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ABSTRACT

We consider a Hamiltonian system that has its origin in a generalization of the exact renormalization group flow of matrix scalar field theory and describes a non-linear generalization of the shock-wave equation that is known to be integrable. Analyzing conserved currents of the system, this paper shows that these follow a nice pattern governed by coefficients of Motzkin polynomials, where each integral of motion corresponds to a path on a unit lattice.

I. INTRODUCTION

The exact renormalization group governed by the Polchinski equation allows us to investigate the dynamics of operators in a field theory under change of scale.[1,2] In Ref. 3, it has been shown that for the simplest case, matrix scalar field theory, these RG flow equations can be written in the Hamiltonian form for some specific choice of operators built from fundamental fields. The corresponding Hamiltonian was shown to be of the following form:

$$H = \int_0^{2\pi} d\sigma\, \Pi(\sigma)^2 J'(\sigma), \tag{1.1}$$

which gives the so-called Burgers–Hopf equations

$$\partial_T \rho = \rho \partial_\sigma \rho, \quad \rho = \frac{\partial_T J}{\partial_\sigma J}, \tag{1.2}$$

where T is related to the physical energy scale at which observations are performed and plays the role of time for the system. This equation describes shock waves and is known to be integrable.

However, on the field theory side, the above discussion is only valid in the IR limit, where a significant amount of information is allowed to be dropped off, hence leading to RG equations written in such an ultra-local form. Attempts to go beyond this approximation and to obtain a more general Hamiltonian have been made in Ref. 4 which, however, were not completely successful. Despite the technical difficulties, which so far have not allowed us to arrive at an ultra-local Hamiltonian and which are related to the cut-off function defining the physical energy scale, some speculations lead to the Hamiltonian given by

$$H = \int_0^{2\pi} d\sigma \left[\Pi^2 J' + \Pi J'^2\right], \tag{1.3}$$

whose limit $J \ll \Pi$ gives the previous system. Since the currents J should vanish at low energies, this is in agreement with the IR nature of the previous Hamiltonian. For example, one of the ways to properly arrive at the above Hamiltonian would be to redefine fundamental fields of the initial system in order to hide the cut-off function inside. This in turn leads to other technical difficulties and, hence, will not be considered in the present paper.

Instead, we start from Hamiltonian (1.3) as it is and investigate its integrability properties motivated by the above discussion. In the area of our interest here are integrals of motions, equivalently conserved currents which can be explicitly constructed and parameterized by an integer number. All these stay in involution with respect to the naturally defined Poisson brackets that might be a good sign of (classical) integrability of the system. Another argument in favor of this conjecture is that taking the limit $J \ll \Pi$ and, hence, dropping the second term in the Hamiltonian one, it returns back to the Burgers–Hopf shock wave equations that are known to be integrable. However, after trying to apply the known integrability criteria, such as constructing a Lax pair, and performing the Painlevé test, we were not able to show this explicitly so far and, hence, leave this task for a future work.

The main result of this paper is the observation of the so-called Motzkin numbers in this Hamiltonian system, which usually appears in the description of unit paths (see sequence A055151 in Ref. 5 and references therein). We show that in the considered system, these numbers appear as coefficients in the obtained integrals of motion, thus rendering them precisely as Motzkin polynomials and allowing us to rewrite them as a very nice expression in terms of hypergeometric ${}_2F_1$ functions. This might be an interesting result from a purely mathematical point of view. In Sec. IV, we discuss possible extensions of the current result.

II. MOTZKIN NUMBERS

In this section, we give a short review describing Motzkin paths and related polynomials that appear in many contexts in the mathematical literature, e.g., Refs. 6–8, and physics, e.g., Refs. 9–11.

