Supporting information for "Autocovariance estimation in regression with a discontinuous signal and m-dependent errors: A difference-based approach"

Appendix A. Proofs and auxiliary results for Section 3

Throughout this supplementary materials we use the following notation: $n_m := n-2(m+1)$, $n_h := n-h$; f_i denotes $f(x_i)$, $v_{i:(i+h)}$ denotes the vector $(v_i\, v_{i+1}\, \cdots\, v_{i+h})^\top \in \mathbb{R}^{h+1}$ (we use this notation with $v \in \{y, f, \varepsilon\}$), $\langle \boldsymbol{a}, \boldsymbol{b}\rangle$ denotes the inner product between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\mathbf{1}$ denotes the vector of ones, and for $x \in \mathbb{R}^d$, $\|x\|$ denotes its Euclidean norm.

A.1. **Proofs for Section 3.1.**

We begin with some preliminary results. For $l < n$, let $\mathbf{Q}_1(Y, \boldsymbol{w}_l)$ be a difference-based estimator of order l and gap 1, cf. Eq. (2.2). With $\widetilde{D}$ as defined in (3.1) define the $(l+2) \times (l+2)$ matrix $D := \widetilde{D}^\top\, \widetilde{D}$. Observe that the identity

$$\sum_{i=j}^{j+1} \left(d_0 y_i + d_1 y_{i+1} + \cdots + d_l\, y_{i+l}\right)^2 = y_{j:(j+l+1)}^\top\, D\, y_{j:(j+l+1)}, \quad j \leq n-2l-3,$$

implies that

$$2n_l\, P(\boldsymbol{w}_l)\, \mathbf{Q}_1(Y, \boldsymbol{w}_l) = \langle \boldsymbol{w}_l, y_{1:(1+l)}\rangle^2 + \sum_{j=1}^{n_l} y_{j:(j+l+1)}^\top\, D\, y_{j:(j+l+1)} + \langle \boldsymbol{w}_l, y_{(n-l):n}\rangle^2, \tag{A.1}$$

where $P(\boldsymbol{w}_l) = \sum_{i=0}^{l} d_i^2$. In Eq. (A.1) it is not difficult to see that $\mathsf{E}[\langle \boldsymbol{w}_l, y_{k:(k+l)}\rangle^2] = o(n)$ for $k = 1, n-l$ and that for $j < n-l$, $\mathsf{E}[y_{j:(j+l+1)}^\top\, D\, y_{j:(j+l+1)}] = \|\widetilde{D} f_{j:(j+l+1)}\|^2 + \mathsf{E}[\varepsilon_{j:(j+h+1)}^\top\, D\, \varepsilon_{j:(j+h+1)}]$. Next, we combine this with Proposition A.1 and get that,

$$\mathsf{E}\left[P(\boldsymbol{w}_l)\mathbf{Q}_1(Y, \boldsymbol{w}_l)\right] = \boldsymbol{w}_l^\top\, \Sigma_{l+1}\, \boldsymbol{w}_l + \frac{1}{2n_l} \sum_{j=1}^{n_l} \|\widetilde{D} f_{j:(j+l+1)}\|^2 + o(1), \tag{A.2}$$

where Σ_{l+1} is the $(l+1) \times (l+1)$ autocovariance matrix $\Sigma_{l+1} = \left(\gamma_{|i-j|}\right)_{i,j=1,\ldots,l}$.

Proof of Theorem 5. It suffices to consider the difference-based estimator of order $l < n$ and gap 1, $\mathbf{Q}_1(Y, \boldsymbol{w}_l)$. For $l \leq m$, due to Lemma 1, (A.2) becomes

$$\mathsf{E}\left[P(\boldsymbol{w}_l)\mathbf{Q}_1(Y, \boldsymbol{w}_l)\right] = B_l + \mathcal{O}\left(n^{-1}\, J_K \sum_{k=1}^{l} \left(\sum_{j=k}^{l} d_j\right)^2\right), \tag{A.3}$$

where

$$B_l = \boldsymbol{w}_l^\top\, \Sigma_{l+1}\, \boldsymbol{w}_l = \gamma_0(d_0^2 + \cdots + d_l^2) + 2\gamma_1(d_0 d_1 + \cdots + d_{l-1} d_l) + \cdots + 2\gamma_l d_0 d_l. \tag{A.4}$$

From now on in this proof we assume that $J_K = o(n)$ and disregard the second summand in the right-hand side of (A.3). Since constraint $\langle \boldsymbol{w}_l, \mathbf{1}\rangle = 0$ implies that at least one of the weights d_j is nonzero, for simplicity from now on we assume that $d_0 \neq 0$. According to Lemma A.1 in this case there does not exist any constant c such that $B_l = c\,\gamma_0$, that is, no difference-based estimate of the form $\mathbf{Q}_1(Y, \boldsymbol{w}_l)$ can be an asymptotically unbiased estimate for γ_0 for $l \leq m$.

Next, suppose that $m < l \leq n$. In what follows, for simplicity, let us assume that $n = N(g+1)$ for some integer $N \geq 1$ and $g = m$. Observe that in this case Proposition A.1 and (A.3) yield, $\mathsf{E}\left[p(\boldsymbol{w}_l)\mathbf{Q}_1(Y, \boldsymbol{w}_l)\right] = B_l + o(1)$. Due to m-dependency, the covariance matrix Σ_{l+1} appearing in B_l is an $(m+1)$-banded Toeplitz matrix, i.e., the (i,j) entry of Σ_{l+1} is given by $\gamma_{|j-i|} \neq 0$ if $|j-i| \leq m$, and outside the $(m+1)$-diagonal the entries of Σ_{l+1} are equal to 0. Suppose that $\boldsymbol{w}_l$ has entries $d_0 = 1$ and $d_j \neq 0$, for $j = k(g+1)$ with $k = 1, \ldots, N-1$, and $d_j = 0$ otherwise. Clearly, $B_l = P(\boldsymbol{w}_l)\,\gamma_0$. Now we show that any asymptotically unbiased estimate for γ_0 is necessarily of the form just described. Indeed, it suffices to consider the vector $\boldsymbol{w}_l^*$, whose entries are identical to those of $\boldsymbol{w}_l$ except for $d_\kappa \neq 0$ for some $\kappa \in \{1, \ldots, m\}$. In this case, due to the form of the covariance matrix Σ_{l+1}, $(\boldsymbol{w}_l^*)^\top\, \Sigma_{l+1}\, \boldsymbol{w}_l^* = c_1\gamma_0 + 2\,d_\kappa\,\gamma_\kappa$, for some constant c_1. Since $\gamma_\kappa \neq 0$, no difference-based estimate of the form $\mathbf{Q}_1(Y, \boldsymbol{w}_l^*)$ can be an asymptotically unbiased estimate for γ_0.

