Published for SISSA by Springer

Received: August 2, 2017
Accepted: September 26, 2017
Published: October 12, 2017

# O($d$+1, $d$+1) enhanced double field theory

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Abstract: Double field theory yields a formulation of the low-energy effective action of bosonic string theory and half-maximal supergravities that is covariant under the T-duality group O($d, d$) emerging on a torus $T^d$. Upon reduction to three spacetime dimensions and dualisation of vector fields into scalars, the symmetry group is enhanced to O($d$+1, $d$+1). We construct an enhanced double field theory with internal coordinates in the adjoint representation of O($d + 1, d + 1$). Its section constraints admit two inequivalent solutions, encoding in particular the embedding of $D = 6$ chiral and non-chiral theories, respectively. As an application we define consistent generalized Scherk-Schwarz reductions using a novel notion of generalized parallelization. This allows us to prove the consistency of the truncations of $D = 6$, $\mathcal{N} = (1, 1)$ and $D = 6$, $\mathcal{N} = (2, 0)$ supergravity on $\mathrm{AdS}_3 \times \mathbb{S}^3$ .

Keywords: Bosonic Strings, M-Theory, String Duality, String Field Theory

ArXiv ePrint: 1707.06693



https://doi.org/10.1007/JHEP10(2017)086

## Contents





## 1 Introduction

The T-duality property of closed string theory implies the emergence of an $\mathrm{O}(d,d,\mathbb{R})$ symmetry upon reduction of the low-energy effective actions on a torus $T^d$. This holds for bosonic string theory but also for the maximal and half-maximal supergravities in $D=10$ and their lower-dimensional descendants. The $\mathrm{O}(d,d)$ invariance is a 'hidden' symmetry from the point of view of conventional (super-)gravity in that it cannot be explained in terms of the symmetries present before compactification. Double field theory (DFT) is the framework that makes $\mathrm{O}(d,d)$ manifest before reduction by working on a suitably generalized, doubled space [1–4]. DFT can be defined for the universal NS sector consisting of metric, $b$-field and dilaton, including bosonic string theory in $D = 26$ and minimal supergravity in $D = 10$, but also for type II string theory [5, 6].

The group $\mathrm{O}(d, d)$ is the universal duality symmetry arising for toroidal compactification of any string theory, but for special theories or backgrounds this symmetry may be enhanced further. For instance, for half-maximal supergravity coupled to $n$ vector multiplets (or heterotic string theory with $n = 16$) the symmetry is enhanced to $\mathrm{O}(d, d+n)$, for which there is a DFT formulation [1, 7, 8]. Moreover, compactifications of half-maximal supergravity to $D = 4$ also exhibit an $\mathrm{SL}(2)$ duality, for which a DFT formulation has been obtained recently [9]. The case of interest for the present paper is the compactification to three spacetime dimensions. In this case, $D = 10$ supergravity yields an $\mathrm{O}(7, 7)$ symmetry that, however, is enhanced to $\mathrm{O}(8, 8)$ for half-maximal and to $\mathrm{E}_{8(8)}$ for maximal supergravity. Similarly, heterotic string theory exhibits an enhanced $\mathrm{O}(8, 24)$ duality [10], while the T-duality group of $D = 26$ bosonic string theory on $T^{23}$ is enhanced to $\mathrm{O}(24, 24)$. More generally, a string theory compactified on $T^d$ to three spacetime dimensions exhibits an $\mathrm{O}(d + 1, d + 1)$ symmetry. This comes about because vector fields in three dimensions can be dualized into scalars which join the universal scalars to combine into a larger coset model [11, 12].

Our goal in this paper is to define an 'enhanced double field theory' that makes the larger duality group $\mathrm{O}(d + 1, d + 1)$ manifest before compactification by working on a suitable extended internal space. More generally, we will define the theory for any pseudo-orthogonal group $\mathrm{O}(p, q)$. In this we closely follow the construction of the maximal $\mathrm{E}_{8(8)}$ exceptional field theory [13] and the $\mathrm{SL}(2, \mathbb{R})$ covariant formulation of $D = 4$ Einstein gravity [14]. Concretely, we generalize the formulation of [15] to an enhanced double field theory, with external and (extended) internal coordinates, but the internal coordinates now live in the adjoint representation of $\mathrm{O}(p, q)$.[1] The coordinates thus read $Y^{\mathcal{M}} = Y^{[MN]}$ with fundamental indices $M, N, \ldots = 1, \ldots, p + q$, subject to section constraints that generalize the level-matching constraint of DFT. A novel feature of this theory compared to the original DFT is that the section constraint has inequivalent solutions. As a consequence, we can embed in particular both the chiral and non-chiral theories in $D = 6$.

One of the conceptually most intriguing aspects of double and exceptional field theories with three external dimensions is that they require the inclusion of 'dual graviton' degrees of freedom. Indeed, in dimensional reduction the three-dimensional vector fields need to be dualized into scalars in order to realize the enhanced symmetry, and these vectors include the Kaluza-Klein vector fields originating from the higher-dimensional metric. Thus, their duals would be part of a higher-dimensional dual graviton, whose existence within a more or less conventional field theory is excluded by strong no-go theorems [16]. This is reflected in the observation that the generalized Lie derivatives supposed to unify the internal diffeomorphisms and tensor gauge transformations do not define a consistent gauge algebra for duality groups associated to three dimensions such as $\mathrm{O}(8, 8)$ [17]. Within exceptional field theory this obstacle shows up in the gauge transformations of the tensor hierarchy in any

[1] Since the coordinates are split into external and internal, with the latter not only being doubled but embedded into the adjoint representation of $\mathrm{O}(p, q)$, this theory could also be referred to as an exceptional field theory in the sense of [13]. We thank the referee for pointing this out.

dimension $n$, among the gauge symmetries associated to the $(n-2)$-forms [18–21]. Nevertheless, consistent double and exceptional field theories can be defined upon including an additional gauge symmetry (subject to somewhat unusual constraints) and its associated gauge potential. Three external dimensions are special because the need for additional gauge symmetries is apparent already at the level of the 'scalar' fields, and the additional gauge potential features among the 'vectors' participating in the gauging and the needed Chern-Simons action.

Concretely, the internal (generalized) diffeomorphisms parameterized by $\Lambda^{\mathcal{M}}$ have to be augmented by new gauge symmetries with parameters $\Sigma_{\mathcal{M}}$ that are subject to 'extended sections constraints' requiring that they behave like a derivative in that, e.g., $\Sigma^{\mathcal{M}}\partial_{\mathcal{M}}=0$. Nevertheless, this additional gauge parameter *cannot* be reduced to the derivative of a (singlet) gauge parameter, nor can the associated gauge vector be eliminated in terms of (derivatives of) the other gauge fields. In the present paper we will confirm that precisely the same construction applies to enhanced DFT with duality group $\mathrm{O}(p,q)$. Moreover, we use the opportunity to clarify the properties of these enhanced gauge symmetries by showing that on the space of 'doubled' gauge parameters $\Upsilon \equiv (\Lambda^{\mathcal{M}}, \Sigma_{\mathcal{M}})$ one has a generalized Dorfman product that shares all properties familiar from, say, DFT. In particular, we will show that the Chern-Simons action can be naturally defined in terms of a similarly 'doubled' gauge field $\mathfrak{A}_\mu \equiv (\mathcal{A}_\mu{}^{\mathcal{M}}, \mathcal{B}_{\mu\,\mathcal{M}})$.

As one of our main applications we will use the $\mathrm{O}(p,q)$ DFT to define consistent generalized Scherk-Schwarz compactifications as in [22, 23], employing a novel notion of generalized parallelization. For a generalized Scherk-Schwarz reduction, the compactification data are entirely encoded in a group matrix ('twist matrix') $U^{\mathcal{N}}{}_{\bar{\mathcal{M}}}$ and a singlet $\rho$, both depending only on the internal coordinates $Y^{\mathcal{M}}$. For duality group $\mathrm{O}(p,q)$ the twist matrix can be decomposed into fundamental matrices $U^N{}_{\bar{M}}$, and we define a 'doubled' twist matrix as for the gauge parameters and gauge fields:

$$\mathfrak{U}_{\bar{M}\bar{N}} \equiv \left(\rho^{-1}U^K{}_{[\bar{M}}U^L{}_{\bar{N}]}\;,\; -\frac{1}{4}\,\rho^{-1}(\partial_{KL}U^P{}_{\bar{M}})U_{P\bar{N}}\right) . \tag{1.1}$$

Although at the level of elementary gauge fields and parameters the additional (covariantly constrained) components cannot be eliminated in terms of (derivatives of) the other fields, for the Scherk-Schwarz ansatz the corresponding component $\mathfrak{U}_{\bar{M}\bar{N}\,KL}$ can be written in terms of derivatives of the twist matrix. Note that with its indices being carried by a derivative, the above form is manifestly consistent with the constraint. We will show that a twist matrix gives rise to a consistent compactification provided the doubled tensor (1.1) satisfies the following algebra with respect to the (generalized) Dorfman product $\circ$:

$$\mathfrak{U}_{\bar{M}\bar{N}} \circ \mathfrak{U}_{\bar{K}\bar{L}} = -X_{\bar{M}\bar{N},\bar{K}\bar{L}}{}^{\bar{P}\bar{Q}}\,\mathfrak{U}_{\bar{P}\bar{Q}}\;, \tag{1.2}$$

where the $X$ are constant and define the embedding tensor of gauged supergravity. For the 'geometric component' this relation encodes the familiar Lie algebra of Killing vector fields. The above defines a notion of generalized parallelizability. Writing the compactification ansatz in terms of the twist matrix, for instance for the 'doubled' gauge vector as $\mathfrak{A}_\mu(x,Y) = \mathfrak{U}_{\bar{M}\bar{N}}(Y)A_\mu{}^{\bar{M}\bar{N}}(x)$, we will show that the $U$-matrices and hence the $Y$-dependence factors out homogeneously, thus proving consistency of the compactification. We will thereby prove the consistency of a large class of compactifications to three dimensions, including the truncations of $D = 6$, $\mathcal{N} = (1,1)$ and $D = 6$, $\mathcal{N} = (2,0)$ supergravity on $\mathrm{AdS}_3 \times \mathbb{S}^3$.

This paper is organized as follows. In section 2 we introduce the $\mathrm{O}(p,q)$ generalized diffeomorphisms, the generalized Dorfman product and the associated gauge vectors. Based on this, we construct in section 3 the complete $\mathrm{O}(p,q)$ enhanced DFT, and discuss its relation, upon solving the section constraint, to (super-)gravity theories in various dimensions. In section 4 we discuss generalized Scherk-Schwarz compactifications in terms of generalized parallelizability and analyze the 'twist equations' (1.2). These results are then applied in section 5 in order to establish the consistency of various Kaluza-Klein truncations to three dimensions. We conclude in section 6 with a general outlook on further applications and generalizations. Appendix A collects some $\mathrm{O}(p,q)$ identities, and in appendix B we give for completeness the details of the generalized Dorfman product for (doubled) vectors in the case of $\mathrm{E}_{8(8)}$.

## 2 $\mathrm{O}(p,q)$ generalized diffeomorphisms and tensor hierarchy

In this section we introduce the $\mathrm{O}(p,q)$ covariant generalized Lie derivatives that define generalized diffeomorphisms. Their structure follows [13, 14] and is conceptually different from theories with external dimension $n \geq 4$: they are defined with respect to a pair of gauge parameters, one of which is subject to further constraints. We clarify their algebraic structure by defining a generalized Dorfman product on the space of such pairs. This significantly simplifies the subsequent constructions, including the tensor hierarchy and the definition of the Chern-Simons action.

