

Fisher Metric, Geometric Entanglement and Spin Networks

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We introduce the geometric formulation of Quantum Mechanics in the quantum gravity context, and we use it to give a tensorial characterization of entanglement on spin network states. Starting from the simplest case of a single-link graph (Wilson line), we define a dictionary to construct a Riemannian metric tensor and a symplectic structure on the space of spin network states, showing how they fully encode the information about separability and entanglement, and, in particular, an entanglement monotone interpreted as a distance with respect to the separable state. In the maximally entangled gauge-invariant case, the entanglement monotone is proportional to a power of the area of the surface dual to the link thus supporting a connection between entanglement and the (simplicial) geometric properties of spin network states. We extend then such analysis to the study of non-local correlations between two non-adjacent regions of a generic spin network. In the end, our analysis shows that the same spin network graph can be understood as an information graph whose connectivity encodes, both at the local and non-local level, the quantum correlations among its parts. This gives a further connection between entanglement and geometry.

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I. INTRODUCTION

Background-independent candidates to a full theory of Quantum Gravity, such as Loop Quantum Gravity [1–5], the modern incarnation of the canonical quantization programme for the gravitational field, together with its covariant counterpart (spin foam models), and Group Field Theory (GFT) [6–9], a closely related formalism sharing the same type of fundamental degrees of freedom, propose a radically new picture of the microscopic quantum structure of spacetime, in which at very small scales continuum space, time and geometry dissolve into non-geometric, purely combinatorial and algebraic (group-theoretic) entities. These entities can be described in terms of spin networks, graphs coloured by irreducible representations of the local gauge group (the Lorentz group, then

usually gauge-fixed to $SU(2)$). Quantum spin network states represent elementary excitations of spacetime itself, and geometric observables are operators acting on them. For example, areas and volumes correspond to quantum operators which are diagonalized on spin network states. The heuristic picture for spin networks is therefore that of “grains of space”.

The key issue, then, becomes the reconstruction or “emergence” of continuum spacetime and geometry from such microscopic building blocks, in some approximate regime of the quantum dynamics. This issue is intertwined with, but goes also much beyond, the difficulties with defining a notion of locality in spacetime due to diffeomorphism invariance.

Various strategies for addressing this open issue are being explored. At a more formal level, they all aim at a better control over the regime of the fundamental theory involving a large number of fundamental degrees of freedom, and therefore rest on the renormalization of quantum gravity models. This line of research has witnessed a tremendous progress in the context of renormalization of group field theory models [10], which amounts automatically to a renormalization of the corresponding spin foam models, as well as in the context of spin foam models understood as generalised lattice gauge theory techniques [11]. At a more directly physical level, the effective continuum dynamics emerging from quantum gravity models has been studied, for example, in the formalism of group field theory condensate cosmology in [12], so far limited to spatially homogenous universes.

There are many hints that entanglement and tools from quantum information theory should play a crucial role both in the characterization of the intrinsic properties of the quantum texture of spacetime and in the reconstruction of its geometry. For instance, even if based on completely different grounds both at the conceptual and technical level, recent developments in AdS/CFT have shown that the entanglement of particles on the boundary is directly related to the connectivity of the bulk regions thus suggesting that our three-dimensional space is held together by quantum entanglement [13]. Later works based on the so-called *Ryu-Takayanagi formula*, which relates the entanglement entropy in a conformal field theory to the area of a minimal surface in its holographic dual [14], has also shown that the stress-energy tensor near the boundary of a bulk spacetime region can be reconstructed from the entanglement on the boundary [15, 16].

Also on the side of fundamental quantum gravity formalisms based on spin networks, there are many proposals to use quantum information to reconstruct geometrical notions such as distance in terms of the entanglement on spin network states [17–19]. The idea is that in a purely relational, background independent context only correlations have a physical meaning and it seems reasonable to regard spin networks themselves as networks of quantum correlations between regions of space and then derive geometrical properties from the intrinsic informational content of the theory. In fact, there has been also a lot of activity, recently, in connecting spin network states and tensor networks, which are a crucial tool for controlling the entanglement structure of many-body quantum states, and in using the same link to extract information about the entanglement entropy encoded in spin network states [20, 21]. In particular, in [21], a precise dictionary between tensor networks, spin networks and group field theory states has been established, and used as a basis for a derivation of the Ryu-Takayanagi formula in a quantum gravity context. Other work on entanglement in spin network states can be found in [22, 23]. Moreover, part of the cited work on spin foam renormalization [11] relies as well on tensor network techniques, providing another route for understanding how the entanglement of spin network states affects the continuum limit of quantum gravity models.

This work aims at providing further insights on the transition “*from pregeometry to geometry*” by introducing the machinery of the geometric formulation of quantum mechanics (GQM) in the quantum gravity context. Indeed, the usual Hilbert space of a quantum mechanical system can be equipped with a Kähler manifold structure inheriting both a Riemannian metric tensor and a symplectic structure from the underlying complex projective space of rays. This allows to im-

port the powerful machinery of differential geometry in QM and, in particular, to characterize the entanglement properties of a composite system in a purely tensorial fashion [24, 25].

Along the same line, we can use quantum tensors to characterize the entanglement on spin network states. The advantages of this formalism are both computational and conceptual. Indeed, unlike the calculations involving entanglement entropy, it does not require the explicit knowledge of the Schmidt coefficients. Moreover, the key structures of the formalism are built purely from the space of states without introducing additional external structures. We will give further motivations for the use of such geometric techniques in a quantum gravity context, in the following.

The paper is organized as follows. In section II we give a review of the role of spin networks in QG and of their interpretation as quanta of space which, because of diffeomorphism invariance, offer a realization of the pregeometric scenario in terms of building blocks of combinatorial and algebraic nature. A notion of entanglement on spin network states can be given in terms of the local correlations coming from gluing links (or surfaces in the dual picture) of open spin network vertices. In particular, the gauge-invariance requirement at the gluing nodes implies a locally maximally entangled state [26].

Section III introduces to the basics of Geometric Quantum Mechanics. The space of rays is recognized to be the proper setting for the description of quantum systems and its main geometric structures are pointed out. In particular, we focus on the structures on the Hilbert space which can be regarded as the pull-back of the tensors defined on the ray space. This is the so-called Fubini-Study Hermitian tensor whose real and imaginary parts provide us with a metric and a symplectic structure, respectively. For pure states, this metric is the well-known Fisher-Rao metric.

Such a pull-back procedure turns out to be very useful in the analysis of bipartite composite systems. Indeed, the pulled-back Hermitian tensor on orbit submanifolds of quantum states related by unitary transformations decomposes in block matrices which encode all the information about the separability or entangled nature of the fiducial state of the given orbit. Of particular interest are the off-diagonal blocks of the metric part which encode the information on quantum correlations between the subsystems and define an entanglement measure interpreted as a distance with respect to the separable case.

Finally, we set up a correspondence between the GQM formalism and spin networks by focusing first on the most simple case of a single Wilson line (Sec. IV-VI) and then extending the analysis to the case of two non-adjacent regions of a generic spin network state (Sec. VII). The analysis of the tensorial structures leads us to rethink of the spin network graph structure itself as an information graph whose connectivity is encoded, both at the local and non-local level, on the choice of the entangled state. The entanglement measure involving the off-diagonal blocks of the metric tensor can be therefore interpreted ultimately as a measure of graph connectivity. By exploiting the geometric information carried by a spin network state we are able to define a connection between entanglement and geometry.

II. QUANTUM STATES OF GEOMETRY: SPIN NETWORKS

We provide here a concise introduction to spin network states, focussing on their interpretation as elementary building blocks of quantum spacetime. For a more detailed discussion we refer to the literature cited in the introduction, as well as to [27–35].

A. Spin networks from canonical Loop Quantum Gravity

Loop Quantum Gravity starts from Einstein's General Relativity (GR) recast into the form of a *gauge theory* with structure group $Spin(1,3)$, plus the additional gauge symmetries resulting from the space-time diffeomorphism invariance. A partial fixing of the $Spin(1,3)$ invariance leads to a phase space description of the classical theory in terms of connections A of a principal $SU(2)$ -bundle over spacelike hypersurfaces Σ , embedded in a spacetime manifold \mathcal{M} , and sections E of the associated vector bundle over Σ , whose pull back are Lie algebra valued pseudo two forms.

The two forms E encode the information about the 3d geometry on Σ , while the A carries the information about the extrinsic curvature of Σ in \mathcal{M} .

The conjugated variables (A, E) , with standard Poisson brackets, define a (Yang–Mills like) phase space, which is then reduced by the imposition of the $SU(2)$ Gauss constraint, the spatial diffeomorphism constraint and the Hamiltonian constraint, respectively implementing the internal local gauge symmetry and the symmetry under diffeomorphisms.

The Dirac quantization procedure, before the imposition of the diffeomorphism constraints, leads to Hilbert spaces \mathcal{H}_Γ associated to graphs embedded in the canonical manifold. Let us describe the structure of the corresponding states.

Let $\Gamma \subset \Sigma$ be a graph, i.e., a finite and ordered collection of smooth oriented paths $\gamma_\ell \in \Sigma$ with $\ell = 1, \dots, L$ meeting at most at their endpoints (such paths will be called the *links* or the *edges* of the graph, while the intersection points will be called *nodes* or *vertices*), and let $\psi : SU(2)^L \rightarrow \mathbb{C}$ be a (smooth) (cylindrical) function $\psi_\Gamma(h_1, \dots, h_L)$ of L group elements. These group elements are interpreted as parallel transports $h_\ell(A) \equiv h_{\gamma_\ell}(A)$ of the connection A along the links γ_ℓ of the graph Γ , embedded in the canonical hypersurface. The linear space of such cylindrical functionals w.r.t. a given graph Γ can be turned into a Hilbert space by equipping it with the following scalar product

$$\langle \psi_{(\Gamma)} | \psi'_{(\Gamma)} \rangle \equiv \int \prod_{\ell=1}^L dh_\ell \overline{\psi(h_1, \dots, h_L)} \psi'(h_1, \dots, h_L), \quad (1)$$

where dh_ℓ are L copies of the (left- and right-invariant) Haar measure of $SU(2)$. The inner product (1) is invariant under $SU(2)$ gauge transformations acting as left or right multiplications on the arguments of the wave functions ψ_Γ , depending on whether the gauge transformation is associated to the starting or end point of the link to which each argument is referring to. This is a direct consequence of the invariance of the Haar measure. One then needs to construct a Hilbert space out of the space of all cylindrical functions for all graphs $\Gamma \subset \Sigma$:

$$\bigcup_{\Gamma \subset \Sigma} \mathcal{H}_\Gamma. \quad (2)$$

To do this, we need to define a scalar product for cylindrical functions based on different graphs. Such a scalar product can be deduced from that on \mathcal{H}_Γ as follows. The construction is based on the introduction of the so-called *cylindrical equivalence relations* which reflect properties of the underlying continuum connection field, and it is therefore directly inspired by the continuum embedding of the graphs Γ , and thus on the origin of the quantum states ψ_Γ as coming from the canonical quantization of a continuum field theory. The details of the construction are not so important for our purposes. Essentially, we define equivalence classes of graphs that can be

regarded as subgraphs of a bigger one, i.e.:

$$[\Gamma] = \{ \Gamma_1 \sim \Gamma_2 \text{ iff } \exists \Gamma \subset \Sigma : \Gamma \supset \Gamma_1, \Gamma_2 \} . \quad (3)$$

and the inner product between states associated to different graphs can be defined to be like in (1) for such bigger graph by defining

$$\langle \psi_{\Gamma_1} | \psi'_{\Gamma_2} \rangle \equiv \langle \psi_{\Gamma} | \psi'_{\Gamma} \rangle , \quad (4)$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$ and the functions $\psi_{\Gamma_1}, \psi'_{\Gamma_2}$ are trivially extended on Γ by setting them constant over the links which do not belong to Γ_1, Γ_2 , respectively¹. The (unconstrained) kinematical Hilbert space will be therefore given by:

$$\mathcal{H}_{kin} = \frac{\bigcup_{\Gamma \subset \Sigma} \mathcal{H}_{\Gamma}}{\sim} . \quad (5)$$

In the spin representation, the equivalence condition \sim which allows us to define the scalar product (4) amounts to take all spins zero on the “virtual” links of the extended graph which do not belong to the starting one [35]. This implies that the Hilbert space (5) can be recasted as a direct sum of single graph-based Hilbert spaces

$$\mathcal{H}_{kin} = \frac{\bigcup_{\Gamma \subset \Sigma} \mathcal{H}_{\Gamma}}{\sim} = \bigoplus_{\Gamma \subset \Sigma} \tilde{\mathcal{H}}_{\Gamma} , \quad (6)$$

but the individual graph-based Hilbert spaces $\tilde{\mathcal{H}}_{\Gamma}$ correspond to \mathcal{H}_{Γ} without zero modes, i.e., where the spins j_{ℓ} never take the value zero. This space equipped with the above inner product can then be understood as a Hilbert space over “generalized” connections on Σ with the so-called Ashtekar-Lewandowski measure $d\mu_{AL}$ [36–38],

$$\mathcal{H}_{kin} \cong L^2(A, d\mu_{AL}) . \quad (7)$$

In the following, we will be mainly concerned with the space associated to each graph Γ , thus with \mathcal{H}_{Γ} , and the cylindrical equivalence conditions will not play much of a role.

Finally, let us introduce a basis for the kinematical Hilbert space, and more precisely for each \mathcal{H}_{Γ} . According to the Peter-Weyl theorem [39], a cylindrical function $\psi_{\Gamma} \in \mathcal{H}_{\Gamma} \cong L^2(SU(2)^L, d\mu_{Haar})$ can be decomposed as

$$\psi_{\Gamma} = \sum_{j_{\ell}, m_{\ell}, n_{\ell}} f_{m_1, \dots, m_L, n_1, \dots, n_L} D_{m_1 n_1}^{(j_1)}(h_1) \dots D_{m_L n_L}^{(j_L)}(h_L) , \quad (8)$$

where $D_{m_{\ell} n_{\ell}}^{(j_{\ell})}(h_{\ell})$ are the Wigner matrices which give the spin- j irreducible representations of the group elements $h_{\ell} \in SU(2)$. An orthonormal basis for the Hilbert space \mathcal{H}_{Γ} is thus provided by

$$\langle \vec{h} | \Gamma; \vec{j}, \vec{m}, \vec{n} \rangle \equiv D_{m_1 n_1}^{(j_1)}(h_1) \dots D_{m_L n_L}^{(j_L)}(h_L) , \quad (9)$$

where we have used a compact vectorial notation $\vec{j}, \vec{m}, \vec{n}$ to denote the spin labels of the unitary

¹ Obviously, this new product reduces to the previous one if Γ_1 and Γ_2 coincide.

irreducible representations of $SU(2)$ associated with each link of the graph, and similarly for the corresponding group elements. Given a notion of kinematical unconstrained Hilbert space \mathcal{H}_{kin} , constructing the physical Hilbert space \mathcal{H}_{phys} would then amount to a series of reduction processes under the imposition of the Gauss and diffeomorphism constraints.

In this work we content ourselves with the kinematical structure of LQG, encoded in \mathcal{H}_{kin}^0 , obtained from \mathcal{H} after the imposition of the Gauss constraint only, which is common to a large extent also to the group field theory formalism, as we will discuss.

B. Gauge-invariant states and spin network basis

The solutions of the quantum Gauss constraint form the Hilbert space \mathcal{H}_{kin}^0 of $SU(2)$ -gauge invariant states, i.e.:

$$\mathcal{H}_{kin}^0 \equiv \text{Inv}_{SU(2)}[\mathcal{H}_{kin}] , \quad (10)$$

which can be defined by the same construction outlined in the previous section, but starting from gauge invariant spaces \mathcal{H}_Γ^0 associated to all possible graphs Γ . From the transformation of continuum parallel transports under $SU(2)$ transformations, it follows that a gauge transformation acts only on the nodes of the graph. Therefore, the gauge-invariance requirement for cylindrical functions translates into the requirement of invariance under the action of the group at the nodes, i.e.:

$$\psi_\Gamma^0(h_1, \dots, h_L) = \psi_\Gamma^0(g(\gamma_1(0))h_1g^{-1}(\gamma_1(1)), \dots, g(\gamma_L(0))h_Lg^{-1}(\gamma_L(1))) , \quad (11)$$

where $\gamma_i(0)$ (resp. $\gamma_i(1)$) indicates the starting (resp. end) point of the link i of the graph Γ . The above invariance can be implemented by group averaging

$$\psi_\Gamma^0(h_1, \dots, h_L) = \int \prod_{v=1}^V dg_v f(g(\gamma_1(0))h_1g^{-1}(\gamma_1(1)), \dots, g(\gamma_L(0))h_Lg^{-1}(\gamma_L(1))) , \quad (12)$$

where V is the number of nodes (vertices) of the graph Γ . In the spin representation of each function, this corresponds to inserting on each node v of the graph the following projector,

$$\mathcal{I}_v = \int dg \prod_{\ell \in v} D^{(j_\ell)}(g) . \quad (13)$$

with

$$\prod_{\ell \in v} D_{m_\ell n_\ell}^{(j_\ell)}(g) \in \bigotimes_{\ell \in v} \mathcal{V}^{(j_\ell)} \quad (14)$$

$\mathcal{V}^{(j_\ell)}$ denoting the $SU(2)$ irreducible spin- j_ℓ representation spaces. Therefore, by using the decomposition of the tensor product $\bigotimes_{\ell \in v} \mathcal{V}^{(j_\ell)}$ into irreducible representations

$$\bigotimes_{\ell \in v} \mathcal{V}^{(j_\ell)} = \bigoplus_i \mathcal{V}^{(j_i)} , \quad (15)$$

we find that \mathcal{I}_v projects onto the gauge invariant part of $\bigotimes_{\ell \in v} \mathcal{V}^{(j_\ell)}$, namely the singlet space $\mathcal{V}^{(0)}$:

$$\mathcal{I}_v : \bigotimes_{\ell \in v} \mathcal{V}^{(j_\ell)} \longrightarrow \mathcal{V}^{(0)}. \quad (16)$$

Being \mathcal{I}_v a projector, it can be decomposed in terms of a basis $\{i_\alpha\}$ of $\mathcal{V}^{(0)}$ and its dual as

$$\mathcal{I}_v = \sum_{\alpha=1}^{\dim \mathcal{V}^{(0)}} i_\alpha i_\alpha^* \in \mathcal{V}^{(0)} \otimes \mathcal{V}^{(0)*}, \quad (17)$$

from which, together with the decomposition of $\bigotimes_{\ell \in v} \mathcal{V}^{(j_\ell)} = (\bigotimes_{\ell \text{ in}} \mathcal{V}^{(j_\ell)*}) \otimes (\bigotimes_{\ell \text{ out}} \mathcal{V}^{(j_\ell)})$ between ingoing and outgoing links of the vertex v , it follows that \mathcal{I}_v is the invariant map between the representation spaces associated with the edges joined at the node v , i.e.:

$$\mathcal{I}_v : \bigotimes_{\ell \text{ in}} \mathcal{V}^{(j_\ell)} \longrightarrow \bigotimes_{\ell \text{ out}} \mathcal{V}^{(j_\ell)}. \quad (18)$$

Such invariants are called *intertwiners*. Hence, if we have an p -valent node, the intertwiner is an element of the invariant subspace $\text{Inv}_{SU(2)}[\mathcal{V}^{(j_1)} \otimes \dots \otimes \mathcal{V}^{(j_p)}]$ of the tensor product space between the p irreducible representations associated to the links joining that node. However, such a procedure is possible only if some conditions necessary to have an invariant subspace are satisfied. For instance, in the case of a 3-valent node, there exists an intertwiner space only if the spin numbers j_1, j_2, j_3 labelling the representations associated to the three links satisfy the Clebsch-Gordan condition:

$$|j_1 - j_2| \leq j_3 \leq j_1 + j_2. \quad (19)$$

For a p -valent node (with $p > 3$) the space $\mathcal{V}^{(0)}$ can have a larger dimension and the construction consists of adding first two irreducible representations, then the third, and so on, thus giving rise to a decomposition in virtual 3-valent nodes in which virtual links are labelled by spins k satisfying the condition (19) and represent the intertwiners as showed in Fig. 1 for a 4-valent node. Since the

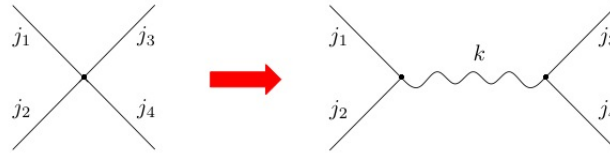


Figure 1: Construction of an intertwiner for a 4-valent vertex.

projector (17) acts only on the nodes of the graph that labels the basis of \mathcal{H}_{kin} , we can write the result of the action of \mathcal{I}_v on elements of \mathcal{H}_{kin} as a linear combination of products of representation matrices $D_{m_\ell n_\ell}^{(j_\ell)}(h_\ell(A))$ contracted with intertwiners. This leads us to give the following

Definition: A triplet $(\Gamma, \vec{j}, \vec{i})$ representing a graph Γ embedded in Σ whose L links are colored by the spins $\vec{j} = (j_1, \dots, j_L)$ and whose V nodes are labelled by intertwiners $\vec{i} = (i_1, \dots, i_V)$ is called a

spin network S embedded in Σ associated with the graph Γ . A **spin network state** $|S\rangle \equiv |\Gamma; \vec{j}, \vec{i}\rangle$ is the cylindrical function over the spin network S associated with the graph Γ which can be written as

$$\langle A|\Gamma; \vec{j}, \vec{i}\rangle = \psi_{\Gamma, \vec{j}, \vec{i}}[A] = \bigotimes_{\ell} D^{(j_{\ell})}(h_{\ell}(A)) \cdot \bigotimes_v i_v, \quad (20)$$

where $D^{(j_{\ell})}(h_{\ell}(A))$ are the spin irreducible representations of the holonomy along each link and \cdot denotes the contraction with the intertwiners whose indices (hidden for simplicity) can be reconstructed from the connectivity of the graph.

