Exponential Stability and Stabilization of Extended Linearizations via Continuous Updates of Riccati Based Feedback

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Many recent works on stabilization of nonlinear systems target the case of locally stabilizing an unstable steady state solution against small perturbations. In this work we explicitly address the goal of driving a system into a nonattractive steady state starting from a well developed state for which the linearization based local approaches will not work. Considering extended linearizations or state-dependent coefficient representations of nonlinear systems, we develop sufficient conditions for stability of solution trajectories. We find that if the coefficient matrix is uniformly stable in a sufficiently large neighborhood of the current state, then the state will eventually decay. Based on these analytical results we propose a scheme that is designed to maintain the stabilization property of Riccati based feedback constant during a certain period of the state evolution. We illustrate the general applicability of the resulting algorithm for setpoint stabilization of nonlinear autonomous systems and its numerical efficiency in two examples.

1 Introduction

We consider the general task to find an input u that drives the state ζ of a nonlinear autonomous input-affine system of type

$$\dot{\zeta}(t) = f(\zeta(t)) + Bu(t), \quad \zeta(0) = z \in \mathbb{R}^n,$$

towards a steady state z^*, i.e., a state z^* for which $f(z^*) = 0$. This problem is commonly known as *set point stabilization*. It is equivalent to considering $\xi = \zeta - z^*$ and the task

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to drive the difference state ξ, that satisfies

$$\dot{\xi}(t) = \tilde{f}(\xi(t)) + Bu, \quad \xi(0) = x_0, \tag{1}$$

to zero, where $\tilde{f}(\xi(t)) := f(\xi(t) + z^*)$ and $x_0 = z - z^*$. If f is Lipshitz continuous and since $\tilde{f}(0) = 0$, there exists [9] a matrix valued function $A\colon \mathbb{R}^n \to \mathbb{R}^{n,n}$ such that (1) can be written as

$$\dot{\xi}(t) = A(\xi(t))\xi(t) + Bu(t), \quad \xi(0) = x_0. \tag{2}$$

Thus, *extended linearizations* or *state dependent coefficient* (SDC) systems like (2) are a suitable starting point for general nonlinear set point stabilization problems. Then the question is, how to define a feedback gain $F(\xi(t))$ such that solutions of the closed loop system

$$\dot{\xi}(t) = [A(\xi(t)) - BF(\xi(t))]\xi(t), \quad \xi(0) = x_0, \tag{3}$$

or, equivalently,

$$\dot{\zeta}(t) = f(\zeta(t)) - BF(\zeta(t) - z^*)[\zeta(t) - z^*], \quad \zeta(0) = z,$$

decay asymptotically to zero or to z^*, respectively. One approach is to define the feedback gain as $F(x) = R^{-1}B^T P(x)$ for a given state $x = \xi(t)$, where $P(x)$ is the solution to the *state dependent Riccati equation (SDRE)*

$$P(x)A(x) + A(x)^\mathsf{T} P(x) - P(x)BR^{-1}B^\mathsf{T} P(x) + Q = 0, \tag{4}$$

for given weighting matrices $R \succ 0$ and $Q \succcurlyeq 0$[1].

Known results [9, 2, 16] on the stabilization via SDRE feedback base on the assumption that the initial state x_0 is close to zero such that the nonlinear terms can be considered as a perturbation of a linear system. Precisely, one considers the SDRE (4) for the extended linear system (2) and defines $P(x) =: P(x_0) + \Delta P(x)$ and $A(x) =: A(x_0) + \Delta A(x)$. Then, if $-BF_0 := -BR^{-1}B^\mathsf{T} P(x_0)$ is stabilizing for $A_0 := A(x_0)$ and if the considered matrix functions are Lipshitz continuous in x, then one can show that the solution to

$$\dot{\xi}(t) = (A_0 - BF_0)\xi(t) + h(t), \quad \xi(0) = x_0,$$

where $h(t) := (\Delta A(\xi(t)) - BR^{-1}B^\mathsf{T}\Delta P(\xi(t)))\xi(t)$, goes to zero as $t \to \infty$ with an exponential decay rate [2], provided that x_0 is sufficiently small.

Our goal, however, is to drive a system from a developed state towards the zero state, which contradicts the smallness assumption on the initial value. Once the system's state is close to the origin, stabilization strategies that base on smallness of the deviation from the zero state and that have been proven successful can be applied; see [3, 5, 7] for numerical studies considering nonlinear PDEs and [17] for a theoretical analysis. For completeness, we mention the earlier works on feedback synthesis for nonlinear systems based on extended linearizations [4, 18], where families of feedback gains parametrized

[1]For an overview of notations and definitions see Table 1 at the end of the introduction

by set points of the considered plants were considered. There again, the analysis of the stabilizing properties bases on smallness of the deviations from the targeted operating points.

The manuscript is organized as follows. In Section 2, we extend the results that were reported in [13] on stability of linear time-varying systems like

$$\dot{\xi}(t) = \tilde{A}(t)\xi(t),$$

to give sufficient conditions for stability of SDC systems like system (2). The basic idea is that for a given trajectory ξ, one can consider $\tilde{A}(t) := A(\xi(t))$. However, this approach leads to sufficient conditions that are very restrictive and probably not easy to confirm for most applications. In view of practical use, in Section 3, we provide localized conditions taking advantage of the observation that with controlling the state ξ, one also controls the coefficients. By means of an example, we show the practicability of the derived estimates.

The general result is that one can achieve an exponential decay of the solutions if, at a fixed state x, the local transient behavior is well balanced with the decay rate of the current coefficient $A(x)$ and if this balance holds true uniformly in a sufficiently large neighborhood. In Section 4, we will introduce conditions and an algorithm for a feedback gain F that ensures uniform bounds on the transitive behavior and a constant decay rate in a neighborhood of the current state via continuously updating an initial feedback. The resulting algorithm is theoretically well founded and generally applicable for set point control of any nonlinear autonomous system that can be written in SDC form. In Section 5, we investigate the proposed update scheme for two numerical examples and show its feasibility and efficiency in comparison to the SDRE feedback. We conclude with summarizing remarks and an outlook.

Symbol	Definition/Explanation
$\mathbb{R}_{>0}, \mathbb{R}_{\geq 0}$	Set of real numbers that are greater (or equal) than zero
$\lVert\cdot\rVert$, $\lVert\cdot\rVert_F$	A generic norm, the *Frobenius* norm
$R \succ 0$, $Q \succcurlyeq 0$	The matrices R and Q are symmetric and positive (semi)definite
$\mathcal{S}_{(K,L,M,\omega;X)}$	Set of matrix valued functions that are uniformly stable, bounded, Lipshitz continuous on a set X, cf. Definition 2.8
$\mathcal{S}_{(K,L,M_T,\omega;X_T)}$	Set of matrix valued functions that are uniformly stable, bounded, Lipshitz continuous on a set X_T, cf. Definition 3.1
$\mathcal{S}_{K,\omega}$	Set of stable matrices with decay rate ω and transients bounded by K, cf. Definition 4.1

Table 1: Symbols and notations used in this paper.

