_ν denotes the modified Bessel function. Now we use the Gegenbauer's addition theorem for Bessel functions which says [61]

$$\frac{K_\nu(kw)}{(kw)^\nu} = 2^\nu \Gamma(\nu) \sum_{n=0}^{\infty} (\nu + n) \frac{K_{\nu+n}(kz)}{(kz)^\nu} \frac{I_{\nu+n}(kx'_\perp)}{(kx'_\perp)^\nu} C_n^{\Delta-1/2}(\tanh x) \quad (\text{A.8})$$

for $z > |x'_\perp|$. Whereas for $z < |x'_\perp|$ the arguments of the Bessel functions should be exchanged. In Lorentzian signature we would need [61]

$$\frac{H_\nu(kw)}{(kw)^\nu} = 2^\nu \Gamma(\nu) \sum_{n=0}^{\infty} (\nu + n) \frac{H_{\nu+n}(kz)}{(kz)^\nu} \frac{J_{\nu+n}(kx'_\perp)}{(kx'_\perp)^\nu} C_n^{\Delta-1/2}(\tanh x). \quad (\text{A.9})$$

¹⁰Recall that the propagator is normalized in such a way that

$$\lim_{x_0 \rightarrow 0} x_0^{\Delta-2} K_\Delta(x_0, \vec{x}, \vec{x}') = \delta(\vec{x} - \vec{x}'). \quad (\text{A.2})$$

¹¹To evaluate this multi-dimensional Fourier transform we note that for a spherically symmetric function $f(\vec{y}) = f(|\vec{y}|)$ the Fourier transform is related to the so-called Hankel transform through

$$\int d^n k e^{-i\vec{k} \cdot \vec{y}} f(\vec{y}) = (2\pi)^{n/2} k^{1-n/2} \int_0^\infty r^{n/2} f(r) J_{\frac{n}{2}-1}(kr) dr. \quad (\text{A.5})$$

Thus the bulk-to-boundary propagator in pure AdS can be expressed as

$$\begin{aligned}
 K_{\Delta}(x_0, \vec{x}, \vec{x}') &= \frac{2}{\sqrt{\pi}} \frac{\Gamma(\Delta - d/2 + 1/2)}{\Gamma(\Delta - d/2)} \frac{z^{\Delta}}{\cosh^{\Delta} x} \frac{1}{(2\pi)^{d-1}} \int d^{d-1} k e^{-i\vec{k}\cdot(\vec{y}-\vec{y}')} k^{2\Delta+1-d} \times \\
 &\quad \sum_{n=0}^{\infty} \left(\Delta + \frac{1-d}{2} + n \right) \left[\theta(z - x'_{\perp}) \frac{K_{\Delta+\frac{1-d}{2}+n}(kz)}{(kz)^{\Delta+\frac{1-d}{2}}} \frac{I_{\Delta+\frac{1-d}{2}+n}(kx'_{\perp})}{(kx'_{\perp})^{\Delta+\frac{1-d}{2}}} + \right. \\
 &\quad \left. + \theta(x'_{\perp} - z) \frac{I_{\Delta+\frac{1-d}{2}+n}(kz)}{(kz)^{\Delta+\frac{1-d}{2}}} \frac{K_{\Delta+\frac{1-d}{2}+n}(kx'_{\perp})}{(kx'_{\perp})^{\Delta+\frac{1-d}{2}}} \right] C_n^{\Delta+\frac{1-d}{2}}(\tanh x).
 \end{aligned}$$

Finally notice that for integer $\Delta - d/2$ the Gegenbauer function is related to the associated Legendre functions by [55]

$$C_n^{\Delta-1/2}(\tanh x) = \frac{\Gamma(\Delta)}{\Gamma(2\Delta-1)} (-2 \cosh x)^{\Delta-1} P_{n+\Delta-1}^{\Delta-1}(\tanh x). \quad (\text{A.10})$$

(For non-integer Δ there is also $Q_{n+\Delta-1}^{\Delta-1}(\tanh x)$ appearing in the final equality). Thus we see that the bulk-to-boundary propagator can be expressed using the defect modes derived before.