First, let us define a lattice path. A lattice path L in $\mathbb{Z}^d$ of the length n is a sequence $v_0, \ldots, v_n \in \mathbb{Z}^d$ with corresponding steps $s_1, \ldots, s_n \in \mathbb{Z}^d$ defined by consecutive difference $s_i = v_i - v_{i-1}$. A Motzkin path of the length n is a lattice path on $\mathbb{N} \times \mathbb{N}$ consisting of up steps $(1, 1)$, down steps $(1, -1)$, and flat steps $(1, 0)$. The number of Motzkin paths from $(0, 0)$ to $(n, 0)$ is given by the Motzkin number (note that historically the Motzkin numbers appeared in a circle chording setting[12]) (sequence A001006 in Ref. 5) m_n, which can be written in the following form:

$$m_n = \sum_{k=0}^{[n/2]} \binom{n}{2k} c_k, \tag{2.1}$$

where c_k are the Catalan numbers (sequence A000108 in Ref. 5) defined as

$$c_k = \frac{1}{k+1}\binom{2k}{k}. \tag{2.2}$$

We are particularly interested in the Motzkin polynomial associated with a Motzkin path. In order to define the Motzkin polynomial, one needs to assign a weight keeping track of the number of up steps (or flat steps). Then, one gets the following polynomial with the corresponding coefficients (sequence A055151 in Ref. 5):

$$m_n(a) = \sum_{k=0}^{[n/2]} T_{n,k}\, a^k, \tag{2.3}$$

where a stands for up steps. The coefficient $T_{n,k}$ is the number of Motzkin paths of the length n with k up steps. Note that the polynomial (2.3) is also known as the Jacobi–Rogers polynomial.[8]

It might now be reasonable to expect the occurrence of other lattice path polynomials such that the Riordan polynomials and Dick polynomials in the context of integrable models.

III. HAMILTONIAN SYSTEM FOR MOTZKIN NUMBERS

Consider a dynamical system defined by the following Hamiltonian:

$$H = \int_{-\pi}^{\pi} d\sigma \left[\Pi^2 J' + \Pi J'^2\right], \tag{3.1}$$

which is motivated by the renormalization group procedure as discussed in the Introduction. Given the direct relation between integrable properties of this system and Motzkin paths and that the system does not look recognizable to the knowledge of the authors, we suggest calling it the Motzkin system. Note, however, that quantum spin chains called Motzkin chains have been considered in Refs. 13 and 14 where authors also use the terminology "the Motzkin Hamiltonian" for the specific nearest neighbor Hamiltonian.[10] So far, there is no an apparent correspondence between this systems; however, one notices that the conserved quantities we obtain below closely correspond to configuration in the ground state of the Motzkin spin chain.

Let us start with equations of motion

$$\begin{aligned}\dot{J} &= 2\Pi J' + J'^2,\\ \dot{\Pi} &= 2\Pi\Pi' + 2\Pi' J' + 2\Pi J'',\end{aligned} \tag{3.2}$$

which can be easily solved with respect to Π providing the following Lagrangian formulation of the theory:

$$\mathcal{L} = \frac{\left(\dot{J} - J'^2\right)^2}{4J'}. \tag{3.3}$$

The equations of motion (EOM) for the field $J(t,\sigma)$ can be found by either varying Lagrangian (3.3) with respect to $J(t,\sigma)$ or substituting an expression for $\Pi(t,\sigma)$ into the second equation in (3.2),

$$\begin{aligned}&\dot{\rho} - \frac{1}{2}\partial_\sigma \rho^2 = 5\dot{J}' - 6J''J',\\ &\rho := \frac{\dot{J}}{J'}.\end{aligned} \tag{3.4}$$

One notices that in the IR limit of the corresponding field theory, the sources become infinitesimally small $J \to 0$, and hence, the above equation drops to the Burgers–Hopf equation in agreement with Ref. 3.

However, in general, the above equation does not immediately correspond to one of the commonly known types of non-linear equations. On the other hand, if one is lucky to find the explicit Lax pair for these equations, it would be possible either to compare the system to one of the known systems (e.g., KdV) or to prove that it is a new integrable system. However, so far, we have not been able to find the corresponding Lax pair and we cannot claim if there exists one. In general, the process of finding of the Lax pair is always some kind of an art.