Note that the arguments above hold also for $g \geq m$. Thus, we have shown that for $\mathbf{Q}_1(Y, \boldsymbol{w}_l)$ to be an (asymptotically unbiased) estimate for γ_0, the vector of weights $\boldsymbol{w}_l$ must have the form $\boldsymbol{w}_l = (\boldsymbol{v}_0 \quad \boldsymbol{v}_1 \quad \cdots \quad \boldsymbol{v}_{N-1})^\top$, where $\boldsymbol{v}_i = (d_{i\cdot g} \quad 0 \quad \cdots \quad 0)^\top \in \mathbb{R}^{g+1}$, $i = 0, \ldots, N-1$; $d_0 \neq 0$, $d_{k\cdot g} \neq 0$ for some $1 \leq k \leq N-1$, and $\sum_{i=0}^{N-1} d_{i\cdot g} = 0$. This completes the proof. $\square$

Lemma A.1. *Suppose that the conditions of Theorem 5 hold. Let $\mathbf{Q}_1(Y, \boldsymbol{w}_l)$ be the difference-based estimator of order $l < n$ and gap 1, cf.* (2.2). *Then for $l \leq m$, there does not exist any constant $c \neq 0$ such that $B_l = \boldsymbol{w}_l^\top\, \Sigma_{l+1}\, \boldsymbol{w}_l = c\,\gamma_0$, where Σ_{l+1} is defined by* (A.2).

Proof. We use induction over l. For $l = 1$, $B_1 = \gamma_0(d_0^2 + d_1^2) + 2\gamma_1 d_0 d_1$, cf. (A.4). Since by assumption $\gamma_1 \neq 0$ (and $d_0 \neq 0$), d_1 is necessarily equal to zero but this implies that $\gamma_0 = 0$ to fulfill (2.3). This contradiction shows that the claim holds for $h = 1$. Then we assume that the claim holds for $l = k \leq m-1$. Let $\tilde{k} = k+1$ and now note that for $B_{\tilde{k}} = c\,\gamma_0$ to hold for some $c \neq 0$, necessarily $\gamma_{\tilde{k}} d_0 d_{\tilde{k}} = 0$. Since $\gamma_{\tilde{k}} \neq 0$ and $d_0 \neq 0$, necessarily $d_{\tilde{k}} = 0$. The latter shows that $B_{\tilde{k}} = B_k$. Hence, the claim holds for $l = k+1$ and this completes the proof. $\square$

Proof of Lemma 1. For $j < n-l$ define $s_{j,l} = \langle \boldsymbol{w}_l, f_{j:(j+l)}\rangle^2$ and note that $\|\widetilde{D} f_{j:(j+l+1)}^\top\|^2 = s_{j,l} + s_{j+1,l}$, see (3.1). In what follows we only consider $s_{j,l}$ since $s_{j+1,l}$ can be handled similarly.

Under the convention $t_k = \lfloor n\tau_k \rfloor$, $f(x_i) = a_k$ if and only if $t_{k-1} \le i < t_k$. Then

$$\sum_{j=1}^{n_l} s_{j,l} \le \sum_{i=0}^{K-1} \left[\sum_{j=t_i}^{t_{i+1}-l-1} s_{j,l} + \sum_{j=t_{i+1}-l}^{t_{i+1}-1} s_{j,l} \right]. \tag{A.5}$$

Then note that for any $i \in \{0, \dots, K-1\}$,

$$\sum_{j=t_i}^{t_{i+1}-l-1} s_{j,l} = a_i^2 \langle \boldsymbol{w}_l, \mathbf{1} \rangle^2 (t_{i+1} - t_i - l - 1)$$

and utilizing that $d_0 + \cdots + d_l = 0$,

$$\sum_{j=t_{i+1}-l}^{t_{i+1}-1} s_{j,l} = \sum_{j=t_{i+1}-l}^{t_{i+1}-1} (d_0 f_j + \cdots + d_l f_{j+l})^2 = (a_i - a_{i+1})^2 \sum_{k=1}^{l} \left(\sum_{j=k}^{l} d_j \right)^2.$$

Substituting these expressions into (A.5) we obtain the right-hand side of (3.2). This completes the proof. $\square$

Proposition A.1. *For $l < n$, let $\boldsymbol{w}_l \in \mathbb{R}^{l+1}$ be the vector of weights in $\mathbf{Q}_1(Y, \boldsymbol{w}_l)$, cf. (2.2). Let $D = \widetilde{D}^\top \widetilde{D}$ where $\widetilde{D}$ is defined by (3.1). Then*

$$\mathsf{E}[\varepsilon_{j:(j+l+1)}^\top D \, \varepsilon_{j:(j+l+1)}] = 2 \, \boldsymbol{w}_l^\top \Sigma_{l+1} \boldsymbol{w}_l.$$

Proof. Let Σ_{l+1} be the matrix defined in (A.2) and note that

$$\mathsf{E}[\varepsilon_{j:(j+l+1)}^\top D \, \varepsilon_{j:(j+l+1)}] = \mathrm{tr}\{D \Sigma_{l+2}\} = \mathrm{tr}\{\widetilde{D} \, \Sigma_{l+2} \, \widetilde{D}^\top\}.$$

Then, observe that the 2×2 matrix $\widetilde{D} \, \Sigma_{l+2} \, \widetilde{D}^\top$ can be written as:

$$\widetilde{D} \, \Sigma_{l+2} \, \widetilde{D}^\top = \begin{pmatrix} \boldsymbol{w}_l^\top \Sigma_{l+1} & \langle \boldsymbol{w}_l, \gamma_{(l+1):1} \rangle \\ \langle \boldsymbol{w}_l, \gamma_{(l+1):1} \rangle & \boldsymbol{w}_l^\top \Sigma_{l+1} \end{pmatrix} \begin{pmatrix} \boldsymbol{w}_l & 0 \\ 0 & \boldsymbol{w}_l \end{pmatrix},$$

where $\gamma_{(l+1):1} := (\gamma_{l+1} \quad \gamma_l \quad \cdots \quad \gamma_2 \quad \gamma_1)^\top \in \mathbb{R}^{l+1}$. A straightforward calculation yields, $\mathrm{tr}\{\widetilde{D} \, \Sigma_{l+2} \, \widetilde{D}^\top\} = 2 \boldsymbol{w}_l^\top \Sigma_{l+1} \boldsymbol{w}_l$. $\square$

A.2. **Proofs for Section 3.2.**

We will assume that the conditions of Theorem 1 hold. Also we will use the following notation: $\chi_h(f_i) := f_i - f_{i+h}$, for $d \in \mathbb{R}$, $\delta_i(d) := s_0(i) + d \, s_{2(m+1)}(i)$ where for $k \ge 0$, $s_k(i) := f_{i+k} - f_{i+m+1}$, $\eta_i(d) := \varepsilon_i - \varepsilon_{i+m+1} + d(\varepsilon_{i+2(m+1)} - \varepsilon_{i+m+1})$, set $P(d) = 2(d^2 + d + 1)$ and for given $\tau_i, \tau_j \in [0, 1]$, $1 \le L, R \le n$,

$$I_{\tau_i, \tau_j}^{L,R} := [\tau_i - L/n, \tau_j - R/n). \tag{A.6}$$

Also, $I_{\tau_i}^{L,R} := I_{\tau_i, \tau_i}^{L,R}$. Under the convention $t_j = \lfloor n\tau_j \rfloor$, we will denote $I_{t_i, t_j}^{L,R} := [t_i - L, t_j - R)$ and use that $i/n \in I_{\tau_i, \tau_j}^{L,R}$ if and only if $i \in I_{t_i, t_j}^{L,R}$ without further mention.