2 Stability of State-dependent Coefficient Systems

To describe exponential stability for the considered type of SDC systems

$$\dot{\xi}(t) = A(\xi(t))\xi(t), \tag{5}$$

we adjust the definition for time varying systems as given in [19, Def. 6.5].

Definition 2.1. System (5) is called *uniformly exponentially stable* if there exist positive constants K and ω such that for any $x_0 \in \mathbb{R}$, a solution ξ of (5) with $\xi(0) = x_0$ satisfies

$$\|\xi(t)\| \leq K e^{-\omega t}\|x_0\|, \quad \text{for } t \geq 0. \tag{6}$$

It is called *uniformly exponentially stable on* X, if, for some $X \subset \mathbb{R}^n$, relation (6) holds for any $x_0 \subset X$.

Note that the definition in [19] is for linear systems but (6) solely bases on solution trajectories and, thus, applies also for nonlinear systems.

Assumption 2.2. *Regarding equation* (5)*, we suppose that*

(1) the map $A\colon \mathbb{R}^n \to \mathbb{R}^{n,n}$ *is Lipshitz-continuous,*

(2) there is a bounded nonempty set $X \subset \mathbb{R}^n$ *such that* $\xi(t) \in X$*, for* $t \geq 0$*, where* ξ *is a solution to* (5)*, with* $\xi(0) = x_0 \in X$*.*

Remark 2.3 (Concerning Lipshitz-continuity of A). In the context of *extended linearizations* of a general nonlinear system $\dot{\xi} = \tilde{f}(\xi)$ with $\tilde{f}(0) = 0$, cf. (1) and (2), the SDC matrix of A can be constructed [9] via

$$A(\xi) = \int_0^1 D\tilde{f}(s\xi)\,\mathrm{d}s, \tag{7}$$

where $D\tilde{f}$ denotes the Jacobian of $\tilde{f}$, provided that $\tilde{f}$ is at least *absolute continuous*. If constructed through formula (7), the function A has the same regularity as the Jacobian of $\tilde{f}$. Moreover, in general, *Lipshitz continuity* of $\tilde{f}$ does not imply *Lipshitz continuity* of A as can be seen from the example of $\tilde{f}\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} : x \to e^{-x}x^{3/2}$ that has the unique SDC factorization $\tilde{f}(x) = A(x)\cdot x := e^{-x}\sqrt{x}\cdot x$, where $\tilde{f}$ is Lipshitz continuous but A is not.

Remark 2.4 (Concerning the existence of X). From the one hand side, the existence of a system invariant set X is a necessary assumption for uniform stability of a set $X_0 \subset X$, since, by autonomy of the system, for a stable trajectory ξ starting in x_0, the set $\{\xi(t) : t \geq 0\}$ fulfills the conditions of Assumption 2.2. On the other hand, this assumption is difficult to establish in the general case unless the existence of physical bounds, attractors like in the *Chafee Infante* equation (cf. Section 5), or limit cycles like in *Predator-Prey* models is known. In Section 3 we will replace this assumption by a localized and computable version.

The following lemma states that in order to state exponential stability for trajectories that start in X, the existence of a global unique solution is a necessary prerequisite.

Lemma 2.5. *Consider equation* (5) *and let Assumption 2.2 hold. Then, for any $x_0 \in X$, there is a unique solution $\xi\colon [0,\infty) \to \mathbb{R}^n$ to* (5) *with $\xi(0) = x_0$.*

Proof. By Lipshitz-continuity of A, it follows that $x \mapsto A(x)x$ is locally Lipshitz continuous. Accordingly, by the *Picard-Lindelöf* theorem, there exists a unique solution ξ locally in time. Since, by assumption, ξ stays in the bounded set X, it can be extended to a global solution. □

We introduce a class of SDC matrices similar to the class of time-dependent coefficient matrices used in [13] via the following assumption.

Assumption 2.6. *For a given bounded set $X \subset \mathbb{R}^n$, the function $A\colon X \to \mathbb{R}^{n,n}$ is* Lipshitz continuous, *i.e. there exists a constant $L \in \mathbb{R}_{>0}$ such that*

$$\|A(x_1) - A(x_2)\| \leq L\|x_1 - x_2\|, \quad \textit{for all } x_1, x_2 \in X, \tag{8}$$

and uniformly stable *on X, i.e. there exist constants ω, $K \in \mathbb{R}_{>0}$ such that*

$$\|e^{A(x)s}\| \leq Ke^{-\omega s}, \quad \textit{for all } x \in X \textit{ and for } s > 0. \tag{9}$$

Lemma 2.7. *Consider equation* (5) *and let Assumption 2.2 and Assumption 2.6 hold. Then*

$$M := \sup_{x\in X}\|A(x)x\| < \infty$$

and any solution to (5) *that starts in X is Lipshitz continuous with Lipshitz constant M.*

Proof. Since A is Lipshitz continuous and X is bounded, $\|A(x)\|$ and, thus, $\|A(x)x\|$ is bounded away from ∞ for all $x \in X$. By assumption, a solution ξ to (5) that starts in X stays in X so that we can estimate

$$\|\xi(t_2) - \xi(t_1)\| = \|\int_{t_1}^{t_2} \dot{\xi}(s)\ \mathrm{d}s\| = \|\int_{t_1}^{t_2} A(\xi(s))\xi(s)\ \mathrm{d}s\| \leq M|t_2 - t_1|, \tag{10}$$

for t_1, $t_2 > 0$. □

By virtue of Lemma 2.7, the following definition, which we use for later reference, is well posed.

Definition 2.8. The matrix-valued function $A\colon X \subset \mathbb{R}^n \to \mathbb{R}^{n,n}$ is an element of the class $\mathcal{S}_{(K,L,M,\omega;X)}$ for some constants K, L, M, ω and a bounded set X, if A and K, L, and ω are such that Assumption 2.2 and Assumption 2.6 hold on X and if $\sup_{x\in X}\|A(x)x\| \leq M$.

Remark 2.9 (Outer approximations to X). As laid out in Remark 2.4, the existence of such a system invariant set X is a necessary condition for stability but strong and hardly checkable. For concrete systems, it might be possible to check for an outer approximation $\bar{X} \supseteq X$ and estimate the constants of Definition 2.8 replacing X by $\bar{X}$. Since this will only give a more conservative estimate of the constants K, L, M, and ω the following results, that provide sufficient stability conditions based on sharper estimates, still apply.

We can now provide an estimate on the exponential growth of solutions of the SDC system (5).

Theorem 2.10. *Consider Equation* (5) *and let Assumption 2.2 and Assumption 2.6 hold. Then, for any $x_0 \in X$, the unique solution $\xi\colon [0,\infty) \to \mathbb{R}^n$ to* (5) *with $\xi(0) = x_0$ satisfies*

$$\|\xi(t)\| \leq K e^{t\cdot(\sqrt{KLM \ln 2} - \omega)} \|x_0\|, \quad \textit{for all } t > 0, \tag{11}$$

where $M := \sup_{x\in X} \|A(x)x\|$.