### 2.1 Generalized diffeomorphisms

We begin by spelling out our conventions for the group $\mathrm{O}(p,q)$. Its fundamental representations is indicated by indices $M, N, \ldots = 1, \ldots, p+q$. Hence, objects living in the adjoint representation, like the coordinates $Y^{\mathcal{M}}$, are labelled by index pairs:

$$Y^{\mathcal{M}} \equiv Y^{[MN]} \equiv Y^{MN} \,. \tag{2.1}$$

The structure constants are given by

$$f^{MN,KL}{}_{PQ} = 8\, \delta_{[P}{}^{[M} \eta^{N][K} \delta_{Q]}{}^{L]} \,, \tag{2.2}$$

with the $\mathrm{O}(p,q)$ invariant metric $\eta_{MN}$, which we use in the following to raise and lower indices. In addition, for $\mathrm{O}(p,q)$ we use two more invariant tensors:

$$s^{PQ,MN}{}_{KL} = 8\, \delta_{(K}{}^{[P} \eta^{Q][M} \delta_{L)}{}^{N]} \,, \tag{2.3}$$

which is symmetric under exchange of $[PQ]$ with $[MN]$, and

$$\mathbb{A}^{PQRS}{}_{KLMN} \equiv \delta_{KLMN}{}^{PQRS} \equiv \frac{1}{24} \left( \delta_K{}^P \delta_L{}^Q \delta_M{}^R \delta_N{}^S \pm \text{ permutations} \right) \,, \tag{2.4}$$

which is totally antisymmetric in the lower and upper sets of indices.



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We can now define section constraints for the derivatives $\partial_{\mathcal{M}} = \partial_{MN}$ dual to the adjoint coordinates (2.1) in analogy to other double and exceptional field theories. In terms of the above defined $\mathrm{O}(p,q)$ tensors, we impose

$$\begin{aligned} s^{MNKL}{}_{PQ}\,\partial_{MN}\otimes\partial_{KL} &= 0 \qquad & f^{MNKL}{}_{PQ}\,\partial_{MN}\otimes\partial_{KL} &= 0\,, \\ \mathbb{A}^{MNKL}{}_{PQRS}\,\partial_{MN}\otimes\partial_{KL} &= 0\,, \qquad & \eta^{MK}\eta^{NL}\,\partial_{MN}\otimes\partial_{KL} &= 0\,. \end{aligned} \tag{2.5}$$

Writing out the invariant tensors in terms of $\eta$ and Kronecker deltas it is easy to see that the section constraints are equivalent to

$$\partial_{[MN}\otimes\partial_{KL]} = 0 = \eta^{NK}\,\partial_{MN}\otimes\partial_{KL}\,, \tag{2.6}$$

which is the form we will use from now on. We recall that as for higher-dimensional DFTs and ExFTs these constraints are meant to hold for arbitrary functions and their products, so that for instance for fields $A, B$ we impose $\partial_{[MN}A\,\partial_{KL]}B = 0$ and $\partial_M{}^K A\,\partial_{NK}B = 0$. The constraints simplify when the second-order differential operator acts on a single object $A$ as follows

$$0 = \partial_{M[N}\partial_{PQ]}A \quad\Rightarrow\quad \partial_{MN}\partial_{PQ}A = -2\,\partial_{M[P}\partial_{Q]N}A\,. \tag{2.7}$$

This can be verified by using that partial derivatives commute, $\partial_{MN}\partial_{KL}A = \partial_{KL}\partial_{MN}A$.

We now turn to the definition of generalized Lie derivatives acting on arbitrary $\mathrm{O}(p,q)$ tensors. For a tensor $V^{MN}$ in the adjoint representation it is defined as

$$\begin{aligned} \mathcal{L}_{(\Lambda,\Sigma)}V^{MN} \equiv\, & \Lambda^{KL}\partial_{KL}\,V^{MN} + 2\,(p+q-2)\,\mathbb{P}^{PQ}{}_{RS}{}^{MN}{}_{KL}\,\partial_{PQ}\Lambda^{RS}\,V^{KL} + \lambda\,\partial_{KL}\Lambda^{KL}\,V^{MN} \\ & -\Sigma_{PQ}\,f^{PQ,MN}{}_{KL}\,V^{KL}\,, \end{aligned} \tag{2.8}$$

where $\mathbb{P}^{\mathcal{M}}{}_{\mathcal{N}}{}^{\mathcal{K}}{}_{\mathcal{L}}$ is the projector to the adjoint representation, explicitly given in (A.3), and we have also allowed for an arbitrary density weight $\lambda$. While these terms capture the generic structure of generalized diffeomorphisms [24, 25] the last term describes a local adjoint $\mathrm{O}(p,q)$ transformation with parameter $\Sigma_{MN}$ which, subject to constraints, will be seen momentarily to be necessary for consistency. Its presence is typical for ExFTs with three external dimensions [13, 14]. The projector $\mathbb{P}$ can be written in terms of the above invariant $\mathrm{O}(p,q)$ tensors, so that one obtains for the generalized Lie derivative

$$\begin{aligned} \mathcal{L}_{(\Lambda,\Sigma)}V^{MN} =\, & \Lambda^{KL}\partial_{KL}\,V^{MN} - V^{KL}\,\partial_{KL}\Lambda^{MN} + (\lambda-1)\,\partial_{PQ}\Lambda^{PQ}\,V^{MN} \\ & +\Big(6\,\mathbb{A}^{PQMN}{}_{RSKL} + \frac{1}{16}\,s^{PQ,MN}{}_{UV}\,s_{RS,KL}{}^{UV} \\ & + \frac{1}{16}\,f^{PQ,MN}{}_{UV}\,f_{RS,KL}{}^{UV}\Big)\partial_{PQ}\Lambda^{RS}\,V^{KL} \\ & -\Sigma_{PQ}\,f^{PQ,MN}{}_{KL}\,V^{KL}\,. \end{aligned} \tag{2.9}$$

Let us emphasize that in the following we will always refer to $\lambda$ as the density weight of a field, as opposed to the 'effective weight' $(\lambda-1)$ emerging in the first line of (2.9).

In the following we will show that the generalized Lie derivatives form a closed algebra, which in turn fixes the coefficient $2\,(p+q-2)$ in front of the projector in (2.8).

More precisely, the $\mathcal{L}_\Lambda$ for $\Sigma = 0$ do not close separately, but closure follows upon including a 'covariantly constrained' parameter $\Sigma_{MN}$ satisfying the same constraints as the derivatives $\partial_{MN}$:

$$\Sigma_{[MN} \otimes \partial_{KL]} = 0 = \eta^{NK}\, \Sigma_{MN} \otimes \partial_{KL}\ , \qquad \text{etc.}\ . \tag{2.10}$$

Indeed, defining the gauge variations of a generic tensor field $V$ by the generalized Lie derivative, $\delta_{\Lambda,\Sigma} V \equiv \mathcal{L}_{(\Lambda,\Sigma)} V$, and provided the section conditions (2.6) are satisfied, one finds for the gauge algebra

$$\big[\, \delta_{(\Lambda_1,\Sigma_1)},\, \delta_{(\Lambda_2,\Sigma_2)} \,\big] = \delta_{[(\Lambda_2,\Sigma_2),(\Lambda_1,\Sigma_1)]}\ , \qquad [(\Lambda_2,\Sigma_2),(\Lambda_1,\Sigma_1)] \equiv (\Lambda_{12},\Sigma_{12})\ , \tag{2.11}$$

with the effective parameters

$$\begin{aligned}
\Lambda_{12}{}^{MN} &= 2\,\Lambda_{[2}{}^{KL}\partial_{KL}\Lambda_{1]}{}^{MN} - 6\,\mathbb{A}^{MNKL}{}_{PQRS}\,\Lambda_{[2}{}^{PQ}\,\partial_{KL}\Lambda_{1]}{}^{RS} \\
&\quad - \frac{1}{16}\left(s_{PQRS}{}^{UV} s^{MNKL}{}_{UV} + f_{PQRS}{}^{UV} f^{MNKL}{}_{UV}\right)\Lambda_{[2}{}^{PQ}\,\partial_{KL}\Lambda_{1]}{}^{RS}\ , \\
\Sigma_{12\,MN} &= -2\,\Sigma_{[2\,|MN}\partial_{KL}\Lambda_{1]}{}^{KL} + 2\,\Lambda_{[2}{}^{KL}\partial_{KL}\Sigma_{1]MN} - 2\,\Sigma_{[2\,|KL}\partial_{MN}\Lambda_{1]}{}^{KL} \\
&\quad - \frac{1}{8}\, f_{RSKL}{}^{PQ}\,\Lambda_{[2}{}^{RS}\partial_{MN}\partial_{PQ}\Lambda_{1]}{}^{KL}\ .
\end{aligned} \tag{2.12}$$

In order to prove the above closure result it is convenient (and sufficient) to work with the Lie derivative acting on an object in the fundamental representation of $\mathrm{O}(p,q)$, i.e., a vector $V^M$, for which we write

$$\mathcal{L}_{(\Lambda,\Sigma)} V^M = \Lambda^{KL}\partial_{KL} V^M + K^M{}_N(\Lambda,\Sigma)\, V^N + \lambda\, \partial_{KL}\Lambda^{KL}\, V^M\ , \tag{2.13}$$

where we defined

$$K^{MN}(\Lambda,\Sigma) = 4\left(\partial^{[M}{}_K \Lambda^{N]K} + \Sigma^{MN}\right)\ . \tag{2.14}$$

The action of the generalized Lie derivative on a tensor with an arbitrary number of fundamental $\mathrm{O}(p,q)$ indices is then defined straightforwardly, with a $K$ term for each index. In particular, one may verify that this definition reproduces the above form of the generalized Lie derivative acting on an adjoint vector $V^{MN}$.

Closure of the gauge transformations given by the generalized Lie derivatives (2.13) can now be proved by a direct computation. Specifically, one may quickly verify that closure is equivalent to the following condition on $K$:

$$K^M{}_N(\Lambda_{12},\Sigma_{12}) = \Lambda_2^{KL}\partial_{KL} K^M{}_N(\Lambda_1,\Sigma_1) + K^M{}_K(\Lambda_2,\Sigma_2) K^K{}_N(\Lambda_1,\Sigma_1) - (1 \leftrightarrow 2)\ , \tag{2.15}$$

where $\Lambda_{12}$ and $\Sigma_{12}$, given in (2.12), can be simplified by writing out the invariant tensors in terms of (2.2)–(2.4):

$$\begin{aligned}
\Lambda_{12}^{MN} &= 2\,\Lambda_{[2}{}^{KL}\partial_{KL}\Lambda_{1]}{}^{MN} - 6\,\Lambda_{[2}{}^{[MN}\partial_{KL}\Lambda_{1]}{}^{KL]} - 4\,\Lambda_{[2K}{}^{[M}\partial_L{}^{N]}\Lambda_{1]}{}^{KL}\ , \\
\Sigma_{12MN} &= -\,2\,\Sigma_{[2\,|MN}\partial_{KL}\Lambda_{1]}{}^{KL} + 2\,\Lambda_{[2}{}^{KL}\partial_{KL}\Sigma_{1]MN} - 2\,\Sigma_{[2\,|KL}\partial_{MN}\Lambda_{1]}{}^{KL} \\
&\quad - \Lambda_{[2}{}^P{}_K\partial_{MN}\partial_{PL}\Lambda_{1]}{}^{KL}\ .
\end{aligned} \tag{2.16}$$



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As a help for the reader and an illustration of the use of the section constraints (2.6) and the analogous constraints (2.10) on $\Sigma$, we display the relevant terms involving $\Sigma$. It is easy to see that, as a consequence of the constraints, they are linear in $\Sigma$ and vanish by use of the first constraint in (2.10) in the form

$$0 \equiv 6\,\Sigma_{[MK}\partial_{NP]} = 2\,\Sigma_{M[K}\,\partial_{|N|P]} + 2\,\Sigma_{N[P}\,\partial_{|M|K]} - \Sigma_{MN}\partial_{KP} - \Sigma_{KP}\partial_{MN}\,. \tag{2.17}$$