The following identities are of great use in what follows: for any integers r, s, u and v,

$$\mathsf{E}[\varepsilon_r^2\,\varepsilon_s^2] = \gamma_0^2 + 2\,\gamma_{|r-s|}^2 \tag{A.7}$$

$$\mathsf{E}[\varepsilon_r^2\,\varepsilon_u\,\varepsilon_v] = \gamma_0\gamma_{|u-v|} + 2\gamma_{|r-u|}\gamma_{|r-v|} \tag{A.8}$$

$$\mathsf{E}[\varepsilon_r\,\varepsilon_s\,\varepsilon_u\,\varepsilon_v] = \gamma_{|r-s|}\gamma_{|u-v|} + \gamma_{|r-u|}\gamma_{|s-v|} + \gamma_{|r-v|}\gamma_{|s-u|}, \tag{A.9}$$

cf. Theorem 3.1 of Triantafyllopoulos (2003).

Lemma A.2. *Let $i \geq 1$, $l \geq 1$, $0 \leq h \leq (m+1)$ and define $E_{i,l,h} = d_0\,\varepsilon_i + d_1\varepsilon_{i+h} + d_2\varepsilon_{i+2h} + \cdots + d_l\varepsilon_{i+lh}$. Then,*

$$\mathsf{E}[E_{i,l,h}^2] = \gamma_0(d_0^2 + d_1^2 + d_2^2 + \cdots + d_l^2) + 2\sum_{j=0}^{l-1}\sum_{k=j+1}^{l} d_j\,d_k\,\gamma_{|j-k|\,h}$$

Proof. Write $E_{i,l,h}^2 = A_{i,l,h} + 2B_{i,l,h}$, where

$$A_{i,l,h} = \sum_{j=0}^{l} d_j^2\varepsilon_{i+jh}^2, \quad B_{i,l,h} = \sum_{j=0}^{l-1} x_j(i,l,h), \quad x_j(i,l,h) = d_j\varepsilon_{i+jh}\sum_{k=j+1}^{l} d_k\,\varepsilon_{i+kh}. \tag{A.10}$$

The result follows by noticing that due to stationarity, for any integers i, j, k and h, $\mathsf{E}[\varepsilon_{i+jh}^2] = \gamma_0$ and $\mathsf{E}[\varepsilon_{i+jh}\varepsilon_{i+kh}] = \gamma_{|j-k|h}$. □

Corollary A.1. *For $l = 2$, $d_0 = 1$, $d_1 = -(d+1)$ and $d_2 = d$, $\mathsf{E}[\eta_i^2(d)] = 2(d^2 + d + 1)\,\gamma_0$.*

Lemma A.3. *For $i \geq 1$, $l \geq 1$, $\mathsf{E}[E_{i,l,m+1}^4] = 3\,\gamma_0^2\,(d_0^2 + d_1^2 + \cdots + d_l^2)^2$.*

Proof. From (A.10), $E_{i,l,m+1}^4 = A_{i,l,m+1}^2 + 4A_{i,l,m+1}B_{i,l,m+1} + 4B_{i,l,m+1}^2$. Note that (A.8) implies that $\mathsf{E}[A_{i,l,m+1}\,B_{i,l,m+1}] = 0$. That is,

$$\mathsf{E}[E_{i,l,m+1}^4] = \mathsf{E}[A_{i,l,m+1}^2] + 4\mathsf{E}[B_{i,l,m+1}^2]. \tag{A.11}$$

Eq. (A.7) yields that for any integers i, j and k, $\mathsf{E}[\varepsilon_{i+j(m+1)}] = 3\gamma_0^2$ and $\mathsf{E}[\varepsilon_{i+j(m+1)}\,\varepsilon_{i+k(m+1)}] = \gamma_0^2$. Consequently,

$$\mathsf{E}[A_{i,l,m+1}^2] = \gamma_0^2\left(3\sum_{j=0}^{l} d_j^4 + 2\sum_{k,j} d_k^2\,d_j^2\right). \tag{A.12}$$

Next, Eq. (A.9) implies that for $r \neq s$, $\mathsf{E}[x_r(i,l,m+1)x_s(i,l,m+1)] = 0$, see (A.10) for definition of $x_{(\cdot)}(i,l,m+1)$. Hence,

$$\mathsf{E}[B_{i,l,m+1}^2] = \sum_{j=0}^{l}\mathsf{E}[x_j^2(i,l,m+1)] = \gamma_0^2\sum_{k,j} d_k^2\,d_j^2. \tag{A.13}$$

The last equality follows from (A.7). The result follows by plugging A.12 and A.13 into A.11. □

Lemma A.4. *Suppose that the conditions of Theorem 1 hold. Then,*

$$\frac{\mathbf{A}}{8} := \sum_{i=1}^{n_m-1} \sum_{j=i+1}^{n_m} \delta_i(d)\delta_j(d)\mathsf{E}[\eta_i(d)\eta_j(d)] = \left[-d(d+1)^2(m+1) + \sum_{h=1}^{m} q_h^{(m)}(d)\rho_h\right] \gamma_0\, J_K, \tag{A.14}$$

where $q_h^{(m)}(d) = [2(m+1) - 3h](d^4+1) + d^2 h$ *and* $\rho_h = \gamma_h/\gamma_0$.

***Proof*.** For $d \in \mathbb{R}$, let $S_r(d) = \sum_{i=1}^{n_m - r} \delta_i(d)\delta_{i+r}(d)$, $1 \le r < n_m$. First observe that due to stationarity, for any $i \ge 1$ and $h \ge 1$, $\Psi(h,d) := \mathsf{E}[\eta_i(d)\,\eta_{i+h}(d)] = 2\,(d^2+d+1)\gamma_h - (d+1)^2\gamma_{|h-(m+1)|} + d\gamma_{|h-2(m+1)|}$. Then due to m-dependency, $\Psi(h,d) = 0$ for all $h > 3m+2$. Consequently, we can write

$$\frac{\mathbf{A}}{8} = \sum_{r=1}^{3m+2} \Psi(r)\, S_r. \tag{A.15}$$

Next, we present the details on how to compute $S_0(d)$. Utilizing (2.8) and (A.6), we can show that for given τ_j,

$$\sum_{x_i \in I_{\tau_j}^{2(m+1),m+1}} \delta_i^2(d) = (m+1)d^2(a_{j+1} - a_j)^2, \qquad \sum_{x_i \in I_{\tau_j}^{m+1,0}} \delta_i^2(d) = (m+1)(a_{j+1} - a_j)^2.$$

Observe that $S_0(d) = \sum_{j=0}^{K-1} \sum_{x_i \in I_{\tau_j}^{2(m+1),m+1} \cup I_{\tau_j}^{m+1,0}} \delta_i^2(d) = (m+1)(d^2+1)\, J_K$. The key part in obtaining the summation $S_r(d)$, $r \ge 1$, consists of splitting it as shown above (and using (2.8)-(A.6)). Thus, we can show that $S_r(d) = T_r(d)\, J_K$ where

$$T_r(d) = \begin{cases} (m+1-r)d^2 + rd + (m+1-r) & \text{for } r = 0, \dots, m \\ d\,(2(m+1) - r) & \text{for } r = m+1, \dots, 2m+1\,. \\ 0 & \text{for } r \ge 2(m+1) \end{cases} \tag{A.16}$$

In order to get (A.14), substitute (A.16) into (A.15) and arrange terms. This completes the proof. □