Spin network states form a complete orthonormal basis for \mathcal{H}_{Γ}^0 [40]. Indeed, using the Peter-Weyl theorem according to which the Wigner matrices form an orthonormal basis of $L^2(SU(2))$, and the definition of the scalar product (4), we have:

$$\langle \Gamma'; \vec{j}', \vec{i}' | \Gamma; \vec{j}, \vec{i} \rangle \equiv \langle \psi_{\Gamma'; \vec{j}', \vec{i}'} | \psi_{\Gamma; \vec{j}, \vec{i}} \rangle = \delta_{\Gamma', \Gamma} \delta_{\vec{j}', \vec{j}} \delta_{\vec{i}', \vec{i}}. \quad (21)$$

The $SU(2)$ constraint is implemented by choosing an intertwiner at each node as discussed before. Thus, the gauge-invariant Hilbert space on a fixed graph Γ with L links and V nodes is given by

$$\mathcal{H}_{\Gamma}^0 = \bigotimes_{j_{\ell}} \left(\bigotimes_v \text{Inv}_{SU(2)} \left[\bigotimes_{\ell \in v} \mathcal{V}^{(j_{\ell})} \right] \right) \cong L^2(SU(2)^L / SU(2)^V). \quad (22)$$

Spin networks will be the main focus of our attention in this paper.

As we mentioned, spin network states diagonalize geometric operators such as area and volume. In particular, only nodes contribute to the spectrum of the volume operator implying that the volume of a given region of a spin network is actually the sum of v terms each of which is associated with one of the nodes inside the region. This matches also the identification of spin networks with states of quantum polyhedra dual to the nodes of the graph [27], where the algebraic data associated to a spin network define a notion of quantum geometry where each face dual to a link ℓ has an area proportional to the spin label j_{ℓ} , and each region around a node v has a volume determined by the intertwiner i_v as well as the spins of the links sharing that node. Such an interpretation can be traduced into the heuristic picture of Fig. 2 where space is represented by a collection of “chunks” (the polyhedra dual to the nodes) with quantized volume. Neighbouring chunks share surfaces whose area is determined by the spin carried by the dual link which intersects it.

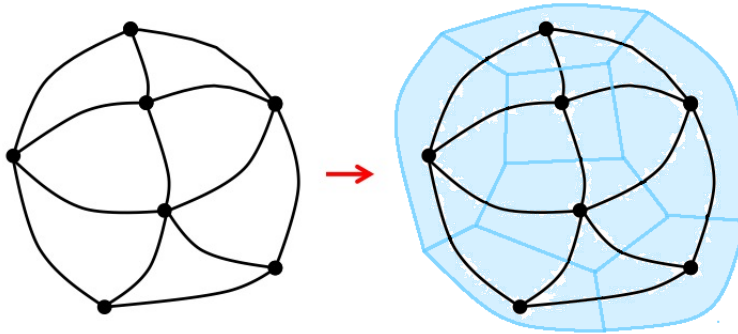


Figure 2: *Heuristic picture of the quantum geometry of space described by spin network states.*

These discrete building blocks can also be understood independently of any embedding into a continuum manifold. In fact, even in the canonical LQG formulation of the theory, the imposition of diffeomorphism invariance forces upon us to avoid relying on any continuum background structure. Moreover, being the fundamental structure of space a quantum superposition of abstract non-embedded entities each of which having a different connectivity (i.e., a different graph structure), what is local in one term of the superposition will in general not be local in others [41, 42]. Therefore, the lack of a background implies that adjacent regions of a spin network do not necessarily correspond to close space regions. Indeed, there is no metric structure that allows us to define a notion of distance and there is no absolute position at all. But a given region of a spin network can be localized with respect to other parts of the graph. This picture implies that notions as “close” and “far” should be understood in terms of relations between parts of a spin network. The picture of the microscopic structure of space(-time) proposed by LQG nicely realizes Penrose’s original idea to find a description of quantum geometry which is at the same time discrete and relational (that is, built up from purely combinatorial and algebraic (i.e., the group data) structures without any reference to background notions of space, time or geometry) [43]. This is also where quantum information tools may become crucial.

C. Spin networks as many-body quantum states in group field theory

Before proceeding with our analysis of entanglement in spin network states, and the application of geometric quantum mechanics, we provide another picture of the same spin network states, which takes their ‘pre-geometric’ nature as given, and does not rely on any continuum structure for the construction of a corresponding Hilbert space (in particular, it does not proceed to the imposition of any cylindrical equivalence condition) [44].² This is the way spin networks appear in the group field theory formalism. However, there are some works on the possibility of defining spin network states in a more abstract, combinatorial way also within the canonical LQG approach [46, 47].

The key point of the construction is to understand how spin network states can be reformulated as “many-body” states. Let us consider a closed graph Γ with V d -valent vertices labelled by the index $i = 1, \dots, V$ and denote the set of its edges by

$$L(\Gamma) = (\{1, \dots, V\} \times \{1, \dots, d\})^2 \quad (23)$$

such that

$$[(ia)(ia)] \notin L(\Gamma) \quad , \quad [(ia)(jb)] \in L(\Gamma) \quad (24)$$

where the last condition specifies the connectivity of the graph telling us the existence of a directed edge connecting the a -th link at the i -th node to the b -th link at the j -th node, with source i and target j . A generic cylindrical function based on the graph Γ will be a function of the group elements $h_{ij}^{ab} \in \mathbb{G}$ ($\mathbb{G} \equiv SU(2)$ in LQG) assigned to each link $\ell := [(ia)(jb)] \in L(\Gamma)$ ³

$$\psi_{\Gamma}(h_{12}^{11}, h_{13}^{21}, \dots) = \psi_{\Gamma}(\{h_{ij}^{ab}\} \equiv \{h_{\ell}\}) \in \mathcal{H}_{\Gamma} \cong L^2(\mathbb{G}^L/\mathbb{G}^V) \quad , \quad (25)$$

² For a general conceptual discussion on the meaning of “pregeometry”, see for instance [45].

³ Since in what follows it is important to distinguish different nodes and the links joining them, we admit a certain excessive complexity of notation using more indices to label links (ab) and their source and target nodes (ij).

with $h_{ij} = h_{ji}^{-1}$ and we impose gauge invariance at each vertex i of the graph, i.e.:

$$\psi_\Gamma(\{h_{ij}\}) = \psi_\Gamma(\{g_i h_{ij} g_j^{-1}\}) \quad \forall g_i \in \mathbb{G}. \quad (26)$$

Consider now a new Hilbert space given by

$$\mathcal{H}_V \cong L^2(\mathbb{G}^{d \times V} / \mathbb{G}^V), \quad (27)$$

whose generic element will be a function of $d \times V$ group elements

$$\varphi(\{g_i^a\}) = \varphi(g_1^1, \dots, g_d^1, \dots, g_1^V, \dots, g_d^V) \in \mathcal{H}_V \quad (28)$$

satisfying the gauge invariance at the vertices of the graph, i.e.:

$$\varphi(\dots, g_a^i, \dots, g_b^j, \dots) = \varphi(\dots, \alpha_i g_a^i, \dots, \alpha_j g_b^j, \dots) \quad , \quad \forall \alpha \in \mathbb{G}. \quad (29)$$

As in LQG, the measure of the Hilbert space is taken to be the Haar measure. The interpretation of such functions is that each φ is associated to a d -valent graph formed by V disconnected components, each corresponding to a single d -valent vertex and d 1-valent vertices, which are called *open spin network vertices*.

Given a closed d -valent graph Γ with V vertices specified by $L(\Gamma)$, a cylindrical function ψ_Γ can be obtained by group averaging a wave function φ

$$\psi_\Gamma(\{h_{ij}^{ab}\}) = \int_{\mathbb{G}} \prod_{[(ia)(jb)] \in L(\Gamma)} d\alpha_{ij}^{ab} \varphi(\{g_i^a \alpha_{ij}^{ab}; g_j^b \alpha_{ij}^{ab}\}) = \psi_\Gamma(\{g_i^a (g_j^b)^{-1}\}) \quad (30)$$

in such a way that each edge is associated with two group elements $g_i^a, g_j^b \in \mathbb{G}$. The integrals over α operate a “gluing” of the open spin network vertices corresponding to φ , pairwise along common links, thus forming the closed spin network represented by the closed graph Γ . Such a gluing can be interpreted as a symmetry requirement. Essentially, what we are saying is that we impose the function φ to depend on the group elements g_i^a, g_j^b only through the combination $g_i^a (g_j^b)^{-1} = h_{ij}^{ab}$ which is invariant under the group action, by the same group element, at the endpoint of two open edges to which these group elements are associated as showed in Fig. 3 for the simple example of the tetrahedral graph. This shows that, only using functions φ , it is always possible to construct a generic function ψ with all the right variables and symmetry properties, i.e., the space of functions ψ is a subset of the space of functions φ .

Moreover, using the Peter-Weyl decomposition theorem, we can give the corresponding formula in the spin representation which expresses the gluing of open spin network vertices and defines cylindrical functions for closed graphs as special cases of functions associated to a given number of them. Indeed, a cylindrical function ψ_Γ can be decomposed as

$$\begin{aligned} \psi_\Gamma(\{h_{ij}^{ab}\}) &= \sum \psi_{\{m_{ij}^{ab} k_{ij}^{ab}\}}^{J_{ij}^{ab}} \prod_i \overline{C_{m_{ij}^{ab}}^{J_{ij}^{ab} \mathcal{I}_i}} C_{n_{ij}^{ab}}^{J_{ij}^{ab} \mathcal{I}_i} \prod_{[(ia)(jb)]} D_{s_{ij}^{ab} n_{ij}^{ab}}^{(J_{ij}^{ab})}(\{h_{ij}^{ab}\}) \\ &= \sum_{\{J\}, \mathcal{I}} \tilde{\psi}_{\{J_{ij}^{ab}\}, \mathcal{I}_i} \prod_i C_{n_{ij}^{ab}}^{J_{ij}^{ab} \mathcal{I}_i} \prod_{[(ia)(jb)]} D_{s_{ij}^{ab} n_{ij}^{ab}}^{(J_{ij}^{ab})}(\{h_{ij}^{ab}\}) \end{aligned} \quad (31)$$

where

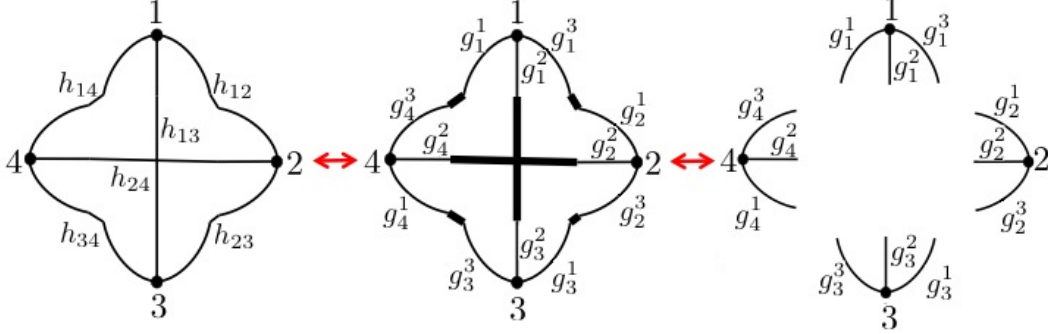


Figure 3: *Gluing of open spin network vertices to form a spin network closed graph.*

- J_{ij}^{ab} label the representations of the group \mathbb{G} and $D^{(J)}$ are the corresponding representation matrices whose indices refer to the start and end vertex of the edge $[(ia)(jb)]$ to which the group element h_{ij}^{ab} is attached;
- $C^{\{J\}, \mathcal{I}}$ are the normalized intertwiners for the group \mathbb{G} , attached in pairs to the vertices, resulting from the gauge-invariance requirement, a basis of which is labelled by additional quantum numbers \mathcal{I} . These intertwiners contract all indices of both nodes and of the representation functions, leaving a gauge-invariant function of spin variables only.

By using a similar decomposition for the function φ , the group averaging expression of ψ in terms of φ can be written as

$$\begin{aligned} \psi_{\Gamma}(\{h_{ij}^{ab} = g_i^a (g_j^b)^{-1}\}) &= \int \prod_{[(ia)(jb)]} d\alpha_{ij}^{ab} \sum_{\{\vec{J}_i, \vec{m}_i\}, \mathcal{I}_i} \varphi_{\vec{m}_i}^{\vec{J}_i, \mathcal{I}_i} \prod_i \left(\prod_{j \neq i} D_{m_i^a n_i^a}^{(J_i^a)}(g_i^a \alpha_{ij}^{ab}) \right) C_{\vec{n}_j}^{\vec{J}_j, \mathcal{I}_j} \quad (32) \\ &= \sum_{\{J_{ij}^{ab}\}, \{m_i^j\}, \mathcal{I}_i} \varphi_{\vec{m}_i}^{\vec{J}_i, \mathcal{I}_i} \prod_i C_{n_{ij}^{ab}}^{J_{ij}^{ab}, \mathcal{I}_i} \prod_{[(ia)(jb)]} \delta_{J_i^a, J_j^b} \delta_{m_i^a, m_j^b} D_{s_{ij}^{ab} n_{ij}^{ab}}^{(J_{ij}^{ab})} (g_i^a (g_j^b)^{-1}) \quad (33) \end{aligned}$$

from which, comparing with (31), we get the gluing formula in spin representation

$$\psi^{\{J_{ij}\}, \mathcal{I}_i} = \sum_{\{\vec{m}\}} \varphi_{\vec{m}_i}^{\vec{J}_i, \mathcal{I}_i} \prod_{[(ia)(jb)]} \delta_{J_i^a, J_j^b} \delta_{m_i^a, m_j^b}. \quad (34)$$

This means that LQG states can be regarded as linear combinations of disconnected open spin network states with additional conditions enforcing the gluing and encoding the connectivity of the graph. Explicitly, Eq. (34) shows that such conditions basically correspond to insert intertwiners given by the identity map at the bivalent vertices where the open links are pairwise glued.

In order to deal with graphs with an arbitrary number of vertices, we consider the Hilbert space

$$\mathcal{H} = \bigoplus_{V=0}^{\infty} \mathcal{H}_V. \quad (35)$$

Eq. (30), or equivalently (32), shows that there is a correspondence between LQG states and states

in \mathcal{H} . This is actually more than a correspondence at the level of sets of states since it is possible to prove that the scalar product in \mathcal{H}_V for the special class of states corresponding to closed graphs induces the standard LQG kinematical scalar product for cylindrical functions $\psi_\Gamma \in \mathcal{H}_\Gamma$ based on a fixed graph (see [44] for details). This means that, assuming that the graph Γ has V vertices, \mathcal{H}_Γ can be embedded into \mathcal{H}_V faithfully, i.e., preserving the scalar product.

Still, even though they agree exactly for each \mathcal{H}_Γ , it is important to stress the main differences between the new Hilbert space \mathcal{H} and \mathcal{H}_{kin}^0 .

- 1) The Hilbert space \mathcal{H} in (35) is defined by taking the direct sum over all the Hilbert spaces $\mathcal{H}_V \supset \mathcal{H}_\Gamma$ with fixed number of vertices without introducing any cylindrical equivalence class. As such, unlike the LQG case, zero modes are now included in the Hilbert space.
- 2) In the new Hilbert space, states associated to different graphs are organized in a different way w.r.t. the LQG space. Indeed, states associated to graphs with different number of vertices are orthogonal, but those associated to different graphs but with the same number of vertices are not orthogonal.

The functions $\varphi(\vec{g}_1, \dots, \vec{g}_V)$ can be understood as “many-body” wave functions for V quanta corresponding to the V open spin network vertices to which the function refers. Indeed, each state can be decomposed into products of “single-particle”/“single-vertex” states

$$|\varphi\rangle = \sum_{\{\vec{x}_i\}_{i=1, \dots, V}} \varphi^{\vec{x}_1 \dots \vec{x}_V} |\vec{x}_1\rangle \otimes \dots \otimes |\vec{x}_V\rangle, \quad (36)$$

which in the group representation reads as

$$\varphi(g) \equiv \langle g | \varphi \rangle = \sum_{\{\vec{x}_i\}} \varphi^{\vec{x}_1 \dots \vec{x}_V} \langle \vec{g}_1 | \vec{x}_1 \rangle \dots \langle \vec{g}_V | \vec{x}_V \rangle \quad (37)$$

where the complete basis of single-vertex wave functions is given by wave functions for individual spin network vertices, i.e.

$$|\vec{x}\rangle = |\vec{J}, \vec{m}, \mathcal{I}\rangle \quad : \quad \psi_{\vec{x}}(\vec{g}) = \langle \vec{g} | \vec{x} \rangle = \left(\prod_{\ell=1}^d D_{m_\ell n_\ell}^{(J_\ell)} \right) C_{n_1 \dots n_d}^{J_1 \dots J_d, \mathcal{I}}. \quad (38)$$

The normalization condition for the φ is provided by

$$\int \prod_{v=1}^V d\vec{g}_v \bar{\varphi}(\vec{g}_1, \dots, \vec{g}_V) \varphi(\vec{g}_1, \dots, \vec{g}_V) = \sum_{\{\chi_v\}} \bar{\varphi}^{\{\chi_v\}} \varphi^{\{\chi_v\}}, \quad (39)$$

where we have used the normalization condition of single-particle wave functions

$$\int d\vec{g} \bar{\psi}_{\vec{x}'}(\vec{g}) \psi_{\vec{x}}(\vec{g}) = \delta_{\vec{x}', \vec{x}}. \quad (40)$$

The functions φ are exactly the many-body wave functions for point particles living on the group manifold \mathbb{G}^d , whose classical phase space is $(T^*\mathbb{G})^d \cong (\mathbb{G} \times \mathcal{G}^*)^d$ which is also the classical phase space of a single polyhedron dual to a d -valent spin network vertex. The resulting picture of the

microstructure of spacetime is thus based on glued pre-geometric fundamental building blocks. This is the general picture underlying the GFT formalism. This is even more evident in a 2nd quantized, Fock space reformulation for the same Hilbert space \mathcal{H} , where building blocks are created and annihilated and their gluing corresponds to interactions of combinatorial nature [44, 48]. We will see also in the following that the gluing of open spin networks can be naturally understood in terms of entanglement of their spin network degrees of freedom, so that the GFT Hilbert space fits very well our pre-geometric approach to quantum spacetime and the idea of reconstructing geometry from entanglement.

III. ENTANGLEMENT IN GEOMETRIC QUANTUM MECHANICS

One of the most appealing reasons to construct a geometric formulation of Quantum Mechanics (QM) concerns the opportunity of making available “classical methods” of Riemannian and symplectic geometry in a quantum mechanical framework [49]. The geometrization program for Quantum Mechanics can be synthesized as the replacement of the usual description of a quantum system in terms of Hilbert spaces with a description in terms of Hilbert manifolds [50–53]. Moreover, according to the probabilistic interpretation of QM, we usually identify (pure) states with equivalence classes (*rays*) of state vectors $|\psi\rangle$ with respect to multiplication by a non-zero complex number [54]. The space of rays $\mathcal{R}(\mathcal{H})$ is a differential manifold identified with the complex projective space $\mathbb{C}P(\mathcal{H})$ associated with \mathcal{H} [55]. The manifold structure of this space requires that we replace all objects, whose definition depends on the linear structure on \mathcal{H} , with tensorial geometrical entities which preserve their meaning under general transformations and not just linear ones [52].

In this section we briefly recall how to construct tensorial quantities on the space of states of a quantum system by focusing on those tensors on \mathcal{H} which can be identified with the pull-back of tensorial objects defined on the underlying complex projective space. Furthermore, a pull-back procedure on orbit submanifolds of quantum states with respect to the action of unitary representations of Lie groups enables us to give a tensorial characterization of quantum entanglement for composite systems [25].

A. Classical Tensors on Quantum states

Let $\mathcal{H} \cong \mathbb{C}^N$ be a finite-dimensional Hilbert space of dimension N and denote by $\{|e_j\rangle\}_{j=1,\dots,N}$ its (orthonormal) basis. We can introduce complex coordinate functions $\{c_j\}$ on \mathcal{H} by setting $\langle e_j|\psi\rangle = c_j(\psi)$, for any $|\psi\rangle \in \mathcal{H}$. By replacing functions with their exterior differentials we may associate with the Hermitian inner product on quantum state vectors $\langle \cdot|\cdot\rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ a Hermitian covariant tensor on quantum-state-valued sections of the tangent bundle $T\mathcal{H}$ defined by [50, 51, 53]

$$h = \langle d\psi \otimes d\psi \rangle := \sum_j d\bar{c}_j \otimes dc_j, \quad (41)$$

such that

$$\langle d\psi \otimes d\psi \rangle (X_\psi, X_{\psi'}) = \langle \psi|\psi'\rangle \quad (42)$$

for any vector field $X_\psi : \phi \mapsto (\phi, \psi)$, $\forall \phi \in \mathcal{H}$. This essentially amounts to identify \mathcal{H} with the tangent space $T_\psi\mathcal{H}$ at each point of the base manifold.

