

# Dimensional regularization of the IR divergences in the Fokker action of point-particle binaries at the fourth post-Newtonian order

Laura Bernard,<sup>1,\*</sup> Luc Blanchet,<sup>2,†</sup> Alejandro  
Bohé,<sup>3,‡</sup> Guillaume Faye,<sup>2,§</sup> and Sylvain Marsat<sup>3,¶</sup>

<sup>1</sup>*CENTRA, Departamento de Física,  
Instituto Superior Técnico – IST, Universidade de Lisboa – UL,  
Avenida Rovisco Pais 1, 1049 Lisboa, Portugal*

<sup>2</sup>*G<sub>R</sub>εCO, Institut d’Astrophysique de Paris — UMR 7095 du CNRS,  
Université Pierre & Marie Curie, 98<sup>bis</sup> boulevard Arago, 75014 Paris, France*

<sup>3</sup>*Albert Einstein Institut, Am Muehlenberg 1, 14476 Potsdam-Golm, Germany*

(Dated: July 6, 2017)

## Abstract

The Fokker action of point-particle binaries at the fourth post-Newtonian (4PN) approximation of general relativity has been determined previously. However two ambiguity parameters associated with infra-red (IR) divergencies of spatial integrals had to be introduced. These two parameters were fixed by comparison with gravitational self-force (GSF) calculations of the conserved energy and periastron advance for circular orbits in the test-mass limit. In the present paper we determine one of these ambiguities from first principle, by means of dimensional regularization. Our computation is thus entirely defined within the dimensional regularization scheme, for treating at once the IR and ultra-violet (UV) divergencies. In particular, we obtain crucial contributions coming from the non-local tail term in arbitrary dimensions. The remaining ambiguity parameter is equivalent to the one appearing in the ADM Hamiltonian approach.

PACS numbers: 04.25.Nx, 04.30.-w, 97.60.Jd, 97.60.Lf

---

\*Electronic address: laura.bernard@tecnico.ulisboa.pt

†Electronic address: blanchet@iap.fr

‡Electronic address: alejandro.bohe@aei.mpg.de

§Electronic address: faye@iap.fr

¶Electronic address: sylvain.marsat@aei.mpg.de

## I. INTRODUCTION

In previous works [1, 2] (respectively referred to as Papers I and II), we determined the Fokker Lagrangian of the motion of compact binary systems (without spins) in harmonic coordinates at the fourth post-Newtonian (4PN) approximation of general relativity.<sup>1</sup> Equivalent results had been previously achieved using the ADM Hamiltonian formalism, in ADM-like coordinates, developed at 4PN order [3–7]. Partial results have been obtained at the 4PN order using the effective field theory (EFT) approach [8–11]. A prominent feature of the 4PN order is the non-locality (in time) due to the imprint of gravitational wave tails starting at that approximation (see also [12]).

Carefully choosing and implementing regularizations plays a crucial role in this field. In Papers I and II we adopted a dimensional regularization scheme for treating the ultra-violet (UV) divergences associated with point-particles, as well as a Hadamard regularization for curing the infra-red (IR) divergences occurring at the bound at infinity of integrals in the gravitational part of the Fokker action (as we know, IR divergences start occurring precisely at the 4PN order). Unfortunately, we had to introduce in Paper I an “*ambiguity parameter*” reflecting some incompleteness in the Hadamard treatment of the IR divergences. This ambiguity was then fixed by matching the conserved energy in the case of circular orbits to known results obtained from gravitational self-force (GSF) calculations in the test-mass limit [13–16]. An equivalent ambiguity parameter had also to be included in the ADM Hamiltonian formalism [6]. Furthermore, we were forced to add in Paper II a *second* ambiguity parameter in order to match the periastron advance for circular orbits with the results from GSF calculations as well. The GSF results for the periastron are known from numerical [17–19] and analytical [7, 12, 20, 21] studies. As we conjectured in Paper II, this second ambiguity parameter was in fact mandatory, since the difference between different prescriptions for the IR regularization of integrals at infinity can be reduced, after a suitable shift of the world-lines, to two and only two offending terms at the 4PN order in the Lagrangian.

The aim of the present paper is to resolve the issue of the second ambiguity parameter, *i.e.*, to compute its value from *first principles*. We shall indeed prove that this value yields the correct periastron advance of circular orbits (once the conserved circular energy has been adjusted by choosing the value of the first ambiguity). Therefore, we conclude that our 4PN dynamics based on the Fokker action in harmonic coordinates is plagued by one and only one ambiguity parameter, in complete agreement with the 4PN Hamiltonian formalism [6, 7]. Furthermore, after having fixed this ambiguity parameter by comparing the circular energy with GSF computations, the periastron advance for circular orbits turns out to be correct as well (of course, one may also determine the first ambiguity from the periastron and then find that the energy is correct as well).

In this paper we shall employ the powerful dimensional regularization [22–24] (instead of Hadamard’s) for resolving the IR divergences of the Fokker action occurring at the bound at infinity of spatial integrals. Therefore our Fokker action will now be entirely based on dimensional regularization, for both the IR and UV divergences. We have two main tasks:

1. Computing the *difference* between the dimensional regularized and the Hadamard regularized gravitational (*i.e.*, Einstein-Hilbert) parts of the Fokker action. For this

---

<sup>1</sup> As usual the  $n$ PN order means the terms of order  $(v/c)^{2n}$  in the equations of motion relatively to the Newtonian acceleration.

calculation we shall use known formulas for the “difference” between these two regularizations coming from Refs. [25, 26]. The post-Newtonian calculation and needed accuracy of this calculation will follow precisely the rules of the method called “ $n + 2$ ” in Paper I (see Sec. IV A there);

2. Evaluating the non-local tail term in  $d$  dimensions or, rather, an associated homogeneous solution that is to be added to the “difference” computed from the  $n + 2$  method. As we shall see (and in agreement with EFT works [9, 10, 27, 28]), such tail-induced homogeneous solution contains a UV-like pole in  $d$  dimensions. We shall prove that this pole precisely cancels the IR-like pole remaining from the  $n + 2$  method after applying suitable shifts, while the finite part gives a supplementary contribution of the form of the ambiguity parameters of Papers I and II.

Adding up the contributions from the latter two steps (and also, subtracting off a particular surface term in our previous Hadamard IR regularization scheme), we finally obtain that the modification of the Lagrangian takes exactly the form postulated in Paper II, but is still parametrized by an unknown parameter, called  $\kappa$ , and corresponding to the “first” ambiguity of Paper I. Adjusting  $\kappa$  so that the energy for circular orbits at 4PN order agrees with GSF calculations, we find that the two ambiguity parameters  $\delta_1$  and  $\delta_2$  (following exactly their definition in Sec. II of Paper II) are in complete agreement with the result of Paper II, see Eq. (2.6) there, so that the periastron advance for circular orbits at 4PN order is also correct.

The plan of this paper is as follows. In Sec. II we obtain the difference between the dimensional and Hadamard IR regularizations for the gravitational part of the Fokker action. After application of shifts we find that such a difference contains a residual IR pole. In Sec. III we investigate general technical formulas for the computation of the near zone expansion of the solution of the wave equation in  $d$  dimensions. These formulas are then applied in Sec. IV to the derivation of the tail term in the near zone metric and then in the Fokker action. We obtain a UV pole and find that it exactly cancels the IR pole coming from the gravitational part of the Fokker action. This determines the second ambiguity (in Sec. V) but we are not able at this stage to recover the first ambiguity, and more work should be done on the matching between the near zone and the far zone in our formalism. Technical appendices provide important material on: the homogeneous solutions of the wave equation and their PN expansion in App. A; the multipole expansion of elementary functions and potentials in  $d$  dimensions, in App. B; some distributional limits of Green’s functions in App. C; the computation of some particular intricate coefficient in App. D.

## II. DIMENSIONAL REGULARIZATION OF INFRA-RED DIVERGENCES

In Paper I [1] it was shown that IR divergences, due to the behaviour of spatial integrals at infinity, start to appear at the 4PN order in the Fokker action of general matter systems. These IR divergences are associated with non-local tail effects in the dynamics occurring at 4PN order [29, 30]. In Paper I it was found that two arbitrary scales respectively associated with tails (denoted  $s_0$  in Paper I) and the IR cut-off (denoted  $r_0$ ) combine to give an “ambiguity” parameter  $\alpha = \ln(r_0/s_0)$  which could not be determined within the method. Equivalent results had been obtained with the Hamiltonian formalism in Ref. [6]. However, in contrast to the Hamiltonian formalism, we had to introduce in Paper II a second ambiguity parameter and argued that it was due to our particular treatment of the IR divergences based

on the Hadamard “partie finie” integral. On the other hand, the UV divergences associated with point particles were cured by dimensional regularization.

In the present paper we shall employ dimensional regularization for both the IR and UV divergences. As we shall see, using dimensional regularization does modify the end result for the Fokker Lagrangian (and associated Hamiltonian), but in a way that is fully consistent with the conjecture put forward in Paper II. Therefore this justifies the final 4PN dynamics obtained in Paper II and in particular, we confirm that the 4PN dynamics is compatible with existing GSF computations of the energy and periastron advance for circular orbits.

We want to regularize the three-dimensional divergent integral

$$I = \int d^3\mathbf{x} F(\mathbf{x}), \quad (2.1)$$

where the function  $F$  is obtained by following the PN iteration procedure of the field equations using the method  $n+2$  (see Sec. IV A of Paper I). The integral (2.1) represents a generic term in the gravitational (Einstein-Hilbert) part of the Fokker Lagrangian  $L_g$ . Specifically, since we are dealing with the IR bound at infinity, we consider

$$I_{\mathcal{R}} = \int_{r>\mathcal{R}} d^3\mathbf{x} F(\mathbf{x}), \quad (2.2)$$

where the integration domain is restricted to be  $r = |\mathbf{x}| > \mathcal{R}$ , with  $\mathcal{R}$  being a sufficiently large constant radius. The divergences occur from the expansion of  $F$  when  $r \rightarrow +\infty$ , which is of the type (for any  $N \in \mathbb{N}$ )

$$F(\mathbf{x}) = \sum_{p=-p_0}^N \frac{1}{r^p} \varphi_p(\mathbf{n}) + o\left(\frac{1}{r^N}\right). \quad (2.3)$$

The coefficients  $\varphi_p$  depend on the unit direction  $\mathbf{n} = \mathbf{x}/r$  and on  $p \in \mathbb{Z}$ ; the minimal value of  $p$  corresponds to some highly divergent behaviour with growing power  $\sim r^{p_0}$  of the distance. In what follows we shall write for simplicity some formal expansion series without expliciting the remainder term, that is

$$F(\mathbf{x}) = \sum_{p \geq -p_0} \frac{1}{r^p} \varphi_p(\mathbf{n}). \quad (2.4)$$

In Paper I a regularization factor  $(r/r_0)^B$  was introduced into the integrand and the integral was considered in the sense of analytic continuation in  $B \in \mathbb{C}$ . Then the regularized value of the integral was defined as the finite part (FP), *i.e.*, the coefficient of the zeroth power of  $B$ , in the Laurent expansion of the regularized integral when  $B \rightarrow 0$ . This prescription, which is equivalent to a Hadamard regularization (HR), reads

$$I_{\mathcal{R}}^{\text{HR}} = \text{FP}_{B=0} \int_{r>\mathcal{R}} d^3\mathbf{x} \left(\frac{r}{r_0}\right)^B F(\mathbf{x}). \quad (2.5)$$

A straightforward calculation, plugging (2.4) into (2.2) (where  $\mathcal{R}$  is a large radius), yields the HR-regularized version of the integral as

$$I_{\mathcal{R}}^{\text{HR}} = - \sum_{p \neq 3} \frac{\mathcal{R}^{3-p}}{3-p} \int d\Omega_2 \varphi_p(\mathbf{n}) - \ln\left(\frac{\mathcal{R}}{r_0}\right) \int d\Omega_2 \varphi_3(\mathbf{n}), \quad (2.6)$$

where  $d\Omega_2$  denotes the standard surface element in the direction  $\mathbf{n}$ . As we see the crucial coefficient in the expansion (2.4) is that for  $p = 3$ ; it corresponds to a logarithmic divergence of the original integral (2.2).

