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Double-soft behavior of the dilaton of spontaneously broken conformal invariance

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Abstract

The Ward identities involving the currents associated to the spontaneously broken scale and special conformal transformations are derived and used to determine, through linear order in the two soft-dilaton momenta, the double-soft behavior of scattering amplitudes involving two soft dilatons and any number of other particles. It turns out that the double-soft behavior is equivalent to performing two single-soft limits one after the other. We confirm the new double-soft theorem perturbatively at tree-level in a D-dimensional conformal field theory model, as well as nonperturbatively by using the "gravity dual" of $\mathcal{N} = 4$ super Yang-Mills on the Coulomb branch; i.e. the Dirac-Born-Infeld action on $\mathrm{AdS}_5 \times S^5$.

Contents

1 Introduction

There are generically two main physical and observable consequences of theories with spontaneously broken continuous symmetries; namely i) the appearance of Nambu-Goldstone (NG) bosons and their dynamics, and ii) the existence of so-called *soft theorems*, which fix the behavior of scattering amplitudes when the momentum of one or more NG bosons goes to zero. They are direct consequences of the Ward identities of the theory.

There are, nevertheless, various important differences between spontaneously breaking an internal or a space-time symmetry. In the case of an internal symmetry, the number of NG bosons is equal to the number of broken generators, while in the case of a spontaneously broken space-time symmetry, the number of NG bosons is less [1]; for instance when conformal symmetry is spontaneously broken to Poincaré symmetry only one NG boson appears, although five generators corresponding to dilatations and special conformal transformations are broken [2, 3].

The two kinds of NG bosons also differ in their soft behavior: In the case of a spontanously broken internal symmetry, amplitudes involving the NG bosons vanish when the momentum of one of the NG bosons goes to zero. A famous example is the non-linear σ-model (NLSM) describing the low-energy behaviour of $SU(n) \times SU(n)$ theory spontaneously broken to the vectorial subgroup $SU(n)$. These zeroes are in the literature called Adler zeroes and their discovery, purely based on current algebra, dates back to the 1960s [4, 5, 6].

The situation is different for the NG boson of spontaneously broken conformal invariance, called *the dilaton* [1]. In this case the amplitude involving a number of dilatons together with other particles does not vanish when the momentum, q, of one dilaton goes to zero, but is fixed in terms of the amplitude without the soft dilaton; i.e. the dilaton has a nonvanishing soft theorem. Specifically, since the dilaton couples linearly to the trace of the energy-momentum tensor, it couples in particular linearly to the mass of any massive particles. Therefore there is a nonzero, and in fact divergent, universal contribution to the amplitude associated to the emission of a zero-momentum dilaton from any massive particle, in complete analogy to the emission of soft photons [18] and gravitons [19]. But moreover, it turns out that the regular part of the dilaton soft behavior at orders q^0 and q^1, which is not associated to emission from external legs, is nonzero and also fixed universally. This follows from the Ward identities of spontaneously broken conformal invariance [**?**, 15, 16, 17], and applies to both massive and massless particle interactions, as emphasized in Ref. [17]. This of course applies to conformal theories that are not anomalous [2], and this was in particular tested in the impressive work in Ref. [21] in the Coulomb branch of $\mathcal{N} = 4$ super Yang-Mills, both perturbatively through one-loop and non-perturbatively by considering the one-instanton effective action. In the same work

[1] It is difficult to give a historical account of this case, as the early literature from the 1960s on the subject goes in many directions, not immediately relevant to us. Let us mention, however, that to our knowledge G. Mack is the first that explicitly discusses the dilaton soft behavior and provides its leading soft theorem in Ref. [7], while its subleading behavior is implicit in the work by D. Gross and J. Wess in Ref. [8]. In these papers earlier references on the realization of conformal symmetry in nature is also given, among which the works of F. Gürsey [9], J. Wess [10], and H. Kastrup [11] were frequently cited as well as the early review by T. Fulton, F. Rohrlich and L. Witten [12]. The seminal papers by Callan [13], Coleman and Jackiw [14] diminished these works to some extend, as it was realized that conformal invariance is anomalous in the quantum theory, especially that of strong interactions. The dilaton has reappeared in a more modern context in phenomenological models for electroweak symmetry breaking, inflationary cosmology, as well as in condensed matter applications. The modern take on the dilaton soft theorems, especially in the context of the recent S-matrix program, were put forward recently in Refs. [15, 16, 17].

[2] In generic quantum field theory models of dilatons, the presence of the trace anomaly introduces a mass for the dilaton, which only in certain cases can be controllably small [20].

constraints on dilaton effective actions, new non-renormalization theorems, as well as a recursive proof of conformal invariance following scale invariance of amplitudes, were all given utilizing the soft dilaton theorem of Ref. [17].

In the case of NLSM-type theories also the double-soft behavior has been studied [6, 22, 23, 24, 25, 26]. In particular, it has been shown that the amplitude for the emission of any even number of NG bosons does no longer vanish when the momenta of two of them go simultaneously to zero. Instead, it is fixed in terms of the amplitude without the two NG bosons with vanishing momenta and of the structure constants of the group in consideration.

In this work we detail the main physical consequences of a spontaneously broken conformal theory. While property i) has been studied intensively in the literature, little attention has been given to property ii), and this is the main purpose of this work. Our main new result is the derivation of the double-soft theorem for the NG boson of spontaneously broken conformal symmetry, i.e. *the dilaton.* [3]

We prove that the soft theorem factorizes any amplitude involving two soft dilatons through subleading order in the two soft momenta. We see that, also in this case, the double-soft behavior of the dilaton differs from that of the NG bosons of a spontaneously broken internal symmetry. It turns out that the double-soft behavior of the dilaton, obtained from the Ward identities for the scale and special conformal transformations, is equivalent to the one obtained by making two single-soft limits one after the other. This particular form of the double-soft theorem allows us additionally to conjecture an any-multiplicity soft dilaton theorem.

The paper is organized as follows. In Sections 2 and 3 we summarize some general properties of conformal field theories in D-dimensional space-time. In Sect . 4 we discuss the Ward identities that follow from the conservation of the Nöther currents associated with the scale and special conformal transformations. Then, in a first subsection, we derive their implication for the scattering amplitude involving a single current and an arbitrary number of other states, while, in the two subsequent subsections, we specify our analysis first to the current corresponding to a scale transformation and then to that corresponding to a special conformal transformation. Sect. 5 is devoted to the case of the Ward identities for amplitudes involving two Nöther currents. Then, in the first subsection, we discuss the case of two dilatation currents, and in the second subsection, the case of one dilatation

[3] The subject of single-, double-, and multi-soft theorems has received much interest in recent years due to their proposed relations with asymptotic symmetries put forward recently by A. Strominger [27, 28], and many papers following. (In particular, the relation between the soft dilaton and asymptotic symmetries was recently discussed in Ref. [29].) This has additionally lead to the discovery of many new soft theorems in both field and string theory, and has lead to new developments in the context of the S-matrix and effective field theory programs, where of particular relevance we should point out Ref. [30, 21]. A comprehensive list of references to this literature can be found in Strominger's recent lecture notes in Ref. [31].

current and one current associated to a special conformal transformation. In Sect. 6 we show that the double-soft behavior, derived in Sect. 5) from the Ward identities, can be equivalently obtained by performing two consecutive soft limits, one after the other, and we conjecture that the same behavior is also valid in the case of a multi-soft limit. In Sect. 7 we check the previously derived double-soft behavior with specific amplitudes of a D-dimensional conformal scalar theory that has been recently studied in the literature and of the "gravity dual" of $\mathcal{N} = 4$ super Yang-Mills on the Coulomb branch. Finally, in the Appendix we give some detail on the calculation of the soft behavior in the "gravity dual" of $\mathcal{N} = 4$ super Yang-Mills on the Coulomb branch.

We would like to add a note of caution: The dilaton discussed in this paper should not be confused with the 'gravity dilaton' appearing in the literature on theories of (super)gravity and string theory, where it is parametrizing the spin zero mode of the gravitational field. This gravity dilaton also has a soft theorem similar to the NG dilaton discussed in this work, which was first studied in Ref. [32, 33], where its leading behavior was determined, and recently its subleading behavior was also shown to be fixed [17, 34, 35, 36]. But the two soft theorems are not equal, although very similar [17], and there is still a lack of rigorous understanding of the relation between the two.

2 Prelude

To set our notations, we start by briefly reviewing aspects of conformal symmetry and its representations in field theory. For more details, we refer to the seminal works in Ref. [37, 38] as well as the textbook in Ref. [39].

The conformal group is the group that leaves the metric invariant up to a scale $g_{\mu\nu}(x) \to \Lambda(x) g_{\mu\nu}(x)$ and can be considered an extension by dilatations and special conformal transformations of the Poincaré group, which belong to $\Lambda(x) = 1$. The group is locally isomorphic to $SO(D, 2)$, where by D we denote the number of space-time dimensions. Infinitesimally, the group transforms space-time coordinates and fields as follows

$$\begin{aligned} x^\mu \to x'^\mu &= x^\mu + \epsilon^{MN} f^\mu_{MN}(x) \\ \Phi(x) \to \Phi'(x) &= \Phi(x) + i\epsilon^{MN}\Gamma_{MN}(x)\Phi(x) \end{aligned} \tag{2.1}$$

where ϵ_{MN} are infinitesimal parameters and f^μ_{MN} are functions obeying

$$\partial^\mu f^\nu_{MN} + \partial^\nu f^\mu_{MN} = \frac{2}{D} g^{\mu\nu}\partial_\rho f^\rho_{MN} \,. \tag{2.2}$$

Γ^{MN} are the $(D + 1)(D + 2)/2$ conformal generators, so that Γ^{MN} is imaginary and antisymmetric in $M, N = 0, \ldots, D + 1$. We consider in this work the flat space limit and take $g^{\mu\nu} \to \eta^{\mu\nu} = \text{diag}(-1, +1, \ldots)$, and $D > 2$.

It is useful to decompose the conformal transformations and generators into translations, Lorentz transformations, dilatations and special conformal transformations. First consider the solutions for f_{MN}:

$$
\begin{aligned}
f^{\rho}_{D,\mu}(x) &= \eta^{\rho}_{\mu}\,, \qquad f^{\rho}_{\mu\nu}(x) = \eta^{\rho}_{\mu} x_{\nu} - \eta^{\rho}_{\nu} x_{\mu}\,, \\
f^{\rho}_{D+1,D}(x) &= x^{\rho}\,, \quad f^{\rho}_{D+1,\mu}(x) = 2x_{\mu}x^{\rho} - \eta^{\rho}_{\mu} x^2
\end{aligned}
\tag{2.3}
$$

where, $\mu, \nu, \rho = 0, \ldots, D-1$ are the space-time indices. The corresponding generators read:

$$
\begin{aligned}
\Gamma^{D,\mu} &= \mathcal{P}^{\mu} = i\partial^{\mu}\,, & \Gamma^{\mu\nu} &= \mathcal{J}^{\mu\nu} = -i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}) - \mathcal{S}^{\mu\nu}\,, \\
\Gamma^{D+1,D} &= \mathcal{D} = i(d_{\Phi} + x_{\mu}\partial^{\mu})\,, & \Gamma^{D+1,\mu} &= \mathcal{K}^{\mu} = i(2x^{\mu}x_{\nu}\partial^{\nu} - x^2\partial^{\mu} + 2d_{\Phi}x^{\mu}) + 2x_{\nu}\mathcal{S}^{\mu\nu}\,,
\end{aligned}
\tag{2.4}
$$

where $\mathcal{P}^{\mu}$ are the generators of translation, $\mathcal{J}^{\mu\nu}$ are the generators of Lorentz transformations with $\mathcal{S}^{\mu\nu}$ corresponding to the spin angular momentum operator, $\mathcal{D}$ is the generator of dilatation, and $\mathcal{K}^{\mu}$ are the generators of special conformal transformation. The coefficient d_{Φ} denotes the scaling dimension of the field Φ. The generators obey the commutation relations:

$$
\begin{aligned}
[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] &= i(\eta^{\mu\rho}\mathcal{J}^{\nu\sigma} + \eta^{\nu\sigma}\mathcal{J}^{\mu\rho} - \eta^{\mu\sigma}\mathcal{J}^{\nu\rho} - \eta^{\nu\rho}\mathcal{J}^{\mu\sigma}) \\
[\mathcal{P}^{\rho}, \mathcal{J}^{\mu\nu}] &= i(\eta^{\rho\nu}\mathcal{P}^{\mu} - \eta^{\rho\mu}\mathcal{P}^{\nu}) \\
[\mathcal{K}^{\rho}, \mathcal{J}^{\mu\nu}] &= i(\eta^{\rho\nu}\mathcal{K}^{\mu} - \eta^{\rho\mu}\mathcal{K}^{\nu}) \\
[\mathcal{K}^{\mu}, \mathcal{P}^{\nu}] &= 2i(\mathcal{J}^{\mu\nu} - \eta^{\mu\nu}\mathcal{D}) \\
[\mathcal{D}, \mathcal{P}^{\mu}] &= -i\mathcal{P}^{\mu} \\
[\mathcal{D}, \mathcal{K}^{\mu}] &= i\mathcal{K}^{\mu}\,,
\end{aligned}
\tag{2.5}
$$

with all other commutators vanishing.

The currents associated to the conformal generators can be constructed by varying the conformal invariant action as in Eq. (2.1) assuming that the infinitesimal parameters ϵ_{MN} are arbitrary functions of x. In this way, for the conformal group, one can get:

$$
\delta S = \int d^D x\, \epsilon^{MN}(x)(\partial_{\mu} J^{\mu}_{MN}) = \int d^D x\, \epsilon^{MN}(x)\partial^{\mu}\left(f^{\nu}_{MN} T_{\mu\nu}\right)
\tag{2.6}
$$

where $T_{\mu\nu}$ is the improved energy-momentum tensor. Using Eq. (2.2), it turns out that the Nöther current is conserved if the improved energy-momentum tensor is conserved and traceless:

$$
\partial^{\nu} T_{\mu\nu} = T^{\mu}_{\mu} = 0
\tag{2.7}
$$

when the classical equations of motion are satisfied. It is easy to see by Eq. (2.3) that the currents $J^{\rho}_{\mu\nu}$ and $J^{\rho}_{D,\mu}$ are conserved independently of the zero-trace condition $T^{\mu}_{\mu} = 0$, since $\partial_\rho f^\rho_{\mu\nu} = \partial_\rho f^\rho_{D,\mu} = 0$. The currents $J^\rho_{D+1,\mu}$ and $J^\rho_{D+1,D}$, on the other hand, are only conserved if $T^\mu_\mu = 0$. Specifically,

$$J^\mu_{D+1,D} = J^\mu_{\mathcal{D}} = x_\nu T^{\mu\nu} \,, \qquad J^\mu_{D+1,\rho} = J^\mu_{\mathcal{K},\rho} = (2x_\nu x_\rho - \eta_{\rho\nu} x^2) T^{\mu\nu} \tag{2.8a}$$

$$\partial_\mu J^\mu_{\mathcal{D}} = T^\mu_\mu \,, \qquad \partial_\mu J^\mu_{\mathcal{K},\rho} = 2x_\rho T^\mu_\mu \tag{2.8b}$$

where we stress once more that $T^{\mu\nu}$ is the improved energy-momentum tensor.

3 Hidden conformal symmetry

We consider the situation where conformal symmetry of some underlying conformal field theory is spontaneously broken due to a Lorentz scalar primary operator getting a nonzero vacuum expectation value (vev), i.e.

$$\langle 0 | \mathcal{O}_{\text{scalar}} | 0 \rangle = v^{d_{\mathcal{O}}} \,, \tag{3.1}$$

where $d_{\mathcal{O}}$ is the scaling dimension of $\mathcal{O}$ so that v has mass dimension one. The vev v is the only mass scale of the broken theory. The vacuum remains invariant under Lorentz transformations and translations, but dilatations and special conformal transformations are then no longer symmetries of the vacuum.

It follows from Goldstone's theorem [2, 3] that a massless scalar state of conformal dimension one (for $D = 4$) appears in the spectrum of the broken theory, parametrizing the massless excitations of the vacuum generated by the broken symmetry currents. This Nambu-Goldstone (NG) boson of spontaneously broken scale invariance is also known as *the dilaton.*[4]

As a consequence, the dilaton couples linearly to the energy-momentum tensor

$$T_{\mu\nu} = -f_\xi \left(\frac{\eta_{\mu\nu}\partial^2 - \partial_\mu \partial_\nu}{D-1} \right) \xi(x) + \cdots \,, \tag{3.2}$$

where $\xi(x)$ parametrizes the dilaton field and f_ξ is a dimensionful constant, thus related to v, which can be thought of as the dilaton decay constant. The ellipsis $\cdots$ denote term of higher order in the fields, i.e. the dilaton is the only field that couples linearly to the energy-momentum tensor.

Taking the trace of the above expression and imposing the equation of motions leads to the expression

$$T^\mu_\mu = f_\xi(-\partial^2 \xi) \,, \tag{3.3}$$

[4]Although $D + 1$ generators are broken, only one NG boson appears. This mismatch of degrees of freedom is a consequence of space-time symmetries being broken, as opposed to when global continuous internal symmetries are broken [3].

which is exact on the equations of motion. As expected, the trace of the energy-momentum no longer vanishes in the broken phase. Instead it is simply parametrized by the equation of motion of the dilaton field. The statement can also be reversed; the dilaton equation of motion is described by the trace of the energy-momentum tensor.

To better appreciate the latter statement, and to also comment on the occurrence of the trace anomaly in generic quantum field theories, let us consider a generic renormalized action in D-dimensions. It can be described in a basis of eigenoperators Ψ_i of (renormalized) scaling dimension d_i as follows:

$$S_0(\mu) = \int d^D x \sum_i g_i(\mu) \Psi_i(x) \,, \tag{3.4}$$

where $g_i(\mu)$ are renormalized coupling constants at a renormalization scale μ.