Corollary 2.11. *Under the assumptions of Theorem 2.10, if*

$$KLM \ln 2 < \omega^2, \tag{12}$$

then system (5) *is uniformly exponentially stable on X as defined in Definition 2.1.*

Proof of Theorem 2.10. We extend the arguments used in [13] to prove this result for linear time-varying systems to extended linearizations. The basic idea is that for a given trajectory x, the state-dependent coefficient A can be considered as a time-dependent coefficient $\tilde{A}(t) := A(x(t))$.

With $A \in \mathcal{S}_{(K,L,M,\omega;X)}$, for any t, $\rho \geq 0$, every solution ξ of (5) with $\xi(0) \in X$ satisfies

$$\|\xi(t)\| \leq K e^{-\omega t} \|\xi(0)\| + KLM \int_0^t |s - \rho| e^{-\omega(t-s)} \|\xi(s)\| \,\mathrm{d}s, \tag{13}$$

cf. [13, Lem. 5.2]]. In fact, for a given solution ξ and t, $\rho \geq 0$, we define $A_\rho := A(\xi(\rho))$ and rewrite (5) as

$$\dot{\xi}(t) = A_\rho \xi(t) + (A(\xi(t)) - A_\rho)\xi(t)$$

to get the following representation of ξ:

$$\xi(t) = e^{A_\rho t}\xi(0) + \int_0^t e^{A_\rho (t-s)} (A(\xi(s)) - A_\rho)\xi(s) \,\mathrm{d}s.$$

Then, taking the norm and using the estimates (8), (9), and (10), namely the Lipshitz continuity of A, the stability of A_ρ, and the Lipshitz continuity of ξ, we estimate that

$$\begin{aligned}
\|\xi(t)\| &\leq \|e^{A_\rho t}\| \|\xi(0)\| + \int_0^t \|e^{A_\rho(t-s)}\| \|(A(\xi(s)) - A_\rho)\| \|\xi(s)\| \,\mathrm{d}s \\
&\leq K e^{-\omega t} \|\xi(0)\| + \int_0^t K e^{-\omega(t-s)} L \|\xi(s) - \xi(\rho)\| \|\xi(s)\| \,\mathrm{d}s \\
&\leq K e^{-\omega t} \|\xi(0)\| + KL \int_0^t e^{-\omega(t-s)} M |s - \rho| \|\xi(s)\| \,\mathrm{d}s
\end{aligned} \tag{14}$$

and arrive at inequality (13). Next, we scale the solution ξ and the time t to obtain the function

$$\zeta\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad \zeta(t) := e^{\omega t/\alpha} \frac{\|\xi(t/\alpha)\|}{K\|\xi(0)\|}, \quad \text{where } \alpha := \sqrt{KLM}. \tag{15}$$

As in [13, (5.18)], we define the functions

$$v(t) := e^{t\cdot\sqrt{\ln 2}}, \quad \text{and} \quad r(t) := \max\{0, t - \sqrt{\ln 2}\},$$

which by [13, Lem. 5.8] satisfy the integral comparison

$$v(t) \geq 1 + \int_0^t |s - r(t)|\zeta(s)\,\mathrm{d}s,$$

and, thus, by [13, Lem. 5.5], give an upper bound:

$$\zeta(t) \leq v(t) \quad \text{or} \quad e^{\omega t/\sqrt{KLM}} \frac{\|\xi(t/\sqrt{KLM})\|}{K\|\xi(0)\|} \leq e^{t\cdot\sqrt{\ln 2}},$$

at any time $t \geq 0$. Accordingly, having undone the scalings (15), we arrive at

$$\|\xi(t)\| \leq K e^{t\cdot(\sqrt{KLM \ln 2} - \omega)}\|\xi(0)\|,$$

for all $t \geq 0$. □

Remark 2.12. For $K < 2$, the factor $\ln 2$ in (11) and (12) can be replaced by $\ln K$, see [13, Thm. 2.1].

3 Local Conditions for Exponential Stability

Relation (12) illustrates the nature of the stability results. For the parametrization $\tilde{A}(t) := A(\xi(t))$, the constant LM is the Lipshitz constant of $t \mapsto \tilde{A}(t)$. Accordingly, the requirement that LM must not exceed some value defined by the decay rate ω and the bound K means that the changes in $\tilde{A}$, that may trigger new transient phases faster than the overall decay fades them out, should be limited.

In the linear time varying case, if one considers global constants ω and K, one also needs a global bound on LM, since the overall decay of the solution can be violated by a sudden change in $\tilde{A}$ at any time. Also, in the linear time varying case, the function $t \to \tilde{A}(t)$ is known for all time so that a global bound can be found. Improvements of the results may be obtained by relating K, ω, and LM locally in time. However, due to the arbitrariness of the mapping $t \mapsto \tilde{A}(t)$, such localizations would be very problem dependent.

Things are different for the extended linearizations. The mapping $t \mapsto A(\xi(t))$ is less arbitrary, since $A(\xi(t))$ will be stabilized together with the solution ξ. If the function $x \mapsto A(x)$ is smooth, then, for $\xi(t) \to 0$, the coefficient $A(\xi(t))$ approaches a constant value.

In fact, when having reached or when starting from a state close to zero, exponential decay can be established by the results on *local exponential stability* [7] or on *almost linear systems*, cf. the proof for the SDRE stabilization properties in [2]. On the other hand, for an arbitrary starting value, a global bound on $M(x) = \|A(x)x\|$ might not be available or might be too conservative. Thus, the results provided only apply to particular classes of problems for which the existence (or an outer approximation; cf. Remark 2.9) of an system invariant set X is known or to particular given trajectories.

The following results address sufficient conditions for exponential decay of solution trajectories at discrete time instances that can be locally estimated by means of bounds on the growth of the solution in a certain time interval. This decay at discrete instances will eventually drive the system into a state close to zero from where the linear theory will provide exponential decay. The piecewise in time character of the results that follow can also be used to define feedback laws that act locally.

We drop the global assumption on the existence of a system invariant set $X \subset \mathbb{R}^n$, cf. Assumption 2.2(2), and consider a set of initial values and a set that contains all states that evolve from these initial values within a finite time horizon.

Definition 3.1. Let $X_0 \subset \mathbb{R}^n$ be a connected closed set that contains the origin and let $T \geq 0$.

a.) By $\Xi_{[0,T]}$ we denote the set of all solution trajectories that start in X_0:

$$\Xi_{[0,T]} := \{\xi\colon [0,T] \to \mathbb{R}^n : \xi \text{ solves } (5) \text{ and } \xi(0) \in X_0\}.$$

b.) By X_T we denote the set that contains all final values of the trajectories

$$X_T := \{\xi(T) : \xi \in \Xi_{[0,T]}\}.$$

c.) By $X_{[0,T]}$ we denote the set that contains all values that are achieved by the solution trajectories within the time interval $[0,T]$:

$$X_{[0,T]} := \{\xi(t) : \xi \in \Xi_{[0,T]},\ 0 \leq t \leq T\}.$$

If any solution to (5) that starts in X_0 has a finite escape time $t_f < T$, we set $X_T := X_{[0,T]} := \mathbb{R}^n$.

