

Hyperbolic scattering of spinning particles by a Kerr black hole

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We investigate the scattering of a spinning test particle by a Kerr black hole within the Mathisson-Papapetrou-Dixon model to linear order in spin. The particle’s spin and orbital angular momentum are taken to be aligned with the black hole’s spin. Both the particle’s mass and spin length are assumed to be small in comparison with the characteristic length scale of the background curvature, in order to avoid backreaction effects. We analytically compute the modifications due to the particle’s spin to the scattering angle, the periastron shift, and the condition for capture by the black hole, extending previous results valid for the nonrotating Schwarzschild background. Finally, we discuss how to generalize the present analysis beyond the linear approximation in spin, including spin-squared corrections in the case of a black-hole-like quadrupolar structure for the extended test body.

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I. INTRODUCTION

The scenario of a black hole which scatters matter, radiation and even other black holes is especially attractive in view of the possibility to consider these events as sources of gravitational waves, potentially detectable by current detectors such as LIGO and VIRGO [1–3] or future detectors. While much is known for “elliptic encounters,” or bound motion, namely the spiraling of one body around another up to the merging of the two bodies into a single one, the case of “hyperbolic encounters,” or unbound motion, has been poorly investigated in the literature. In fact, up to now the back-reaction on a Kerr background of a small particle moving on an equatorial hyperbolic-like orbit has not yet been analytically computed. The first attempt concerning a spinless particle in a Schwarzschild background has been presented only very recently in Ref. [4]. Our knowledge of the main features of the process mostly relies on the post-Newtonian approximation (at low orders; see, e.g., Refs. [5–7] and references therein) or numerical relativity simulations [8–11]. This lack of information is due to the computational difficulties associated with the problem, like the fact that the hyperbolic gravitational wave spectrum is a continuum, in contrast to the quasi-circular case, where it is mainly monochromatic. Furthermore, apart from the direct relevance to actual hyperbolic motions, the study of unbound motion is also relevant for the analysis of bound motion. In fact, for instance, computing the scattering angle for a hyperbolic encounter can encode gauge-invariant information characterizing both unbound and bound motion (see, e.g., Ref. [12], where the gauge-invariant “effective-one-body” function $\rho(x)$ was introduced in the discussion of small eccentricity motion).

A necessary pre-requisite to the study of back-reaction effects due to gravitational wave emission is the complete

knowledge of the conservative dynamics of a hyperbolic encounter of a test particle with the (spinning) black hole. When the particle’s internal structure is negligible, it moves along a hyperbolic-like geodesic orbit around the black hole. We study here the problem of the scattering of a particle endowed with spin by a Kerr black hole in the framework of the Mathisson-Papapetrou-Dixon (MPD) model [13–15], generalizing a previous work in the non-rotating Schwarzschild background spacetime [16]. The motion is non-geodesic due to the presence of a spin-curvature coupling force. The two bodies are assumed to have aligned/antialigned spins and the equatorial plane is chosen as the orbital plane. According to the MPD model the extended body is treated as a test body, i.e., backreaction effects on the background metric are neglected, even if it has an internal structure described by its spin. The MPD equations of motion can be solved analytically in terms of elliptic integrals, allowing to explicitly compute the corrections to first-order in spin to the scattering angle, i.e., the most natural gauge-invariant and physical observable associated with the scattering process. We also determine the shift of the periastron position and the modification due to spin to the critical impact parameter for capture by the hole with respect to the well known geodesic case in both ultrarelativistic and non-relativistic regimes. Finally, we show how to extend the present analysis to take into account spin-squared effects as well by suitably modifying the MPD set of equations, qualitatively discussing some features of the second-order-in-spin corrections to the orbital motion in comparison with the linear-in-spin ones, in the simplest case of an extended body endowed with a “black-hole-like” quadrupolar structure.

We use geometrical units and assume that Greek indices run from 0 to 3, whereas Latin indices from 1 to 3.

II. EQUATORIAL PLANE MOTION IN A KERR SPACETIME

The motion of a spinning test particle in any given gravitational background is described by the MPD equations [13–15]

$$\frac{DP^\mu}{d\tau} = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}U^\nu S^{\alpha\beta} \equiv F_{(\text{spin})}^\mu, \quad (1)$$

$$\frac{DS^{\mu\nu}}{d\tau} = 2P^{[\mu}U^{\nu]}. \quad (2)$$

Here $P^\mu \equiv mu^\mu$ (with $u \cdot u \equiv u^\mu u_\mu = -1$) is the total 4-momentum of the body with mass m and unit timelike four-velocity u^μ ; $S^{\mu\nu}$ is the (antisymmetric) spin tensor; and $U^\mu = dx^\mu/d\tau$ is the timelike unit 4-velocity vector tangent to the body's "center-of-mass worldline," parametrized by the proper time τ (with parametric equations $x^\mu = x^\mu(\tau)$), selected to perform the multipole reduction which is at the basis of the MPD approach.

Self-consistency of the model requires that some additional conditions be imposed to the determine reference world line, as Eqs. (1) and (2) represent only 10 equations for the 14 degrees of freedom $x^\mu(\tau)$, $P^\mu(\tau)$ and $S^{\mu\nu}(\tau)$. Here we shall use the Tulczyjew-Dixon conditions [15, 17], which read

$$S^{\mu\nu}u_\nu = 0. \quad (3)$$

The spin tensor can then be fully represented by the spatial vector (with respect to u)

$$S(u)^\alpha = \frac{1}{2}\eta(u)^\alpha{}_{\beta\gamma}S^{\beta\gamma}, \quad S^{\mu\nu} = \eta(u)^{\mu\nu}{}_\alpha S(u)^\alpha, \quad (4)$$

where we have denoted by $\eta(u)_{\alpha\beta\gamma} = \eta_{\mu\alpha\beta\gamma}u^\mu$ the spatial unit volume 3-form (with respect to u) built from the unit volume 4-form $\eta_{\alpha\beta\gamma\delta} = \sqrt{-g}\epsilon_{\alpha\beta\gamma\delta}$ which orients the spacetime, with $\epsilon_{\alpha\beta\gamma\delta}$ ($\epsilon_{0123} = 1$) being the Levi-Civita alternating symbol and g the determinant of the metric.

It is also useful to introduce the signed magnitude s of the spin vector

$$s^2 = S(u)^\beta S(u)_\beta = \frac{1}{2}S_{\mu\nu}S^{\mu\nu}, \quad (5)$$

which is a constant of motion. Implicit in the MPD model is the condition that the length scale $|s|/m$ naturally associated with the spin should be very small compared to the one associated with the background curvature (say M), in order to neglect back reaction effects, namely $|\hat{s}| \equiv |s|/mM \ll 1$. This condition leads to a simplified set of linearized evolution equations around the geodesic motion. In fact, the total 4-momentum P of the particle is still aligned with U in this limit, i.e., $P^\mu = mU^\mu + O(S^2)$, the mass m of the particle remaining constant along the path. Furthermore, Eq. (2) becomes $DS^{\mu\nu}/d\tau = O(S^2)$, implying that the spin vector is parallel transported along U .

Finally, when the background spacetime has Killing vectors, there are conserved quantities along the motion

[18], which can be used to simplify the equations of motion. In the case of stationary axisymmetric spacetimes with coordinates adapted to the spacetime symmetries, they are the energy E and the total angular momentum J associated with the timelike and azimuthal Killing vectors $\xi = \partial_t$ and $\eta = \partial_\phi$, respectively, and read

$$E = -P_t + \frac{1}{2}S^{\alpha\beta}\xi_{\alpha;\beta}, \quad J = P_\phi - \frac{1}{2}S^{\alpha\beta}\eta_{\alpha;\beta}. \quad (6)$$

A. Kerr spacetime and ZAMO adapted frame

Let us consider the Kerr spacetime, whose metric written in standard Boyer-Lindquist coordinates is given by

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4aMr}{\Sigma}\sin^2\theta dt d\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{A}{\Sigma}\sin^2\theta d\phi^2, \quad (7)$$

with $\Delta = r^2 - 2Mr + a^2$, $\Sigma = r^2 + a^2\cos^2\theta$ and $A = (r^2 + a^2)^2 - \Delta a^2\sin^2\theta$. Here a and M denote the specific angular momentum and the total mass of the spacetime solution, so that the quantity $\hat{a} = a/M$ is dimensionless. The inner and outer horizons are located at $r_\pm = M \pm \sqrt{M^2 - a^2}$.