Let us now specialise to $d = 2$ and integer Δ , so that \vec{y} becomes just t and \vec{k} reduces to ω . We get

$$\begin{aligned}
 K_{\Delta}(x_0, \vec{x}, \vec{x}') &= \left(-\frac{1}{2} \right)^{\Delta-1} \frac{1}{\Gamma(\Delta-1)} \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} \times \\
 &\quad \times \sum_{n=0}^{\infty} (n + \Delta - 1/2) \tilde{G}_{n+\Delta-1/2}^{\text{Eucl}}[z, x'_{\perp}] \frac{1}{x'_{\perp}{}^{\Delta}} \frac{P_{n+\Delta-1}^{\Delta-1}(\tanh x)}{\cosh x},
 \end{aligned} \quad (\text{A.11})$$

where $\tilde{G}_{n+\Delta-1/2}^{\text{Eucl}}[z, x'_{\perp}]$ is obtained from (5.3) by analytically continuing $\omega^2 \rightarrow -\omega^2$ which effectively replaces Bessel functions according to $J_{\nu} \rightarrow I_{\nu}$ and $H_{\nu} \rightarrow K_{\nu}$ ($K_{\nu}(|\omega|z)$ is regular at large z). (A.11) indeed coincides with the pure AdS limit of (5.15).

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] J.L. Cardy, *Conformal Invariance and Surface Critical Behavior*, *Nucl. Phys. B* **240** (1984) 514 [[INSPIRE](#)].
- [2] D.M. McAvity and H. Osborn, *Conformal field theories near a boundary in general dimensions*, *Nucl. Phys. B* **455** (1995) 522 [[cond-mat/9505127](#)] [[INSPIRE](#)].
- [3] M. Billò, V. Gonçalves, E. Lauria and M. Meineri, *Defects in conformal field theory*, *JHEP* **04** (2016) 091 [[arXiv:1601.02883](#)] [[INSPIRE](#)].
- [4] A. Gadde, *Conformal constraints on defects*, [arXiv:1602.06354](#) [[INSPIRE](#)].
- [5] D. Friedan and A. Konechny, *On the boundary entropy of one-dimensional quantum systems at low temperature*, *Phys. Rev. Lett.* **93** (2004) 030402 [[hep-th/0312197](#)] [[INSPIRE](#)].