In the present paper, motivated by the success of dimensional regularization when treating the UV divergencies, we treat the IR divergences of the integral (2.1) with the same regularization procedure. In  $d$  spatial dimensions the equivalent of  $F(\mathbf{x})$ , *i.e.*, arising from the same PN iteration of the field equations but performed in  $d$  dimensions, will be a function  $F^{(d)}(\mathbf{x})$  with a more general expansion when  $r \rightarrow +\infty$  of the type<sup>2</sup>

$$F^{(d)}(\mathbf{x}) = \sum_{p \geq -p_0} \sum_{q=-q_0}^{q_1} \frac{1}{r^p} \left( \frac{\ell_0}{r} \right)^{q\varepsilon} \varphi_{p,q}^{(\varepsilon)}(\mathbf{n}). \quad (2.7)$$

The difference with (2.4) is that the powers of  $1/r$  now depend linearly on  $\varepsilon = d - 3$ , with  $p \in \mathbb{Z}$  as before and with also  $q \in \mathbb{Z}$ , bounded from below and from above by  $-q_0$  and  $q_1$ . Here  $\ell_0$  denotes the usual constant scale associated with dimensional regularization. Assuming that the coefficients  $\varphi_{p,q}^{(\varepsilon)}$  have a well-defined limit when  $\varepsilon \rightarrow 0$ , *i.e.*, that they do not contain any pole  $\propto 1/\varepsilon$  (such an assumption is always verified at 4PN order), we obtain the following relation with the coefficients  $\varphi_p$  in the limit  $\varepsilon \rightarrow 0$ ,

$$\varphi_p(\mathbf{n}) = \sum_{q=-q_0}^{q_1} \varphi_{p,q}^{(\varepsilon=0)}(\mathbf{n}). \quad (2.8)$$

The dimensional regularization (DR) prescription, to be considered as usual in the sense of complex analytic continuation in  $d \in \mathbb{C}$ , reads now

$$I_{\mathcal{R}}^{\text{DR}} = \int_{r > \mathcal{R}} \frac{d^d \mathbf{x}}{\ell_0^{d-3}} F^{(d)}(\mathbf{x}). \quad (2.9)$$

Working in the limit where  $\varepsilon \rightarrow 0$ , *i.e.*, keeping only the pole  $\propto 1/\varepsilon$  followed by the finite part  $\propto \varepsilon^0$ , and using also the relation (2.8), we readily obtain<sup>3</sup>

$$I_{\mathcal{R}}^{\text{DR}} = - \sum_{p \neq 3} \frac{\mathcal{R}^{3-p}}{3-p} \int d\Omega_2 \varphi_p(\mathbf{n}) + \sum_q \left[ \frac{1}{(q-1)\varepsilon} - \ln \left( \frac{\mathcal{R}}{\ell_0} \right) \right] \int d\Omega_{2+\varepsilon} \varphi_{3,q}^{(\varepsilon)}(\mathbf{n}) + \mathcal{O}(\varepsilon). \quad (2.10)$$

Very important in this formula, is that the angular integration in the second term, because of the presence of the pole, is to be performed over the  $(d-1)$ -dimensional sphere, with surface element  $d\Omega_{2+\varepsilon}(\mathbf{n})$ , up to order  $\varepsilon$ .

We shall thus add to the computations of Papers I and II the *difference* between the two prescriptions, say  $\mathcal{D}I = I_{\mathcal{R}}^{\text{DR}} - I_{\mathcal{R}}^{\text{HR}}$ . Note that the first term in (2.10) is identical to the corresponding term in (2.6), and thus cancels out in the difference. We thus obtain, to dominant order when  $\varepsilon \rightarrow 0$ ,

$$\mathcal{D}I = \sum_q \left[ \frac{1}{(q-1)\varepsilon} - \ln \left( \frac{r_0}{\ell_0} \right) \right] \int d\Omega_{2+\varepsilon} \varphi_{3,q}^{(\varepsilon)}(\mathbf{n}) + \mathcal{O}(\varepsilon), \quad (2.11)$$

<sup>2</sup> In Appendix B we shall refer to the far zone expansion when  $r \rightarrow +\infty$  as a “multipole” expansion and conveniently denote it as  $\mathcal{M}(F^{(d)})$ .

<sup>3</sup> *A priori* the result also contains terms that diverge at infinity. These terms correspond to the coefficients  $\varphi_{p,q}^{(\varepsilon)}$  with  $q = 1$  and  $p \leq 3$ , but do not appear in our computation.

where, as expected, the scale  $\mathcal{R}$  has disappeared from the difference.

We have applied the formula (2.11) to each of the terms composing the gravitational part  $L_g$  of the Fokker Lagrangian. Thus, we have computed the expansion when  $r \rightarrow +\infty$  of the various potentials parametrizing the metric in  $d$  dimensions as given by Eqs. (4.14)–(4.15) in Paper I.<sup>4</sup> These potentials are those needed at the 4PN order following the method “ $n + 2$ ” described in Sec. IVA of Paper I. For this calculation we use the far-zone expansion of some elementary functions in  $d$  dimensions (notably the elementary Fock kernel  $g$  [32]); this will be described in Appendix B. Once we have computed the expansions of all the potentials we plug them into  $L_g$  and obtain the coefficients  $\varphi_{3,q}^{(\varepsilon)}(\mathbf{n})$  corresponding to all the terms. Then we simply evaluate Eq. (2.11) for each of the terms<sup>5</sup> and obtain the Fokker action with IR divergences correctly regularized by means of DR.

The total difference will actually be called  $\mathcal{D}L_g^{\text{inst}} = \sum \mathcal{D}I$ . Indeed it is composed of all the terms obtained following the method  $n + 2$ , which keeps track of the “instantaneous” terms, but neglects the “tail” term which will be investigated in Sec. IV. Thus,  $\mathcal{D}L_g^{\text{inst}}$  is composed of a pole part  $\propto 1/\varepsilon$  followed by a finite part  $\propto \varepsilon^0$  which depends on the arbitrary IR scale  $r_0$  as well as on  $\ell_0$ . We next look for a (physically irrelevant) shift that will remove most of the poles  $1/\varepsilon$  and eliminate most of the dependence on the constant  $r_0$ . We find, after applying a suitable shift, that the difference becomes (irreducibly)

$$\begin{aligned} \mathcal{D}L_g^{\text{inst}} = & \frac{G^2 m}{5c^8} \left[ \frac{1}{\varepsilon} - 2 \ln \left( \frac{\sqrt{\bar{q}} r_0}{\ell_0} \right) \right] \left( I_{ij}^{(3)} \right)^2 \\ & + \frac{G^4 m m_1^2 m_2^2}{c^8 r_{12}^4} \left( -\frac{2479}{150} (n_{12} v_{12})^2 + \frac{1234}{75} v_{12}^2 \right) + \mathcal{O}(\varepsilon) . \end{aligned} \quad (2.12)$$

We pose  $\bar{q} = 4\pi e^{\gamma_E}$  with  $\gamma_E$  being the Euler constant. The other notations are exactly the same as in Papers I and II, *e.g.*,  $m = m_1 + m_2$  is the total mass and  $(n_{12} v_{12})$  is the Euclidean scalar between the relative direction between the two bodies and their relative velocity.

As we see there is a remaining pole in Eq. (2.12), and we shall prove in Sec. IV that it will be cancelled by a corresponding pole coming from the 4PN tail term evaluated in  $d$  dimensions. The pole is proportional to the square of the third time-derivative of the quadrupole moment  $I_{ij}$ . In a small 4PN term, the quadrupole can be taken to be the Newtonian one; however, consistently with the pole  $1/\varepsilon$  in front, it is to be evaluated in  $d$  dimensions, up to order  $\varepsilon$  included. For completeness we show here the complete expression up to that order,

$$\begin{aligned} \left( I_{ij}^{(3)} \right)^2 = & \frac{G^3 m_1^2 m_2^2}{r_{12}^4} \left( -\frac{88}{3} (n_{12} v_{12})^2 + 32 v_{12}^2 \right) \left[ 1 - \frac{\varepsilon}{2} \ln \left( \frac{\sqrt{\bar{q}} r_{12}}{\ell_0} \right) \right] \\ & + \varepsilon \frac{G^3 m_1^2 m_2^2}{r_{12}^4} \left( -\frac{836}{9} (n_{12} v_{12})^2 + 96 v_{12}^2 \right) + \mathcal{O}(\varepsilon^2) . \end{aligned} \quad (2.13)$$

Gladly, we discover that the two terms in the second line of Eq. (2.12) have exactly the structure of the two “ambiguity” parameters  $\delta_1$  and  $\delta_2$  that were introduced in Paper II. As we shall see, this will permit to confirm the conjecture advocated in Paper II, namely that

<sup>4</sup> Extensive use is made of the software Mathematica together with the tensor package *xAct* [31].

<sup>5</sup> In practical calculations we always verify that the coefficient  $\varphi_{3,1}^{(\varepsilon)}(\mathbf{n})$  averages to zero, so that there is no problem with the value  $q = 1$  in Eq. (2.11).

different IR regularizations have merely the effect of changing the values of two and only two ambiguity parameters  $\delta_1$  and  $\delta_2$  (modulo, of course, irrelevant world-line shifts).

Next, in addition to Eq. (2.12), we must also consider another “instantaneous” contribution when working in full DR. This is due to the fact that in HR it matters if we start from a gravitational Lagrangian at quadratic order of the type  $\sim \partial h \partial h$  or of the type  $\sim h \square h$  (*i.e.*, the propagator form). Indeed the two Lagrangians differ by a surface term  $\sim \partial(h \partial h)$  coming from the integration by part, and this surface term does contribute in HR. On the contrary, in DR it does not matter whether one starts with the Lagrangian in the form  $\sim \partial h \partial h$  or with the Lagrangian in propagator form  $\sim h \square h$ . Indeed the surface term is always zero in DR by analytic continuation in the dimension  $d$ . The fact that the two Lagrangians are equivalent in DR constitutes a very nice feature of DR as opposed to HR. In Paper I we initially performed our HR calculation with the  $\sim \partial h \partial h$  Lagrangian and then corrected it by adding the appropriate surface term so that our HR prescription starts with a Lagrangian having the propagator form  $\sim h \square h$ . On the other hand our calculation of the difference yielding (2.12) has been done with the prescription  $\sim \partial h \partial h$ , so we now have to subtract off the latter surface term. After applying an appropriate shift, this gives the following contribution to be *subtracted* from the HR result in order to control the full DR:

$$\mathcal{D}L_g^{\text{surf}} = \frac{G^4 m m_1^2 m_2^2}{c^8 r_{12}^4} \left[ -\frac{52}{15} (n_{12} v_{12})^2 + \frac{64}{15} v_{12}^2 \right]. \quad (2.14)$$

Again we find it to have the form of the ambiguity parameters modulo shifts.

In the language of EFT (see for instance Ref. [27]) our “instantaneous” calculation which has been done in the present section and yields Eq. (2.12), corresponds to the so-called “potential mode” contribution, say  $V_{\text{pot}}$ . As emphasized in [27, 28], the pole it contains is an IR pole, thus  $\varepsilon \equiv \varepsilon_{\text{IR}}$ . However, there is now to take into account the contribution coming from the conservative part of the 4PN tail effect in  $d$  dimensions, which corresponds in the EFT language to the “radiation” contribution, say  $V_{\text{rad}}$ . As we shall prove in Sec. IV the IR pole in Eq. (2.12) will be cancelled by a corresponding UV pole  $\varepsilon \equiv \varepsilon_{\text{UV}}$  coming from the radiation term in  $d$  dimensions.

### III. FORMULAS FOR THE NEAR-ZONE EXPANSION IN $d$ DIMENSIONS

In this section and the following one we shall prove that there is another contribution in the difference between DR and HR, coming from the tail effect in  $d$  dimensions. Indeed the computation in the previous section was based on the method “ $n + 2$ ” (see Sec. IV A of Paper I) which is valid for *symmetric* terms defined from the usual symmetric propagator. However the tail effect at 4PN order is to be added separately since it is in the form of an hereditary type homogeneous solution of the wave equation, which is of the *anti-symmetric* type (*i.e.*, advanced minus retarded), thus regular when  $r \rightarrow 0$ , and which has not been taken into account in the method  $n + 2$ .

We start by general considerations on the near-zone expansion of the solution of the flat scalar wave equation in  $d + 1$  space-time dimensions (thus, with  $\mathbf{x} \in \mathbb{R}^d$ ),<sup>6</sup>

$$\square h(\mathbf{x}, t) = N(\mathbf{x}, t). \quad (3.1)$$

---

<sup>6</sup> General conventions from earlier works [25, 26] are adopted. We pose  $G = c = 1$  in this section.