Let us introduce the zero angular momentum observer (ZAMO) family of fiducial observers, with 4-velocity

$$n = N^{-1}(\partial_t - N^\phi\partial_\phi), \quad (8)$$

where $N = (-g^{tt})^{-1/2} = \sqrt{\Delta\Sigma/A}$ and $N^\phi = g_{t\phi}/g_{\phi\phi} = -2aMr/A$ are the lapse and shift functions, respectively. The ZAMO adapted form of the metric then reads

$$ds^2 = -N^2 dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}(d\phi + N^\phi dt)^2. \quad (9)$$

A suitable spatial orthonormal frame adapted to ZAMOs is given by

$$e_{\hat{t}} = n, \quad e_{\hat{r}} = \frac{1}{\sqrt{g_{rr}}}\partial_r \equiv \partial_{\hat{r}}, \\ e_{\hat{\theta}} = \frac{1}{\sqrt{g_{\theta\theta}}}\partial_\theta \equiv \partial_{\hat{\theta}}, \quad e_{\hat{\phi}} = \frac{1}{\sqrt{g_{\phi\phi}}}\partial_\phi \equiv \partial_{\hat{\phi}}. \quad (10)$$

The nonzero ZAMO kinematical quantities (i.e., acceleration $a(n) = \nabla_n n$ and expansion vector $\theta_{\hat{\phi}}(n)^\alpha = \theta(n)^\alpha{}_\beta e_{\hat{\phi}}^\beta$) all belong to the r - θ 2-plane of the tangent space [19–22], i.e.,

$$a(n) = a(n)^{\hat{r}}e_{\hat{r}} + a(n)^{\hat{\theta}}e_{\hat{\theta}} \\ \equiv \partial_{\hat{r}}(\ln N)e_{\hat{r}} + \partial_{\hat{\theta}}(\ln N)e_{\hat{\theta}}, \\ \theta_{\hat{\phi}}(n) = \theta_{\hat{\phi}}(n)^{\hat{r}}e_{\hat{r}} + \theta_{\hat{\phi}}(n)^{\hat{\theta}}e_{\hat{\theta}} \\ \equiv -\frac{\sqrt{g_{\phi\phi}}}{2N}(\partial_{\hat{r}}N^\phi e_{\hat{r}} + \partial_{\hat{\theta}}N^\phi e_{\hat{\theta}}). \quad (11)$$

On the equatorial plane ($\theta = \pi/2$) the only nonvanishing components are given by

$$\begin{aligned} a(n)^{\hat{r}} &= \frac{M}{r^2\sqrt{\Delta}} \frac{(r^2 + a^2)^2 - 4a^2Mr}{r^3 + a^2r + 2a^2M}, \\ \theta_{\hat{\phi}}(n)^{\hat{r}} &= -\frac{aM(3r^2 + a^2)}{r^2(r^3 + a^2r + 2a^2M)}. \end{aligned} \quad (12)$$

B. Spinning particle motion on the equatorial plane

Let us consider a spinning particle moving on the equatorial plane of a Kerr spacetime with the spin vector aligned with the spacetime rotation axis, i.e.,

$$S(U) = -se_{\hat{\theta}}. \quad (13)$$

Its 4-velocity U decomposed with respect to the ZAMOs is

$$U = \gamma(U, n)[n + \nu(U, n)], \quad (14)$$

where

$$\nu(U, n) \equiv \nu(U, n)^{\hat{r}}e_{\hat{r}} + \nu(U, n)^{\hat{\phi}}e_{\hat{\phi}}, \quad (15)$$

and $\gamma(U, n) = 1/\sqrt{1 - \|\nu(U, n)\|^2}$ is the associated Lorentz factor. Hereafter we will use the abbreviated notation $\gamma(U, n) \equiv \gamma$ and $\nu(U, n)^{\hat{a}} \equiv \nu^{\hat{a}}$.

The equations of motion (1) then imply

$$\begin{aligned} \frac{d\nu^{\hat{r}}}{d\tau} &= -\gamma \left\{ (\nu^{\hat{\phi}})^2 k_{(\text{lie})} \right. \\ &\quad \left. + (1 - (\nu^{\hat{r}})^2) \left[a(n)^{\hat{r}} + 2\theta_{\hat{\phi}}(n)^{\hat{r}}\nu^{\hat{\phi}} \right] \right\} \\ &\quad + \frac{1}{m\gamma} \left[(1 - (\nu^{\hat{r}})^2) F_{(\text{spin})}^{\hat{r}} - \nu^{\hat{r}}\nu^{\hat{\phi}} F_{(\text{spin})}^{\hat{\phi}} \right], \\ \frac{d\nu^{\hat{\phi}}}{d\tau} &= \gamma\nu^{\hat{r}}\nu^{\hat{\phi}} \left[k_{(\text{lie})} + a(n)^{\hat{r}} + 2\theta_{\hat{\phi}}(n)^{\hat{r}}\nu^{\hat{\phi}} \right] \\ &\quad + \frac{1}{m\gamma} \left[-\nu^{\hat{r}}\nu^{\hat{\phi}} F_{(\text{spin})}^{\hat{r}} + (1 - (\nu^{\hat{\phi}})^2) F_{(\text{spin})}^{\hat{\phi}} \right], \end{aligned} \quad (16)$$

where

$$k_{(\text{lie})} = -\frac{(r^3 - a^2M)\sqrt{\Delta}}{r^2(r^3 + a^2r + 2a^2M)} \quad (17)$$

denotes the Lie relative curvature [20, 21] evaluated at $\theta = \pi/2$, defined by $k_{(\text{lie})} = -\partial_{\hat{r}} \ln \sqrt{g_{\phi\phi}}$. The frame components of the spin force with respect to ZAMOs are given by

$$\begin{aligned} F_{(\text{spin})}^{\hat{r}} &= mM\hat{s}\gamma^2 \left[(E_{\hat{\theta}\hat{\theta}} - E_{\hat{r}\hat{r}})\nu^{\hat{\phi}} + (1 + (\nu^{\hat{\phi}})^2)H_{\hat{r}\hat{\theta}} \right], \\ F_{(\text{spin})}^{\hat{\phi}} &= -mM\hat{s}\gamma^2 \left[(E_{\hat{\theta}\hat{\theta}} - E_{\hat{\phi}\hat{\phi}})\nu^{\hat{r}} + \nu^{\hat{r}}\nu^{\hat{\phi}}H_{\hat{r}\hat{\theta}} \right], \end{aligned} \quad (18)$$

where

$$\begin{aligned} E_{\hat{r}\hat{r}} &= -\frac{M(2r^4 + 5r^2a^2 - 2a^2Mr + 3a^4)}{r^4(r^3 + a^2r + 2a^2M)}, \\ E_{\hat{\theta}\hat{\theta}} &= -E_{\hat{\phi}\hat{\phi}} - E_{\hat{r}\hat{r}}, \\ E_{\hat{\phi}\hat{\phi}} &= \frac{M}{r^3}, \\ H_{\hat{r}\hat{\theta}} &= -\frac{3Ma(r^2 + a^2)\sqrt{\Delta}}{r^4(r^3 + a^2r + 2a^2M)}, \end{aligned} \quad (19)$$

are the nontrivial components of the electric and magnetic parts of the Riemann tensor with respect to ZAMOs, and $\hat{s} \equiv s/mM$ is the dimensionless signed spin magnitude. The remaining nonvanishing component $F_{(\text{spin})}^{\hat{t}}$ follows from the condition $F_{(\text{spin})} \cdot U = 0$. Finally, Eq. (16) must be coupled with the decomposition (14) of the 4-velocity $U = dx^\alpha/d\tau$, i.e.,

$$\begin{aligned} \frac{dt}{d\tau} &= \frac{\gamma}{N}, \\ \frac{dr}{d\tau} &= \frac{\gamma\nu^{\hat{r}}}{\sqrt{g_{rr}}}, \\ \frac{d\phi}{d\tau} &= \frac{\gamma}{\sqrt{g_{\phi\phi}}} \left(\nu^{\hat{\phi}} - \frac{\sqrt{g_{\phi\phi}}N^\phi}{N} \right), \end{aligned} \quad (20)$$

providing the evolution of t , r and ϕ .