The definition of $\mathcal{S}_{(K,L,M,\omega;X)}$, cf. Definition 2.8, readily extends to $\mathcal{S}_{(K,L,M_T,\omega;X_T)}$, if one assumes that for an element $A\colon X \subset \mathbb{R}^n \to \mathbb{R}^{n,n}$ and $T > 0$, there exist constants K, L, M_T, ω such that Assumption 2.2 and Assumption 2.6 hold and such that $M_T := \sup_{x \in X_{[0,T]}} \|A(x)x\| < \infty$ is valid on the set $X_{[0,T]}$. Note that in the case of solutions with finite escape time less than T, the set $X_{[0,T]}$ is not bounded and the latter assumption $M_T < \infty$ does not hold, cf. Definition 3.1.

Remark 3.2. We will assume that the pointwise stability constants ω and K and the Lipshitz constant L are independent of the state. The uniformity of the stability constants will be used to state global convergence and is going to be a design target of a feedback stabilization. The uniformity of the Lipshitz constant is given for the case that A is affine linear in x. Also, a state dependent L can be treated with the same approach illustrated below.

In the following theorem, we provide a local condition for exponential decay at discrete time instances of trajectories that start in X_0. The basic reasoning is that if for a time t^* all trajectories are in a set that is contained in the considered set of initial values X_0 then, because of the autonomy of the system, the system states will be contained in $X_{[0,t^*]}$ thereafter. Accordingly, if one can establish exponential decay for the short time horizon, then the decay will hold on for the whole time axis. Having stated the basic result, we refine it by providing a dynamic bound which can replace the static constant LM_T, which is sharper, and which can be evaluated numerically.

Theorem 3.3. *For given $T > 0$ and X_0 as in Definition 3.1, let $A \in \mathcal{S}_{(K,L,M_T,\omega;X_T)}$ and for $0 \le t \le T$, let $M_t := \sup_{x \in X_t} \|A(x)x\|$. If for a t^*, with $0 < t^* \le T$,*

$$-\,\omega_{t^*} := \sqrt{KLM_{t^*} \ln 2} - \omega \tag{16}$$

and

$$-\,\omega^* := \frac{\ln K}{t^*} - \omega_{t^*} \tag{17}$$

are negative, then the snapshots $\xi(t)$ of any solution ξ to (5) with $\xi(0) = x_0 \in X_0$ taken on the discrete grid $\mathcal{T}^* := \{t \colon t = N \cdot t^*,\ N = 0, 1, \dots\}$ decay exponentially in the sense that

$$\|\xi(t)\| \le \|x_0\| e^{-\omega^* t}, \quad \textit{for all } t \in \mathcal{T}^*.$$

Proof. The assumptions made include that $M_T < \infty$ so that for every $x_0 \in X_0$ the associated solution ξ to (5) that starts in x_0 exists on $[0, T]$. Noting that by definition the bound $M_t = \sup_{x \in X_t} \|A(x)x\|$ grows with t and noting that Theorem 2.10 is also valid on a finite time horizon, any such solution ξ fulfills

$$\|\xi(t)\| \le K e^{-\omega_t t} \|x_0\| = e^{(\frac{\ln K}{t} - \omega_t)t} \|x_0\|, \quad \text{for } 0 < t \le T,$$

with $\omega_t := \sqrt{KLM_t \ln 2} - \omega$. Thus, if there exists a t^* such that $-\omega_{t^*}$ and $-\omega^*$ as defined in (16) and (17) are negative, then at t^* any such solution ξ fulfills

$$\|\xi(t^*)\| \le \|x_0\| e^{-\omega^* t^*},$$

with $e^{-\omega^* t^*} < 1$. Accordingly, the current value $\xi(t^*)$ is in a ball $X_0^* \subset X_0$. Repeating the previous arguments with X_0 replaced by X_0^* and x_0 by $x(t^*)$ and noting that the new constants K, L, and M_t will be smaller than the previous, we can directly state the estimate

$$\|\xi(2t^*)\| \le \|\xi(t^*)\| e^{-\omega^* t^*} \le \|x_0\| e^{-\omega^* 2t^*},$$

which, by induction, holds for any multiple of t^*. □

Next, we replace the static constant LM_t by a dynamic estimate that bases on differential and integral mean values.

Lemma 3.4. *For a given* $T > 0$*, let* $A \in \mathcal{S}_{(K,L,M_T,\omega;X_T)}$ *be smoothly differentiable. If also the chosen norm* $\|\cdot\|$ *is smoothly differentiable, then the constant* LM_t *in Theorem 3.3 can be replaced by*

$$m_t := \inf_{\rho\in\mathbb{R}_{\geq 0}} \sup_{\xi\in\Xi_{[0,t]}} \frac{\int_0^t e^{-\omega(t-s)}\|A(\xi(s)) - A(\xi(\rho))\|\|\xi(s)\| \,\mathrm{d}s}{\int_0^t e^{-\omega(t-s)}|s-\rho|\|\xi(s)\| \,\mathrm{d}s}. \tag{18}$$

Proof. Under the given assumptions, for $\xi \in \Xi_{[0,T]}$ and $\rho \in \mathbb{R}_{\geq 0}$, the function $s \mapsto \|A(\xi(s)) - A(\xi(\rho))\|$ is differentiable so that, by the *Mean-Value Theorem*, the value

$$M_\rho(s) := \frac{\|A(\xi(s)) - A(\xi(\rho))\|}{|s-\rho|}. \tag{19}$$

is well defined through a continuous function on $(0, T)$ and in particular at $s = \rho$. Thus, we can rewrite the estimate (14) as

$$\|\xi(t)\| \leq Ke^{-\omega t}\|\xi(0)\| + K\int_0^t e^{-\omega(t-s)}M_\rho(s)|s-\rho|\|\xi(s)\| \,\mathrm{d}s.$$

Since the function $s \mapsto e^{-\omega(t-s)}|s-\rho|\|\xi(s)\|$ is continuous and positive there exists a constant $\tilde{m}$ such that

$$\int_0^t e^{-\omega(t-s)}M_\rho(s)|s-\rho|\|\xi(s)\| \,\mathrm{d}s = \tilde{m}\int_0^t e^{-\omega(t-s)}|s-\rho|\|\xi(s)\| \,\mathrm{d}s. \tag{20}$$

If $\xi(s) \equiv 0$ for all s, we set $\tilde{m} = 0$. For all other cases, by virtue of (20) and (19), the constant $\tilde{m}$ is given as

$$\tilde{m} = \frac{\int_0^t e^{-\omega(t-s)}\|(A(\xi(s)) - A(\xi(\rho))\|\|\xi(s)\| \,\mathrm{d}s}{\int_0^t e^{-\omega(t-s)}|s-\rho|\|\xi(s)\| \,\mathrm{d}s}.$$

Thus, using relation (19) and (20) for the estimate (13), one can replace the constant LM_t in Theorem 3.3 by m_t, which is the worst case estimate of $\tilde{m}$ with respect to all trajectories $\xi \in \Xi_{[0,t]}$ for a given $\rho \in (0, t)$ that possibly has been optimized in order to make the estimate as small as possible. □

The conditions formulated in Theorem 3.3 as well as the improved bounds introduced in Lemma 3.4 can be checked numerically as we illustrate it by the following example. Note that Lemma 3.4 is valid also if m_t is approximated by

$$m_t(\bar{\rho}) := \sup_{\xi\in\Xi_{[0,t]}} \frac{\int_0^t e^{-\omega(t-s)}\|A(\xi(s)) - A(\xi(\bar{\rho}))\|\|\xi(s)\| \,\mathrm{d}s}{\int_0^t e^{-\omega(t-s)}|s-\bar{\rho}|\|\xi(s)\| \,\mathrm{d}s}.$$

for a delibarate choice of $\bar{\rho} \geq 0$. Clearly, such a suboptimal estimate is easier to compute but less sharp.