The source of such an equation will represent a generic term in the source of the equation (4.2) that we shall solve in the next section. The retarded Green's function  $G_{\text{ret}}(\mathbf{x}, t)$  of that scalar wave equation, thus satisfying  $\square G_{\text{ret}}(\mathbf{x}, t) = \delta(t)\delta^{(d)}(\mathbf{x})$ , explicitly reads

$$G_{\text{ret}}(\mathbf{x}, t) = -\frac{\tilde{k}}{4\pi} \frac{\theta(t-r)}{r^{d-1}} \gamma_{\frac{1-d}{2}}\left(\frac{t}{r}\right), \quad (3.2)$$

where  $\tilde{k} = \pi^{1-\frac{d}{2}}\Gamma(\frac{d}{2}-1)$  (with  $\Gamma$  being the usual Eulerian function) denotes a pure constant so defined that  $\lim_{d \rightarrow 3} \tilde{k} = 1$ , and  $\theta(t-r)$  denotes the usual Heaviside step function. The corresponding advanced Green's function  $G_{\text{adv}}(\mathbf{x}, t)$  is given by the same expression but with  $\theta(-t-r)$  instead of  $\theta(t-r)$ . We have introduced for convenience the function  $\gamma_s(z)$  defined for any  $s \in \mathbb{C}$  and  $|z| \geq 1$  by<sup>7</sup>

$$\begin{aligned} \gamma_s(z) &= \frac{2\sqrt{\pi}}{\Gamma(s+1)\Gamma(-s-\frac{1}{2})} (z^2-1)^s \\ &= \frac{\Gamma(-s)}{2^{2s+1}\Gamma(s+1)\Gamma(-2s-1)} (z^2-1)^s, \end{aligned} \quad (3.3)$$

where the normalisation has been chosen so that

$$\int_1^{+\infty} dz \gamma_s(z) = 1. \quad (3.4)$$

The latter integral converges when  $-1 < \Re(s) < -\frac{1}{2}$  and can be extended to any  $s \in \mathbb{C}$  by complex analytic continuation. For strictly negative integer values (say  $s \in -1 - \mathbb{N}$ ) the function (3.3) is zero in an ordinary sense, but is actually a distribution; for instance we can check that  $\gamma_{-1}(z) = \delta(z-1)$ , see Appendix C. Notice that the Green's function (3.2) depends only on  $t$  and the  $d$ -dimensional Euclidean norm  $r = |\mathbf{x}|$ . Its Fourier transform is also known, see *e.g.* Eq. (2.4) in Ref. [26]. The retarded solution of the wave equation (3.1) is given by

$$h(\mathbf{x}, t) = \int_{-\infty}^{+\infty} dt' \int d^d \mathbf{x}' G_{\text{ret}}(\mathbf{x} - \mathbf{x}', t - t') N(\mathbf{x}', t')$$

---

<sup>7</sup> The function  $\gamma_s(z)$  is the natural generalization of the function  $\gamma_\ell(z)$  (for  $\ell \in \mathbb{N}$ ) introduced in [33, 34] to parametrize ‘‘radiation-reaction’’ STF multipole moments. In a similar way one can introduce a function  $\delta_s(z)$  which would be a generalization of the function parametrizing the ‘‘source-type’’ multipole moments [35],

$$\delta_s(z) = \frac{\Gamma(s + \frac{3}{2})}{\sqrt{\pi}\Gamma(s+1)} (1-z^2)^s,$$

and satisfying  $\int_{-1}^1 dz \delta_s(z) = 1$ . One can show that  $\gamma_s(z) = -(1 + e^{-2i\pi s})\delta_s(z)$ , thus  $\gamma_\ell(z) = -2\delta_\ell(z)$  when  $\ell \in \mathbb{N}$ . Note also that the Riesz [36] kernels  $Z_\alpha(t, r)$  in Minkowski  $d+1$  space-time (satisfying the convolution algebra  $Z_\alpha * Z_\beta = Z_{\alpha+\beta}$ ) are given in terms of the function  $\gamma_s(z)$  by

$$Z_\alpha(t, r) = \frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{\alpha}{2})} \frac{r^{\alpha-d-1}}{2^\alpha \pi^{\frac{d}{2}}} \gamma_{\frac{\alpha-d-1}{2}}\left(\frac{t}{r}\right).$$



$$= -\frac{\tilde{k}}{4\pi} \int_1^{+\infty} dz \gamma_{\frac{1-d}{2}}(z) \int d^d \mathbf{x}' \frac{N(\mathbf{x}', t - z|\mathbf{x} - \mathbf{x}'|)}{|\mathbf{x} - \mathbf{x}'|^{d-2}}. \quad (3.5)$$

Now, we want to identify a piece in this solution, that will be a homogeneous anti-symmetric solution of the wave equation which is regular when  $\mathbf{x} \rightarrow 0$ . It may be obtained by performing the formal near-zone expansion of  $h(\mathbf{x}, t)$ . Later we shall use this homogeneous solution to control the tail effect in the near zone. Thus, for this application we consider that  $N(\mathbf{x}, t)$  represents a particular term in the quadratic part of the Einstein field equations outside the matter source, *i.e.*, a generic term of  $N_2[h_1]$  in Eq. (4.2) below. In particular  $N(\mathbf{x}, t)$  is to be thought as already “multipole-expanded” outside the matter source.

We start from Eq. (3.5) in which we swap the time and space integrals, defining

$$\tilde{N}_{\text{ret}}(\mathbf{x}', |\mathbf{x} - \mathbf{x}'|, t) = -\frac{\tilde{k}}{4\pi} \int_1^{+\infty} dz \gamma_{\frac{1-d}{2}}(z) \frac{N(\mathbf{x}', t - z|\mathbf{x} - \mathbf{x}'|)}{|\mathbf{x} - \mathbf{x}'|^{d-2}}, \quad (3.6)$$

which is a homogeneous solution of the wave equation with respect to the field point  $\mathbf{x}$ :  $\square \tilde{N}_{\text{ret}}(\mathbf{x}', |\mathbf{x} - \mathbf{x}'|, t) = 0$ . Homogeneous solutions of the wave equation are investigated in general terms in Appendix A. The  $d$ -dimensional integral (3.5) is defined by complex analytic continuation in  $d = 3 + \varepsilon$ , and we are looking to the neighbourhood of  $\varepsilon = 0$ , the latter point being excluded. However, we shall find that for some particular terms in our calculation, the analytic continuation cannot be performed as the  $\varepsilon$ 's cancel out. In order to be protected when such cancellation happens, we introduce a regulator  $r'^m$  in factor of the source (where  $r' = |\mathbf{x}'|$ ), and carry out all calculations with some finite parameter  $\eta \in \mathbb{C}$ , invoking the analytic continuation in  $\eta$  when necessary. At the end of our calculation we shall compute the limit when  $\eta \rightarrow 0$ , and find that this limit is finite for any  $\varepsilon$ . Finally we apply the DR prescription on the result, looking for the neighbourhood of  $\varepsilon = 0$  and the presence of poles  $1/\varepsilon$ . From now on we thus consider (with implicit limit when  $\eta \rightarrow 0$ )

$$h(\mathbf{x}, t) = \int d^d \mathbf{x}' r'^m \tilde{N}_{\text{ret}}(\mathbf{x}', |\mathbf{x} - \mathbf{x}'|, t). \quad (3.7)$$

The  $d$ -dimensional integral is then split into two pieces, each of which corresponds to the regions of integration  $|\mathbf{x}'| < R$  and  $|\mathbf{x}'| > R$ , respectively, for some positive  $R$ . If we choose  $R$  equal to the near-zone radius, we are allowed to replace the source  $\tilde{N}_{\text{ret}}(\mathbf{x}', |\mathbf{x} - \mathbf{x}'|, t)$  of the inner integral by its own PN expansion, as given by Eqs. (A7)–(A8) in Appendix A. The result may be written as an integral over the whole space, minus the same integral over the region  $|\mathbf{x}'| < R$ . This yields

$$\begin{aligned} h(\mathbf{x}, t) &= \int d^d \mathbf{x}' r'^m \overline{\tilde{N}_{\text{ret}}(\mathbf{x}', |\mathbf{x} - \mathbf{x}'|, t)} \\ &+ \int_{|\mathbf{x}'| > R} d^d \mathbf{x}' r'^m \left[ \tilde{N}_{\text{ret}}(\mathbf{x}', |\mathbf{x} - \mathbf{x}'|, t) - \overline{\tilde{N}_{\text{ret}}(\mathbf{x}', |\mathbf{x} - \mathbf{x}'|, t)} \right], \end{aligned} \quad (3.8)$$

where the overbar refers to the PN expansion. Next, in the second integral, extending over the exterior zone ( $|\mathbf{x}'| > R$ ), we can perform a formal Taylor expansion when  $|\mathbf{x}'| \rightarrow +\infty$ . After expressing the result in terms of symmetric-trace-free (STF) tensors, we find

$$\tilde{N}_{\text{ret}}(\mathbf{x}', |\mathbf{x} - \mathbf{x}'|, t) = \sum_{q=0}^{+\infty} \frac{(-)^q}{q!} \sum_{j=0}^{+\infty} \Delta^{-j} \hat{x}_Q \left( \hat{\partial}'_Q \tilde{N}_{\text{ret}}^{(2j)}(\mathbf{y}, r', t) \right)_{\mathbf{y}=\mathbf{x}'}, \quad (3.9)$$

where  $\hat{\partial}'_Q$  denotes the STF projection of a product of  $q$  partial derivatives  $\partial'_Q = \partial'_{i_1} \cdots \partial'_{i_q}$  with respect to  $x'^i$  (*i.e.*,  $\partial'_i = \partial/\partial x'^i$ ), where  $Q = i_1 \cdots i_q$  is a multi-index with  $q$  indices, and where the time multi-derivatives are indicated with the superscript index  $(2j)$ . Furthermore we employ the useful short-hand notation (with  $r = |\mathbf{x}|$ ) [33, 34]

$$\Delta^{-j} \hat{x}_Q = \frac{\Gamma(q + \frac{d}{2})}{\Gamma(q + j + \frac{d}{2})} \frac{r^{2j} \hat{x}_Q}{2^{2j} j!}, \quad (3.10)$$

for the iterated inverse Poisson operator acting on the STF product  $\hat{x}_Q$  of  $q$  source points  $x^i$ , such a notation being motivated by the fact that  $\Delta(\Delta^{-j} \hat{x}_Q) = \Delta^{-j+1} \hat{x}_Q$ . Notice that in Eq. (3.9) the point  $\mathbf{y}$  is held constant when applying the partial derivatives, and is to be replaced by  $\mathbf{x}'$  only afterwards. The same treatment applies also for the overbarred quantity in the last term of (3.8). At this stage we obtain the near-zone or PN expansion

$$\begin{aligned} \overline{h(\mathbf{x}, t)} &= \int d^d \mathbf{x}' r'^m \overline{\tilde{N}_{\text{ret}}(\mathbf{x}', |\mathbf{x} - \mathbf{x}'|, t)} \\ &+ \sum_{q=0}^{+\infty} \frac{(-)^q}{q!} \sum_{j=0}^{+\infty} \Delta^{-j} \hat{x}_Q \int_{|\mathbf{x}'| > R} d^d \mathbf{x}' r'^m \left( \hat{\partial}'_Q \tilde{N}_{\text{ret}}^{(2j)} - \overline{\hat{\partial}'_Q \tilde{N}_{\text{ret}}^{(2j)}} \right)_{\mathbf{y}=\mathbf{x}'}. \end{aligned} \quad (3.11)$$

Applying the same idea as before, *i.e.*, decomposing the second term as an integral over the whole space minus the same integral restricted to the inner region  $|\mathbf{x}'| < R$ , we can further rewrite the above expression as

$$\overline{h} = \int d^d \mathbf{x}' r'^m \overline{\tilde{N}_{\text{ret}}(\mathbf{x}', |\mathbf{x} - \mathbf{x}'|, t)} + \sum_{q=0}^{+\infty} \frac{(-)^q}{q!} \sum_{j=0}^{+\infty} \Delta^{-j} \hat{x}_Q \int d^d \mathbf{x}' r'^m \left( \hat{\partial}'_Q \tilde{N}_{\text{ret}}^{(2j)} \right)_{\mathbf{y}=\mathbf{x}'} + \overline{\Delta}. \quad (3.12)$$

This takes almost the requested form, but there is still the last term with a peculiar unwanted form, given by

$$\overline{\Delta} = - \sum_{q=0}^{+\infty} \frac{(-)^q}{q!} \sum_{j=0}^{+\infty} \Delta^{-j} \hat{x}_Q \left[ \int_{|\mathbf{x}'| < R} d^d \mathbf{x}' r'^m \left( \hat{\partial}'_Q \tilde{N}_{\text{ret}}^{(2j)} \right)_{\mathbf{y}=\mathbf{x}'} + \int_{|\mathbf{x}'| > R} d^d \mathbf{x}' r'^m \left( \overline{\hat{\partial}'_Q \tilde{N}_{\text{ret}}^{(2j)}} \right)_{\mathbf{y}=\mathbf{x}'} \right]. \quad (3.13)$$

However, in the near-zone integral, we can again replace the integrand by the PN expansion, so that the two integrals combine to a single integral extending over all space, which *in fine* turns out to be formally zero:

$$\overline{\Delta} = - \sum_{q=0}^{+\infty} \frac{(-)^q}{q!} \sum_{j=0}^{+\infty} \Delta^{-j} \hat{x}_Q \int d^d \mathbf{x}' r'^m \left( \hat{\partial}'_Q \tilde{N}_{\text{ret}}^{(2j)} \right)_{\mathbf{y}=\mathbf{x}'} = 0. \quad (3.14)$$

To prove the last statement we recall from Eqs. (A7)–(A8) that  $\overline{\tilde{N}_{\text{ret}}(\mathbf{y}, |\mathbf{x}'|, t)}$ , which is the PN expansion of a retarded solution of the wave equation, has the form of a sum  $\sum F_{a,b}(t) r'^{a+\varepsilon b}$ . Hence, when integrating this term and after performing the angular integration, we find a radial integral of the type  $\int_0^{+\infty} dr' r'^{a'+b'\varepsilon+\eta}$ , which is thus zero by analytic continuation in  $\varepsilon$ , except for the particular case where  $b' = 0$ ; the latter case is precisely the one where we need the “protection” of the regulator  $r'^m$  in order to complete our proof. Not

only is the regulator important for establishing (3.14) but it permits a complete calculation of all the terms (see the Appendix D). We shall find that the limit  $\eta \rightarrow 0$  is perfectly well defined for the sum of all the terms as the poles  $1/\eta$  cancel out.