In order to solve the full set of equations (16) and (20) we take advantage of the existence of the conserved quantities (6), which become

$$\begin{aligned} \hat{E} &= N\gamma \left\{ 1 - \frac{\sqrt{g_{\phi\phi}}N^\phi}{N}\nu^{\hat{\phi}} \right. \\ &\quad \left. + M\hat{s} \left[a(n)^{\hat{r}}\nu^{\hat{\phi}} + \theta_{\hat{\phi}}(n)^{\hat{r}} \right. \right. \\ &\quad \left. \left. + \frac{\sqrt{g_{\phi\phi}}N^\phi}{N} \left(k_{(\text{lie})} + \theta_{\hat{\phi}}(n)^{\hat{r}}\nu^{\hat{\phi}} \right) \right] \right\}, \\ \hat{J} &= \frac{\sqrt{g_{\phi\phi}}}{M}\gamma \left[\nu^{\hat{\phi}} - M\hat{s} \left(k_{(\text{lie})} + \theta_{\hat{\phi}}(n)^{\hat{r}}\nu^{\hat{\phi}} \right) \right], \end{aligned} \quad (21)$$

where $\hat{E} \equiv E/m$ and $\hat{J} \equiv J/mM$ are dimensionless. Eq. (21) thus provide two algebraic relations for the frame components $\nu^{\hat{r}}$ and $\nu^{\hat{\phi}}$ of the linear velocity, which once inserted in Eq. (20) finally yield

$$\begin{aligned}
\frac{\Delta}{M^2} \frac{dt}{d\tau} &= \hat{E} \left(\frac{r^2}{M^2} + \hat{a}^2 \right) + \frac{M}{r} [(2\hat{a} + 3\hat{s})\hat{E} - (2\hat{a} + \hat{s})\hat{J}] - \frac{M^3}{r^3} \hat{a}^2 \hat{s} (\hat{J} - \hat{a}\hat{E}), \\
\left(\frac{dr}{d\tau} \right)^2 &= \hat{E}^2 - 1 + \frac{2M}{r} + \frac{M^2}{r^2} [\hat{a}^2 (\hat{E}^2 - 1) - \hat{J}(\hat{J} - 2\hat{E}\hat{s})] \\
&\quad + \frac{2M^3}{r^3} (\hat{J} - \hat{a}\hat{E})(\hat{J} - \hat{a}\hat{E} - 3\hat{E}\hat{s}) + \frac{2M^5}{r^5} \hat{a}\hat{s} (\hat{J} - \hat{a}\hat{E})^2, \\
\frac{\Delta}{M} \frac{d\phi}{d\tau} &= \hat{J} - \hat{E}\hat{s} - \frac{2M}{r} (\hat{J} - \hat{a}\hat{E} - \hat{E}\hat{s}) - \frac{M^3}{r^3} \hat{a}\hat{s} (\hat{J} - \hat{a}\hat{E}), \tag{22}
\end{aligned}$$

to first order in spin.

Let us introduce a conical-like representation of the radial variable, i.e.,

$$r = \frac{Mp}{1 + e \cos \chi}, \tag{23}$$

where both the semi-latus rectum p and the eccentricity $e \geq 0$ are dimensionless parameters [23]. Bound orbits have $0 \leq e < 1$ and $0 < \hat{E} < 1$ and oscillate between a minimum radius $r_{(\text{per})}$ (periastron, $\chi = 0$) and a maximum radius $r_{(\text{apo})}$ (apastron, $\chi = \pi$)

$$r_{(\text{per})} = \frac{Mp}{1 + e}, \quad r_{(\text{apo})} = \frac{Mp}{1 - e}, \tag{24}$$

corresponding to the extremal points of the radial motion, i.e.,

$$\left. \frac{dr}{d\tau} \right|_{r=r_{(\text{per})}} = 0 = \left. \frac{dr}{d\tau} \right|_{r=r_{(\text{apo})}}. \tag{25}$$

Unbound (hyperbolic-like) orbits, instead, have eccentricity $e > 1$ and energy parameter $\hat{E} > 1$. In particular, a typical scattering orbit starts far from the hole at radial infinity, reaches a minimum approach distance $r_{(\text{per})}$, and then comes back to radial infinity, corresponding to a scattering angle $\chi \in [-\chi_{(\text{max})}, \chi_{(\text{max})}]$, $\chi_{(\text{max})} = \arccos(-1/e)$ (see Eq. (23)). In this case the apoastron does not exist anymore, as it corresponds to a negative value of the radial variable. However, the parametrization (24) can still be adopted.

The conditions (25) allow one to express \hat{E} and \hat{J} in terms of (p, e) as follows

$$\hat{E} = \hat{E}_0 + \hat{s}\hat{E}_s, \quad \hat{J} = \hat{J}_0 + \hat{s}\hat{J}_s, \tag{26}$$

to first order in spin. The geodesic values are given by

$$\hat{E}_0^2 = \frac{1}{p} \left[\frac{(1 - e^2)^2 \hat{x}_0^2}{p^2} + p - (1 - e^2) \right], \tag{27}$$

where

$$\hat{x}_0^2 = \frac{-N \mp \sqrt{N^2 - 4CF}}{2F}, \tag{28}$$

with F , N and C dimensionless coefficients given by [24]

$$\begin{aligned}
F &= \left(1 - \frac{3 + e^2}{p} \right)^2 - \frac{4\hat{a}^2(1 - e^2)^2}{p^3}, \\
-\frac{N}{2} &= (p - 3 - e^2) + \hat{a}^2 \left(1 + \frac{1 + 3e^2}{p} \right), \\
C &= (\hat{a}^2 - p)^2, \tag{29}
\end{aligned}$$

and

$$N^2 - 4CF = \frac{16\hat{a}^2}{p^3} \{ [p^2 - 2p + \hat{a}^2(1 + e^2)]^2 - 4e^2(p - \hat{a}^2)^2 \}. \tag{30}$$

The upper (lower) sign corresponds to prograde (retrograde) motion. The geodesic angular momentum is simply related to \hat{E}_0 and \hat{x}_0 by $\hat{x}_0 = \hat{J}_0 - \hat{a}\hat{E}_0$. Formulas valid for retrograde orbits can be obtained from those for prograde orbits by $\hat{a} \rightarrow -\hat{a}$ and $\hat{J}_0 \rightarrow -\hat{J}_0$, under which $\hat{x}_0 \rightarrow -\hat{x}_0$.

The first-order corrections to the energy and angular momentum are given by

$$\begin{aligned}
\hat{E}_s &= -\frac{\hat{x}_0}{2p^5} (1 - e^2)^2 [(3 + e^2)\hat{x}_0^2 + p^2], \\
\hat{J}_s &= \hat{E}_0 - \frac{\hat{x}_0}{2p^2} \hat{E}_0 \hat{J}_0 (3 + e^2) - \frac{2\hat{x}_0}{p^3} \hat{a} (1 - e^2). \tag{31}
\end{aligned}$$