Lemma A.5. *Suppose that the conditions of Theorem 1 hold. Then*

$$\mathbf{B} := 2\sum_{i=1}^{n_m-1} \sum_{j=i+1}^{n_m} \mathsf{E}[\eta_i^2(d)\,\eta_j^2(d)] = 8n_m^2\,(d^2+d+1)^2\gamma_0^2 + 2\sum_{r=1}^{3m+2} (n_m - r)\,\Lambda_r(d) \ge 0. \tag{A.17}$$

where

$$\begin{aligned} \Lambda_r(d;\gamma_{(\cdot)}) = {} & 8\,(d^2+d+1)^2\gamma_h^2 + 2(1+d)^4\gamma_{|h-(m+1)|}^2 + 2d^2\gamma_{|h-2(m+1)|}^2 \\ & - 4(1+d)[(1+d)^3 + (d^3+d^2+d+1)]\gamma_h\,\gamma_{|h-(m+1)|} \\ & - 4d(d+1)^2\gamma_{|h-(m+1)|}\,\gamma_{|h-2(m+1)|}. \end{aligned} \tag{A.18}$$

Proof. Straightforward calculations and Eqs. (A.7), (A.8) and (A.9) yield that for $i \geq 1$ and $r \geq 0$,

$$\mathsf{E}[\eta_i^2(d)\,\eta_{i+r}^2(d)] = 4\,(d^2+d+1)^2\gamma_0^2 + \Lambda_r(d).$$

Arrange terms and use m-dependency to get

$$\frac{\mathbf{B}}{2} = \sum_{r=1}^{n_m-1}\sum_{s=1}^{n_m-r} \mathsf{E}[\eta_s^2(d)\,\eta_{s+r}^2(d)].$$

The result now follows by noticing that $\Lambda_r(d) = 0$ for all $r \geq 3(m+1)$. $\square$

Lemma A.6. *Suppose that the assumptions of Theorem 1 hold. Additionally, assume that the correlation function $\rho_h = \gamma_h/\gamma_0$ satisfies that $\rho_h = \rho \in (\max\{-1, -8/(3m^2)\}, 1)$, $1 \leq h \leq m$. Then $p_1(d;\gamma_{(\cdot)})$ and $\mathsf{BIAS}^*[\widehat{\gamma}_0^{(m)}(d)]$ are minimized at $d = 1$.*

Proof. Since $\mathsf{BIAS}[\widehat{\gamma}_0^{(m)}(d)]$ is minimized at $d = 1$, cf. Theorem 2, we only need to focus on minimizing $p_1(\cdot;\gamma_{(\cdot)})$. Let $Q(d) = d^2 + d + 1$. It is easily seen that

$$p_1(d;\gamma_{(\cdot)}) = \frac{m+1}{Q^2(d)}\left[2(d^4+1) + m(d^4+d^2+1)\sum_{h=1}^{m}\rho_h\right].$$

For $\rho = 0$ (indepent observations) the result follows since $\operatorname{argmin}_{d\in\mathbb{R}}(d^4+1)/Q^2(d) = 1$. For $\rho > 0$, we have that $\operatorname{argmin}_{d\in\mathbb{R}}(d^4+d^2+1)/Q^2(d) = 1$ and hence for $d \in \mathbb{R}$, $p_1(d;\gamma_{(\cdot)}) \geq p_1(1;\gamma_{(\cdot)})$. For $\rho < 0$, note that

$$\frac{\partial}{\partial d}\,p_1(d;\gamma_{(\cdot)}) = \frac{2(m+1)(d^2-1)(d^2(\rho\, m^2+2) + d(\rho\, m^2+4) + \rho\, m^2+2)}{Q(d)^3},$$

It is immediate that on $\mathbb{R}$, the critical points of p_1 are -1 and 1. For $\rho \in (-8/(3m^2), 0)$, $\frac{\partial^2}{\partial d^2}\,p_1(-1;\gamma_{(\cdot)}) = -4\rho\, m^2 > 0$ and $\frac{\partial^2}{\partial d^2}\,p_1(d;\gamma_{(\cdot)}) = 4(9\rho\, m^2+24)/81 > 0$, i.e., both critical points are minima. The result follows by noting that $p_1(1;\rho) = p_1(-1;\rho)/9$. $\square$

The following auxiliary results are used in the proof of Theorem 1. Recall that $\widehat{\delta}^{(h)}$ is the ordinary difference-based estimator of gap h, cf. (2.4). Define

$$\mathbf{C}_h^{1/2} = \sum_{i=1}^{n-h}(\varepsilon_i - \varepsilon_{i+h})^2, \quad \mathbf{D}_h^{1/2} = \sum_{i=0}^{K-1}\sum_{j\in I_{t_i}^{h,0}}(a_i - a_{i+1})(\varepsilon_j - \varepsilon_{j+h}). \tag{A.19}$$

We can show that

$$(2(n-h))^2\,\mathsf{E}[(\widehat{\delta}^{(h)})^2] = h^2 J_K^2 + 4hJ_K(n-h)(\gamma_0 - \gamma_h) + \mathsf{E}[\mathbf{C}_h] + \mathsf{E}[\mathbf{D}_h]. \tag{A.20}$$

For $\mathsf{E}[\mathbf{C}_h]$ observe that due to Eqs. (A.7)-(A.8)-(A.9),

$$\mathsf{E}[(\varepsilon_i - \varepsilon_{i+h})^2\,(\varepsilon_j - \varepsilon_{j+h})^2] = 4(\gamma_0 - \gamma_h)^2 + \vartheta_1(i,j) + \vartheta_2(i,j)$$

with

$$\vartheta_1(i,j) = \text{const.}\, \gamma^2_{|j-i+s|}, \quad \vartheta_2(i,j) = \sum_{s,t} \text{const.}\, \gamma_{|j-i+s|}\gamma_{|j-i+t|},$$

where $s,t \in \{0, \pm h\}$. That is,

$$\mathsf{E}[\mathbf{C}_h] = [2(n-h)]^2(\gamma_0 - \gamma_h)^2 + S^{(h)}_{1,n}, \tag{A.21}$$

where $S^{(h)}_{1,n} = \sum_{i,j}^{n-2h}[\vartheta_1(i,j) + \vartheta_2(i,j)]$; note that $S^{(h)}_{1,n} = \mathcal{O}(n)$.

Lemma A.7. *Suppose that the conditions of Theorem 1 hold. Let* $\mathbf{D}_h$ *be defined by* (A.19). *Then,* $\mathsf{E}[\mathbf{D}_h] = F_h(\gamma_{(\cdot)})\, J_K$ *where* $F_1 = 2(\gamma_0 - \gamma_1)$ *and for* $2 \le h \le m$

$$F_h(\gamma_{(\cdot)}) = 2\left[(h-1)(\gamma_0 - \gamma_h) + \sum_{j=2}^{h}\sum_{i=1}^{j+1}\left(2\gamma_{|j-i|} - \gamma_{|j-i-h|} - \gamma_{|j-i+h|}\right)\right] \tag{A.22}$$

Proof. For $h=1$ the result follows by noting that for any t_i, $I^{h,0}_{t_i} = \{\, t_i - 1 \,\}$. For $2 \le h \le m$ note that

$$\mathbf{D}_h = \sum_{i=0}^{K-1}\sum_{k\in I^{h,0}_{t_i}} (a_i - a_{i+1})^2(\varepsilon_k - \varepsilon_{k+h})^2 + \sum_{s=1}^{K-2}\sum_{r=1}^{K-1}(a_s - a_{s+1})(a_r - a_{r+1})(\tilde{\mathbf{D}}_{r,s} + \tilde{\mathbf{D}}_{s,r}),$$

where

$$\tilde{\mathbf{D}}_{s,t} = \sum_{i\in I^{h,0}_{t_r}}\sum_{j\in I^{h,0}_{t_s}} (\varepsilon_i - \varepsilon_{i+h})(\varepsilon_j - \varepsilon_{j+h}).$$