The first term on the right-hand side of (3.12) represents the retarded integral acting directly on the PN expansion of the source, *i.e.*,  $\bar{N}$  (or, rather,  $r^\eta \bar{N}$ ). Thus the PN expansion of the corresponding solution can now be rewritten as

$$\bar{h} = \overline{\square_{\text{ret}}^{-1}}[r^\eta \bar{N}] + \sum_{q=0}^{+\infty} \frac{(-)^q}{q!} \sum_{j=0}^{+\infty} \Delta^{-j} \hat{x}_Q \int d^d \mathbf{x}' r'^\eta \left( \hat{\partial}'_Q \tilde{N}_{\text{ret}}^{(2j)} \right)_{\mathbf{y}=\mathbf{x}'}, \quad (3.15)$$

where the retardations in the inverse d'Alembertian operator are PN expanded. Since the first term is obviously a particular solution of the (PN-expanded) wave equation in the limit  $\eta \rightarrow 0$ , the second term in (3.15) is a homogeneous solution; let us call it  $\bar{h}^{\text{asym}}$  for a reason to soon become clear. In more details it reads

$$\bar{h}^{\text{asym}} = -\frac{\tilde{k}}{4\pi} \sum_{q=0}^{+\infty} \frac{(-)^q}{q!} \sum_{j=0}^{+\infty} \Delta^{-j} \hat{x}_Q \int_1^{+\infty} dz \gamma_{\frac{1-d}{2}}(z) \int d^d \mathbf{x}' r'^\eta \hat{\partial}'_Q \left[ \frac{N^{(2j)}(\mathbf{y}, t - zr')}{r'^{d-2}} \right]_{\mathbf{y}=\mathbf{x}'}. \quad (3.16)$$

This is our looked-for homogeneous solution; it is clearly of the form  $\bar{h}^{\text{asym}} = \sum_{q=0}^{+\infty} \sum_{j=0}^{+\infty} \Delta^{-j} \hat{x}_Q F_Q^{(2j)}(t)$ , on which form we can directly check that  $\square \bar{h}^{\text{asym}} = 0$ . Furthermore, that solution is manifestly regular when  $r \rightarrow 0$ , and so it must be identified with a homogeneous *anti-symmetric* solution of the wave equation in  $d$  dimensions, of the type half-retarded minus advanced. In particular, Eq. (3.16) must be identified with an anti-symmetric solution  $H^{\text{asym}}$  whose general form is given by Eq. (A15). Bearing unimportant factors, this means that we should always be able to find a function  $f_Q(t)$  such that

$$F_Q(t) = \int_0^{+\infty} d\tau \tau^{-\varepsilon} \left[ f_Q^{(2\ell+2)}(t - \tau) - f_Q^{(2\ell+2)}(t + \tau) \right]. \quad (3.17)$$

We prove this statement by going to the Fourier domain. Given the Fourier transform  $\hat{F}_Q(\omega)$  of  $F_Q(t)$ , Eq. (3.17) will be verified provided that the Fourier transform  $\hat{f}_Q(\omega)$  of  $f_Q(t)$  takes the expression

$$\hat{f}_Q(\omega) = \frac{2i(-)^\ell}{\cos(\frac{\pi\varepsilon}{2})\Gamma(1-\varepsilon)} \frac{\text{sign}(\omega)}{|\omega|^{2\ell+1+\varepsilon}} \hat{F}_Q(\omega). \quad (3.18)$$

Next, we consider the case of a source term which has a definite multipolarity  $\ell$ , namely  $N(\mathbf{x}, t) = \hat{n}_L N(r, t)$ , where  $\hat{n}_L$  is the STF projection of the product of  $\ell$  unit vectors  $n_i$ , and like before  $L = i_1 \cdots i_\ell$ . We shall denote the corresponding solution by  $\bar{h}_L^{\text{asym}}(\mathbf{x}, t)$ . Using  $\hat{\partial}'_Q f(r') = \hat{n}'_Q r'^q (r'^{-1} d/dr')^q f(r')$  in (3.16), we can explicitly perform the angular integration in  $d$  dimensions, see *e.g.* Eqs. (B23) in [25], and get

$$\begin{aligned} \bar{h}_L^{\text{asym}} &= \frac{(-)^{\ell+1} \Gamma(\frac{d}{2})}{2^\ell (d-2) \Gamma(\frac{d}{2} + \ell)} \sum_{j=0}^{+\infty} \Delta^{-j} \hat{x}_L \int_1^{+\infty} dz \gamma_{\frac{1-d}{2}}(z) \\ &\quad \times \int_0^{+\infty} dr' r'^{d+\ell-1+\eta} \left( \frac{1}{r'} \frac{d}{dr'} \right)^\ell \left[ \frac{N^{(2j)}(|\mathbf{y}|, t - zr')}{r'^{d-2}} \right]_{|\mathbf{y}|=r'}. \end{aligned} \quad (3.19)$$

Still this formula can be substantially simplified by means of a series of integrations by parts over the  $z$ -variable, and we nicely obtain

$$\bar{h}_L^{\text{asym}} = -\frac{1}{d+2\ell-2} \sum_{j=0}^{+\infty} \Delta^{-j} \hat{x}_L \int_1^{+\infty} dz \gamma_{\frac{1-d}{2}-\ell}(z) \int_0^{+\infty} dr' r'^{-\ell+1+\eta} N^{(2j)}(r', t-zr'). \quad (3.20)$$

We now specialize Eq. (3.20) to the case of a source term made of a quadratic interaction between a monopolar static solution  $\propto r^{d-2}$  and some homogeneous multipolar *retarded* solution, namely, a spatial multi-derivative of a monopolar retarded solution, see Eq. (4.5). Indeed such source term will be the one we meet when computing the tail effect as seen in the near zone ( $r \rightarrow 0$ ). Its generic form is of the type (with  $\varepsilon = d-3$ )

$$N(r, t) = r^{-k-2\varepsilon} \int_1^{+\infty} dy y^p \gamma_{-1-\frac{\varepsilon}{2}}(y) F(t-yr), \quad (3.21)$$

where  $k, p \in \mathbb{N}$  and the function  $F(t)$  stands for some time derivative of a component of a multipole moment, namely the source quadrupole moment  $I_{ij}(t)$  that we shall consider in Sec. IV. Plugging (3.21) into (3.20), and performing the change of integration variable  $r' \rightarrow \tau = (y+z)r'$ , we readily obtain

$$\bar{h}_L^{\text{asym}} = -\frac{C_\ell^{p,k}}{2\ell+1+\varepsilon} \sum_{j=0}^{+\infty} \Delta^{-j} \hat{x}_L \int_0^{+\infty} d\tau \tau^{-\ell-k+1-2\varepsilon} F^{(2j)}(t-\tau), \quad (3.22)$$

with the following purely numerical coefficient (also depending on the dimension)

$$C_\ell^{p,k} = \int_1^{+\infty} dy y^p \gamma_{-1-\frac{\varepsilon}{2}}(y) \int_1^{+\infty} dz (y+z)^{\ell+k-2+2\varepsilon-\eta} \gamma_{-\ell-1-\frac{\varepsilon}{2}}(z). \quad (3.23)$$

We are ultimately interested in the limit  $\varepsilon \rightarrow 0$ , but it is clear that the integral over  $\tau$  in (3.22) becomes ill-defined in this limit because of the bound  $\tau = 0$  of the integral. On the other hand since  $F(t)$  is a time derivative of a multipole moment, we can assume that it is zero in a neighbourhood of  $t = -\infty$  so there is no problem with the bound  $\tau = +\infty$  of the integral. We thus make explicit the generic presence of a pole  $\propto 1/\varepsilon$  when  $\varepsilon \rightarrow 0$  by operating the integral  $\ell+k-1$  times by parts. In contrast with the IR pole in Sec II, such a pole will be an UV-type pole,  $\varepsilon \equiv \varepsilon_{\text{UV}}$ . All surface terms vanish by analytic continuation in  $\varepsilon$  and because  $F(t-\tau)$  is zero when  $\tau \rightarrow \infty$ , so we arrive at

$$\bar{h}_L^{\text{asym}} = \frac{(-)^{\ell+k} C_\ell^{p,k}}{2\ell+1+\varepsilon} \frac{\Gamma(2\varepsilon)}{\Gamma(\ell+k-1+2\varepsilon)} \sum_{j=0}^{+\infty} \Delta^{-j} \hat{x}_L \int_0^{+\infty} d\tau \tau^{-2\varepsilon} F^{(2j+\ell+k-1)}(t-\tau). \quad (3.24)$$

Note the retarded character of this solution, which comes directly from the retarded character of the source term postulated in Eq. (3.21). In our approach, we are iterating the Einstein field equations by means of retarded potentials. Thus, at some given non-linear order, for instance quadratic, we obtain a retarded source term which represents the *physical* solution, containing both conservative and radiation-reaction dissipative effects. Only at this stage do we identify an ‘‘anti-symmetric’’ piece which is a part of the physical retarded solution generated by that source term, and which will be associated with the tail effect in the near zone.

The equation 3.24 is our final formula for this section, with which we can directly control the looked-for limit when  $\varepsilon \rightarrow 0$ . In generic cases a pole will show up due to the presence of the factor  $\Gamma(2\varepsilon) = \frac{\Gamma(1+2\varepsilon)}{2\varepsilon}$ , while the finite part beyond the pole will contain an ordinary tail integral with the usual logarithmic kernel. The numerical coefficient  $C_\ell^{p,k}$  defined by Eq. (3.23) is *a priori* not trivial to control, but fortunately we have found a way to compute it analytically as described in Appendix D.

#### IV. DERIVATION OF THE TAIL TERM IN $d$ DIMENSIONS

We shall compute the tail term in  $d$  dimensions directly in the near zone metric of general matter sources, then obtain its contribution in the equations of motion of compact binaries and finally in the Fokker action. The Einstein field equations in harmonic gauge in the vacuum region outside an isolated source read

$$\square h^{\mu\nu} = \Lambda^{\mu\nu}[h], \quad (4.1a)$$

$$\partial_\nu h^{\mu\nu} = 0, \quad (4.1b)$$

where  $\square$  is the flat d'Alembertian operator,  $h^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}$  is the ‘‘gothic’’ metric deviation from flat space-time, and  $\Lambda^{\mu\nu}$  denotes the non-linear gravitational source term, which is at least quadratic in  $h$  and its derivatives. As we shall see, to control the 4PN tail effect we can limit ourselves to the quadratic non-linear order, say  $h^{\mu\nu} = Gh_1^{\mu\nu} + G^2h_2^{\mu\nu} + \mathcal{O}(G^3)$ . Denoting by  $N^{\mu\nu}[h]$  the quadratic piece in the non-linear source term  $\Lambda^{\mu\nu}$  the equations to be solved are thus

$$\square h_2^{\mu\nu} = N^{\mu\nu}[h_1], \quad (4.2)$$

together with  $\partial_\nu h_2^{\mu\nu} = 0$ . At this stage we know that the tail effect is an interaction between the constant mass of the system  $M$  and its time-varying mass-type STF quadrupole moment  $I_{kl}(t)$ . Accordingly the linearized metric is composed of two pieces, say  $h_1^{\mu\nu} = h_M^{\mu\nu} + h_{I_{kl}}^{\mu\nu}$ . The static one corresponding to the mass reads

$$h_M^{00} = -4\tilde{M}, \quad h_M^{0i} = 0, \quad h_M^{ij} = 0, \quad (4.3)$$

while the dynamical one for the quadrupole moment in harmonic gauge is given by

$$h_{I_{kl}}^{00} = -2\partial_{ij}\tilde{I}_{ij}, \quad (4.4a)$$

$$h_{I_{kl}}^{0i} = 2\partial_j\tilde{I}_{ij}^{(1)}, \quad (4.4b)$$

$$h_{I_{kl}}^{ij} = -2\tilde{I}_{ij}^{(2)}. \quad (4.4c)$$

We are essentially following the notation of Eqs. (3.44) in [26]. In particular we denote an homogeneous retarded solution of the d'Alembertian equation as

$$\begin{aligned} \tilde{I}_{ij}(t, r) &= -4\pi \int_{-\infty}^{+\infty} dt' G_{\text{ret}}(\mathbf{x}, t - t') I_{ij}(t') \\ &= \frac{\tilde{k}}{r^{d-2}} \int_1^{+\infty} dz \gamma_{\frac{1-d}{2}}(z) I_{ij}(t - zr). \end{aligned} \quad (4.5)$$

See the retarded Green's function of the d'Alembertian equation in Eq. (3.2) above. For the static mass this reduces to a homogeneous solution of the Laplace equation,

$$\tilde{M}(r) = -4\pi M \int_{-\infty}^{+\infty} dt' G_{\text{ret}}(\mathbf{x}, t - t') = \frac{\tilde{k}M}{r^{d-2}}. \quad (4.6)$$

The quadratic source term  $N^{\mu\nu}[h_1]$  built out of the linearized metrics (4.3)–(4.4) reads

$$N_{M \times I_{kl}}^{00} = -h_M^{00} \partial_{00} h_{I_{kl}}^{00} - h_{I_{kl}}^{ij} \partial_{ij} h_M^{00} - \frac{3d-2}{2(d-1)} \partial_i h_M^{00} \partial_i h_{I_{kl}}^{00} + \partial_i h_M^{00} \partial_0 h_{I_{kl}}^{0i}, \quad (4.7a)$$

$$N_{M \times I_{kl}}^{0i} = -h_M^{00} \partial_{00} h_{I_{kl}}^{0i} + \frac{d}{2(d-1)} \partial_i h_M^{00} \partial_0 h_{I_{kl}}^{00} + \partial_j h_M^{00} \partial_0 h_{I_{kl}}^{ij} + \partial_j h_M^{00} (\partial_i h_{I_{kl}}^{0j} - \partial_j h_{I_{kl}}^{0i}), \quad (4.7b)$$

$$N_{M \times I_{kl}}^{ij} = -h_M^{00} \partial_{00} h_{I_{kl}}^{ij} + \frac{d-2}{d-1} \partial_{(i} h_M^{00} \partial_{j)} h_{I_{kl}}^{00} - \frac{d-2}{2(d-1)} \delta_{ij} \partial_k h_M^{00} \partial_k h_{I_{kl}}^{00} - \delta_{ij} \partial_k h_M^{00} \partial_0 h_{I_{kl}}^{0k} + 2\partial_{(i} h_M^{00} \partial_0 h_{I_{kl}}^{j)0}. \quad (4.7c)$$