III. HYPERBOLIC-LIKE MOTION

After converting the radial equation into an equation for χ through Eq. (23), the azimuthal equation finally becomes

$$\frac{d\phi}{d\chi} = u_p^{1/2} \frac{\hat{J}_0 - 2u_p \hat{x}_0 (1 + e \cos \chi)}{[1 + u_p^2 \hat{x}_0^2 (e^2 - 2e \cos \chi - 3)]^{1/2} [1 - 2u_p (1 + e \cos \chi) + \hat{a}^2 u_p^2 (1 + e \cos \chi)^2]} - \hat{s} u_p^{5/2} \hat{x}_0 (\hat{x}_0 \hat{E}_0 + \hat{a}) \frac{e \cos \chi + 3}{[1 + u_p^2 \hat{x}_0^2 (e^2 - 2e \cos \chi - 3)]^{3/2}}, \quad (32)$$

to first order in spin, with $u_p = 1/p$. The solution of this equation can be obtained analytically in terms of elliptic functions as

$$\phi(\chi) = \phi_0(\chi) + \hat{s} \phi_{\hat{s}}(\chi), \quad (33)$$

where

$$\begin{aligned} \phi_0(\chi) &= \frac{\kappa}{2\hat{a}^2 e^2 u_p^2 \sqrt{e u_p \hat{x}_0^2 (b_+ - b_-)}} \left\{ [\hat{J}_0 - 2u_p \hat{x}_0 (1 + e b_+)] \left[\Pi\left(\frac{\pi}{2} - \frac{\chi}{2}, k_+, \kappa\right) - \Pi(k_+, \kappa) \right] \right. \\ &\quad \left. - [\hat{J}_0 - 2u_p \hat{x}_0 (1 + e b_-)] \left[\Pi\left(\frac{\pi}{2} - \frac{\chi}{2}, k_-, \kappa\right) - \Pi(k_-, \kappa) \right] \right\}, \\ \phi_{\hat{s}}(\chi) &= \frac{\kappa (\hat{E}_0 \hat{x}_0 + \hat{a})}{2\sqrt{e u_p \hat{x}_0^2}} \left\{ K(\kappa) - F\left(\frac{\pi}{2} - \frac{\chi}{2}, \kappa\right) \right. \\ &\quad \left. - \frac{C + 6\hat{x}_0^2}{C - 2e\hat{x}_0^2} \left[E(\kappa) - E\left(\frac{\pi}{2} - \frac{\chi}{2}, \kappa\right) + \kappa \sqrt{e} \hat{x}_0 \frac{\sin \chi}{(A - 2e\hat{x}_0^2 \cos \chi)^{1/2}} \right] \right\}, \end{aligned} \quad (34)$$

with $\phi(0) = 0$ and

$$\begin{aligned} \kappa^2 &= \frac{4e u_p^2 \hat{x}_0^2}{1 + (e - 1)(e + 3)\hat{x}_0^2}, \\ k_{\pm} &= \frac{2}{1 + b_{\pm}}, \\ b_{\pm} &= \frac{1 - \hat{a}^2 u_p \pm \sqrt{1 - \hat{a}^2}}{\hat{a}^2 e u_p}, \\ u_p^2 C &= 1 + u_p^2 \hat{x}_0^2 (e^2 - 3). \end{aligned} \quad (35)$$

Here $K(k)$ and $F(\varphi, k)$ and $E(k)$ and $E(\varphi, k)$ are the complete and incomplete elliptic integrals of the first kind and of the second kind, respectively, defined by

$$\begin{aligned} F(\varphi, k) &= \int_0^{\varphi} \frac{dz}{\sqrt{1 - k^2 \sin^2 z}}, \quad K(k) = F(\pi/2, k), \\ E(\varphi, k) &= \int_0^{\varphi} \sqrt{1 - k^2 \sin^2 z} dz, \quad E(k) = E(\pi/2, k), \end{aligned} \quad (36)$$

whereas

$$\begin{aligned} \Pi(\phi, n, k) &= \int_0^{\phi} \frac{dz}{(1 - n \sin^2 z) \sqrt{1 - k^2 \sin^2 z}}, \\ \Pi(n, k) &= \Pi(\pi/2, n, k), \end{aligned} \quad (37)$$

are the incomplete and complete elliptic integrals of the third kind, respectively [25].

Unbound orbits which are not captured by the black hole start at an infinite radius at the azimuthal angle $\phi = \phi(-\chi_{(\max)})$, the radius decreases to its periastron value at $\phi = 0$ and then returns back to infinite value at $\phi = \phi(\chi_{(\max)})$, undergoing a total increment of $\Delta\phi = \phi(\chi_{(\max)}) - \phi(-\chi_{(\max)})$. This scattering process is symmetric with respect to the minimum approach ($\chi = 0$) as in the case of a spinless particle, i.e., $\phi(-\chi_{(\max)}) = -\phi(\chi_{(\max)})$, so that $\Delta\phi = 2\phi(\chi_{(\max)})$, and the deflection angle from the original direction of the orbit is

$$\delta(u_p, e, \hat{s}) = \delta_0(u_p, e) + \hat{s} \delta_{\hat{s}}(u_p, e), \quad (38)$$

with $\delta_0(u_p, e) = 2\phi_0(\chi_{(\max)}) - \pi$ and $\delta_{\hat{s}}(u_p, e) = 2\phi_{\hat{s}}(\chi_{(\max)})$.

Figure 1 shows a typical hyperbolic-like orbit of a spinning particle with spin aligned along the positive z -axis and in the opposite direction compared with the corresponding geodesic orbit of a spinless particle. The orbital parameters are chosen as $p = 20$ and $e = 2$, implying that the distance of minimum approach is $r_{(\text{per})} \approx 6.66667M$. The trajectory of the spinning particle thus depends on the same parameters p and e as in the spinless case. Once these parameters have been fixed, orbits with different values of \hat{s} have the same closest approach distance, but different values of energy and angular momentum. On the contrary, setting a pair of values of (\hat{E}, \hat{J}) leads to a shift of the periastron due to spin, as shown below.

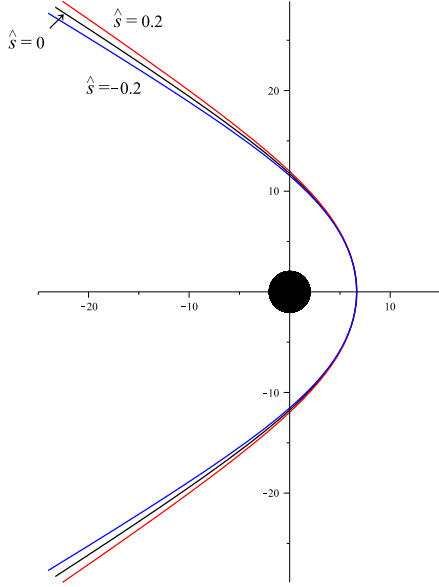


FIG. 1: The hyperbolic-like orbits of a spinning test particle with $\hat{s} = \pm 0.2$ are shown in the r - ϕ plane for the choice of parameters $p = 20$ and $e = 2$ (so that $\chi_{(\max)} = 2\pi/3$) and $\hat{a} = 0.5$ in comparison with the geodesic of a spinless particle ($\hat{s} = 0$). The values of both energy and angular momentum per unit mass are different depending on \hat{s} , whereas the minimum approach distance $r_{(\text{per})} \approx 6.66667M$ is the same for all cases. The deflection angle is $\delta \approx 2.03215$ (i.e., 116.43382 deg) for $\hat{s} = 0$, $\delta \approx 2.08603$ (i.e., 119.52073 deg) for $\hat{s} = -0.2$, and $\delta \approx 1.97828$ (i.e., 113.34690 deg) for $\hat{s} = 0.2$.

To first order in the rotation parameter \hat{a} Eq. (32) becomes

$$\begin{aligned} \frac{d\phi}{d\chi} &= \frac{1}{\sqrt{1 - 6u_p - 2eu_p \cos \chi}} - 4\hat{a} \frac{\hat{E}_0 u_p}{\hat{J}_0} \frac{1}{(1 - 6u_p - 2eu_p \cos \chi)^{3/2} (1 - 2u_p - 2eu_p \cos \chi)} \\ &\quad - \hat{s} \frac{\hat{E}_0 u_p}{\hat{J}_0} \frac{e \cos \chi + 3}{(1 - 6u_p - 2eu_p \cos \chi)^{3/2}} \\ &\equiv \frac{d}{d\chi} \phi_0 + \frac{d}{d\chi} \phi_{\hat{a}} + \frac{d}{d\chi} \phi_{\hat{s}}, \end{aligned} \quad (39)$$

where also terms of the order $\hat{a}\hat{s}$ have been neglected. Integration then gives

$$\phi(\chi) = \phi_0(\chi) + \phi_{\hat{a}}(\chi) + \phi_{\hat{s}}(\chi). \quad (40)$$

The terms ϕ_0 and $\phi_{\hat{s}}$ are given in Eq. (24) of Ref. [16], whereas the term linear in \hat{a} is

$$\begin{aligned} \phi_{\hat{a}}(\chi) &= \hat{a} \frac{\hat{E}_0}{\hat{J}_0} \frac{\kappa}{\sqrt{eu_p}} \left[\frac{E[(\pi - \chi)/2, \kappa] - E(\kappa)}{1 - 6u_p - 2eu_p} - \frac{\Pi[(\pi - \chi)/2, \nu, \kappa] - F[(\pi - \chi)/2, \kappa]}{1 - 2u_p + 2eu_p} \right. \\ &\quad \left. - \frac{\kappa \sqrt{eu_p}}{1 - 6u_p - 2eu_p} \frac{\sin \chi}{\sqrt{1 - 6u_p - 2eu_p \cos \chi}} \right], \end{aligned} \quad (41)$$

with $\nu = 4eu_p/(1 - 2u_p + 2eu_p)$.