Since for any t_i, $\mathsf{E}\left[\sum_{j\in I^{h,0}_{t_i}}(\varepsilon_j - \varepsilon_{j+h})\right]^2 = 2(h-1)(\gamma_0 - \gamma_h) + \Lambda^*(h;\gamma_{(\cdot)})$, where

$$\Lambda^*(h;\gamma_{(\cdot)}) = 2\sum_{j=2}^{h}\sum_{i=1}^{j+1}\left(2\gamma_{|j-i|} - \gamma_{|j-i-h|} - \gamma_{|j-i+h|}\right),$$

the result is established if we show that $\mathsf{E}[\tilde{\mathbf{D}}_{r,s}] = \mathsf{E}[\tilde{\mathbf{D}}_{s,r}] = 0$. To this end, observe that for any $s \in \{1,\ldots,K-2\}$ and $t \in \{s+1,\ldots,K-1\}$:

$$\mathsf{E}[\tilde{\mathbf{D}}_{s,r}] = \sum_{i=t_s-h}^{t_s-1}\sum_{i=t_r-h}^{t_r-1}\left[2\gamma_{|i-j|} - \gamma_{|j-i-h|} - \gamma_{|j-i+h|}\right]$$

let $x = t_r - t_s$ and recall that by assumption, $\min_{1\le i\le K-1}|t_i - t_{i-1}| > 4(m+1)$ to get,

$$= \sum_{i=1}^{h}\sum_{j=1}^{h}\left[2\gamma_{|x+j-i|} - \gamma_{|x+j-i-h|} - \gamma_{|x+j-i+h|}\right] = 0.$$

The last equality follows because $\gamma_{|h|} = 0$ for all $h \ge m+1$. A similar argument shows that $\mathsf{E}[\tilde{\mathbf{D}}_{r,s}] = 0$. This completes the proof. ☐

From now on, $\sum_{i,j} := \sum_{i=1}^{n_m} \sum_{j=1}^{n_h}$.

Lemma A.8. *Supppose that the conditions of Theorem 1 hold. Let* $\widehat{\delta}^{(h)}$ *and* $\widehat{\gamma}_0^{(m)}(d)$ *be given by Eqs.* (2.4)-(2.5)*. Then,*

$$\mathsf{E}[\widehat{\gamma}_0^{(m)}(d) \times \widehat{\delta}^{(h)}] = \frac{1}{2P(d)n_h n_m} \left\{ [I^* + II^* + III^*] \, J_K + S_{2,n}^{(h)}(d) \right\}, \tag{A.23}$$

where

$$I^* = (m+1)(d^2+1) \left[2n_h(\gamma_0 - \gamma_h) + h \, J_K \right]$$

$$II^* = 8(d^2-1)V_h, \qquad V_h = \sum_{s=0}^{m} \sum_{t=1}^{h} \gamma_{s+t} - \sum_{s=1}^{m+1} \sum_{t=1}^{h} \gamma_{|t-s|}, \quad 1 \le h \le m,$$

$$III^* = 2P(d) \, h \, n_m \gamma_0.$$

Here, $S_{2,n}^{(h)}(d) = \mathcal{O}(n)$ *and does not depend on* J_K*.*

Proof. By definition

$$\mathsf{E}[\widehat{\gamma}_0^{(m)}(d) \times \widehat{\delta}^{(h)}] = \frac{\mathsf{E}[I + II + III]}{2P(d) \, n_h \, n_m},$$

where

$$I = \sum_{i,j} \left[\delta_i^2(d)(\chi_j + \varepsilon_j - \varepsilon_{j+h})^2 \right], \quad II = 2 \sum_{i,j} \delta_i(d) \, \eta_i(d) \left[2\chi_j(\varepsilon_j - \varepsilon_{j+h}) + (\varepsilon_j - \varepsilon_{j+h})^2 \right]$$

$$III = \sum_{i,j} \eta_i^2(d) \left[\chi_j + \varepsilon_j - \varepsilon_{j+h} \right]^2$$

Since for all j, $\mathsf{E}[\varepsilon_j] = 0$ and $\mathsf{E}[(\varepsilon_j - \varepsilon_{j+h})^2] = 2(\gamma_0 - \gamma_h)$, we utilize the arguments leading to Eqs. (3.3)-(3.7) and get

$$I^* = \mathsf{E}[I] = (m+1)(d^2+1) \left[2n_h(\gamma_0 - \gamma_h) + h \, J_K \right] J_K. \tag{A.24}$$

Due to Gaussianity for any i and j, $\mathsf{E}[\eta_i(d)(\varepsilon_j - \varepsilon_{j+h})^2] = 0$. Thus, according to Lemma A.9

$$I^{**} = \mathsf{E}[II] = 4 \sum_{i,j} \mathsf{E}[\delta_i(d) \, \eta_i(d) \, \chi_j(\varepsilon_j - \varepsilon_{j+h})] = 8(d^2-1) \, J_K \, V_h. \tag{A.25}$$

Gaussianity and Lemma A.2 yield, $\sum_{i,j} \mathsf{E}[\eta_i^2(d) \chi_j(\varepsilon_j - \varepsilon_{j+h})] = 0$, and $\mathsf{E}[\eta_i^2(d)] = 2P(d)\gamma_0$, respectively. Consequently,

$$I^{***} = \mathsf{E}[III] = 2P(d)\gamma_0 \, n_m \, h \, J_K + S_{2,n}^{(h)}(d), \tag{A.26}$$

where by stationarity, $S_{2,n}^{(h)}(d) = \sum_{i,j} \mathsf{E}[\eta_i^2(d)(\varepsilon_j - \varepsilon_{j+h})^2] = \mathcal{O}(n)$. In order to get Eq. (A.23) sum up Eqs. (A.24)-(A.25)-(A.26) and arrange terms. $\square$

Lemma A.9. *Suppose that the conditions of Theorem 1 hold. Let*

$$\Psi_{K,d} = \sum_{i,j} \delta_i(d)\, \eta_i(d) \chi_j(\varepsilon_j - \varepsilon_{j+h}).$$

Then,

$$\mathsf{E}[\Psi_{K,d}] = 2(d^2 - 1) J_K\, V_h. \tag{A.27}$$

See Lemma A.8 for a definition of V_h.

Proof. Set $c_m = 2(m+1)$. Since for given j,

$$\sum_{i \in I_{t_j}^{c_m, c_m/2}} \delta_i(d)\, \eta_i(d) = d\,(a_j - a_{j+1}) \times \sum_{i \in I_{t_j}^{c_m, c_m/2}} \eta_i := \mathbf{E}_{\tau_j}(d),$$
$$\sum_{i \in I_{t_j}^{c_m/2, 0}} \delta_i(d)\, \eta_i(d) = (a_j - a_{j+1}) \times \sum_{i \in I_{t_j}^{c_m/2, 0}} \eta_i(d) := \mathbf{F}_{\tau_j}(d),$$

it follows that $\sum_i \delta_i(d) \eta_i(d) = \sum_j \big(\mathbf{E}_{\tau_j}(d) + \mathbf{F}_{\tau_j}(d)\big)$.