As we have investigated in Sec. III, the tail effect we are looking for comes from a suitable homogeneous anti-symmetric solution of the wave equations (4.2). We have therefore applied our end result given by Eq. (3.24), together with the explicit method for the computation of the coefficients  $C_\ell^{p,k}$  as explained in Appendix D, to each of the terms of Eqs. (4.7). We consider only the pole part  $\propto 1/\varepsilon$  followed by the finite part when  $\varepsilon \rightarrow 0$ , and re-expand when  $c \rightarrow +\infty$  in order to keep only the terms contributing at the 4PN order. We then obtain the homogeneous solution responsible for the tails as<sup>8</sup>

$$h_{\text{asym}}^{00ii} = \frac{8G^2 M}{15c^{10}} x^{ij} \int_0^{+\infty} d\tau \left[ \ln \left( \frac{c\sqrt{q}\tau}{2\ell_0} \right) - \frac{1}{2\varepsilon} + \frac{49}{60} \right] I_{ij}^{(\tau)}(t - \tau) + \mathcal{O} \left( \frac{1}{c^{12}} \right), \quad (4.8a)$$

$$h_{\text{asym}}^{0i} = -\frac{8G^2 M}{3c^9} x^j \int_0^{+\infty} d\tau \left[ \ln \left( \frac{c\sqrt{q}\tau}{2\ell_0} \right) - \frac{1}{2\varepsilon} + \frac{2}{3} \right] I_{ij}^{(6)}(t - \tau) + \mathcal{O} \left( \frac{1}{c^{11}} \right), \quad (4.8b)$$

$$h_{\text{asym}}^{ij} = \frac{8G^2 M}{c^8} \int_0^{+\infty} d\tau \left[ \ln \left( \frac{c\sqrt{q}\tau}{2\ell_0} \right) - \frac{1}{2\varepsilon} + \frac{1}{2} \right] I_{ij}^{(5)}(t - \tau) + \mathcal{O} \left( \frac{1}{c^{10}} \right), \quad (4.8c)$$

where we have introduced the usual variable  $h^{00ii} = \frac{2}{d-1}[(d-2)h^{00} + h^{ii}]$  (see Paper I). In this standard harmonic gauge the tail integrals and their associated (UV) poles are spread out in all components of the metric.

Alternatively, we can do the calculation starting from the linear quadrupole metric in a transverse-tracefree (TT) harmonic gauge. Thus, instead of Eqs (4.4), we may consider the linear quadrupole TT metric

$$h'_{I_{kl}}{}^{00} = 0, \quad (4.9a)$$

$$h'_{I_{kl}}{}^{0i} = 0, \quad (4.9b)$$

$$h'_{I_{kl}}{}^{ij} = -2\tilde{I}_{ij}^{(2)} + 4\partial_{k(i}\tilde{I}_{j)k} - \frac{2}{d-1}\delta_{ij}\partial_{kl}\tilde{I}_{kl} - 2\frac{d-2}{d-1}\partial_{ijk}\tilde{I}_{kl}^{(-2)}. \quad (4.9c)$$

<sup>8</sup> We suppress the mention “ $M \times I_{kl}$ ”, and restore the factors of  $c$  and  $G$ . Here  $G$  denotes the usual Newtonian constant, such that  $G^{(d)} = G \ell_0^{d-3}$  in  $d$  dimensions. We recall also that  $\bar{q} = 4\pi e^{\gamma_E}$ .

In the TT gauge the quadratic source term is especially simple,

$$N'_{M \times I_{kl}}{}^{00} = -\partial_{ij} h_M^{00} h_{I_{kl}}{}^{ij}, \quad (4.10a)$$

$$N'_{M \times I_{kl}}{}^{0i} = \partial_j h_M^{00} \partial_0 h_{I_{kl}}{}^{ij}, \quad (4.10b)$$

$$N'_{M \times I_{kl}}{}^{ij} = -h_M^{00} \partial_0 h_{I_{kl}}{}^{ij}, \quad (4.10c)$$

and, relaunching our calculation, we readily obtain

$$h_{\text{asym}}{}^{00ii} = -\frac{52}{525} \frac{G^2 M}{c^{10}} x^{ij} I_{ij}^{(6)}(t) + \mathcal{O}\left(\frac{1}{c^{12}}\right), \quad (4.11a)$$

$$h_{\text{asym}}{}^{0i} = \frac{8}{15} \frac{G^2 M}{c^9} x^j I_{ij}^{(5)}(t) + \mathcal{O}\left(\frac{1}{c^{11}}\right), \quad (4.11b)$$

$$h_{\text{asym}}{}^{ij} = \frac{16}{5} \frac{G^2 M}{c^8} \int_0^{+\infty} d\tau \left[ \ln\left(\frac{c\sqrt{q}\tau}{2\ell_0}\right) - \frac{1}{2\varepsilon} + \frac{9}{40} \right] I_{ij}^{(5)}(t - \tau) + \mathcal{O}\left(\frac{1}{c^{10}}\right). \quad (4.11c)$$

In the TT gauge the tail integral and the associated pole appear only in the spatial components of the metric (notice also that  $h_{\text{asym}}{}^{iii} = 0$  in this case).

Finally the tails in the harmonic metric (4.8) or its TT counterpart (4.11) will yield a modification of the equations of motion. To compute it in the simplest way we perform a gauge transformation (this time, at quadratic order), so designed as to transfer all relevant terms in the “00*ii*” component of the metric. In the new gauge the 4PN tail effect is thus entirely described by the single scalar potential  $h''_{\text{asym}}{}^{00ii}$ , or equivalently by the 00 component of the usual covariant metric, given by  $g''_{00}{}^{\text{asym}} = -\frac{1}{2} h''_{\text{asym}}{}^{00ii}$ . We finally obtain

$$g''_{00}{}^{\text{asym}} = -\frac{8G^2 M}{5c^8} x^{ij} \int_0^{+\infty} d\tau \left[ \ln\left(\frac{c\sqrt{q}\tau}{2\ell_0}\right) - \frac{1}{2\varepsilon} + \kappa \right] I_{ij}^{(7)}(t - \tau) + \mathcal{O}\left(\frac{1}{c^{10}}\right). \quad (4.12)$$

This result properly recovers the known tail integral in 3 dimensions, see Eqs. (5.24) in [29]. In addition there is the pole and a certain numerical constant  $\kappa$  (a rational fraction). The (UV-type) pole is in agreement with the result of Ref. [10]. On the other hand the constant  $\kappa$  has the form of the ambiguity  $\alpha$  introduced in Paper I, which is itself equivalent to the ambiguity  $C$  in the Hamiltonian formalism [6].

Although the above procedure gives a definite value for  $\kappa$ , we shall now argue that the exact value of this constant is not determined at this stage. The non-local tail integral and the associated pole in (4.12) are found to be invariant, but the constant  $\kappa$  is found to depend on the particular gauge used to start the iteration with. For instance we find  $\kappa^{\text{harm}} = \frac{11}{40}$  with the linear metric in the original harmonic gauge (4.4), but  $\kappa^{\text{TT}} = \frac{443}{840}$  with the metric in the TT gauge (4.9).<sup>9</sup> Of course this is due to the fact that the split we have done in Sec. IV between a “particular” solution (which has been used conjointly with the method  $n+2$ ) and an “homogeneous” solution (regular when  $r \rightarrow 0$ ) originating from the radiation zone and containing the tail integral is not gauge invariant. Here we argue that the correct value of  $\kappa$  will depend on the details of the matching between the near zone and the radiation zone in the particular coordinate system we use. For the moment such a precise matching has

---

<sup>9</sup> This situation is also well known in 3 dimensions, where the related constant is  $\frac{11}{12}$  in harmonic coordinates but  $\frac{17}{12}$  in Schwarzschild-like coordinates [37].

not been done and we shall thus leave the parameter  $\kappa$  undetermined, which means that we shall still have the first ambiguity of Paper I, equivalent to the one in the Hamiltonian formalism [6]. Of course the value of  $\kappa$  is known from comparison with GSF calculations, and we already note that the correct value, namely  $\kappa = \frac{41}{60}$  (see Sec. V), is in agreement with the published coefficient obtained by Galley *et al* [10] in their computation of the  $d$ -dimensional tail effect using EFT methods, see their Eq. (3.3). Within our formalism, more work is needed to control the value of the first ambiguity parameter  $\kappa$ . However, as we shall prove, the result (4.12) is sufficient to determine the second ambiguity parameter in Paper II.

Once we have the single scalar effect (4.12) at the level of the metric, it is straightforward to obtain the equivalent effect at the level of the Lagrangian or Fokker action. Recall that the corresponding piece in the Fokker action will describe only the conservative part of the dynamics associated with the tail effect (see Paper I for discussion). We thus find the manifestly time-symmetric contribution to the gravitational part of the action,

$$S_g^{\text{tail}} = \frac{G^2 M}{5c^8} \int_{-\infty}^{+\infty} dt I_{ij}^{(3)}(t) \int_0^{+\infty} d\tau \left[ \ln \left( \frac{c\sqrt{q}\tau}{2\ell_0} \right) - \frac{1}{2\varepsilon} + \kappa \right] \left( I_{ij}^{(4)}(t - \tau) - I_{ij}^{(4)}(t + \tau) \right), \quad (4.13)$$

which can elegantly be rewritten by means of the Hadamard partie finie (Pf) integral as

$$S_g^{\text{tail}} = \frac{G^2 M}{5c^8} \text{Pf}_{\tau_0^{\text{DR}}} \iint \frac{dt dt'}{|t - t'|} I_{ij}^{(3)}(t) I_{ij}^{(3)}(t'), \quad (4.14)$$

where  $\tau_0^{\text{DR}} = \frac{2\ell_0}{c\sqrt{q}} e^{\frac{1}{2\varepsilon} - \kappa}$  plays the role of the Hadamard cut-off scale. Finally, when considering the difference between the DR and HR results, we have to correct for the different treatments of the tail term in the two procedures. In Sec. III of Paper I we obtained the tail term in the same form as Eq. (4.14) but with a different Hadamard scale  $\tau_0^{\text{HR}} = 2s_0$ . The difference of Lagrangians to be added to the result of Paper I concerning the tail is thus

$$\begin{aligned} \mathcal{D}L_g^{\text{tail}} &= -\frac{2G^2 M}{5c^8} \ln \left( \frac{\tau_0^{\text{DR}}}{\tau_0^{\text{HR}}} \right) \left( I_{ij}^{(3)} \right)^2 \\ &= \frac{G^2 m}{5c^8} \left[ -\frac{1}{\varepsilon} + 2\kappa + 2 \ln \left( \frac{\sqrt{q} s_0}{\ell_0} \right) \right] \left( I_{ij}^{(3)} \right)^2, \end{aligned} \quad (4.15)$$

where we approximated  $M = m + \mathcal{O}(1/c^2)$  in the second equality. Thus, the pole in (4.15) will indeed cancel out the pole in the instantaneous part of the Fokker action, see Eq. (2.12).

## V. DETERMINATION OF THE SECOND AMBIGUITY PARAMETER

We gather and recapitulate our results from the previous sections. Recall that in Paper I, the 4PN Fokker Lagrangian constructed in harmonic coordinates initially depended on the arbitrary constant parameter

$$\alpha = \ln \left( \frac{r_0}{s_0} \right), \quad (5.1)$$

which was then adjusted to the value  $\alpha = \frac{811}{672}$  by comparison to the circular orbit limit of the binary's conserved energy in the small mass ratio limit. We therefore have to:



1. Restore the arbitrariness of the parameter  $\alpha$  by adding to the end result of Paper I the contribution

$$\mathcal{D}L_g^\alpha = \frac{2G^2m}{5c^8} \left( \alpha - \frac{811}{672} \right) \left( I_{ij}^{(3)} \right)^2 ; \quad (5.2)$$

2. Add the difference between the DR and HR evaluations of the IR divergences in the instantaneous part of the gravitational action, as computed using the method  $n + 2$ , and whose result has been obtained in Eq. (2.12);
3. Subtract off the particular surface term given by Eq. (2.14) and which was necessary in the HR scheme for having a Lagrangian starting at the quadratic order with the propagator form  $\propto h \square h$ ;
4. Finally, add the difference between the radiation non-local tails in DR and HR as obtained in (4.15).