To move forward in the analysis of the system, we present an infinite set of integrals of motion for the system in question, which appears to be a good argument in favors of integrability of the Hamiltonian flow. It is suggestive to consider the notion of integrability in the Liouville sense, which means having a maximal set of Poisson-commuting integrals of motion [i.e., function(al)s on the phase space whose Poisson brackets vanish] which are not trivial, i.e., not zero and not Casimir elements. Certainly, one must be careful when applying the Liouville criterion to infinitely dimensional systems such as field theoretical equations. For this reason, we present the required full set of integrals of motion refraining from the claim that the system is, indeed, completely integrable.

Hence, starting with some obvious integrals of motion of the type

$$\begin{aligned}I_1 &= \int_\sigma \left(\Pi + J'\right),\\ I_2 &= \int_\sigma \Pi J',\\ I_3 &= \int_\sigma \Pi J'(\Pi + J') = H,\end{aligned} \tag{3.5}$$

it is straightforward to see that there exists the following infinite tower of such constructions (see Appendix A):

$$\begin{aligned}I_n &= \int_\sigma \sum_{k=1}^{n} \left(\Pi J'\right)^k \left(\Pi + J'\right)^{n-2k} t_{n,k} + \delta^1_n(\Pi + J'),\\ t_{n,k} &= \frac{(n-2)!}{(n-2k)!k!(k-1)!}, \forall n > 1,\\ t_{1,1} &= 1.\end{aligned} \tag{3.6}$$

It is worth mentioning here that one should not be confused by the fact that we have discrete set of integrals of motion facing "continuous" variables $J(t,\sigma)$. Since σ is compact, one actually has a discrete spectrum of variables.

By making use of the relation between $t_{n,k}$ and the Motzkin polynomial coefficients

$$t_{n,k} = T_{n-2,k-1}, \tag{3.7}$$

we can write

$$I_n = \int_\sigma \sum_{k=1}^{n} \left(\Pi J'\right)^k \left(\Pi + J'\right)^{n-2k} T_{n-2,k-1} + \left(\Pi + J'\right)\delta^1_n. \tag{3.8}$$

Interestingly, the sum in the first term above can be performed explicitly and the result for this term can be written in terms of the hypergeometric function to give (note that $n \neq 1$)

$$\begin{aligned}\tilde{I}_n &= \int_\sigma \sum_{k=1}^{n} \left(\Pi J'\right)^k \left(\Pi + J'\right)^{n-2k} T_{n-2,k-1} \\ &= \int_\sigma \left(\Pi J'\right)\left(\Pi + J'\right)^{n-2} {}_2F_1\left[1 - \frac{n}{2}, \frac{3}{2} - \frac{n}{2}; 2, \frac{4\left(\Pi J'\right)}{\left(\Pi + J'\right)^2}\right].\end{aligned} \tag{3.9}$$

This expression can be further simplified by making use of the following quadratic relation:

$${}_2F_1\left[\frac{a}{2}, \frac{a}{2} + \frac{1}{2}; 1 + a - b, \frac{4z}{(1+z)^2}\right] = (1+z)^a {}_2F_1\left[a, b; 1 + a - b, z\right] \tag{3.10}$$

to obtain

$$\tilde{I}_n = \int_\sigma \Pi J'^{n-1} {}_2F_1\left[2 - n, 1 - n; 2, \frac{\Pi}{J'}\right]. \tag{3.11}$$

The conserved quantities I_k can be shown to be in involution, i.e., they commute with respect to the standard Poisson bracket,

$$\begin{aligned}\{f, g\} &= \frac{\partial f}{\partial J}\frac{\partial g}{\partial \Pi} - \frac{\partial f}{\partial \Pi}\frac{\partial g}{\partial J}, \\ \{I_m, I_n\} &= 0.\end{aligned} \tag{3.12}$$

Finally, it is possible to write the above integrals in terms of conserving currents using the notations $\partial_\mu = (\partial/\partial T, \partial/\partial\sigma)$,

$$\dot{I}_m = 0 \rightarrow \partial_\mu j_m{}^\mu = 0. \tag{3.13}$$

Hence, the currents read

$$\begin{aligned}j_n{}^0 &= \sum_{k=1}^{n} \alpha^k \beta^{n-2k} t_{n,k}, \\ j_n{}^1 &= -2\sum_{k=1}^{n} \alpha^k \beta^{n-2k+1} \frac{n-k}{n-2k+1} t_{n,k},\end{aligned} \tag{3.14}$$

with $\alpha = \Pi J'$ and $\beta = \Pi + J'$. According to the Noether theorem, each conserving current is associated with a global symmetry of the system.