The change of the action under dilatations yields by definition the trace of the energy-momentum tensor, as can be verified from Eq. (2.6) using Eq. (2.3). Specifically, taking $x'_\mu = e^\lambda x_\mu \approx x_\mu + \lambda\, x_\mu$, yields for any action

$$\delta S = \lambda \int d^D x\, T^\mu_\mu(x) \,. \tag{3.5}$$

The explicit variation of the action S_0 is, on the other hand, readily derived by making a scale transformation of the scale $\mu' = \mu e^{-\lambda}$, as well as of the (scalar) operators $\Psi'_i(x) = e^{-\lambda d_i} \Psi_i(\mathrm{e}^{-\lambda} x)$. Then for infinitesimal transformations, we find at linear order in λ:

$$\delta S_0 = -\lambda \int d^D x \sum_i \left((d_i - D) g_i(\mu) + \mu \frac{\partial g_i}{\partial \mu} \right) \Psi_i(x) \equiv \lambda \int d^D x\, T_0{}^\mu_\mu(x) \,, \tag{3.6}$$

where in the second equality we identified $T_0{}^\mu_\mu$. For marginal operators where the scaling dimension $d_i = D$, the first term vanishes, but the β-functions, $\beta_i(g) = \mu \partial g_i / \partial \mu$, for the corresponding coupling constants still contribute to the trace. This is the consequence of the trace-anomaly for general quantum field theories. In a theory with only marginal operators and where the couplings remain unrenormalized, the trace anomaly vanishes. An example is $\mathcal{N} = 4$ super Yang-Mills theory.

Let us now connect this to our previous expressions for a spontaneously broken conformal theory. It is possible to render the action scale invariant by introducing a conformal compensator field [40], $\bar{\xi}$, with canonical kinetic term and free-field scaling dimension $d = (D-2)/2$ by the following formal replacement:

$$g_i(\mu) \Psi_i(x) \to g_i \left(\frac{\mu\, v}{\bar{\xi}^{\frac{1}{d}}(x)} \right) \left(\frac{\bar{\xi}(x)}{v^d} \right)^{\frac{D-d_i}{d}} \Psi_i(x) \tag{3.7}$$

yielding the Lagrangian

$$\mathcal{L}_0(\mu) \to \mathcal{L}(\mu) = -\frac{1}{2}\partial_\nu\bar{\xi}\partial^\nu\bar{\xi} + \sum_i g_i\left(\frac{\mu\, v}{\bar{\xi}^{\frac{1}{d}}(x)}\right)\left(\frac{\bar{\xi}(x)}{v^d}\right)^{\frac{D-d_i}{d}}\Psi_i(x) \tag{3.8}$$

It is easy to check that under the transformations

$$\bar{\xi}(x) \to \mathrm{e}^{-\lambda d}\bar{\xi}(\mathrm{e}^{-\lambda}x) \;\; ; \;\; \Psi_i(x) \to \mathrm{e}^{-\lambda d_i}\Psi_i(\mathrm{e}^{-\lambda}x) \;\; ; \;\; \mu \to \mathrm{e}^{-\lambda}\mu \; , \tag{3.9}$$

the action, corresponding to the Lagrangian in Eq. (3.8), is left invariant. The introduction of the field dependent coupling constants is a formal trick and should be understood as an expansion in the shifted (dilaton) field $\bar{\xi} = v^d + \xi$, which is well-defined only in the broken phase, i.e.

$$g_i\left(\frac{\mu\, v}{\bar{\xi}^{\frac{1}{d}}(x)}\right) = g_i(\mu) - \frac{\xi(x)}{dv^d}\;\mu\frac{\partial g_i}{\partial\mu} + \cdots \tag{3.10}$$

Alternatively, the formal replacement can also be understood through a nonlinear realization, by the replacement of the field $\bar{\xi}(x) = v^d e^{\sigma(x)/v^d}$.

Now it can be checked that the renormalized low-energy action of the broken phase, where $\xi \ll v^d$, is given by

$$S(\mu) = S_0(\mu) + \int d^Dx\left(-\frac{1}{2}\partial_\mu\xi\partial^\mu\xi + \frac{1}{d}\frac{\xi}{v^d}T_0{}^\mu_\mu + \cdots\right) \tag{3.11}$$

where $T_0{}^\mu_\mu$ was defined in Eq. (3.6) and the ellipses $\cdots$ stand for terms of higher order in ξ/v_d. Finally, we see that the equation of motion of the dilaton ξ is given by:

$$d\,v^d(-\partial^2\xi) = T_0{}^\mu_\mu + \cdots \tag{3.12}$$

This is equivalent to the general expression in Eq. (3.3), with the identification of the decay constant $f_\xi = dv^d$. We furthermore learn that this expression contains the effects of renormalization, or, in other words, of the trace anomaly of the theory without the dilaton. The low-energy effective action of the dilaton can also be obtained by integrating out the massive fields in the broken phase, and can be constructed using anomaly matching considerations, put forward in Ref. [41], and since studied in the recent a-theorem literature [42, 43, 44].

The simplest example of the above construction is to consider a free massive scalar field in D dimensions. Its Lagrangian reads:

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{m^2}{2}\chi^2 \tag{3.13}$$

with $[\chi] = d = (D-2)/2$. Introducing the conformal compensator, and defining a dimensionless coupling constant λ through the relation $m^2 = v^2\lambda^{2/d}$, the resulting theory reads:

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\chi)^2 - \frac{1}{2}(\partial_\mu\bar{\xi})^2 - \frac{v^2}{2}\left(\frac{\lambda\bar{\xi}}{v^d}\right)^{2/d}\chi^2 \tag{3.14}$$

This is a classically scale invariant theory in D dimensions. As we have argued, it stays scale invariant in the renormalized theory by substituting $\lambda(\mu) \to \lambda(\mu v/\bar{\xi}^{1/d}(x))$. This model has been considered in recent works [45, 46, 47, 15], where its validity as a quantum conformal theory has been studied (see also the early related work [48]). We will later come back to this model for computing tree-level scattering amplitudes of the theory, and thus only its classical scale invariance is of importance to us.

4 Current algebra and soft theorems from Ward identities

An observable consequence in scattering processes of spontaneously broken symmetries is the so-called *soft theorems*. These are identities relating S-matrices with NG bosons to S-matrices without the NG bosons, and they exist as a consequence of the Ward identities of the broken symmetry currents.

In this section we detail the relationship between Ward identities of the broken conformal currents and their implications on scattering amplitudes, leading to soft theorems for the dilaton. The main observation of the previous section that we will draw on, is that, by Eq. (2.8b) and Eq. (3.3), the divergence of the broken conformal currents are parametrized solely by the equation of motion of the dilaton, i.e.

$$\partial_\mu J^\mu_{\mathcal{D}} = f_\xi(-\partial^2\xi)\,, \qquad \partial_\mu J^\mu_{\mathcal{K},\rho} = 2f_\xi x_\rho(-\partial^2\xi)\,. \tag{4.1}$$

The dilatation current is broken by a dimension $d+2$ operator, while the special conformal transformation currents are broken by dimension $d+1$ operators, where $d = [\xi] = (D-2)/2$. In both cases, the dimensions are below the space-time dimension $D = 2d+2$, and the currents can thus be considered *partially conserved* [49]. It is due to this that we can proceed and formulate a current algebra for the spontaneously broken symmetries in analogy to the PCAC method.

The main object one must study to derive low energy theorems is the matrix element

$$T^*\langle 0|J_1^{\mu_1}(y_1)\cdots J_m^{\mu_m}(y_m)\phi_1(x_1)\cdots\phi_n(x_n)|0\rangle \tag{4.2}$$

where J_i represent some broken symmetry currents, ϕ_i are generic fields with scaling dimension d_i, which for simplicity will all be taken to be scalars, and T^* denotes the

Lorentz covariantized time-ordered product, which for our concern implies that derivatives act outside of the time ordering. This is a modified definition of the usual T-product, which importantly leads to the removal of the so-called *Schwinger terms* in the Ward identities, when the currents are partially conserved [50, 49].

It will be useful to define the Fourier transformed field operators:

$$\begin{aligned} \tilde{J}_i^\mu(q) &= \int d^D x \; e^{-iq\cdot x} J_i^\mu(x)\,, \quad \text{for } i = 1, \ldots, m \\ \tilde{\phi}_i(k_i) &= \int d^D x \; e^{-ik_i\cdot x_i} \phi_i(x_i)\,, \quad \text{for } i = 1, \ldots, n \end{aligned} \tag{4.3}$$

It is also useful to remember that the charge associated to a current is given by:

$$Q_i = \int d^{D-1}\mathbf{x} \; J_i^0(t, \mathbf{x}) = \int d^D x \; J_i^0(x)\delta(x^0 - t) \tag{4.4}$$

and that the infinitesimal symmetry transformation of a field associated with a conserved current is given by the equal-time commutator:

$$\delta_i\phi(x) = i[Q_i, \phi(x)] = i\int d^D y \; [J_i^0(y), \phi(x)]\delta(x^0 - y^0) \tag{4.5}$$

By Eqs. (2.1) and (4.5) the charges and the generators are simply related by:

$$[Q_i, \phi(x)] = \Gamma_i(x)\phi(x)\,. \tag{4.6}$$

A basic assumption for current algebra is that we can make use of the following distributional identification even for partially conserved currents:

$$[J_i^0(y), \phi(x)]\delta(x^0 - y^0) := [Q_i, \phi(x)]\delta^{(D)}(x - y) \tag{4.7}$$

This identity assumes that there are no boundary terms that vanish upon integrations, which are the would-be Schwinger terms. We assume that such terms vanish, as is generally true for partially conserved currents when considered in T^*-ordered correlation functions.

This assumption becomes useful when considering the derivative of the matrix-element with respect to the current coordinates. For instance:

$$\partial_\mu^x T^*\langle 0|J_1^\mu(x)\Phi(y)|0\rangle = T^*\langle 0|\partial \cdot J_1(x)\Phi(y)|0\rangle + \delta^{(D)}(x - y)\langle 0|[Q_1, \Phi(y)]|0\rangle \tag{4.8}$$

where the second term on the right-hand side arises from taking the derivative of the step-function θ, and we used the distributional identity Eq. (4.7).

In the following subsections we study the Ward identity implications for the case of spontaneously broken symmetries, and in particular the specific cases of theories with broken dilatation and special conformal transformations.

4.1 Single-soft Ward identity: General treatment

Considering one derivative acting on the matrix element of the T^*-ordered product of operators with one current insertion, we get:

$$\begin{aligned}&\partial_\mu^x T^*\langle 0|J^\mu(x)\phi(x_1)\cdots\phi(x_n)|0\rangle \\ &= T^*\langle 0|\partial_\mu J^\mu(x)\phi(x_1)\cdots\phi(x_n)|0\rangle - i\sum_{i=1}^n \delta^{(D)}(x-x_i)T^*\langle 0|\phi(x_1)\cdots\delta\phi(x_i)\cdots\phi(x_n)|0\rangle\end{aligned} \tag{4.9}$$

If J^μ parametrizes an unbroken symmetry, its divergence vanishes, and thus the first term on the right-hand side is zero, leading to the usual Ward identity of conserved currents.

If the symmetry is, on the other hand, spontaneously broken one may proceed in two different ways: One can either define and work with a new current, whose divergence also vanishes (as we will briefly explain below) or parametrize the divergence of the current in terms of the associated NG boson. In these notes we are taking the latter approach.

Due to Eq. (4.1), we assume the divergence of the current to be parametrized in terms of the NG boson, ξ, i.e. the dilaton, as follows:

$$\partial_\mu J^\mu(x) = g_J(x)(-\partial^2)\xi(x) \tag{4.10}$$

where g_J is some function that is determined by the broken symmetry current, J. The function g_J may at most be of dimension less than D to obey partial conservation. Furthermore, if g_J satisfies $\partial^2 g_J = 0$, which is the case for dilatations and special conformal transformations, then, as mentioned before, we may define a new conserved current $j^\mu(x) = J^\mu + g_J(x)(\partial^\mu\xi) - (\partial^\mu g_J)\xi(x)$, and work with this instead, by standard methods.

Now, let us consider Eq. (4.9) for small transferred momentum of the current. We will in general assume that the Fourier transform of the correlation functions in Eq. (4.2) have no pole in the momentum variables of the current. This implies that the left-hand side of Eq. (4.9) vanishes in the soft limit of transferred momentum, i.e.

$$iq_\mu\langle 0|\tilde{J}^\mu(q)\phi(x_1)\cdots\phi(x_n)|0\rangle = 0 + \mathcal{O}(q) \tag{4.11}$$

This leads to what we call the *single-soft Ward identity*:

$$\begin{aligned}&\int d^Dx\, \mathrm{e}^{-iqx}T^*\langle 0|\partial_\mu J^\mu(x)\phi(x_1)\cdots\phi(x_n)|0\rangle \\ &= -\sum_{i=1}^n \mathrm{e}^{-iqx_i}T^*\langle 0|\phi(x_1)\cdots\Gamma_J(x_i)\phi(x_i)\cdots\phi(x_n)|0\rangle + \mathcal{O}(q)\end{aligned} \tag{4.12}$$

where we used the relation in Eq. (4.6) between the charge commutator and the infinitesimal generators, which, on more general ground, may induce a linear combination of other fields, and this should be understood implicitly.

The Fourier transform on all variables of Eq. (4.12) leads to the momentum space version of the single-soft Ward identity:

$$\tilde{g}_J(q)\left(q^2\langle\tilde{\xi}(q)\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle\right) = -\sum_{i=1}^{n}\tilde{\Gamma}_J(k_i+q)\langle\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_i+q)\cdots\tilde{\phi}(k_n)\rangle + \mathcal{O}(q) \tag{4.13}$$

where by $\langle\cdots\rangle$ we denote the Fourier transform of the T^*-ordered matrix element, and we remember that the Fourier transform of functions of x are operators in the dual momentum space. Amputating the correlation functions reduces the expression further. The amputated correlation function is defined as:

$$\langle\tilde{\phi}_1\cdots\tilde{\phi}_n\rangle_{\mathrm{amp}} = \frac{\langle\tilde{\phi}_1\cdots\tilde{\phi}_n\rangle}{\langle\tilde{\phi}_1\tilde{\phi}_1\rangle\cdots\langle\tilde{\phi}_n\tilde{\phi}_n\rangle}\,. \tag{4.14}$$

The two-point correlator (the propagator) of a scalar field reads:

$$\Delta_i(k) \equiv \langle\tilde{\phi}_i(k)\tilde{\phi}_i(k)\rangle = \left(\frac{(-i)}{k^2+m^2}\right)^{\frac{D}{2}-d_i} \tag{4.15}$$

where m is the mass of the scalar field, d_i is its scaling dimension, D is the number of space-time dimensions, and we defined Δ_i.

Since ξ is massless and $[\xi]=(D-2)/2$, it follows from Eq. (4.13) that

$$\begin{aligned} & i\prod_{i=1}^{n}\Delta_i(k_i)\,\tilde{g}_J(q)\langle\tilde{\xi}(q)\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}} \\ & = \prod_{j\neq i}^{n}\Delta_j(k_j)\sum_{i=1}^{n}\tilde{\Gamma}_J(k_i+q)\Delta_i(k_i+q)\langle\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_i+q)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}} + \mathcal{O}(q) \end{aligned} \tag{4.16}$$

It is useful to define the commutator of $\tilde{\Gamma}_J$ with the propagator as the propagator multiplying a new operator $\tilde{F}_J(k_i+q,m_i)$, i.e.:

$$\left[\tilde{\Gamma}_J(k_i+q)\,,\;\Delta_i(k_i+q)\right] = \Delta_i(k_i+q)\tilde{F}_J(k_i+q,m_i) \tag{4.17}$$

This allows us to finally write the soft Ward identity as an identity among amputated correlators:

$$\begin{aligned} & i\tilde{g}_J(q)\langle\tilde{\xi}(q)\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}} \\ & = \sum_{i=1}^{n}\left(\tilde{F}_J(k_i+q,m_i)+\tilde{\Gamma}_J(k_i+q)\right)\langle\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_i+q)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}} + \mathcal{O}(q) \end{aligned} \tag{4.18}$$

It shows explicitly the relation between the correlation functions with a soft Nambu-Goldstone boson inserted and the correlation functions without the Nambu-Goldstone boson. The expansions in q of the right-hand side should be considered with care.

We can proceed further and derive the consequences of this identity on amplitudes. According to LSZ reduction, the amplitude is the on-shell residue of correlation functions in Fourier space, or equivalently in terms of the amputated correlation functions it is the on-shell T-matrix element of those functions. Therefore the previous expression yields the relation:

$$\begin{aligned} & i\tilde{g}_J(q)\delta^{(D)}(\textstyle\sum_i k_i + q)T_{n+1}(q; k_1, \ldots, k_n) \\ &= \sum_{i=1}^{n} \left(\tilde{F}_J(k_i + q, m_i) + \tilde{\Gamma}_J(k_i + q)\right) \delta^{(D)}(\textstyle\sum_i k_i + q) T_n(k_1, \ldots, k_i + q, \ldots k_n) + \mathcal{O}(q) \end{aligned} \tag{4.19}$$

To remove the delta functions on both sides, we need to commute the momentum-conserving delta-functions through the $\tilde{F}_J$ and $\tilde{\Gamma}_J$ operators. We assume that this commutator is a function multiplying the delta-function over the momenta and we thus define:

$$\begin{aligned} \left[\tilde{F}_J(k_i + q, m_i)\,, \delta^{(D)}(\textstyle\sum_i k_i + q)\right] &= f_J(k_i + q, m_i)\delta^{(D)}(\textstyle\sum_i k_i + q) \\ \left[\tilde{\Gamma}_J(k_i + q)\,, \delta^{(D)}(\textstyle\sum_i k_i + q)\right] &= \gamma_J(k_i + q)\delta^{(D)}(\textstyle\sum_i k_i + q) \end{aligned} \tag{4.20}$$

In the next sections we will see that this assumption is satisfied in the case of the scale and special conformal transformations.