Concerning the matter part  $L_m$  of the Fokker Lagrangian, nothing is to be changed with respect to the result of Paper I since there are no IR divergences therein and  $L_m$  stands correct in DR. Finally our full DR Lagrangian reads

$$L = L^{\text{Paper I}} + \mathcal{D}L_g^\alpha + \mathcal{D}L_g^{\text{inst}} - \mathcal{D}L_g^{\text{surf}} + \mathcal{D}L_g^{\text{tail}}. \quad (5.3)$$

Inserting our explicit results we find that the poles properly cancel out as announced; furthermore the constants  $r_0$ ,  $s_0$  and  $\ell_0$  also correctly disappear, and so does the irrational number  $\bar{q} = 4\pi e^{\gamma_E}$ . The modification of the Lagrangian then takes exactly the form postulated in Eq. (2.4) of Paper II, namely

$$L = L^{\text{Paper I}} + \frac{G^4 m m_1^2 m_2^2}{c^8 r_{12}^4} \left( \delta_1 (n_{12} v_{12})^2 + \delta_2 v_{12}^2 \right), \quad (5.4)$$

but where the two ambiguity parameters  $\delta_1$  and  $\delta_2$  are now determined solely in terms of the *single* parameter  $\kappa$  coming from our  $d$ -dimensional computation of the tails in Secs. III–IV. Such a parameter is actually equivalent to the ambiguity  $\alpha$  in Paper I, and we get

$$\delta_1 = \frac{1733}{1575} - \frac{176}{15} \kappa, \quad \delta_2 = -\frac{1712}{525} + \frac{64}{5} \kappa. \quad (5.5)$$

In Paper II we demanded that the conserved energy and periastron advance for circular orbits recover the GSF calculations in the small mass-ratio limit, and this gave  $\delta_1 = -\frac{2179}{315}$  and  $\delta_2 = \frac{192}{35}$ . We thus have the unique choice

$$\kappa = \frac{41}{60}. \quad (5.6)$$

This result confirms the soundness of the postulated form of the ambiguities in Paper II and shows the power of dimensional regularization for handling both UV and IR divergences in the problem of motion in classical GR. However, still in our approach, as well as in the Hamiltonian approach [6, 7], remains the problem of better understanding the first ambiguity parameter. As we have argued, this parameter  $\kappa$  should be determined from first principles (*i.e.*, without external help from GSF calculations), by a careful matching between the near-zone field and the radiation field. We shall postpone this investigation to future work.

Remarkably, the value (5.6) of  $\kappa$  agrees with the result found by Galley *et al* [10] in their computation of the tail term in  $d$  dimensions (including both conservative and dissipative effects) by means of EFT methods. This indicates that when the EFT calculation will be fully completed at the 4PN order [8–11], the problem of the first ambiguity will be solved as well. This also strongly supports our expectation that the parameter  $\kappa$  should be determined from first principles within our present Fokker Lagrangian approach.

## Acknowledgments

It is a pleasure to thank Tanguy Marchand for having checked the calculation of the tail effect in Sec. IV. L.Bl. and G.F. acknowledge a very useful and productive “Workshop on analytical methods in General Relativity” organized by Rafael Porto and Riccardo Sturani at ICTP/SAIFR in São Paulo, Brazil. L.Be. acknowledges financial support provided under the European Union’s H2020 ERC Consolidator Grant “Matter and strong-field gravity: New frontiers in Einstein’s theory” grant agreement no. MaGRaTh646597.

## Appendix A: Homogeneous solutions of the wave equation in $d + 1$ dimensions

The general “monopolar” homogeneous retarded solution of the wave equation in  $d + 1$  dimensions (where  $d = 3 + \varepsilon$ ), such that  $\square \tilde{f}_{\text{ret}}(t, r) = 0$ , reads, following the notation (4.5),

$$\begin{aligned} \tilde{f}_{\text{ret}}(r, t) &= -4\pi \int_{-\infty}^{+\infty} dt' G_{\text{ret}}(\mathbf{x}, t - t') f(t') \\ &= \frac{\tilde{k}}{r^{d-2}} \int_1^{+\infty} dy \gamma_{\frac{1-d}{2}}(y) f\left(t - \frac{ry}{c}\right), \end{aligned} \quad (\text{A1})$$

or, in more details, recalling  $\tilde{k} = \Gamma(\frac{d}{2} - 1)/\pi^{\frac{d}{2}-1}$  and the function  $\gamma_s(y)$  displayed in Eq. (3.3),

$$\tilde{f}_{\text{ret}}(r, t) = \frac{2}{\pi^{\frac{\varepsilon}{2}} \Gamma(-\frac{\varepsilon}{2})} \int_1^{+\infty} dy (y^2 - 1)^{-1-\frac{\varepsilon}{2}} f\left(t - \frac{ry}{c}\right). \quad (\text{A2})$$

In this Appendix we shall mostly investigate the post-Newtonian expansion of that solution. We notice that by posing  $\tau = ry/c$  we are fixing the argument of the function  $f$  in (A2), and then the formal PN expansion  $c \rightarrow +\infty$  becomes equivalent to a formal expansion when  $y \rightarrow +\infty$ , which can simply be evaluated by inserting into (A2) the series

$$(y^2 - 1)^{-1-\frac{\varepsilon}{2}} = \sum_{k=0}^{+\infty} \frac{(-)^k}{k!} \frac{\Gamma(-\frac{\varepsilon}{2})}{\Gamma(-k - \frac{\varepsilon}{2})} y^{-2-2k-\varepsilon}. \quad (\text{A3})$$

In this way we readily obtain

$$\tilde{f}_{\text{ret}} = \frac{2}{\pi^{\frac{\varepsilon}{2}} c^{1+\varepsilon}} \sum_{k=0}^{+\infty} \frac{(-)^k}{k!} \frac{(r/c)^{2k}}{\Gamma(-k - \frac{\varepsilon}{2})} \int_{r/c}^{+\infty} d\tau \tau^{-2-2k-\varepsilon} f(t - \tau). \quad (\text{A4})$$

At this stage we split the integral according to  $\int_{r/c}^{+\infty} = -\int_0^{r/c} + \int_0^{+\infty}$ . The two pieces will respectively yield the decomposition of Eq. (A4) into “even” and “odd” pieces in the limit

$\varepsilon \rightarrow 0$ , where we are following the standard PN terminology, *i.e.*, meaning the parity of the power of  $1/c$  in front. Thus,

$$\tilde{f}_{\text{ret}} = \tilde{f}_{\text{even}} + \tilde{f}_{\text{ret}}^{\text{odd}}. \quad (\text{A5})$$

In the even piece, corresponding to (minus) the integral from 0 to  $r/c$ , we are allowed to formally expand the integrand when  $\tau \rightarrow 0$ , since by definition  $r/c \rightarrow 0$  for the PN expansion. At first sight, this yields a complicated double infinite summation, but which can be drastically simplified thanks to the formula

$$\sum_{k=0}^{+\infty} \frac{(-)^k}{k!} \frac{1}{\left(k + \frac{1-p+\varepsilon}{2}\right) \Gamma(-k - \frac{\varepsilon}{2})} = \frac{\Gamma(\frac{1-p+\varepsilon}{2})}{\Gamma(\frac{1-p}{2})}. \quad (\text{A6})$$

Although it is valid for any  $p \in \mathbb{N}$ , this formula gives zero whenever  $p$  is an odd integer. Thus only will contribute the even values  $p = 2j$ , reflecting the even character, in the PN sense, of that term. Furthermore we get a ‘‘local’’ expansion in any dimensions, given by

$$\tilde{f}_{\text{even}} = \frac{r^{-1-\varepsilon}}{\pi^{\frac{1+\varepsilon}{2}}} \sum_{j=0}^{+\infty} \frac{(-)^j}{2^{2j} j!} \Gamma\left(\frac{1+\varepsilon}{2} - j\right) \left(\frac{r}{c}\right)^{2j} f^{(2j)}(t). \quad (\text{A7})$$

As for the odd piece, corresponding to the integral from 0 to  $+\infty$ , it will irreducibly be given by a non-local integral, except when  $\varepsilon = 0$ . We perform a series of integrations by parts to arrive at an expression which is manifestly finite in the limit  $\varepsilon \rightarrow 0$ :

$$\tilde{f}_{\text{ret}}^{\text{odd}} = -\frac{1}{2\pi^{\frac{\varepsilon}{2}} c^{1+\varepsilon}} \frac{\Gamma(\frac{1+\varepsilon}{2})}{\Gamma(1 - \frac{\varepsilon}{2})} \sum_{j=0}^{+\infty} \frac{1}{2^{2j} j!} \frac{(r/c)^{2j}}{\Gamma(j + \frac{3+\varepsilon}{2})} \int_0^{+\infty} d\tau \tau^{-\varepsilon} f^{(2j+2)}(t - \tau). \quad (\text{A8})$$

Notice that this expression, unlike (A7), is regular when  $r \rightarrow 0$ , *i.e.*,  $\tilde{f}_{\text{ret}}^{\text{odd}} \in C^\infty(\mathbb{R})$ . We straightforwardly check that Eqs. (A7) and (A8) recover in the limit  $\varepsilon \rightarrow 0$  the usual even and odd parts of the PN expansion of the monopolar wave (in particular,  $\tilde{f}_{\text{ret}}^{\text{odd}}$  becomes local in this limit):

$$\tilde{f}_{\text{ret}}(r, t) \Big|_{\varepsilon=0} = \frac{f(t - r/c)}{r}, \quad (\text{A9a})$$

$$\tilde{f}_{\text{even}}(r, t) \Big|_{\varepsilon=0} = \sum_{j=0}^{+\infty} \frac{r^{2j-1}}{(2j)! c^{2j}} f^{(2j)}(t) \quad (\text{A9b})$$

$$\tilde{f}_{\text{ret}}^{\text{odd}}(r, t) \Big|_{\varepsilon=0} = -\sum_{j=0}^{+\infty} \frac{r^{2j}}{(2j+1)! c^{2j+1}} f^{(2j+1)}(t). \quad (\text{A9c})$$

The same analysis but done for the *advanced* monopolar homogeneous solution, *i.e.*, using the advanced Green’s function [given by Eq. (3.2) with  $\theta(-t - r)$  in place of  $\theta(t - r)$ ], gives

$$\tilde{f}_{\text{adv}} = \tilde{f}_{\text{even}} + \tilde{f}_{\text{adv}}^{\text{odd}}, \quad (\text{A10})$$

where the even part is the same as before, and with the advanced odd part

$$\tilde{f}_{\text{adv}}^{\text{odd}} = -\frac{1}{2\pi^{\frac{\varepsilon}{2}} c^{1+\varepsilon}} \frac{\Gamma(\frac{1+\varepsilon}{2})}{\Gamma(1 - \frac{\varepsilon}{2})} \sum_{j=0}^{+\infty} \frac{1}{2^{2j} j!} \frac{(r/c)^{2j}}{\Gamma(j + \frac{3+\varepsilon}{2})} \int_0^{+\infty} d\tau \tau^{-\varepsilon} f^{(2j+2)}(t + \tau). \quad (\text{A11})$$

In the limit  $\varepsilon \rightarrow 0$  we evidently get  $\tilde{f}_{\text{adv}}^{\text{odd}}|_{\varepsilon=0} = -\tilde{f}_{\text{ret}}^{\text{odd}}|_{\varepsilon=0}$ . Further, we define the associated symmetric and anti-symmetric solutions,

$$\tilde{f}_{\text{sym}} = \frac{1}{2}(\tilde{f}_{\text{ret}} + \tilde{f}_{\text{adv}}) = \tilde{f}_{\text{even}} + \frac{1}{2}(\tilde{f}_{\text{ret}}^{\text{odd}} + \tilde{f}_{\text{adv}}^{\text{odd}}), \quad (\text{A12a})$$

$$\tilde{f}_{\text{asym}} = \frac{1}{2}(\tilde{f}_{\text{ret}} - \tilde{f}_{\text{adv}}) = \frac{1}{2}(\tilde{f}_{\text{ret}}^{\text{odd}} - \tilde{f}_{\text{adv}}^{\text{odd}}). \quad (\text{A12b})$$

In particular, the anti-symmetric solution is non-local (except when  $\varepsilon = 0$ ), regular when  $r \rightarrow 0$ , and becomes purely odd in the PN sense when  $\varepsilon = 0$ ,

$$\tilde{f}_{\text{asym}} = -\frac{1}{4\pi^{\frac{\varepsilon}{2}}c^{1+\varepsilon}} \frac{\Gamma(\frac{1+\varepsilon}{2})}{\Gamma(1-\frac{\varepsilon}{2})} \sum_{j=0}^{+\infty} \frac{1}{2^{2j}j!} \frac{(r/c)^{2j}}{\Gamma(j+\frac{3+\varepsilon}{2})} \int_0^{+\infty} d\tau \tau^{-\varepsilon} \left[ f^{(2j+2)}(t-\tau) - f^{(2j+2)}(t+\tau) \right]. \quad (\text{A13})$$

The most general ‘‘multipolar’’ homogeneous retarded solution will be obtained by repeatedly applying spatial differentiations on the latter monopolar solution, hence

$$\tilde{H}_{\text{ret}}(\mathbf{x}, t) = \sum_{\ell=0}^{+\infty} \hat{\partial}_L \tilde{f}_{\text{ret}}^L(r, t), \quad (\text{A14})$$

where  $\hat{\partial}_L$  denotes the STF product of  $\ell$  spatial derivatives (and  $L = i_1 \cdots i_\ell$ ). Similarly one can define the advanced, symmetric and anti-symmetric multipolar solutions. For instance, the anti-symmetric solution can be re-written in the manifestly regular form

$$\begin{aligned} \tilde{H}_{\text{asym}} = & -\frac{1}{4\pi^{\frac{\varepsilon}{2}}} \frac{\Gamma(\frac{1+\varepsilon}{2})}{\Gamma(1-\frac{\varepsilon}{2})} \sum_{\ell=0}^{+\infty} \frac{1}{2^\ell \Gamma(\ell + \frac{3+\varepsilon}{2})} \sum_{j=0}^{+\infty} \frac{\Delta^{-j} \hat{x}_L}{c^{2j+2\ell+1+\varepsilon}} \\ & \times \int_0^{+\infty} d\tau \tau^{-\varepsilon} \left[ f_L^{(2j+2\ell+2)}(t-\tau) - f_L^{(2j+2\ell+2)}(t+\tau) \right], \end{aligned} \quad (\text{A15})$$

where we recall the short-hand notation

$$\Delta^{-j} \hat{x}_L = \frac{\Gamma(\ell + \frac{3+\varepsilon}{2})}{\Gamma(\ell + j + \frac{3+\varepsilon}{2})} \frac{r^{2j} \hat{x}_L}{2^{2j} j!}. \quad (\text{A16})$$

The homogeneous solution investigated in Sec. III, and that we computed directly from a near-zone expansion, is precisely of the previous anti-symmetric type (A15). We showed this by going to the Fourier domain, see Eqs. (3.17) and (3.18).