We will next discuss the transformation rules for partial derivatives of tensor fields. Let us compute the variation of the partial derivative of a vector of weight $\lambda$:

$$\begin{aligned}\delta_{\Lambda,\Sigma}(\partial_{MN}V_K) &= \partial_{MN}\big(\Lambda^{PQ}\partial_{PQ}V_K + K_K{}^P(\Lambda,\Sigma)V_P + \lambda\,\partial_{PQ}\Lambda^{PQ}\,V_K\big)\\ &= \Lambda^{PQ}\partial_{PQ}\partial_{MN}V_K + \partial_{MN}\Lambda^{PQ}\,\partial_{PQ}V_K + K_K{}^P\partial_{MN}V_P\\ &\quad + \lambda\,\partial_{PQ}\Lambda^{PQ}\,\partial_{MN}V_K + \partial_{MN}K_K{}^PV_P + \lambda\,\partial_{MN}\partial_{PQ}\Lambda^{PQ}\,V_K\,.\end{aligned} \tag{2.18}$$

In order to compare this with the expression for a generalized Lie derivative, we use the section constraint as in (2.17), which yields

$$\partial_{MN}\Lambda^{PQ}\partial_{PQ}V_K = 2\,K_{[M}{}^P\partial_{|P|N]}V_K - \partial_{PQ}\Lambda^{PQ}\partial_{MN}V_K\,. \tag{2.19}$$

Thus, using this in (2.18), we have shown

$$\delta_{\Lambda,\Sigma}(\partial_{MN}V_K) = \mathcal{L}^{[\lambda-1]}_{(\Lambda,\Sigma)}(\partial_{MN}V_K) + \partial_{MN}K_K{}^PV_P + \lambda\,\partial_{MN}\partial_{PQ}\Lambda^{PQ}\,V_K\,, \tag{2.20}$$

where the notation in the first term indicates that the generalized Lie derivative acts now with weight $(\lambda-1)$. [We will use similar notations in the following whenever it is convenient.] The additional terms involving second derivatives of the gauge parameter are referred to as non-covariant variations. The non-covariant variations for the (first) partial derivatives of arbitrary tensors take the analogous form, with a $\partial K$ term for each index and one term proportional to $\lambda$ involving $\partial_{MN}(\partial_{PQ}\Lambda^{PQ})$ (which, of course, vanishes for zero density weight).

We close this subsection by discussing trivial gauge parameters or gauge symmetries of gauge symmetries, i.e., choices of $(\Lambda,\Sigma)$ whose generalized Lie derivative (2.8) gives zero on all fields as a consequence of the constraints. The simplest example is

$$\Lambda^{MN} = \partial_{KL}\chi^{[MNKL]}\,, \tag{2.21}$$

with $\Sigma_{MN}=0$. Indeed, the transport term vanishes by the section constraint, and $K^{MN}=0$ as a consequence of the section constraint in the form (2.7). There are more subtle trivial gauge parameters, involving both $\Lambda$ and $\Sigma$, parameterized by an arbitrary $\chi^{KL}$:

$$\Lambda^{MN} = \partial^{[M}{}_K\chi^{N]K}\,, \qquad \Sigma_{MN} = -\frac{1}{4}\partial_{MN}\partial_{KL}\chi^{KL}\,. \tag{2.22}$$

Again, triviality follows from the section constraints, which immediately imply that transport (and density) terms vanish, as well as $K^{MN}=0$ by a quick computation with (2.7). Note that $\chi^{MN}$ can be symmetric, in which case the $\Sigma$ parameter vanishes. In particular, this contains as a special case the familiar DFT trivial parameter $\Lambda^{MN}=\partial^{MN}\chi$ via $\chi^{MN}\equiv\chi\,\eta^{MN}$. There is a more general trivial parameter for the latter:

$$\Lambda^{MN} = \Omega^{MN}\,, \quad \text{with } \Omega^{MN} \text{ covariantly constrained}\,, \tag{2.23}$$

by which we mean $\Omega^{MN}\partial_{MN} = 0$, etc. Finally, there is a trivial parameter that generalizes (2.22) for $\chi^{MN}$ antisymmetric. Indeed, the $\mathrm{E}_{8(8)}$ case suggests that $\Lambda^{MN} = f^{MN,KL}{}_{PQ}\Omega_{KL}{}^{PQ}$, where $\Omega$ is covariantly constrained in the first index, is trivial. Here we find that

$$\begin{aligned} &\Lambda^{MN} = \Omega^{[M}{}_K{}^{N]K}\,, \qquad \Sigma_{MN} = -\frac{1}{8}\partial_{MN}\Omega_{KL}{}^{KL} - \frac{1}{8}\partial_{KL}\Omega_{MN}{}^{KL} \\ &\text{with } \Omega_{KL}{}^{PQ} \text{ \textit{covariantly constrained} (2.10) in the first index pair}\,, \end{aligned} \tag{2.24}$$

is indeed trivial.

### 2.2 Generalized Dorfman structure

We will now introduce a new notation that allows us to exhibit an algebraic structure on the space of gauge parameters $\Lambda^{MN}$, $\Sigma_{MN}$ that is analogous to the Dorfman product appearing for DFTs and ExFTs with external dimension $n \geq 4$. We introduce for the gauge parameters the pair notation

$$\Upsilon \equiv (\Lambda, \Sigma)\,, \tag{2.25}$$

and we treat the first entry as an adjoint vector $\Lambda^{MN}$ of weight $\lambda = 1$ and the second entry as a co-adjoint vector $\Sigma_{MN}$ of weight zero that is covariantly constrained according to (2.10).[2]

Our goal is to define a product for such doubled objects such that its antisymmetric part coincides with the gauge algebra structure introduced in the previous subsection and its symmetric part is a trivial gauge parameter. It turns out these conditions are satisfied for

$$\begin{aligned} \Upsilon_1 \circ \Upsilon_2 &\equiv (\Lambda_1, \Sigma_1) \circ (\Lambda_2, \Sigma_2) \\ &\equiv \left( \mathcal{L}^{[1]}_{\Upsilon_1}\Lambda_2^{MN}\,,\; \mathcal{L}^{[0]}_{\Upsilon_1}\Sigma_{2MN} + \frac{1}{4}\Lambda_2^{KL}\partial_{MN}K(\Upsilon_1)_{KL} \right)\,, \end{aligned} \tag{2.26}$$

where we used the notation (2.14) for $K(\Upsilon_1) \equiv K(\Lambda_1, \Sigma_1)$. Moreover, the Lie derivatives in here act as defined in the previous subsection, with $\Lambda$ carrying weight one and $\Sigma$ weight zero. Specifically, using that $\Sigma$ is constrained one computes

$$\mathcal{L}_{\Upsilon_1}\Sigma_{2MN} = \mathcal{L}^{[0]}_{(\Lambda_1,\Sigma_1)}\Sigma_{2MN} = \Lambda_1^{KL}\partial_{KL}\Sigma_{2MN} + \partial_{MN}\Lambda_1^{KL}\Sigma_{2KL} + \partial_{KL}\Lambda_1^{KL}\Sigma_{2MN}\,. \tag{2.27}$$

Note that, curiously, the 'anomalous' terms in the $\Sigma$ component of (2.26) have the order of 1 and 2 such that we *cannot* think of this as a deformed Lie derivative of $\Sigma_2$ w.r.t. $\Upsilon_1$, because $\Lambda_2$ enters explicitly. This ordering turns out to be crucial for the following construction.

We first verify that the antisymmetrized product defines the expected bracket:

$$\big[\Upsilon_1, \Upsilon_2\big] \equiv \frac{1}{2}(\Upsilon_1 \circ \Upsilon_2 - \Upsilon_2 \circ \Upsilon_1) \equiv \big[(\Lambda_1, \Sigma_1), (\Lambda_2, \Sigma_2)\big] \equiv (\Lambda_{[1,2]}, \Sigma_{[1,2]})\,, \tag{2.28}$$

[2] Of course, since we have a metric to raise and lower indices, adjoint and co-adjoint representations are actually equivalent, but it is sometimes useful to make this distinction in order to keep track of the two components.

where

$$
\begin{aligned}
\Lambda^{MN}_{[1,2]} \equiv\, & \Lambda_1{}^{KL}\partial_{KL}\Lambda_2{}^{MN} - 3\,\Lambda_1{}^{[MN}\partial_{KL}\Lambda_2{}^{KL]} - 2\,\Lambda_{1K}{}^{[M}\partial_L{}^{N]}\Lambda_2{}^{KL} \\
& + 4\,\Sigma_1^{[M}{}_K\Lambda_2^{|K|N]} - (1 \leftrightarrow 2)\,, \\
\Sigma_{[1,2]MN} \equiv\, & \frac{1}{2}(\Lambda_1^{KL}\partial_{KL}\Sigma_{2MN} + \partial_{MN}\Lambda_1^{KL}\Sigma_{2KL} + \partial_{KL}\Lambda_1^{KL}\Sigma_{2MN} \\
& - \Lambda_1^{KL}\partial_{MN}\Sigma_{2KL} - \Lambda_1{}^P{}_K\partial_{MN}\partial_{PL}\Lambda_2{}^{KL} - (1 \leftrightarrow 2))\,.
\end{aligned}
\tag{2.29}
$$

This is not quite of the form (2.16), but is equivalent to it upon adding trivial gauge parameters. Indeed, the gauge algebra is only well-defined up to trivial gauge parameters, and adding a trivial parameter of the form (2.24), with

$$
\Omega_{MN}{}^{KL} = -4\,\Sigma_{1MN}\Lambda_2^{KL} - (1 \leftrightarrow 2)\,,
\tag{2.30}
$$

shows that the above indeed defines the gauge algebra bracket. Next we have to prove that the symmetric part of the product is trivial. We compute:

$$
\begin{aligned}
\frac{1}{2}(\Upsilon_1 \circ \Upsilon_2 + \Upsilon_2 \circ \Upsilon_1) = \Big(& 3\,\partial_{KL}\big(\Lambda_2{}^{[MN}\Lambda_1{}^{KL]}\big) + \Omega^{[M}{}_K{}^{N]K} + \partial^{[M}{}_K\chi^{N]K}\,, \\
& -\frac{1}{8}\partial_{MN}\Omega_{KL}{}^{KL} - \frac{1}{8}\partial_{KL}\Omega_{MN}{}^{KL}\Big)\,,
\end{aligned}
\tag{2.31}
$$

where

$$
\begin{aligned}
\Omega_{MN}{}^{KL} &\equiv -4\,\Sigma_{1MN}\Lambda_2{}^{KL} - 2\,\partial_{MN}\Lambda_1{}^{[K}{}_P\Lambda_2{}^{L]P} + (1 \leftrightarrow 2)\,, \\
\chi^{MN} &\equiv 2\,\Lambda_2{}^{(M}{}_K\,\Lambda_1{}^{N)K}\,.
\end{aligned}
\tag{2.32}
$$

We infer that the result is indeed of the trivial form (2.21), (2.22) and (2.24).