Example 3.5. Consider the following parametrized SDC system

$$\begin{bmatrix} \dot{\xi}_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} -1 & -(1+\xi_1^2) \\ 1+\xi_1^2 & \alpha \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \quad \xi(0) = x_0 \in X_0, \tag{21}$$

with a system matrix $A(x)$ that for any $x = \begin{bmatrix} \xi_1 & \xi_2 \end{bmatrix}^\mathsf{T} \in \mathbb{R}^2$ and for $\alpha \in [-1, 1]$ has the two eigenvalues λ_1, λ_2 with real parts $\Re(\lambda_1) = \Re(\lambda_2) = \frac{1}{2}(-1+\alpha)$. Moreover, since the imaginary parts of λ_1, λ_2 are distinct, the matrix is diagonalizable. Accordingly, the exponential growth rate is given as $-\omega = \frac{1}{2}(-1+\alpha)$ and, for given ξ, we can compute

$$K(\xi(t)), \quad \text{such that} \quad \|e^{A(\xi(t))s}\| \leq K(\xi(t))e^{-\omega s} \quad \text{for } s > 0,$$

as the condition number of the matrix formed by the eigenvectors of $A(\xi(t))$. The value $K(\xi(t))$ gives the local bound on the transient behavior at a fixed state $x = \xi(t)$ and, for given X_0 and T, the global constant K used in Theorem 3.3 is obtained as $K = \max_{\xi \in \Xi_{[0,T]}} K(\xi(t))$.

Finally, given the set of initial values X_0, one can estimate m_t, cf. (18), through examining the solution trajectories to (21) that start on a discrete grid in X_0. Thus, one can numerically check the existence of a t^*, such, that for given α and X_0 it holds that

$$-\omega^* := \frac{\ln K}{t^*} + \sqrt{K m_{t^*} \ln 2} - \omega \tag{22}$$

is negative, which is a sufficient condition for the stability of the considered system in the considered range of initial values.

For the presented example on how the above estimates can detect stability, we set $\alpha = 0.4$, which results in $\omega = 0.3$, and we set $X_0 \subset \mathbb{R}^2$ to be the closed ball around the origin of radius $r = 0.25$. The grid for X_0 uses 12 equally distributed points on the circle with radius $r = 0.25$, another 8 points on the circle with $r = 0.17$, and 4 points at $r = 0.08$.

From the computed trajectories we compute $K(\xi(t))$ (Fig. 1(a)), m_t (Fig. 1(b)) with the manually optimized $\rho := 0.55t$ and evaluate $-\omega^*$ as in (22) (Fig. 1(c)). Since for $t^* \approx 6.0$, the value of $-\omega^*$ becomes negative, the sufficient conditions for stability as described in Theorem 3.3 and Lemma 3.4 are fulfilled. Obviously, the computed trajectories approach zero as $t \to \infty$ (Fig. 1(d)).

Note that the estimates depend strongly on the chosen set of initial values. In fact, if one extends the set of initial values X_0, the set of trajectories may include some that are not stable and also $-\omega^*$ does not become negative, see Fig. 2.

4 Stabilization by Updating Riccati Based Feedback

As can be inferred from the sufficient conditions in Theorem 2.10 and 3.3 for exponential decay of solutions, a feedback designed for stabilization should be such that the closed loop matrix $A(x) - BF(x)$, cf. (3), is uniformly stable with respect to the state x. In this

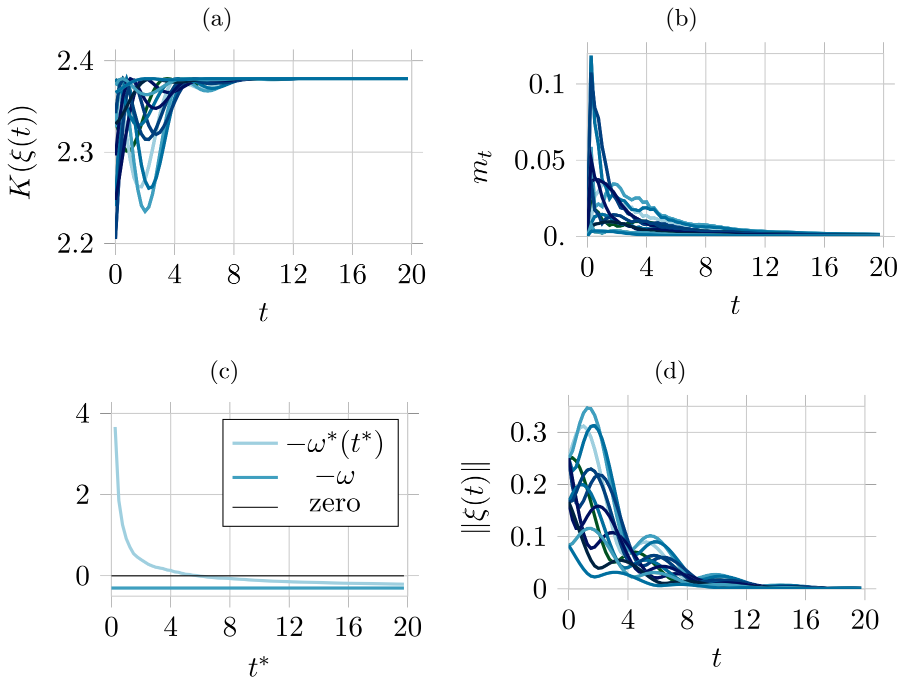

Figure 1: Computed bounds $K(\xi(t))$ for the transient behavior (a), the estimate m_t (b) and the resulting decay rates ω^* (c), and the norm of the trajectories over time and for various initial data in X_0 (d).