- [6] K. Jensen and A. O’Bannon, *Constraint on Defect and Boundary Renormalization Group Flows*, *Phys. Rev. Lett.* **116** (2016) 091601 [[arXiv:1509.02160](#)] [[INSPIRE](#)].
- [7] H. Casini, I.S. Landea and G. Torroba, *The g-theorem and quantum information theory*, *JHEP* **10** (2016) 140 [[arXiv:1607.00390](#)] [[INSPIRE](#)].
- [8] C. Bachas and M. Petropoulos, *Anti-de Sitter D-branes*, *JHEP* **02** (2001) 025 [[hep-th/0012234](#)] [[INSPIRE](#)].
- [9] C. Bachas, J. de Boer, R. Dijkgraaf and H. Ooguri, *Permeable conformal walls and holography*, *JHEP* **06** (2002) 027 [[hep-th/0111210](#)] [[INSPIRE](#)].
- [10] T. Takayanagi, *Holographic Dual of BCFT*, *Phys. Rev. Lett.* **107** (2011) 101602 [[arXiv:1105.5165](#)] [[INSPIRE](#)].
- [11] M. Fujita, T. Takayanagi and E. Tonni, *Aspects of AdS/BCFT*, *JHEP* **11** (2011) 043 [[arXiv:1108.5152](#)] [[INSPIRE](#)].
- [12] J. Erdmenger, M. Flory and M.-N. Newrzella, *Bending branes for DCFT in two dimensions*, *JHEP* **01** (2015) 058 [[arXiv:1410.7811](#)] [[INSPIRE](#)].
- [13] J. Erdmenger, C. Hoyos, A. O’Bannon and J. Wu, *A Holographic Model of the Kondo Effect*, *JHEP* **12** (2013) 086 [[arXiv:1310.3271](#)] [[INSPIRE](#)].
- [14] A. O’Bannon, I. Papadimitriou and J. Probst, *A Holographic Two-Impurity Kondo Model*, *JHEP* **01** (2016) 103 [[arXiv:1510.08123](#)] [[INSPIRE](#)].
- [15] J. Erdmenger, M. Flory, C. Hoyos, M.-N. Newrzella and J.M.S. Wu, *Entanglement Entropy in a Holographic Kondo Model*, *Fortsch. Phys.* **64** (2016) 109 [[arXiv:1511.03666](#)] [[INSPIRE](#)].
- [16] J. Erdmenger, C. Hoyos, A. O’Bannon, I. Papadimitriou, J. Probst and J.M.S. Wu, *Holographic Kondo and Fano Resonances*, [arXiv:1611.09368](#) [[INSPIRE](#)].
- [17] J. Erdmenger, C. Hoyos, A. O’Bannon, I. Papadimitriou, J. Probst and J.M.S. Wu, *Two-point Functions in a Holographic Kondo Model*, *JHEP* **03** (2017) 039 [[arXiv:1612.02005](#)] [[INSPIRE](#)].
- [18] D. Bak, M. Gutperle and S. Hirano, *A Dilatonic deformation of AdS₅ and its field theory dual*, *JHEP* **05** (2003) 072 [[hep-th/0304129](#)] [[INSPIRE](#)].
- [19] I. Papadimitriou and K. Skenderis, *Correlation functions in holographic RG flows*, *JHEP* **10** (2004) 075 [[hep-th/0407071](#)] [[INSPIRE](#)].
- [20] A.B. Clark, D.Z. Freedman, A. Karch and M. Schnabl, *Dual of the Janus solution: An interface conformal field theory*, *Phys. Rev.* **D 71** (2005) 066003 [[hep-th/0407073](#)] [[INSPIRE](#)].
- [21] E. D’Hoker, J. Estes and M. Gutperle, *Ten-dimensional supersymmetric Janus solutions*, *Nucl. Phys.* **B 757** (2006) 79 [[hep-th/0603012](#)] [[INSPIRE](#)].
- [22] E. D’Hoker, J. Estes and M. Gutperle, *Exact half-BPS Type IIB interface solutions. I. Local solution and supersymmetric Janus*, *JHEP* **06** (2007) 021 [[arXiv:0705.0022](#)] [[INSPIRE](#)].
- [23] E. D’Hoker, J. Estes and M. Gutperle, *Exact half-BPS Type IIB interface solutions. II. Flux solutions and multi-Janus*, *JHEP* **06** (2007) 022 [[arXiv:0705.0024](#)] [[INSPIRE](#)].
- [24] O. Aharony, L. Berdichevsky, M. Berkooz and I. Shamir, *Near-horizon solutions for D3-branes ending on 5-branes*, *Phys. Rev.* **D 84** (2011) 126003 [[arXiv:1106.1870](#)] [[INSPIRE](#)].
- [25] R. Benichou and J. Estes, *Geometry of Open Strings Ending on Backreacting D3-branes*, *JHEP* **03** (2012) 025 [[arXiv:1112.3035](#)] [[INSPIRE](#)].