IV. CONCLUSION

In this paper, the main result is the analysis of integrability properties of the corresponding Hamiltonian system and observation that these are related to Motzkin paths.

We start from a Hamiltonian of the form

$$H = \int_{-\pi}^{\pi} d\sigma \left[\Pi^2 J' + \Pi J'^2\right], \tag{4.1}$$

which is motivated by studies of the exact renormalization group flows of $N \times N$ matrix scalar field theories.[3] In particular, in the limit $J \ll \Pi$ (together with all its derivatives), equations of motion for the above system become

$$\partial_T \rho = \rho \partial_\sigma \rho, \; \rho = \frac{\partial_T J}{\partial_\sigma J}, \tag{4.2}$$

which is known as the Burgess–Hopf equation describing dynamics of shock waves. On the field theory side, this corresponds to taking the IR limit and the large N limit as it has been shown in Ref. 3. So far, a direct path has not been found from renormalization group equations to the Hamiltonian in question; hence, we consider it a motivated toy-model.

Starting with this setup, we show that, given the fundamental fields $J(\sigma, T)$ and their conjugates $\Pi(\sigma, T)$, this system has an infinite number of conserved quantities I_n of the form

$$I_n = \int_\sigma \sum_{k=1}^{n} \left(\Pi J'\right)^k \left(\Pi + J'\right)^{n-2k} T_{n-2,k-1} + \left(\Pi + J'\right)\delta_n^1, \tag{4.3}$$

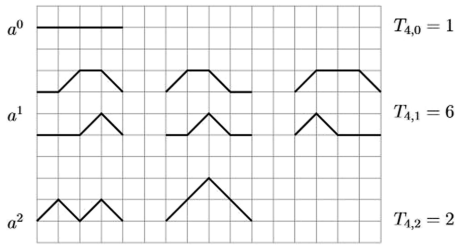

FIG. 1. Motzkin paths of length four between the points (0, 0) and (4, 0). The powers of a in the left column denote the number of up steps. The number of paths of length n with k up step is given by the Motzkin polynomial coefficient $T_{n,k}$.

which Poisson-commute with each other. This allows us to conjecture that the non-linear system in question might be, indeed, integrable. A way to show that explicitly would be to find the corresponding Lax pair or to satisfy the Painlevé criterion.

What is more interesting about the presented result is the appearance of the so-called Motzkin number $T_{n,k}$ (sequence A055151 in Ref. 5) in the integrals of motion I_n. Initially, these numbers and the corresponding polynomials appear in the problem of counting all routes on a lattice with a given number of vertical and horizontal steps (see Fig. 1). Hence, each integral of motion I_n is given by a Motzkin polynomial corresponding to a path of length $n-2$ with $k-1$ horizontal steps.

An interesting problem would be to consider other polynomials corresponding to various paths on a lattice and reversely build a set of expressions understood as integrals of motion for some system. Having such a procedure would be a fascinating way of generating dynamical systems. If it becomes possible to strictly prove classical integrability of the system, one may speculate on whether having the description of the corresponding integrals of motion in terms of some polynomials is equivalent to integrability. The most straightforward application of this logic would be to start with Catalan numbers (2.2), corresponding to the so-called Dyck path, and to investigate whether one is able to build a Hamiltonian. Such a reversal construction is beyond the present paper and is reserved for future work.

On the other hand, the above logic may be reverted and one may ask whether integrals of motion of a given Hamiltonian system correspond to some polynomials not known before. This conjecture can be tested explicitly on some simple examples. Along the same lines would be to deviate from the construction of integrals of motion and to consider, instead, the Lax pair for a system as a more strict integrability criterion.