The soft-identity for amplitudes now reads:

$$\begin{aligned} i\tilde{g}_J(q)T_{n+1}(q; k_1, \ldots, k_n) = \sum_{i=1}^{n} \Big[& f_J(k_i + q, m_i) + \gamma_J(k_i + q) + \tilde{F}_J(k_i + q, m_i) \\ & + \tilde{\Gamma}_J(k_i + q)\Big] T_n(k_1, \ldots, k_i + q, \ldots k_n) + \mathcal{O}(q) \end{aligned} \tag{4.21}$$

where momentum conservation is implicit on both sides. More precisely, the identity holds once momentum conservation is imposed to fix the same momentum on both sides of the equation. To make this statement explicit in our expression, we introduce the notation for the nth momentum:

$$\bar{k}_n = -\sum_{i=1}^{n-1} k_i - q \tag{4.22}$$

meaning that one hard momentum is kept fixed. The expansion in q on both sides should be done carefully, once the functions and operators are specified. Whether this leads to a soft theorem depends on the Fourier transform of g_J which may be an operator valued function acting on the dual momentum variables.

4.2 Soft Ward identity of the dilatation current, $J^\mu_{\mathcal{D}}$

We consider the construction in the previous section for the specific case of dilatations $J^\mu = J^\mu_{\mathcal{D}}$ and $\Gamma_{\mathcal{D}} = \mathcal{D}$. Following the notation of Sec. 2 we have

$$\mathcal{D}_\varphi(x)\varphi(x) = i(d_\varphi + x^\mu\partial_\mu)\varphi(x)\,, \quad \partial_\mu J^\mu_{\mathcal{D}} = T^\mu_\mu = f_\xi(-\partial^2)\xi\,, \tag{4.23}$$

where φ is any field and d_φ is its scaling dimension. This defines $g_{\mathcal{D}}(x) = f_\xi$. The Fourier transforms are:

$$\tilde{g}_{\mathcal{D}} = f_\xi\,, \quad \tilde{\mathcal{D}}_i(k) = i(d_i - D - k\cdot\partial_k) \tag{4.24}$$

and the commutator with the scalar propagator reads:

$$\left[\tilde{\mathcal{D}}_i(k_i+q)\,,\,\Delta_i(k_i+q)\right] = \Delta_i(k_i+q)\left[i(D-2d_i)\left(1-\frac{m_i^2}{(k_i+q)^2+m_i^2}\right)\right] \tag{4.25}$$

This defines the operator $\tilde{F}_i(k_i+q,m_i)$, which is simply a function because $\mathcal{D}_i$ is a linear operator. In the massless case it is simply a number $\tilde{F}_i(k_i+q,0) = i(D-2d_i)$. The term $m_i^2/((k_i+q)^2+m_i^2)$ should not be expanded in small q, since it then blows up on shell, where $k_i^2 = -m_i^2$. Instead, as explained in Ref. [17], these terms should be kept through LSZ reduction, and taken on-shell yielding $m_i^2/(2k_i\cdot q)$. It was then shown in Ref. [17] that this procedure reproduces the correct mass-dependence of amplitude in the soft limit. For the sake of simplicity, we will here, and throughout this work, neglect such 'Laurent' terms in the soft expansion and only focus on the 'Taylor' terms. To be precise, we define

$$\tilde{F}_i^{\mathrm{T}}(k_i+q,m_i) = \tilde{F}_i(k_i+q,m_i) - \tilde{F}_i^{\mathrm{L}}(k_i+q,m_i) \tag{4.26}$$

where $\tilde{F}_i^{\mathrm{L}}$ is the part of $\tilde{F}_i$ which on-shell has all the soft momentum poles of the form

$$\tilde{F}_i^{\mathrm{L}} \sim \sum_{n=1}^{\infty}\frac{L_n}{(k_i\cdot q)^n} \tag{4.27}$$

and thus $\tilde{F}_i^{\mathrm{T}}$ represents the part of $\tilde{F}_i$ which has a well-defined Taylor expansion on-shell.

We now have all the ingredients to write down the soft Ward identity. Considering for simplicity only the *finite* parts of the soft limit as just described, i.e. neglecting parts belonging to the Laurent expansion, we get from Eq. (4.18) for $\tilde{F}_i \to \tilde{F}_i^{\mathrm{T}}$:

$$\begin{aligned}
&if_\xi\langle\tilde{\xi}(q)\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}} \\
&= i\sum_{i=1}^{n}\left(D-2d_i+(d_i-D-(k_i+q)\cdot\partial_{k_i})\right)\langle\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_i+q)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}}+\mathcal{O}(q) \\
&= i\sum_{i=1}^{n}\left(-d_i-k_i\cdot\partial_{k_i}\right)\langle\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_i+q)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}}+\mathcal{O}(q)
\end{aligned} \tag{4.28}$$

Using furthermore the commutation relation:

$$\left[\sum_{i=1}^{n} k_i \cdot \partial_{k_i}\,, \delta^{(D)}(\textstyle\sum_i k_i + q)\right] = -D\delta^{(D)}(\textstyle\sum_i k_i + q) \tag{4.29}$$

which according to Eq. (4.20) defines the function $\gamma_{\mathcal{D}}(k_i + q) = iD$, we arrive at the soft theorem:

$$T_{n+1}(q; k_1, \ldots, \bar{k}_n) = \frac{1}{f_\xi}\left[D - \sum_{i=1}^{n}(d_i + k_i \cdot \partial_{k_i})\right] T_n(k_1, \ldots, \bar{k}_n) + \mathcal{O}(q) \tag{4.30}$$

This is a well-known expression dating back to works by G. Mack [7]. It is worth observing that, due to the momentum conservation, the T-matrix, in Eq. (4.30), depends only on $(n-1)$ momenta. Therefore in the definition of dilatation operator, one of the momentum derivatives does not give any contribution when evaluated on the amplitude. The $\sum_{i=1}^{n} k_i \cdot \partial_{k_i}$ is thus a sum on only $(n-1)$ momenta. This observation will be used in the sect. 7 where the soft theorems will be verified on specific amplitudes computed in models with spontaneously broken conformal symmetry.

The complete treatment given in simplified form here, where all terms including those belonging to the soft Laurent expansion were taken into account, was performed in Ref. [17], where it was shown to also yield a soft factorizing theorem. The additional Laurent contributions automatically yield the terms that one can explicitly derive by Feynman diagram techniques, when noting that the dilaton couples linearly on the legs of massive external states. This was indeed the route taken in Ref. [15], but by our method it follows automatically from the Ward identity, as shown in Ref. [17].

4.3 Soft Ward identity of special conformal transformations, $J^\mu_{\mathcal{K},\lambda}$

We specify in this section the general treatment to the case of special conformal transformations with $J^\mu = J^\mu_{\mathcal{K},\lambda}$ and $\Gamma_{\mathcal{K},\lambda} = \mathcal{K}_\lambda$. Following the notation of Sec. 2 we have

$$\mathcal{K}_{\lambda,\varphi}(x)\varphi(x) = i\left((2x_\lambda x_\nu - \eta_{\lambda\nu}x^2)\partial^\nu + 2\,d_\varphi\,x_\lambda + 2ix^\nu \mathcal{S}_{\nu\lambda}\right)\varphi(x) \tag{4.31a}$$

$$\partial_\mu J^\mu_{\mathcal{K},\lambda} = = 2\,x_\lambda\,T^\mu{}_\mu = 2\,f_\xi\,x_\lambda(-\partial^2)\,\xi(x) \tag{4.31b}$$

where φ is any field and d_φ is its scaling dimension. The second expression defines $g_\lambda(x) = 2f_\xi x_\lambda$.

To derive the Fourier transformed operators, we simply replace every x_μ with a derivative $i\frac{\partial}{\partial k^\mu}$, while the derivative ∂^ν can be replaced with ik^ν. Then after passing k-derivatives through k^ν, one finds:

$$\begin{aligned} \tilde{g}_\lambda(q) &= i2f_\xi\partial_{q,\lambda}\,, \\ \tilde{\mathcal{K}}_{\lambda,\varphi}(k) &= 2k_\nu\partial_k^\nu\partial_{k,\lambda} - k_\lambda\partial_k^2 - 2(d_\varphi - D)\partial_{k,\lambda} + 2i\mathcal{S}_{\lambda\nu}\partial_k^\nu \end{aligned} \tag{4.32}$$

To derive the commutation relations with the propagators, we need to specify the spin of the hard states to define the form of their propagator. Assuming for simplicity that the hard states are spin 0 scalar fields, we should neglect the spin operator. Then it can be checked that the commutator with the scalar propagator reads:

$$\begin{aligned}
&\Big[\tilde{\mathcal{K}}_{\lambda,i}(k_i+q)\,,\ \Delta_i(k_i+q)\Big] \\
&\quad= \Delta_i(k_i+q)\left[\frac{2(D-2d_i)(k_i+q)_\lambda}{(k_i+q)^2+m_i^2}\left(\frac{\left(\frac{D}{2}-d_i+1\right)m_i^2}{(k_i+q)^2+m_i^2}\right)\right] \\
&\quad\quad+ \Delta_i(k_i+q)\left[-2(D-2d_i)\left(1-\frac{m_i^2}{(k_i+q)^2+m_i^2}\right)\partial_{k,\lambda}\right]
\end{aligned} \tag{4.33}$$

where the first term is coming from the full action of $\tilde{\mathcal{K}}_{\lambda,i}$ on the propagator, while the second term arises due to the non-linearity of $\tilde{\mathcal{K}}_{\lambda,i}$, i.e. terms where one derivative hits the propagator and the other goes through. This expression defines the operator $\tilde{F}_{\lambda,i}(k_i+q,m_i)$, which due to the non-linearity of $\tilde{\mathcal{K}}_{\lambda,i}$ has a part which is not just a function, but a derivative operator. In reduced form:

$$\begin{aligned}
\tilde{F}_{\lambda,i}(k_i+q,m_i) =& -2(D-2d_i)\partial_{k,\lambda} \\
&+ \frac{2(D-2d_i)m_i^2}{(k_i+q)^2+m_i^2}\left[\left(\frac{D}{2}-d_i+1\right)\frac{(k_i+q)_\lambda}{(k_i+q)^2+m_i^2}+\partial_{k,\lambda}\right]
\end{aligned} \tag{4.34}$$

As in the previous section, we will here restrict our analysis to the part only belonging to the soft Taylor expansion, and refer to Ref. [17] for the full treatment. Thus, according to the definition in Eq. (4.26), we simply consider:

$$\tilde{F}_{\lambda,i} \to \tilde{F}^{\mathrm{T}}_{\lambda,i}(k_i+q,m_i) = 2(2d_i-D)\partial_{k,\lambda}\,, \quad \text{for spinless } \varphi_i \tag{4.35}$$

We note again that this is equivalent to the massless case, however, this restriction being more general. By this prescription, we find from Eq. (4.18) the following single-soft Ward identity:

$$\begin{aligned}
&-2f_\xi\partial_{q,\lambda}\langle\tilde{\xi}(q)\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}} \\
&= \sum_{i=1}^{n}\Big(2(2d_i-D)\partial_{k_i,\lambda}+\tilde{\mathcal{K}}_{\lambda,i}(k_i+q)\Big)\langle\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_i+q)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}}+\mathcal{O}(q) \\
&= \sum_{i=1}^{n}\Big(2d_i\partial_{k_i,\lambda}+2(k_i+q)_\nu\partial^\nu_{k_i}\partial_{k_i,\lambda}-(k_i+q)_\lambda\partial^2_{k_i}\Big)\langle\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_i+q)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}}+\mathcal{O}(q)
\end{aligned} \tag{4.36}$$

It is useful to define the operator

$$\hat{K}^\lambda_{k_i} = \frac{1}{2}k_i^\lambda\partial^2_{k_i}-(d_i+k_i\cdot\partial_{k_i})\partial^\lambda_{k_i}\,, \tag{4.37}$$

and then the single-soft Ward identity of special conformal transfomations reads:

$$f_\xi \partial_{q,\lambda} \langle \tilde{\xi}(q) \tilde{\phi}(k_1) \cdots \tilde{\phi}(k_n) \rangle_{\text{amp}} = \sum_{i=1}^{n} \hat{K}_{k_i+q,\lambda} \langle \tilde{\phi}(k_1) \cdots \tilde{\phi}(k_i+q) \cdots \tilde{\phi}(k_n) \rangle_{\text{amp}} + \mathcal{O}(q) \tag{4.38}$$

Imposing LSZ reduction on this expression, and noting that the various operators involved all commute with the momentum conserving delta-function, this expression readily yields:

$$f_\xi \partial_{q,\lambda} T_{n+1}(q; k_1, \ldots, \bar{k}_n) = \sum_{i=1}^{n} \hat{K}_{k_i+q,\lambda} T_n(k_1, \ldots, k_i+q, \ldots, \bar{k}_n) + \mathcal{O}(q) \tag{4.39}$$

Since both sides of this expression should be evaluated for $q \sim 0$, it is clear that the left-hand side, when multiplied by q_λ is the first order term in the Taylor expansion of T_{n+1} around $q = 0$. Thus:

$$\begin{aligned} T_{n+1}(q, k_1, \ldots, \bar{k}_n) &= T_{n+1}(0, k_1, \ldots, \bar{k}_n) + q^\mu \frac{\partial}{\partial q^\mu} T_{n+1}(0, k_1, \ldots, \bar{k}_n) + \mathcal{O}(q^2) \\ &= \frac{1}{f_\xi} \left[D - \sum_{i=1}^{n} (d_i + k_i \cdot \partial_{k_i}) + q^\lambda \sum_{i=1}^{n} \hat{K}_{k_i,\lambda} \right] T_n(k_1, \ldots, \bar{k}_n) + \mathcal{O}(q^2) \end{aligned} \tag{4.40}$$

where we used Eq. (4.30) for the leading term in the expansion and Eq. (4.39) for the subleading term. For the Laurent terms, the order of soft limit and on-shell limit is subtle and must be performed with care, nevertheless it is possible to show that one can derive the soft theorem through subleading order [17] following the same procedure, including all the correct terms of the Laurent expansion.

5 Double-Soft Ward Identity and double-soft dilaton theorem

In this section we apply the same current algebra procedures as defined and performed in the preceeding section, but with the complication of inserting two currents in the matrix element of T^*-ordered product of operators. This leads to new soft Ward identities as well as a new double-soft theorem for the dilaton.

We consider the forementioned matrix element and take space-time derivatives on the space-time variables of the two currents. In addition to the single-soft assumption of Eq. (4.11), we similarly assume

$$\int d^D y \ \mathrm{e}^{-iky} \int d^D x \ \mathrm{e}^{-iqx} \partial_\nu^y \partial_\mu^x T^* \langle 0 | J_1^\mu(x) J_2^\nu(y) \phi(x_1) \cdots \phi(x_n) | 0 \rangle = 0 + \mathcal{O}(k_\nu q_\mu) \tag{5.1}$$

This follows from taking the Fourier transform of the derivatives, and assuming that the correlation function has no poles in the momentum variables of the currents.

Considering instead the action of the derivatives on the matrix element we find:

$$
\begin{aligned}
&\int d^D y \ \mathrm{e}^{-iky} \int d^D x \ \mathrm{e}^{-iqx} \partial_\nu^y \partial_\mu^x T^* \langle 0 | J_1^\mu(x) J_2^\nu(y) \phi(x_1) \cdots \phi(x_n) | 0 \rangle \\
&= \int d^D y \ \mathrm{e}^{-iky} \partial_\nu^y \Bigg[\int d^D x \ \mathrm{e}^{-iqx} T^* \langle 0 | (\partial_\mu J_1^\mu(x)) J_2^\nu(y) \phi(x_1) \cdots \phi(x_n) | 0 \rangle \\
&\quad + \mathrm{e}^{-iqy} T^* \langle 0 | [Q_1, J_2^\nu(y)] \phi(x_1) \cdots \phi(x_n) | 0 \rangle \\
&\quad + \sum_{i=1}^n \mathrm{e}^{-iqx_i} T^* \langle 0 | J_2^\nu(y) \phi(x_1) \cdots [Q_1, \phi(x_i)] \cdots \phi(x_n) | 0 \rangle \Bigg] \\
&= \int d^D y \ \mathrm{e}^{-iky} \int d^D x \ \mathrm{e}^{-iqx} T \langle 0 | (\partial_\mu J_1^\mu(x)) (\partial_\nu^y J_2^\nu(y)) \phi(x_1) \cdots \phi(x_n) | 0 \rangle \\
&\quad + \int d^D x \ \mathrm{e}^{-i(q+k)x} T^* \langle 0 | \ [Q_2, \partial_\mu J_1^\mu(x)] \ \phi(x_1) \cdots \phi(x_n) | 0 \rangle \\
&\quad + \sum_{i=1}^n \int d^D x \ \mathrm{e}^{-i(qx+kx_i)} T^* \langle 0 | (\partial_\mu J_1^\mu(x)) \phi(x_1) \cdots [Q_2, \phi(x_i)] \cdots \phi(x_n) | 0 \rangle \\
&\quad + \int d^D y \ \mathrm{e}^{-iky} \partial_\nu^y \left(\mathrm{e}^{-iqy} T^* \langle 0 | [Q_1, J_2^\nu(y)] \phi(x_1) \cdots \phi(x_n) | 0 \rangle \right) \\
&\quad + \sum_{i=1}^n \int d^D y \ \mathrm{e}^{-i(ky+qx_i)} T^* \langle 0 | (\partial_\nu J_2^\nu(y)) \phi(x_1) \cdots [Q_1, \phi(x_i)] \cdots \phi(x_n) | 0 \rangle \\
&\quad + \sum_{i \neq j} \mathrm{e}^{-iqx_i} \mathrm{e}^{-ikx_j} T^* \langle 0 | \phi(x_1) \cdots [Q_2, \phi(x_j)] \cdots [Q_1, \phi(x_i)] \cdots \phi(x_n) | 0 \rangle \\
&\quad + \sum_{i=1}^n \mathrm{e}^{-i(q+k)x_i} T^* \langle 0 | \phi(x_1) \cdots [Q_2, [Q_1, \phi(x_i)]] \cdots \phi(x_n) | 0 \rangle
\end{aligned} \tag{5.2}
$$

where we are assuming $x \neq y$. This expression can be further reduced by using the single-soft Ward identity in Eq. (4.12), as well as the identities in Eq. (4.10) and Eq. (4.6). Let us remark that the left-hand side of this Ward identity is manifestly symmetric under $q \leftrightarrow k$ and $J_1 \leftrightarrow J_2$. This means that our end result for the right-hand side must as well possesses this symmetry. For simplicity, we impose this at the end, but in principle the above expression could be already symmetrized.