- [26] A.B. Clark, N. Crossette, G.M. Newman and A. Rommal, *AdS-Sliced Flavor Branes and Adding Flavor to the Janus Solution*, *Phys. Rev. D* **89** (2014) 026014 [[arXiv:1309.7872](#)] [[INSPIRE](#)].
- [27] A. Clark and A. Karch, *Super Janus*, *JHEP* **10** (2005) 094 [[hep-th/0506265](#)] [[INSPIRE](#)].
- [28] M. Suh, *Supersymmetric Janus solutions in five and ten dimensions*, *JHEP* **09** (2011) 064 [[arXiv:1107.2796](#)] [[INSPIRE](#)].
- [29] D. Gaiotto and E. Witten, *S-duality of Boundary Conditions In $N = 4$ Super Yang-Mills Theory*, *Adv. Theor. Math. Phys.* **13** (2009) 721 [[arXiv:0807.3720](#)] [[INSPIRE](#)].
- [30] D. Gaiotto and E. Witten, *Janus Configurations, Chern-Simons Couplings, And The theta-Angle in $N = 4$ Super Yang-Mills Theory*, *JHEP* **06** (2010) 097 [[arXiv:0804.2907](#)] [[INSPIRE](#)].
- [31] E. D'Hoker, J. Estes, M. Gutperle and D. Krym, *Exact Half-BPS Flux Solutions in M-theory. I: Local Solutions*, *JHEP* **08** (2008) 028 [[arXiv:0806.0605](#)] [[INSPIRE](#)].
- [32] E. D'Hoker, J. Estes, M. Gutperle and D. Krym, *Exact Half-BPS Flux Solutions in M-theory II: Global solutions asymptotic to $AdS_7 \times S^4$* , *JHEP* **12** (2008) 044 [[arXiv:0810.4647](#)] [[INSPIRE](#)].
- [33] L. Berdichevsky and B.-e. Dahan, *Local gravitational solutions dual to M2-branes intersecting and/or ending on M5-branes*, *JHEP* **08** (2013) 061 [[arXiv:1304.4389](#)] [[INSPIRE](#)].
- [34] E. D'Hoker, J. Estes, M. Gutperle, D. Krym and P. Sorba, *Half-BPS supergravity solutions and superalgebras*, *JHEP* **12** (2008) 047 [[arXiv:0810.1484](#)] [[INSPIRE](#)].
- [35] T. Azeyanagi, A. Karch, T. Takayanagi and E.G. Thompson, *Holographic calculation of boundary entropy*, *JHEP* **03** (2008) 054 [[arXiv:0712.1850](#)] [[INSPIRE](#)].
- [36] M. Chiodaroli, M. Gutperle and D. Krym, *Half-BPS Solutions locally asymptotic to $AdS_3 \times S^3$ and interface conformal field theories*, *JHEP* **02** (2010) 066 [[arXiv:0910.0466](#)] [[INSPIRE](#)].
- [37] M. Chiodaroli, M. Gutperle, L.-Y. Hung and D. Krym, *String Junctions and Holographic Interfaces*, *Phys. Rev. D* **83** (2011) 026003 [[arXiv:1010.2758](#)] [[INSPIRE](#)].
- [38] M. Chiodaroli, E. D'Hoker and M. Gutperle, *Open Worldsheets for Holographic Interfaces*, *JHEP* **03** (2010) 060 [[arXiv:0912.4679](#)] [[INSPIRE](#)].
- [39] M. Chiodaroli, M. Gutperle and L.-Y. Hung, *Boundary entropy of supersymmetric Janus solutions*, *JHEP* **09** (2010) 082 [[arXiv:1005.4433](#)] [[INSPIRE](#)].
- [40] M. Chiodaroli, E. D'Hoker, Y. Guo and M. Gutperle, *Exact half-BPS string-junction solutions in six-dimensional supergravity*, *JHEP* **12** (2011) 086 [[arXiv:1107.1722](#)] [[INSPIRE](#)].
- [41] M. Chiodaroli, E. D'Hoker and M. Gutperle, *Simple Holographic Duals to Boundary CFTs*, *JHEP* **02** (2012) 005 [[arXiv:1111.6912](#)] [[INSPIRE](#)].
- [42] M. Chiodaroli, E. D'Hoker and M. Gutperle, *Holographic duals of Boundary CFTs*, *JHEP* **07** (2012) 177 [[arXiv:1205.5303](#)] [[INSPIRE](#)].
- [43] D. Bak, M. Gutperle and R.A. Janik, *Janus Black Holes*, *JHEP* **10** (2011) 056 [[arXiv:1109.2736](#)] [[INSPIRE](#)].
- [44] J. Estes, *Finite temperature holographic duals of 2-dimensional BCFTs*, *JHEP* **07** (2015) 020 [[arXiv:1503.07375](#)] [[INSPIRE](#)].
- [45] D. Bak and H. Min, *Multi-faced Black Janus and Entanglement*, *JHEP* **03** (2014) 046 [[arXiv:1311.5259](#)] [[INSPIRE](#)].