Exploiting the similarity in the used numbers, one may be interested in considering the Motzkin spin chain (Refs. 13 and 14) whose ground state configuration is similar to the conserved currents of the present paper. In this case, higher values of k in $T_{n,k}$ correspond to long-range entanglement of the chain; however, the field–theoretical interpretation of this relation is obscure.

Finally, on the field–theoretical side, one is still interested in finding a way to show that the Motzkin Hamiltonian indeed describes exact renormalization group flow of a theory, probably under some assumptions.

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APPENDIX A: DERIVATION OF INTEGRALS OF MOTION

Let us start by listing few first integrals of motion for which purpose we first introduce the notations

$$\begin{aligned} \alpha &= \Pi J', \\ \beta &= \Pi + J'. \end{aligned} \tag{A1}$$

The equations of motion for these variables are rather simple,

$$\begin{aligned} \dot{\alpha} &= 2(\alpha\beta)', \\ \dot{\beta} &= \left(\beta^2 + 2\alpha\right)', \end{aligned} \tag{A2}$$

using which one easily checks that the following expressions represent conserved charges:

$$\begin{aligned} I_1 &= \int_\sigma \beta, \\ I_2 &= \int_\sigma \alpha, \\ I_3 &= \int_\sigma \alpha\beta, \\ I_4 &= \int_\sigma \alpha^2 + \alpha\beta, \\ I_5 &= \int_\sigma 3\alpha^2\beta + \alpha\beta^3, \\ I_6 &= \int_\sigma 2\alpha^3 + 6\alpha^2\beta^2 + \alpha\beta. \end{aligned} \tag{A3}$$

One immediately notices that all terms in each expression are of the same power in the fields Π and J'. After observing some other patterns above, one conjectures the following general expressions for an integral of motion:

$$I_n = \int_\sigma \sum_{k=1}^{n} \alpha^k \beta^{n-k} t_{n,k}, \tag{A4}$$

where the coefficients $t_{n,k}$ are constrained to satisfy certain conditions.

Indeed, let us show that these expressions represent an infinite number of conserving charges and find the coefficients explicitly. Hence, we consider time derivative

$$\begin{aligned} \dot{I}_n &= \int_\sigma \sum_{k=1}^{n} 2k\alpha^{k-1}\alpha'\beta^{n-2k+1} t_{n,k} + \sum_{k=2}^{n+1} 2(n-2k+2)\alpha^{k-1}\beta^{n-2k+1}\alpha' t_{n,k-1} \\ &\quad + \sum_{k=1}^{n} 2(n-2k)\alpha^k\beta^{n-2k}\beta' t_{n,k} \\ &= \int_\sigma \sum_{k=1}^{n} (2kt_{n,k} + 2(n-2k+2)t_{n,k-1})\alpha^{k-1}\alpha'\beta^{n-2k+1} - 2n\beta^{n-1}\alpha' t_{n,0} \\ &\quad + \sum_{k=1}^{n} 2(n-k)\alpha^k\beta^{n-2k}\beta' t_{n,k} - 2n\alpha^n\alpha'\beta^{-n-1} t_{n,n}. \end{aligned} \tag{A5}$$

Here, in the first line, we used the equations of motion for α and β and shifted the summation index in the second term, while in the second line, we added and subtracted the term with $k = 1$ needed to complete the second sum. Now, to form a full derivative and to make the additional terms vanish, one imposes the following conditions for the coefficients:

$$\begin{aligned} \frac{k-1}{n-2k+1} t_{n,k} &= \frac{n-2k+2}{k} t_{n,k-1}, \\ t_{n,0} &= 0, \\ t_{n,n} &= 0. \end{aligned} \tag{A6}$$

The second condition above ensures that all terms in the charges I_n always have at least one power of α as it can be explicitly seen from (A3), while the last condition removes terms of negative powers from (A5). In what follows, this will be extended to $\{t_{n,k} = 0 \text{ for all } 2k > n\}$.