The left-hand side of Eq. (5.2) is by Eq. (5.1) zero up to $\mathcal{O}(k^\nu q^\mu)$. The first term on the right-hand side can by Eq. (4.10) be reduced to:

$$
\begin{aligned}
&\int d^D y \ \mathrm{e}^{-iky} \int d^D x \ \mathrm{e}^{-iqx} T^* \langle 0 | (\partial_\mu J_1^\mu(x)) (\partial_\nu J_2^\nu(y)) \phi(x_1) \cdots \phi(x_n) | 0 \rangle \\
&= \int d^D y \ \mathrm{e}^{-iky} g_2(y) \int d^D x \ \mathrm{e}^{-iqx} g_1(x) T^* (-\partial_x^2)(-\partial_y^2) \langle 0 | \xi(x) \xi(y) \phi(x_1) \cdots \phi(x_n) | 0 \rangle \\
&= \tilde{g}_1(q) \tilde{g}_2(k) \left(k^2 q^2 \langle \tilde{\xi}(q) \tilde{\xi}(k) \phi(x_1) \cdots \phi(x_n) \rangle \right)
\end{aligned} \tag{5.3}
$$

Performing the Fourier transform of the remaining fields gives:

$$\begin{aligned} &\tilde{g}_1(q)\tilde{g}_2(k)\, k^2\, q^2 \int \prod_{j=1}^{n} \Big[\frac{d^D k_i}{(2\pi)^D} e^{-ik_j x_j}\Big] \langle \tilde{\xi}(q)\tilde{\xi}(k)\phi(x_1)\cdots\phi(x_n)\rangle \\ &= -\prod_{i=1}^{n} \Delta_i(k_i)\tilde{g}_1(q)\tilde{g}_2(k)\langle\tilde{\xi}(q)\tilde{\xi}(k)\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\text{amp}} \end{aligned} \tag{5.4}$$

where the correlation function on the right-hand side is amputated, and Δ_i are the two-point correlation functions of the fields ϕ_i, defined in Eq. (4.15).

The second term can be simplified as follows

$$\begin{aligned} &\int d^D x\ \mathrm{e}^{-i(q+k)x} T^* \langle 0|\, [Q_2, \partial_\mu J_1^\mu(x)]\ \phi(x_1)\cdots\phi(x_n)|0\rangle \\ &= \int d^D x\ \mathrm{e}^{-i(q+k)x} T^* \langle 0|\, [Q_2, g_1(x)(-\partial^2)\xi(x)]\ \phi(x_1)\cdots\phi(x_n)|0\rangle \\ &= \int d^D x\ \mathrm{e}^{-i(q+k)x} \Gamma_{2,g_1\partial^2\xi}(x) g_1(x) T^* \langle 0|\ (-\partial^2)\xi(x)\,\phi(x_1)\cdots\phi(x_n)|0\rangle \\ &= \tilde{\Gamma}_{2,g_1\partial^2\xi}(q+k)\tilde{g}_1(q+k)\Big((q+k)^2\langle\tilde{\xi}(q+k)\phi(x_1)\cdots\phi(x_n)\rangle\Big) \end{aligned} \tag{5.5}$$

where $\tilde{\Gamma}_{2g_1\partial^2\xi}(q+k)$ is the Fourier transform of the generator of infinitesimal transformations related to Q_2 and $g_1\partial^2\xi$, as defined in Eq. (4.5). Again, performing the Fourier transform of the remaining fields gives:

$$\begin{aligned} &\tilde{\Gamma}_{2,g_1\partial^2\xi}(q+k)\tilde{g}_1(q+k)(q+k)^2 \int \prod_{j=1}^{n} \Big[\frac{d^D k_i}{(2\pi)^D} e^{-ik_j x_j}\Big] \langle\tilde{\xi}(q+k)\phi(x_1)\cdots\phi(x_n)\rangle \\ &= -i\prod_{i=1}^{n}\Delta_i(k_i)\tilde{\Gamma}_{2,g_1\partial^2\xi}(q+k)\tilde{g}_1(q+k)\langle\tilde{\xi}(q+k)\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\text{amp}} \end{aligned} \tag{5.6}$$

We cannot reduce this expression further, since we need to know the explicit form of the operator $\tilde{\Gamma}_{2,g_1\partial^2\xi}$, which acts on both $\tilde{g}_1$ and the $(n+1)$-point amputated correlation function, involving the dilaton. We will later see that when one of the associated currents is the dilatation current, this expression can be further reduced by using the single-soft theorem of the previous section.

The third term on the right-hand side of Eq. (5.2) can similarly be reduced to:

$$\begin{aligned} &\sum_{i=1}^{n}\int d^D x\ \mathrm{e}^{-i(qx+kx_i)} T^*\langle 0|(\partial_\mu J_1^\mu(x))\phi(x_1)\cdots[Q_2,\phi(x_i)]\cdots\phi(x_n)|0\rangle \\ &= \sum_{i=1}^{n}\mathrm{e}^{-ikx_i}\tilde{g}_1(q)\int d^D x\ \mathrm{e}^{-iqx} T^*\langle 0|(-\partial^2)\xi(x)\phi(x_1)\cdots\Gamma_{2,\phi_i}(x_i)\phi(x_i)\cdots\phi(x_n)|0\rangle \\ &= \sum_{i=1}^{n}\mathrm{e}^{-ikx_i}\tilde{g}_1(q)\Big(q^2\langle\tilde{\xi}(q)\phi(x_1)\cdots\Gamma_{2,\phi_i}(x_i)\phi(x_i)\cdots\phi(x_n)\rangle\Big) \end{aligned} \tag{5.7}$$

This expression can be further reduced by making use of the singe-soft Ward identity given in Eq. (4.13), after Fourier transforming also the x_i variables. Thus, the previous expression transforms to

$$\begin{aligned}
&\sum_{i=1}^{n} \tilde{g}_1(q) \left(q^2 \langle \tilde{\xi}(q) \tilde{\phi}(k_1) \cdots \tilde{\Gamma}_{2,\phi_i}(k_i + k) \tilde{\phi}(k_i + k) \cdots \tilde{\phi}(k_n) \rangle \right) \\
&= - \sum_{i=1}^{n} \sum_{j \neq i}^{n} \tilde{\Gamma}_{1,\phi_j}(k_j + q) \tilde{\Gamma}_{2,\phi_i}(k_i + k) \langle \tilde{\phi}(k_1) \cdots \tilde{\phi}(k_j + q) \cdots \tilde{\phi}(k_i + k) \cdots \tilde{\phi}(k_n) \rangle \\
&\quad - \sum_{i=1}^{n} \tilde{\Gamma}_{1,\phi_i}(k_i + k + q) \tilde{\Gamma}_{2,\phi_i}(k_i + k + q) \langle \tilde{\phi}(k_1) \cdots \tilde{\phi}(k_i + k + q) \cdots \tilde{\phi}(k_n) \rangle + \mathcal{O}(q)
\end{aligned} \tag{5.8}$$

where care was taken on using the soft Ward identity for $j = i$. We may now amputate the correlation function, which can be expressed using the definition for $\tilde{F}$ in Eq. (4.17)

$$\begin{aligned}
&\sum_{i=1}^{n} \tilde{g}_1(q) \left(q^2 \langle \tilde{\xi}(q) \tilde{\phi}(k_1) \cdots \tilde{\Gamma}_{2,\phi_i}(k_i + k) \tilde{\phi}(k_i + k) \cdots \tilde{\phi}(k_n) \rangle \right) \\
&= - \prod_{l=1}^{n} \Delta_l(k_l) \sum_{i=1}^{n} \left(\tilde{F}_{2,\phi_i}(k_i + k, m_i) + \tilde{\Gamma}_{2,\phi_i}(k_i + k) \right) \sum_{j \neq 1}^{n} \left(\tilde{F}_{1,\phi_j}(k_j + q, m_j) + \tilde{\Gamma}_{1,\phi_j}(k_j + q) \right) \\
&\quad \times \langle \tilde{\phi}(k_1) \cdots \tilde{\phi}(k_i + k) \cdots \tilde{\phi}(k_j + q) \cdots \tilde{\phi}(k_n) \rangle_{\mathrm{amp}} \\
&\quad - \prod_{l=1}^{n} \Delta_l(k_l) \sum_{i=1}^{n} \left(\tilde{F}_{1,\phi_i}(k_i + k + q, m_i) + \tilde{\Gamma}_{1,\phi_i}(k_i + k + q) \right) \\
&\quad \times \left(\tilde{F}_{2,\phi_i}(k_i + k + q, m_i) + \tilde{\Gamma}_{2,\phi_i}(k_i + k + q) \right) \\
&\quad \times \langle \tilde{\phi}(k_1) \cdots \tilde{\phi}(k_i + k + q) \cdots \tilde{\phi}(k_n) \rangle_{\mathrm{amp}} + \mathcal{O}(q, k)
\end{aligned} \tag{5.9}$$

where we took the limit $k, q \to 0$ in the propagators Δ_l.

In the fourth term of the right-hand side of Eq. (5.2) we did not act with the derivative ∂^y_ν, because we instead Fourier transform it to show that the term is of $\mathcal{O}(k)$ by assumption:

$$\begin{aligned}
&\int d^D y \ \mathrm{e}^{-iky} \partial^y_\nu \left(\mathrm{e}^{-iqy} T^* \langle 0 | [Q_1, J^\nu_2(y)] \phi(x_1) \cdots \phi(x_n) | 0 \rangle \right) \\
&= ik_\nu \int d^D y \ \mathrm{e}^{-i(k+q)y} \Gamma_1(y) T^* \langle 0 | J^\nu_2(y) \phi(x_1) \cdots \phi(x_n) | 0 \rangle \\
&= ik_\nu \tilde{\Gamma}_1(k + q) \int d^D y \ \mathrm{e}^{-i(k+q)y} T^* \langle 0 | J^\nu_2(y) \phi(x_1) \cdots \phi(x_n) | 0 \rangle \\
&= 0 + \mathcal{O}(k)
\end{aligned} \tag{5.10}$$

where the last line follows from Eq. (4.11) (as well as assuming no pole in $\tilde{\Gamma}_1$).

The fifth term is equivalent to the third term, but with the symmetry indices interchanged $1 \leftrightarrow 2$ and the soft-momenta likewise interchanged $q \leftrightarrow k$. Thus the fifth term gives:

$$
\begin{aligned}
&\int \prod_{j=1}^{n} \left[d^D x_j \ \mathrm{e}^{-ik_j x_j} \right] \sum_{i=1}^{n} \int d^D y \ \mathrm{e}^{-i(ky+qx_i)} T^* \langle 0 | (\partial_\nu J_2^\nu(y)) \phi(x_1) \cdots [Q_1, \phi(x_i)] \cdots \phi(x_n) | 0 \rangle \\
&= -\prod_{l=1}^{n} \Delta_l(k_l) \sum_{i=1}^{n} \left(\tilde{F}_{2,\phi_i}(k_i + k, m_i) + \tilde{\Gamma}_{2,\phi_i}(k_i + k) \right) \sum_{j \neq 1}^{n} \left(\tilde{F}_{1,\phi_j}(k_j + q, m_j) + \tilde{\Gamma}_{1,\phi_j}(k_j + q) \right) \\
&\quad \times \langle \tilde{\phi}(k_1) \cdots \tilde{\phi}(k_i + k) \cdots \tilde{\phi}(k_j + q) \cdots \tilde{\phi}(k_n) \rangle_{\text{amp}} \\
&\quad - \prod_{l=1}^{n} \Delta_l(k_l) \sum_{i=1}^{n} \left(\tilde{F}_{2,\phi_i}(k_i + k + q, m_i) + \tilde{\Gamma}_{2,\phi_i}(k_i + k + q) \right) \\
&\quad \times \left(\tilde{F}_{1,\phi_i}(k_i + k + q, m_i) + \tilde{\Gamma}_{1,\phi_i}(k_i + k + q) \right) \\
&\quad \times \langle \tilde{\phi}(k_1) \cdots \tilde{\phi}(k_i + k + q) \cdots \tilde{\phi}(k_n) \rangle_{\text{amp}} + \mathcal{O}(q, k)
\end{aligned}
\tag{5.11}
$$

The terms with the double sum, where $j \neq i$, are the same as before since the operators here commute. The operators in the single sum, on the other hand, do not commute. Instead these terms, together with the similar ones in Eq. (5.9), add up to ensure the symmetry $q \leftrightarrow k$ and $J_1 \leftrightarrow J_2$, which is manifest on the left-hand side of the Ward identity.

The sixth term leads to

$$
\begin{aligned}
&\sum_{i \neq j} \mathrm{e}^{-iqx_i} \mathrm{e}^{-ikx_j} T^* \langle 0 | \phi(x_1) \cdots [Q_2, \phi(x_j)] \cdots [Q_1, \phi(x_i)] \cdots \phi(x_n) | 0 \rangle \\
&= \sum_{i \neq j} \mathrm{e}^{-ikx_j} \Gamma_{2,\phi_j}(x_j) \mathrm{e}^{-iqx_i} \Gamma_{1,\phi_i}(x_i) T^* \langle 0 | \phi(x_1) \cdots \phi(x_n) | 0 \rangle
\end{aligned}
\tag{5.12}
$$

It is easy to see that by taking the Fourier transform and amputating the correlation function, this expression exactly cancels the similar terms with double sums in, either the third expression in Eq. (5.9) or the fifth expression in Eq. (5.11).

Finally, for the seventh term we make use of the Jacobi identity:

$$
[Q_2, [Q_1, \phi_i]] = [[Q_2, Q_1], \phi_i] + [Q_1, [Q_2, \phi_i]] \tag{5.13}
$$

As mentioned earlier, the left-hand side of the Ward identity is manifestly symmetric under $q \leftrightarrow k$, $J_1 \leftrightarrow J_2$. To ensure the symmetry on the right-hand side we should symmetrize the seventh term. This symmetrization gets rid of the commutator $[Q_2, Q_1]$ above and sends:

$$
[Q_2, [Q_1, \phi_i]] \to \frac{1}{2} \left(\Gamma_{2,\phi_i} \Gamma_{1,\phi_i} + \Gamma_{1,\phi_i} \Gamma_{2,\phi_i} \right) \phi_i \tag{5.14}
$$

Thus the seventh term by symmetrization is the sum of the two terms

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^{n}\mathrm{e}^{-i(q+k)x_i}T^*\langle 0|\phi(x_1)\cdots[Q_2,[Q_1,\phi(x_i)]]\cdots\phi(x_n)|0\rangle+(1\leftrightarrow 2)\\&=\sum_{i=1}^{n}\mathrm{e}^{-i(q+k)x_i}\frac{1}{2}\left(\Gamma_{2,\phi_i}\Gamma_{1,\phi_i}+\Gamma_{1,\phi_i}\Gamma_{2,\phi_i}\right)T^*\langle 0|\phi(x_1)\cdots\phi(x_i)\cdots\phi(x_n)|0\rangle\end{aligned} \tag{5.15}$$

It is readily seen that after Fourier transforming and amputating, this expression cancels one half of the similar terms in Eq. (5.9) and Eq. (5.11).

Finally, taking into account the symmetrization, we can express the full double-soft Ward identity on amputated correlation functions in momentum space:

$$\begin{aligned}&\tilde{g}_1(q)\tilde{g}_2(k)\langle\tilde{\xi}(q)\tilde{\xi}(k)\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}}\,|_{q,k\sim 0}\\&=-\frac{i}{2}\left[\tilde{\Gamma}_{2,g_1\partial^2\xi}(q+k)\tilde{g}_1(q+k)+\tilde{\Gamma}_{1,g_2\partial^2\xi}(q+k)\tilde{g}_2(q+k)\right]\langle\tilde{\xi}(q+k)\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}}\\&\quad-\Big[\sum_{i=1}^{n}\Big(\tilde{F}_{1,\phi_i}(k_i+k,m_i)+\tilde{\Gamma}_{1,\phi_i}(k_i+k)\Big)\sum_{j\neq i}^{n}\Big(\tilde{F}_{2,\phi_j}(k_j+q,m_j)+\tilde{\Gamma}_{2,\phi_j}(k_j+q)\Big)\\&+\frac{1}{2}\sum_{i=1}^{n}\Big(\tilde{F}_{1,\phi_i}(k_i+k+q,m_i)+\tilde{\Gamma}_{1,\phi_i}(k_i+k+q)\Big)\Big(\tilde{F}_{2,\phi_i}(k_i+k+q,m_i)+\tilde{\Gamma}_{2,\phi_i}(k_i+k+q)\Big)\\&+\frac{1}{2}\sum_{i=1}^{n}\Big(\tilde{F}_{2,\phi_i}(k_i+k+q,m_i)+\tilde{\Gamma}_{2,\phi_i}(k_i+k+q)\Big)\Big(\tilde{F}_{1,\phi_i}(k_i+k+q,m_i)+\tilde{\Gamma}_{1,\phi_i}(k_i+k+q)\Big)\Big]\\&\qquad\times\langle\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_i+k)\cdots\tilde{\phi}(k_j+q)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}}+\mathcal{O}(q,k)\end{aligned} \tag{5.16}$$

where in the last correlator it is implicitly assumed that for the single-sum expressions one should understand $\tilde{\phi}(k_i+k)\cdots\tilde{\phi}(k_i+q)\sim\tilde{\phi}(k_i+k+q)$.