A. Periastron shift

A different (equivalent) parametrization of the orbit can be adopted in terms of energy \hat{E} and angular momentum \hat{J} instead of p and e . In this case the periastron distance depends on \hat{s} , and can be determined from the turning points for radial motion.

In the case of a spinless particle the radial and azimuthal equations can be conveniently written in the following factorized form in terms of the dimensionless inverse radial variable $u = M/r$

$$\left(\frac{du}{d\tau}\right)^2 = \frac{2\hat{x}^2}{M^2}u^4(u - u_{1(0)})(u - u_{2(0)})(u - u_{3(0)}),$$

$$\frac{d\phi}{d\tau} = \frac{2\hat{x}}{M\hat{a}^2}u^2 \frac{u_{4(0)} - u}{(u - u_+)(u - u_-)}, \quad (42)$$

which can be combined to yield

$$\left(\frac{du}{d\phi}\right)^2 = \frac{\hat{a}^4}{2} \frac{(u - u_+)^2(u - u_-)^2}{(u_{4(0)} - u)^2} \times (u - u_{1(0)})(u - u_{2(0)})(u - u_{3(0)}). \quad (43)$$

Here $u_{1(0)} < u_{2(0)} < u_{3(0)}$ are the ordered roots of the equation

$$u^3 - (\hat{x}^2 + 2\hat{a}\hat{E}\hat{x} + \hat{a}^2)\frac{u^2}{2\hat{x}^2} + \frac{u}{\hat{x}^2} + \frac{\hat{E}^2 - 1}{2\hat{x}^2} = 0, \quad (44)$$

with $\hat{x} = \hat{J} - \hat{a}\hat{E}$, whereas

$$u_{\pm} = \frac{M}{r_{\pm}}, \quad u_{4(0)} = \frac{\hat{J}}{2\hat{x}}. \quad (45)$$

For hyperbolic orbits we have $u_{1(0)} < 0 < u \leq u_{2(0)} < u_{3(0)}$, with $u_{2(0)}$ corresponding to the closest approach distance, i.e., $r_{(\text{per})} = M/u_{2(0)}$ [23].

The orbital equation for a spinning particle can be cast in a similar form as Eq. (43), i.e.,

$$\left(\frac{du}{d\phi}\right)^2 = A \frac{\hat{a}^4}{2} (1 - \hat{s}Bu) \frac{(u - u_+)^2(u - u_-)^2}{(u - u_4)^2} \times (u - u_1)(u - u_2)(u - u_3), \quad (46)$$

to first order in \hat{s} , where

$$A = 1 + \frac{\hat{s}}{4\hat{x}^4} (\hat{E}\hat{x} + \hat{a}) [\hat{a}^3(3\hat{E}\hat{x} + \hat{a}) - 2\hat{x}^2(2 - \hat{a}^2)]$$

$$\equiv 1 + \hat{s}A_{\hat{s}},$$

$$B = -\frac{\hat{a}^2}{2\hat{x}^2} (\hat{E}\hat{x} + \hat{a}), \quad (47)$$

with $u_i = u_{i(0)} + \hat{s}u_{i(1)}$. The corrections to the geodesic values are

$$u_{1(1)} = -\frac{1 - 2u_{1(0)} + \hat{a}^2u_{1(0)}^2}{\hat{x}^2[\hat{x}(1 - 2u_{1(0)}) + \hat{a}\hat{E}]} \frac{\hat{E}(\hat{E}^2 - 1 + 2u_{1(0)}) - \hat{x}(\hat{E}\hat{x} + \hat{a})u_{1(0)}^3}{(u_{1(0)} - u_{3(0)})(u_{1(0)} - u_{2(0)}),$$

$$u_{2(1)} = -\frac{1 - 2u_{2(0)} + \hat{a}^2u_{2(0)}^2}{\hat{x}^2[\hat{x}(1 - 2u_{2(0)}) + \hat{a}\hat{E}]} \frac{\hat{E}(\hat{E}^2 - 1 + 2u_{2(0)}) - \hat{x}(\hat{E}\hat{x} + \hat{a})u_{2(0)}^3}{(u_{2(0)} - u_{1(0)})(u_{2(0)} - u_{3(0)}),$$

$$u_{3(1)} = -\frac{1 - 2u_{3(0)} + \hat{a}^2u_{3(0)}^2}{\hat{x}^2[\hat{x}(1 - 2u_{3(0)}) + \hat{a}\hat{E}]} \frac{\hat{E}(\hat{E}^2 - 1 + 2u_{3(0)}) - \hat{x}(\hat{E}\hat{x} + \hat{a})u_{3(0)}^3}{(u_{3(0)} - u_{1(0)})(u_{3(0)} - u_{2(0)}),$$

$$u_{4(1)} = -\frac{\hat{a}}{16\hat{x}^3} [(\hat{E}\hat{x} + \hat{a})^3 - 8\hat{E}^2\hat{x}]. \quad (48)$$

The periastron shift is then given by

$$r_{(\text{per})} = \frac{M}{u_2} = \frac{M}{u_{2(0)}} \left(1 - \hat{s} \frac{u_{2(1)}}{u_{2(0)}}\right), \quad (49)$$

to first order in spin.

Let us turn to the orbital equation (46), which yields

$$\frac{d\phi}{du} = \pm \frac{\sqrt{2}}{\hat{a}^2\sqrt{A}} \frac{f(u)}{\sqrt{(u - u_1)(u_2 - u)(u_3 - u)}}, \quad (50)$$

with

$$f(u) = \left(1 + \frac{B}{2}\hat{s}u\right) \frac{u_4 - u}{(u - u_+)(u - u_-)}, \quad (51)$$

where the \pm sign should be chosen properly during the whole scattering process (with $u_1 < 0 < u \leq u_2 < u_3$), depending on the choice of initial conditions. For instance, by choosing $\phi(u_2) = 0$ at periastron, integration between u_2 and u gives

$$\begin{aligned}
\phi(u) = & \pm \frac{2\sqrt{2}}{\hat{a}^2(u_+ - u_-)\sqrt{u_3 - u_1}} \left\{ \left[1 - \frac{1}{2}(A_{\hat{s}} - Bu_+) \hat{s} \right] \frac{u_4 - u_+}{u_1 - u_+} (\Pi(\alpha, \beta_+, m) - \Pi(\beta_+, m)) \right. \\
& - \left. \left[1 - \frac{1}{2}(A_{\hat{s}} - Bu_-) \hat{s} \right] \frac{u_4 - u_-}{u_1 - u_-} (\Pi(\alpha, \beta_-, m) - \Pi(\beta_-, m)) \right. \\
& \left. + \frac{1}{2} B \hat{s} (u_+ - u_-) [K(m) - F(\alpha, m)] \right\}, \tag{52}
\end{aligned}$$