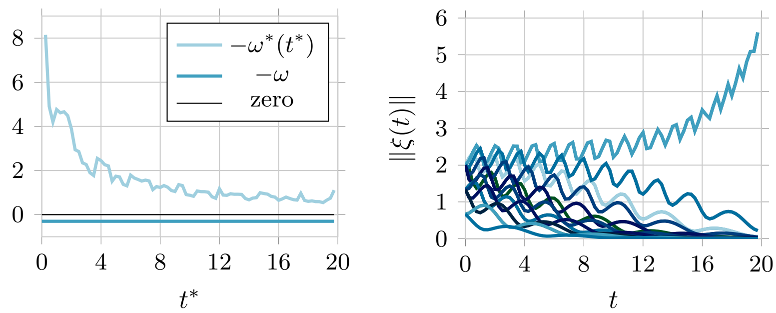

Figure 2: Estimate of ω^* and the trajectories for a set of initial values that are not uniformly stable.

section we show how one can continuously update an SDRE feedback so that the bounds on the transient behavior and the decay for the closed loop matrix stay constant in a neighborhood. More precisely, if for a given state x, an SDRE based feedback renders the system stable with certain stability constants K and ω, the introduced approach can maintain these constants for small changes in x in the course of the time evolution of the system.

For further reference, we define an abbreviation for the class of considered matrices.

Definition 4.1. We say that $A \in \mathbb{R}^{n,n}$ is in the class $\mathcal{S}_{K,\omega}$ for given constants K and ω, if

$$\|e^{A\tau}\| \leq Ke^{-\omega\tau},$$

for $\tau > 0$.

Assume that at the current state x, we have $A(x) - BF(x) \in \mathcal{S}_{K,\omega}$, where $F(x) = R^{-1}B^\mathsf{T}P$ and where $P = P(x)$ solves the Riccati equation (4) for given $B \in \mathbb{R}^{n,p}$, $R \succ 0 \in \mathbb{R}^{p,p}$, and $Q \succcurlyeq 0 \in \mathbb{R}^{n,n}$. Then, we have that

$$\begin{bmatrix} A(x) & -BR^{-1}B^\mathsf{T} \\ -Q & -A(x)^\mathsf{T} \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} Z, \tag{23}$$

where $Z = A - BR^{-1}B^\mathsf{T}P \in \mathcal{S}_{K,\omega}$. The following theorem proposes an update of F to account for changes in the system matrix $A(x + x_\Delta) =: A(x) + A_\Delta$ induced by a change x_Δ in the current state x.

Theorem 4.2. *Consider relation (23) with $Z \in \mathcal{S}_{K,\omega}$. If for an $A_\Delta \in \mathbb{R}^{n,n}$, there exist $Q_\Delta \in \mathbb{R}^{n,n}$, R_Δ, and $E \in \mathbb{R}^{n,n}$ such that*

$$\begin{bmatrix} A(x) + A_\Delta & -B[R^{-1} + R_\Delta]B^\mathsf{T} \\ -Q - Q_\Delta & -A(x)^\mathsf{T} - A_\Delta^\mathsf{T} \end{bmatrix} \begin{bmatrix} I + E \\ P \end{bmatrix} = \begin{bmatrix} I + E \\ P \end{bmatrix} Z, \tag{24}$$

and if $\|E\| < 1$, then $(I + E)$ is invertible and with $P_\Delta := P(I + E)^{-1}$ it holds that

$$A(x) + A_\Delta - B[R^{-1} + R_\Delta]B^\mathsf{T}P_\Delta \in \mathcal{S}_{\tilde{K},\omega},$$

with $\tilde{K} = \frac{1+\|E\|}{1-\|E\|}K$.

Proof. Using the *Neumann series* [14, Exa. I.4.5], one can infer from $\|E\| < 1$ that $(I + E)$ is invertible and that $\|(I + E)^{-1}\| \leq \frac{1}{1-\|E\|}$. By multiplying the first block line in (24) by $(I + E)^{-1}$ from the left, taking the norm on both sides, recalling that $Z \in \mathcal{S}_{K,\omega}$, and estimating $\|I + E\| \leq 1 + \|E\|$, we prove the theorem. □

As a consequence of Theorem 4.2, as long as for given A_Δ, one can find Q_Δ, R_Δ, and E, with $\|E\| < \epsilon < 1$ small enough, one can stabilize $A(x) + A_\Delta$ through the feedback gain

$$F_+ = (R^{-1} + R_\Delta)B^\mathsf{T}P(I + E)^{-1}$$

in a neighborhood of $A(x)$ with a constant decay rate ω and a constant bound on the transient behavior.

We will use the result of Theorem 4.2 to define updates for a given feedback. For further reference, we formulate the situation as a problem.

Problem 4.3. *Consider the SDC system (2) at time $t \geq 0$ and $\xi(t) =: x$. Let $R \succ 0$ and $Q \succcurlyeq 0$ be given and P satisfy the SDRE (4) so that*

$$\begin{bmatrix} A(x) & -BR^{-1}B^\mathsf{T} \\ -Q & -A(x)^\mathsf{T} \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} Z, \tag{25}$$

holds for a $Z \in \mathcal{S}_{K,\omega}$, and let $0 < \epsilon < 1$. For a given $A_\Delta \in \mathbb{R}^{n,n}$, find Q_Δ, R_Δ, and a corresponding E so that

$$\begin{bmatrix} A(x) + A_\Delta & -B[R^{-1} + R_\Delta]B^\mathsf{T} \\ -Q - Q_\Delta & -A(x)^\mathsf{T} - A_\Delta^\mathsf{T} \end{bmatrix} \begin{bmatrix} I + E \\ P \end{bmatrix} = \begin{bmatrix} I + E \\ P \end{bmatrix} Z, \quad \text{and} \quad \|E\| < \epsilon. \tag{26}$$

In what follows, we will address sufficient conditions for the existence of such updates E and how they can be computed.

Lemma 4.4. *Consider Problem 4.3. Any solution (Q_Δ, R_Δ, E) satisfies*

$$(A(x) + A_\Delta)E - EZ = -A_\Delta + BR_\Delta B^\mathsf{T} P. \tag{27}$$

and

$$-QE - Q_\Delta(I + E) - A_\Delta^\mathsf{T} P = 0. \tag{28}$$

Conversely, for given $0 < \epsilon < 1$, if there exist R_Δ and E with $\|E\| < \epsilon$ that fulfill (27), then (28) can be solved for Q_Δ and (Q_Δ, R_Δ, E) satisfy (26).

Proof. With P solving (23), the updated system (24) is equivalent to (27) and (28). Conversely, if there is a solution E to (27), then the first block line in (26) is satisfied. If also $\|E\| < 1$, then $I + E$ is invertible and there is a unique Q_Δ so that (28) and, thus, the second block line of (26) are fulfilled. □

According to Lemma 4.4, a desired solution E to (24), namely an E with $\|E\| < 1$, is always solely defined by (27). Thus, solvability of (27) is the key for applying the approach of updating the initial Riccati based feedback.

Equation (27) is a *Sylvester* equation [12, Ch. 16] that can be written as

$$\mathcal{P}(A(x) + A_\Delta, -Z)\,\mathrm{vec}(E) = \mathrm{vec}(-A_\Delta + BR_\Delta B^\mathsf{T} P), \tag{29}$$

where $\mathcal{P}(A_1, A_2) := A_1 \otimes I - I \otimes A_2$ and vec is the operator that stacks the columns of a matrix into a long vector. For given A_1 and A_2, the *Sylvester* operator $\mathcal{P}$ is invertible, if and only if A_1 and A_2 do not have a common eigenvalue.