The decomposition of the coordinate functions c_j into real and imaginary part, say $c_j = x_j + iy_j$, that is to replace \mathcal{H} with its realification $\mathcal{H}_{\mathbb{R}} := \mathbb{R}e(\mathcal{H}) \oplus \mathbb{I}m(\mathcal{H}) \cong \mathbb{R}^{2N}$, allows to identify an Euclidean metric and a symplectic structure on $\mathcal{H}_{\mathbb{R}}$ respectively with the real and imaginary part of the Hermitian tensor (41), i.e.

$$h = g + i\omega = \delta_{jk}(dx^j \otimes dx^k + dy^j \otimes dy^k) + i\delta_{jk}(dx^j \otimes dy^k - dy^j \otimes dx^k) . \quad (43)$$

These two tensors are related by a $(1,1)$ -tensor field $J = \delta_{jk}(dx^j \otimes \frac{\partial}{\partial y^k} - dy^j \otimes \frac{\partial}{\partial x^k})$ playing the role of a complex structure. The real differential manifold $\mathcal{H}_{\mathbb{R}}$ is thus equipped with a Kähler manifold structure [53].

Coming back to the space of rays, it is well known [54, 55] that the equivalence classes of state vectors identifying points of the complex projective space $\mathbb{C}P(\mathcal{H}) \cong \mathcal{R}(\mathcal{H})$ can be represented by rank-one projectors $\rho = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} \in D^1(\mathcal{H}) \subset \mathfrak{u}^*(\mathcal{H})$ called *pure states* which satisfy the properties $\rho^\dagger = \rho, \rho^2 = \rho, \text{Tr}\rho = 1$. Inheriting the differential calculus from $\mathfrak{u}^*(\mathcal{H})$, we define an operator-valued $(0,2)$ -tensor $d\rho \otimes d\rho$ which may be turned into a covariant tensor by evaluating it on the state ρ itself, i.e.

$$\text{Tr}(\rho d\rho \otimes d\rho) . \quad (44)$$

The pull-back of the tensor (44) from $\mathcal{R}(\mathcal{H})$ to $\mathcal{H}_0 \equiv \mathcal{H} - \{0\}$ along the (momentum) map

$$\mu : \mathcal{H}_0 \ni |\psi\rangle \longmapsto \rho = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} \in \mathcal{R}(\mathcal{H}) \cong D^1(\mathcal{H}) \subset \mathfrak{u}^*(\mathcal{H}) \quad (45)$$

gives the so-called Fubini-Study Hermitian tensor [56]

$$h_{FS} = \frac{\langle d\psi \otimes d\psi \rangle}{\langle \psi | \psi \rangle} - \frac{\langle \psi | d\psi \rangle}{\langle \psi | \psi \rangle} \otimes \frac{\langle d\psi | \psi \rangle}{\langle \psi | \psi \rangle} , \quad (46)$$

whose real and imaginary parts define a metric and a symplectic structure on \mathcal{H}_0 .

Consider now a manifold \mathbb{M} and an embedding $i_{\mathbb{M}} : \mathbb{M} \hookrightarrow \mathcal{H}$ of \mathbb{M} into \mathcal{H} . The induced pull-back $i_{\mathbb{M}}^*$ of the Hermitian tensor (41) or (46) defines a covariant Riemannian metric tensor and a closed (symplectic in a non-degenerate case) 2-form on \mathbb{M} . In this spirit, it has been shown in [57] that the Fisher-Rao metric tensor, used in statistics and information theory [58], may be obtained from the Fubini-Study tensor on the space of pure quantum states.

In particular, if \mathbb{M} admits the structure of a Lie group \mathbb{G} , we may identify submanifolds of states as the orbits generated by the action of a unitary representation of the group upon a normalized fiducial state $|0\rangle \in \mathcal{H}_0$, i.e.

$$\mathcal{O} \cong \mathbb{G}/\mathbb{G}_0 = \{|g\rangle = U(g)|0\rangle \mid g \in \mathbb{G}\} / \sim , \quad (47)$$

where \mathbb{G}_0 is the isotropy group of the state $|0\rangle$. Correspondingly on $\mathcal{R}(\mathcal{H})$ we identify orbit submanifolds of pure quantum states with respect to the co-adjoint action on some fiducial pure state

$$\rho_0 = \frac{|0\rangle\langle 0|}{\langle 0|0\rangle}$$

$$\rho(g) = U(g)\rho_0U^{-1}(g) . \quad (48)$$

The operator-valued 1-form $d\rho$ can be thus written as

$$d\rho = dU\rho_0U^{-1} + U\rho_0dU^{-1} + Ud\rho_0U^{-1} = U[U^{-1}dU, \rho_0]_U U^{-1}, \quad (49)$$

being $d\rho_0 = 0$ for any pure state ρ_0 . The pull-back of the Hermitian tensor (44) on the orbit submanifold embedded in $D^1(\mathcal{H}) \cong \mathcal{R}(\mathcal{H})$ yields the following tensor [25, 51]

$$\mathcal{K} = \left\{ \text{Tr}(\rho_0 R(X_j) R(X_k)) - \text{Tr}(\rho_0 R(X_j)) \text{Tr}(\rho_0 R(X_k)) \right\} \theta^j \otimes \theta^k, \quad (50)$$

where $R(X_j)$ denotes the Lie algebra representation defined by the unitary representation of the Lie group \mathbb{G} and θ^j the dual basis of left-invariant 1-forms such that $U^{-1}dU = iR(X_j)\theta^j$. Again, the real symmetric and imaginary skewsymmetric part of the tensor \mathcal{K} ⁴

$$\mathcal{K}_+ = \left\{ \frac{1}{2} \text{Tr}(\rho_0 [R(X_j), R(X_k)]_+) - \text{Tr}(\rho_0 R(X_j)) \text{Tr}(\rho_0 R(X_k)) \right\} \theta^j \odot \theta^k, \quad (51)$$

$$\mathcal{K}_- = \frac{1}{2} \text{Tr}(\rho_0 [R(X_j), R(X_k)]_-) \theta^j \wedge \theta^k, \quad (52)$$

provide a Riemannian metric tensor and a symplectic structure on the orbit \mathcal{O}_{ρ_0} .

Such a pull-back procedure can be extended also to the case of a composite system [24, 25]. The correlation properties of the fiducial state ρ_0 are captured by the tensorial structures induced on the orbits of the action of local unitary groups which define submanifolds of states with fixed amount of entanglement.

B. Quantum Fisher tensor for bipartite N-level systems

Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \cong \mathbb{C}^{N_A} \otimes \mathbb{C}^{N_B}$ be the Hilbert space of a composite system consisting of two N level systems A and B with number of levels respectively given by $N_A = \dim \mathcal{H}_A$ and $N_B = \dim \mathcal{H}_B$. For the sake of clarity, in what follows we will denote by \otimes the usual tensor product of spaces and by \otimes_F the product of forms. So let ρ_0 be a fiducial pure state in $D^1(\mathcal{H}_A \otimes \mathcal{H}_B)$ and \mathbb{G}_0 its isotropy group, we want to compute now the pull-back of the Hermitian quantum Fisher tensor

$$\text{Tr}(\rho d\rho \otimes_F d\rho) \quad (53)$$

on the orbits

$$\mathcal{O} \cong U(N_A) \times U(N_B) / \mathbb{G}_0 \quad (54)$$

of unitarily related (pure) quantum states

$$\rho = U\rho_0U^{-1}, \quad (55)$$

⁴ Here $[\cdot, \cdot]_{\pm}$ respectively denote the anticommutator and the commutator and we use the shorthand notation

$$\theta^a \odot \theta^b = \frac{1}{2}(\theta^a \otimes \theta^b + \theta^b \otimes \theta^a) \quad , \quad \theta^a \wedge \theta^b = \frac{1}{2}(\theta^a \otimes \theta^b - \theta^b \otimes \theta^a) .$$

for the symmetrized and antisymmetrized product of forms.

induced by the co-adjoint action of the unitary group on ρ_0 with respect to the product representation

$$U = U_A \otimes U_B = (U_A \otimes \mathbb{1}_B) \cdot (\mathbb{1}_A \otimes U_B). \quad (56)$$

The corresponding Lie algebra representation $\mathfrak{u}(\mathcal{H}_A) \oplus \mathfrak{u}(\mathcal{H}_B)$ is provided by means of the following realization

$$R(X_j) = \begin{cases} \sigma_j^{(A)} \otimes \mathbb{1}_B & \text{for } 1 \leq j \leq N_A^2 \\ \mathbb{1}_A \otimes \sigma_{j-N_A^2}^{(B)} & \text{for } N_A^2 + 1 \leq j \leq N_A^2 + N_B^2 \end{cases} \quad (57)$$

of the infinitesimal generators of the one-dimensional subgroup of $U(N_A) \times U(N_B)$. In the following, for both subsystems we will adopt a short-hand notation of indices a, b without specifying their range of values. Therefore, being $U^{-1}dU$ a left-invariant 1-form, it can be decomposed as⁵

$$U^{-1}dU = i\sigma_a^{(A)}\theta_A^a \otimes \mathbb{1}_B + \mathbb{1}_A \otimes i\sigma_b^{(B)}\theta_B^b, \quad (58)$$

where $\{\theta_A\}$ and $\{\theta_B\}$ denote a basis of left-invariant 1-forms on the corresponding Lie group representation acting on the subsystem A, B respectively. The operator-valued 1-form (49) can be thus written as

$$d\rho = U[i\sigma_a^{(A)}\theta_A^a \otimes \mathbb{1}_B, \rho_0]_- U^{-1} + U[\mathbb{1}_A \otimes i\sigma_b^{(B)}\theta_B^b, \rho_0]_- U^{-1}, \quad (59)$$

from which it follows that

$$\begin{aligned} d\rho \otimes_F d\rho &= \left(U[i\sigma_a^{(A)} \otimes \mathbb{1}_B, \rho_0]_- [i\sigma_b^{(A)} \otimes \mathbb{1}_B, \rho_0]_- U^{-1} \right) \theta_A^a \otimes_F \theta_B^b \\ &+ \left(U[i\sigma_a^{(A)} \otimes \mathbb{1}_B, \rho_0]_- [\mathbb{1}_A \otimes i\sigma_b^{(B)}, \rho_0]_- U^{-1} \right) \theta_A^a \otimes_F \theta_B^b \\ &+ \left(U[\mathbb{1}_A \otimes i\sigma_a^{(B)}, \rho_0]_- [i\sigma_b^{(A)} \otimes \mathbb{1}_B, \rho_0]_- U^{-1} \right) \theta_B^a \otimes_F \theta_A^b \\ &+ \left(U[\mathbb{1}_A \otimes i\sigma_a^{(B)}, \rho_0]_- [\mathbb{1}_A \otimes i\sigma_b^{(B)}, \rho_0]_- U^{-1} \right) \theta_B^a \otimes_F \theta_B^b. \end{aligned} \quad (60)$$

The pull-back of the Hermitian tensor (53) on the orbit submanifold starting from the fiducial state ρ_0 then reads as

$$\mathcal{K} = \mathcal{K}_{ab}^{(A)} \theta_A^a \otimes_F \theta_A^b + \mathcal{K}_{ab}^{(AB)} \theta_A^a \otimes_F \theta_B^b + \mathcal{K}_{ab}^{(BA)} \theta_B^a \otimes_F \theta_A^b + \mathcal{K}_{ab}^{(B)} \theta_B^a \otimes_F \theta_B^b \quad (61)$$

with

$$\begin{cases} \mathcal{K}_{ab}^{(A)} = -\text{Tr}(\rho_0 [\sigma_a^{(A)} \otimes \mathbb{1}_B, \rho_0]_- [\sigma_b^{(A)} \otimes \mathbb{1}_B, \rho_0]_-) \\ \mathcal{K}_{ab}^{(AB)} = -\text{Tr}(\rho_0 [\sigma_a^{(A)} \otimes \mathbb{1}_B, \rho_0]_- [\mathbb{1}_A \otimes \sigma_b^{(B)}, \rho_0]_-) \\ \mathcal{K}_{ab}^{(BA)} = -\text{Tr}(\rho_0 [\mathbb{1}_A \otimes \sigma_a^{(B)}, \rho_0]_- [\sigma_b^{(A)} \otimes \mathbb{1}_B, \rho_0]_-) \\ \mathcal{K}_{ab}^{(B)} = -\text{Tr}(\rho_0 [\mathbb{1}_A \otimes \sigma_a^{(B)}, \rho_0]_- [\mathbb{1}_A \otimes \sigma_b^{(B)}, \rho_0]_-) \end{cases} \quad (62)$$

⁵ Here we use the decomposition of the exterior differential operator $d = d_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes d_B$ acting on a product representation (56).

Now, being ρ_0 pure, we have that $\rho_0^3 = \rho_0^2 = \rho_0$. Hence a direct computation yields

$$\mathcal{K}_{ab}^{(A)} = \text{Tr}(\rho_0 \sigma_a^{(A)} \sigma_b^{(A)} \otimes \mathbb{1}_B) - \text{Tr}(\rho_0 \sigma_a^{(A)} \otimes \mathbb{1}_B) \text{Tr}(\rho_0 \sigma_b^{(A)} \otimes \mathbb{1}_B), \quad (63)$$

$$\mathcal{K}_{ab}^{(AB)} = \text{Tr}(\rho_0 \sigma_a^{(A)} \otimes \sigma_b^{(B)}) - \text{Tr}(\rho_0 \sigma_a^{(A)} \otimes \mathbb{1}_B) \text{Tr}(\rho_0 \mathbb{1}_A \otimes \sigma_b^{(B)}), \quad (64)$$

$$\mathcal{K}_{ab}^{(BA)} = \text{Tr}(\rho_0 \sigma_a^{(B)} \otimes \sigma_b^{(A)}) - \text{Tr}(\rho_0 \sigma_a^{(B)} \otimes \mathbb{1}_A) \text{Tr}(\rho_0 \mathbb{1}_B \otimes \sigma_b^{(A)}), \quad (65)$$

$$\mathcal{K}_{ab}^{(B)} = \text{Tr}(\rho_0 \mathbb{1}_A \otimes \sigma_a^{(B)} \sigma_b^{(B)}) - \text{Tr}(\rho_0 \mathbb{1}_A \otimes \sigma_a^{(B)}) \text{Tr}(\rho_0 \mathbb{1}_A \otimes \sigma_b^{(B)}). \quad (66)$$

Therefore, the pulled-back Hermitian tensor \mathcal{K} decomposes into a Riemannian metric \mathcal{K}_+ and a symplectic structure \mathcal{K}_-

$$\mathcal{K} = \mathcal{K}_+ + i \mathcal{K}_-, \quad (67)$$

respectively given by

$$\mathcal{K}_+ = \mathcal{K}_{(ab)}^{(A)} \theta_A^a \odot \theta_A^b + \mathcal{K}_{(ab)}^{(AB)} \theta_A^a \odot \theta_B^b + \mathcal{K}_{(ab)}^{(BA)} \theta_B^a \odot \theta_A^b + \mathcal{K}_{(ab)}^{(B)} \theta_B^a \odot \theta_B^b, \quad (68)$$

$$\mathcal{K}_- = \mathcal{K}_{[ab]}^{(A)} \theta_A^a \wedge \theta_A^b + \mathcal{K}_{[ab]}^{(B)} \theta_B^a \wedge \theta_B^b, \quad (69)$$

with

$$\begin{cases} \mathcal{K}_{(ab)}^{(A)} = \frac{1}{2} \text{Tr} \left(\rho_0 [\sigma_a^{(A)}, \sigma_b^{(A)}]_+ \otimes \mathbb{1}_B \right) - \text{Tr} \left(\rho_0 \sigma_a^{(A)} \otimes \mathbb{1}_B \right) \text{Tr} \left(\rho_0 \sigma_b^{(A)} \otimes \mathbb{1}_B \right) \\ \mathcal{K}_{(ab)}^{(B)} = \frac{1}{2} \text{Tr} \left(\rho_0 \mathbb{1}_A \otimes [\sigma_a^{(B)}, \sigma_b^{(B)}]_+ \right) - \text{Tr} \left(\rho_0 \mathbb{1}_A \otimes \sigma_a^{(B)} \right) \text{Tr} \left(\rho_0 \mathbb{1}_A \otimes \sigma_b^{(B)} \right) \\ \mathcal{K}_{(ab)}^{(AB)} = \text{Tr} \left(\rho_0 \sigma_a^{(A)} \otimes \sigma_b^{(B)} \right) - \text{Tr} \left(\rho_0 \sigma_a^{(A)} \otimes \mathbb{1}_B \right) \text{Tr} \left(\rho_0 \mathbb{1}_A \otimes \sigma_b^{(B)} \right) \\ \mathcal{K}_{(ab)}^{(BA)} = \text{Tr} \left(\rho_0 \sigma_a^{(B)} \otimes \sigma_b^{(A)} \right) - \text{Tr} \left(\rho_0 \sigma_a^{(B)} \otimes \mathbb{1}_A \right) \text{Tr} \left(\rho_0 \mathbb{1}_B \otimes \sigma_b^{(A)} \right) \\ \mathcal{K}_{[ab]}^{(A)} = \frac{1}{2} \text{Tr} \left(\rho_0 [\sigma_a^{(A)}, \sigma_b^{(A)}]_- \otimes \mathbb{1}_B \right) \\ \mathcal{K}_{[ab]}^{(B)} = \frac{1}{2} \text{Tr} \left(\rho_0 \mathbb{1}_A \otimes [\sigma_a^{(B)}, \sigma_b^{(B)}]_- \right) \end{cases} \quad (70)$$

where we used the bracket notation $(ab), [ab]$ for the matrix indices of the symmetric and antisymmetric part, respectively. Thus, the coefficient matrix of the pulled-back Hermitian tensor splits into different blocks carrying the information about the separable or entangled nature of the fiducial state ρ_0 , say:

$$\mathcal{K}_{ab} = \left(\begin{array}{c|c} \mathcal{K}_{(ab)}^{(A)} & \mathcal{K}_{(ab)}^{(AB)} \\ \hline \mathcal{K}_{(ab)}^{(BA)} & \mathcal{K}_{(ab)}^{(B)} \end{array} \right) + i \left(\begin{array}{c|c} \mathcal{K}_{[ab]}^{(A)} & 0 \\ \hline 0 & \mathcal{K}_{[ab]}^{(B)} \end{array} \right). \quad (71)$$

Indeed, by means of Fano decomposition of ρ_0 , it is easy to see that [25]

- when ρ_0 is separable the off-diagonal blocks of the metric component vanish and the pulled-back Hermitian tensor \mathcal{K} decomposes into a direct sum $\mathcal{K}_A \oplus \mathcal{K}_B$ of Hermitian tensors associated with the two subsystems;
- when ρ_0 is maximally entangled the symplectic component vanishes.

In particular, the information about (quantum) correlations between the two subsystems is encoded in the off-diagonal block-coefficient $N_A^2 \times N_B^2$ and $N_B^2 \times N_A^2$ matrices $\mathcal{K}^{(AB)}$ and $\mathcal{K}^{(BA)}$ which allow

us to define an entanglement monotone given by [59]

$$\mathcal{E} = \frac{N_{<}^2}{4(N_{<}^2 - 1)} \text{Tr} \left(\mathcal{K}^{(AB)T} \mathcal{K}^{(AB)} \right) = \frac{N_{<}^2}{4(N_{<}^2 - 1)} \text{Tr} \left(\mathcal{K}^{(BA)T} \mathcal{K}^{(BA)} \right), \quad (72)$$

where $N_{<} = \min(N_A, N_B)$. Such entanglement monotone is directly related to a measure constructed with the reduced states $\rho_0^{(A,B)} = \text{Tr}_{B,A}(\rho_0)$ [60]

$$\text{Tr} (R^\dagger R) = \frac{1}{N_{<}^4} \text{Tr} \left(\mathcal{K}^{(AB)T} \mathcal{K}^{(AB)} \right), \quad R := \rho_0 - \rho_0^{(A)} \otimes \rho_0^{(B)} \quad (73)$$

which allows to interpret $\text{Tr} \left(\mathcal{K}^{(AB)T} \mathcal{K}^{(AB)} \right)$ geometrically as a distance between entangled and separable states.

IV. QUANTUM METRIC AND ENTANGLEMENT ON SPIN NETWORKS

We are now willing to exploit the information-geometrical approach to Quantum Mechanics outlined in the previous section to extract geometric structures such as a metric tensor and a symplectic form directly from states.

As a preliminary step we shall first apply this language to the simple case of a single link (Wilson line), in order to write down the Fubini-Study metric on the corresponding Hilbert manifold and use it to characterize the entanglement resulting from the gluing of two lines into one. We shall see that our analysis leads to interpret (the presence of) the link as a result of entanglement and to characterize connectivity (i.e., the existence of the link) by means of the entanglement measure constructed from the off-diagonal block matrices of the metric tensor on orbits of unitarily related quantum states. In particular, and this is our main result, we shall identify the maximally entangled state with the gauge-invariant Wilson loop state and in this case the associated entanglement measure turns out to be proportional to a power of the area (of a surface dual to the link). Hence, although we consider only a simple case, this approach suggests the possibility of reconstructing the geometry of quantum spacetime by looking at the correlation structure and entanglement properties of spin-network states giving for instance a proper definition of the concept of a “*quantum distance measure*”.