Our next goal is to show that the product satisfies a certain Jacobi or Leibniz-type identity that will be instrumental for our subsequent construction. To this end it is convenient to extend the notion of generalized Lie derivative slightly so as to act on doubled objects $\mathfrak{A} \equiv (\mathcal{A}^{MN}, \mathcal{B}_{MN})$ of the same type as $\Upsilon$:

$$
\mathcal{L}_\Upsilon\mathfrak{A} \equiv \Upsilon \circ \mathfrak{A}\,.
\tag{2.33}
$$

From the definition (2.26) of the product we infer that for the first component (the '$\Lambda$ component') this reduces to the conventional generalized Lie derivative, but for the $\Sigma$ component there is an additional contribution due to the 'anomalous' term in (2.26). We will next prove, however, that these extended generalized Lie derivatives still close according to the same bracket:

$$
\big[\mathcal{L}_{\Upsilon_1}, \mathcal{L}_{\Upsilon_2}\big]\,\mathfrak{A} = \mathcal{L}_{[\Upsilon_1,\Upsilon_2]}\mathfrak{A}\,.
\tag{2.34}
$$

Again, for the $\Lambda$ component this reduces to the closure of standard generalized Lie derivatives established in the previous subsection, but for the $\Sigma$ component one obtains additional contributions, so that after a brief computation

$$
[\mathcal{L}_{\Upsilon_1}, \mathcal{L}_{\Upsilon_2}]\mathfrak{A} = \Big(\mathcal{L}_{[\Upsilon_1,\Upsilon_2]}\mathcal{A}^{MN}\,,\; \mathcal{L}_{[\Upsilon_1,\Upsilon_2]}\mathcal{B}_{MN} + \frac{1}{4}\mathcal{A}^{KL}\mathcal{L}_{\Upsilon_1}(\partial_{MN}K(\Upsilon_2)_{KL}) - (1 \leftrightarrow 2)\Big)\,.
\tag{2.35}
$$



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On the other hand, the right-hand side of (2.34) equals

$$\mathcal{L}_{[\Upsilon_1,\Upsilon_2]}\mathfrak{A} = \left(\mathcal{L}_{[\Upsilon_1,\Upsilon_2]}\mathcal{A}^{MN}\ ,\ \mathcal{L}_{[\Upsilon_1,\Upsilon_2]}\mathcal{B}_{MN} + \frac{1}{4}\mathcal{A}^{KL}\partial_{MN}K([\Upsilon_1,\Upsilon_2])_{KL}\right) . \tag{2.36}$$

In order to prove that the above two right-hand sides are actually identical we use

$$\partial_{MN}(\mathcal{L}_{\Upsilon_1}K(\Upsilon_2)_{KL}) = \mathcal{L}_{\Upsilon_1}(\partial_{MN}K(\Upsilon_2)_{KL}) + 2\,\partial_{MN}K(\Upsilon_1)_{[K}{}^{P}K(\Upsilon_2)_{|P|L]}\ . \tag{2.37}$$

This follows as in (2.20), using that the Lie derivative acts on $K$, defined in (2.14), as a tensor of zero density weight. With this one can quickly establish

$$\left[\mathcal{L}_{\Upsilon_1},\mathcal{L}_{\Upsilon_2}\right]\mathfrak{A} = \mathcal{L}_{[\Upsilon_1,\Upsilon_2]}\mathfrak{A} + \left(0\ ,\ \frac{1}{4}\mathcal{A}^{KL}\partial_{MN}X_{KL}\right) , \tag{2.38}$$

where

$$X_{KL} \equiv \mathcal{L}_{\Upsilon_1}K(\Upsilon_2)_{KL} - K(\Upsilon_1)_K{}^{P}K(\Upsilon_2)_{PL} - (1\leftrightarrow 2) - K([\Upsilon_1,\Upsilon_2])_{KL}\ . \tag{2.39}$$

Using (2.15) it is easy to see that this is actually zero, completing the proof of (2.34).

We now derive a Leibniz identity for the product from the closure relation (2.34). We first note that for $\Upsilon$ trivial the extended generalized Lie derivative (2.33) vanishes:

$$\Upsilon\quad\text{trivial}\qquad\Rightarrow\qquad \Upsilon\circ\mathfrak{A} = 0\ . \tag{2.40}$$

This holds by definition for the $\Lambda$ component and for the $\Sigma$ component follows from the fact that the $K(\Upsilon)$ entering the anomalous term of (2.26) is zero for trivial parameters. Thus, using that the symmetric part (2.31) of the product is trivial, the closure relation can also be written as

$$\left[\mathcal{L}_{\Upsilon_1},\mathcal{L}_{\Upsilon_2}\right]\mathfrak{A} = \mathcal{L}_{\Upsilon_1\circ\Upsilon_2}\mathfrak{A}\ . \tag{2.41}$$

Using (2.33) twice we can write this as

$$\Upsilon_1\circ(\Upsilon_2\circ\mathfrak{A}) - \Upsilon_2\circ(\Upsilon_1\circ\mathfrak{A}) = (\Upsilon_1\circ\Upsilon_2)\circ\mathfrak{A}\ . \tag{2.42}$$

Upon renaming the doubled objects entering here and reordering the equations, we have thus established the Leibniz identity

$$\mathfrak{A}\circ(\mathfrak{B}\circ\mathfrak{C}) = (\mathfrak{A}\circ\mathfrak{B})\circ\mathfrak{C} + \mathfrak{B}\circ(\mathfrak{A}\circ\mathfrak{C})\ . \tag{2.43}$$

Let us finally note that formally all relations that hold for conventional Dorfman products are then also satisfied for the product defined here, except that the relevant objects are doubled in the sense of (2.25). In particular, the Jacobiator of the bracket (2.28) can then be proved to be trivial in precise analogy to the original DFT and ExFTs for $\mathrm{E}_{n(n)}$ with $n \leq 7$.

### 2.3 Gauge fields, tensor hierarchy, and Chern-Simons action

We will now introduce gauge fields that, roughly speaking, take values in the algebra given by the Dorfman product defined above. More precisely, we introduce gauge fields ${\mathcal{A}_\mu}^{MN}$ and $\mathcal{B}_{\mu\,MN}$ and combine them into a pair or doubled object as above:

$$\mathfrak{A}_\mu \equiv \left({\mathcal{A}_\mu}^{MN}, \mathcal{B}_{\mu\,MN}\right) . \tag{2.44}$$

In particular, $\mathcal{A}$ carries weight one and $\mathcal{B}$ weight zero while being constrained according to (2.10), i.e.

$$\mathcal{B}_{\mu\,[MN} \otimes \partial_{KL]} = 0 = \eta^{NK}\,\mathcal{B}_{\mu\,MN} \otimes \partial_{KL} \,, \qquad \text{etc. .} \tag{2.45}$$

Their transformation rules receive inhomogeneous terms as to be expected for gauge fields. Indeed, in analogy to Yang-Mills theories we postulate the following gauge transformations w.r.t. doubled parameters (2.25)

$$\delta_\Upsilon \mathfrak{A}_\mu \equiv \mathfrak{D}_\mu \Upsilon \,, \tag{2.46}$$

where we defined the covariant derivative

$$\mathfrak{D}_\mu = \partial_\mu - \mathfrak{A}_\mu \circ \ . \tag{2.47}$$

It should be emphasized that the covariant derivative as written is only defined on doubled objects, which is indicated by the mathfrak notation. We can, however, define covariant derivatives for any field with a well-defined action of the generalized Lie derivatives in section 2.1. For a generic (undoubled) tensor field $V$ we define

$$D_\mu V \equiv \partial_\mu V - \mathcal{L}_{(\mathcal{A}_\mu,\mathcal{B}_\mu)} V \ . \tag{2.48}$$

For instance, for a vector $V^M$ of zero weight this reads explicitly

$$\begin{aligned} D_\mu V^M \equiv\ & \partial_\mu V^M - {\mathcal{A}_\mu}^{KL}\partial_{KL}\, V^M + 2\left(\partial^{MP}\mathcal{A}_{\mu\,PK} - \partial_{KP}{\mathcal{A}_\mu}^{PM}\right) V^K \\ & - 4\,\eta^{ML}\,\mathcal{B}_{\mu\,LK}\, V^K \ . \end{aligned} \tag{2.49}$$

Despite $V$ not being a doubled object we can prove in an index-free fashion that the covariant derivative indeed transforms covariantly:

$$\begin{aligned} \delta_{(\Lambda,\Sigma)}(D_\mu V) &= \delta_\Upsilon(\partial_\mu V - \mathcal{L}_{\mathfrak{A}_\mu} V) = \partial_\mu(\mathcal{L}_\Upsilon V) - \mathcal{L}_{\partial_\mu \Upsilon - \mathfrak{A}_\mu\circ\Upsilon} V - \mathcal{L}_{\mathfrak{A}_\mu}(\mathcal{L}_\Upsilon V) \\ &= \mathcal{L}_{\partial_\mu \Upsilon} V + \mathcal{L}_\Upsilon(\partial_\mu V) - \mathcal{L}_{\partial_\mu\Upsilon} V + \mathcal{L}_{\mathfrak{A}_\mu\circ\Upsilon} V - \mathcal{L}_{\mathfrak{A}_\mu}(\mathcal{L}_\Upsilon V) \\ &= \mathcal{L}_\Upsilon(\partial_\mu V - \mathcal{L}_{\mathfrak{A}_\mu} V) + ([\mathcal{L}_\Upsilon, \mathcal{L}_{\mathfrak{A}_\mu}] + \mathcal{L}_{\mathfrak{A}_\mu\circ\Upsilon})V \\ &= \mathcal{L}_\Upsilon(D_\mu V) \,, \end{aligned} \tag{2.50}$$

where we used (2.41) in the last step. This proves the covariance of the covariant derivative under combined tensor transformations given by generalized Lie derivatives and vector gauge transformations, whose component form is with (2.46) and (2.26) found to be

$$\begin{aligned} \delta_{(\Lambda,\Sigma)}{\mathcal{A}_\mu}^{MN} &= D_\mu \Lambda^{MN} \,, \\ \delta_{(\Lambda,\Sigma)}\mathcal{B}_{\mu\,MN} &= D_\mu \Sigma_{MN} - \Lambda^{KL}\partial_{MN}\mathcal{B}_{\mu KL} - \Lambda^K{}_L\partial_{MN}\partial_{KP}{\mathcal{A}_\mu}^{LP} \,, \end{aligned} \tag{2.51}$$

which of course may also be verified with a direct component computation. This clarifies the seemingly 'non-covariant' terms in the gauge transformations of $\mathcal{B}_\mu$, first identified for the $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{E}_{8(8)}$ cases [13, 14], and explains them as a consequence of the necessary 'anomalous' terms of the Dorfman product.

Let us next discuss the gauge structure and invariant field strengths for the gauge vectors. With the Leibniz identity (2.43) it is straightforward to compute the commutator of two gauge transformations (2.46):

$$\left[\delta_{\Upsilon_1}, \delta_{\Upsilon_2}\right] \mathfrak{A}_\mu = \delta_{\Upsilon_1 \circ \Upsilon_2} \mathfrak{A}_\mu + 2\left\{\Upsilon_{[1}, \mathfrak{D}_\mu \Upsilon_{2]}\right\} , \tag{2.52}$$

where we introduced the notation

$$\{\mathfrak{A}, \mathfrak{B}\} \equiv \mathfrak{A} \circ \mathfrak{B} + \mathfrak{B} \circ \mathfrak{A} . \tag{2.53}$$

We infer from (2.52) that the vector gauge transformations do not quite close, but the failure of closure involves the symmetrized product, which is trivial, cf. (2.31). This implies that the extra terms can be absorbed into higher-form (here 2-form) gauge transformations, as is standard in the tensor hierarchy. Thus, the combined one- and two-form transformations close. Another way to see the need for 2-forms is by inspection of the naive field strength for the gauge vectors:

$$\mathfrak{F}_{\mu\nu} \equiv \partial_\mu \mathfrak{A}_\nu - \partial_\nu \mathfrak{A}_\mu - \left[\mathfrak{A}_\mu, \mathfrak{A}_\nu\right] + \cdots , \tag{2.54}$$

where the ellipsis denotes 2-form terms to be added momentarily. In components, writing $\mathfrak{F}_{\mu\nu} \equiv (F_{\mu\nu}, G_{\mu\nu}) + \cdots$, this reads

$$\begin{aligned} F_{\mu\nu}{}^{MN} &\equiv 2\,\partial_{[\mu}\mathcal{A}_{\nu]}{}^{MN} - 2\,\mathcal{A}_{[\mu}{}^{KL}\,\partial_{KL}\mathcal{A}_{\nu]}{}^{MN} + 6\,\mathcal{A}_{[\mu}{}^{[MN}\partial_{KL}\mathcal{A}_{\nu]}{}^{KL]} - 4\,\mathcal{A}_{[\mu}{}^{K[M}\partial^{N]L}\mathcal{A}_{\nu]\,KL}, \\ G_{\mu\nu\,MN} &\equiv 2\,D_{[\mu}\mathcal{B}_{\nu]\,MN} - \mathcal{A}_{[\mu\,K}{}^{P}\,\partial_{PQ}\partial_{MN}\mathcal{A}_{\nu]}{}^{KQ} . \end{aligned} \tag{2.55}$$