In the case of massless hard states, the limit $q,k\to 0$ may be well-behaved. If that is so, and if furthermore $[\Gamma_1,\Gamma_2]=0$, then the above expression simplifies to:

$$\begin{aligned}&\tilde{g}_1(q)\tilde{g}_2(k)\langle\tilde{\xi}(q)\tilde{\xi}(k)\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}}\,|_{q,k\sim 0}\\&=-\frac{i}{2}\left[\tilde{\Gamma}_{2,g_1\partial^2\xi}(q+k)\tilde{g}_1(q+k)+\tilde{\Gamma}_{1,g_2\partial^2\xi}(q+k)\tilde{g}_2(q+k)\right]\langle\tilde{\xi}(q+k)\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}}\\&\quad-\sum_{i=1}^{n}\Big(\tilde{F}_{1,\phi_i}(k_i)+\tilde{\Gamma}_{1,\phi_i}(k_i)\Big)\sum_{j=i}^{n}\Big(\tilde{F}_{2,\phi_j}(k_j)+\tilde{\Gamma}_{2,\phi_j}(k_j)\Big)\langle\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}}+\mathcal{O}(q,k)\end{aligned} \tag{5.17}$$

5.1 Double-soft Ward identity of two dilatation currents

We specialize the previous analysis to the case of two dilatation current insertions in the matrix element. Following the definitions and expressions in Sec. 4.2, we have

$$\begin{aligned} \tilde{g}_1 = \tilde{g}_2 = f_\xi\,, \qquad & \tilde{\mathcal{D}}_i(k) = i(d_i - D - k\cdot\partial_k)\,, \\ \tilde{F}_i(k_i+q, m_i) = i(D-2d_i)&\left(1 - \frac{m_i^2}{(k_i+q)^2+m_i^2}\right) \\ f_i(k_i+q, m_i) = 0\,, \qquad & \gamma_i(k_i+q) = iD\ . \end{aligned} \tag{5.18}$$

We will in this work only focus on the parts of the double-soft Ward identities belonging to the Taylor expansion in the soft momenta, as described and prescribed in Eq. (4.26). In this case, this is equivalent to setting $\tilde{F}_i \to \tilde{F}_i^{\mathrm{T}} = \tilde{F}_i(k_i+q, 0)$. Due to this restriction and since $[\mathcal{D}, \mathcal{D}] = 0$, we need only to consider the simpler form of the double-soft Ward identity in Eq. (5.17).

Let us first consider the first term on the right-hand side of Eq. (5.17), which under the above specifications takes the form:

$$\begin{aligned} &- i\tilde{\Gamma}_{2,g_1\partial^2\xi}(q+k)\tilde{g}_1(q+k)\langle\tilde{\xi}(q+k)\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}} \\ &= (d_{\partial^2\xi} - D - (k+q)\cdot\partial_{k+q})f_\xi\langle\tilde{\xi}(q+k)\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}} \end{aligned} \tag{5.19}$$

Now using the single-soft Ward identities given in Eq. (4.28) it follows that the right-hand side of Eq. (5.19) is equal to:

$$(d_{\partial^2\xi} - D)\sum_{i=1}^{n}(-d_i - k_i\cdot\partial_{k_i})\langle\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_i+k+q)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}} + \mathcal{O}(k+q) \tag{5.20}$$

Then it is straightforward to write the full expression for Eq. (5.17):

$$\begin{aligned} &f_\xi^2\langle\tilde{\xi}(q)\tilde{\xi}(k)\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}} \\ &= (d_{\partial^2\xi} - D)\sum_{i=1}^{n}(-d_i - k_i\cdot\partial_{k_i})\langle\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_i+k+q)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}} \\ &\quad - \sum_{i=1}^{n}\left(i(D-2d_i) + i(d_i - D - k_i\cdot\partial_{k_i})\right)\sum_{j=1}^{n}\big(i(D-2d_j) + i(d_j - D - k_j\cdot\partial_{k_j})\big) \\ &\quad\ \times \langle\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}} + \mathcal{O}(q,k) \\ &= \sum_{i=1}^{n}(-d_i - k_i\cdot\partial_{k_i})\left[(d_{\partial^2\xi} - D) + \sum_{j=1}^{n}\big(-d_j - k_j\cdot\partial_{k_j}\big)\right]\langle\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}} + \mathcal{O}(q,k) \end{aligned} \tag{5.21}$$

Since $\partial^2\xi$ is the second descendant of the primary field, ξ, the dilaton, it follows that

$$d_{\partial^2\xi} = 2 + d_\xi = D - d_\xi \tag{5.22}$$

Thus

$$\begin{aligned} & f_\xi^2 \langle \tilde{\xi}(q)\tilde{\xi}(k)\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\text{amp}} \\ &= \sum_{i=1}^n (-d_i - k_i\cdot\partial_{k_i}) \left(-d_\xi + \sum_{j=1}^n (-d_j - k_j\cdot\partial_{k_j})\right) \langle\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\text{amp}} + \mathcal{O}(q,k) \end{aligned} \tag{5.23}$$

This expression is nothing but two consecutive applications of the single-soft Ward identity, where in the first application, one of the dilatons is taken to be hard. This shows that there is no difference at leading order between the two limits: $q \sim k \ll k_i$ and $q \ll k \ll k_i$.

We can go on and express this in terms of amplitudes by performing the LSZ reduction. This gives us the double-soft theorem:

$$\begin{aligned} f_\xi^2 T_{n+2}(q,k,k_1,\ldots,\bar{k}_n) = & \left[D - d_\xi - \sum_{j=1}^n \left(d_j + k_j\cdot\partial_{k_j}\right)\right]\left[D - \sum_{i=1}^n (d_i + k_i\cdot\partial_{k_i})\right] \\ & \times T_n(k_1,\ldots,\bar{k}_n) + \mathcal{O}(q,k) \end{aligned} \tag{5.24}$$

This is again nothing but two single-soft dilaton theorems applied consecutively. Thus there is no distinction between two soft dilatons emitted consecutively with two soft dilatons emitted simultaneously. The bar on k_n means that we keep one of the hard momenta, say k_n, fixed by momentum conservation, as in Eq. (4.22).

In the case where all fields have free scalar field dimension $d_i = d_\xi = d = (D-2)/2$, then

$$\begin{aligned} f_\xi^2 T_{n+2}(q,k,k_1,\ldots,\bar{k}_n) = & \left[D - (n+1)d - \sum_{j=1}^n k_j\cdot\partial_{k_j}\right]\left[D - nd - \sum_{i=1}^n k_i\cdot\partial_{k_i}\right] \\ & \times T_n(k_1,\ldots,\bar{k}_n) + \mathcal{O}(q,k) \end{aligned} \tag{5.25}$$

Eq. (5.24) is, however, more general, since the hard states can be interacting fields carying anomalous dimension. We can parametrize this by denoting $d_i = d + \eta_i$, while still $d_\xi = d$, then:

$$\begin{aligned} f_\xi^2 T_{n+2}(q,k,k_1,\ldots,\bar{k}_n) = & \left[D - (n+1)d - \sum_{j=1}^n (\eta_j + k_j\cdot\partial_{k_j})\right] \\ & \times \left[D - nd - \sum_{i=1}^n (\eta_i + k_i\cdot\partial_{k_i})\right] T_n(k_1,\ldots,\bar{k}_n) + \mathcal{O}(q,k) \end{aligned} \tag{5.26}$$

where η_i are the anomalous dimensions of the scalar fields ϕ_i.

5.2 Double-soft Ward identity of the two currents, $J^{\mu}_{\mathcal{D}}$ and $J^{\mu}_{\mathcal{K},\lambda}$

We consider the double-soft Ward identity in Eq. (5.16), following insertions of a dilatation current, $J^{\mu}_{\mathcal{D}}$, and a special conformal transformation current, $J^{\mu}_{\mathcal{K},\lambda}$, in the matrix element Eq. (4.2). Following the definitions and expressions in Sect. 2, 4.2 and 4.3, as well as the restriction described at Eq. (4.26), we take

$$\begin{aligned} \tilde{g}_1 = f_\xi\,, &\qquad \tilde{g}_{2,\lambda}(k) = i2f_\xi\partial_{k,\lambda}\,, \\ \tilde{\mathcal{D}}_i(k) = i(d_i - D - k\cdot\partial_k)\,, &\qquad \tilde{\mathcal{K}}_{\lambda,i}(k) = 2k^\nu\partial_{k,\nu}\partial_{k,\lambda} - k_\lambda\partial_k^2 - 2(d_i - D)\partial_{k,\lambda} \\ \tilde{F}^{\mathrm{T}}_{1,\phi_i}(k_i+q,m_i) = \ i(D-2d_i)\,, &\qquad \tilde{F}^{\mathrm{T},\lambda}_{2,\phi_i}(k_i+q,m_i) = -2(D-2d_i)\partial_k^\lambda\,, \\ f_{1,\phi_i}(k_i+q,m_i) = 0\,, &\qquad \gamma_{1,\phi_1}(k_i+q) = D\,, \\ f^\lambda_{2,\phi_i}(k_i+q,m_i) = 0\,, &\qquad \gamma^\lambda_{2,\phi_i}(k_i+q) = 0\,, \end{aligned} \tag{5.27}$$

For consistency it can be checked that:

$$[\tilde{\mathcal{D}},\tilde{\mathcal{K}}_\lambda] = i\tilde{\mathcal{K}}_\lambda \tag{5.28}$$

This is in fact true for any value of d_i and thus this term in $\tilde{\mathcal{K}}_\lambda$ can take any prefactor and still preserve the commutation relation above.

Let us consider the first line on the right-hand side of Eq. (5.16), reading:

$$\begin{aligned} &-\frac{i}{2}\left[\tilde{\Gamma}_{2,g_1\partial^2\xi}(q+k)\tilde{g}_1(q+k) + \tilde{\Gamma}_{1,g_2\partial^2\xi}(q+k)\tilde{g}_2(q+k)\right]\langle\tilde{\xi}(q+k)\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}} \\ &\quad = -\frac{i}{2}\left[f_\xi\mathcal{K}_{\lambda,\partial^2\xi}(q+k) + i2f_\xi\mathcal{D}_{x\partial^2\xi}(q+k)\partial_{k+q,\lambda}\right]\langle\tilde{\xi}(q+k)\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}} \\ &\quad = if_\xi\left[d_{\partial^2\xi} + d_{x\partial^2\xi} - 2D\right]\partial_{k+q,\lambda}\langle\tilde{\xi}(q+k)\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}} + \mathcal{O}(k+q) \end{aligned} \tag{5.29}$$

The last expression can be further reduced by making use of the single-soft Ward identity for special conformal transformations, given in Eq. (4.38), getting

$$= i(d_{\partial^2\xi} + d_{x\partial^2\xi} - 2D)\sum_{i=1}^n \hat{K}_{k_i,\lambda}\langle\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\mathrm{amp}} + \mathcal{O}(k+q) \tag{5.30}$$

where $\hat{K}_{k_i,\lambda}$ was defined in Eq. (4.37), and differs from $\tilde{\mathcal{K}}_{i,\lambda}$ only in the term with a single derivative and an overall factor $-1/2$. It therefore obeys the same commutation relations as $\tilde{\mathcal{K}}_{i,\lambda}$, i.e. $[\tilde{\mathcal{D}},\hat{K}_\lambda] = i\hat{K}_\lambda$.

Considering the remaining terms, let us notice that we have:

$$\begin{aligned} &\tilde{F}^{\mathrm{T}}_{1,\phi_i}(k_i,0) + \tilde{\Gamma}_{1,\phi_i}(k_i) = i(-d_i - k_i\cdot\partial_{k_i}) = i\hat{D}_i \\ &\tilde{F}^{\mathrm{T}}_{2,\phi_i}(k_i,0) + \tilde{\Gamma}_{2,\phi_i}(k_i) = -2(D-2d_i)\partial_{k_i,\lambda} + \tilde{\mathcal{K}}_{\lambda,\phi_i} = -2\hat{K}_{k_j,\lambda} \end{aligned} \tag{5.31}$$

where for brevity we also defined $\hat{D}_i$, thus $[\hat{D}, \hat{K}_\lambda] = \hat{K}_\lambda$. From this we find that Eq. (5.16) reads:

$$\begin{aligned}
&i2f_\xi^2 \partial_{k,\lambda} \langle \tilde{\xi}(q)\tilde{\xi}(k)\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\text{amp}} \,|_{q,k\sim 0} \\
&= \Bigg[i(d_{\partial^2\xi} + d_{x\partial^2\xi} - 2D)\sum_{i=1}^n \hat{K}_{k_i,\lambda} + 2\sum_{i=1}^n i\hat{D}_i \sum_{j\neq i} \hat{K}_{k_j,\lambda} \\
&\quad + \sum_{i=1}^n i\hat{D}_i\hat{K}_{k_i,\lambda} + \sum_{i=1}^n i\hat{K}_{k_i,\lambda}\hat{D}_i \Bigg] \langle \tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\text{amp}} + \mathcal{O}(q,k) \\
&= i\sum_{j=1}^n \hat{K}_{k_j,\lambda} \Bigg[d_{\partial^2\xi} + d_{x\partial^2\xi} - 2D + 1 + 2\sum_{i=1}^n \hat{D}_i \Bigg] \langle \tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\text{amp}} + \mathcal{O}(q,k)
\end{aligned} \tag{5.32}$$

In going from the first equality to the second equality, we used the commutation relation between $\hat{D}$ and $\hat{K}_\lambda$.

Using that $d_{\partial^2\xi} = d + 2 = D - d$ and $d_{x\partial^2\xi} = d + 1 = D - d - 1$, we arrive at:

$$\begin{aligned}
&f_\xi^2 \partial_{k,\lambda} \langle \tilde{\xi}(q)\tilde{\xi}(k)\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\text{amp}} \,|_{q,k\sim 0} \\
&\quad = \sum_{j=1}^n \hat{K}_{k_j,\lambda} \left(-d + \sum_{i=1}^n \hat{D}_i \right) \langle \tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\text{amp}} + \mathcal{O}(q,k)
\end{aligned} \tag{5.33}$$

It follows that by studying instead the Ward identity of $Q_1^\mu = \mathcal{K}^\mu$ and $Q_2 = \mathcal{D}$, we equivalently find an expression reading

$$\begin{aligned}
&f_\xi^2 \partial_{q,\lambda} \langle \tilde{\xi}(q)\tilde{\xi}(k)\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\text{amp}} \,|_{q,k\sim 0} \\
&\quad = \sum_{j=1}^n \hat{K}_{k_j,\lambda} \left(-d + \sum_{i=1}^n \hat{D}_i \right) \langle \tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\text{amp}} + \mathcal{O}(q,k)
\end{aligned} \tag{5.34}$$

which differs only from Eq. (5.33) by the soft-momentum derivative on the left-hand side. Contracting either expression with the respective soft momentum k^λ and q^λ, it follows that these expressions provide the $\mathcal{O}(q,k)$ terms in the Taylor series of the double-soft Ward identity.

Reducing these to relations among amplitudes, we use that only the dilatations give a contribution by acting on the momentum-conserving delta-function, thus yielding:

$$f_\xi^2 \partial_{k,\lambda} T_{n+2}(q,k,k_1,\ldots,\bar{k}_n) = \sum_{j=1}^n \hat{K}_{k_j,\lambda} \left(D - d + \sum_{i=1}^n \hat{D}_i \right) T_n(k_1,\ldots,\bar{k}_n) + \mathcal{O}(q,k) \tag{5.35}$$

and similarly for $\partial_{q,\lambda}$ acting on T_{n+2}. By contracting these identities with k^λ and q^λ yields the soft expansion of T_{n+2},

$$T_{n+2}(q,k;k_i) = T_{n+2}(0,0;k_i) + q\cdot\partial_q T_{n+2}(0,0;k_i) + k\cdot\partial_k T_{n+2}(0,0;k_i) + \cdots \tag{5.36}$$

which together with the result of the previous subsection explicitly reads:

$$\begin{aligned} f_\xi^2 T_{n+2}(q,k,k_1,\ldots,\bar{k}_n) = \Bigg[&\left(D - d + \sum_{i=1}^n \hat{D}_i\right)\left(D + \sum_{i=1}^n \hat{D}_i\right) \\ &+ (q^\lambda + k^\lambda)\sum_{i=1}^n \hat{K}_{k_i,\lambda}\left(D - d + \sum_{i=1}^n \hat{D}_i\right)\Bigg] T_n(k_1,\ldots,\bar{k}_n) + \mathcal{O}(q^2,k^2,qk) \end{aligned} \tag{5.37}$$

5.3 Double-soft Ward identity of two special conformal currents: A no-go for higher-order soft factorization

We finally consider the double-soft Ward identity following two insertions of special conformal currents in the matrix element. We restrict again our attention to the part belonging only to the Taylor series of the soft expansion. Then since $[\mathcal{K}_\mu,\mathcal{K}_\nu]=0$ we may simply study Eq. (5.17). Using the identities in Eq. (5.27) and Eq. (5.31) for the special conformal current, we can immediately write the double-soft Ward identity following from Eq. (5.17):

$$\begin{aligned} &-4f_\xi^2\partial_{q,\lambda}\partial_{k,\gamma}\langle\tilde{\xi}(q)\tilde{\xi}(k)\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\text{amp}}\,|_{q,k\sim 0} \\ &= -i\tilde{\mathcal{K}}_{\gamma,x_\nu\partial^2\xi}(q+k)(i2f_\xi\partial_{q+k,\lambda})\langle\tilde{\xi}(q+k)\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\text{amp}} \\ &\quad - 4\sum_{i=1}^n \hat{K}_{\lambda,i}(k_i)\sum_{j=1}^n \hat{K}_{\gamma,i}(k_j)\langle\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\rangle_{\text{amp}} + \mathcal{O}(q,k) \end{aligned} \tag{5.38}$$

where $\tilde{\mathcal{K}}_{\gamma,x_\nu\partial^2\xi}(q+k)$ is defined in Eq. (4.32).