## Appendix B: Multipole expansion of elementary functions in $d$ dimensions

For our computation of the difference between the DR and HR prescriptions for the IR regularization of integrals at infinity in Sec. II, we need to control the expansion at infinity ( $r \rightarrow +\infty$ ) of non-linear potentials in  $d$  dimensions. These potentials are defined by means of elementary solutions of the Poisson or d’Alembert equation in  $d$  dimensions, the simplest one being the famous Fock kernel obeying in  $d$  dimensions

$$\Delta g = r_1^{2-d} r_2^{2-d}. \quad (\text{B1})$$

The exact expression in 3 dimensions is  $g^{(\varepsilon=0)} = \ln(r_1 + r_2 + r_{12})$  [32]. The explicit form of the solution in  $d$  dimensions has been obtained in the Appendix C of Ref. [25]. In the Appendix B of Paper I we have given the local expansion of that function in  $d$  dimensions near the singularities (when  $r_1$  or  $r_2 \rightarrow 0$ ). Here we compute the far zone expansion when  $r \rightarrow +\infty$ , that we shall refer to as a multipole expansion denoted  $\mathcal{M}(g)$ .

Suppose we want to compute the multipole expansion  $\mathcal{M}(P)$  of some elementary potential  $P$ , solution of the wave equation  $\square P = \sigma$ , where  $\sigma$  is some source term with non compact support like in (B1). In the usual post-Newtonian (or near zone) iteration scheme, neglecting time-odd contributions, the potential is given by  $P = \mathcal{I}^{-1}\sigma$  where the usual symmetric propagator reads

$$\mathcal{I}^{-1} = \sum_{p=0}^{+\infty} \left( \frac{1}{c} \frac{\partial}{\partial t} \right)^p \Delta^{-p-1}. \quad (\text{B2})$$

Now the far-zone expansion  $\mathcal{M}(P)$  will be obtained from the far-zone expansion  $\mathcal{M}(\sigma)$  of the corresponding source term by application of (B2), but for a non-compact support source it is known that there is also a homogeneous solution of the *symmetric* type to be added, and which is specified by Eq. (3.23) of Ref. [33]. Generalizing the formula to  $d$  dimensions, this means that the solution is the sum of a particular solution obtained by application of Eq. (B2), plus a specific homogeneous symmetric one,

$$\mathcal{M}(P) = \mathcal{I}^{-1}[\mathcal{M}(\sigma)] - \frac{1}{4\pi} \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \overline{\partial_L \tilde{\sigma}_{\text{sym}}^L}, \quad (\text{B3})$$

where the overbar on the homogeneous solution means the PN or near-zone expansion, and, following the Appendix A, the homogeneous symmetric solution reads

$$\tilde{\sigma}_{\text{sym}}^L(r, t) = \frac{\tilde{k}}{r^{d-2}} \int_1^{+\infty} dy \gamma_{\frac{1-d}{2}}(y) \left[ \sigma_L(t - ry/c) + \sigma_L(t + ry/c) \right]. \quad (\text{B4})$$

Here  $\sigma_L$  denotes the  $\ell$ -th multipole moment of the source  $\sigma$  given (in non-STF guise) by

$$\sigma_L(t) = \int d^d \mathbf{x}' x'_L \sigma(\mathbf{x}', t). \quad (\text{B5})$$

Note that we are performing a full DR calculation, so the multipole moment  $\sigma_L$  is defined without invoking a finite part regularization (based on some regulator  $(r/r_0)^B$  with  $B \in \mathbb{C}$ ); instead, DR is taking care of the IR divergences, appearing here due to the fact that the source  $\sigma$  has a non-compact spatial support. Similarly, the particular solution or first term in Eq. (B3), is defined in a pure DR way, with the iterated Poisson operator  $\Delta^{-p-1}$  in (B2) acting on each term of the multipole expansion of the source  $\mathcal{M}(\sigma)$ , whose general structure in  $d$  dimensions is provided by Eq. (2.7). It is clear that the Poisson operator and its iterated version make sense when applied to such terms [see, *e.g.*, (A16)].

Finally, because of the overbar prescription in Eq. (B3), we need the post-Newtonian or near-zone expansion of the object  $\tilde{\sigma}_{\text{sym}}^L$ . The PN expansion of the homogeneous symmetric solution has been investigated in the previous App. A. It consists essentially of even contributions but also, in  $d$  dimensions, or some residual non-local odd terms, see Eqs. (A12). The odd terms will disappear in 3 dimensions; we neglect these since they are dissipative

contributions. Thus we simply assimilate the symmetric part with the even part, and we get, from Eq. (A7),

$$\tilde{\sigma}_{\text{sym}}^L = \frac{r^{-1-\varepsilon}}{\pi^{\frac{1+\varepsilon}{2}}} \sum_{j=0}^{+\infty} \frac{(-)^j}{2^{2j} j!} \Gamma\left(\frac{1+\varepsilon}{2} - j\right) \left(\frac{r}{c}\right)^{2j} \sigma_L^{(2j)}(t). \quad (\text{B6})$$

We have applied the previous formulas to the source term  $\sigma = r_1^{2-d} r_2^{2-d}$  in Eq. (B1). Defining  $g$  and  $f$  such that, up to the 1PN order,

$$P = g + \frac{1}{2c^2} \partial_t^2 f + \mathcal{O}\left(\frac{1}{c^4}\right), \quad (\text{B7})$$

we have  $\Delta g = \sigma$  and  $\Delta f = 2g$  in this convention, and obtain<sup>10</sup>

$$\begin{aligned} \mathcal{M}(g) &= \frac{r_{12}^{1-\varepsilon}}{1-\varepsilon} \sum_{\ell=0}^{+\infty} \frac{2^{\ell-1}}{(\ell+1)!} \frac{\Gamma(\ell + \frac{\varepsilon+1}{2})}{\Gamma(\frac{\varepsilon+1}{2})} \frac{\hat{n}^L}{r^{\ell+1+\varepsilon}} \sum_{s=0}^{\ell} y_1^{\langle L-S, S \rangle} y_2^S \\ &+ \frac{1}{\left[\Gamma\left(\frac{1+\varepsilon}{2}\right)\right]^2} \sum_{m=0}^{+\infty} \frac{2^{m-2}}{r^{m+2\varepsilon}} \sum_{s=0}^{\lfloor \frac{m}{2} \rfloor} \frac{\Gamma\left(\frac{3+\varepsilon}{2} + m - 2s\right)}{\Gamma\left(\frac{3+\varepsilon}{2} + m - s\right)} \frac{\hat{n}^{M-2S}}{(m-s+\varepsilon)(s+\frac{\varepsilon-1}{2})(2s)!!} \\ &\times \sum_{\ell=0}^m \frac{\Gamma(\ell + \frac{\varepsilon+1}{2})}{(\ell-s)!} \frac{\Gamma(m-\ell + \frac{\varepsilon+1}{2})}{(m-\ell-s)!} \hat{y}_1^{L-S, S'} \hat{y}_2^{M-LS, S'}, \end{aligned} \quad (\text{B8a})$$

$$\begin{aligned} \mathcal{M}(f) &= \frac{r_{12}^{1-\varepsilon}}{(1-\varepsilon)^2} \sum_{\ell=0}^{+\infty} \frac{2^{\ell-2}}{(\ell+1)!} \frac{\Gamma(\ell + \frac{\varepsilon-1}{2})}{\Gamma(\frac{\varepsilon-1}{2})} \left[ \sum_{s=0}^{\ell} y_1^{\langle L-S, S \rangle} y_2^S \left( r^2 - \frac{(2\ell + \varepsilon - 1)}{(2\ell + \varepsilon + 3)(\ell + 2)} \right. \right. \\ &\times \left. \left. \left( y_1^2(\ell - s + 1) + y_2^2(s + 1) - \frac{2r_{12}^2}{3-\varepsilon}(\ell - s + 1)(s + 1) \right) \right) \right] \frac{\hat{n}^L}{r^{\ell+1+\varepsilon}} \\ &+ \frac{1}{\left[\Gamma\left(\frac{1+\varepsilon}{2}\right)\right]^2} \sum_{m=0}^{+\infty} \frac{2^{m-4}}{r^{m-2+2\varepsilon}} \sum_{s=0}^{\lfloor \frac{m}{2} \rfloor} \frac{\Gamma\left(\frac{3+\varepsilon}{2} + m - 2s\right) \Gamma(m-s-1+\varepsilon) \Gamma\left(s + \frac{\varepsilon-3}{2}\right)}{\Gamma\left(\frac{3+\varepsilon}{2} + m - s\right) \Gamma(m-s+1+\varepsilon) \Gamma\left(s + \frac{\varepsilon+1}{2}\right)} \frac{\hat{n}^{M-2S}}{(2s)!!} \\ &\times \sum_{\ell=0}^m \frac{\Gamma(\ell + \frac{\varepsilon+1}{2})}{(\ell-s)!} \frac{\Gamma(m-\ell + \frac{\varepsilon+1}{2})}{(m-\ell-s)!} \hat{y}_1^{L-S, S'} \hat{y}_2^{M-LS, S'}. \end{aligned} \quad (\text{B8b})$$

Similarly, in our calculations we have also to consider the potentials  $f_{12}$  and  $f_{21}$  obeying

$$\Delta f_{12} = r_1^{4-d} r_2^{2-d}, \quad \Delta f_{21} = r_1^{2-d} r_2^{4-d}, \quad (\text{B9})$$

and we obtain, for instance,

$$\mathcal{M}(f_{12}) = \frac{r_{12}^{3-\varepsilon}}{3-\varepsilon} \sum_{\ell=0}^{+\infty} \frac{2^{\ell-1}}{(\ell+2)!} \frac{\Gamma(\ell + \frac{\varepsilon+1}{2})}{\Gamma(\frac{\varepsilon+1}{2})} \frac{\hat{n}^L}{r^{\ell+1+\varepsilon}} \sum_{s=0}^{\ell} (s+1) y_1^{\langle L-S, S \rangle} y_2^S$$

<sup>10</sup> With our notation for multi-indices meaning, for instance,

$$\begin{aligned} \hat{n}^{M-2S} &= \text{STF}[n^{i_1} \dots n^{i_{m-2s}}], \\ \hat{n}^{M-2S} \hat{y}_1^{L-S, S'} \hat{y}_2^{M-LS, S'} &= \hat{n}^{i_1 \dots i_{m-2s}} \hat{y}_1^{i_1 \dots i_{\ell-s} j_1 \dots j_s} \hat{y}_2^{i_{\ell-s+1} \dots i_{m-2s} j_1 \dots j_s}. \end{aligned}$$

$$\begin{aligned}
& - \frac{1}{(1-\varepsilon)\left[\Gamma\left(\frac{\varepsilon-1}{2}\right)\right]^2} \sum_{m=0}^{+\infty} \frac{2^{m-1}}{r^{m+2\varepsilon}} \sum_{s=0}^{\lfloor \frac{m}{2} \rfloor} \frac{\Gamma\left(\frac{3+\varepsilon}{2} + m - 2s\right)}{\Gamma\left(\frac{3+\varepsilon}{2} + m - s\right)} \frac{\hat{n}^{M-2s}}{(2s)!!} \\
& \quad \times \sum_{\ell=0}^m \frac{\Gamma\left(\ell + \frac{\varepsilon-1}{2}\right)}{(\ell-s)!} \frac{\Gamma\left(m-\ell + \frac{\varepsilon+1}{2}\right)}{(m-\ell-s)!} \hat{y}_1^{L-S,S'} \hat{y}_2^{M-LS,S'} \\
& \quad \times \left[ \frac{r^2}{(m-s-1+\varepsilon)(s+\frac{\varepsilon-3}{2})} - \frac{(2\ell+\varepsilon-1)y_1^2}{(2\ell+\varepsilon+3)(m-s+\varepsilon)(s+\frac{\varepsilon-1}{2})} \right]. \quad (\text{B10})
\end{aligned}$$

The above formulas have been extensively used to control the IR divergences in the gravitational part of the Fokker action in Sec. (II). However we have found that in fact, the result of our computation of the difference DR–HR does not depend on the detailed prescription we followed to control the homogeneous anti-symmetric solution in Eq. (B3). The independence with respect to the added homogeneous solution in Eq. (B3) is certainly a good sign of the solidness of our result.