We consider now the general variation under an arbitrary $\delta\,\mathfrak{A}_\mu$, for which we compute

$$\delta\,\mathfrak{F}_{\mu\nu} = 2\,\mathfrak{D}_{[\mu}\,\delta\,\mathfrak{A}_{\nu]} + \{\mathfrak{A}_{[\mu}, \delta\,\mathfrak{A}_{\nu]}\} + \cdots . \tag{2.56}$$

We do not quite obtain the expected identity with only the covariant curl of $\delta\,\mathfrak{A}_\mu$, but the additional terms are trivial and can hence be absorbed into the 2-forms. More precisely, 2-forms are introduced in precise correspondence with the trivial terms in the symmetrized product (2.31). We thus define the full field strength to be $\mathfrak{F}_{\mu\nu} \equiv (\mathcal{F}_{\mu\nu}, \mathcal{G}_{\mu\nu})$, where

$$\begin{aligned} \mathcal{F}_{\mu\nu}{}^{MN} &= F_{\mu\nu}{}^{MN} + \partial_{KL}C_{\mu\nu}{}^{[KLMN]} + \partial_K{}^{[M}\,C_{\mu\nu}{}^{N]K} + 8\,\mathcal{C}_{\mu\nu\,KL}{}^{K[M}\,\eta^{N]L} , \\ \mathcal{G}_{\mu\nu\,MN} &= G_{\mu\nu\,MN} + \partial_{KL}\mathcal{C}_{\mu\nu\,MN}{}^{KL} + \partial_{MN}\mathcal{C}_{\mu\nu\,KL}{}^{KL} , \end{aligned} \tag{2.57}$$

and the two-forms $\mathcal{C}_{\mu\nu\,MN}{}^{KL}$ are covariantly constrained in its indices $[MN]$. After adding the appropriate 2-forms to the field strength, we can show its complete gauge covariance. To this end, we use the identity

$$\left[\mathfrak{D}_\mu, \mathfrak{D}_\nu\right]\Upsilon = -\mathfrak{F}_{\mu\nu} \circ \Upsilon , \tag{2.58}$$



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which follows immediately from (2.47) and the fact that the 2-form contributions are of the trivial form and hence drop out of this relation by (2.40). We similarly have for the covariant derivatives (2.48)

$$[D_\mu, D_\nu]V^M = -\mathcal{L}_{(F_{\mu\nu},G_{\mu\nu})}V^M = -\mathcal{L}_{(\mathcal{F}_{\mu\nu},\mathcal{G}_{\mu\nu})}V^M \,. \tag{2.59}$$

This is contained in (2.58), which can be evaluated on the first ($\Lambda$) component of a doubled object, thereby reproducing this equation. We then compute with (2.58)

$$\delta_\Upsilon \mathfrak{F}_{\mu\nu} = \Upsilon \circ \mathfrak{F}_{\mu\nu} \,, \tag{2.60}$$

using that up to trivial contributions taken care of by the 2-forms the order of the product can be exchanged up to a sign.

**Chern-Simons term**

We will now define a Chern-Simons action for the gauge vectors $\mathfrak{A}_\mu$. To this end we need an invariant inner product. The naive ansatz for the 'off-diagonal' inner product between adjoint and co-adjoint vector needs to be deformed by a derivative term in order to account for the 'anomalous' term in the $\Sigma$ component of the product. One finds:

$$\langle\langle \mathfrak{A}_1, \mathfrak{A}_2\rangle\rangle \equiv \langle\langle(\mathcal{A}_1,\mathcal{B}_1),(\mathcal{A}_2,\mathcal{B}_2)\rangle\rangle \equiv 2\,\mathcal{A}_{(1}{}^{MN}\mathcal{B}_{2)MN} + \mathcal{A}_{(1}{}^{MN}\partial_{MK}\mathcal{A}_{2)N}{}^K \,. \tag{2.61}$$

The invariance condition means, more precisely, invariance up to total derivatives:

$$\langle\langle \Upsilon\circ\mathfrak{A}_1, \mathfrak{A}_2\rangle\rangle + \langle\langle \mathfrak{A}_1, \Upsilon\circ\mathfrak{A}_2\rangle\rangle = \partial_{MN}(\Lambda^{MN}\langle\langle \mathfrak{A}_1,\mathfrak{A}_2\rangle\rangle) \,, \tag{2.62}$$

which can be verified by an explicit computation. Thus, a truly invariant inner product involves the $N$-dimensional $Y$ integration (where $N = \frac{1}{2}(p+q)(p+q-1)$):

$$\langle \mathfrak{A}_1, \mathfrak{A}_2\rangle \equiv \int \mathrm{d}^N Y \big(\mathcal{A}_1{}^{MN}\mathcal{B}_{2MN} + \mathcal{A}_2{}^{MN}\mathcal{B}_{1MN} + \mathcal{A}_1{}^{MN}\partial_{MK}\mathcal{A}_{2N}{}^K\big) \,, \tag{2.63}$$

where we used that one can integrate by parts in the terms with derivatives to combine two terms into one. We can then also write, using the notation (2.14),

$$\langle \mathfrak{A}_1, \mathfrak{A}_2\rangle = \int \mathrm{d}^N Y \left(\frac{1}{4}\mathcal{A}_1^{MN}K_{MN}(\mathfrak{A}_2) + \mathcal{A}_2^{MN}\mathcal{B}_{1MN}\right) . \tag{2.64}$$

Although no longer manifest, the inner product defined in this way is of course still symmetric in the two arguments, up to boundary terms. An important consequence is that the inner product is zero whenever one argument is trivial:

$$\mathfrak{T} \;\; \text{trivial} \qquad \Rightarrow \qquad \langle \mathfrak{A}\,, \mathfrak{T}\rangle = 0 \quad \forall\, \mathfrak{A} \,. \tag{2.65}$$

This follows directly from (2.64),

$$\langle \mathfrak{A}\,, \mathfrak{T}\rangle = \int \mathrm{d}^N Y \left(\frac{1}{4}\mathcal{A}^{MN}K_{MN}(\mathfrak{T}) + \mathcal{T}^{MN}\mathcal{B}_{MN}\right) = 0 \,, \tag{2.66}$$

using that for trivial $\mathfrak{T} = (\mathcal{T}^{MN}, \tau_{MN})$ we have $K(\mathfrak{T}) = 0$ and that the contraction of $\mathcal{T}^{MN}$ with any covariantly constrained object such as $\mathcal{B}_{MN}$ vanishes.

Having established the existence of an invariant inner product, a natural ansatz for the Chern-Simons action is its familiar three-dimensional form:

$$S_{\rm CS} = \int \mathrm{d}^3x\, \varepsilon^{\mu\nu\rho}\Big(\big\langle\, \mathfrak{A}_\mu\,,\, \partial_\nu \mathfrak{A}_\rho\,\big\rangle - \frac{1}{3}\big\langle\, \mathfrak{A}_\mu\,,\, \mathfrak{A}_\nu \circ \mathfrak{A}_\rho\,\big\rangle\Big)\,, \tag{2.67}$$

where the internal integration is implicit in the inner product. Using the Leibniz identity, its gauge variation up to total derivatives can be written as

$$\delta_\Upsilon S_{\rm CS} = -\frac{2}{3}\int \mathrm{d}^3x\, \varepsilon^{\mu\nu\rho}\, \big\langle \mathfrak{A}_\mu, \{\mathfrak{A}_\nu, \partial_\rho \Upsilon\}\big\rangle = 0\,, \tag{2.68}$$

which vanishes as a consequence of (2.65) since the symmetric pairing $\{\,,\,\}$ is trivial. Using the Leibniz identity (2.43) again, one can show that under an arbitrary variation $\delta\,\mathfrak{A}_\mu$

$$\delta S_{\rm CS} = \int \mathrm{d}^3x\, \varepsilon^{\mu\nu\rho}\big\langle \delta\,\mathfrak{A}_\mu\,,\, \mathfrak{F}_{\nu\rho}\big\rangle\,. \tag{2.69}$$

Because of the degeneracy (2.65) this does not imply that the field equations are $\mathfrak{F}_{\mu\nu} = 0$, but only a suitably projected version of the field strength is zero. In the following we will couple such a Chern-Simons action to charged matter, such that the field equations relate a projection of the field strength to scalar currents. We can now use this result to compare with the more familiar form of this variation. We first recall the identification

$$\mathfrak{F}_{\mu\nu} = \big(\mathcal{F}_{\mu\nu}{}^{MN}, \mathcal{G}_{\mu\nu MN}\big)\,. \tag{2.70}$$

We then read off from (2.69) and (2.63)

$$\delta S_{\rm CS} = \int \mathrm{d}^3x\, \mathrm{d}^NY\, \varepsilon^{\mu\nu\rho}\Big(\delta\mathcal{B}_{\mu MN}\mathcal{F}_{\nu\rho}{}^{MN} + \delta\mathcal{A}_\mu{}^{MN}\big(\mathcal{G}_{\nu\rho\, MN} + \partial_{MK}\mathcal{F}_{\nu\rho N}{}^K\big)\Big)\,. \tag{2.71}$$

### 2.4 Covariant derivatives and variations

For completeness and in order to relate to the 'covariant variations' employed for the supersymmetric $\mathrm{E}_{8(8)}$ ExFT in [26], we will now discuss some aspects of the '$\mathrm{O}(p,q)$ covariant' geometry, notably the notion of connections and torsion. We begin by introducing derivatives that covariantize the internal partial derivatives w.r.t. the internal generalized diffeomorphisms. For a (co-adjoint) vector of weight zero we define

$$\nabla_{\mathcal{M}} V_{\mathcal{N}} \equiv \partial_{\mathcal{M}} V_{\mathcal{N}} - \Gamma_{\mathcal{M},\mathcal{N}}{}^{\mathcal{K}}\, V_{\mathcal{K}}\,, \tag{2.72}$$

with connections $\Gamma_{\mathcal{M},\mathcal{N}}{}^{\mathcal{K}}$ that take values in the Lie algebra $\mathfrak{so}(p,q)$. We can thus introduce $\Gamma_{\mathcal{M},\mathcal{N}}$ by

$$\Gamma_{\mathcal{M},\mathcal{N}}{}^{\mathcal{K}} \equiv\; \Gamma_{\mathcal{M},\mathcal{L}}\, f^{\mathcal{L}\mathcal{K}}{}_{\mathcal{N}}\,, \tag{2.73}$$

which reads in index pairs

$$\Gamma_{MN,KL}{}^{PQ} = \frac{1}{4}\,\Gamma_{MN,RS}\, f^{RS,PQ}{}_{KL}\,, \tag{2.74}$$



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with a pre-factor for later convenience. This implies for fundamental vectors

$$\begin{aligned}\nabla_{MN}V_K &= \partial_{MN}V_K - \Gamma_{MN,K}{}^{L}V_L\ ,\\ \nabla_{MN}V^K &= \partial_{MN}V^K + \Gamma_{MN,L}{}^{K}V^L\ .\end{aligned} \tag{2.75}$$