The recurrence relations (A6) can be used to determine explicit expressions for the coefficients $t_{n,k}$ as follows:

$$\begin{aligned} t_{n,k} &= \frac{(n-2k+1)(n-2k+2)}{k(k-1)} t_{n,k-1} \\ &= \frac{(n-2k+1)(n-2k+2)(n-2k+3)(n-2k+4)\cdots(n-3)(n-2)}{k(k-1)(k-1)(k-2)\cdots 2\cdot 1} t_{n,1} \\ &= \frac{(n-2)!}{(n-2k)!k!(k-1)!} t_{n,1}. \end{aligned} \tag{A7}$$

Finally, using the freedom to choose the overall normalization of each of I_n, one is allowed to fix $t_{n,1} = 1$,

$$t_{n,k} = \frac{(n-2)!}{(n-2k)!k!(k-1)!}. \tag{A8}$$

Since this formula does not cover the case $n = 1$ and the cases $2k > n$, we set the additional constraints based on the explicit form of the integrals of motion (A3),

$$\begin{aligned} t_{1,1} &= 1, \\ t_{n,k} &= 0, \forall k > 2n. \end{aligned} \tag{A9}$$

APPENDIX B: POISSON BRACKETS

For proper integrable systems, one observes all integrals of motion in involution meaning that Poisson brackets $\{I_n, I_m\}$ vanishes for any I_n and I_m. Since the fundamental variables for our theory are $\Pi(T,\sigma)$ and $J(T,\sigma)$, the Poisson bracket is written as

$$\{F,G\} = \int_\sigma \frac{\delta F}{\delta J(T,\sigma)}\frac{\delta G}{\delta \Pi(T,\sigma)} - \frac{\delta G}{\delta J(T,\sigma)}\frac{\delta F}{\delta \Pi(T,\sigma)}, \tag{B1}$$

where F and G are some functionals in Π and J. In what follows, we will not mention dependence on T and σ for the sake of space.

Let us now show that $\{I_n, I_m\} = 0$ for any n, m and start with derivatives of I_n with respect to the fundamental variables. Denoting variation with respect to a function f by δ_f, we note the following:

$$\begin{aligned} \delta_J I_m &= -\partial_\sigma \delta_{J'} I_m, \\ \delta_{J'} I_m &= \delta_\alpha I_m \Pi + \delta_\beta I_m, \\ \delta_\Pi I_m &= \delta_\alpha I_m J' + \delta_\beta I_m. \end{aligned} \tag{B2}$$

Given these, we can write for the Poisson bracket

$$\begin{aligned} \{I_m, I_n\} &= 2\int_\sigma \delta_J I_{[m} \delta_\Pi I_{n]} = -2\int_\sigma \partial_\sigma\big(\delta_{J'} I_{[m}\big)\delta_\Pi I_{n]} \\ &= -2\int_\sigma \partial_\sigma\big(\delta_\alpha I_m \Pi + \delta_\beta I_m\big)\big(\delta_\alpha I_n J' + \delta_\beta I_n\big) \\ &= -2\int_\sigma \alpha(\delta_\alpha I_m)'\delta_\alpha I_n + \Pi'\delta_\alpha I_m \delta_\beta I_n + J'(\delta_\beta I_m)'\delta_\alpha I_n \\ &+ (\delta_\beta I_m)'\delta_\beta I_n + \Pi(\delta_\alpha I_m)'\delta_\beta I_n \\ &= -2\int_\sigma \big[\alpha(\delta_\alpha I_m)'\delta_\alpha I_n + \beta'\delta_\alpha I_m \delta_\beta I_n + \beta(\delta_\alpha I_m)'\delta_\beta I_n + (\delta_\beta I_m)'\delta_\beta I_n\big], \end{aligned} \tag{B3}$$

where antisymmetrization in $\{m, n\}$ is always undermined. Here, we used integration by parts and the antisymmetry properties to recollect terms with J, Π and their derivatives back into α and β.