In the considered case, there is no guarantee that the spectra of $A(x) + A_\Delta$ and $-Z$ are disjoint. Thus, we can not state unique existence of solutions. If $A(x) + A_\Delta$ and $-Z$ share an eigenvalue, then the associated $\mathcal{P}$ is rank-deficient. Then Equation (27) has a solution, or better infinitely many solutions, only if the inhomogeneity is consistent. Based on these considerations, we propose two practical approaches to obtain such a solution E.

1. Solve (27) with $R_\Delta = 0$. If this fails, then the linear operator $\mathcal{P}$ is not invertible and A_Δ is not in the range of $\mathcal{P}$. One can try whether for a small second summand, $-A_\Delta + BR_\Delta B^T P$ is consistent. However, since B typically has only a few columns, this is only a low-rank update which is unlikely to fix the inconsistency in general.

2. If (27) is not solvable, one may solve the perturbed system

$$(A(x) + A_\Delta - BR^{-1}B^\mathsf{T}P)E + EZ = -A_\Delta + BR_\Delta B^\mathsf{T}P, \tag{30}$$

   which is hopefully a slight perturbation, if E is small. If A_Δ is small, then Equation (30) has a unique solution since $BR^{-1}B^\mathsf{T}P$ was stabilizing A and also Z has only eigenvalues with negative real part.

Another issue is the smallness of the update E – a second crucial ingredient of the approach. If we assume that in (29) the operator $\mathcal{P}$ is invertible, then the norm of the update is readily estimated by

$$\|E\|_F \leq \|\mathcal{P}^{-1}\|_2 \|Y\|_F, \tag{31}$$

where we have used the abbreviation $Y := -A_\Delta + BR_\Delta B^\mathsf{T}P$. Relation (31) is also what the general perturbation estimates given in [12, Eq. (16.23), (16.25)] reduce to in the considered case.

At a first glance, the smallness of $Y = -A_\Delta + BR_\Delta B^\mathsf{T}P$ induces a small E. The freedom in the choice of R_Δ can be used to further optimize the solution. Either through minimizing the norm of Y, which is probably not optimal in terms of a minimal norm E but which comes with the a-priori estimate (31), or through minimizing the solution in an optimization setup. The latter optimization approach may also be employed if $\mathcal{P}$ is not invertible, provided that one can guarantee a consistent right hand side for all considered choices of parameters.

Estimates for $\|\mathcal{P}^{-1}\|_2$ may be obtained as follows. The direct approach would be to compute the largest singular value of $\mathcal{P}^{-1}$ that defines the considered spectral norm of $\mathcal{P}^{-1}$, e.g., via the *power method* [10]. Alternative ways are given by virtue of the equality of the smallest singular value of $\mathcal{P}(A_1, A_2)$ to the so called *separation* of A_1 and A_2:

$$\operatorname{sep}(A_1, A_2) = \min_X \frac{\|A_1 X - X A_2\|_F}{\|X\|_F},$$

cf. [20], e.g., via an algorithm reported in [8] that bases on *Schur* decompositions and that has been implemented, e.g., in the SB04OD subroutine of *SLICOT* [6].

5 Numerical Examples

We consider the *5D example* that was considered in [2, Ch. 3.4] and which writes as an SDC system $\dot{\xi} = A\xi + Bu$ like

$$\begin{bmatrix} \dot{\xi}_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \xi_4^2 & 0 \\ -\xi_1 & 0 & 0 & \xi_4^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u, \quad \xi(0) = x_0 \in \mathbb{R}^5. \tag{32a}$$

We add the observation $\eta = C\xi$, defined as

$$\eta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{bmatrix}. \tag{32b}$$

Note that with the chosen input and output operators, the system – considered as a linear system pointwise at any staty $\xi(t)$ – is controllable and observable so that, in particular, at every state x there exists a feedback that stabilizes the matrix $A(x)$. We compute stabilizing feedbacks by means of the SDRE (4) and the update scheme that was defined through Theorem 4.2.

In the first approach, that we will denote by sdre, we only use the SDRE based feedback which requires the solution of a Riccati equation at every step of the numerical integration. In the second approach, referred to as p-update, we update the initial SDRE feedback according to Theorem 4.2. If the norm of the current update E exceeds a threshold $\epsilon < 1$, we reset the base feedback P with the solution of the SDRE at the current state x.

The parameters for the definition of the SDRE feedback and the updates are set to

$$R = 10^{-3} I_{2\times 2}, \quad Q = C^\mathsf{T} C, \quad \text{and} \quad R_\Delta = 0.$$

We use SciPy's built-in integrator odeint with the absolute and relative accuracy tolerances set to 10^{-6} to integrate the closed loop system on $(0, 3]$, starting from the initial value

$$x_0 = \begin{bmatrix} -1.3 & -1.4 & -1.1 & -2.0 & 0.3 \end{bmatrix}^\mathsf{T}.$$

This initial value is different from the one used in [2] for which the initial solution of the SDRE applied as a static feedback already stabilizes the trajectory.

As illustrated in Figure 3(b), without stabilization, the system blows up in a short time, while with stabilization, the trajectories approach zero. This successful stabilization was achieved for the sdre case as well as for the p-update case for varying update thresholds ϵ. In the p-update approach, during the time integration, Sylvester equations are solved in order to update the feedback to bound the variation in K, cf. Theorem 4.2 and Lemma 4.4, and to keep the decay rate piecewise constant, cf. Figure 3(a). Note that $\epsilon = 0$ corresponds to the sdre scenario and that the jumps occur where $\|E\|$ exceeds ϵ and where the p-update scheme is reinitiated with the current SDRE solution.

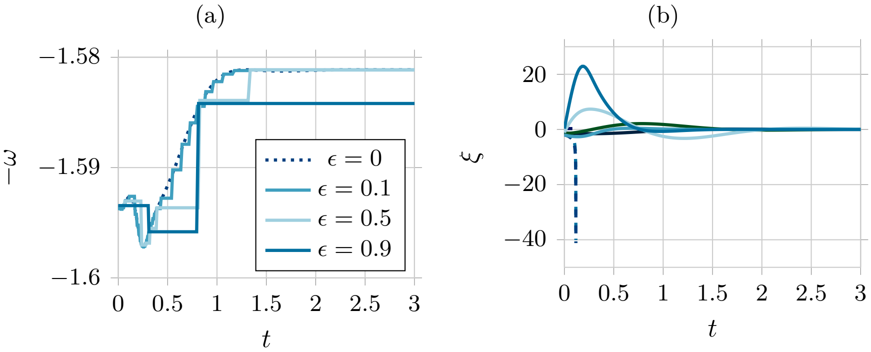

Figure 3: (a): The decay rate $-\omega$ of the closed loop matrix over time t for varying ϵ and (b): the trajectories ξ of the stabilized (solid lines) and of the uncontrolled (dashed lines) system (32).