This time we have run into a problem: There is no single-soft Ward identity that relates the first term on the right-hand side to an expression in terms of the n-point correlation function. We have not been able to circumvent this problem, and it thus looks like a no-go theorem for obtaining soft factorization at the order $q_\mu k_\nu$. We furthermore note that we have no Ward identities that could potentially lead to soft factorization of terms with $q_\mu q_\nu$ and $k_\mu k_\nu$, which would be required to establish a full soft theorem at the order $q\,k$. We note, however, that the second term does takes the form of a soft theorem, relating the $n+2$ point correlation function to the n-point function acted upon by two special conformal transformation. One may be able to express this for amplitudes as a relation between $n+2$-, $n+1$- and n-point function, but we do not attempt to do so here.

6 Multi-soft dilatons

In Sec. 4 we have derived the soft theorem for the emission of a single soft dilaton, through $\mathcal{O}(q)$ in the soft momentum, q, while in Sec. 5 we have obtained a soft theorem for two soft dilatons through $\mathcal{O}(q_1^\mu q_2^\nu)$ with q_1 and q_2 the momenta of the two soft dilatons taken to be $q_1 \sim q_2 \ll k_i$, where k_i is any of the hard momenta involved in the amplitude. In this section we will first show that the double-soft theorem is equivalent to what one would get by making two consecutive emissions of the soft dilatons, one after the other, with $q_1 \ll q_2 \ll k_i$. From this observation we can make the conjecture that the amplitude for the emission of any number of soft dilatons is fixed by the consecutive soft limit of single dilatons emitted one after the other, that is:

$$\lim_{q_1,\ldots,q_m\to 0} A_{m+n}(q_1,\ldots,q_m;k_1,\ldots,k_n) = \lim_{q_1\to 0}\lim_{q_2\to 0}\cdots\lim_{q_m\to 0} A_{m+n}(q_1,\ldots,q_m;k_1,\ldots,k_n) \tag{6.1}$$

where on the left-hand side it is assumed that all soft momenta scale simultaneously to zero, while on the right-hand side it is assumed that $q_m \ll q_{m-1} \ll \cdots \ll q_1 \ll k_i$.

To see that this conjecture holds for the double-soft case, let us first summarize our previous results. The soft theorem for the emission of a single soft dilaton reads:

$$T_{n+1}(q,k_1,\ldots,\bar{k}_n) = \frac{1}{f_\xi}\left[D + \sum_{i=1}^n \hat{D}_i + q^\mu \sum_{i=1}^n \hat{K}_{k_i,\mu}\right] T_n(k_1,\ldots,\bar{k}_n) + \mathcal{O}(q^2) \tag{6.2}$$

The soft theorem for the simultaneous emission of two soft dilatons reads:

$$\begin{aligned} f_\xi^2 T_{n+2}(q_1,q_2,k_1,\ldots,\bar{k}_n) = &\Bigg[\left(D - d + \sum_{i=1}^n \hat{D}_i\right)\left(D + \sum_{i=1}^n \hat{D}_i\right) \\ &+ (q_1^\lambda + q_2^\lambda)\sum_{i=1}^n \hat{K}_{k_i,\lambda}\left(D - d + \sum_{i=1}^n \hat{D}_i\right)\Bigg] T_n(k_1,\ldots,\bar{k}_n) + \mathcal{O}(q_1^2, q_2^2, q_1 q_2) \end{aligned} \tag{6.3}$$

where

$$\hat{D}_i = -\left(d_i + k_i\cdot\partial_{k_i}\right), \qquad \hat{K}_{k_i,\mu} = \frac{1}{2}k_{i\mu}\partial_{k_i}^2 - (k_i\cdot\partial_{k_i})\partial_{k_i,\mu} - d_i\,\partial_{k_i,\mu} \tag{6.4}$$

Now let us consider an $(n+2)$-point amplitude, which involves at least two dilatons, carying momenta q_1 and q_2. If we take the momentum q_1 to be soft compared to the other momenta, i.e. $q_1 \ll q_2, k_i$, then the single soft theorem gives us:

$$\begin{aligned} &f_\xi T_{n+2}(q_1,q_2,k_1,\ldots,\bar{k}_n) \\ &= \left[D + \sum_{i=1}^n \hat{D}_i - (d + q_2\cdot\partial_{q_2}) + q_1^\lambda \sum_{i=1}^n \hat{K}_{k_i,\lambda} + q_1^\lambda \hat{K}_{q_2,\lambda}\right] T_{n+1}(q_2,k_1,\ldots,\bar{k}_n) + \mathcal{O}(q_1^2) \end{aligned} \tag{6.5}$$

If $q_2 \ll k_i$ in the above expression, the behavior of the $(n+1)$-point amplitude is also fixed through $\mathcal{O}(q_2^2)$, i.e.

$$T_{n+1}(q_2, k_1, \ldots, \bar{k}_n) = \frac{1}{f_\xi}\left[D + \sum_{i=1}^{n} \hat{D}_i + q_2^\mu \sum_{i=1}^{n} \hat{K}_{k_i,\mu}\right] T_n(k_1, \ldots, \bar{k}_n) + \mathcal{O}(q_2^2) \tag{6.6}$$

Inserting this expression in Eq. (6.5) we find

$$\begin{aligned}
&f_\xi^2 T_{n+2}(q_1, q_2, k_1, \ldots, \bar{k}_n)\\
&\quad= \left[D + \sum_{i=1}^{n} \hat{D}_i - (d + q_2 \cdot \partial_{q_2}) + q_1^\lambda \sum_{i=1}^{n} \hat{K}_{k_i,\lambda} + q_1^\lambda \hat{K}_{q_2,\lambda}\right]\\
&\qquad \times \left[D + \sum_{i=1}^{n} \hat{D}_i + q_2^\lambda \sum_{i=1}^{n} \hat{K}_{k_i,\lambda}\right] T_n(k_1, \ldots, \bar{k}_n) + \mathcal{O}(q_1^2, q_2^2, q_1 q_2)\\
&\quad= \Bigg[\left(D - d + \sum_{i=1}^{n} \hat{D}_i\right)\left(D + \sum_{i=1}^{n} \hat{D}_i\right) + \left(D + \sum_{i=1}^{n} \hat{D}_i - d - 1\right) q_2^\lambda \sum_{i=1}^{n} \hat{K}_{k_i,\lambda}\\
&\qquad + q_1^\lambda \sum_{i=1}^{n} \hat{K}_{k_i,\lambda}\left(D + \sum_{i=1}^{n} \hat{D}_i\right) - d\, q_1^\lambda \sum_{i=1}^{n} \hat{K}_{k_i,\lambda}\Bigg] T_n(k_1, \ldots, \bar{k}_n) + \mathcal{O}(q_1^2, q_2^2, q_1 q_2)
\end{aligned} \tag{6.7}$$

After the second equaltiy, the first three terms are just an organized expansion of the multiplication, where the form of $\hat{D}_i$ and $\hat{K}_{k_i,\lambda}$ is unimportant and one only needs to use in the second term the identity $q_2 \cdot \partial_{q_2} q_2^\lambda = q_2^\lambda$. The last term is obtained by using $q_1^\lambda \hat{K}_{q_2,\lambda}\, q_2^\rho = q_1^\rho(-d)$. The term of order $q_1^\lambda q_2^\rho$ has been neglected.

Using the commutation relation $[\hat{D}_i, \hat{K}_{k_i,\lambda}] = \hat{K}_{k_i,\lambda}$, the expression reduces to:

$$\begin{aligned}
f_\xi^2 T_{n+2}(q_1, q_2, k_1, \ldots, \bar{k}_n) = &\Bigg[\left(D - d + \sum_{i=1}^{n} \hat{D}_i\right)\left(D + \sum_{i=1}^{n} \hat{D}_i\right)\\
&+ (q_1^\lambda + q_2^\lambda) \sum_{i=1}^{n} \hat{K}_{k_i,\lambda}\left(D - d + \sum_{i=1}^{n} \hat{D}_i\right)\Bigg] T_n(k_1, \ldots, \bar{k}_n) + \mathcal{O}(q_1^2, q_2^2)
\end{aligned} \tag{6.8}$$

thus exactly reproducing the double-soft theorem Eq. (5.37) derived from current algebra. Based on this result, we conjecture that multi-soft dilaton amplitudes are fixed by the consecutive soft limit of single dilatons emitted one after the other, as just detailed for the consecutive double-soft emission.

7 Examples of dilaton amplitudes

7.1 The simplest D-dimensional conformally broken field theory

We consider amplitudes of the simplest D-dimensional conformal model presented in Sec. 3, and specifically given by Eq. (3.14). In the spontaneously broken phase, the Lagrangian is expanded around a nonzero vacuum expectation value for the conformal compensator field $\bar{\xi} = f_\xi/d + \xi$, where ξ is the dilaton field, and $f_\xi = dv^d$,

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \chi)^2 - \frac{1}{2}(\partial_\mu \xi)^2 - \frac{1}{2}m^2\chi^2 - \frac{m^2}{f_\xi}\chi^2\xi - \frac{c_2}{2}\frac{m^2}{f_\xi^2}\chi^2\xi^2 - \frac{c_3}{3!}\frac{m^2}{f_\xi^3}\chi^2\xi^3 - \frac{c_4}{4!}\frac{m^2}{f_\xi^4}\chi^2\xi^4 + \dots \tag{7.1}$$

where the mass is related to the dimensionless coupling constant and vev in the following manner:

$$m^2 = v^2\lambda^{2/d} \tag{7.2}$$

and the first few coefficients read:

$$c_2 = \frac{6-D}{2}\,, \quad c_3 = \frac{(6-D)(4-D)}{2}\,, \quad c_4 = \frac{(6-D)(4-D)(10-3D)}{4} \tag{7.3}$$

having used that $d = [\xi] = (D-2)/2$.

We have expanded the Lagrangian up to the sixth order in the fields, since we would now like to compute the three-, four-, five- and six-point amplitudes involving two massive external states χ, and one, two, three and four dilatons, respectively. The three point amplitude is given by the only three point vertex, reading:

$$T_3^{2\chi,\xi} = -\frac{2m^2}{f_\xi} = -\frac{4}{D-2}\frac{m^2}{v^d}\,. \tag{7.4}$$

There are no derivative couplings in the Lagrangian. Thus momenta enter amplitudes only from internal propagators. For amplitudes with two massive external states, only massive internal propagators enter. It is useful to define the variables

$$s_{i_1,i_2,\dots i_n} = (k_{i_1} + \dots + k_{i_n})^2 + m^2 \tag{7.5}$$

where the indices enumerate the external states. We will take the two massive state to be states 1 and 2, thus entering amplitudes with momenta k_1 and k_2, while states $3,\dots,n$ are taken to be dilatons entering amplitudes with momenta $k_3,\dots,k_n$.

The four-point amplitude then reads:

$$T_4^{2\chi,2\xi} = -\frac{2m^2}{f_\xi^2}\left(c_2 - \frac{2m^2}{s_{13}} - \frac{2m^2}{s_{23}}\right) \tag{7.6}$$

which has contributions both from the four-point vertex and two three-point amplitudes attached by an internal massive propagator. Momentum conservation is implicit in this expression, e.g. $s_{13} = s_{24}$.

The five-point amplitude reads:

$$\begin{aligned} T_5^{(2\chi,3\xi)} =& -c_3 \frac{2m^2}{f_\xi^3} + c_2 \frac{\left(2m^2\right)^2}{f_\xi^3} \left[\frac{1}{s_{13}} + \frac{1}{s_{23}} + \frac{1}{s_{14}} + \frac{1}{s_{24}} + \frac{1}{s_{15}} + \frac{1}{s_{25}} \right] \\ & - \left(\frac{2m^2}{f_\xi} \right)^3 \left[\frac{1}{s_{14}s_{23}} + \frac{1}{s_{24}s_{13}} + \frac{1}{s_{24}s_{15}} + \frac{1}{s_{14}s_{25}} + \frac{1}{s_{25}s_{13}} + \frac{1}{s_{15}s_{23}} \right] \end{aligned} \tag{7.7}$$

Finally, the six-point amplitude reads:

$$\begin{aligned} T_6^{(2\chi,4\xi)} =& - \frac{2c_4}{f_\xi^4} + c_3 \frac{(2m^2)^2}{f_\xi^4} \sum_{i=3}^{6} \left[\frac{1}{s_{1i}} + \frac{1}{s_{2i}} \right] + c_2^2 \frac{(2m^2)^2}{f_\xi^4} \sum_{i=4}^{6} \left[\frac{1}{s_{13i}} + \frac{1}{s_{23i}} \right] \\ & - c_2 \frac{(2m^2)^3}{2f_\xi^4} \sum_{i=3}^{6} \left[\frac{1}{s_{1i}} \sum_{j\neq 1,2,i} \left(\frac{1}{s_{2j}} + \frac{2}{s_{1ij}} \right) + \frac{1}{s_{2i}} \sum_{j\neq 1,2,i} \left(\frac{1}{s_{1j}} + \frac{2}{s_{2ij}} \right) \right] \\ & + \frac{(2m^2)^4}{f_\xi^4} \sum_{i=3}^{6} \frac{1}{s_{1i}} \sum_{j\neq 1,2,i} \frac{1}{s_{2j}} \sum_{k\neq 1,2,i,j} \frac{1}{s_{1ik}} \end{aligned} \tag{7.8}$$

The soft theorems provided in this work can now all be explicitly checked. Some details should be noted. First, one must fix an overall momentum variable by momentum conservation. Since we are interested in the expansion of the soft momenta, we do not impose momentum conservation on these variables, but instead impose it on one of the hard dilaton momenta. For instance, taking the momenta k_5 and k_6 to be soft for relating the 6-, 5- and 4-point amplitudes, a consistent choice is to take

$$k_4 \to \bar{k}_4 \tag{7.9}$$

where $\bar{k}_4$ is replaced by minus the sum of all other momenta of the 4-, 5- and 6-point amplitudes. This is already explicit in Eq. (7.6) for T_4, and is trivially imposed on the 5- and 6-point amplitudes, e.g. $s_{14} \to s_{235}$ in T_5 or $s_{14} \to s_{2356}$ in T_6.

The next important step one must make to check our expressions, is to subtract from the amplitudes all terms that belong to the Laurent series in the soft expansion, as defined in Eq. (4.27). For instance, considering the single soft limit of T_5 when $k_5 \ll k_i$ for $i = 1, \ldots, 4$, the part of T_5 that gives the Taylor series in k_5 is:

$$T_{5,\text{Taylor}}^{(2\chi,3\xi)} = -c_3 \frac{2m^2}{f_\xi^3} + c_2 \frac{\left(2m^2\right)^2}{f_\xi^3} \left[\frac{1}{s_{13}} + \frac{1}{s_{135}} \right] - \left(\frac{2m^2}{f_\xi} \right)^3 \left[\frac{1}{s_{13}s_{135}} \right] + (s_{1\ldots} \leftrightarrow s_{2\ldots}) \tag{7.10}$$

It is now obvious that at leading order in k_5, this expression reads:

$$T^{(2\chi,3\xi)}_{5,\text{Taylor}} = -\frac{2m^2}{f_\xi^3}\left[c_3 - 2c_2\frac{2m^2}{s_{13}} + \frac{(2m^2)^2}{s_{13}^2} + (s_{1\ldots} \leftrightarrow s_{2\ldots})\right] + \mathcal{O}(k_5) \tag{7.11}$$

It is a straightforward exercise from here to check that:

$$\frac{1}{f_\xi}\left[D - 4d - \sum_{i=1}^{3} k_i \cdot \partial_{k_i}\right] T_4^{(2\chi,2\xi)}(k_1, k_2, k_3, \bar{k}_4) = T^{(2\chi,3\xi)}_{5,\text{Taylor}} + \mathcal{O}(k_5) \tag{7.12}$$

where $d = (D-2)/2$, in agreement with Eq. (4.30). We remark that this expression also takes into account the massive terms in Eq. (7.11). The full expression for T_5 also has contributions at $\mathcal{O}(k_5^0)$ from expanding terms such as

$$\frac{1}{s_{15}s_{135}} = \frac{1}{2(k_1\cdot k_5)s_{13}}\left(1 + \frac{2(k_1+k_3)\cdot k_5}{s_{13}}\right) + \mathcal{O}(k_5) \tag{7.13}$$

however, these terms belong to the Laurent series of the soft expansion, and thus not part of Eq. (4.30).

The check of the single soft theorem is now extended to the subleading order of the five point amplitude. The $O(k_5)$ terms of the five point amplitude read:

$$T^{(2\chi,\,3\xi)}_{5\ \text{Taylor}}\Big|_{O(k_5)} = -2\frac{(2m^2)^2}{f_\xi^3}\frac{k_5\cdot(k_1+k_3)}{s_{13}^2}\left[\frac{(6-D)}{2} - \frac{2m^2}{s_{13}}\right] + (1 \leftrightarrow 2) \tag{7.14}$$

and it is straightforward to verify that it satisfies the identity:

$$\frac{k_5^\mu}{f_\xi}\sum_{i=1}^{3}\left[\frac{1}{2}k_{i\mu}\frac{\partial^2}{\partial k_{i\nu}\partial k_i^\nu} - k_i^\nu\frac{\partial}{\partial k_i^\mu}\frac{\partial}{\partial k_i^\nu} - d\frac{\partial}{\partial k_i^\mu}\right]T_4^{2\chi;2\xi}(k_1,k_2,k_3,\bar{k}_n) = T^{(2\chi,\,3\xi)}_{5\ \text{Taylor}}\Big|_{O(k_5)} \tag{7.15}$$

in agreement with the single soft theorem in Eq. (4.39), as originally proposed in Ref. [17].

The single-soft dilaton relations between T_6 and T_5 can be checked in a similar fashion.