A. A first dictionary via Wilson line states

The formalism developed in section III provides us with a general algorithmic procedure to construct tensorial geometric structures on the space of states of a given quantum theory. This has some important advantage both from the technical and computational point of view. Indeed, this geometric language not only makes available tools of differential geometry in the framework of Quantum Mechanics but also provides a way to characterize entanglement in a purely tensorial fashion with no need to explicitly compute the Schmidt coefficients or the entanglement entropy which may enter the discussion only in a second time. The procedure can be synthesized in full generality by means of the following steps:

- 1) The space of rays $\mathcal{R}(\mathcal{H}) \cong D^1(\mathcal{H})$ associated to the Hilbert space \mathcal{H} of the system is recognized to be the proper setting for Quantum Mechanics. Therefore, the relevant structures are those

available on the ray space. In particular, we focus on the pull-back to the Hilbert space of the Hermitian structure (44) naturally available on $D^1(\mathcal{H})$ along the momentum map (45), according to the diagram

$$\begin{array}{ccc} \mathcal{H}_0 & \xrightarrow{\mu} & \mathfrak{u}^*(\mathcal{H}) \\ \pi \downarrow & & \uparrow i \\ \mathcal{R}(\mathcal{H}) & \xrightarrow[\cong]{} & D^1(\mathcal{H}) \end{array} \quad (74)$$

The real and imaginary parts of this tensor provide us with a metric tensor and a symplectic structure, respectively.

- 2) We consider a stratification of the Hilbert manifold by means of the orbits with respect to the action of a Lie group \mathbb{G} on \mathcal{H} . By choosing a fiducial state $|0\rangle \in \mathcal{H}_0$ and a unitary representation $U(g)$ of \mathbb{G} , the orbit \mathcal{O} starting from $|0\rangle$ identifies a submanifold of quantum states $|g\rangle = U(g)|0\rangle$ when we consider an embedding map via the group action

$$\phi_0 : \mathbb{G} \ni g \mapsto |g\rangle = U(g)|0\rangle \in \mathcal{H}_0, \quad (75)$$

and correspondingly on the space of rays

$$\tilde{\phi}_0 : \mathbb{G} \ni g \mapsto \rho(g) = U(g)\rho_0 U^{-1}(g) \in \mathcal{R}(\mathcal{H}). \quad (76)$$

We can thus restrict ourself to the Hermitian tensor on this submanifold by noticing that it is completely described by the pull-back tensor on the Lie group \mathbb{G} according to the following diagrams

$$\begin{array}{ccc} & & \mathbb{G} \xrightarrow{\phi_0} S(\mathcal{H}) \\ \mathbb{G} \xrightarrow{\phi_0} \mathcal{H}_0 & & \begin{array}{ccc} U(1) \downarrow & & \downarrow U(1) \\ \mathbb{G}/U(1) \xrightarrow{\tilde{\phi}_0} \mathcal{R}(\mathcal{H}) & & \\ \pi_0 \downarrow & & \uparrow i_{\mathcal{O}} \\ \mathbb{G}/\mathbb{G}_0^{U(1)} \xrightarrow[\cong]{} \mathcal{O} & & \end{array} \\ \pi_0 \downarrow & \uparrow i_{\mathcal{O}} & \implies \end{array} \quad (77)$$

where $S(\mathcal{H}) = \{|\psi\rangle \in \mathcal{H}_0 : \langle\psi|\psi\rangle = 1\} \subset \mathcal{H}_0$ is the unit sphere of normalized state vectors and $\mathbb{G}_0^{U(1)}$ is an enlarged isotropy group taking into account the $U(1)$ -degeneracy directions for the Hermitian tensor on pure states.

- 3) The application of the pull-back strategy of the point 2) to the case of a composite bipartite system ($\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$) gives us a Hermitian tensor defined on orbits of unitarily related quantum states, which encodes the information about the entanglement or separability of the fiducial state we start with. In particular, the off-diagonal blocks can be used to define an entanglement measure (72) interpreted as a distance from the corresponding separable state.

In order to import the information-geometric machinery in the context of Quantum Gravity, let us consider the case of a single link (Wilson line) and let us apply the various steps of the above procedure.

B. Step 1: Metric tensor on the space of states

A few observations are in order. Let us consider a generic Wilson line state

$$|\psi_\gamma^{(j)}\rangle = \sum_{mn} c_{mn}^{(j)} |j, m, n\rangle \in \mathcal{H}_\gamma^{(j)} , \quad (78)$$

where we assume for simplicity that j is fixed. As pointed out in [61], Eq. (78) is the expansion of the single-link state in the spin basis. Indeed, starting from the classical phase space $T^*SU(2) \cong SU(2) \times \mathfrak{su}^*(2)$, the $SU(2)$ -valued holonomies $h(A)$ along the path γ and the $\mathfrak{su}(2)$ -valued fluxes of the triad fields through the surface crossed by γ play the role of canonically conjugate variables. Passing to the quantum level, we consider the group basis given by a complete set of orthonormal states $\{|h\rangle = |h(A)\rangle \mid h \in SU(2)\}$, labelled by group elements, such that

$$\langle h' | h \rangle = \delta((h')^{-1}h) \quad , \quad \int_{SU(2)} dh |h\rangle \langle h| = \mathbb{1} \quad (79)$$

where dh denotes the Haar measure on the link. Note that we are considering the group elements themselves as operators instead of some coordinate functions on $SU(2)$. By this we mean that $|h\rangle$ are the eigenstates of the operator \hat{h} in the sense that, for any coordinate system on the group, $|h\rangle$ are the eigenstates of the coordinates as guaranteed by the property $f(\hat{h})|h\rangle \equiv f(h)|h\rangle$, for any function $f \in \mathcal{F}(SU(2))$. We therefore define the Hilbert space \mathcal{H}_γ to consist of those states $|\psi_\gamma\rangle$ which decompose in the group basis as

$$|\psi_\gamma\rangle = \int_{SU(2)} dh \psi_\gamma(h) |h\rangle , \quad (80)$$

where

$$\psi_\gamma(h) \equiv \langle h | \psi_\gamma \rangle \in L^2(SU(2), dh) \quad (81)$$

is the (cylindrical) wave function which depends on the holonomy h along γ . By using now the Peter-Weyl decomposition of functions on $SU(2)$ in terms of spin representations, we define the spin $|j, m, n\rangle$ in $\mathcal{H}_\gamma^{(j)} \cong L^2(SU(2))$ by setting

$$\langle h | j, m, n \rangle := \sqrt{2j+1} D_{mn}^{(j)}(h) , \quad (82)$$

and the orthogonality relations of the Wigner representation matrices $D_{mn}^{(j)}$ ensure the normalization of the basis states

$$\langle j', m', n' | j, m, n \rangle = \delta_{jj'} \delta_{mm'} \delta_{nn'} , \quad (83)$$

together with the decomposition of the identity

$$\mathbb{1}_j = \sum_{mn} |j, m, n\rangle \langle j, m, n| . \quad (84)$$

Any state in \mathcal{H}_γ can be therefore expanded in the spin basis as in (78) with coefficients given by

$$c_{mn}^{(j)} \equiv \langle j, m, n | \psi_\gamma \rangle = \int dh \psi_\gamma[h] \langle j, m, n | h \rangle = \sqrt{2j+1} \int dh \psi_\gamma[h] \overline{D_{mn}^{(j)}(h)}. \quad (85)$$

Remark: Let us stress that the states $|j, m, n\rangle$ are different from the usual spherical harmonics states $|j, m\rangle$, even though they are related by means of the definition of the Wigner matrix elements $D_{mn}^{(j)}(h) \equiv \langle j, m | D^{(j)}(h) | j, n \rangle$. The states $|j, m\rangle$ are not states on the group but elements of the vector space $\mathcal{V}^{(j)}$ corresponding to the $(2j+1)$ -dimensional representation, whereas $|j, m, n\rangle$ can be thought of as elements of $\mathcal{H}_\gamma^{(j)} \cong \mathcal{V}^{(j)} \otimes \mathcal{V}^{(j)*}$. Indeed, the spherical harmonics are given by [62]

$$\langle h | j, m \rangle := Y_m^j(h) \equiv \sqrt{\frac{2j+1}{4\pi}} \overline{D_{m0}^{(j)}(h)} \quad (86)$$

and, being invariant under the right multiplication by elements of the group $U(1)$, they are functions on the 2-sphere $SU(2)/U(1) \cong S^2$. Therefore, they do not form a complete basis for $L^2(SU(2))$ and we need to use the states $|j, m, n\rangle$ which instead provide us with a full representation of both right and left multiplication. We will come back on this point later in this section.

We choose to work with the spin basis because in this way our results will depend explicitly on the algebraic data of the spin-network graph (j, m and n in this specific situation). Thus, remembering the notations of section III, we have the following correspondences:

$$|e_a\rangle \quad \longleftrightarrow \quad |j, m, n\rangle \equiv |e_{mn}^{(j)}\rangle \quad (87)$$

$$c_a(\psi) \quad \longleftrightarrow \quad c_{mn}^{(j)} \equiv \langle j, m, n | \psi_\gamma \rangle \quad (88)$$

$$|d\psi\rangle = \sum_a dc_a |e_a\rangle \quad \longleftrightarrow \quad |d\psi_\gamma^{(j)}\rangle = \sum_{mn} dc_{mn}^{(j)} |j, m, n\rangle \quad (89)$$

Hence the constructions of Sec. III can be now repeated and we find:

$$\langle d\psi_\gamma^{(j)} \otimes d\psi_\gamma^{(j)} \rangle = d\bar{c}_{mn}^{(j)} \otimes dc_{mn}^{(j)} \quad (\text{sum over } m, n). \quad (90)$$

Indeed:

$$\begin{aligned} \langle d\psi_\gamma^{(j)} \otimes d\psi_\gamma^{(j)} \rangle &= \sum_{mn} \langle d\psi_\gamma^{(j)} | j, m, n \rangle \langle j, m, n | d\psi_\gamma^{(j)} \rangle \\ &= \sum_{mn} \int_{SU(2)} dh \langle d\psi_\gamma^{(j)} | j, m, n \rangle \langle j, m, n | h(A) \rangle \langle h(A) | d\psi_\gamma^{(j)} \rangle \\ &= \sum_{mnm'n'} \int_{SU(2)} dh \left(\langle d\psi_\gamma^{(j)} | j, m, n \rangle \langle j, m, n | h(A) \rangle \langle h(A) | j, m', n' \rangle \langle j, m', n' | d\psi_\gamma^{(j)} \rangle \right) \\ &= (2j+1) \sum_{mnm'n'} \left(\int_{SU(2)} dh \overline{D_{mn}^{(j)}(h(A))} D_{m'n'}^{(j)}(h(A)) \right) d\bar{c}_{mn}^{(j)} \otimes dc_{m'n'}^{(j)} \\ &= (2j+1) \sum_{mnm'n'} d\bar{c}_{mn}^{(j)} \otimes dc_{m'n'}^{(j)} \frac{\delta_{mm'} \delta_{nn'}}{2j+1} = \sum_{mn} d\bar{c}_{mn}^{(j)} \otimes dc_{mn}^{(j)}. \end{aligned} \quad (91)$$

Similarly:

$$\langle \psi_\gamma^{(j)} | d\psi_\gamma^{(j)} \rangle = \bar{c}_{mn}^{(j)} dc_{mn}^{(j)} \quad (\text{sum over } m, n). \quad (92)$$

Finally, the pull-back to the Hilbert space of the Fubini-Study Hermitian tensor is given by:

$$\begin{aligned} \mathcal{K}_{\mathcal{H}_\gamma} &= \frac{\langle d\psi_\gamma^{(j)} \otimes d\psi_\gamma^{(j)} \rangle}{\langle \psi_\gamma^{(j)} | \psi_\gamma^{(j)} \rangle} - \frac{\langle d\psi_\gamma^{(j)} | \psi_\gamma^{(j)} \rangle \otimes \langle \psi_\gamma^{(j)} | d\psi_\gamma^{(j)} \rangle}{\langle \psi_\gamma^{(j)} | \psi_\gamma^{(j)} \rangle^2} \\ &= \frac{d\bar{c}_{mn}^{(j)} \otimes dc_{mn}^{(j)}}{\sum_{mn} |c_{mn}^{(j)}|^2} - \frac{d\bar{c}_{mn}^{(j)} c_{mn}^{(j)} \otimes \bar{c}_{mn}^{(j)} dc_{mn}^{(j)}}{\left(\sum_{mn} |c_{mn}^{(j)}|^2\right)^2}. \end{aligned} \quad (93)$$

C. Step 2: Pull-back on orbits of quantum states

To pull-back the Hermitian tensor (93) on orbit submanifolds of quantum states we need to understand what are the objects entering the diagram (77) in the specific case under examination. Let us therefore choose in $\mathcal{H}_\gamma^{(j)}$ a fiducial Wilson line state given by:

$$|0\rangle \equiv |\psi_\gamma^{(j)}\rangle = \sum_{mn} c_{mn}^{(j)} |j, m, n\rangle. \quad (94)$$

Since we are considering spin basis states $|j, m, n\rangle$ constructed with the common eigenstates of the operator J^2 and one of the J 's (say J_z), i.e., with a fixed orientation (say the z -axis) of the magnetic moments at the endpoints of the link⁶, the only transformations that we can perform on such states are those generated by the operators J_1, J_2, J_3 which have a well-defined action on the basis states. The group \mathbb{G} acting on \mathcal{H}_γ is thus given by the group $SU(2)$. Therefore, the diagram (77) which explain the various levels at which the (co-adjoint) orbit \mathcal{O} is embedded in the projective Hilbert space $\mathcal{R}(\mathcal{H}_\gamma)$ now becomes

$$\begin{array}{ccc} SU(2) & \xrightarrow{\phi_0} & S(\mathcal{H}_\gamma) \\ U(1) \downarrow & & \downarrow U(1) \\ SU(2)/U(1) & \xrightarrow{\tilde{\phi}_0} & \mathcal{R}(\mathcal{H}_\gamma) \\ \pi_0 \downarrow & & \uparrow i_{\mathcal{O}} \\ SU(2)/\mathbb{G}_0^{U(1)} & \xrightarrow{\cong} & \mathcal{O} \end{array} \quad (95)$$

By considering the $SU(2)$ action on the fiducial state $|0\rangle$, we realize the embedding of the Lie group into $\mathcal{H}_\gamma - \{0\}$ as

$$\phi_0 : SU(2) \ni h \longmapsto |h\rangle = U^{(j)}(h) |0\rangle \in \mathcal{H}_\gamma - \{0\}, \quad (96)$$

⁶ We may also consider a more general situation in which we have an additional degree of freedom to take into account a different direction of the magnetic moment. As discussed in [63], in this case the basis states are given by $|j, \hat{m}, \hat{n}\rangle$, where \hat{m} simply denotes the new direction $(\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$ obtained by rotating the direction $\hat{z} = (0, 0, 1)$. This kind of states can be used for instance to account a non-completely precise face matching of polyhedra glued along faces dual to the graph edges which will give some torsion thus providing a generalization of Regge geometries as twisted geometries [64, 65].

where the spin- j -representation is given by

$$U^{(j)} : SU(2) \longrightarrow \text{Aut}(\mathcal{H}_\gamma) \quad , \quad h \longmapsto U^{(j)}(t) = e^{iR^{(j)}(X^k)t_k} . \quad (97)$$

with $R^{(j)}(X^k) \equiv J_k$ denoting the set of Hermitian operators which represent the $SU(2)$ generators. The corresponding embedding of $\mathbb{G} \equiv SU(2)$ into the space of rays is given by the co-adjoint action map

$$\tilde{\phi}_0 : h \longmapsto U^{(j)}(h)\rho_0U^{(j)\dagger}(h) \quad , \quad \rho_0 \in \mathcal{R}(\mathcal{H}_\gamma) . \quad (98)$$

As showed in Eq. (50), for a pure fiducial state ρ_0 , the pull-back of the Hermitian tensor (93) to the co-adjoint orbit starting from it is given by

$$\mathcal{K} = \mathcal{K}_{k\ell}\theta^k \otimes \theta^\ell , \quad (99)$$

with coefficients

$$\mathcal{K}_{k\ell} = \text{Tr}(\rho_0 J_k J_\ell) - \text{Tr}(\rho_0 J_k)\text{Tr}(\rho_0 J_\ell) . \quad (100)$$

Moreover, in the case of a pure state, by using the explicit expression for the fiducial state

$$\rho_0 = \frac{|0\rangle\langle 0|}{\langle 0|0\rangle} = \frac{|\psi_\gamma^{(j)}\rangle\langle\psi_\gamma^{(j)}|}{\langle\psi_\gamma^{(j)}|\psi_\gamma^{(j)}\rangle} , \quad (101)$$

we find the pulled-back tensor on the corresponding orbits in the Hilbert space \mathcal{H}_γ :

$$\begin{aligned} \mathcal{K}_{k\ell} &= \frac{\langle 0|J_k J_\ell|0\rangle}{\langle 0|0\rangle} - \frac{\langle 0|J_k|0\rangle\langle 0|J_\ell|0\rangle}{\langle 0|0\rangle^2} \\ &= \frac{\langle\psi_\gamma^{(j)}|J_k J_\ell|\psi_\gamma^{(j)}\rangle}{\langle\psi_\gamma^{(j)}|\psi_\gamma^{(j)}\rangle} - \frac{\langle\psi_\gamma^{(j)}|J_k|\psi_\gamma^{(j)}\rangle\langle\psi_\gamma^{(j)}|J_\ell|\psi_\gamma^{(j)}\rangle}{\langle\psi_\gamma^{(j)}|\psi_\gamma^{(j)}\rangle^2} \\ &= \langle J_k J_\ell \rangle_{\psi_\gamma^{(j)}} - \langle J_k \rangle_{\psi_\gamma^{(j)}} \langle J_\ell \rangle_{\psi_\gamma^{(j)}} . \end{aligned} \quad (102)$$

We then see that the Hermitian tensor on the orbits embedded in \mathcal{H}_γ coincides with the covariance matrix of the $SU(2)$ generators. Indeed, starting from the definition of the covariance matrix whose entry in the k th row and ℓ th column is

$$\text{Cov}(J)_{k\ell} = \langle (J_k - \langle J_k \rangle)(J_\ell - \langle J_\ell \rangle) \rangle , \quad (103)$$

we have

$$\begin{aligned} \langle (J_k - \langle J_k \rangle)(J_\ell - \langle J_\ell \rangle) \rangle &= \langle (J_k J_\ell - J_k \langle J_\ell \rangle - \langle J_k \rangle J_\ell + \langle J_k \rangle \langle J_\ell \rangle) \rangle \\ &= \langle J_k J_\ell \rangle - \langle J_k \rangle \langle J_\ell \rangle - \langle J_k \rangle \langle J_\ell \rangle + \langle J_k \rangle \langle J_\ell \rangle = \langle J_k J_\ell \rangle - \langle J_k \rangle \langle J_\ell \rangle . \end{aligned} \quad (104)$$

The tensor (102) therefore will measure the correlations in the fluctuations of the operators J . The non-commutativity of such operators implies that the covariance matrix (103) is not symmetric, but if we remember the decomposition of the Hermitian tensor in its real symmetric and imaginary

skewsymmetric part, we find a metric tensor

$$\mathcal{K}_{(k\ell)} = \frac{1}{2} \langle [J_k, J_\ell]_+ \rangle_0 - \langle J_k \rangle_0 \langle J_\ell \rangle_0 \equiv \mathbb{R}e[\langle (J_k - \langle J_k \rangle)(J_\ell - \langle J_\ell \rangle) \rangle] , \quad (105)$$

and a symplectic structure

$$\mathcal{K}_{[k\ell]} = \mathbb{I}m\left(\frac{1}{2} \langle [J_k, J_\ell]_- \rangle_0\right) = \frac{1}{2} \langle \varepsilon_{k\ell r} J_r \rangle_0 , \quad (106)$$

where we have used the commutation relations $[J_k, J_\ell]_- = i\varepsilon_{k\ell r} J_r$ of the Lie algebra $\mathfrak{su}(2)$.

D. Step 3: Link as entanglement of semi-links

The simplest bipartite system is provided by regarding the single link Hilbert space $\mathcal{H}_\gamma^{(j)}$ (for fixed j) as the tensor product Hilbert space

$$\mathcal{H}_\gamma^{(j)} \cong \mathcal{V}^{(j)} \otimes \mathcal{V}^{(j)*} , \quad (107)$$

where $\mathcal{V}^{(j)}$ denotes the $(2j+1)$ -dimensional linear space carrying the irreducible representation of $SU(2)$ and for any $j \in \frac{\mathbb{N}}{2}$, the system $\{|j, m\rangle\}_{-j \leq m \leq j}$ is orthonormal, i.e.

$$\mathcal{V}^{(j)} = \text{span}\{|j, m\rangle\}_{-j \leq m \leq j} , \quad (108)$$

while $\mathcal{V}^{(j)*}$ is its dual vector space. Indeed, as already mentioned (see remark in Sec. V.A), the Wilson line state (78) can be regarded as

$$|\psi_\gamma^{(j)}\rangle = \sum_{mn} c_{mn}^{(j)} |j, n\rangle \otimes |j, m\rangle^* = \sum_{mn} c_{mn}^{(j)} |j, n\rangle \otimes \langle j, m| = \sum_{mn} c_{mn}^{(j)} \underbrace{|j, n\rangle \langle j, m|}_{|j, m, n\rangle} \quad (109)$$

in such a way that

$$\psi_\gamma^{(j)}[A] \equiv \langle h(A) | \psi_\gamma^{(j)} \rangle = \sum_{mn} c_{mn}^{(j)} \langle h(A) | j, m, n \rangle = \sqrt{2j+1} \sum_{mn} c_{mn}^{(j)} D_{mn}^{(j)}(h(A)) , \quad (110)$$

where $D_{mn}^{(j)}(h(A)) = \langle j, m | D^{(j)}(h(A)) | j, n \rangle$ are the Wigner D-matrix elements corresponding to the spin- j irreducible representation labelling the link.