Let $\sum_{j, I_{\tau_j}^{L,R}} := \sum_{j=0}^{K-1} \sum_{i \in I_{\tau_j}^{L,R}}$. Note that $\chi_i = (a_j - a_{j+1}) \mathbb{1}_{I_{\tau_j}^{h,0}}(i)$ and this, in turn, implies that $\sum_i \chi_i(\varepsilon_i - \varepsilon_{i+h}) = \sum_{j,i} (a_j - a_{j+1})(\varepsilon_i - \varepsilon_{i+h}) := \mathbf{G}_h$. Consequently,

$$\Psi_{K,d} = \sum_{j=0}^{K-1} \big(\mathbf{E}_{\tau_j}(d) + \mathbf{F}_{\tau_j}(d)\big) \times \mathbf{G}_h = T_1 + T_2 + T_3 + U_1 + U_2 + U_3,$$

where

$$T_1 = d \sum_{j, I_{t_j}^{(c_m, m/2)}} (a_j - a_{j+1}) \varepsilon_i\, \mathbf{G}_h, \quad T_2 = -d(1+d) \sum_{j, I_{t_j}^{(c_m, m/2)}} (a_j - a_{j+1}) \varepsilon_{i + c_m/2}\, \mathbf{G}_h$$
$$T_3 = d^2 \sum_{j, I_{t_j}^{(c_m, m/2)}} (a_j - a_{j+1}) \varepsilon_{i + c_m}\, \mathbf{G}_h, \quad U_1 = \sum_{j, I_{t_j}^{(m/2, 0)}} (a_j - a_{j+1}) \varepsilon_i\, \mathbf{G}_h,$$
$$U_2 = -(1+d) \sum_{j, I_{t_j}^{(m/2, 0)}} (a_j - a_{j+1}) \varepsilon_{i + c_m/2}\, \mathbf{G}_h, \quad U_3 = d \sum_{j, I_{t_j}^{(m/2, 0)}} (a_j - a_{j+1}) \varepsilon_{i + c_m}\, \mathbf{G}_h.$$

The result follows by computing the expected value of T's and U's. Now we compute $\mathsf{E}[T_1]$ and $\mathsf{E}[U_1]$, the remaining terms of $\mathsf{E}[\Psi_{K,d}]$ can be treated similarly. In what follows,

$$\sum_{r,s} \mathcal{S}_{i,j}^{L_1, R_1, L_2, R_2} := \sum_{r \in I_{t_i}^{L_1, R_1}} \sum_{s \in I_{t_j}^{L_2, R_2}} \varepsilon_r(\varepsilon_s + \varepsilon_{s+h}).$$

We begin by writing $\mathsf{E}[T_1] = d\mathsf{E}[T_{1,1} + T_{1,2}]$, where $T_{1,1} = \sum_{j=0}^{K-1} (a_j - a_{j+1})^2 \sum_{r,s} \mathcal{S}_{j,j}^{c_m, c_m/2, h, 0}$ and $T_{1,2} = \sum_{i=0}^{K-2} \sum_{j=i+1}^{K-1} (a_i - a_{i+1})\, (a_j - a_{j+1}) \Big(\sum_{r,s} \mathcal{S}_{i,j}^{c_m, c_m/2, h, 0} + \sum_{r,s} \mathcal{S}_{j,i}^{c_m, c_m/2, h, 0}\Big)$. Note now that for any τ_j and due to m-dependency,

$$\mathsf{E}\left[\sum_{r,s} \mathcal{S}_{j,j}^{c_m,c_m/2,h,0}\right] = \sum_{r=\tau_j-c_m}^{\tau_j-(m+2)} \sum_{s=\tau_j-h}^{\tau_j-1} [\gamma_{|r-s|} - \gamma_{|r-(s+h)|}] = \sum_{s=m+2}^{2(m+1)} \sum_{t=1}^{h} \gamma_{|s-t|} = \sum_{s=m+1}^{m+h} \sum_{t=1}^{h} \gamma_{|s-t|}, \tag{A.28}$$

These calculations hold independently of the value of τ_j, and consequently we get that $\mathsf{E}[T_{1,1}] = \sum_{r=m+1}^{m+h} \sum_{s=1}^{h} \gamma_{|r-s|} J_K$. Similar calculations along with the m-dependency and (2.8) allows us to get that $\mathsf{E}[T_{1,2}] = 0$. All in all, we have shown that

$$\mathsf{E}[T_1] = d \sum_{s=m+1}^{m+h} \sum_{t=1}^{h} \gamma_{|h-(s+t)|} J_K. \tag{A.29}$$

Similar arguments yield,

$$\mathsf{E}[T_2] = -d(1+d) \sum_{s=1}^{m+1} \sum_{t=1}^{h} [\gamma_{|t-s|} - \gamma_{|t-(s+h)|}] J_K \tag{A.30}$$

$$\mathsf{E}[T_3] = d^2 \sum_{s=0}^{m} \sum_{t=1}^{h} [\gamma_{s+t} - \gamma_{|h-(s+t)|}] J_K. \tag{A.31}$$

Now, we consider $\mathsf{E}[U_1]$. Write $U_1 = U_{1,1} + U_{1,2}$, where

$$U_{1,1} = \sum_{j=0}^{K-1} (a_j - a_{j+1})^2 \sum_{r,s} \mathcal{S}_{j,j}^{c_m/2,0,h,0}$$

$$U_{1,2} = \sum_{i=0}^{K-2} \sum_{j=i+1}^{K-1} (a_i - a_{i+1})(a_j - a_{j+1}) \left(\sum_{r,s} \mathcal{S}_{i,j}^{c_m/2,0,h,0} + \sum_{r,s} \mathcal{S}_{j,i}^{h,0,c_m/2,0} \right)$$

Following (A.28) we can show that for any τ_j, $\mathsf{E}\left[\sum_{r,s} \mathcal{S}_{j,j}^{c_m/2,0,h,0}\right] = \sum_{s=1}^{m+1} \sum_{t=1}^{h} [\gamma_{|t-s|} - \gamma_{|h-(t+s)|}]$. Again m-dependency and (2.8) allow us to show that $\mathsf{E}[U_{1,2}] = 0$. Therefore,

$$\mathsf{E}[U_1] = \sum_{r=1}^{m+1} \sum_{s=1}^{h} [\gamma_{|r-s|} - \gamma_{|s-(r+h)|}] J_K. \tag{A.32}$$

Similar arguments yield,

$$\mathsf{E}[U_2] = -(1+d) \sum_{r=0}^{m+1} \sum_{s=1}^{h} [\gamma_{r+s} - \gamma_{|h-(r+s)|}] J_K \tag{A.33}$$

$$\mathsf{E}[U_3] = -d \sum_{r=m+1}^{m+h} \sum_{s=1}^{h} \gamma_{|h-(r+s)|} J_K. \tag{A.34}$$

Since $\sum_{r=0}^{m} \sum_{s=1}^{h} \gamma_{|h-(r+s)|} = \sum_{r=1}^{m+1} \sum_{s=1}^{h} \gamma_{|s-r|}$, $\sum_{r=1}^{m+1} \sum_{s=1}^{h} \gamma_{|s-(r+h)|} = \sum_{r=0}^{m} \sum_{s=1}^{h} \gamma_{r+s}$, (A.27) follows after summing up (A.29)-(A.34). $\square$

Appendix B. Proofs and auxiliary results for Section 4

In this Appendix we will assume that the conditions of Theorem 7 hold and use the notation introduced at the beginning of Appendices A and A.2. Also throughout this section $c_m = 2(m+1)$ and the symbols κ_1, κ_2, etc., will denote constants which do not depend on n. We will write $\sum_{i,j}$ to denote $\sum_{i=1}^{n_m-1} \sum_{i=j+1}^{n_m}$.