The double-soft relations between the 5- and 3-point amplitudes can be easily verified. Choosing the soft momenta to be k_4, $k_5 \ll k_i$, $i = 1,2,3$, and using momentum conservation to replace k_3 with the other momenta, we first notice that only the first term in Eq. (7.7) is regular in the double-soft limit; i.e. all other terms (which carry the momentum dependence) belong to the Laurent series of the soft-expansion and should not be considered. It is then easy to see that

$$T^{(2\chi,3\xi)}_{5;\text{Taylor}} = \frac{(6-D)(4-D)}{2\,f_\xi^2}T_3^{(2\chi,\xi)} = \frac{1}{f_\xi^2}(D-4d)(D-3d)T_3^{(2\chi,\xi)} \tag{7.16}$$

where $d = (D-2)/2$ is the scaling dimension of all the fields. Since $T_3^{(2\chi,\xi)}$ is momentum independent, this expression is exactly the prediction of the double-soft theorems, both the one coming from the Ward identity of two dilatation currents, but also (trivially) the one coming from a dilatation current and a special conformal transformation current, since $\partial^\mu_{4,5} T^{(2\chi,3\xi)}_{5;\text{Taylor}} = \hat{K}^\mu_i T_3^{(2\chi,\xi)} = 0$. This example also shows, how reversibly one can predict coefficients of effective actions from the soft theorems, here a 5-point interaction coefficient from knowledge of the three-point interaction.

Before making the similar checks on the much less trivial case of 6- and 4-point amplitudes, let us note that the rest of T_5, which are on-shell singular for $k_4 = k_5 = 0$, read

$$
\begin{aligned}
T^{(2\chi,3\xi)}_{5;\text{Laurent}} &= \left(\frac{1}{s_{15}} + \frac{1}{s_{14}} + \frac{1}{s_{245}} + \frac{1}{s_{24}} + \frac{1}{s_{25}} + \frac{1}{s_{245}}\right) T_3^{(2\chi,\xi)} V_4^{(2\chi,2\xi)} \\
&+ (T_3^{(2\chi,\xi)})^3 \left[\frac{1}{s_{14}}\left(\frac{1}{s_{23}} + \frac{1}{s_{25}}\right) + \frac{1}{s_{15}}\left(\frac{1}{s_{24}} + \frac{1}{s_{23}}\right)\frac{1}{s_{245}}\left(\frac{1}{s_{24}} + \frac{1}{s_{25}}\right)\right]
\end{aligned}
\tag{7.17}
$$

where we identified the 4-point vertex $V_4^{(2\chi,2\xi)} = -\frac{2m^2}{f_\xi^2} c_2$. In this form, it is easy to see that all terms belonging to the Laurent series of the double-soft expansion are simply coming from processes where two soft dilatons are directly emitted from the hard external legs in different ways. This observation applies generally to all tree-amplitudes and trivializes thus the Laurent part of the soft-expansion.

We now consider the double-soft expansion of T_6 in terms of the soft momenta k_5 and k_6 through order $\mathcal{O}(k_5, k_6)$. The softness of the two momenta are taken to be equal, and we should thus consider the Taylor expansion of T_6 around $(k_5, k_6) = (0, 0)$. As prescribed we need to replace $k_4 \to \bar{k}_4$ and remove terms that belong to the Laurent series. From Eq. (7.8) we then find:

$$
\begin{aligned}
T^{(2\chi,4\xi)}_{6,\text{Taylor}} &= -c_4 \frac{2m^2}{f_\xi^4} + \left(\frac{2m^2}{f_\xi^2}\right)^2 \Bigg[\frac{c_3}{s_{13}} + \frac{c_3}{s_{1356}} + \frac{c_2^2}{s_{135}} + \frac{c_2^2}{s_{136}} + \frac{(2m^2)^2}{s_{13}s_{1356}}\Big(\frac{1}{s_{135}} + \frac{1}{s_{136}}\Big) \\
&- c_2\Big(\frac{2m^2}{s_{13}s_{1356}} + \frac{2m^2}{s_{13}s_{135}} + \frac{2m^2}{s_{13}s_{136}} + \frac{2m^2}{s_{136}s_{1356}} + \frac{2m^2}{s_{135}s_{1356}}\Big) + (s_{1\ldots} \leftrightarrow s_{2\ldots})\Bigg]
\end{aligned}
\tag{7.18}
$$

From here it is straightforward to show that the Taylor-expansion of this expression through first order around $(k_5, k_6) = (0, 0)$ exactly match the double-soft theorem in Eq. (5.37), by using the four-point amplitude in Eq. (7.6).

For completeness, we note again that the on-shell singular terms for $(k_5,\, k_6) = 0$; i.e. those belonging to the Laurent expansion of the amplitude, can be compactly written as:

$$
\begin{aligned}
T^{(2\chi,4\xi)}_{6,\text{Laurent}} &= \left[T_3^{(2\chi,\xi)} \frac{1}{s_{15}} T_5^{(2\chi;3xi)}(k_1 + k_5, k_2, k_3, k_4, k_6) + (5 \leftrightarrow 6) + (1 \leftrightarrow 2)\right] \\
&+ \left[\frac{V_4^{(2\chi,2\xi)}}{s_{256}} T_4^{(2\chi,2\xi)}(s_{13}, s_{14}) - \frac{T_3^{(2\chi,\xi)}}{s_{15}} T_4^{(2\chi,2\xi)}(s_{135}, s_{263}) \frac{T_3^{(2\chi,\xi)}}{s_{26}} + (1 \leftrightarrow 2)\right]
\end{aligned}
\tag{7.19}
$$

The terms in the first line corresponds to the cases where a soft dilaton is directly emitted from one of the hard, massive, external states, through the 3-point interaction vertex, which is equivalent to the amplitude T_3. The similar type of process where two soft dilatons are emitted from the hard, massive legs are given in the second line, involving two factors of T_3, while finally the case corresponding to the process where two soft dilatons are emitted simultaneously and from the same point from a hard, massive external state is also present and involves the 4-point interaction vertex, $V_4 = -c_2(2m^2)/f_\xi^2$.

7.2 $\mathcal{N} = 4$ super Yang-Mills theory on the Coulomb branch

The $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory is a (super)conformal field theory, where the gauge coupling stays nonperturbatively unrenormalized. Its action in component fields of the supermultiplet reads:

$$S = \int d^4x \mathrm{Tr}\left(-\frac{1}{4}G_{\mu\nu}G^{\mu\nu} - \frac{1}{2}(D_\mu\phi_i)^2 + \frac{i}{2}\bar{\psi}^r\gamma^\mu D_\mu\psi_r + \frac{g}{2}\bar{\psi}^r\Gamma^i_{rr'}[\phi_i, \psi^{r'}] + \frac{g^2}{4}\left([\phi_i, \phi_j]^2\right)\right) \tag{7.20}$$

where $r, r' = 1, \ldots, 4$, $i, j = 1 \ldots 6$, $D_\mu = \partial_\mu - ig[A_\mu, \cdot]$ and Γ_i are Euclidean six-dimensional Dirac matrices satifying the anti-commutation relations $\{\Gamma_i, \Gamma_j\} = \pm 2\delta_{ij}$. All fields are in the adjoint representation of the gauge group. The theory possesses an $SU(4)$ global R-symmetry, under which the fermions transform in the fundamental, **4**, representation and the scalars transform in the antisymmetric rank two, **6**, representation. The potential is given by

$$\mathrm{Tr}([\phi_i, \phi_j][\phi_i, \phi_j]) = -f^{abe}\ f^{cde}\ \phi_i^a\phi_j^b\phi_i^c\phi_j^d \tag{7.21}$$

where we have used $\phi_i = \phi_i^a T^a$, $[T^a, T^b] = if^{abc}T^c$, and $\mathrm{Tr}(T^aT^b) = \delta^{ab}$. If $a = b$ or $c = d$ then this expression vanishes, due to antisymmetry of the structure constant f^{abe}. This is independent of the value of ϕ_i and thus there is an $O(6)$ symmetry of this minimum. Any vev acquired by one of the scalars, breaks spontaneously the conformal symmetry and the $SU(4)$ global R-symmetry, isomorphic to $SO(6)$ (under which the scalars transform as vectors), is broken to $SU(4) \to Sp(4)$ (or equivalently $SO(6) \to SO(5)$). This is the so-called Coulomb branch of the theory. There will be 5 Nambu-Goldstone (NG) bosons belonging to the breaking of the global group, and one additional NG boson belonging to the breaking of conformal symmetry, i.e. *the dilaton*.

The gauge symmetry is also broken, but the additional gauge degrees of freedom of the scalars will be eaten up by the corresponding gauge bosons. To be specific, consider the $SU(N+1)$ gauge theory. The Coulomb branch induce $SU(N+1) \to SU(N) \times U(1)$. At low energies where massive states decouple, the $SU(N)$ and $U(1)$ sectors are two separate SYM theories, where the 6 NG bosons form the 6 massless scalars of the $U(1)$ supermultiplet.

The Ward identities and soft theorems presented in this work, can be checked to be satisfied by explicit computation of amplitudes in the weakly coupled regime of the above action on the Coulomb branch. For the single-soft dilaton, the check has been performed in Ref. [21] through one loop. Here we will instead consider the strongly coupled regime of the theory on the Coulomb branch by utilizing its gravity dual, for instance described in Section 6 of Ref. [43].

The gravity dual of the Coulomb branch is modeled by a D3-probe brane in the gravitational background of N D3-branes. In the large N limit backreaction on the background can be neglected. The dynamics of the D3 brane is governed by the Dirac-Born-Infeld (DBI) action on $\mathrm{AdS}_5 \times S^5$, which including the Wess-Zumino term for the zero-force condition (the pullback of the 5-form flux), is given by:

$$S = -\frac{1}{\kappa^2}\int d^4x \frac{r^4}{L^4}\left(\sqrt{-\det\left(\eta_{\mu\nu} + \frac{L^4}{r^4}\frac{\partial x^i}{\partial x^\mu}\frac{\partial x^i}{\partial x^\nu} + \kappa\frac{L^2}{r^2}F_{\mu\nu}\right)} - 1\right) \tag{7.22}$$

where $\kappa = (2\pi)^{3/2}\alpha'\sqrt{g_s}$, L is the AdS_5 radius, $r^2 = \sum_{i=1}^{6} x_i^2$ is the S^5 radius, $\eta_{\mu\nu}$ is the metric on the D3-brane with indices $\mu,\nu = 0,\ldots,3$ and x_i are the bulk coordinates with $i = 4,\ldots 9$. The scalar field dynamics on the D3-brane is given by correctly normalizing the bosonic coordinates

$$x_i = \kappa\phi_i\,, \quad \phi^2 = \sum_{i=1}^{6}\phi_i^2 \tag{7.23}$$

leading to

$$S = -\frac{1}{\lambda^2}\int d^4x\ \phi^4\left(\sqrt{-\det\left(\eta_{\mu\nu} + \frac{\lambda^2}{\phi^4}\frac{\partial \phi^i}{\partial x^\mu}\frac{\partial \phi^i}{\partial x^\nu} + \frac{\lambda}{\phi^2}F_{\mu\nu}\right)} - 1\right) \tag{7.24}$$

where

$$\lambda \equiv \frac{L^2}{\kappa} \tag{7.25}$$

We note that λ is a dimensionless constant. Using the dictionary relating the AdS_5 radius with the gauge coupling constant, one finds that λ is fixed by the $SU(N)$ gauge group of the $\mathcal{N} = 4$ SYM dual as follows

$$\frac{L^4}{\alpha'^2} = 4\pi N g_s \quad \Rightarrow\ \lambda = \frac{\sqrt{2N}}{2\pi} \tag{7.26}$$

The previous action is conformally invariant and is well-defined locally only if one of the scalar fields gets a non-vanishing vacuum expectation value that breaks spontaneously the conformal symmetry. Such a field with a non-vanishing vev will be the dilaton, while the other five scalar fields should describe the NG bosons corresponding to the breaking of the R-symmetry group $SO(6) \to SO(5)$.

In this setup, the Ward identities and soft theorems proposed in this work should be satisfied. We will here describe the check on the relations between the 4-, 5-, and 6-point dilaton tree amplitudes. (We note that as an effective field theory, only tree amplitudes of this theory are supposed to describe the $\mathcal{N} = 4$ SYM theory in the strongly coupled regime.) It is to this end only necessary to consider the part of the Lagrangian involving the dilaton field up to six-point interactions. We choose to take the following Coulomb branch:

$$\phi_i = v\delta_{i6} + \tilde{\phi}_i \,, \quad \tilde{\phi}_6 \equiv \xi \tag{7.27}$$

Then expanding the action, we find the following interaction Lagrangian for the dilaton

$$\mathcal{L}^{\xi}_{4,5,6} = \frac{\lambda^2}{8v^4}\left[1 - \frac{4\xi}{v} + 10\frac{\xi^2}{v^2}\right](\partial_\mu\xi\partial^\mu\xi)^2 - \frac{\lambda^4}{16v^8}(\partial_\mu\xi\partial^\mu\xi)^3 \tag{7.28}$$

describing dilaton self-interactions up to six-points.

It is straightforward to compute the four-point amplitude simply given by the contact interaction above. It reads:

$$A_4 = \frac{\lambda^2}{4v^4}[s_{12}s_{34} + s_{13}s_{24} + s_{14}s_{23}] = \frac{4\Delta a}{v^4}[s^2 + t^2 + u^2] \tag{7.29}$$

where $s_{ij} = (k_i + k_j)^2$ and in the second equality we identified the so-called $\Delta a = 16\lambda^2 = N^2/(8\pi)^2$ parameter of the works on the dilaton effective action and a-theorem [41, 42, 43], as well as the Mandelstam variables, $s = -s_{12}$, $t = -s_{13}$, $u = -s_{23}$, after imposing momentum conservation. The five-point amplitude is also straightforwardly computed from the contact interaction only, and is simply related to the four-point amplitude as follows:

$$\begin{aligned} A_5(1,2,3,4,5) = -\frac{4}{v}\Big[&A_4(1,2,3,4) + A_4(1,2,3,5) + A_4(1,2,4,5) \\ &+ A_4(1,3,4,5) + A_4(2,3,4,5)\Big] \end{aligned} \tag{7.30}$$

Finally, we provide the expression for the six-point amplitude. The computation is more involved, since there are contributions from three different interactions, where two involve the two different six-point contact interactions and one involve two four-point interactions where one dilaton is exchanged between them, thus containing an on-shell pole. Accordingly, we divide the amplitude in three partial expressions in the following way:

$$A_6 = \lambda^2 A_6^{\partial^4} + \lambda^4\left(A_6^{\partial^6} + A_6^{\text{pole}}\right) \tag{7.31}$$

where we defined the partial amplitudes without the coupling constant, to make explicit the different powers it enters with. It follows that since A_5 and A_4 only contain terms with λ^2 couplings, only the first partial amplitude is related to the lower-point amplitudes through the soft theorems. The soft theorems thus immediately predict that the two other partial amplitudes should either cancel or vanish in the soft limits.

The pole terms are straightforwardly given in terms of the four-point amplitude as follows:

$$\lambda^4 A_6^{\text{pole}} = \sum_{\text{ineq. perm.}} \frac{A_4(1,2,3,-[123])A_4([123],4,5,6)}{s_{123}} \tag{7.32}$$

where the entry [123] indicates that the momentum variable is equal to $(k_1 + k_2 + k_3)$, which due to momentum conservation is the momentum exchanged between the two vertices, explaining also the denominator (propagator). The sum is over the 10 inequivalent ways of choosing three out of the 6 momenta modulo the complement. The order is unimportant, since A_4 is totally symmetric in the four momenta. We can denote the 10 terms by their pole structure, given by:

$$\{s_{123},\ s_{124},\ s_{125},\ s_{126},\ s_{134},\ s_{135},\ s_{136},\ s_{145},\ s_{146},\ s_{156}\} \tag{7.33}$$

The partial amplitude $A_6^{\partial^4}$ can also be given in terms of A_4 in the following way:

$$\lambda^2 A_6^{\partial^4} = \frac{20}{v^2} \sum_{\text{cycl.perm}}^{1,\ldots 6} \left[A_4(1,2,3,4) + A_4(1,2,3,5) + \frac{1}{2} A_4(1,2,4,5) \right] \tag{7.34}$$

where the sum is over cyclic permutations of the indices $1,2,3,4,5,6$ generating six terms from each of the above three terms. The factor $1/2$ on the last term is due to the extra symmetry of that term, and thus takes care of overcounting of the sum.

Finally, the expression for the partial amplitude $A_6^{\partial^6}$ reads:

$$\lambda^4 A_6^{\partial^6} = \frac{3\lambda^4}{8v^8} \sum_{\text{cycl.perm}}^{1,\ldots 6} \left[\frac{s_{14}s_{25}s_{36}}{6} + \frac{s_{12}s_{34}s_{56}}{3} + \frac{s_{14}s_{23}s_{56}}{2} + \frac{s_{15}s_{24}s_{36}}{2} + s_{13}s_{24}s_{56} \right] \tag{7.35}$$

where the denominators of the terms in the bracket indicate the permutation symmetry of the terms to avoid overcounting, e.g. the first term reproduces itself by any of the 6 cyclic permutations.

We now study the single-soft and double-soft dilaton properties of these amplitudes. To study the relations between the 5- and 4-point amplitudes, we first fix momentum conservation and replace overall the momentum k_4 with minus the sum of the other momenta. It then becomes a straightforward exercise to check the following relations:

$$\begin{aligned}\lim_{k_5\to 0} A_5(1,2,3,\bar{4},5) &= \frac{1}{v}\left[4-\sum_{i=1}^{4}(d_i+k_i\cdot\partial_{k_i})\right]A_4(1,2,3,\bar{4}) \\ &= -\frac{1}{v}\sum_{i=1}^{3}k_i\cdot\partial_{k_i}A_4(s,t,u) = -\frac{4}{v}A_4(s,t,u)\end{aligned} \tag{7.36}$$

$$\begin{aligned}\lim_{k_5\to 0}\partial_5^\mu A_5(1,2,3,\bar{4},5) &= \frac{1}{v}\sum_{i=1}^{4}\hat{K}^\mu_{k_i}A_4(1,2,3,\bar{4}) = \frac{1}{v}\sum_{i=1}^{3}\hat{K}^\mu_{k_i}A_4(s,t,u) \\ &= -\frac{2\lambda^2}{v^5}[s_{23}\,k_1^\mu + s_{13}\,k_2^\mu + s_{12}\,k_3^\mu]\end{aligned} \tag{7.37}$$

To study the similar relations between the 6- and 5-point amplitudes we take k_6 to be soft. It is readily seen that $A_6^{\partial^6}$ and A_6^{pole} do not contribute to the soft limit $k_6 \to 0$ of A_6, since they contain in each term the soft momentum k_6. This is consistent with the observation made before that these two contributions should either vanish or cancel in the soft limits. The leading order single-soft relation between A_6 and A_5 is easiest to check by not imposing momentum conservation. It is then easy to confirm that:

$$\begin{aligned}&\lim_{k_6\to 0} A_6(1,2,3,4,5,6) = \lambda^2 \lim_{k_6\to 0} A_6^{\partial^4} = \frac{1}{v}\left[-1-\textstyle\sum_{i=1}^{5}k_i\cdot\partial_{k_i}\right]A_5(1,2,3,4,5) \\ &= \frac{20}{v^2}\Big[A_4(1,2,3,4)+A_4(1,2,3,5)+A_4(1,2,4,5)+A_4(1,3,4,5)+A_4(2,3,4,5)\Big]\end{aligned} \tag{7.38}$$

where the second equality readily follows from $\sum_{i=1}^{5} k_i\cdot\partial_{k_i}A_5 = 4A_5$. This works without the need to impose momentum conservation, because every term is linear in each momentum.