Substituting the explicit form of the integrals I_m and introducing a new variable $\gamma = \alpha\beta^{-2}$ for convenience, the integrand of the above expression can be written as (all terms are antisymmetric in $\{m, n\}$)

$$\begin{aligned} &\sum_{k,l} t_{n,k}t_{m,l}\Big[kl\gamma^k\beta^n(\gamma^{l-1}\beta^{m-2})' + (m-2l)k\gamma^{k-1}\beta^{n-1}(\gamma^l\beta^{m-1})' \\ &\qquad +(n-2k)(m-2l)(\gamma^l\beta^{m-1})'\gamma^k\beta^{n-1}\Big] - (n \leftrightarrow m) \\ &= \sum_{k,l} t_{n,k}t_{m,l}\Big[kl\gamma^{k-1}\beta^{n-1}(\gamma^l\beta^{m-1})' + (m-2l)k\gamma^{k-1}\beta^{n-1}(\gamma^l\beta^{m-1})' \\ &\qquad +(n-2k)(m-2l)(\gamma^l\beta^{m-1})'\gamma^k\beta^{n-1}\Big] - (n \leftrightarrow m) \\ &= \sum_{k,l} t_{n,k}t_{m,l}\Big[(m-l)(m-1)k\gamma^{k+l-1} + (n-2k)(m-2l)(m-1)\gamma^{k+l}\Big]\beta^{n+m-3}\beta' \\ &\quad +t_{n,k}t_{m,l}\Big[(m-l)kl\gamma^{k+l-2} + l(n-2k)(m-2l)\gamma^{k+l-1}\Big]\gamma'\beta^{n+m-2} - (n \leftrightarrow m), \end{aligned} \tag{B4}$$

where in the second line, we the antisymmetry to shift powers of β and γ out of the derivative in the first term. Noticing that the power of γ in the second term in each line is just that of the first term shifted as $k \to k+1$, we can use the property of the Motzkin coefficients

$k(k-1)t_{n,k} = (n-2k+1)(n-2k+2)t_{n,k-1}$ to write the above expression as

$$\{I_n, I_m\} = \sum_{k,l} t_{n,k}t_{m,l}\Big[A_{n,m,k,l}\gamma^{k+l-2}\gamma'\beta^{n+m-2} + B_{n,m,k,l}\gamma^{k+l-1}\beta^{n+m-3}\beta'\Big] - (n \leftrightarrow m),$$

where

$$A_{n,m,k,l} = \frac{kl(m-l)(n+2k+1) + kl(k-1)(m-2l)}{n-2k+1},$$
$$B_{n,m,k,l} = \frac{k(m-l)(m-1)(n+2k+1) + k(k-1)(m-2l)(m-1)}{n-2k+1}. \tag{B5}$$

This sum does not form a full derivative term by term and one must turn to summation over $p = k + l$ to actually get cancellation. Hence, we write

$$\{I_n, I_m\} = \sum_{p=2}^{N}\sum_{k=1}^{p-1}\big[\hat{A}_{n,N,k,p}\gamma^{p-2}\gamma'\beta^{N-2} + \hat{B}_{n,N,k,p}\gamma^{p-1}\beta^{N-3}\beta'\big],$$

with

$$\hat{A}_{n,N,k,p} = A_{n,N-n,k,p-k} - A_{N-n,n,p-k,k},$$
$$\hat{B}_{n,N,k,p} = B_{n,N-n,k,p-k} - B_{N-n,n,p-k,k} \tag{B6}$$

and $N = n + m$. Although each term in the sum has now the same power of the variables γ and β, the full derivative can be obtained only after taking the summation along k explicitly. This is a tough calculational task, and it is much easier to check

$$\sum_{k=1}^{p-1}\frac{\hat{A}_{n,N,k,p}}{p-1} - \sum_{k=1}^{p-1}\frac{\hat{B}_{n,N,k,p}}{N-2} = 0. \tag{B7}$$

Indeed, using Wolfram Mathematica and performing the calculation explicitly, one gets the desired cancellation for any p, N, and n. Obviously, this ensures that the expression (B6) is, indeed, a full derivative, and hence, all the integrals of motion are in involution.

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