### Appendix C: Distributional limits of the function $\gamma_s(z)$

The function  $\gamma_s(z)$  defined by Eq. (3.3) is zero in an ordinary sense for strictly negative integer values  $s = -1 - \ell$  (where  $\ell \in \mathbb{N}$ ). In this Appendix we compute  $\gamma_{-1-\ell}(z)$  in the sense of distributions. From Eq. (3.3) we have

$$\gamma_{-1-\ell-\frac{\varepsilon}{2}}(z) = \frac{2\sqrt{\pi}}{\Gamma(-\ell-\frac{\varepsilon}{2})\Gamma(\ell+\frac{1+\varepsilon}{2})} (z^2-1)^{-1-\ell-\frac{\varepsilon}{2}} \theta(z-1). \quad (\text{C1})$$

We added the Heaviside step function  $\theta(z-1)$  to recall that this expression is defined only for  $z > 1$ . Considered as a distribution (indexed by a parameter  $\varepsilon \in \mathbb{C}$ ), Eq. (C1) is to be applied on test functions  $\varphi(z)$  that are at once smooth, *i.e.*,  $\varphi \in C^\infty(\mathbb{R})$ , and with compact support. Hence,

$$\langle \gamma_{-1-\ell-\frac{\varepsilon}{2}}, \varphi \rangle = \frac{2\sqrt{\pi}}{\Gamma(-\ell-\frac{\varepsilon}{2})\Gamma(\ell+\frac{1+\varepsilon}{2})} \int_1^{+\infty} dz (z^2-1)^{-1-\ell-\frac{\varepsilon}{2}} \varphi(z). \quad (\text{C2})$$

Under this form we see that the limit  $\varepsilon \rightarrow 0$  is ill-defined at the bound  $z = 1$ , but can be made finite by performing some integrations by parts. The surface terms will always be zero by analytic continuation in  $\varepsilon$  at the bound  $z = 1$ , and because the test function has a compact support. After  $\ell + 1$  integrations by parts we obtain

$$\langle \gamma_{-1-\ell-\frac{\varepsilon}{2}}, \varphi \rangle = (-)^{\ell+1} \frac{2\sqrt{\pi}}{\Gamma(1-\frac{\varepsilon}{2})\Gamma(\ell+\frac{1+\varepsilon}{2})} \int_1^{+\infty} dz (z-1)^{-\frac{\varepsilon}{2}} \left( \frac{d}{dz} \right)^{\ell+1} \left[ (z+1)^{-1-\ell-\frac{\varepsilon}{2}} \varphi(z) \right], \quad (\text{C3})$$

and, under that form, we can directly take the limit  $\varepsilon \rightarrow 0$  with result

$$\langle \gamma_{-1-\ell}, \varphi \rangle = \frac{(-)^{\ell} 2^{\ell+1}}{(2\ell-1)!!} \left( \frac{d}{dz} \right)^{\ell} \left[ \frac{\varphi(z)}{(z+1)^{\ell+1}} \right] \Big|_{z=1}. \quad (\text{C4})$$

More explicitly this gives

$$\langle \gamma_{-1-\ell}, \varphi \rangle = \sum_{i=0}^{\ell} (-)^i \alpha_i^{\ell} \varphi^{(i)}(1), \quad (\text{C5a})$$

$$\text{where } \alpha_i^\ell = \frac{2^{i-\ell}}{(2\ell-1)!!} \frac{(2\ell-i)!}{i!(\ell-i)!}. \quad (\text{C5b})$$

So, finally the result for  $\gamma_{-1-\ell}$  when viewed as a distribution reads

$$\gamma_{-1-\ell}(z) = \sum_{i=0}^{\ell} \alpha_i^\ell \delta^{(i)}(z-1), \quad (\text{C6})$$

with  $\delta^{(i)}$  being the  $i$ -th derivative of the Dirac function. In particular  $\gamma_{-1}(z) = \delta(z-1)$  recovers the fact that the Green's function (3.2) reduces in  $3+1$  dimensions to the usual

$$G_{\text{ret}}^{(\varepsilon=0)}(\mathbf{x}, t) = -\frac{\delta(t-r)}{4\pi r}. \quad (\text{C7})$$

#### Appendix D: Computation of the coefficients $C_\ell^{p,k}$

These coefficients, defined in  $d = 3 + \varepsilon$  dimensions by Eq. (3.23), are written in the form

$$C_\ell^{p,k} = \frac{4\pi}{\Gamma(-\frac{\varepsilon}{2})\Gamma(-\ell-\frac{\varepsilon}{2})\Gamma(\frac{1+\varepsilon}{2})\Gamma(\ell+\frac{1+\varepsilon}{2})} L_{a,b,c}^p, \quad (\text{D1})$$

together with the following definition of the double integral,

$$L_{a,b,c}^p = \int_1^{+\infty} dy y^p (y^2-1)^a \int_1^{+\infty} dz (z^2-1)^b (y+z)^c, \quad (\text{D2})$$

and the particular set of coefficients  $a = -1 - \frac{\varepsilon}{2}$ ,  $b = -\ell - 1 - \frac{\varepsilon}{2}$ , and  $c = \ell + k - 2 + 2\varepsilon - \eta$ , where the parameter  $\eta$  was introduced in Eq. (3.7).

The integral (D2) is computed by first relating it to the simpler integral corresponding to  $p = 0$ , namely  $K_{a,b,c} = L_{a,b,c}^0$  or

$$K_{a,b,c} = \int_1^{+\infty} dy (y^2-1)^a \int_1^{+\infty} dz (z^2-1)^b (y+z)^c. \quad (\text{D3})$$

The latter integral in turn converges for  $\Re(a) > -1$ ,  $\Re(b) > -1$ ,  $\Re(2a+c) < -1$  and  $\Re(2b+c) < -1$ . Moreover, it admits an explicit closed-form expression in terms of Eulerian  $\Gamma$ -functions,

$$K_{a,b,c} = \frac{\Gamma(a+1)\Gamma(b+1)\Gamma(-a-\frac{c}{2}-\frac{1}{2})\Gamma(-b-\frac{c}{2}-\frac{1}{2})\Gamma(-a-b-\frac{c}{2}-1)}{4\sqrt{\pi}\Gamma(-\frac{c}{2}+\frac{1}{2})\Gamma(-a-b-c-1)}, \quad (\text{D4})$$

so that, regarded as a function of  $a$ ,  $b$  and  $c$ , it can be extended to the complex plane by analytic continuation, except for a countable number of isolated points.

Finally it is very easy to relate  $L_{a,b,c}^p$  to  $K_{a,b,c}$ . When  $p = 2q$  is an even integer, we have

$$L_{a,b,c}^{2q} = \sum_{i=0}^q \binom{q}{i} K_{a+i,b,c}, \quad (\text{D5})$$



where  $\binom{q}{i}$  is the usual binomial coefficient. And, when  $\ell = 2q + 1$  is an odd integer, we go back to the even case (D5) thanks to the formula

$$L_{a,b,c}^{2q+1} = \frac{1}{2} \left[ L_{a+1,b,c-1}^{2q} - L_{a,b+1,c-1}^{2q} + L_{a,b,c+1}^{2q} \right]. \quad (\text{D6})$$

With those formulas we can compute the  $C_\ell^{p,k}$  for all required values of  $\ell$ ,  $p$  and  $k$ . Note that there are some combinations of  $a$ ,  $b$  and  $c$  for which the  $\varepsilon$ 's disappear. In these cases it is crucial to keep the parameter  $\eta$  finite, and to compute the expansion series when  $\eta \rightarrow 0$  (for any  $\varepsilon$ , *i.e.*, before applying the limit  $\varepsilon \rightarrow 0$ ). We find that many individual terms behave like  $1/\eta$  and are thus ill-defined, but that these divergences always cancel out from the sum of all these terms. Thus, at the end we always get a finite result when  $\eta = 0$ , which can then be evaluated in the limit  $\varepsilon \rightarrow 0$ .

- 
- [1] L. Bernard, L. Blanchet, A. Bohé, G. Faye, and S. Marsat, Phys. Rev. D **93**, 084037 (2016), arXiv:1512.02876 [gr-qc].
  - [2] L. Bernard, L. Blanchet, A. Bohé, G. Faye, and S. Marsat, Phys. Rev. **95**, 044026 (2017), arXiv:1610.07934 [gr-qc].
  - [3] P. Jaranowski and G. Schäfer, Phys. Rev. D **86**, 061503(R) (2012), arXiv:1207.5448 [gr-qc].
  - [4] P. Jaranowski and G. Schäfer, Phys. Rev. D **87**, 081503(R) (2013), arXiv:1303.3225 [gr-qc].
  - [5] P. Jaranowski and G. Schäfer, Phys. Rev. D **92**, 124043 (2015), arXiv:1508.01016 [gr-qc].
  - [6] T. Damour, P. Jaranowski, and G. Schäfer, Phys. Rev. D **89**, 064058 (2014), arXiv:1401.4548 [gr-qc].
  - [7] T. Damour, P. Jaranowski, and G. Schäfer, Phys. Rev. D **93**, 084014 (2016), arXiv:1601.01283 [gr-qc].
  - [8] S. Foffa and R. Sturani, Phys. Rev. D **87**, 064011 (2012), arXiv:1206.7087 [gr-qc].
  - [9] S. Foffa and R. Sturani, Phys. Rev. D **87**, 044056 (2013), arXiv:1111.5488 [gr-qc].
  - [10] C. R. Galley, A. K. Leibovich, R. A. Porto, and A. Ross, Phys. Rev. D **93**, 124010 (2016), 1511.07379.
  - [11] S. Foffa, P. Mastrolia, R. Sturani, and C. Sturm (2016), 1612.00482.
  - [12] L. Blanchet and A. Le Tiec (2017), arXiv:1702.06839 [gr-qc].
  - [13] L. Blanchet, S. Detweiler, A. Le Tiec, and B. Whiting, Phys. Rev. D **81**, 084033 (2010), arXiv:1002.0726 [gr-qc].
  - [14] A. Le Tiec, L. Blanchet, and B. Whiting, Phys. Rev. D **85**, 064039 (2012), arXiv:1111.5378 [gr-qc].
  - [15] A. Le Tiec, E. Barausse, and A. Buonanno, Phys. Rev. Lett. **108**, 131103 (2012), arXiv:1111.5609 [gr-qc].
  - [16] D. Bini and T. Damour, Phys. Rev. D **87**, 121501(R) (2013), arXiv:1305.4884 [gr-qc].
  - [17] L. Barack, T. Damour, and N. Sago, Phys. Rev. D **82**, 084036 (2010), arXiv:1008.0935 [gr-qc].
  - [18] A. Le Tiec, A. Mroué, L. Barack, A. Buonanno, H. Pfeiffer, N. Sago, and A. Taracchini, Phys. Rev. Lett. **107**, 141101 (2011), arXiv:1106.3278 [gr-qc].
  - [19] M. van de Meent, Phys. Rev. Lett. **118**, 011101 (2017).
  - [20] T. Damour, Phys. Rev. D **81**, 024017 (2010), arXiv:0910.5533 [gr-qc].
  - [21] T. Damour, P. Jaranowski, and G. Schäfer, Phys. Rev. D **91**, 084024 (2015), arXiv:1502.07245 [gr-qc].

- [22] G. 't Hooft and M. Veltman, Nucl. Phys. **B44**, 139 (1972).
- [23] C. G. Bollini and J. J. Giambiagi, Phys. Lett. B **40**, 566 (1972).
- [24] P. Breitenlohner and D. Maison, Comm. Math. Phys. **52**, 11 (1977).
- [25] L. Blanchet, T. Damour, and G. Esposito-Farèse, Phys. Rev. D **69**, 124007 (2004), gr-qc/0311052.
- [26] L. Blanchet, T. Damour, G. Esposito-Farèse, and B. R. Iyer, Phys. Rev. D **71**, 124004 (2005), gr-qc/0503044.
- [27] R. Porto and I. Rothstein (2017), arXiv:1703.06433 [gr-qc].
- [28] R. A. Porto (2017), arXiv:1703.06434 [gr-qc].
- [29] L. Blanchet and T. Damour, Phys. Rev. D **37**, 1410 (1988).
- [30] L. Blanchet, Phys. Rev. D **47**, 4392 (1993).
- [31] J. M. Martín-García, A. García-Parrado, A. Stecchina, B. Wardell, C. Pitrou, D. Brizuela, D. Yllanes, G. Faye, L. Stein, R. Portugal, et al., *xAct: Efficient tensor computer algebra for Mathematica* (GPL 2002–2012), <http://www.xact.es/>.
- [32] V. Fock, *Theory of space, time and gravitation* (Pergamon, London, 1959).
- [33] O. Poujade and L. Blanchet, Phys. Rev. D **65**, 124020 (2002), gr-qc/0112057.
- [34] L. Blanchet, G. Faye, and S. Nissanke, Phys. Rev. D **72**, 044024 (2005).
- [35] L. Blanchet, Class. Quant. Grav. **15**, 1971 (1998), gr-qc/9801101.
- [36] M. Riesz, Acta Math. **81**, 1 (1949).
- [37] L. Blanchet and G. Schäfer, Class. Quant. Grav. **10**, 2699 (1993).