Scheme	ϵ	#fb-switches	#f-eva	comp-time
sdre	0	—	245	$0.054s$
p-update	0.1	32	1287	$0.271s$
p-update	0.5	7	521	$0.110s$
p-update	0.9	2	374	$0.078s$

Table 2: Influence of ϵ on the number of switches #fb-switches in the feedback definition, on the number of function evaluations #f-eva in the time integrator, and on the overall computation time comp-time for the simulation of the *5D example* (32).

Apart from allowing for application of the theoretical results of Section 2, the p-update approach comes with the advantage over sdre that mainly Sylvester equations are solved instead of Riccati equations. In the considered five dimensional setup, the solution of the Sylvester equation (27) using scipy.linalg.solve_sylvester takes about $100\mu s$ which is much less time than $182\mu s$ that is needed by scipy.linalg.solve_continuous_are to solve the associated Riccati equation (4). The additional effort to compute $B^\mathsf{T} P(I+E)^{-1}$ in each time step is $12\mu s$ and comparatively small.

In terms of the overall computation time, however, the sdre approach outperforms the p-update procedure in the presented example. Here, the generally faster computation of the feedback is compensated by the additional number of time steps that was required by the integrator to achieve the same accuracy. We observe that for smaller thresholds ϵ, which cause more sudden changes in the feedback matrix, the integrator needs more function evaluations due to less smoothness in the system, cf. Table 2. Nevertheless, as we show in a second example, for larger systems, for which the differences in the computational complexity between the linear Sylvester and the nonlinear Riccati equation is much more significant, the p-update will be more economic also in the overall costs.

As a second example, we consider the *Chafee Infante* equation, which is an autonomous

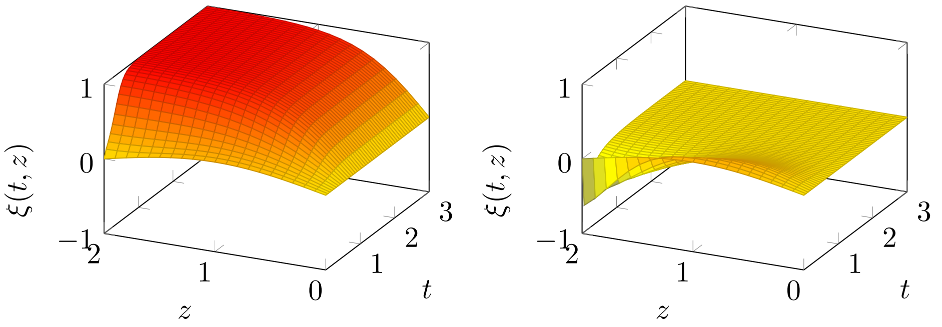

Figure 4: The uncontrolled (left) and the stabilized (right) evolution of the solution to the *Chaffee Infante* equation (33).

PDE. Precisely, for the spatial coordinate $z \in (0, 2)$ and time $t \in (0, 3]$, we consider

$$\dot{\xi} = \partial_{zz}\xi + 5(1 - \xi^2)\xi \tag{33a}$$

with boundary conditions

$$\xi(t)\big|_{z=0} = 0 \quad \text{and} \quad \partial_z \xi(t)\big|_{z=2} = u(t) \tag{33b}$$

and the initial value

$$x_0 = 0.2 \sin(0.5\pi z). \tag{33c}$$

It is known that the equilibrium point $\xi = 0$ of (33) is unstable and that the solution for any $x_0 \neq 0$ converges to one of two stable equilibria; cf. [1]. We discretize (33) by a finite-element scheme using *FEniCS* [15] and N equally distributed linear hat functions which leads to an SDC system with N degrees of freedom in the state and a single input. The output matrix $C \in \mathbb{R}^{5,N}$ is defined to observe the solution at the spatial locations $z = 0$, $z = 0.5$, $z = 1$, $z = 1.5$, and $z = 2$. The parameters are chosen as $Q = C^\mathsf{T} C$, $R = 10^{-1}$, and $R_\Delta = 0$. We use scipy.integrate.odeint to integrate the closed-loop system as in the previous examples. Since one deals with a finite element discretization, one should use the norm induced by the corresponding mass matrix to compare the errors independently of the discretization. We mimic this scaling in the norms by scaling the prescribed tolerances 10^{-6} with the inverse of the elements length $2/N$.

Both the p-update and the sdre stabilization successfully force the system into the unstable zero state as illustrated in Figure 4. As expected, for ever larger N, i.e. ever larger system sizes, the advantage of solving linear updates in the p-update scheme over solving nonlinear Riccati equations in the sdre scheme becomes increasingly evident; cf. Table 3.

The code and information on the system architecture used for the tests is available from the public *git* repository [11].

Scheme	ϵ	#fb-switches	#f-eva	comp-time
		$N = 20$		
sdre	0	—	442	$1.921s$
p-update	0.5	2	838	$3.266s$
p-update	0.9	0	451	$1.756s$
		$N = 40$		
sdre	0	—	849	$6.267s$
p-update	0.5	3	1936	$10.000s$
p-update	0.9	1	1186	$6.140s$
		$N = 60$		
sdre	0	—	1194	$15.426s$
p-update	0.5	4	2240	$18.379s$
p-update	0.9	2	1770	$14.140s$
		$N = 80$		
sdre	0	—	1589	$42.088s$
p-update	0.5	6	2953	$35.840s$
p-update	0.9	3	2096	$25.486s$
		$N = 100$		
sdre	0	—	2106	$90.148s$
p-update	0.5	7	3778	$68.080s$
p-update	0.9	4	2423	$43.816s$

Table 3: Influence of ϵ on the number of switches #fb-switches in the feedback definition, on the number of function evaluations #f-eva in the time integrator, and on the overall computation time comp-time for the simulation of the stabilized *Chaffee Infante* equation (33) with finite element discretizations on varying mesh sizes N.

6 Conclusions

We analysed the stability of trajectories ξ of an SDC system like (2) based on properties of the spectrum of $A(\xi(t))$. The straight-forward adaptation of known sufficient conditions for linear time-varying systems came with strong global assumptions that are unlikely to be fulfilled or confirmed. Taking into account that the coefficient function $\xi(t) \mapsto A(\xi(t))$ is stabilized together with the trajectory, we derived sufficient conditions for stability that can be checked locally. In view of using the obtained theoretical results for feedback stabilization, we developed an update scheme that ensures uniform decay rates and bounds on the transient behavior of the closed-loop SDC system matrix.

In terms of achieved stabilization, the well-established SDRE approach worked as good as the newly developed update scheme which, however, comes with two major advantages. Firstly, the update scheme allows for controlling the current spectral properties of the closed loop matrix so that the developed stability conditions can be applied. Secondly, in every time instance, only a linear equation is solved, which pays off for larger systems in particular. Both the usability of the sufficient conditions and the efficiency of the approach to stabilization via updating an initial feedback was illustrated in numerical examples.

By now, in the numerical examples as well as in the theoretical investigations, we have not considered the potential for optimization within the derived approaches. For example, the freedom in the choice of the weighting matrix perturbation R_Δ may well be used to optimize the feedback update E. Additionally, it might be worth investigating whether structural assumptions on the changes A_Δ in the coefficient matrices can be exploited to provide feedback updates of, e.g., low-rank.

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