Lemma B.1. *Suppose that the conditions of Theorem 7 hold. Then, for $h = 0, \ldots, m$,*

$$\mathsf{E}[\widehat{\gamma}_h^{(m)}(d_{h,m})] = \gamma_h + \mathcal{O}(S_n), \qquad S_n = \sum_{j=1}^{K_n} \frac{\vartheta_j^2}{n} + \sum_{j=1}^{K_n} n^{-2(\alpha_j+1/2)}.$$

Proof. We begin with the case $h = 0$. Since $\widehat{\gamma}_0^{(m)}(d) = (2(d^2+d+1)\,n_m)^{-1} \sum_{i=1}^{n_m} (\delta_i(d) + \eta_i(d))^2$, see Appendix A.2 for definition of $\delta_i(d)$ and $\eta_i(d)$, and $\mathsf{E}[\eta_i^2(d)] = 2(d^2+d+1)\gamma_0$, cf. Corollary A.1, in order to analyze the asymptotic bias of $\widehat{\gamma}_0^{(m)}(d)$ it suffices to focus on $\sum_i \delta_i^2(d)$. To this end we write $\delta_i^2(d) = s_0^2(i) + 2d\, s_0(i)\, s_{c_m}(i) + d^2\, s_{c_m}^2(i)$ and note that for given i there exists a unique τ_j such that

$$s_0(i) = \begin{cases} t(i,j) := a_j(i/n) - a_j((i+m+1)/n) & \text{for } i \in I_{\tau_{j-1},\tau_j}^{(0,c_m/2)} \\ u(i,j) := a_j(i/n) - a_{j+1}((i+m+1)/n) & \text{for } i \in I_{\tau_j}^{(c_m/2,0)} \end{cases},$$

and a similar characterization holds for $s_{c_m}(\cdot)$, see Eq. (A.6) in Appendix A.2 for definition of $I_{\tau_{j-1},\tau_j}^{(0,c_m/2)}$ and $I_{\tau_j}^{(c_m/2,0)}$. This implies that the dominat terms in $\delta_i^2(d)$ are of the form $t^2(i,j) + u^2(i,j)$ and in what follows we provide bounds for these terms.

Recall that $\vartheta_j = a_j(\tau_{j+1}^-) - a_{j+1}(\tau_{j+1})$ and by assumption there exists a number $c > 0$ such that $|\vartheta_j| > c$ for all j. Utilizing the Hölder condition of f we get

$$\sum_{j=1}^{K_n-1} \sum_{i \in I_{\tau_{j-1},\tau_j}^{(0,c_m/2)}} t^2(i,j) \le \kappa_{1,m} \sum_{j=1}^{K_n-1} n^{-2\alpha_j}, \tag{B.1}$$

as well as

$$\sum_{j=1}^{K_n-1} \sum_{i \in I_{\tau_j}^{(c_m/2,0)}} u^2(i,j) \le \sum_{j=1}^{K_n} \vartheta_j^2 + \kappa_{2,m} \sum_{j=1}^{K_n} n^{-\alpha_j}\, \vartheta_j + \kappa_{1,m} \sum_{j=1}^{K_n} n^{-2\alpha_j}.$$

Here $\kappa_{1,m} = \sup_j (m+1)^{\alpha_j} < \infty$, $\kappa_{2,m} = 2\,\kappa_{1,m}$. Define the set of indices $I_{K_n} := \{j \in \{1,\ldots,K_n\} : |\vartheta_j| \ge 1\}$ and observe that

$$\Big|\sum_{j=1}^{K_n} n^{-\alpha_j}\, \vartheta_j\Big| \le \{\sum_{j \in I_{K_n}} \vartheta_j^2 + \sum_{j \notin I_{K_n}} |\vartheta_j|\} \le \{\sum_{j \in I_{K_n}} \vartheta_j^2 + \sum_{j \notin I_{K_n}} \frac{1}{c}\, \vartheta_j^2\} \le (1 + c^{-1}) \sum_{j=1}^{K_n} \vartheta_j^2. \tag{B.2}$$

From (B.1) and (B.2) follows that $\sum_{i=1}^{n_m} \delta_i^2(d) = \mathcal{O}\left(\sum_{j=1}^{K_n} \vartheta_j^2 + \sum_{j=1}^{K_n} n^{-2\alpha_j}\right)$. Consequently,

$$\mathsf{E}[\widehat{\gamma}_0^{(m)}(d)] = \gamma_0 + \mathcal{O}\left(S_n\right), \quad S_n = \sum_{j=1}^{K_n} \frac{\vartheta_j^2}{n} + \sum_{j=1}^{K_n} n^{-2(\alpha_j+1/2)}. \tag{B.3}$$

For $h \geq 1$, firstly note that by writing $\widehat{\delta}^{(h)} = (2(n-h))^{-1} \sum_{i=1}^{n_h}(s_0(i) + \eta_i(0))^2$ we can mimick the calculations above and get

$$\mathsf{E}[\widehat{\delta}^{(h)}] = \gamma_0 - \gamma_h + \mathcal{O}(S_n). \tag{B.4}$$

Then, by definition, $\widehat{\gamma}_h^{(m)}(d) = \widehat{\gamma}_0^{(m)}(d) - \widehat{\delta}^{(h)}$, cf. (2.6) in the Introduction, and the result follows by adding Eqs. (B.3) and (B.4). □

Lemma B.2. *Suppose that the assumptions of Lemma B.1 hold. Then*

$$\mathsf{VAR}(\widehat{\gamma}_0^{(m)}(d)) = \mathcal{O}(\sum_{j=1}^{K_n}(\vartheta_j^2/n^2 + n^{-(\alpha_j+2)})) + \mathcal{O}(n^{-1}) \tag{B.5}$$

the same result holds for $\mathsf{VAR}(\widehat{\delta}^{(h)})$. *Moreover,*

$$\mathsf{VAR}(\widehat{\gamma}_h^{(m)}(d)) = \mathcal{O}(\mathsf{VAR}(\widehat{\gamma}_0^{(m)}(d)) + (\sum_{j=1}^{K_n} |\vartheta_j|/n)^2 + (\sum_{j=1}^{K_n} n^{-(\alpha_j+1)})^2). \tag{B.6}$$

Proof. We write $\widehat{\gamma}_0^{(m)}(d) = n^{-1} \sum b_i^2(d)$, where $b_i(d) = \delta_i(d) + \eta_i(d)$, see Appendix A.2 for notation. It is easily seen that $\mathsf{VAR}(b_i^2(d)) = 4\delta_i^2(d)\,\mathsf{VAR}(\eta_i(d)) + \mathsf{VAR}(\eta_i^2(d))$. From Corollary A.1 and Lemma A.3, $\mathsf{VAR}(\eta_i(d))$ and $\mathsf{VAR}(\eta_i^2(d))$ are uniformly bounded. This implies that the order of magnitude of $\sum_i \mathsf{VAR}(b_i^2(d))$ depends solely on $\sum_i \delta_i^2(d)$. From arguments in the proof of Lemma B.1 we get that

$$\sum_{i=1}^{n_m} \mathsf{VAR}(b_i^2(d)) = \mathcal{O}\left(\sum_{j=1}^{K_n} \vartheta_j^2 + \sum_{j=1}^{K_n} n^{-2\alpha_j}\right) + \mathcal{O}(n). \tag{B.7}$$