In (2.20) we computed the non-covariant gauge transformation for a partial derivative of a vector. From this result and the first equation above we infer that the covariant derivative indeed transforms covariantly provided the connection transforms as tensor of weight $\lambda = -1$, plus the usual inhomogeneous term:

$$\delta_\Upsilon \Gamma_{MN,KL} = \partial_{MN}K_{KL}(\Upsilon) + \mathcal{L}_\Upsilon^{[-1]}\Gamma_{MN,KL}\ , \tag{2.76}$$

with gauge parameter (2.25), and $K_{KL}$ defined in (2.14). We can also define the covariant derivative of a tensor of arbitrary density weight $\lambda$, using that the above implies for the non-covariant variation

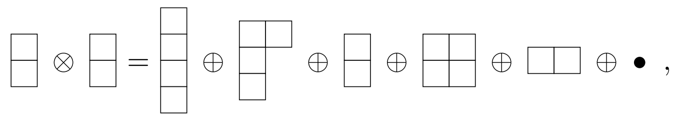
$$\Delta_\Lambda^{\rm nc}\big(\Gamma_{[M}{}^{K}{}_{,N]K}\big) = \partial_{MN}\big(\partial_{KL}\Lambda^{KL}\big)\ . \tag{2.77}$$

Thus, for a vector of weight $\lambda$,

$$\nabla_{MN}V_K = \partial_{MN}V_K - \Gamma_{MN,K}{}^{L}V_L - \lambda\,\Gamma_{[M}{}^{L}{}_{,N]L}\,V_K\ . \tag{2.78}$$

We next aim to define a torsion tensor as a particular projection of the connection that transforms tensorially. In general, the connection lives in the tensor product

$$\Gamma_{MN,KL}\ :$$



(2.79)

where the Hook and window tableaux are traceless, with the antisymmetric and symmetric tableaux carrying two boxes denoting their trace parts. The latter is traceless itself with its trace give by the singlet $\bullet$. We next use that the section constraint implies

$$\partial_{[MN}K_{KL]} = 0\ , \qquad \partial_{(M}{}^{K}K_{N)K} = 0\ , \tag{2.80}$$

as may be quickly verified by a direct computation. We then infer with (2.76) that the following projections have tensor character:

$$\mathcal{T}_{MNKL} \equiv 6\,\Gamma_{[MN,KL]}\ , \qquad \mathcal{T}_{MN} \equiv 2\,\Gamma_{(M}{}^{K}{}_{,N)K}\ , \tag{2.81}$$

corresponding to the totally antisymmetric and the symmetric trace tableaux. We may also combine this into a reducible torsion tensor:

$$\mathcal{T}_{MN,KL} \equiv \mathcal{T}_{MNKL} + 2\,\mathcal{T}_{[M[\underline{K}}\,\eta_{N]\underline{L}]}\ . \tag{2.82}$$

In the following, we will thus impose torsionlessness of the connection $\Gamma$ as

$$\mathcal{T}_{MN,KL} = 0\ . \tag{2.83}$$

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As usual in generalized geometries, this condition does not fully determine the connection [1] but all the parts that are required in order to formulate the field equations and transformation rules. For DFTs and ExFTs with external dimension $n \geq 4$ the torsion tensor is such that for vanishing torsion the manifestly covariant Lie derivative in which all partial derivatives have been replaced by covariant derivatives equals the original generalized Lie derivative. The same is not quite true for ExFTs with $n = 3$ [26, 27], but we have the following close analogue: for

$$\widetilde{\Sigma}_{MN} \equiv \Sigma_{MN} - \frac{1}{4}\,\Gamma_{MN,KL}\,\Lambda^{KL}\,, \tag{2.84}$$

one can write for a vector $V_M$ or arbitrary density weight

$$\big(\mathcal{L}^{\nabla}_{(\Lambda,\widetilde{\Sigma})} - \mathcal{L}_{(\Lambda,\Sigma)}\big)V_M = -\mathcal{T}_{MN,KL}\,V^N\Lambda^{KL}\,, \tag{2.85}$$

in terms of (2.82). This follows by a direct computation. Useful intermediate results are (recalling that $\Lambda$ has weight $\lambda = 1$)

$$\begin{aligned}
\nabla_{MN}\Lambda^{MN} &= \partial_{MN}\Lambda^{MN} + \Gamma_M{}^K{}_{,NK}\Lambda^{MN}\,, \\
K^{\nabla}_{MN}(\Lambda,\Sigma) &= K_{MN}(\Lambda,\Sigma) + \Lambda^{KL}\big(\Gamma_{MN,KL} + \Gamma_{KL,MN} - \mathcal{T}_{MNKL}\big) - 2\,\mathcal{T}_{[M}{}^K\Lambda_{|K|N]}\,.
\end{aligned} \tag{2.86}$$

With these relations we can relate the general variation (2.71) of the Chern-Simons term to its ‘covariant variation’ as used in [26]. Indeed, one quickly sees, upon adding and subtracting connection terms, that

$$\delta S_{\rm CS} = \int \mathrm{d}^3x\,\mathrm{d}^NY\,\varepsilon^{\mu\nu\rho}\Big(\,\Delta\mathcal{B}_{\mu MN}\,\mathcal{F}_{\nu\rho}{}^{MN} + \delta\mathcal{A}_\mu{}^{MN}\big(\tilde{\mathcal{G}}_{\nu\rho\,MN} + \nabla_{MK}\mathcal{F}_{\nu\rho\,N}{}^K\big)\Big)\,, \tag{2.87}$$

where we introduced

$$\begin{aligned}
\Delta\mathcal{B}_{\mu MN} &\equiv \delta\mathcal{B}_{\mu MN} - \frac{1}{4}\,\Gamma_{MN,KL}\,\delta\mathcal{A}_\mu{}^{KL}\,, \\
\tilde{\mathcal{G}}_{\mu\nu\,MN} &\equiv \mathcal{G}_{\mu\nu\,MN} - \frac{1}{4}\,\Gamma_{MN,KL}\,\mathcal{F}_{\mu\nu}{}^{KL}\,.
\end{aligned} \tag{2.88}$$

Let us also note with (2.76) that $\tilde{\mathcal{G}}$ transforms as

$$\delta_\Upsilon\tilde{\mathcal{G}}_{\mu\nu\,MN} = \mathcal{L}_\Upsilon\tilde{\mathcal{G}}_{\mu\nu\,MN}\,, \tag{2.89}$$

i.e., it transforms covariantly in the more conventional sense of covariance.

## 3 Construction of $\mathrm{O}(p,q)$ enhanced DFT

Having set up the formalism we can now construct the enhanced DFT invariant under $\mathrm{O}(p,q)$ generalized diffeomorphisms. The field content of the $\mathrm{O}(p,q)$ enhanced DFT is given by the gauge fields (2.44) together with an external $3\times3$ metric $g_{\mu\nu}$ (or vielbein $e_\mu{}^a$), and an internal $\mathrm{O}(p,q)$ valued metric $\mathcal{M}_{MN}$.

### 3.1 Building blocks of the DFT action

The field equations of $\mathrm{SO}(p,q)$ enhanced DFT are most compactly derived from a Lagrangian whose different terms are of the form generic for exceptional field theory with three external dimensions [13, 14]

$$\mathcal{L} = \mathcal{L}_{\rm EH} + k_1\,\mathcal{L}_{\rm kin} + k_2\,\mathcal{L}_{\rm CS} + k_3\,\mathcal{L}_{\rm pot}\;, \tag{3.1}$$

each term being separately invariant under generalized internal diffeomorphisms. The modified Einstein-Hilbert term and the scalar kinetic term have the following form

$$\begin{aligned}\mathcal{L}_{\rm EH} &= \sqrt{-g}\,e_a{}^\mu e_b{}^\nu \left(R_{\mu\nu}{}^{ab} + \mathcal{F}_{\mu\nu}{}^{MN} e^{a\,\rho}\partial_{MN} e_\rho{}^b\right) \equiv \sqrt{-g}\,\hat{R}\;,\\ \mathcal{L}_{\rm kin} &= \frac{1}{16}\sqrt{-g}\;g^{\mu\nu}\,D_\mu\mathcal{M}^{MN}D_\nu\mathcal{M}_{MN},\end{aligned} \tag{3.2}$$

with the covariant derivatives (2.48) and the Riemann tensor $R_{\mu\nu}{}^{ab}$ computed from the external vielbein $e_\mu{}^a$ with derivatives covariantized under internal diffeomorphisms under which $e_\mu{}^a$ transforms as a scalar density (of weight $\lambda=1$). By construction, both these terms are invariant under generalized internal diffeomorphisms with the second term in $\hat{R}$ moreover ensuring invariance under local $\mathrm{SO}(1,2)$ Lorentz transformations.[3]

The Chern-Simons term in (3.1) is given by the standard non-abelian form (2.67) based on the gauge invariant inner product (2.61) on the gauge algebra of internal diffeomorphisms. For concreteness, we spell out its explicit form

$$\begin{aligned}\mathcal{L}_{\rm CS} = \varepsilon^{\mu\nu\rho}\Big(&F_{\mu\nu}{}^{MN}\mathcal{B}_{\rho\,MN} + \partial_\mu\mathcal{A}_{\nu\,N}{}^K\partial_{KM}\mathcal{A}_\rho{}^{MN} - \frac{2}{3}\,\partial_{MN}\partial_{KL}\mathcal{A}_\mu{}^{KP}\mathcal{A}_\nu{}^{MN}\mathcal{A}_{\rho\,P}{}^L\\ &+\frac{2}{3}\,\mathcal{A}_\mu{}^{LN}\partial_{MN}\mathcal{A}_\nu{}^M{}_P\partial_{KL}\mathcal{A}_\rho{}^{PK} - \frac{4}{3}\,\mathcal{A}_\mu{}^{LN}\partial_{MP}\mathcal{A}_\nu{}^M{}_N\partial_{KL}\mathcal{A}_\rho{}^{PK}\Big)\;,\end{aligned} \tag{3.3}$$

with its variation given by (2.71).

The last term in (3.1) is referred to as the *potential term* (from a three-dimensional point of view) as it does not carry any external derivatives $\partial_\mu$, but is bilinear in the internal currents

$$\begin{aligned}\mathcal{J}_{MN,KL} &\equiv \mathcal{M}^{PQ}\eta_{QK}\partial_{MN}\mathcal{M}_{LP}\;\;,\\ (\mathcal{J}_{MN})_\mu{}^\nu &\equiv g^{\nu\rho}\,\partial_{MN}g_{\mu\rho}\;,\end{aligned} \tag{3.4}$$

such that the full expression is invariant under generalized internal diffeomorphisms up to total derivatives. It is useful to note the non-covariant transformation behavior of the currents (3.4)

$$\begin{aligned}\mathcal{L}_{(\Lambda,\Sigma)}\mathcal{J}_{MN,KL} &= \delta^{\rm cov}\mathcal{J}_{MN,KL} + \left(\mathcal{M}_{P[K}\mathcal{M}_{L]Q} - \eta_{P[K}\eta_{L]Q}\right)\partial_{MN}K^{PQ}\;,\\ \mathcal{L}_{(\Lambda,\Sigma)}\mathcal{J}_\mu{}^\nu &= \delta^{\rm cov}\mathcal{J}_\mu{}^\nu + 2\,\partial_{MN}\partial_{KL}\Lambda^{KL}\,\delta_\mu{}^\nu\;,\end{aligned} \tag{3.5}$$

[3]Note the absence of the factor of 1/2 with respect to the expression in [28] that is due to our different sum conventions for sums over pairs of antisymmetric indices.

with $K^{PQ}$ from (2.14). It is then straightforward to verify by explicit calculation that the following combination of currents

$$\begin{aligned} V \equiv & -\frac{1}{8}\,\mathcal{M}^{KP}\mathcal{M}^{LQ}\,\partial_{KL}\mathcal{M}_{MN}\,\partial_{PQ}\mathcal{M}^{MN} - \frac{1}{2}\,\partial_{MK}\mathcal{M}^{NP}\partial_{NL}\mathcal{M}^{MQ}\,\mathcal{M}^{KL}\mathcal{M}_{PQ} \\ & -\frac{1}{4}\,\partial_{MN}\mathcal{M}^{PK}\,\partial_{KL}\mathcal{M}^{QM}\,\mathcal{M}_P{}^L\mathcal{M}_Q{}^N + 2\,\partial_{MK}\mathcal{M}^{NK}\,\partial_{NL}\mathcal{M}^{ML} \\ & - g^{-1}\partial_{MN}g\,\partial_{KL}\mathcal{M}^{MK}\mathcal{M}^{NL} - \frac{1}{4}\,\mathcal{M}^{MK}\mathcal{M}^{NL}\,g^{-2}\partial_{MN}g\,\partial_{KL}g \\ & - \frac{1}{4}\,\mathcal{M}^{MK}\mathcal{M}^{NL}\,\partial_{MN}g^{\mu\nu}\partial_{KL}g_{\mu\nu}\,, \end{aligned} \tag{3.6}$$

is such that $\mathcal{L}_{\rm pot} \equiv -\sqrt{-g}\,V$ is indeed invariant under generalized internal diffeomorphisms up to total derivatives.