More precisely, the Wilson line cylindrical basis functions correspond to the Weyl symbols of the holonomy operator $\hat{h}(A)$ with respect to the quantization map $|j, n\rangle \langle j, m|$ defined by [66]

$$\mathcal{W}(\hat{h}(A)) \equiv \text{Tr}(|j, n\rangle \langle j, m| \hat{h}(A)) = D_{mn}^{(j)}(h(A)) , \quad (111)$$

where, for any $j \in \frac{\mathbb{N}}{2}$, the set $\{\hat{v}_{nm}^j\} \equiv \{|j, n\rangle \langle j, m|\}_{-j \leq n, m \leq j}$ of $(2j+1) \times (2j+1)$ linear maps $|j, n\rangle \langle j, m| \in \mathcal{V}^{(j)} \otimes \mathcal{V}^{(j)*}$ is the canonical basis of the algebra $\text{End}(\mathcal{V}^{(j)})$ of endomorphisms of $\mathcal{V}^{(j)}$, orthonormal with respect to the scalar product $\langle \hat{v}_{nm}^j, \hat{v}_{n'm'}^j \rangle = \text{Tr}((\hat{v}_{nm}^j)^\dagger \hat{v}_{n'm'}^j)$.

Let us now try to characterize the entanglement for the bipartite system (107) where essentially we are thinking of the Wilson line state as the composite state of two semilink states (roughly,

spherical harmonics $Y_m^j(h) = \langle h|j, m\rangle$). Notice this is basically the same idea used in the GFT construction of link-connected, thus spin network states associated to connected graphs, from the basic ‘many-body’ states of the formalism, associated to ‘open spin network vertices’(nodes with outgoing semi-links). According to the diagram (95), we select a fiducial pure state

$$\rho_0 \in D^1(\mathcal{V}^{(j)} \otimes \mathcal{V}^{(j)*}) \cong \mathcal{R}(\mathcal{V}^{(j)} \otimes \mathcal{V}^{(j)*}) = \mathcal{R}(\mathcal{H}_\gamma^{(j)}), \quad (112)$$

and then we consider the product representation

$$\phi_0 : \mathbb{G} \equiv SU(2) \times SU(2) \longrightarrow \text{Aut}(\mathcal{V}^{(j)} \otimes \mathcal{V}^{(j)*}), \quad (113)$$

providing the following embedding map

$$\mathbb{G} \ni g \longmapsto \rho_g = U(g)\rho_0U^\dagger(g) \in \mathcal{R}(\mathcal{V}^{(j)} \otimes \mathcal{V}^{(j)*}) \quad (114)$$

where $U(g) = e^{iR(X_k)t}$. Infinitesimal generators $R(X_k)$ are realized as the tensor products between the identity of a subsystem and the spin operators J_k representing the $\mathfrak{su}(2)$ algebra in terms of selfadjoint operators on the Hilbert space $\mathcal{V}^{(j)}$ (cfr. Eq. (57)). Thus, according to Sec. III.B, we find that the pull-back of the Hermitian Fisher tensor $\text{Tr}(\rho d\rho \otimes d\rho)$ from $\mathcal{R}(\mathcal{H}_\gamma^{(j)}) = \mathcal{R}(\mathcal{V}^{(j)} \otimes \mathcal{V}^{(j)*})$ to the co-adjoint orbit

$$\mathcal{O}_{\rho_0} := SU(2) \times SU(2)/\mathbb{G}_{\rho_0}, \quad (115)$$

where \mathbb{G}_{ρ_0} is the isotropy group of the fiducial state (111)⁷, decomposes into a symmetric Riemannian and a skewsymmetric (pre-)symplectic component

$$\mathcal{K}_{k\ell} = \mathcal{K}_{(k\ell)} + i\mathcal{K}_{[k\ell]} = \left(\begin{array}{c|c} A & C \\ \hline C & B \end{array} \right) + i \left(\begin{array}{c|c} D_A & 0 \\ \hline 0 & D_B \end{array} \right), \quad (116)$$

with 3×3 blocks given by

$$\begin{cases} A_{ab} = \frac{1}{2}\text{Tr}(\rho_0[J_a, J_b]_+ \otimes \mathbb{1}) - \text{Tr}(\rho_0 J_a \otimes \mathbb{1})\text{Tr}(\rho_0 J_b \otimes \mathbb{1}) \\ B_{ab} = \frac{1}{2}\text{Tr}(\rho_0 \mathbb{1} \otimes [J_a, J_b]_+) - \text{Tr}(\rho_0 \mathbb{1} \otimes J_a)\text{Tr}(\rho_0 \mathbb{1} \otimes J_b) \\ C_{ab} = \text{Tr}(\rho_0 J_a \otimes J_b) - \text{Tr}(\rho_0 J_a \otimes \mathbb{1})\text{Tr}(\rho_0 \mathbb{1} \otimes J_b) \\ (D_A)_{ab} = \frac{1}{2}\text{Tr}(\rho_0[J_a, J_b]_- \otimes \mathbb{1}) \\ (D_B)_{ab} = \frac{1}{2}\text{Tr}(\rho_0 \mathbb{1} \otimes [J_a, J_b]_-) \end{cases} \quad (117)$$

Therefore, we see that if ρ_0 is maximally entangled, that is the reduced states are maximally mixed

$$\rho_0^{(A)} = \rho_0^{(B)} = \frac{1}{\dim \mathcal{V}^{(j)}} \mathbb{1}_{A,B} = \frac{\mathbb{1}_{\mathcal{V}^{(j)}}}{2j+1}, \quad (118)$$

⁷ The topology of the orbit will thus depend on the isotropy group of the selected fiducial state. We refer to [67, 68] for a general discussion.

then

$$(D_A)_{ab} = \frac{1}{2} \text{Tr}(\rho_0^{(B)} [J_a, J_b]_-) \propto \text{Tr}([J_a, J_b]_-) = 0, \quad (119)$$

and similarly for $(D_B)_{ab}$. On the other hand, if ρ_0 is separable, i.e., $\rho_0 = \rho_0^{(A)} \otimes \rho_0^{(B)}$, then we have

$$\begin{aligned} C_{ab} &= \text{Tr}(\rho_0^{(A)} J_a \otimes \rho_0^{(B)} J_b) - \text{Tr}(\rho_0^{(A)} J_a \otimes \rho_0^{(B)}) \text{Tr}(\rho_0^{(A)} \otimes \rho_0^{(B)} J_b) \\ &= \text{Tr}(\rho_0^{(A)} J_a) \text{Tr}(\rho_0^{(B)} J_b) - \underbrace{\text{Tr}(\rho_0^{(A)} J_a)}_1 \underbrace{\text{Tr}(\rho_0^{(B)})}_1 \text{Tr}(\rho_0^{(A)} \otimes \rho_0^{(B)} J_b) = 0. \end{aligned} \quad (120)$$

Thus, as stated in Sec. III.B, information about the separability or entanglement of the fiducial state ρ_0 is encoded into the different blocks of the pulled-back Hermitian tensor on the orbit of unitarily related states starting from ρ_0 . Indeed, the vanishing of the symplectic tensor for a maximally entangled state ρ_0 corresponds to a vanishing separability while the off-diagonal blocks of the Riemannian tensor are responsible for the entanglement degree of the state ρ_0 and allow us to define an associated entanglement monotone $\text{Tr}(C^T C)$ which identifies an entanglement measure geometrically interpreted as a distance between entangled and separable states. As we will discuss later in this work, since we are regarding the link as resulting from the entanglement of semilinks, such entanglement monotone gives us a measure of the existence of the link itself and so of the graph connectivity.

E. Two limiting cases: maximally entangled and separable states

In order to visualize the considerations of the previous section, let us focus on the two extreme cases respectively given by a maximally entangled and a separable Wilson line state, and compute explicitly the pull-back of the Hermitian tensor on the orbit having that state as fiducial state. To this aim, we start by considering the Schmidt decomposition [69] of the normalized state (109):

$$|\psi_\gamma^{(j)}\rangle = \sum_k \lambda_k |j, k\rangle \otimes \langle j, k|. \quad (121)$$

In the maximally entangled case all Schmidt coefficients are equal and, according to the normalization condition $\langle \psi_\gamma^{(j)} | \psi_\gamma^{(j)} \rangle = 1$, they are given by:

$$\lambda_k = \frac{1}{\sqrt{2j+1}} \quad \forall k \in [-j, +j], \quad (122)$$

thus yielding a maximally entangled state

$$|\psi_\gamma^{(j)}\rangle = \frac{1}{\sqrt{2j+1}} \sum_k |j, k\rangle \otimes \langle j, k|, \quad (123)$$

which is nothing but the gauge-invariant Wilson loop state $|\psi_{WL}\rangle$. Indeed, as discussed in Sec. 2.2, such a state corresponds to glue the two endpoints of the link in a bivalent vertex and contract

their magnetic moments with an intertwiner provided by the normalized identity in $\mathcal{V}^{(j)}$, i.e.:

$$|\psi_{WL}\rangle = \sum_{k,k'} \frac{\delta_{k,k'}}{\sqrt{2j+1}} |j, k\rangle \otimes \langle j, k'| \equiv \sum_{k,k'} i_{k,k'} |j, k\rangle \otimes \langle j, k'| . \quad (124)$$

Therefore, concerning the open single line state regarded as an entangled state of two semilinks, there is a correspondence between maximal entanglement and gauge-invariance which is actually realized by identifying the maximally entangled state (123) with the closed Wilson loop state, i.e.

$$\mathcal{H}_{max.ent.} \equiv \mathcal{H}_{loop} = \text{Inv}_{SU(2)}[\mathcal{V}^{(j)} \otimes \mathcal{V}^{(j)*}] \subset \mathcal{V}^{(j)} \otimes \mathcal{V}^{(j)*} \cong \mathcal{H}_{link} . \quad (125)$$

By taking the maximally entangled Wilson loop state as our fiducial state, we are interested in the corresponding pulled-back Hermitian tensor on the orbit starting from it. The pure state density matrix $\rho_0 \in D^1(\mathcal{V}^{(j)} \otimes \mathcal{V}^{(j)*})$ associated with it is given by

$$\rho_0 = |\psi_{WL}\rangle \langle \psi_{WL}| = \frac{1}{2j+1} \sum_{k,k'} (|j, k\rangle \langle j, k'|) \otimes (|j, k'\rangle \langle j, k|) , \quad (126)$$

such that the reduced states are diagonal with eigenvalues exactly given by the square of the Schmidt coefficients, e.g.

$$(\rho_0)_A = \text{Tr}_B(\rho_0) = \frac{1}{2j+1} \sum_k |j, k\rangle \langle j, k| = \frac{\mathbf{1}_j}{\dim \mathcal{V}^{(j)}} . \quad (127)$$

Hence, by using Eqs. (116,117), after lengthy but straightforward calculations, the pull-back of the Hermitian tensor \mathcal{K} on the orbit \mathcal{O}_{ρ_0} of Eq. (115) takes the following form (see the appendix of [70] for details)

$$\begin{pmatrix} \frac{1}{3}j(j+1) & 0 & 0 & \frac{1}{3}j(j+1) & 0 & 0 \\ 0 & \frac{1}{3}j(j+1) & 0 & 0 & \frac{1}{3}j(j+1) & 0 \\ 0 & 0 & \frac{1}{3}j(j+1) & 0 & 0 & \frac{1}{3}j(j+1) \\ \frac{1}{3}j(j+1) & 0 & 0 & \frac{1}{3}j(j+1) & 0 & 0 \\ 0 & \frac{1}{3}j(j+1) & 0 & 0 & \frac{1}{3}j(j+1) & 0 \\ 0 & 0 & \frac{1}{3}j(j+1) & 0 & 0 & \frac{1}{3}j(j+1) \end{pmatrix} \quad (128)$$

from which, using the decomposition $\mathcal{K}_{k\ell} = \mathcal{K}_{(k\ell)} + i\mathcal{K}_{[k\ell]}$, we see that the real symmetric part $\mathcal{K}_{(k\ell)}$ decomposes in the block-diagonal matrices A, B and the two equal block-off-diagonal matrices C , according to

$$\mathcal{K}_{(k\ell)} = \begin{pmatrix} A & C \\ C & B \end{pmatrix} \quad (129)$$

with

$$A = B = \begin{pmatrix} \frac{1}{3}j(j+1) & 0 & 0 \\ 0 & \frac{1}{3}j(j+1) & 0 \\ 0 & 0 & \frac{1}{3}j(j+1) \end{pmatrix}, C = \begin{pmatrix} \frac{1}{3}j(j+1) & 0 & 0 \\ 0 & \frac{1}{3}j(j+1) & 0 \\ 0 & 0 & \frac{1}{3}j(j+1) \end{pmatrix}, \quad (130)$$

while the imaginary skewsymmetric part $\mathcal{K}_{[k\ell]}$

$$\mathcal{K}_{[k\ell]} = \left(\begin{array}{c|c} D_A & 0 \\ \hline 0 & D_B \end{array} \right), \text{ with } D_A = D_B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (131)$$

gives a vanishing symplectic structure, as expected for the maximally entangled case. Moreover, by using the off-diagonal blocks (130) of the Riemannian symmetric part, the associated entanglement monotone is given by

$$\text{Tr}(C^T C) = \sum_{a,b=1}^3 C_{ab}^2 = \frac{1}{3}[j(j+1)]^2. \quad (132)$$

On the other extreme, if we consider a separable fiducial state, the two spin states do not talk with each other and may have in general different spins, i.e.:

$$|0\rangle = |j_1, k_1\rangle \otimes |j_2, k_2\rangle. \quad (133)$$

The corresponding pure state density matrix is given by

$$\rho_0 = \rho_0^{(A)} \otimes \rho_0^{(B)} = (|j_1, k_1\rangle \langle j_1, k_1|) \otimes (|j_2, k_2\rangle \langle j_2, k_2|). \quad (134)$$

Hence, the pull-back of the Hermitian tensor \mathcal{K} on the orbit \mathcal{O}_{ρ_0} will take the following form [70]

$$\begin{pmatrix} \frac{1}{2}[j_1(j_1+1) - k_1^2] & \frac{i}{2}k_1 & 0 & 0 & 0 & 0 \\ -\frac{i}{2}k_1 & \frac{1}{2}[j_1(j_1+1) - k_1^2] & 0 & 0 & 0 & 0 \\ 0 & 0 & k_1(k_1 - k_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}[j_2(j_2+1) - k_2^2] & \frac{i}{2}k_2 & 0 \\ 0 & 0 & 0 & -\frac{i}{2}k_2 & \frac{1}{2}[j_2(j_2+1) - k_2^2] & 0 \\ 0 & 0 & 0 & 0 & 0 & k_2(k_2 - k_1) \end{pmatrix} \quad (135)$$

from which we see that, as expected for the separable case, we have vanishing off-diagonal block matrices C and a direct sum

$$\underbrace{\begin{pmatrix} \frac{1}{2}[j_1(j_1+1) - k_1^2] & \frac{i}{2}k_1 & 0 \\ -\frac{i}{2}k_1 & \frac{1}{2}[j_1(j_1+1) - k_1^2] & 0 \\ 0 & 0 & k_1(k_1 - k_2) \end{pmatrix}}_{\mathcal{K}_A} \oplus \underbrace{\begin{pmatrix} \frac{1}{2}[j_2(j_2+1) - k_2^2] & \frac{i}{2}k_2 & 0 \\ -\frac{i}{2}k_2 & \frac{1}{2}[j_2(j_2+1) - k_2^2] & 0 \\ 0 & 0 & k_2(k_2 - k_1) \end{pmatrix}}_{\mathcal{K}_B} \quad (136)$$

of two decoupled Hermitian tensors \mathcal{K}_A and \mathcal{K}_B one for each subsystem. Moreover, a further decomposition of the Hermitian tensor (135) as $\mathcal{K}_{k\ell} = \mathcal{K}_{(k\ell)} + i\mathcal{K}_{[k\ell]}$, gives a symmetric real part

$$\mathcal{K}_{(k\ell)} = \left(\begin{array}{c|c} A & C \\ \hline C & B \end{array} \right) \quad (137)$$

with

$$A = \begin{pmatrix} \frac{1}{2}[j_1(j_1+1) - k_1^2] & 0 & 0 \\ 0 & \frac{1}{2}[j_1(j_1+1) - k_1^2] & 0 \\ 0 & 0 & k_1(k_1 - k_2) \end{pmatrix}, \quad (138)$$

$$B = \begin{pmatrix} \frac{1}{2}[j_2(j_2 + 1) - k_2^2] & 0 & 0 \\ 0 & \frac{1}{2}[j_2(j_2 + 1) - k_2^2] & 0 \\ 0 & 0 & k_2(k_2 - k_1) \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (139)$$

and an imaginary skewsymmetric part

$$\mathcal{K}_{[k\ell]} = \left(\begin{array}{c|c} D_A & 0 \\ \hline 0 & D_B \end{array} \right) \quad (140)$$

with

$$D_A = \begin{pmatrix} 0 & \frac{1}{2}k_1 & 0 \\ -\frac{1}{2}k_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D_B = \begin{pmatrix} 0 & \frac{1}{2}k_2 & 0 \\ -\frac{1}{2}k_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (141)$$

Finally, we have

$$\text{Tr}(C^T C) = 0, \quad (142)$$

i.e., coherently with its interpretation as a distance from the separable state, the entanglement measure associated with the block-off-diagonal matrices C is zero in the unentangled case.

V. ENTANGLEMENT IN EXTENDED GRAPH STRUCTURES

A. Gluing links by entanglement

Let us proceed a little step further with respect to what shown in section III, and consider the description of the entanglement resulting from the gluing of two lines into one. The bipartite Hilbert space is given by two copies of a single link Hilbert space with fixed but different spin labels, i.e.

$$\mathcal{H} = \mathcal{H}_{\gamma_1}^{(j_1)} \otimes \mathcal{H}_{\gamma_2}^{(j_2)}, \quad (143)$$

and the fiducial state is chosen to be a Wilson line state (one link state) coming from the gluing of two other links, thus admitting the following expression

$$|0\rangle \equiv |\psi_\gamma\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m,n,k,\ell} c_{mn} |j, m, k\rangle \otimes |j, \ell, n\rangle \delta_{k,\ell}. \quad (144)$$

The local $SU(2)$ gauge-invariance requirement at the gluing point $v \equiv \gamma_1(1) = \gamma_2(0)$, implemented by the bivalent intertwiner $\delta_{k,\ell}/\sqrt{2j+1}$ contracting the magnetic numbers of the glued endpoints, forces the two spins to be equal (i.e., $j_1 = j_2 = j$). In other words, $|0\rangle$ is a locally $SU(2)$ -invariant state in \mathcal{H} , that is

$$|0\rangle \in \mathcal{H}_\gamma^{(j)} \subset \mathcal{H}, \quad \gamma = \gamma_1 \circ \gamma_2. \quad (145)$$

However, in order to compute an entanglement measure which can be interpreted as the distance of our fiducial state from the separable one, we need to consider the action ϕ of a Lie group \mathbb{G} on \mathcal{H} and not only on the gauge-reduced level $\tilde{\phi} : \mathbb{G}/SU(2) \rightarrow \mathcal{H}_\gamma^{(j)}$. Therefore, the underlying scheme of the construction of the pulled-back Hermitian tensor on the orbit of states with fixed amount of entanglement will be given by the following diagram

$$\begin{array}{ccc}
 \mathbb{G} & \xrightarrow{\phi_0} & \mathcal{H}_{\gamma_1}^{(j_1)} \otimes \mathcal{H}_{\gamma_2}^{(j_2)} \\
 \downarrow SU(2) & \text{“gluing”} & \downarrow SU(2) \\
 \mathbb{G}/SU(2) & \xrightarrow{\tilde{\phi}_0} & \mathcal{H}_{\gamma_1 \circ \gamma_2}^{(j)} \\
 \downarrow \pi_0 & & \uparrow i_{\mathcal{O}} \\
 \mathbb{G}/\mathbb{G}_0 & \xrightarrow{\cong} & \mathcal{O}
 \end{array} \tag{146}$$

We recall that the group \mathbb{G} is a group of local unitary transformations which as such do not modify the degree of entanglement along the orbit starting at the selected fiducial state. In the specific case under consideration, the group \mathbb{G} is $SU(2)$ and its action on the bipartite Hilbert space (143) is realized through a product representation

$$U(\mathcal{H}) = U(\mathcal{H}_{\gamma_1}) \otimes U(\mathcal{H}_{\gamma_2}) , \tag{147}$$

whose infinitesimal generators are given by the $SU(2)$ -generators tensored by the identity of one of the subsystems. Indeed, each subsystem Hilbert space reads as

$$\mathcal{H}_{\gamma_i}^{(j)} \cong \mathcal{V}^{(j_i)} \otimes \mathcal{V}^{(j_i)*} \quad (i = 1, 2) , \tag{148}$$

and so the bipartite Hilbert space (143) can be regarded as

$$\mathcal{H} \cong (\mathcal{V}^{(j_1)} \otimes \mathcal{V}^{(j_1)*}) \otimes (\mathcal{V}^{(j_2)} \otimes \mathcal{V}^{(j_2)*}) . \tag{149}$$

The gluing operation $\gamma = \gamma_1 \circ \gamma_2$ corresponds to select the subspace

$$\mathcal{V}^{(j_1)} \otimes \text{Inv}_{SU(2)}[\mathcal{V}^{(j_1)} \otimes \mathcal{V}^{(j_2)*}] \otimes \mathcal{V}^{(j_2)} \subset \mathcal{H} , \tag{150}$$

which coincides with the space

$$\mathcal{V}^{(j)} \otimes \mathcal{V}^{(j)*} \cong \mathcal{H}_\gamma^{(j)} \quad , \quad j = j_1 = j_2 \tag{151}$$

since, according to the *Schur's lemma* [39], when we have only two spin representations the invariant bivalent intertwining operator $\mathcal{V}^{(j_1)} \rightarrow \mathcal{V}^{(j_2)}$ is either proportional to the identity if $j_1 = j_2$ or zero if $j_1 \neq j_2$, i.e., the invariant subspace is trivial.