It can be shown that $\mathsf{COV}(b_i^2(d), b_j^2(d)) = 4\delta_i(d)\delta_j(d)\mathsf{E}[\eta_i(d)\,\eta_j(d)] + 2\delta_i(d)\mathsf{E}[\eta_i(d)\,\eta_j^2(d)] + 2\delta_j(d)\mathsf{E}[\eta_j(d)\,\eta_i^2(d)] + \mathsf{COV}(\eta_i^2(d), \eta_j^2(d))$. Due to m-dependency and stationarity of moments up to 4-th order we get for any i and j, $\mathsf{E}[\eta_i(d)\,\eta_j(d)] = \kappa_2\,\mu_2(|j-i|)$, $\mathsf{E}[\eta_i(d)\,\eta_j^2(d)] = \kappa_3\,\mu_3(|j-i|)$ and $\mathsf{COV}(\eta_i^2(d), \eta_j^2(d)) = \kappa_4\,\mu_4(|j-i|)$, where $\mu_2(\cdot)$, $\mu_3(\cdot)$ and $\mu_4(\cdot)$ are functions which depend only on sums of moments of second, third and fourth order of ε, respectively. With the same arguments used in Lemma A.4, we can establish that $\sum_{i,j} \delta_i(d)\,\delta_j(d)\mu_2(j-i) = \mathcal{O}(n)$. Similar arguments allow us to get that $\sum_{i,j} \delta_i(d)\,\mu_3(j-i) = \mathcal{O}(\sum_i \delta_i(d))$ and $\sum_{i,j} \mathsf{COV}(\eta_i^2(d), \eta_j^2(d)) = \mathcal{O}(n)$.

All in all, we have proven that

$$\sum_{i,j} \mathsf{COV}(b_i^2(d), b_j^2(d)) = \mathcal{O}\left(\sum_i \delta_i(d)\right) + \mathcal{O}(n). \tag{B.8}$$

Since $\delta_i(d) = s_0(i) + d s_{c_m}(i)$, see Appendix A.2 for notation, from the characterization of $s_0(\cdot)$ and $s_{c_m}(\cdot)$ given in Lemma B.1 it follows that the order of magnitude of (B.8) is driven by $\sum_i s_0(i)$. We get,

$$\left|\sum_{i=1}^{n_m} s_0(i)\right| = \left|\sum_{i=1}^{K_n-1}\left(\sum_{j\in I^{(0,c_m/2)}_{\tau_{i-1},\tau_i}} t(j,i) + \sum_{j\in I^{(c_m/2,0)}_{\tau_i}} u(j,i)\right)\right| = \mathcal{O}\left(\sum_{j=1}^{K_n}(n^{-\alpha_j} + |\vartheta_j|^2)\right). \tag{B.9}$$

The latter follows because of Hölder condition on f and calculations leading to (B.2). Observe that Eq. (B.5) follows by a combination of Eqs. (B.7)-(B.8)-(B.9).

For $h \geq 1$, we write $\widehat{\delta}^{(h)} = (2(n-h))^{-1}\sum_{i=1}^{n_h}(s_0(i) + \eta_i(0))^2$ and mimick the calculations above to deduce that $\widehat{\gamma}_0^{(m)}(d)$ and $\widehat{\delta}^{(h)}$ have variances of the same order. Then, since $\widehat{\gamma}_h^{(m)} = \widehat{\gamma}_0^{(m)} - \widehat{\delta}^{(h)}$, cf. (2.6) in the Introduction, we combine Eq. (B.5) and Lemma B.3 below to show the validity of Eq. (B.6). This completes the proof. □

Lemma B.3. *Suppose that the assumptions of Lemma B.1 hold. Then*

$$\mathsf{COV}(\widehat{\gamma}_0^{(m)}(d_{h,m}), \delta^{(h)}) = \mathcal{O}\left((\sum_{j=1}^{K_n}|\vartheta_j|/n)^2 + (\sum_{j=1}^{K_n} n^{-(\alpha_j+1)})^2\right) + \mathcal{O}(n^{-1}). \tag{B.10}$$

***Proof*.** We begin with the case $d_{h,m} = 1$. Throughout this proof $k \in \{h, c_m\}$. By definition,

$$\mathsf{COV}(\widehat{\gamma}_0^{(m)}(1), \widehat{\delta}^{(h)}) = \frac{1}{12\, n_m\, n_h}\sum_{i=1}^{n_m}\sum_{j=1}^{n_h}\mathsf{COV}(z_{i,c_m}, z_{j,h}) + \mathcal{O}(n^{-1}), \tag{B.11}$$

where for given index i, $z_{i,k} = y_{i:(i+k+1)}^\top D_{k+2}\, y_{i:(i+k+1)}$, see Appendix A for notation of $y_{i:(i+k+1)}$ and Eq. (3.1) for definition of the $(k+2)\times(k+2)$ matrix D_{k+2}. We also write $c_i(k) = f_{i:(i+k+1)}^\top D_{k+2}\,\varepsilon_{i:(i+k+1)}$ and $d_i(k) = \varepsilon_{i:(i+k+1)}^\top D_{k+2}\,\varepsilon_{i:(i+k+1)}$ which by standard calculations yield $\mathsf{COV}(z_{i,c_m}, z_{j,h}) = 4\mathsf{E}[c_i(c_m)\, c_j(h)] + 2\mathsf{E}[c_i(c_m)\, d_j(h)] + 2\mathsf{E}[c_j(h)\, d_i(c_m)] - \mathrm{tr}(D_{c_m+2}\,\Sigma_{c_m+2})\,\mathrm{tr}(D_{h+2}\,\Sigma_{h+2})$. Stationarity and m-dependency, arguments also used in Lemma B.2, allow us to get that the second, third and fourth summands above are sums of stationary moments of second, third and fourth order, respectively. Hence the contribution of these terms to (B.11) is of order $\mathcal{O}(n^{-1})$. It is not difficult to see that for given indeces i and j, $c_i(m)\, c_j(h)$ is the sum of 8 terms of the form $(f_i - f_{i+(m+1)})(f_j - f_{j+h})(\varepsilon_i - 2\varepsilon_{i+m+1} + \varepsilon_{i+2(m+1)})(\varepsilon_j - \varepsilon_{j+h})$ and due to stationarity and m-dependency, $\mathsf{E}[c_i(m)\, c_j(h)]$ is bounded by $|f_i - f_{i+(m+1)})(f_j - f_{j+h})|$. Now, since $s_k(i) = f_{i+k} - f_{i+m+1}$, see notation in Appendix A.2, we utilize the ideas leading to the bound of $\mathsf{VAR}(\widehat{\gamma}_0^{(m)}(1))$, cf. (B.5), and obtain that

$$\Big|\sum_{i=1}^{n_m}\sum_{j=1}^{n_h}(f_i - f_{i+(m+1)})(f_j - f_{j+h})\Big| = \mathcal{O}\left((\sum_{j=1}^{K_n}|\vartheta_j|)^2 + (\sum_{j=1}^{K_n} n^{-\alpha_j})^2 \right). \tag{B.12}$$

Thus for $d_{h,m} = 1$, the result follows by a combination of Eqs. (B.11) and (B.12). For the other values of $d_{h,m}$, cf. (2.14) in the Introduction, we mimick the calculations above to complete the proof. $\square$

References

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