The Lagrangian (3.1) thus is (term by term) invariant under internal generalized diffeomorphisms up to total derivatives $\sqrt{-g}^{-1}\partial_\mu\left(\sqrt{-g}I^\mu\right)$. It remains to fix the relative coefficients $k_1$, $k_2$, $k_3$. This will be a consequence of the invariance under external diffeomorphisms.

### 3.2 External diffeomorphisms

The full Lagrangian (3.1) should also be invariant under a suitable definition of external diffeomorphisms with parameter $\xi^\mu(x,Y)$. This fixes all remaining constants in the Lagrangian. The calculation closely follows the analogous cases of maximal $\mathrm{E}_{8(8)}$ ExFT [13] and minimal SL(2) ExFT [14], such that here we only briefly sketch the pertinent cancellations in order to determine the constants $k_1$, $k_2$, $k_3$. For the external dreibein field and the scalar matrix external diffeomorphisms take the usual form

$$\begin{aligned} \delta e_\mu{}^a &= \xi^\mu D_\nu e_\mu{}^a + D_\mu\xi^\nu e_\nu{}^a\,, \\ \delta\mathcal{M}_{MN} &= \xi^\mu D_\mu\mathcal{M}_{MN}, \end{aligned} \tag{3.7}$$

of properly covariantized three-dimensional diffeomorphisms. For the gauge fields, we start from an ansatz following [13, 14]

$$\begin{aligned} \delta_\xi^{(0)}\mathcal{A}_\mu{}^{MN} &= \xi^\nu F_{\nu\mu}{}^{MN} + g_{\mu\nu}\mathcal{M}^{MK}\mathcal{M}^{NL}\partial_{KL}\xi^\nu, \\ \delta_\xi^{(0)}\mathcal{B}_{\mu\,MN} &= \xi^\nu G_{\nu\mu\,MN} + \beta_1 g_{\mu\nu}\mathcal{J}_{MN}{}^{KL}\partial_{KL}\xi^\nu + \beta_2\sqrt{-g}\,\varepsilon_{\mu\nu\lambda}g^{\lambda\rho}\mathcal{D}^\nu(g_{\rho\sigma}\partial_{MN}\xi^\sigma), \end{aligned} \tag{3.8}$$

which reduces to standard three-dimensional diffeomorphisms in case the parameter $\xi^\mu$ does not depend on the internal coordinates. The coefficients $\beta_1,\beta_2$ will be fixed in the following.

In what follows it proves useful to have the explicit form of variation of the full Lagrangian with respect to a variation of the gauge fields which we put in the form

$$\delta_{(\mathcal{A},B)}\,\mathcal{L} = \varepsilon^{\mu\nu\rho}\left(\mathcal{E}^{(\mathcal{A})MN}_{\mu\nu}\,\delta\mathcal{B}_{\rho\,MN} + \mathcal{E}^{(\mathcal{B})}_{\mu\nu\,MN}\,\delta\mathcal{A}_\rho{}^{MN}\right), \tag{3.9}$$

with the coefficients

$$\begin{aligned} \mathcal{E}^{(\mathcal{A})MN}_{\mu\nu} &= k_2\,F_{\mu\nu}{}^{MN} - \frac{1}{2}\sqrt{-g}\,k_1\,\varepsilon_{\mu\nu\sigma}\,j^{\sigma\,MN}\,, \\ \mathcal{E}^{(\mathcal{B})}_{\mu\nu\,MN} &= k_2\,G_{\mu\nu MN} + \sqrt{-g}\,\varepsilon_{\mu\nu\sigma}\,\hat{J}^\sigma{}_{MN} - \frac{1}{16}\,k_1\sqrt{-g}\,\varepsilon_{\mu\nu\sigma}j^{\sigma K}{}_L\mathcal{J}_{MN}{}^L{}_K \\ &\quad + \partial_{MK}\mathcal{E}^{(\mathcal{A})}_{\mu\nu}{}_N{}^K\,. \end{aligned} \tag{3.10}$$



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Here, the internal current $\mathcal{J}_{MN}{}^{L}{}_{K}$ has been defined in (3.4), the external currents are defined as

$$
\begin{aligned}
j_{\mu}{}^{MN} &= \eta_{KL}\,\mathcal{M}^{K[M}\mathcal{D}_{\mu}\mathcal{M}^{N]L}\,,\\
\hat{J}^{\mu}{}_{MN} &= -2\,e^{\mu}{}_{a}\,e^{\nu}{}_{b}\left(\partial_{MN}\omega_{\nu}{}^{ab} - D_{\nu}(e^{\rho[a}\partial_{MN}e_{\rho}{}^{b]})\right)\,,
\end{aligned}
\tag{3.11}
$$

and related to the sources from the Einstein-Hilbert and the kinetic scalar term, respectively, cf. (3.14) below. Note that the first equation of (3.10) does not appear as a full field equation of the theory, since the field $\mathcal{B}_{\mu\,MN}$ w.r.t. which we vary in (3.9) is constrained according to (2.45).

With the variation (3.7), (3.8) and the general variation (2.71) of the Chern-Simons term, we find that under external diffeomorphisms this term transforms non-trivially as

$$
\begin{aligned}
\delta_{\xi}^{(0)}\mathcal{L}_{\rm CS} =\;& \varepsilon^{\mu\nu\rho}\left(\mathcal{M}^{MP}\mathcal{M}^{NQ}\partial_{MK}F_{\mu\nu N}{}^{K}g_{\rho\sigma}\partial_{PQ}\xi^{\sigma} + G_{\mu\nu PQ}\mathcal{M}^{PK}\mathcal{M}^{QL}g_{\rho\sigma}\partial_{KL}\xi^{\sigma}\right)\\
&+\beta_1\,\varepsilon^{\mu\nu\rho}F_{\mu\nu}{}^{PQ}\mathcal{J}_{PQ}{}^{KL}g_{\rho\sigma}\partial_{KL}\xi^{\sigma} - 2\,\beta_2\,\sqrt{-g}\,F^{\mu\nu MN}\mathcal{D}_{\mu}(\partial_{MN}\xi^{\rho}g_{\rho\nu})\\
&-\frac{1}{2}\,\varepsilon^{\mu\nu\rho}\partial_{MK}\xi^{\sigma}F_{\sigma\rho}{}^{MN}F_{\mu\nu N}{}^{K}\,,
\end{aligned}
\tag{3.12}
$$

up to total derivatives. Using (3.10), the last term here can be written as

$$
\begin{aligned}
-\frac{1}{2}\,\varepsilon^{\mu\nu\rho}\,\partial_{MK}\xi^{\sigma}F_{\sigma\rho}{}^{MN}F_{\mu\nu N}{}^{K} =\;& -\frac{1}{2\,k_2^2}\,\varepsilon^{\mu\nu\rho}\partial_{MK}\xi^{\sigma}\mathcal{E}^{(\mathcal{A})MN}_{\sigma\rho}\mathcal{E}^{(\mathcal{A})}_{\mu\nu}{}_{N}{}^{K}\\
&-\left(\frac{k_1}{k_2^2}\sqrt{-g}\,F_{\nu\rho}{}^{MN} - \frac{k_1^2}{4\,k_2^2}\,\varepsilon_{\mu\nu\rho}\,j^{\mu\,MN}\right)j^{\nu}{}_{N}{}^{K}\partial_{MK}\xi^{\rho}\,.
\end{aligned}
\tag{3.13}
$$

Next we proceed with variation of the Einstein-Hilbert term. With its variation under a general variation of the gauge field $\mathcal{A}_{\mu}{}^{MN}$ given by

$$
\delta_{\mathcal{A}}\mathcal{L}_{\rm EH} = \hat{J}^{\mu}{}_{MN}\,\delta\mathcal{A}_{\mu}{}^{MN}\,,
\tag{3.14}
$$

the full diffeomorphism variation of the covariantized EH term becomes up to total derivatives

$$
\delta_{\xi}^{(0)}(\sqrt{-g}\hat{R}) = \sqrt{-g}\,F^{\mu\nu MN}\mathcal{D}_{\mu}(\partial_{MN}\xi^{\rho}g_{\rho\nu}) + \sqrt{-g}\,\mathcal{M}^{MK}\mathcal{M}^{NL}\hat{J}_{\mu\,KL}\partial_{MN}\xi^{\mu}\,.
\tag{3.15}
$$

The first term in this variation has been computed in [14] and cancels the corresponding term in the variation of the Chern-Simons term if we choose $\beta_2 = 1/(2\,k_2)$.

Also the variation of the scalar kinetic term follows [14]. We find

$$
\begin{aligned}
\delta\mathcal{L}_{\rm kin} =\;& \delta^{\rm cov}\,\delta\mathcal{L}_{\rm kin} + \sqrt{-g}\,j^{\mu R}{}_{Q}\partial_{PR}\big(g_{\mu\rho}\mathcal{M}^{PK}\mathcal{M}^{QL}\partial_{KL}\xi^{\rho}\big)\\
&-\frac{1}{8}\sqrt{-g}\,g_{\mu\nu}\mathcal{M}^{PK}\mathcal{M}^{QL}\partial_{PQ}\xi^{\nu}\mathcal{J}_{KL}{}^{MN}j_{\nu MN} - \sqrt{-g}\,F_{\mu\nu}{}^{KQ}\partial_{KL}\xi^{\nu}j^{\mu L}{}_{Q}\\
&+\sqrt{-g}\,\beta_1 e\mathcal{J}_{KL}{}^{PQ}j_{\nu}{}^{KL}\partial_{PQ}\xi^{\nu} + \frac{1}{2k_2}\,\varepsilon^{\mu\nu\rho}j_{\mu}{}^{KL}\mathcal{D}_{\nu}\big(g_{\rho\sigma}\partial_{KL}\xi^{\sigma}\big)\,.
\end{aligned}
\tag{3.16}
$$