We are thus brought back to the situation of the previous section. The pulled-back Hermitian tensor \mathcal{K} is again given by the pull-back of (117) with a fiducial state now given by (144) and the spin operators J act non-trivially only at the free endpoints of the resulting new link. Hence, there is no

need to repeat our calculations and we only notice that, coherently with the general considerations of Sec. III, we have:

- For the skewsymmetric part:

$$\begin{aligned}
(D_A)_{ab} &= \langle 0|[J_a, J_b]_- \otimes \mathbb{1}|0\rangle = \frac{1}{2j+1} \sum_{mnlm'n'\ell'} \overline{c_{m'n'}} c_{mn} \langle j, m', \ell' |[J_a, J_b]_- |j, m, \ell\rangle \delta_{\ell', \ell} \delta_{n', n} \\
&= \frac{1}{2j+1} \sum_{mnlm'} \overline{c_{m'n}} c_{mn} \langle j, m', \ell |[J_a, J_b]_- |j, m, \ell\rangle = \sum_{mnm'} \overline{c_{m'n}} c_{mn} \langle j, m' |[J_a, J_b]_- |j, m\rangle
\end{aligned} \tag{152}$$

being $\langle j, \ell | j, \ell\rangle = 1$. We then see that when the fiducial state is maximally entangled, i.e., all the Schmidt coefficients are equal, we end up with the trace of the commutator which is zero.

- By similar arguments, when $|0\rangle$ is separable, we see that the block-off-diagonal matrices of the symmetric part vanish:

$$C_{ab} = \langle 0|J_a \otimes J_b|0\rangle - \langle 0|J_a \otimes \mathbb{1}|0\rangle \langle 0|\mathbb{1} \otimes J_b|0\rangle = \langle J_a\rangle \langle J_b\rangle - \langle J_a\rangle \langle J_b\rangle = 0. \tag{153}$$

B. Fisher quantum metric for a bounded region of space

We now proceed to a more interesting case of a closed connected graph, thus made of several links and nodes. This is considerably more involved, of course, than a single link. However, we must point out that, from the point of view of both LQG and GFT, it remains a drastic truncation of the set of (kinematical) degrees of freedom, and of the possibly relevant states, particularly from the point of view of a reconstruction of an approximate continuum spacetime and geometry, since the latter would most likely require quantum superpositions of graph structures. Indeed, it implies truncating the theory to a finite set of degrees of freedom loosing the functional aspects of the theory. It should also be noted, though, that most work in the LQG context is limited to such fixed-graph computations, due to the complexity of the more general case. In particular, the following analysis is framed in a very similar way as the one in [?].

Let us consider then a generic closed connected spin network graph Γ describing a region of quantum space and let us single out of Γ two (non-adjacent) vertices v_A and v_B , corresponding to two generic subregions of space A and B . Such a cut defines an open subgraph Γ_R dual to an ideal connected *bounded* region R . The boundary of the region ∂R is comprised by two set of edges, which have only one end vertex laying in R , respectively defining the boundary ∂A and ∂B of the two regions cut out. Consistently, bulk edges are paths connecting vertices in R . We can picture R as a 3-ball and ∂R as its boundary 2-sphere punctured by the boundary edges.

We are interested in studying the non-local quantum correlations induced among the two regions A and B by the presence of R . Following [71], we expect the wave function on R to play the role of a quantum *background metric* correlating A to B . Our goal is then to provide an interpretation of such a metric in terms of the Fisher quantum metric. Therefore, we focus on the *open* spin network state with support on Γ_R . Due to the presence of open edges, the wave functional defined on the boundary and bulk holonomies $\{g_e\}$ will not be gauge invariant but rather covariant under $SU(2)$

gauge symmetry,

$$\varphi_{\Gamma_R}(\{g_e\}) \equiv \int_{SU(2)} dg \varphi_{\Gamma_R}(\{g g_{e \in \partial R}, g_{e \in R}\}) \quad (154)$$

Nevertheless, the gauge invariance of the bulk graph allows us to drastically simplify the form of φ_{Γ_R} by means of a gauge fixing procedure (see [71, 72] for full details). This amounts to choosing a (reference) vertex v_0 in Γ_R and a maximal tree T therein.⁸ Hence, to fixing to the identity the group elements on the edges belonging to the tree, $g_{e \in T} = \mathbb{1}$.

The functional built on the resulting graph is invariant under the diagonal action of $SU(2)$ on the open external edges and under the adjoint action of $SU(2)$ on the interior graph,

$$\varphi_{\Gamma_R}^T(\{g_{e \in \Gamma_R}\}) = \varphi_{\Gamma_R}^T(\{\mathbb{1}\}_{e \in T}, \{\tilde{g}_e\}_{e \in \partial R}, \{\mathcal{G}_e^T\}_{e \in R \setminus T}) \quad (155)$$

with $\mathcal{G}_e^T = (G_{t(e)}^{-1} g_e G_{s(e)})$ gauge fixed holonomies around the (non-contractible) loops of the internal graph and $\{\tilde{g}_e\}$ holonomies on the boundary edges, given either by $(G_{t(e)}^{-1} g_e)$ or $(g_e G_{s(e)})$, depending on whether their internal vertex is source or target.

As a result of the gauge fixing, Γ_R reduces to a single vertex flower graph with E open edges and L loops hence the wave functional $\varphi_{\Gamma_R}^T$ can be written as a superposition of single vertex intertwiner basis states $|i_R\rangle$ spanning the intertwiners Hilbert space

$$\mathcal{H}_R = \bigoplus_{\{j_e, j_l\}} \text{Inv}_{SU(2)} [\mathcal{V}_{j_e}^{\otimes E} \otimes (\mathcal{V}_{j_l}^{\otimes L} \otimes \mathcal{V}_{j_l}^{* \otimes L})], \quad (156)$$

where j_e, j_l respectively label the spin representations attached to the boundary edges and bulk loops. Notice that the closure constraint is satisfied, as the sum of the flux-vectors living on the external edges is balanced by the sum of the flux-vectors on the internal loops.

The wave-function $\varphi_{\Gamma_R}^T$ provides us with the most concise description for a region of quantum space with non-trivial internal degrees of freedom. The gauge reduction isomorphism does not produce any coarse graining, nevertheless in the procedure one discards the combinatorial information about the internal subgraph encoded in the choice of a specific maximal tree T ⁹.

C. The bipartite system

In order to avoid the complication of dealing with loopy curvature defects [72], we restrict the analysis to the case of a region of quantum space without non-trivial internal curvature degrees of freedom. The absence of loops carrying curvature excitations at the reduced, gauge-fixed vertex allows us to dually think of such region as a portion of 3d space flatly embedded in 4d. In this case, the flower graph reduces to a n -valent intertwiner. The irreducible representations carried by the open edges are dual to boundary patches comprising the quantum surface of the convex flat polyhedron dual to the intertwiner. In particular, we want to divide n in the two sets $n_A = (\{j_e\} | e \in \partial A)$ and $n_B = (\{j_e\} | e \in \partial B)$.

⁸ The maximal tree is the set of edges in Γ_R going through all vertices in the region without ever making any loop. It has $|T| = V_R - 1$ elements. If $|T| = E_R$, this means that Γ_R has no loops, hence trivial topology. The number of non-trivial loops of Γ_R is given by $L = E_R - |T|$.

⁹ Notice that this combinatorial information is understood to encode physical information. This is manifest, for example, in the GFT formalism where the number of nodes in a graph is counted by a simple observable of the theory, the number operator. This means that the gauge fixing procedure, albeit not a coarse graining, as we emphasized, implies shifting the attention from a subset of observables to another.

Starting from such separation of degrees of freedom, we can think of the generic pure state $\varphi_{\Gamma_R}^T$ as a *bipartite* quantum mechanical system constrained by an overall $SU(2)$ gauge invariance. In these terms, we can see the intertwiner space as embedded in the larger unconstrained space

$$\mathcal{H}_R \subseteq \mathcal{H} \equiv \mathcal{H}_A \otimes \mathcal{H}_B, \quad (157)$$

where the two boundary spaces are respectively defined by the tensor product of the $SU(2)$ irreps labelling the edges

$$\mathcal{H}_A \equiv \bigotimes_{e \in \partial A} \mathcal{V}^{j_e} \quad \text{and} \quad \mathcal{H}_B \equiv \bigotimes_{e \in \partial B} \mathcal{V}^{j_e}. \quad (158)$$

The two boundary sets of degrees of freedom are ‘‘coupled’’ by the request of gauge invariance at the vertex. Such coupling generates quantum entanglement. Therefore we have,

$$\mathcal{H}_R = \bigoplus_{\{j_e\}} \text{Inv}_{SU(2)} [\mathcal{H}_A \otimes \mathcal{H}_B] \quad (159)$$

Now, for what concerns our analysis, we further simplify the problem by considering the case of a single intertwiner graph space, namely we work at fixed $\{j_e\}$, avoiding the full sum $\bigoplus_{\{j_e\}}$ in the above (159). This is another strong simplification, since it implies a further reduction of the degrees of freedom being considered, which were already truncated to a finite number by fixing a graph Γ .

For convenience, we proceed by taking the decomposition of the two tensor product states in direct sums of irreducible representations: we write each subspace $\mathcal{H}_{A,B}$ as $\bigoplus_k \mathcal{V}^k \otimes \mathcal{D}_k$, where \mathcal{V}^k is a $(2k+1)$ -dimensional re-coupled spin- k irrep of $SU(2)$, coming with its degeneracy space \mathcal{D}_k . In these terms, the single intertwiner Hilbert space can be explicitly written in terms of the subsystems A and B , as follows

$$\begin{aligned} \mathcal{H}_R^{\{j_e\}} &= \text{Inv}_{SU(2)} [\mathcal{H}_A \otimes \mathcal{H}_B] = \text{Inv}_{SU(2)} \left[\bigoplus_{k,y} (\mathcal{V}_A^k \otimes \{j_{e \in \partial A}\} \mathcal{D}_k) \otimes (\mathcal{V}_B^y \otimes \{j_{e \in \partial B}\} \mathcal{D}_y) \right] \\ &= \bigoplus_{k,y} \delta_{k,y} \left[(\mathcal{V}_A^k \otimes \mathcal{V}_B^y) \otimes (\{j_{e \in \partial A}\} \mathcal{D}_k \otimes \{j_{e \in \partial B}\} \mathcal{D}_y) \right] \\ &= \bigoplus_k (\mathcal{V}_A^k \otimes \{j_{e \in \partial A}\} \mathcal{D}_k) \otimes (\mathcal{V}_B^k \otimes \{j_{e \in \partial B}\} \mathcal{D}_k) = \bigoplus_k \{j_{e \in \partial A}\} \mathcal{H}_A^{(k)} \otimes \{j_{e \in \partial B}\} \mathcal{H}_B^{(k)}. \end{aligned} \quad (160)$$

Notice that, by gauge invariance, the two re-coupled spin irreps appearing in the second line are tensored to form a trivial representation for each k . Consistently, the dimension of the single intertwiner space reduces to

$$\begin{aligned} N_R \equiv \dim(\mathcal{H}_R^{\{j_e\}}) &= \dim \left[\bigoplus_k (\mathcal{V}_A^k \otimes \mathcal{V}_B^k) \otimes (\mathcal{D}_k^A \otimes \mathcal{D}_k^B) \right] \\ &= \sum_k \dim \mathcal{D}_k^A \cdot \dim \mathcal{D}_k^B = \sum_k N_A^k N_B^k. \end{aligned} \quad (161)$$

where we define $N_{A,B}^k \equiv d_k^{(n_{A,B})} = \dim \mathcal{D}_k^{(A,B)}$, the dimensions of the degeneracy spaces. Starting from the decomposition in (160), a convenient basis in the two subsystems A and B is labeled by three numbers, respectively $|k, m, \alpha_k\rangle$ and $|k, m, \beta_k\rangle$, with α_k, β_k giving the number of the different irreducible representations $\mathcal{V}_{A,B}^k$ for given k [71]. A basis for the single intertwiner space is then written as

$$|k, \alpha_k, \beta_k\rangle = \sum_{m=-k}^k \frac{(-1)^{k-m}}{\sqrt{2k+1}} |k, -m, \alpha_k\rangle_A \otimes |k, m, \beta_k\rangle_B \quad (162)$$

Given the peculiar tensor structure of the unfolded intertwiner space, each basis state can be represented as a tensor product state on three subspaces [71],

$$|k, \alpha_k, \beta_k\rangle \equiv |k\rangle_{\mathcal{V}_A^k \otimes \mathcal{V}_B^k} \otimes |\alpha_k\rangle_{\mathcal{D}_k^A} \otimes |\beta_k\rangle_{\mathcal{D}_k^B} \quad (163)$$

where the generic $|\zeta_k\rangle$ labels a basis vector of \mathcal{D}_k , with ζ_k running from 1 to $N_k = \dim \mathcal{D}_k$.

A generic state vector in $\mathcal{H}_R^{\{j_e\}}$ will be denoted by a superposition of product basis states of the three subspaces, say

$$|\psi_{\Gamma_R}\rangle = \sum_{\substack{k, \\ \alpha_k, \beta_k}} c_{k, \alpha_k, \beta_k}^{\{j_e\}} |k\rangle_{\mathcal{V}_A^k \otimes \mathcal{V}_B^k} \otimes |\alpha_k\rangle_{\mathcal{D}_k^A} \otimes |\beta_k\rangle_{\mathcal{D}_k^B}. \quad (164)$$

However, we are going to focus our attention to states with correlations entangling the two degeneracy spaces only. This translates in the choice of a specific class of states, generically written as

$$|\psi_{\Gamma_R}\rangle = \sum_{\alpha_k, \beta_k} c_{\alpha_k, \beta_k}^{(k, \{j_e\})} |k\rangle_{\mathcal{V}_A^k \otimes \mathcal{V}_B^k} \otimes |\alpha_k\rangle_{\mathcal{D}_k^A} \otimes |\beta_k\rangle_{\mathcal{D}_k^B}, \quad (165)$$

with no sum over k . Such a choice amounts to discard any quantum correlation among the tensored irreps space and the degeneracy spaces. This partially reduces the complexity of the problem. In particular, for fixed spin k of the virtual link, the unfolded intertwiner space (160) admits the following tensor product structure

$$\begin{aligned} \mathcal{H}_R^{k, \{j_e\}} &= \{j_e \in \partial A\} \mathcal{H}_A^{(k)} \otimes \{j_e \in \partial B\} \mathcal{H}_B^{(k)} = (\mathcal{V}_A^k \otimes \{j_e \in \partial A\} \mathcal{D}_k) \otimes (\mathcal{V}_B^k \otimes \{j_e \in \partial B\} \mathcal{D}_k) \\ &= (\mathcal{V}_A^k \otimes \mathcal{V}_B^k) \otimes \mathcal{D}_k^A \otimes \mathcal{D}_k^B. \end{aligned} \quad (166)$$

A generic pure state density matrix $\rho_R = |\psi_R\rangle \langle \psi_R| \in D^1(\mathcal{H}_R^{k, \{j_e\}})$ will then have the following simplified form

$$\rho_R \equiv |\psi_R\rangle \langle \psi_R| = \rho^{(k)} \otimes \rho_{AB}^{(k)}, \quad (167)$$

with

$$\rho^{(k)} = |k\rangle \langle k| \in D^1(\mathcal{V}_A^k \otimes \mathcal{V}_B^k), \quad (168)$$

and

$$\rho_{AB}^{(k)} = \sum_{\substack{\alpha_k, \beta_k \\ \alpha'_k, \beta'_k}} c_{\alpha_k, \beta_k}^{(k, \{j_e\})} \bar{c}_{\alpha'_k, \beta'_k}^{(k, \{j_e\})} |\alpha_k\rangle \langle \alpha'_k| \otimes |\beta_k\rangle \langle \beta'_k| \in D^1(\mathcal{D}_k^A \otimes \mathcal{D}_k^B), \quad (169)$$

i.e., with respect to the total space $\mathcal{H}_R^{k, \{j_e\}}$ for fixed k , the pure state ρ_R is separable and can be factorized into a tensor product between a pure state $\rho^{(k)}$ involving only the spin k irreps and a pure state $\rho_{AB}^{(k)}$ possibly entangling the degeneracy spaces. Since there are no correlations among the tensored irreps space and the degeneracy spaces, we can trace out $\rho^{(k)}$ and focus our attention only on the state $\rho_{AB}^{(k)}$ over the degeneracy spaces.

Therefore, for each k , we can effectively treat the intertwiner state as an entangled pure state on the bipartite degeneracy space $\mathcal{D}_k^A \otimes \mathcal{D}_k^B$. In other words, we can describe the degeneracy structure of the unfolded intertwiner state as a couple of entangled quantum N -level systems with number of levels provided by the degeneracy factors $N_X^k = \dim \mathcal{D}_k^X \equiv d_k^{(n_X)}$, $X = A, B$, respectively.

We would like to stress again that we are interested in studying the quantum correlations between A and B induced by the spin network state outside A and B , i.e., by the outside space geometry defined by the complementary region R to $A \cup B$ in the graph Γ . The idea is that, in the quantum geometry defined by the spin network, such correlations could be related to a notion of distance between two of its parts A and B , that somehow transcends the direct connectivity information encoded in the graph by the links connecting the two regions and the associated quantum geometric labels (the group-theoretic data), while at the same time being compatible with them. Once more, the need to encode geometric properties in algebraic and combinatorial data, first, and then reconstruct them also via the quantum correlations between the subsystems described by the quantum states (as subregions of the graph underlying them) stems from the fundamental background independence of the theory. The latter, as we have discussed, implies not only the absence of a background metric field defining a notion of position and distance, but also of any physically significant embedding manifold, that can support such field. Thus everything has to be encoded in quantum “relations” between parts of the spin network.

In what follows we study the above-mentioned correlations by means of the tensorial structures intrinsically defined on the space of pure quantum states describing the degeneracy structure of the unfolded intertwiner state. In particular, we compute the full quantum Fisher tensor on the orbit submanifolds identified by a fiducial bipartite state (169) and analyze the behaviour of the corresponding entanglement measure as a function of the virtual spin k which, as discussed before, is expected to capture the correlations induced among the regions A and B by the *quantum metric* defined by the outside spin network geometry.