The subleading single-soft relation between A_6 and A_5 implies the two relations:

$$\lim_{k_6\to 0}\lambda^2\partial_6^\mu A_6^{\partial^4} = \frac{1}{v}\sum_{i=1}^{5}\hat{K}^\mu_{k_i}A_5 \tag{7.39a}$$

$$\lim_{k_6\to 0}\lambda^4\partial_6^\mu(A_6^{\partial^6}+A_6^{\text{pole}}) = 0 \tag{7.39b}$$

As explained before, the reason for having two relations is clear by noting that A_5 only involves terms with λ^2 couplings. The first relation can be seen as a constraint on the four-derivative interaction term from the five-point interaction. The second relation can be seen as a constraint on the six-derivative interaction term from the four-point interaction, because the pole terms are composed of two four-point vertices. The latter relation, which involves cancellation of poles, is nontrivially satisfied, and we have shown this in detail in the Appendix. We will here show in some detail the validity of the first relation. By expanding Eq. (7.34) at the first order in the soft momentum k_6, we get after some rewriting:

$$
\begin{aligned}
\lambda^2 A_6^{\partial^4}(1,2,3,\bar{4},5,6)\big|_{O(k_6)} &= -\frac{20}{v^2}\Bigg[A_4(1,2,3,6) + A_4(1,2,5,6) + A_4(1,3,5,6) \\
&\quad + A_4(2,3,5,6) + \frac{\lambda^2}{v^4}(k_1+k_2+k_3+k_5)^2(k_1+k_2+k_3+k_5)k_6\Bigg] \\
&= -\frac{20}{v^2}\Big[A_4(1,2,3,6) + A_4(1,2,5,6) + A_4(1,3,5,6) + A_4(2,3,5,6)\Big]
\end{aligned}
\tag{7.40}
$$

where the second equality follows from the identity $(k_1+k_2+k_3+k_5)^2(k_1+k_2+k_3+k_5)k_6 = -2(k_4k_6)(k_4k_6) = 0 + O(k_6^2)$.

On the other hand, the action of the subleading soft operator on the five point amplitude can be seen to give:

$$
\begin{aligned}
\frac{k_6^\mu}{v}\sum_{i\neq 4}^{5}\hat{K}_{k_i,\mu}\, A_5(1,2,3,\bar{4},5,6) &= -\frac{16\lambda^2}{v^6}\Big[k_6(k_1+k_2+k_3+k_5)(k_4k_6)\Big] \\
&\quad - \frac{20}{v^2}\Big[A_4(1,2,3,6) + A_4(1,2,5,6) + A_4(1,3,5,6) + A_4(2,3,5,6)\Big] \\
&= -\frac{20}{v^2}\Big[A_4(1,2,3,6) + A_4(1,2,5,6) + A_4(1,3,5,6) + A_4(2,3,5,6)\Big] + O(k_6^2)
\end{aligned}
\tag{7.41}
$$

We observe that, as predicted, Eq. (7.40) and Eq. (7.41) are identical.

Moving on to the double-soft theorems, we here check the newly obtained relations between the 6- and 4-point amplitudes. We fix k_4 by momentum conservation in both amplitudes, and take k_5 and k_6 to be soft momenta. We note that $A_6^{\partial^6}$ and A_6^{pole} (except for Laurent terms) do not contribute to the soft limit $k_5, k_6 \to 0$ of A_6 nor $\partial^\mu_{5,6}A_6$, since they contain in each term both momenta k_5 and k_6. The Laurent terms in A_6^{pole} are the non-regular ones in the soft limit, and to order k_5, k_6, they read:

$$
A_{6,\text{Laurent}} = \sum_{m=1}^{3}\frac{A_4(m,5,6,-[m56])A_4(\text{complement})}{s_{m56}} = -\frac{\lambda^4}{v^8}\sum_{m=1}^{3}\frac{(k_mk_5)(k_mk_6)}{k_m(k_5+k_6)} + \mathcal{O}(k_5^2, k_6^2)
\tag{7.42}
$$

where by the 'complement' we mean the other three momenta of the six-point amplitude on the external legs of A_4 and $[m56]$ on the internal leg. These are the lowest order terms in the soft expansion of A_6^{pole}, and correspond to the physical case where two soft dilatons are emitted simultaneously from one hard external leg. As such they are trivial.

Focusing on the nontrivial soft part of the six-point amplitude coming from $A_6^{\partial^4}$ it is straightforward to check that

$$\begin{aligned}\lim_{k_5,k_6\to 0} A_{6,\text{Taylor}}(1,2,3,\bar{4},5,6) &= \lambda^2 \lim_{k_5,k_6\to 0} A_6^{\partial^4}(1,2,3,\bar{4},5,6) = \frac{20}{v^2} A_4(1,2,3,\bar{4}) \\ &= \frac{1}{v^2}\left(-1-\sum_{i=1}^{3} k_i\cdot\partial_{k_i}\right)\left(-\sum_{i=1}^{3} k_i\cdot\partial_{k_i}\right) A_4(1,2,3,\bar{4})\end{aligned} \tag{7.43}$$

where the last line readily follows from Eq. (7.36). It is likewise easy to check the second double-soft identity.

$$\begin{aligned}\lim_{k_5,k_6\to 0} \partial_{5,6}^{\mu} A_{6,\text{Taylor}}(1,2,3,\bar{4},5,6) &= \lambda^2 \lim_{k_5,k_6\to 0} \partial_{5,6}^{\mu} A_6^{\partial^4}(1,2,3,\bar{4},5,6) \\ &= -\frac{10\lambda^2}{v^6}\left(s_{12}\, k_3^{\mu} + s_{13}\, k_2^{\mu} + s_{23}\, k_1^{\mu}\right) \\ &= \frac{1}{v^2}\sum_{i=1}^{3} \hat{K}_{k_i}^{\mu}\left(-1-\sum_{i=1}^{3} k_i\cdot\partial_{k_i}\right) A_4(1,2,3,\bar{4})\end{aligned} \tag{7.44}$$

where the last line follows immediately from Eq. (7.36) and (7.37).

8 Conclusions

In this paper we have studied the Ward identities of spontaneously broken scale and special conformal invariance, and from them derived the consequences for scattering amplitudes describing the interaction between the dilaton (the Nambu-Goldstone boson of the spontaneously broken conformal symmetry) and other spinless particles.

We have shown that the Ward identities give rise to soft theorems for the dilaton, which fix the behavior of scattering amplitudes involving soft dilatons, when scattering on other spinless states. The results are straightforward to generalized to scattering on spin-carrying states, namely one should simply include the spin-projection part in the analysis of special conformal transformations and amputate correlation functions accordingly.

Our main new result is the derivation of a double-soft theorem for the dilaton, which extends the single soft theorem found in Ref. [17] to the case of double-soft scattering of dilatons. It turns out that the amplitudes factorize in a soft and a hard part through linear order in the soft dilaton momenta, be there one or two soft dilatons involved. The soft part is given by operators related to the generators of the dilatation and special conformal transformation acting on the hard part, which is just the amplitude involving only the hard states. The new double-soft theorem turns out to be equivalent to performing two single-soft limits one after the other, and we like to point out that this is different from the case of double-soft scattering of pions. This observation allows us to propose that multi-soft scattering of dilatons should behave in the same way.

The dilaton soft theorems, being consequences of symmetries, are independent of a specific microscopic description and as such are universal. This means that any (quantum) theory of spontaneously broken conformal symmetry must obey the soft theorems put forward in this work. Consequently, this puts constraints on any effective description, for instance on the possible interactions and coupling in a low-energy effective action of spontaneously broken conformal invariance. We have specifically demonstrated this by checking explicitly the single- and double-soft theorems relating 4-, 5-, and 6-point amplitudes in two models; one that is valid semiclassically in any number of dimensions, and another that is fully valid in the quantum theory but only in four dimensions; namely the Coulomb branch in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, which we studied in the strongly coupled regime. Both theories are frequently studied in the literature, and our detailed checks may serve as new relations among amplitudes of the theories that were not noticed before.

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A Single-soft limit of A_6 of section 7.2

In this appendix we show that Eq. (7.39b) is fulfilled. Let us summarize the expressions for the amplitudes in Sec. 7.2:

$$A_4(1,2,3,4) = \frac{\lambda^2}{4v^4}[s_{12}s_{34} + s_{13}s_{24} + s_{14}s_{23}] \tag{A.1}$$

$$\begin{aligned} A_5(1,2,3,4,5) = -\frac{4}{v}\Big[&A_4(1,2,3,4) + A_4(1,2,3,5) + A_4(1,2,4,5) \\ &+ A_4(1,3,4,5) + A_4(2,3,4,5)\Big] \end{aligned} \tag{A.2}$$

$$A_6 = \lambda^2 A_6^{\partial^4} + \lambda^4 \left(A_6^{\partial^6} + A_6^{\text{pole}}\right) \tag{A.3}$$

with

$$\lambda^2 A_6^{\partial^4} = \frac{20}{v^2} \sum_{\text{cycl.perm}}^{1,\ldots 6} \left[A_4(1,2,3,4) + A_4(1,2,3,5) + \frac{1}{2} A_4(1,2,4,5) \right] \tag{A.4}$$

$$\lambda^4 A_6^{\partial^6} = -\frac{3\lambda^4}{8v^8} \sum_{\text{cycl.perm}}^{1,\ldots 6} \left[\frac{s_{14}s_{25}s_{36}}{6} + \frac{s_{12}s_{34}s_{56}}{3} + \frac{s_{14}s_{23}s_{56}}{2} + \frac{s_{15}s_{24}s_{36}}{2} + s_{13}s_{24}s_{56} \right] \tag{A.5}$$

$$\lambda^4 A_6^{\text{pole}} = \sum_{\text{ineq. perm.}} \frac{A_4(1,2,3,-[123]) A_4([123],4,5,6)}{s_{123}} \tag{A.6}$$

where the last sum over inequivalent permutations are given by the denominator structures:

$$\{s_{123},\ s_{124},\ s_{125},\ s_{126},\ s_{134},\ s_{135},\ s_{136},\ s_{145},\ s_{146},\ s_{156}\} \tag{A.7}$$

As explained in the main text, the soft limit $k_6 \to 0$ of A_6 reproduces the correct soft theorem, since $A_6^{\partial^6}$ and A_6^{pole} both vanish in this limit. At subleading order they do not vanish, but should instead cancel each other, since they cannot contribute to the soft theorem due to the coupling being λ^4, while A_5 has only terms with coupling λ^2. This cancellation can only occur if the denominators in A_6^{pole} cancel out at subleading order. Let us first show this.

To show that the denominators of A_6^{pole} cancel out at subleading order, we first rewrite all denominators explicitly in terms of k_6:

$$\begin{aligned} &\to \{s_{456},\ s_{356},\ s_{346},\ s_{126},\ s_{256},\ s_{246},\ s_{136},\ s_{236},\ s_{146},\ s_{156}\} \\ &\overset{k_6 \to 0}{\to} \{s_{45},\ s_{35},\ s_{34},\ s_{12},\ s_{25},\ s_{24},\ s_{13},\ s_{23},\ s_{14},\ s_{15}\} \end{aligned} \tag{A.8}$$

Now consider the numerator corresponding to the first term above:

$$\begin{aligned} &A_4(1,2,3,[456]) A_4(-[456],4,5,6) = \left(\frac{\lambda^2}{4v^4}\right)^2 \\ &\times [s_{12}(s_{34}+s_{35}+s_{36}) + s_{13}(s_{24}+s_{25}+s_{26}) + s_{23}(s_{14}+s_{15}+s_{16})] \\ &\times [-(s_{45}+s_{46})s_{56} - (s_{45}+s_{56})s_{46} - (s_{46}+s_{56})s_{45}] \end{aligned} \tag{A.9}$$

To linear order in k_6 this expression reduces to:

$$\begin{aligned} &A_4(1,2,3,[456]) A_4(-[456],4,5,6) = -\left(\frac{\lambda^2}{4v^4}\right)^2 \\ &\times [s_{12}(s_{34}+s_{35}) + s_{13}(s_{24}+s_{25}) + s_{23}(s_{14}+s_{15})] \\ &\times 2[s_{46}+s_{56}]s_{45} + \mathcal{O}(k_6^2) \end{aligned} \tag{A.10}$$

We observe that s_{45} factorizes and exactly cancels the denominator, which is also equal to s_{45}. We may also observe that the second line is simply:

$$[s_{12}(s_{34}+s_{35})+s_{13}(s_{24}+s_{25})+s_{23}(s_{14}+s_{15})] = \frac{4v^4}{\lambda^2}\left(A_4(1,2,3,4)+A_4(1,2,3,5)\right) \tag{A.11}$$

Summarizing, we have shown that:

$$\frac{A_4(1,2,3,[456])A_4(-[456],4,5,6)}{s_{456}} = -\frac{\lambda^2}{v^4}k_6\cdot(k_4+k_5)\Big[A_4(1,2,3,4)+A_4(1,2,3,5)\Big]+\mathcal{O}(k_6^2) \tag{A.12}$$

By summing over all ten inequivalent permutation terms we find (for short we denote $A_4(i,j,k,l) = A_{ijkl}$)

$$\begin{aligned}
\lambda^4 A_6^{\text{pole}} =& -\frac{\lambda^2}{v^4}k_6\cdot k_1\Big[A_{1234}+A_{1235}+A_{1245}+A_{1345}+4A_{2345}\Big]\\
&-\frac{\lambda^2}{v^4}k_6\cdot k_2\Big[A_{1234}+A_{1235}+A_{1245}+A_{2345}+4A_{1345}\Big]\\
&-\frac{\lambda^2}{v^4}k_6\cdot k_3\Big[A_{1234}+A_{1235}+A_{1345}+A_{2345}+4A_{1245}\Big]\\
&-\frac{\lambda^2}{v^4}k_6\cdot k_4\Big[A_{1234}+A_{1245}+A_{1345}+A_{2345}+4A_{1235}\Big]\\
&-\frac{\lambda^2}{v^4}k_6\cdot k_5\Big[A_{1235}+A_{1245}+A_{1345}+A_{2345}+4A_{1234}\Big]+\mathcal{O}(k_6^2)\\
=& -\frac{\lambda^2}{v^4}k_6\cdot(k_1+k_2+k_3+k_4+4k_5)A_{1234}+\cdots\\
=& -\frac{\lambda^2}{v^4}k_6\cdot(3k_5)A_{1234}+\cdots
\end{aligned} \tag{A.13}$$

where the $\cdots$ in the last and next to last line should be understood as the 5 other terms, which are simply the 5 cyclic permutations of the indices 2345. To get the last expression we used momentum conservation $k_1+k_2+k_3+k_4 = -k_5-k_6$ where k_6 gives rise to a higher order term and can be neglected. Notice that we are not fixing one momentum by momentum conservation, rather we use it to simplify expressions. One may fix a momentum in the end after all rewritings. Explicitly, we have found:

$$\lambda^4 A_6^{\text{pole}} = -3\frac{\lambda^2}{v^4}\Big[A_{1234}\,k_5+A_{1235}\,k_4+A_{1245}\,k_3+A_{1345}\,k_4+A_{2345}\,k_1\Big]\cdot k_6+\mathcal{O}(k_6^2) \tag{A.14}$$

Let us now consider $A_6^{\partial^6}$ which is linear in k_6 (in fact, in any momenta):

$$\lambda^4 A_6^{\partial^6} = \frac{3\lambda^4}{8v^8}\Big[s_{14}s_{25}s_{36} + s_{12}s_{34}s_{56} + s_{23}s_{45}s_{61} + s_{14}s_{23}s_{56} + s_{25}s_{34}s_{16} + s_{36}s_{45}s_{12} + s_{15}s_{24}s_{36} + s_{26}s_{35}s_{14} + s_{13}s_{46}s_{25} + s_{13}s_{24}s_{56} + s_{24}s_{35}s_{61} + s_{35}s_{46}s_{12} + s_{46}s_{51}s_{23} + s_{51}s_{62}s_{34} + s_{62}s_{13}s_{45}\Big] \tag{A.15}$$

It is easy to see that by factorizing k_6 in each term and collecting together the k_i terms it multiplies we get:

$$\lambda^4 A_6^{\partial^6} = \frac{3\lambda^2}{v^4}\Big[A_{1234}\, k_5 + A_{1235}\, k_4 + A_{1245}\, k_3 + A_{1345}\, k_4 + A_{2345}\, k_1\Big] \cdot k_6 \tag{A.16}$$

Comparing this expression with that in Eq. (A.14), we observe that they are identical but with opposite sign. Thus at linear order in k_6 the terms proportional to λ^4 in Eq. (A.3) do not contribute, which as explained is an expected consequence of the soft theorem at subleading order in the soft momentum k_6. This reversibly illustrates the strong constraints that soft theorems put on effective field theories.

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