- [46] J. Estes, K. Jensen, A. O’Bannon, E. Tsatis and T. Wrase, *On Holographic Defect Entropy*, *JHEP* **05** (2014) 084 [[arXiv:1403.6475](#)] [[INSPIRE](#)].
- [47] M. Gutperle and J.D. Miller, *Entanglement entropy at holographic interfaces*, *Phys. Rev. D* **93** (2016) 026006 [[arXiv:1511.08955](#)] [[INSPIRE](#)].
- [48] M. Gutperle and A. Trivella, *Note on entanglement entropy and regularization in holographic interface theories*, *Phys. Rev. D* **95** (2017) 066009 [[arXiv:1611.07595](#)] [[INSPIRE](#)].
- [49] S. Ryu and T. Takayanagi, *Holographic derivation of entanglement entropy from AdS/CFT*, *Phys. Rev. Lett.* **96** (2006) 181602 [[hep-th/0603001](#)] [[INSPIRE](#)].
- [50] K. Jensen and A. O’Bannon, *Holography, Entanglement Entropy and Conformal Field Theories with Boundaries or Defects*, *Phys. Rev. D* **88** (2013) 106006 [[arXiv:1309.4523](#)] [[INSPIRE](#)].
- [51] Y. Korovin, *First order formalism for the holographic duals of defect CFTs*, *JHEP* **04** (2014) 152 [[arXiv:1312.0089](#)] [[INSPIRE](#)].
- [52] L. Bianchi, S. Chapman, X. Dong, D.A. Galante, M. Meineri and R.C. Myers, *Shape dependence of holographic Rényi entropy in general dimensions*, *JHEP* **11** (2016) 180 [[arXiv:1607.07418](#)] [[INSPIRE](#)].
- [53] O. Aharony, O. DeWolfe, D.Z. Freedman and A. Karch, *Defect conformal field theory and locally localized gravity*, *JHEP* **07** (2003) 030 [[hep-th/0303249](#)] [[INSPIRE](#)].
- [54] P. Liendo, L. Rastelli and B.C. van Rees, *The Bootstrap Program for Boundary CFT_d*, *JHEP* **07** (2013) 113 [[arXiv:1210.4258](#)] [[INSPIRE](#)].
- [55] BATEMAN MANUSCRIPT PROJECT, H. Bateman et al., *Higher transcendental functions*, volume 1, U.S. Office of Naval Research, McGraw-Hill (1953).
- [56] E. Witten, *On the conformal field theory of the Higgs branch*, *JHEP* **07** (1997) 003 [[hep-th/9707093](#)] [[INSPIRE](#)].
- [57] C. Vafa, *Instantons on D-branes*, *Nucl. Phys. B* **463** (1996) 435 [[hep-th/9512078](#)] [[INSPIRE](#)].
- [58] R. Dijkgraaf, *Instanton strings and hyperKähler geometry*, *Nucl. Phys. B* **543** (1999) 545 [[hep-th/9810210](#)] [[INSPIRE](#)].
- [59] J.R. David, G. Mandal and S.R. Wadia, *Microscopic formulation of black holes in string theory*, *Phys. Rept.* **369** (2002) 549 [[hep-th/0203048](#)] [[INSPIRE](#)].
- [60] S.B. Giddings, *The Boundary S matrix and the AdS to CFT dictionary*, *Phys. Rev. Lett.* **83** (1999) 2707 [[hep-th/9903048](#)] [[INSPIRE](#)].
- [61] G. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Mathematical Library, Cambridge University Press (1995).
- [62] W. Magnus, F. Oberhettinger and R. Soni, *Formulas and theorems for the special functions of mathematical physics*, Grundlehren der mathematischen Wissenschaften, Berlin (1966).
- [63] R. Szmtykowski, *Some integrals and series involving the Gegenbauer polynomials and the Legendre functions on the cut $(-1, 1)$* , [arXiv:1107.2680](#).
- [64] A. Petkou and K. Skenderis, *A nonrenormalization theorem for conformal anomalies*, *Nucl. Phys. B* **561** (1999) 100 [[hep-th/9906030](#)] [[INSPIRE](#)].
- [65] C. Bachas and J. Estes, *Spin-2 spectrum of defect theories*, *JHEP* **06** (2011) 005 [[arXiv:1103.2800](#)] [[INSPIRE](#)].