D. Quantum Fisher tensor for bipartite degeneracy spaces

In order to explicitly calculate the entanglement monotone (72) we need to put the block-coefficient matrices (70) of the Hermitian tensor \mathcal{K} in a more manageable form. To this aim it is convenient to choose the *standard* (or *natural*) basis for the $\mathfrak{u}(N)$ Lie algebras associated with the two subsystems, that is

$$\sigma_a^{(A)} \mapsto \tau_{aa'}^{(A)} \equiv |a_k\rangle \langle a'_k| \quad , \quad \sigma_b^{(B)} \mapsto \tau_{bb'}^{(B)} \equiv |b_k\rangle \langle b'_k| \quad (170)$$

with $a_k, a'_k = 1, \dots, N_A^k \equiv d_k^{(n_A)}$ and $b_k, b'_k = 1, \dots, N_B^k \equiv d_k^{(n_B)}$ such that

$$\left(\tau_{aa'}^{(A)} \right)_{cc'} = \delta_{ac} \delta_{a'c'} , \quad (171)$$

and similarly for the $\tau^{(B)}$'s. Such a change of basis essentially amounts to replace the σ 's with the τ 's in the expressions (70). Indeed, the Lie algebra-valued left-invariant 1-form $U^{-1}dU$ can still be decomposed as in Eq. (58), say

$$U^{-1}dU = U_A^{-1}d_A U_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes U_B^{-1}d_B U_B = i\tau_{aa'}^{(A)}\theta_A^{aa'} \otimes \mathbb{1}_B + \mathbb{1}_A \otimes i\tau_{bb'}^{(B)}\theta_B^{bb'} , \quad (172)$$

where, for each subsystem $X = A, B$, the $\tau^{(X)}$ can be expressed as linear combinations of the infinitesimal generators $\sigma^{(X)}$ with complex coefficients and, correspondingly, also the new bases of left-invariant 1-forms (i.e., $\{\theta_A^{aa'}\}$ and $\{\theta_B^{bb'}\}$) are now given in terms of \mathbb{C} -linear combinations of the previous ones¹⁰. With this choice of basis, the block-coefficient matrices (70) then become:

$$\left\{ \begin{array}{l} \mathcal{K}_{(aa')_{(bb')}}^{(A)} = \frac{1}{2} \text{Tr} \left(\rho_0 [\tau_{aa'}^{(A)}, \tau_{bb'}^{(A)}]_+ \otimes \mathbb{1}_B \right) - \text{Tr} \left(\rho_0 \tau_{aa'}^{(A)} \otimes \mathbb{1}_B \right) \text{Tr} \left(\rho_0 \tau_{bb'}^{(A)} \otimes \mathbb{1}_B \right) \\ \mathcal{K}_{(aa')_{(bb')}}^{(B)} = \frac{1}{2} \text{Tr} \left(\rho_0 \mathbb{1}_A \otimes [\tau_{aa'}^{(B)}, \tau_{bb'}^{(B)}]_+ \right) - \text{Tr} \left(\rho_0 \mathbb{1}_A \otimes \tau_{aa'}^{(B)} \right) \text{Tr} \left(\rho_0 \mathbb{1}_A \otimes \tau_{bb'}^{(B)} \right) \\ \mathcal{K}_{(aa')_{(bb')}}^{(AB)} = \text{Tr} \left(\rho_0 \tau_{aa'}^{(A)} \otimes \tau_{bb'}^{(B)} \right) - \text{Tr} \left(\rho_0 \tau_{aa'}^{(A)} \otimes \mathbb{1}_B \right) \text{Tr} \left(\rho_0 \mathbb{1}_A \otimes \tau_{bb'}^{(B)} \right) \\ \mathcal{K}_{(aa')_{(bb')}}^{(BA)} = \text{Tr} \left(\rho_0 \tau_{aa'}^{(B)} \otimes \tau_{bb'}^{(A)} \right) - \text{Tr} \left(\rho_0 \tau_{aa'}^{(B)} \otimes \mathbb{1}_A \right) \text{Tr} \left(\rho_0 \mathbb{1}_B \otimes \tau_{bb'}^{(A)} \right) \\ \mathcal{K}_{[aa']_{[bb']}}^{(A)} = \frac{1}{2} \text{Tr} \left(\rho_0 [\tau_{aa'}^{(A)}, \tau_{bb'}^{(A)}]_- \otimes \mathbb{1}_B \right) \\ \mathcal{K}_{[aa']_{[bb']}}^{(B)} = \frac{1}{2} \text{Tr} \left(\rho_0 \mathbb{1}_A \otimes [\tau_{aa'}^{(B)}, \tau_{bb'}^{(B)}]_- \right) \end{array} \right. \quad (173)$$

The fiducial state (169) can be thus written as

$$\rho_0 = \sum_{\alpha\alpha'\beta\beta'} c_{\alpha\beta} \bar{c}_{\alpha'\beta'} \tau_{\alpha\alpha'}^{(A)} \otimes \tau_{\beta\beta'}^{(B)} . \quad (174)$$

¹⁰ For instance, in the $U(2)$ case we have:

$$\begin{aligned} \sigma_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \tau_{11} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} , \quad \tau_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad \tau_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \quad \tau_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} . \end{aligned}$$

Therefore

$$\sigma_0 = \tau_{11} + \tau_{22} , \quad \sigma_1 = \tau_{12} + \tau_{21} , \quad \sigma_2 = i(\tau_{21} - \tau_{12}) , \quad \sigma_3 = \tau_{11} - \tau_{22}$$

from which, by imposing that $U^{-1}dU = i\sigma_k \theta^k = i\tau_{kk'} \theta^{kk'}$, it is easy to see that

$$\theta^{11} = \theta^0 + \theta^3 , \quad \theta^{12} = \theta^1 - i\theta^2 , \quad \theta^{21} = \theta^1 + i\theta^2 , \quad \theta^{22} = \theta^0 - \theta^3 .$$

where we do not explicitly write the superscripts $(k, \{j_e\})$ to simplify the notation. As discussed in appendix A, a direct computation of the expressions (173) yields

$$\mathcal{K}_{(aa')(bb')}^{(AB)} = c_{a'b'} \bar{c}_{ab} - \sum_{\beta} c_{a'\beta} \bar{c}_{a\beta} \cdot \sum_{\gamma} c_{\gamma b'} \bar{c}_{\gamma b} , \quad (175)$$

$$\mathcal{K}_{(aa')(bb')}^{(A)} = \frac{1}{2} \left(\delta_{a'b} \sum_{\beta} c_{b'\beta} \bar{c}_{a\beta} + \delta_{ab'} \sum_{\beta} c_{b\beta} \bar{c}_{a'\beta} \right) - \sum_{\beta} c_{a'\beta} \bar{c}_{a\beta} \cdot \sum_{\delta} c_{b'\delta} \bar{c}_{b\delta} , \quad (176)$$

$$\mathcal{K}_{[aa'][bb']}^{(A)} = \frac{1}{2} \left(\delta_{a'b} \sum_{\beta} c_{b'\beta} \bar{c}_{a\beta} - \delta_{ab'} \sum_{\beta} c_{b\beta} \bar{c}_{a'\beta} \right) . \quad (177)$$

Similar results hold for $\mathcal{K}^{(BA)}$ and the $\mathcal{K}^{(B)}$ blocks of the symmetric and antisymmetric part, respectively.

Finally, omitting for the moment the constant factor in front of the trace in Eq. (72), the entanglement monotone \mathcal{E} is given by (cfr. Eqs. (A6,A7)):

$$\mathcal{E} = \sum_{aa'cc'} \left(c_{a'c'} \bar{c}_{ac} - \sum_{\beta} c_{a'\beta} \bar{c}_{a\beta} \sum_{\gamma} c_{\gamma c'} \bar{c}_{\gamma c} \right)^2 . \quad (178)$$

Remark: Here we see the advantage of choosing the standard basis. Indeed, in this basis, our measure of entanglement \mathcal{E} , as well as all the block-coefficient matrices of the tensor \mathcal{K} , are written directly in terms of the coefficients c of the fiducial state.

E. Some special cases

Let us finally check that the blocks of the tensor \mathcal{K} actually encode the information about separability or entanglement by considering some explicit choice of the fiducial state ρ_0 . In particular we have to check that the off-diagonal blocks, and hence the entanglement monotone \mathcal{E} , vanish when ρ_0 is separable, while the symplectic part vanishes when ρ_0 is maximally entangled. So if ρ_0 is separable, the coefficients $c_{\alpha\beta}$ factorize as $\lambda_{\alpha}\bar{\lambda}_{\beta}$ and the fiducial state (174) can be written as:

$$\rho_0 = \left(\sum_{\alpha, \alpha'=1}^{N_A} \lambda_{\alpha} \bar{\lambda}_{\alpha'} \tau_{\alpha\alpha'}^{(A)} \right) \otimes \left(\sum_{\beta, \beta'=1}^{N_B} \lambda_{\beta} \bar{\lambda}_{\beta'} \tau_{\beta\beta'}^{(B)} \right) \equiv \rho_0^{(A)} \otimes \rho_0^{(B)} . \quad (179)$$

Therefore, when this is the case, the matrix elements of the off-diagonal blocks (175) are given by

$$\begin{aligned} \mathcal{K}_{(aa')(bb')}^{(AB)} &= \lambda_{a'} \lambda_{b'} \bar{\lambda}_a \bar{\lambda}_b - \lambda_{a'} \bar{\lambda}_a \left(\sum_{\beta} \lambda_{\beta} \bar{\lambda}_{\beta} \right) \cdot \lambda_{b'} \bar{\lambda}_b \left(\sum_{\gamma} \lambda_{\gamma} \bar{\lambda}_{\gamma} \right) \\ &= \lambda_{a'} \lambda_{b'} \bar{\lambda}_a \bar{\lambda}_b - \lambda_{a'} \bar{\lambda}_a \lambda_{b'} \bar{\lambda}_b = 0 \end{aligned} \quad (180)$$

where we have used the normalization condition $\sum_{\beta} |\lambda_{\beta}|^2 = 1$. Thus, being $\mathcal{K}^{(AB)} = 0$ in the separable case, also the entanglement measure \mathcal{E} defined in (72) obviously vanishes as it can be directly checked from Eq. (178).

On the other hand, if ρ_0 is maximally entangled, then the coefficients $c_{\alpha\beta}$ are given by $\delta_{\alpha\beta}/\sqrt{N}$

with $N_{<} = \min(N_A, N_B)$ and the fiducial state can be written as:

$$\rho_0 = \frac{1}{N_{<}} \sum_{\alpha, \alpha'=1}^{N_{<}} \tau_{\alpha\alpha'}^{(A)} \otimes \tau_{\alpha\alpha'}^{(B)}. \quad (181)$$

From the expression (A5) we then see that

$$\mathcal{K}_{\begin{smallmatrix} [aa'] \\ [bb'] \end{smallmatrix}}^{(A)} = \frac{1}{2N_{<}} \left(\delta_{a'b} \sum_{\beta} \delta_{b'\beta} \delta_{a\beta} - \delta_{ab'} \sum_{\beta} \delta_{b\beta} \delta_{a'\beta} \right) = \frac{1}{2N_{<}} \left(\delta_{a'b} \delta_{b'a} - \delta_{ab'} \delta_{ba'} \right) = 0, \quad (182)$$

i.e., as expected, the symplectic part vanishes in the maximally entangled case. Moreover, in this case the entanglement measure (72) is given by

$$\begin{aligned} \text{Tr} \left(\mathcal{K}^{(AB)T} \mathcal{K}^{(AB)} \right) &= \sum_{aa'cc'} \left(\frac{1}{N_{<}} \delta_{a'c'} \delta_{ac} - \frac{1}{N_{<}^2} \sum_{\beta} \delta_{a'\beta} \delta_{a\beta} \sum_{\gamma} \delta_{\gamma c'} \delta_{\gamma c} \right)^2 \\ &= \sum_{aa'cc'} \left(\frac{1}{N_{<}} \delta_{a'c'} \delta_{ac} - \frac{1}{N_{<}^2} \delta_{aa'} \delta_{cc'} \right)^2 \\ &= \sum_{aa'cc'} \left(\frac{1}{N_{<}^2} (\delta_{a'c'} \delta_{ac})^2 + \frac{1}{N_{<}^4} (\delta_{aa'} \delta_{cc'})^2 - \frac{2}{N_{<}^3} \delta_{a'c'} \delta_{ac} \delta_{aa'} \delta_{cc'} \right) \\ &= 1 + \frac{1}{N_{<}^2} - \frac{2}{N_{<}^2} = \frac{N_{<}^2 - 1}{N_{<}^2}, \end{aligned} \quad (183)$$

i.e., restoring the constant factor in Eq. (178)

$$\mathcal{E} = \frac{N_{<}^2}{4(N_{<}^2 - 1)} \text{Tr} \left(\mathcal{K}^{(AB)T} \mathcal{K}^{(AB)} \right) = \frac{1}{4}, \quad (184)$$

and the distance with respect to the separable state (73) takes the following value:

$$\text{Tr} (R^\dagger R) = \frac{1}{N_{<}^4} \text{Tr} \left(\mathcal{K}^{(AB)T} \mathcal{K}^{(AB)} \right) = \frac{N_{<}^2 - 1}{N_{<}^6}. \quad (185)$$

Finally, let us consider the intermediate case of a generic entangled fiducial state, that is

$$c_{\alpha\beta} = f(\alpha) \delta_{\alpha\beta}, \quad (186)$$

with $f(\alpha)$ complex smooth functions satisfying the normalization condition

$$\sum_{\alpha} |f(\alpha)|^2 = 1. \quad (187)$$

The fiducial state (174) can be thus written as

$$\rho_0 = \sum_{\alpha, \alpha'} f(\alpha) \bar{f}(\alpha') \tau_{\alpha\alpha'}^{(A)} \otimes \tau_{\alpha\alpha'}^{(B)}, \quad (188)$$

and we have

$$\begin{aligned}
\text{Tr} \left(\mathcal{K}^{(AB)T} \mathcal{K}^{(AB)} \right) &= \sum_{aa'cc'} \left(f(a') \bar{f}(a) \delta_{a'c'} \delta_{ac} - \sum_{\beta} f(a') \bar{f}(a) \delta_{a'\beta} \delta_{a\beta} \sum_{\gamma} f(c') \bar{f}(c) \delta_{\gamma c'} \delta_{\gamma c} \right)^2 \\
&= \sum_{aa'cc'} \left(f(a') \bar{f}(a) \delta_{a'c'} \delta_{ac} - f(a') \bar{f}(a) \delta_{a'a} f(c') \bar{f}(c) \delta_{cc'} \right)^2 \\
&= \sum_a \bar{f}(a)^2 \sum_{a'} f(a')^2 + \sum_{aa'cc'} f(a')^2 \bar{f}(a)^2 \delta_{aa'} \delta_{aa'} f(c')^2 \bar{f}(c)^2 \delta_{cc'} \delta_{cc'} + \\
&\quad - 2 \sum_{aa'cc'} f(a')^2 \bar{f}(a)^2 f(c') \bar{f}(c) \delta_{aa'} \delta_{a'c'} \delta_{ac} \delta_{aa'} \delta_{cc'} \\
&= \sum_a \bar{f}(a)^2 \sum_{a'} f(a')^2 - \sum_{ac} |f(a)|^4 |f(c)|^2 \delta_{ac} \\
&= \sum_a \bar{f}(a)^2 \sum_{a'} f(a')^2 - \sum_a |f(a)|^6 .
\end{aligned} \tag{189}$$

Therefore, if the functions f are real¹¹, Eq. (189) reduces to:

$$\text{Tr} \left(\mathcal{K}^{(AB)T} \mathcal{K}^{(AB)} \right) = 1 - \sum_a |f(a)|^6 . \tag{190}$$

In particular, we see that when $f(a) = 1/\sqrt{N_{<}}$, $\forall a = 1, \dots, N_{<}$, we recover the result (183) for the maximally entangled case.

To sum up, we collect the above results in the following table:

$\rho_{\mathbf{0}}$	$\mathbf{c}_{\alpha\beta}$	$\text{Tr} \left(\mathcal{K}^{(\mathbf{AB})T} \mathcal{K}^{(\mathbf{AB})} \right)$
separable	$\lambda_{\alpha} \lambda_{\beta}$	0
maximally entangled	$\frac{\delta_{\alpha\beta}}{\sqrt{N_{<}}}$	$1 - \frac{1}{N_{<}^2}$
entangled	$f(\alpha) \delta_{\alpha\beta}, f(\alpha) \in \mathbb{C}$	$\sum_{\alpha} \bar{f}(\alpha)^2 \sum_{\alpha'} f(\alpha')^2 - \sum_{\alpha} f(\alpha) ^6$
	$f(\alpha) \delta_{\alpha\beta}, f(\alpha) \in \mathbb{R}$	$1 - \sum_{\alpha} f(\alpha) ^6$

F. Spin- $\frac{1}{2}$ graph and large n correlations

As an explicit example let us consider the case of a spin network graph Γ whose edges are all labeled by spins fixed at the fundamental representation, i.e., $j_e = \frac{1}{2} \forall e \in \partial A \cup \partial B \equiv \partial R$. The specific structure of the starting graph is not relevant for our analysis and the only assumption

¹¹ This is actually the case of a Schmidt decomposition.

we make is that the gauge reduction procedure leads to a single intertwiner graph Γ_R between the $2n$ ($n \in \mathbb{N}$) $SU(2)$ -representations defining the boundary ∂R with no loops carrying curvature excitations. The number of boundary edges must be necessarily even since there does not exist any intertwiner between an odd number of $\frac{1}{2}$ -spin representations.

With such a choice of spin labels, we may unfold the single $2n$ -valent intertwiner into two $(n+1)$ -valent vertices coupling the two boundary sets of n spin- $\frac{1}{2}$ edges with a virtual link labeled by a fixed spin k . As long as $k \leq n$, the dimension of the corresponding degeneracy spaces \mathcal{D}_k^A and \mathcal{D}_k^B can be then expressed in terms of binomial coefficients as [72]:

$$N = N_A^k = N_B^k = d_k^{(n)} = \binom{2n}{n+k} - \binom{2n}{n+k+1} = \frac{2k+1}{n+k+1} \binom{2n}{n+k}. \quad (191)$$

Therefore, with this separation of boundary degrees of freedom, the gauge-reduced unfolded intertwiner state can be actually described as a couple of entangled N -level systems with the same number of levels given by (191).

Let us analyze the large n behaviour of the quantum correlations between the two subsystems as a function of k . We focus on a fiducial state $|0\rangle \equiv |\psi_{\Gamma_R}\rangle$ with all coefficients $c_{\alpha_k \beta_k}^{(k, \{j_e = \frac{1}{2}\})}$ equal to $\frac{1}{\sqrt{N}}$ whose corresponding pure state density matrix is given by

$$\rho_0 = \frac{1}{N} \sum_{\alpha_k, \alpha'_k=1}^N \tau_{\alpha_k \alpha'_k}^{(A)} \otimes \tau_{\alpha_k \alpha'_k}^{(B)}. \quad (192)$$

In the previous section we found that, for such a choice of the fiducial state, the entanglement measure constructed with the off-diagonal blocks of the pulled-back metric tensor on the orbit starting at ρ_0 is given by

$$\text{Tr} \left(\mathcal{K}^{(AB)T} \mathcal{K}^{(AB)} \right) = 1 - \frac{1}{N^2}. \quad (193)$$

For large n , say $n \gg 1$, the degeneracy factors (191) admit the following asymptotic expression [71]

$$N = d_k^{(n)} \sim \frac{2^{2n+1}}{\sqrt{\pi n}} \frac{x}{(1+x)\sqrt{1-x^2}} e^{-n\varphi(x)}, \quad (194)$$

with

$$\varphi(x) = (1+x) \log(1+x) + (1-x) \log(1-x) \quad , \quad x = \frac{k}{n} \in [0, 1]. \quad (195)$$

Hence, when k goes to zero, or equivalently $x \rightarrow 0$, then $\varphi(x) \rightarrow 0$ and $d_{k=0}^{(n)}$ goes to zero linearly with x (i.e., with k). On the other hand, when k becomes comparable with n (that is $x \rightarrow 1$), $\varphi(x) \sim 2 \log 2$ and $d_{k=n}^{(n)}$ goes to infinity. Explicitly, we have:

$$N = d_k^{(n)} \underset{\text{large } n}{\sim} \frac{2}{\sqrt{\pi}} 2^{2n} \sqrt{n} \frac{k}{(n+k)\sqrt{n^2 - k^2}} \longrightarrow \begin{cases} 0 & \text{for } k \rightarrow 0 \\ \infty & \text{for } k \rightarrow n \end{cases}. \quad (196)$$

Thus, the entanglement monotone (193) exhibits the following large n behaviours

$$\mathrm{Tr} \left(\mathcal{K}^{(AB)T} \mathcal{K}^{(AB)} \right) = 1 - \frac{1}{N^2} = 1 - \frac{1}{(d_k^{(n)})^2} \sim \begin{cases} \infty & \text{for } k \rightarrow 0 \\ 1 & \text{for } k \rightarrow n \end{cases} . \quad (197)$$

Finally, the corresponding behaviour of the associated distance with respect to the separable state is

$$\frac{1}{N^4} \mathrm{Tr} \left(\mathcal{K}^{(AB)T} \mathcal{K}^{(AB)} \right) \sim \begin{cases} \infty & \text{for } k \rightarrow 0 \\ 0 & \text{for } k \rightarrow n \end{cases} . \quad (198)$$

VI. DISCUSSION

As stressed in Section 1, spin network states are supposed to be the fundamental degrees of freedom representing the quantum structure of spatial three-dimensional geometry¹². According to [18], the degrees of freedom of a spin network flow through the network itself which can be then considered as a quantum circuit, i.e., the $SU(2)$ -representation vectors living on the edges of the graph evolve along the edges then meet and intertwine at the nodes, the latter playing the role of quantum gates. This is a quite interesting interpretation from a relational point of view. Indeed, as already remarked, in a diffeomorphism invariant context, only relations between objects have physical meaning and the physical content of the theory should be understood in a purely relation way. A quantum space state essentially defines a set of correlations between the (sub-)regions of the spin network and ultimately between regions of space. In the spirit of Penrose's original idea [43, 73], spin networks then come to be networks of quantum correlations (entanglements) between regions of space. The 3-dimensional quantum space can be therefore identified with the set of processes that can occur on it, processes represented by the various quantum channels forming the network, and its properties (e.g., connectivity and geometry) should be reconstructed from the quantum information encoded in the fundamental degrees of freedom. It is therefore crucial to understand and properly characterize the intrinsic informational content of spin networks.

In this work we have tried to address this issue using the tensorial structures available in the geometric formulation of Quantum Mechanics and their characterization of entanglement. Specifically, we implemented a three-steps procedure to build up these structures and we have applied it on a single link state which may be regarded as the most simple circuit consisting of two "one-valent nodes" (semilink states) and a link connecting them. Indeed, even at the level of the basis states we associate a state $|j, m, n\rangle$ with an irreducible representation of the group $SU(2)$ whose Wigner matrix $D_{mn}^{(j)}$ is actually an amplitude $\langle j, m | \hat{h}(A) | j, n \rangle$ from an initial to a final spin state given by the transformed state under the group action. Hence, a Wilson line must be intended as the description of the transformation relating the two (quantum) references represented by the endpoints, i.e., it can be regarded as a process connecting two spin states and generating the minimal element of geometry¹³. Such a relational interpretation is well described in our formalism where, for any fixed spin j , the single link Hilbert space is regarded as a bipartite space $\mathcal{H}^{(j)} \cong \mathcal{V}^{(j)} \otimes \mathcal{V}^{(j)*}$ of

¹² This point of view is shared by most of the background-independent QG approaches, not only LQG. From the spin foam point of view [29], for instance, a spin network can be thought of as a trivial case of a spin foam with no spacetime vertices thus representing a static spacetime set-up.

¹³ Note that the two states are characterized by the same spin j since the action of $SU(2)$ on such states can modify the magnetic moments but not the spin quantum number.

two spin- j irreducible representation Hilbert spaces (the presence of the dual space is due to the opposite ingoing orientation at the second endpoint), and the link itself is actually an entangled state of the two semilink spin states.

Now the point is to understand which informations can be extracted from the tensorial structures constructed on the space of states by means of the geometric formalism developed so far. First of all, let us notice that, being tensors on the state space, they depend on the algebraic and combinatorial data carried by the spin-network thus concerning the pregeometric level of the theory. The latter acquires a geometric character when we define gauge-invariant operators acting on the quantum states which are then interpreted as the building blocks of a *quantum geometry* in the dual simplicial picture. In a background-independent framework, such a “microscopic” quantum geometry is at the root of space(-time) but how its geometry can be reconstructed in general remains to be understood. The inbuilt diffeomorphism invariance of the formalism (e.g. no role for any background manifold to start with), makes things trickier.