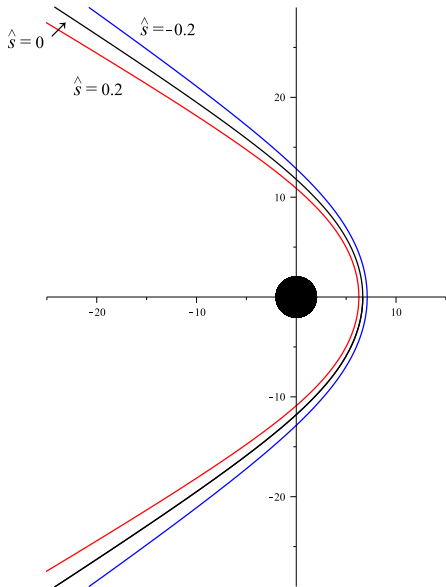


FIG. 2: The hyperbolic-like orbits of a spinning test particle with $\hat{s} = \pm 0.2$ are shown in the r - ϕ plane for the choice of parameters $\hat{a} = 0.5$, $\hat{E} \approx 1.08411$ and $\hat{J} \approx 5.28307$ (so that $\hat{x} \approx 4.74102$). The reference geodesic is the same as in Fig. 1. The periastron is at $u_2 \approx 0.14066$ (i.e., $r_{(\text{per})} \approx 7.10950M$) for $\hat{s} = -0.2$, $u_2 = 0.15$ (i.e., $r_{(\text{per})} \approx 6.66667M$) for $\hat{s} = 0$ and $u_2 \approx 0.15934$ (i.e., $r_{(\text{per})} \approx 6.27577M$) for $\hat{s} = 0.2$. The deflection angle is $\delta \approx 2.03215$ (i.e., 116.43382 deg) for $\hat{s} = 0$, $\delta \approx 1.95264$ (i.e., 111.87803 deg) for $\hat{s} = -0.2$, and $\delta \approx 2.10363$ (i.e., 120.52895 deg) for $\hat{s} = 0.2$.

to first order in \hat{s} , where $\Pi(\alpha, 0, m) = F(\alpha, m)$ and $\Pi(0, m) = K(m)$, and

$$m = \sqrt{\frac{u_2 - u_1}{u_3 - u_1}}, \quad \alpha = \sqrt{\frac{u - u_1}{u_2 - u_1}}, \quad \beta_{\pm} = \frac{u_2 - u_1}{u_{\pm} - u_1}. \tag{53}$$

The total change in ϕ for the complete scattering process is then given by $2\phi(0)$ determined by Eq. (52) with $\alpha = \alpha(0) = \sqrt{-u_1/(u_2 - u_1)}$, so that the deflection angle is $\delta(\hat{E}, \hat{J}, \hat{s}) = 2\phi(0) - \pi$.

A typical orbit using this parametrization is shown in Fig. 2. The two branches (upper +, lower -) join at the periastron on the horizontal axis.

B. Capture by the hole

The condition for capture by the black hole is $u_2(\hat{E}, \hat{J}, \hat{s}) \geq u_3(\hat{E}, \hat{J}, \hat{s})$, i.e., the cubic at the right hand side of Eq. (46) has a double root at the critical point $u_2 = u_3$ [26, 27]. Solving this condition for \hat{J} then yields the critical value of the dimensionless total angular momentum \hat{J}_{crit} for capture, with associated dimensionless critical impact parameter $\hat{b}_{\text{crit}} = b_{\text{crit}}/M$ defined by¹

$$\hat{b}_{\text{crit}} = \frac{\hat{J}_{\text{crit}}}{\sqrt{\hat{E}^2 - 1}}. \tag{54}$$

The case of a spinless particle moving on a hyperbolic-like geodesic orbit in the equatorial plane is well known in both ultrarelativistic ($\hat{E} \gg 1$) and non-relativistic ($\beta^2 \equiv \hat{E}^2 - 1 \ll 1$) regimes (see, e.g., Ref. [28], Sec. 3.4.5, and references therein). For example, the critical impact parameter for corotating orbits and $\hat{a} > 0$ in the limit $\hat{E} \rightarrow \infty$ is given by (see Eq. (3.4.37) there)

$$\hat{b}_{(0)} = \hat{a} + 8 \cos^3 \left(\frac{1}{3} \arccos(-\hat{a}) \right). \tag{55}$$

For a spinning particle we find

$$\begin{aligned}
\hat{b}_{\text{crit}(0)}|_{\text{u.r.}} = & \hat{b}_{(0)} \left[1 + \frac{1}{3\hat{E}^2} \left(1 + \frac{\hat{a}}{\hat{b}_{(0)}} - \hat{s}k_{\hat{s}} \right) \right. \\
& \left. + O\left(\frac{1}{\hat{E}^4}\right) \right], \tag{56}
\end{aligned}$$

where

$$\begin{aligned}
k_{\hat{s}} = & 2\hat{b}_{(0)} \frac{(2w - 2\hat{a}^2 - \hat{b}_{(0)}^2)[6w - (\hat{b}_{(0)} + \hat{a})^2]^3}{(\hat{b}_{(0)} + \hat{a})^6(\hat{b}_{(0)} - \hat{a})(\hat{b}_{(0)} - 2\hat{a})} \times \\
& \frac{w - \hat{a}(\hat{b}_{(0)} + \hat{a})}{2w - \hat{b}_{(0)}(\hat{b}_{(0)} + \hat{a})}, \tag{57}
\end{aligned}$$

¹ The definition of the "impact parameter" should be viewed simply as a convenient rescaling of J . It can be formulated equivalently in terms of the particle's world line at infinity as $J = L + \hat{E}s = m\sqrt{\hat{E}^2 - 1}b + \hat{E}s$, (cf. Eqs. (21) (as $r \rightarrow \infty$) and (68)), where L denotes the orbital angular momentum at infinity, see below.

with $w = \sqrt{3(\hat{b}_{(0)}^2 - \hat{a}^2)}$. To linear order in the rotational parameter \hat{a} (also neglecting terms $\hat{a}\hat{s}$) the previous expression becomes

$$\begin{aligned} \hat{b}_{\text{crit}(0)}|_{\text{n.r.}} &\simeq 3\sqrt{3} \left(1 - \frac{2}{3\sqrt{3}}\hat{a}\right) \times \\ &\left[1 + \frac{1}{3\hat{E}^2} \left(1 + \frac{1}{3\sqrt{3}}(\hat{a} + \hat{s})\right)\right. \\ &\left. + O\left(\frac{1}{\hat{E}^4}\right)\right]. \end{aligned} \quad (58)$$

In the non-relativistic limit the critical impact parameter for corotating geodesic orbits and $\hat{a} > 0$ is given by (see Eq. (3.4.33) of Ref. [28])

$$\hat{b}_{(0)} = \frac{2}{\beta} \left(1 + \sqrt{1 - \hat{a}}\right). \quad (59)$$

For a spinning particle we find

$$\hat{b}_{\text{crit}(0)}|_{\text{n.r.}} = \hat{b}_{(0)} \left[1 + \hat{s} \frac{\sqrt{1 - \hat{a}}}{2(1 + \sqrt{1 - \hat{a}})^2}\right] + O(\beta^0), \quad (60)$$

which in the limit of small rotation parameter reduces to

$$\hat{b}_{\text{crit}(0)}|_{\text{n.r.}} \simeq \frac{4}{\beta} \left(1 - \frac{1}{4}\hat{a}\right) \left(1 + \frac{1}{8}\hat{s}\right) + O(\beta^0). \quad (61)$$

IV. INCLUDING QUADRATIC-IN-SPIN CORRECTIONS

In the previous section we have seen how (linear) corrections due to spin affect the geodesic scattering angle. According to the positive/negative sign of the spin the scattering angle is larger/smaller than the corresponding geodesic value. It is interesting to consider quadratic-in-spin corrections too, not sensitive to the sign of the spin. This requires generalizing the above discussion to include quadrupolar terms in the MPD equations. The presence of a completely general quadrupolar modification to the MPD equations and subsequent derivation of the scattering angle is beyond the scope of this work. We will limit here our analysis to the simpler case of a “black-hole-like” extended body only, i.e., to the case of a body with special spin-squared quadrupolar structure, as specified below.

Let us recall that the MPD equations for bodies with general quadrupolar structure read [29–32]

$$\begin{aligned} \frac{DP_\mu}{d\tau} &= -\frac{1}{2}R_{\mu\nu\alpha\beta}U^\nu S^{\alpha\beta} - \frac{1}{6}\nabla_\mu R_{\nu\rho\alpha\beta}J^{\nu\rho\alpha\beta}, \\ \frac{DS^{\mu\nu}}{d\tau} &= 2P^{[\mu}U^{\nu]} + \frac{4}{3}R^{[\mu}{}_{\rho\alpha\beta}J^{\nu]\rho\alpha\beta}, \end{aligned} \quad (62)$$

where $J^{\mu\nu\alpha\beta}$ denotes the relativistic quadrupole tensor, which has all the algebraic symmetries of a Riemann tensor. When specialized to the “spin-squared” case it reads

$$J^{\mu\nu\alpha\beta} = \frac{3P \cdot U}{P^2} u^{[\mu} S^{\nu]\rho} u^{[\alpha} S^{\beta]\rho}, \quad (63)$$

corresponding to an extended body with a “black-hole-like quadrupole” tensor². As before, it is useful to trade $S^{\mu\nu}$ for the spin vector $S(u)^\mu$ obtained by the spatial duality operation of Eq. (4). The relationship between U and $P = mu$ then follows from differentiating along U the spin supplementary conditions and using Eqs. (62). For the “black-hole-like” extended body a direct calculation shows that P and U remain aligned, i.e., $u^\mu = U^\mu + O(S^3)$ (see, e.g., Eq. 4.21 of Ref. [33] with $C_Q = 1$). However, the mass $m = \|P\|$ is no longer a conserved quantity, and is given by

$$\begin{aligned} m &= m_0 + \frac{1}{6}R_{\mu\nu\alpha\beta}J^{\mu\nu\alpha\beta} \\ &= m_0 - \frac{1}{2m_0}E(U)_{\nu\beta}S(u)^\nu S(u)^\beta + O(S^3), \end{aligned} \quad (64)$$

where $E(U)_{\nu\beta} = R_{\mu\nu\alpha\beta}U^\mu U^\alpha$ is the electric part of the Riemann tensor with respect to U and m_0 denotes the (constant) “bare” mass of the body.

For equatorial motion with aligned spin (see Eq. (13)) the spin evolution equations are satisfied with constant signed spin magnitude s , and the orbital motion is fully determined by the conservation of energy and total angular momentum still written in the form (6). The orbital equation then reads

$$\begin{aligned} \left(\frac{du}{d\phi}\right)^2 &= \mathcal{V}(u; \hat{E}, \hat{L}, \hat{s}) + O(\hat{s}^3) \\ &= \mathcal{V}_0(u; \hat{E}, \hat{L}) + \hat{s}\mathcal{V}_1(u; \hat{E}, \hat{L}) \\ &\quad + \hat{s}^2\mathcal{V}_2(u; \hat{E}, \hat{L}) + O(\hat{s}^3), \end{aligned} \quad (65)$$

with

² A more general expression uses a polarizability parameter C_Q as a proportionality constant in the rhs of Eq. (63), related to the shape of the test body, and can be found for example in [31];

see also Ref. [33], Eqs. (2.6) and (2.7) where $C_Q = 1$ to compare with the present case.

$$\begin{aligned} \mathcal{V}(u; \hat{E}, \hat{L}, \hat{s}) = & \frac{\hat{\Delta}^2}{\hat{w}^2} \left\{ (\hat{E}^2 - 1)(1 + \hat{a}^2 u^2) - \hat{L}^2 u^2 + (u + \hat{y}^2 u^3) (2 + 3\hat{\Delta} u^2 \hat{s}^2) \right. \\ & \left. + \frac{\hat{\Delta} u^3}{\hat{w}} \left[-(\hat{E} \hat{y} + \hat{a}) \left[2\hat{y} \hat{s} + \left(2\hat{E} + \frac{3\hat{a} \hat{y}^2 u^3}{\hat{w}} \right) \hat{s}^2 \right] + \hat{y}^3 u^3 \hat{s}^2 \right] \right\}, \end{aligned} \quad (66)$$

where

$$\hat{\Delta} = 1 - 2u + \hat{a}^2 u^2, \quad \hat{y} = \hat{L} - \hat{a} \hat{E}, \quad \hat{w} = \hat{L} - 2\hat{y}u. \quad (67)$$

The dimensionless energy and angular momentum \hat{E} and \hat{J} as well as the test-body's orbital angular momentum \hat{L} (at infinity) are now defined by using the conserved bare mass m_0 , namely

$$\begin{aligned} \hat{E} &= \frac{E}{m_0}, & \hat{L} &= \hat{J} - \hat{E} \hat{s}, \\ \hat{L} &= \frac{L}{m_0 M}, & \hat{s} &= \frac{s}{m_0 M}. \end{aligned} \quad (68)$$

Equivalently, one may take as parameters \hat{E} , \hat{s} and \hat{J} (in place of \hat{L}) so that the above equation (formally) becomes

$$\begin{aligned} \left(\frac{du}{d\phi} \right)^2 &= \mathcal{W}(u; \hat{E}, \hat{J}, \hat{s}) + O(\hat{s}^3) \\ &= \mathcal{W}_0(u; \hat{E}, \hat{J}) + \hat{s} \mathcal{W}_1(u; \hat{E}, \hat{J}) \\ &\quad + \hat{s}^2 \mathcal{W}_2(u; \hat{E}, \hat{J}) + O(\hat{s}^3). \end{aligned} \quad (69)$$

The solution of Eq. (69) can then be formally written as

$$\phi(u) = \phi_0(u; \hat{E}, \hat{J}) + \hat{s} \phi_1(u; \hat{E}, \hat{J}) + \hat{s}^2 \phi_2(u; \hat{E}, \hat{J}) + O(\hat{s}^3). \quad (70)$$

It turns out that the \hat{s}^2 corrections are more involved with respect to the linear ones and the discussion best proceeds by working numerically, or by using approximation schemes such as the post-Newtonian and post-Minkowskian ones. Fig. 2 can be redone then to include the \hat{s}^2 modifications to the orbit; however, with the same choice of parameters as in Fig. 2 the differences cannot be visually appreciated. It is interesting to study the behavior of the various separated contributions $\phi_n(u; \hat{E}, \hat{J})$ ($n = 0, 1, 2$) as shown in Fig. 3. At the minimum approach distance the geodesic term ϕ_0 vanishes (as the full quantity $\phi(u)$), while both the linear and quadratic corrections diverge, one positively (ϕ_2) and the other negatively (ϕ_1), leading to compensating effects.

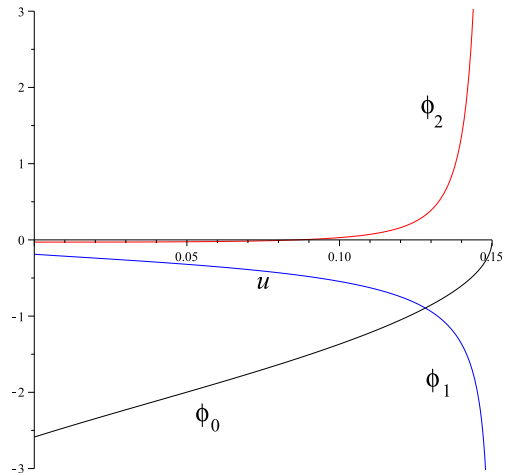


FIG. 3: The behavior of each individual contribution ϕ_0 , ϕ_1 and ϕ_2 to the azimuthal angle ϕ is shown as a function of u for the same choice of parameters as in Fig. 2 (corresponding to the lower choice of parameters there), namely $\hat{a} = 0.5$, $\hat{E} \approx 1.08411$ and $\hat{J} \approx 5.28307$. The spin dependent term are both indefinitely growing as the minimum approach distance is reached, but with opposite behavior. With these values of parameters one can compute the minimum approach distance and the result is $u_2 \approx 0.15 + 0.0467\hat{s} + 0.0124\hat{s}^2$.

V. CONCLUDING REMARKS

We have studied here the hyperbolic scattering of a two body system consisting of a rotating (Kerr) black hole and a spinning test body described according to the Mathisson-Papapetrou-Dixon model. The motion has been assumed to be equatorial, i.e., confined on the reflection-symmetry plane of the Kerr spacetime, with the spin of the body aligned/anti-aligned with that of the hole, implying meaningful simplifications and allowing for a handy treatment. Backreaction effects on the background metric are neglected because of the smallness condition of the characteristic spin length scale with respect to the background curvature length scale implicit in the MPD model.