Therefore, concerning the single link case, it seems more natural to look at the notion of connectivity rather than of distance. The metric tensor constructed in this work indeed lives on the space of states not on space(-time). More precisely, it is given by the pull-back of the Fubini-Study metric tensor from the ray space to the Hilbert space. From this point of view, the entanglement monotone $\text{Tr}(C^T C)$ involving the off-diagonal blocks of pulled-back tensor can be interpreted as a measure of the existence of the link itself, i.e., as a measure of connectivity in the sense of graph topology. Indeed, as discussed in Sec. 3.8, in general it can be regarded as a distance with respect to the separable case, i.e.:

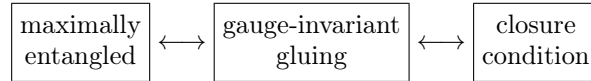
$$\text{Tr}(C^T C) \propto \text{Tr}(R^\dagger R) \quad , \quad R := \rho - \rho_A \otimes \rho_B . \quad (199)$$

Thus, it is zero when ρ is separable and it is maximum when the state is maximally entangled. By focusing now on the case of a single link reinterpreted as an entangled state of two semilinks, we may conclude that

$\text{Tr}(C^T C) = 0$	\leftrightarrow	unentangled semilinks/no link	\leftrightarrow	not connected
$\text{Tr}(C^T C) \neq 0$	\leftrightarrow	link as entangled semilinks	\leftrightarrow	connected

This is even more clear in the case of two links where the entanglement is directly associated to their gluing into a single link. Hence, we have a relation between entanglement and graph connectivity as we expect from the relational point of view discussed before where the network graph itself is regarded as describing the correlations between its nodes and hence there will be a link/a process connecting two points only if they are correlated (correlations which translate into entanglement at the quantum level). Moreover, in the maximally entangled case, which corresponds to the gauge-invariant Wilson loop state, we find that the entanglement measure (132) depends only on the spin number j labelling the link through the eigenvalue $j(j+1)$ of the area operator. Therefore, already at the level of the simplest example of a single link state, the analysis developed so far seems to suggest a connection between entanglement on spin networks states and their geometric properties. For instance, the gluing of two link in a gauge-invariant way resulting into a (locally) maximally entangled state ensures the matching of their dual surfaces into a unique surface dual to the single link resulting from the gluing. Pushing this interpretation further we may argue that the *connectivity* and *geometricity* of the fundamental degrees of freedom might emerge from their entanglement properties. In order to check this statement we need to extend our analysis to more complicated spin networks. Next step may be for instance to consider the gluing of 4 Wilson lines into a 4-valent node. Only in the maximally entangled case in which we associate the resulting node

with an invariant tensor (intertwiner) corresponds to a well-defined geometric figure (tetrahedron), i.e.:



Moreover, since there are more degrees of freedom involved, the analysis may reveal some new connection with other geometric observables (e.g., volume) which are trivial in the single link case. We stress again that however we are still at a pregeometric level. By this we mean that the advocated entanglement-geometry connection concerns abstract non-embedded objects and so the resulting geometric features are not yet those of a continuum spatial geometry, like we are accustomed to use in effective field theory and in GR. Some ideas concerning this and further possibilities will be outlined in the section of conclusions.

VII. CONCLUSIONS AND OUTLOOK

Motivated by the idea that, in the background independent framework of a quantum theory of gravity, entanglement is expected to play a key role in the reconstruction of spacetime geometry, this dissertation is a preliminary investigation towards the possibility of using the formalism of Geometric Quantum Mechanics (GQM) to give a tensorial characterization of entanglement on spin network states. Our analysis focuses on the simple case of a single link graph state for which we define a dictionary to construct a Riemannian metric tensor and a symplectic structure on the space of states. The manifold of (pure) quantum states is then stratified in terms of orbits of equally entangled states and the block-coefficient matrices of the corresponding pulled-back tensors fully encode the information about separability and entanglement. In particular, the off-diagonal blocks C define an entanglement monotone $\text{Tr}(C^T C)$ interpreted as a distance with respect to the separable state.

The main achievements of our construction are:

1. A formalism which fits well to a purely relational interpretation of the link as an elementary process describing the quantum correlations between its endpoints.
2. A quantitative characterization of graph connectivity by means of the entanglement monotone $\text{Tr}(C^T C)$ which comes to be a measure of the existence of the process/link.
3. A connection between the GQM formalism and the (simplicial) geometric properties of the quantum states through entanglement. In the maximally entangled case, which for the single link corresponds to a gauge-invariant loop, the entanglement monotone is actually proportional to a power of the corresponding expectation value of the area operator.
4. Assuming bulk flatness, i.e. the absence of curvature degrees of freedom, a generic region of a spin network can be thought as an intertwiner between the N links puncturing the dual surface. From the point of view of the surface, a state of geometry of that region is then described by a superposition of the possible N -valent intertwiners [18]. It has been proposed in [19] that a notion of distance between two regions of space should be derived in terms of the entanglement between the two regions A and B of the underlying spin network induced by the rest of the network. By regarding the whole system as a bipartite system in which each

subsystem is a n -level system, where the number of levels is determined by the degeneracy of the new intertwiner space resulting from the coarse-graining and as such it will depend also on the spin labelling the virtual link, we have studied in some detail such quantum correlations using our GQM formalism. In particular, we have obtained several analytic results for the entanglement monotone $\text{Tr}(C^T C)$, which support its interpretation as a measure of spatial connectivity.

These results may be intended as the starting point of a long-term program whose final goal should be to understand in full generality how the tensorial structures defined on the space of spin network states can be used to characterize their geometric features and, in particular, how they can provide new insights to reconstruct the (quantum) geometry of spacetime. We have seen that even at the simplest level of a single link we can already grasp some connections between entanglement and geometry. Obviously, being this case so simple, it does not enable us to explore the full spectrum of possibilities and, as already mentioned, we need to extend our construction to more general cases. Let us then close by sketching some future perspective:

- The general setup we have used to compute entanglement properties of the bipartite system associated to two regions of a spin network state should now be extended, and the calculations generalised. This can be done in several directions, corresponding to the progressive removal of the various approximations we had imposed on our system, in this work. One is the introduction of curvature degrees of freedom, or of a proper coarse graining procedure to deal with them. Another is the inclusion of the sum over spin degrees of freedom in the calculations, i.e. allowing for a superposition of spin network states while keeping the underlying combinatorial structure fixed. More ambitiously, we need to learn to control the entanglement properties of superpositions over graph structures. For the latter goal, the GFT formalism may be the one with the greatest potential. Only such progressive generalization of our results can truly give strong support for the conjectured entanglement-connectivity relation and, in perspective, for the entanglement-geometry one. In order to generalise our entanglement calculation to superpositions of spin network graphs, one possibility is to adopt the tensor model techniques already used for the calculation of the entanglement entropy of horizon states built out of spin networks in [23].
- We may focus on coherent states and exploit their interpretation as semiclassical states to study the classical limit of the metric tensor. Let us also notice that in this case we are selecting a particular family of states with their own parameter space. Moreover, since the parameter space of coherent states is isomorphic to the classical phase space of discrete geometries associated to spin network states, establishing a correspondence between the (entanglement) properties measured in terms of information geometry and the same simplicial geometry should be rather direct. This should also enable us to exploit the connection between the Fubini-Study and the Fisher-Rao metrics and the related tools of information geometry (see for example [74] where the quantum metric is derived from relative entropy).
- The analysis of entanglement with classical tensors has been extended also to the case of mixed states [75]. The case of Gibbs states, in which the expectation values of (geometric) observables such as area play the role of the parameters of the exponential family of maximally mixed states, would be interesting to study black holes. In particular, once we generalise our construction to include curvature degrees of freedom in the bulk of our spatial region, any subsequent coarse-graining of the same curvature degrees of freedom will result in mixed states. In other words, one could associate the mixed nature of the quantum states being considered to the curvature degrees of freedom having been traced out.

- A further interesting aspect concerns the very interpretation of these tensors in those cases where the space of states is a tensor product of boundary states spaces of a process. The case of the single link, where the Hermitian tensor can be associated with an amplitude from an initial to a final spin state, may be generalized to a full (spin foam) path integral amplitude, meant as a process generating a region of space-time. In this case, the Fubini-Study metric would provide a metric for the space-time region. This setting has interesting formal analogies with the general boundary formalism [76, 77].

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Appendix A: Computation in the Standard Basis

In this appendix we report the explicit computations of the block-coefficient matrices (173) of the Hermitian quantum Fisher tensor on the orbit generated from the fiducial state

$$\rho_0 = \sum_{\alpha\alpha'\beta\beta'} c_{\alpha\beta} \bar{c}_{\alpha'\beta'} \tau_{\alpha\alpha'}^{(A)} \otimes \tau_{\beta\beta'}^{(B)}. \quad (\text{A1})$$

Let us start with the off-diagonal blocks:

$$\begin{aligned} \mathcal{K}_{(aa')(bb')}^{(AB)} &= \sum_{\alpha\alpha'\beta\beta'} c_{\alpha\beta} \bar{c}_{\alpha'\beta'} \text{Tr} \left(\tau_{\alpha\alpha'}^{(A)} \tau_{aa'}^{(A)} \otimes \tau_{\beta\beta'}^{(B)} \tau_{bb'}^{(B)} \right) + \\ &\quad - \sum_{\alpha\alpha'\beta\beta'} c_{\alpha\beta} \bar{c}_{\alpha'\beta'} \text{Tr} \left(\tau_{\alpha\alpha'}^{(A)} \tau_{aa'}^{(A)} \otimes \tau_{\beta\beta'}^{(B)} \right) \cdot \sum_{\gamma\gamma'\delta\delta'} c_{\gamma\delta} \bar{c}_{\gamma'\delta'} \text{Tr} \left(\tau_{\gamma\gamma'}^{(A)} \otimes \tau_{\delta\delta'}^{(B)} \tau_{bb'}^{(B)} \right) \\ &= \sum_{\alpha\alpha'\beta\beta'} c_{\alpha\beta} \bar{c}_{\alpha'\beta'} \text{Tr}_A \left(\tau_{\alpha\alpha'}^{(A)} \tau_{aa'}^{(A)} \right) \text{Tr}_B \left(\tau_{\beta\beta'}^{(B)} \tau_{bb'}^{(B)} \right) + \\ &\quad - \sum_{\alpha\alpha'\beta\beta'} c_{\alpha\beta} \bar{c}_{\alpha'\beta'} \text{Tr}_A \left(\tau_{\alpha\alpha'}^{(A)} \tau_{aa'}^{(A)} \right) \text{Tr}_B \left(\tau_{\beta\beta'}^{(B)} \right) \cdot \sum_{\gamma\gamma'\delta\delta'} c_{\gamma\delta} \bar{c}_{\gamma'\delta'} \text{Tr}_A \left(\tau_{\gamma\gamma'}^{(A)} \right) \text{Tr}_B \left(\tau_{\delta\delta'}^{(B)} \tau_{bb'}^{(B)} \right) \\ &= \sum_{\alpha\alpha'\beta\beta'} c_{\alpha\beta} \bar{c}_{\alpha'\beta'} \delta_{\alpha\alpha'} \delta_{\alpha'a} \delta_{\beta\beta'} \delta_{\beta'b} - \sum_{\alpha\alpha'\beta\beta'} c_{\alpha\beta} \bar{c}_{\alpha'\beta'} \delta_{\alpha\alpha'} \delta_{\alpha'a} \delta_{\beta\beta'} \cdot \sum_{\gamma\gamma'\delta\delta'} c_{\gamma\delta} \bar{c}_{\gamma'\delta'} \delta_{\gamma\gamma'} \delta_{\delta\delta'} \delta_{\delta'b} \\ &= c_{a'b'} \bar{c}_{ab} - \sum_{\beta} c_{a'\beta} \bar{c}_{a\beta} \cdot \sum_{\gamma} c_{\gamma b'} \bar{c}_{\gamma b} \end{aligned} \quad (\text{A2})$$

where in the third equality we used the relations $\text{Tr}_A(\tau_{\alpha\alpha'}^{(A)} \tau_{aa'}^{(A)}) = \delta_{\alpha\alpha'} \delta_{\alpha'a}$ and $\text{Tr}_A(\tau_{\gamma\gamma'}^{(A)}) = \delta_{\gamma\gamma'}$ (and similarly for the $\tau^{(B)}$'s) which can be easily deduced from Eq. (171). A similar result holds for the block $\mathcal{K}^{(BA)}$.

As regards the other blocks, by using the commutation and anti-commutation relations

$$[\tau_{aa'}^{(X)}, \tau_{bb'}^{(X)}]_{\pm} = \delta_{a'b} \tau_{ab'}^{(X)} \pm \delta_{ab'} \tau_{a'b}^{(X)} \quad , \quad X = A, B \quad (\text{A3})$$

we get

$$\begin{aligned} \mathcal{K}_{\substack{(aa') \\ (bb')}}^{(A)} &= \frac{1}{2} \sum_{\alpha\alpha'\beta\beta'} c_{\alpha\beta} \bar{c}_{\alpha'\beta'} \text{Tr}_A \left(\tau_{\alpha\alpha'}^{(A)} [\tau_{aa'}^{(A)}, \tau_{bb'}^{(A)}]_{+} \right) \text{Tr}_B \left(\tau_{\beta\beta'}^{(B)} \right) + \\ &\quad - \sum_{\alpha\alpha'\beta\beta'} c_{\alpha\beta} \bar{c}_{\alpha'\beta'} \text{Tr}_A \left(\tau_{\alpha\alpha'}^{(A)} \tau_{aa'}^{(A)} \right) \text{Tr}_B \left(\tau_{\beta\beta'}^{(B)} \right) \cdot \\ &\quad \cdot \sum_{\gamma\gamma'\delta\delta'} c_{\gamma\delta} \bar{c}_{\gamma'\delta'} \text{Tr}_A \left(\tau_{\gamma\gamma'}^{(A)} \tau_{bb'}^{(A)} \right) \text{Tr}_B \left(\tau_{\delta\delta'}^{(B)} \right) \\ &= \frac{1}{2} \sum_{\alpha\alpha'\beta} c_{\alpha\beta} \bar{c}_{\alpha'\beta} \text{Tr}_A \left(\tau_{\alpha\alpha'}^{(A)} [\tau_{aa'}^{(A)}, \tau_{bb'}^{(A)}]_{+} \right) - \sum_{\beta} c_{a'\beta} \bar{c}_{a\beta} \cdot \sum_{\delta} c_{b'\delta} \bar{c}_{b\delta} \\ &= \frac{1}{2} \sum_{\alpha\alpha'\beta} c_{\alpha\beta} \bar{c}_{\alpha'\beta} \left[\delta_{a'b} \text{Tr}_A \left(\tau_{\alpha\alpha'}^{(A)} \tau_{ab'}^{(A)} \right) + \delta_{ab'} \text{Tr}_A \left(\tau_{\alpha\alpha'}^{(A)} \tau_{a'b}^{(A)} \right) \right] + \\ &\quad - \sum_{\beta} c_{a'\beta} \bar{c}_{a\beta} \cdot \sum_{\delta} c_{b'\delta} \bar{c}_{b\delta} \\ &= \frac{1}{2} \sum_{\alpha\alpha'\beta} c_{\alpha\beta} \bar{c}_{\alpha'\beta} (\delta_{a'b} \delta_{\alpha b'} \delta_{\alpha' a} + \delta_{ab'} \delta_{\alpha b} \delta_{\alpha' a'}) - \sum_{\beta} c_{a'\beta} \bar{c}_{a\beta} \cdot \sum_{\delta} c_{b'\delta} \bar{c}_{b\delta} \\ &= \frac{1}{2} \left(\delta_{a'b} \sum_{\beta} c_{b'\beta} \bar{c}_{a\beta} + \delta_{ab'} \sum_{\beta} c_{b\beta} \bar{c}_{a'\beta} \right) - \sum_{\beta} c_{a'\beta} \bar{c}_{a\beta} \cdot \sum_{\delta} c_{b'\delta} \bar{c}_{b\delta} \quad , \end{aligned} \quad (\text{A4})$$

and

$$\begin{aligned} \mathcal{K}_{\substack{[aa'] \\ [bb']}}^{(A)} &= \frac{1}{2} \sum_{\alpha\alpha'\beta\beta'} c_{\alpha\beta} \bar{c}_{\alpha'\beta'} \text{Tr}_A \left(\tau_{\alpha\alpha'}^{(A)} [\tau_{aa'}^{(A)}, \tau_{bb'}^{(A)}]_{-} \right) \text{Tr}_B \left(\tau_{\beta\beta'}^{(B)} \right) \\ &= \frac{1}{2} \sum_{\alpha\alpha'\beta} c_{\alpha\beta} \bar{c}_{\alpha'\beta} \text{Tr}_A \left(\tau_{\alpha\alpha'}^{(A)} [\tau_{aa'}^{(A)}, \tau_{bb'}^{(A)}]_{-} \right) \\ &= \frac{1}{2} \sum_{\alpha\alpha'\beta} c_{\alpha\beta} \bar{c}_{\alpha'\beta} \left[\delta_{a'b} \text{Tr}_A \left(\tau_{\alpha\alpha'}^{(A)} \tau_{ab'}^{(A)} \right) - \delta_{ab'} \text{Tr}_A \left(\tau_{\alpha\alpha'}^{(A)} \tau_{a'b}^{(A)} \right) \right] \\ &= \frac{1}{2} \sum_{\alpha\alpha'\beta} c_{\alpha\beta} \bar{c}_{\alpha'\beta} (\delta_{a'b} \delta_{\alpha b'} \delta_{\alpha' a} - \delta_{ab'} \delta_{\alpha b} \delta_{\alpha' a'}) \\ &= \frac{1}{2} \left(\delta_{a'b} \sum_{\beta} c_{b'\beta} \bar{c}_{a\beta} - \delta_{ab'} \sum_{\beta} c_{b\beta} \bar{c}_{a'\beta} \right) \quad . \end{aligned} \quad (\text{A5})$$

Similar results hold for the $\mathcal{K}^{(B)}$ blocks of the symmetric and antisymmetric part, respectively. Finally, let us consider the entanglement measure (72). First of all, let us compute the product

$\mathcal{K}^{(AB)}\mathcal{K}^{(AB)T}$ whose matrix elements, according to Eq. (A2), are given by

$$\begin{aligned}
\left(\mathcal{K}^{(AB)}\mathcal{K}^{(AB)T}\right)_{\substack{(aa') \\ (bb')}} &= \sum_{cc'} \mathcal{K}_{\substack{(aa') \\ (cc')}}^{(AB)} \mathcal{K}_{\substack{(cc') \\ (bb')}}^{(AB)T} = \sum_{cc'} \mathcal{K}_{\substack{(aa') \\ (cc')}}^{(AB)} \mathcal{K}_{\substack{(bb') \\ (cc')}}^{(AB)} \\
&= \sum_{cc'} \left[\left(c_{a'c'} \bar{c}_{ac} - \sum_{\beta} c_{a'\beta} \bar{c}_{a\beta} \cdot \sum_{\gamma} c_{\gamma c'} \bar{c}_{\gamma c} \right) \cdot \right. \\
&\quad \left. \cdot \left(c_{b'c'} \bar{c}_{bc} - \sum_{\delta} c_{b'\delta} \bar{c}_{b\delta} \cdot \sum_{\zeta} c_{\zeta c'} \bar{c}_{\zeta c} \right) \right] \\
&= \sum_{cc'} \left(c_{a'c'} \bar{c}_{ac} c_{b'c'} \bar{c}_{bc} + \sum_{\beta} c_{a'\beta} \bar{c}_{a\beta} \sum_{\gamma} c_{\gamma c'} \bar{c}_{\gamma c} \sum_{\delta} c_{b'\delta} \bar{c}_{b\delta} \cdot \right. \\
&\quad \cdot \sum_{\zeta} c_{\zeta c'} \bar{c}_{\zeta c} - c_{b'c'} \bar{c}_{bc} \sum_{\beta} c_{a'\beta} \bar{c}_{a\beta} \sum_{\gamma} c_{\gamma c'} \bar{c}_{\gamma c} + \\
&\quad \left. - c_{a'c'} \bar{c}_{ac} \sum_{\delta} c_{b'\delta} \bar{c}_{b\delta} \sum_{\zeta} c_{\zeta c'} \bar{c}_{\zeta c} \right). \tag{A6}
\end{aligned}$$

Now taking the trace of Eq.(A6) amounts to set $a = b, a' = b'$ and to sum over a, a' . Therefore, omitting for the moment the constant factor in front of the trace in Eq. (72), we have:

$$\begin{aligned}
\mathcal{E} &= \sum_{aa'cc'} \left[c_{a'c'}^2 \bar{c}_{ac}^2 + \left(\sum_{\gamma} c_{\gamma c'} \bar{c}_{\gamma c} \right)^2 \left(\sum_{\delta} c_{a'\delta} \bar{c}_{a\delta} \right)^2 - 2c_{a'c'} \bar{c}_{ac} \sum_{\beta} c_{a'\beta} \bar{c}_{a\beta} \sum_{\gamma} c_{\gamma c'} \bar{c}_{\gamma c} \right] \\
&= \sum_{aa'cc'} \left(c_{a'c'} \bar{c}_{ac} - \sum_{\beta} c_{a'\beta} \bar{c}_{a\beta} \sum_{\gamma} c_{\gamma c'} \bar{c}_{\gamma c} \right)^2. \tag{A7}
\end{aligned}$$

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