Taking advantage of the existence of conserved quan-

tities related to Killing symmetries, we have been able to analytically solve the “linearized in spin” equations of motion in terms of elliptic integrals. We have computed the scattering angle to linear order in spin by using two equivalent parametrizations of the orbit: in terms of either eccentricity and (inverse) semi-latus rectum, or total energy and angular momentum (which are gauge-invariant variables). In the former case orbits with different values of the spin magnitude have the same closest approach distance, but different values of energy and angular momentum. On the contrary, a given pair of values of energy and angular momentum (the same for both geodesic and accelerated orbits) leads to a shift of the periastron due to spin. In this case the geodesic value of the scattering angle is modified in such a way that it increases/decreases according to the spin orientation being aligned/anti-aligned with the black hole rotation axis, respectively.

Finally, we have shown how to generalize the present

analysis to the case of extended bodies with quadrupolar structure, even if in the simplified situation of spin-squared quadrupole interactions for a “black-hole-like” body. The presence of a quadrupolar force term in the equations of motion complicates matters, so that obtaining a closed form expression for the orbit is not as simple as in the linear-in-spin case. We leave to a forthcoming paper the discussion of more general quadrupolar situations, which plays a role in the current analysis of binary dynamics.

Acknowledgments

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- [1] See the LIGO website at <http://www.ligo.org>
- [2] See the VIRGO website at <http://www.virgo-gw.eu>
- [3] B. P. Abbott *et al.* [LIGO Scientific and Virgo Collaborations], “Observation of Gravitational Waves from a Binary Black Hole Merger,” *Phys. Rev. Lett.* **116**, 061102 (2016) [arXiv:1602.03837 [gr-qc]].
- [4] S. Hopper and V. Cardoso, “Scattering of point particles by black holes: gravitational radiation,” arXiv:1706.02791 [gr-qc].
- [5] T. Damour, F. Guercilena, I. Hinder, S. Hopper, A. Nagar and L. Rezzolla, “Strong-Field Scattering of Two Black Holes: Numerics Versus Analytics,” *Phys. Rev. D* **89**, no. 8, 081503 (2014) [arXiv:1402.7307 [gr-qc]].
- [6] T. Damour, “Gravitational scattering, post-Minkowskian approximation and Effective One-Body theory,” *Phys. Rev. D* **94**, no. 10, 104015 (2016) [arXiv:1609.00354 [gr-qc]].
- [7] D. Bini and T. Damour, “Gravitational scattering of two black holes at the fourth post-Newtonian approximation,” arXiv:1706.06877 [gr-qc].
- [8] M. Shibata, H. Okawa and T. Yamamoto, “High-velocity collision of two black holes,” *Phys. Rev. D* **78**, 101501 (2008) [arXiv:0810.4735 [gr-qc]].
- [9] U. Sperhake, V. Cardoso, F. Pretorius, E. Berti and J. A. Gonzalez, “The High-energy collision of two black holes,” *Phys. Rev. Lett.* **101**, 161101 (2008) [arXiv:0806.1738 [gr-qc]].
- [10] U. Sperhake, V. Cardoso, F. Pretorius, E. Berti, T. Hinderer and N. Yunes, “Cross section, final spin and zoom-whirl behavior in high-energy black hole collisions,” *Phys. Rev. Lett.* **103**, 131102 (2009) [arXiv:0907.1252 [gr-qc]].
- [11] U. Sperhake, E. Berti, V. Cardoso and F. Pretorius, “Universality, maximum radiation and absorption in high-energy collisions of black holes with spin,” *Phys. Rev. Lett.* **111**, no. 4, 041101 (2013) [arXiv:1211.6114 [gr-qc]].
- [12] T. Damour, “Gravitational Self Force in a Schwarzschild Background and the Effective One Body Formalism,” *Phys. Rev. D* **81**, 024017 (2010) [arXiv:0910.5533 [gr-qc]].
- [13] M. Mathisson, “Neue mechanik materieller systemes,” *Acta Phys. Polon.* **6**, 163 (1937).
- [14] A. Papapetrou, “Spinning test particles in general relativity. 1,” *Proc. Roy. Soc. Lond. A* **209**, 248 (1951).
- [15] W. G. Dixon, “Dynamics of extended bodies in general relativity. I. Momentum and angular momentum,” *Proc. Roy. Soc. Lond. A* **314**, 499 (1970).
- [16] D. Bini and A. Geralico, “Hyperbolic-like elastic scattering of spinning particles by a Schwarzschild black hole,” *Gen. Rel. Grav.* **49**, no. 6, 84 (2017).
- [17] W. Tulczyjew, “Motion of multipole particles in general relativity theory,” *Acta Phys. Polon.* **18**, 393 (1959).
- [18] J. Ehlers and E. Rudolph, “Dynamics of extended bodies in general relativity center-of-mass description and quasirigidity,” *Gen. Relativ. Gravit.* **8**, 197 (1977).
- [19] R.T. Jantzen, P. Carini and D. Bini, “The many faces of gravitoelectromagnetism,” *Ann. Phys.* **215**, 1 (1992) [gr-qc/0106043].
- [20] D. Bini, P. Carini and R.T. Jantzen, “The intrinsic derivative and centrifugal forces in general relativity. 1. Theoretical foundations,” *Int. J. Mod. Phys. D* **6**, 1 (1997) [gr-qc/0106013].
- [21] D. Bini, P. Carini and R. T. Jantzen, “The intrinsic derivative and centrifugal forces in general relativity. 2. Applications to circular orbits in some familiar stationary axisymmetric space-times,” *Int. J. Mod. Phys. D* **6**, 143 (1997) [gr-qc/0106014].
- [22] D. Bini, F. de Felice and R. Jantzen, “Absolute and relative Frenet-Serret and Fermi-Walker transport,” *Class. Quant. Grav.* **16**, 2105 (1999).
- [23] S. Chandrasekhar, “The mathematical theory of black holes,” Clarendon, Oxford, UK, 1985.
- [24] K. Glampedakis and D. Kennefick, “Zoom and whirl: Eccentric equatorial orbits around spinning black holes and their evolution under gravitational radiation reaction,” *Phys. Rev. D* **66**, 044002 (2002) [gr-qc/0203086].
- [25] I. S. Gradshteyn and I. M. Ryzhik, “Table of Integrals,

- Series, and Products,” Ed. by A. Jeffrey and D. Zwillinger, Academic Press, New York, 6th edition, 2000.
- [26] C. W. Misner, K. S. Thorne and J. A. Wheeler, “Gravitation,” Freeman, San Francisco, 1973.
- [27] P. J. Young, “Capture of particles from plunge orbits by a black hole,” *Phys. Rev. D* **14**, 3281 (1976).
- [28] V. P. Frolov and I. D. Novikov, “Black hole physics: Basic concepts and new developments,” (Fundamental theories of physics. 96)
- [29] J. Steinhoff and D. Puetzfeld, “Influence of internal structure on the motion of test bodies in extreme mass ratio situations,” *Phys. Rev. D* **86**, 044033 (2012) [arXiv:1205.3926 [gr-qc]].
- [30] D. Bini and A. Geralico, “Dynamics of quadrupolar bodies in a Schwarzschild spacetime,” *Phys. Rev. D* **87**, no. 2, 024028 (2013) [arXiv:1408.5261 [gr-qc]].
- [31] T. Hinderer *et al.*, “Periastron advance in spinning black hole binaries: comparing effective-one-body and Numerical Relativity,” *Phys. Rev. D* **88**, no. 8, 084005 (2013) [arXiv:1309.0544 [gr-qc]].
- [32] D. Bini and A. Geralico, “Deviation of quadrupolar bodies from geodesic motion in a Kerr spacetime,” *Phys. Rev. D* **89**, no. 4, 044013 (2014) [arXiv:1311.7512 [gr-qc]].
- [33] D. Bini, G. Faye and A. Geralico, “Dynamics of extended bodies in a Kerr spacetime with spin-induced quadrupole tensor,” *Phys. Rev. D* **92**, no. 10, 104003 (2015) [arXiv:1507.07441 [gr-qc]].