Upon integrating by parts the derivative $\mathcal{D}_\nu$ in the last term above it can be rewritten in the following form

$$
\begin{aligned}
&-\frac{1}{2k_2}\,D_\nu\big(\varepsilon^{\mu\nu\rho}j_\mu{}^{KL}\big)g_{\rho\sigma}\partial_{KL}\xi^\sigma = \\
&\quad=\frac{1}{2k_2}\,\varepsilon^{\mu\nu\rho}j_\nu{}^K{}_N j_\mu{}^{NL}g_{\rho\sigma}\partial_{KL}\xi^\sigma+\frac{1}{4k_2}\,\varepsilon^{\mu\nu\rho}\mathcal{M}^K{}_M\mathcal{L}_{(F_{\nu\mu},G_{\nu\mu})}\mathcal{M}^{ML}g_{\rho\sigma}\partial_{KL}\xi^\sigma \\
&\quad=\frac{1}{2k_2}\,\varepsilon^{\mu\nu\rho}j_\nu{}^K{}_N j_\mu{}^{NL}g_{\rho\sigma}\partial_{KL}\xi^\sigma+\frac{1}{4k_2}\varepsilon^{\mu\nu\rho}F_{\nu\mu}{}^{PQ}\,\mathcal{J}_{PQ}{}^{KL}g_{\rho\sigma}\partial_{KL}\xi^\sigma \\
&\qquad+\frac{1}{k_2}\,\varepsilon^{\mu\nu\rho}\,\mathcal{M}^{MK}\mathcal{M}^{NL}\partial_M{}^P F_{\nu\mu\,NP}\,g_{\rho\sigma}\,\partial_{KL}\xi^\sigma \\
&\qquad+\frac{1}{k_2}\,\varepsilon^{\mu\nu\rho}G_{\nu\mu PQ}\mathcal{M}^{PK}\,\mathcal{M}^{QL}g_{\rho\sigma}\partial_{KL}\xi^\sigma\,.
\end{aligned}
\tag{3.17}
$$

Here, the terms linear in the field strengths cancel the corresponding terms from the variation of the Chern-Simons term (3.12) if the following holds true

$$
k_1=k_2^2\,,\qquad \beta_1=\frac{1}{4}\,. \tag{3.18}
$$

The remaining contributions coming from the Einstein-Hilbert, the scalar kinetic and the Chern-Simons terms can be collected in the following expression

$$
\begin{aligned}
&\delta_\xi^{(0)}\Big(\mathcal{L}_{\rm EH}+k_1\,\mathcal{L}_{\rm kin}+k_2\,\mathcal{L}_{\rm CS}\Big)= \\
&\quad=\sqrt{-g}\,\mathcal{M}^{MK}\mathcal{M}^{NL}\hat{\jmath}_{\mu\,KL}\partial_{MN}\xi^\mu+k_1\,\sqrt{-g}\,j^{\mu L}{}_Q\partial_{PL}\big(g_{\mu\rho}\mathcal{M}^{PK}\mathcal{M}^{QL}\partial_{KL}\xi^\rho\big) \\
&\qquad-\frac{1}{8}k_1\,\sqrt{-g}\,g_{\mu\nu}\mathcal{M}^{PK}\mathcal{M}^{QL}\partial_{PQ}\xi^\nu j_{KL}{}^{MN}j_{\nu MN} \\
&\qquad+k_1\beta_1\,\sqrt{-g}\,\mathcal{J}_{KL}{}^{PQ}j_\mu{}^{KL}\partial_{PQ}\xi^\mu-\frac{1}{2\,k_2}\varepsilon^{\mu\nu\rho}\partial_{MK}\xi^\sigma\mathcal{E}_{\sigma\rho}^{(\mathcal{A})MN}\mathcal{E}_{\mu\nu}^{(\mathcal{A})}{}_N{}^K \\
&\qquad+\frac{k_1^2}{4\,k_2}(-g)\,\varepsilon_{\mu\nu\rho}j^{\mu\,MN}j^\nu{}_N{}^K\partial_{MK}\xi^\rho+\frac{k_1}{2\,k_2}\,\varepsilon^{\mu\nu\rho}j_\nu{}^K{}_N j_\mu{}^{NL}g_{\rho\sigma}\partial_{KL}\xi^\sigma
\end{aligned}
\tag{3.19}
$$

Terms in the last line cannot be cancelled by any contribution coming from the scalar potential and hence must cancel each other, for which we must choose $k_1=2$.

To see the cancellations coming from the variation of the scalar potential let us look only at the relevant terms inside variation of the potential (3.6). First it is useful to write first variations of the scalar current $\mathcal{J}_{MN}{}^{KL}$ and of the derivative $\partial_{MN}g_{\mu\nu}$ that read

$$
\begin{aligned}
\delta_\xi\big(\mathcal{J}_{MN}{}^K{}_L\big)&=\xi^\mu\mathcal{D}_\mu\big(\mathcal{J}_{MN}{}^K{}_L\big)+\partial_{MN}\xi^\mu\,j_\mu{}^K{}_L, \\
\delta_\xi\big(\partial_{MN}g_{\mu\nu}\big)&=\mathcal{L}_\xi\big(\partial_{MN}g_{\mu\nu}\big)+\partial_{MN}\,\xi^\rho\mathcal{D}_\rho g_{\mu\nu}+2\big(\partial_{MN}\mathcal{D}_{(\mu}\xi^\rho\big)g_{\nu)\rho}.
\end{aligned}
\tag{3.20}
$$

The first term in each line is a covariant variation, while the remaining parts give the non-covariant variation of the scalar potential. Since the full cancellations work precisely like in the $E_{8(8)}$ theory [13] there is no need to repeat the full derivation here. Let us check the most indicative terms to fix the coefficients and to check the consistency. For that we consider the following contribution from the non-covariant variation of $\mathcal{L}_{\rm pot}$

$$
-k_3\,\delta\left(\sqrt{-g}\,V\right)=-k_3\,\delta^{\rm cov}\left(\sqrt{-g}\,V\right)-\frac{k_3}{2}\sqrt{-g}\,\partial_{KL}\xi^\mu j_\mu{}^{MN}\mathcal{J}_{MN}{}^{KL}+\ldots, \tag{3.21}
$$

whose cancellation against the corresponding term in (3.19) forces us to set $k_3 = 2k_1\beta_1 = 1$. We have now fixed all the unknown coefficients in (3.1) and (3.8)

$$k_1 = 2\,, \quad k_2 = \sqrt{2}\,, \quad k_3 = 1\,, \qquad \beta_1 = \frac{1}{4}\,, \quad \beta_2 = \frac{1}{2\sqrt{2}}\,. \tag{3.22}$$

After these numerical values ensure all the above cancellations to take place we are finally left with the following variation of the full Lagrangian

$$\begin{aligned}\delta_\xi^{(0)}\Big(\mathcal{L}_{\rm EH} + 2\,\mathcal{L}_{\rm kin} + \sqrt{2}\,\mathcal{L}_{\rm CS} + \mathcal{L}_{\rm pot}\Big) &= -\frac{1}{2\sqrt{2}}\varepsilon^{\mu\nu\rho}\partial_{MK}\xi^\sigma\mathcal{E}^{(\mathcal{A})MN}_{\sigma\rho}\mathcal{E}^{(\mathcal{A})}_{\mu\nu}{}_N{}^K \\ &= \frac{1}{\sqrt{2}}\varepsilon^{\mu\nu\rho}\xi^\sigma\partial_{MK}\mathcal{E}^{(\mathcal{A})MN}_{\sigma\rho}\mathcal{E}^{(\mathcal{A})}_{\mu\nu}{}_N{}^K\,,\end{aligned} \tag{3.23}$$

up to total derivatives. To get rid of this remnant, we perform the same trick as in [13] and define the full diffeomorphism transformation of the gauge fields as the following deformation of the initial ansatz (3.8)

$$\begin{aligned}\delta_\xi\mathcal{A}_\mu{}^{MN} &= \delta_\xi^{(0)}\mathcal{A}_\mu{}^{MN} + \frac{1}{\sqrt{2}}\xi^\nu\mathcal{E}^{(\mathcal{A})MN}_{\mu\nu} \\ \delta_\xi\mathcal{B}_{\mu\,MN} &= \delta_\xi^{(0)}\mathcal{B}_{\mu\,MN} + \frac{1}{\sqrt{2}}\xi^\nu\Big(\mathcal{E}^{(\mathcal{B})}_{\mu\nu}{}_{MN} - \frac{1}{8}f_{MN,KL}{}^{PQ}\partial_{PQ}\mathcal{E}^{(\mathcal{A})KL}_{\mu\nu}\Big) \\ &= \delta_\xi^{(0)}\mathcal{B}_{\mu\,MN} + \frac{1}{\sqrt{2}}\xi^\nu\Big(\mathcal{E}^{(\mathcal{B})}_{\mu\nu}{}_{MN} - \partial_{K[M}\mathcal{E}^{(\mathcal{A})K}_{\mu\nu}{}_{N]}\Big).\end{aligned} \tag{3.24}$$

Indeed, according to (3.9) and the above discussion, the variations $\delta_\xi^{(0)}$ provide the contribution (3.23) which cancels against the term coming from the $\partial\mathcal{E}^{(\mathcal{A})}$ in the second line. The new contributions of the form $\mathcal{E}^{(\mathcal{A})}\cdot\mathcal{E}^{(\mathcal{B})}$ cancel each other as they form an expression totally antisymmetric in four space-time indices. The mutual factor in the brackets of the second and the last line above was chosen in such a way as to keep $\delta_\xi\mathcal{B}_{\mu\,MN}$ satisfying the same section constraints as the field $\mathcal{B}_{\mu\,MN}$ does.

Hence, the full diffeomorphism transformations leaving the theory invariant can be collected as follows

$$\begin{aligned}\delta e_\mu^a &= \xi^\mu\mathcal{D}_\nu e_\mu^a + \mathcal{D}_\mu\xi^\nu e_\nu^a\,, \qquad \delta\mathcal{M}_{MN} = \xi^\mu\mathcal{D}_\mu\mathcal{M}_{MN}\,, \\ \delta_\xi\mathcal{A}_\mu{}^{MN} &= -\sqrt{-g}\,\xi^\nu\varepsilon_{\mu\nu\sigma}j^{\sigma MN} + g_{\mu\nu}\,\mathcal{M}^{MK}\mathcal{M}^{NL}\partial_{KL}\xi^\nu\,, \\ \delta_\xi\mathcal{B}_{\mu\,MN} &= \sqrt{-g}\,\varepsilon_{\mu\nu\rho}\Big(\frac{1}{2\sqrt{2}}g^{\lambda\rho}\mathcal{D}^\nu(g_{\lambda\sigma}\partial_{MN}\xi^\sigma) + \xi^\nu\hat{j}^\rho{}_{MN} - \frac{1}{8}\xi^\nu j^\rho{}_{KL}\mathcal{J}_{MN}{}^{KL}\Big) \\ &\quad + \frac{1}{2}\,g_{\mu\nu}\,\mathcal{J}_{MN}{}^{KL}\partial_{KL}\xi^\nu\,,\end{aligned} \tag{3.25}$$

that have precisely the same form as the ones in [13] as expected. The final Lagrangian then becomes

$$\mathcal{L} = \mathcal{L}_{\rm EH} + 2\,\mathcal{L}_{\rm kin} + \sqrt{2}\,\mathcal{L}_{\rm CS} + \mathcal{L}_{\rm pot}\,, \tag{3.26}$$

with all relative coefficients now fixed by invariance under external diffeomorphisms (3.25).

### 3.3 Solutions of the section constraint

Let us now discuss the explicit solutions of the section constraint (2.6). We will identify two inequivalent solutions that essentially correspond to the embedding of $D = 6$ non-chiral and chiral theories, respectively.

For the first solution, we start from the theory based on $\mathrm{O}(d{+}1, d{+}1{+}n)$ and consider its decomposition under $\mathrm{GL}(d)$ embedded as

$$\mathrm{O}(d+1, d+1+n) \supset \mathrm{O}(d, d) \supset \mathrm{GL}(d)\ , \tag{3.27}$$

with fundamental vectors breaking into

$$\{V^M\} \longrightarrow \left\{V^i, V^0, V_i, V_0, \tilde{V}^p\right\}\ , \quad i = 1, \dots, d\ , \quad p = 1, \dots, n\ , \tag{3.28}$$

and a Cartan-Killing form

$$\eta_{MN} = \begin{pmatrix} 0_{d\times d} & 0 & \delta_i{}^j & 0 & 0_{d\times n} \\ 0 & 0 & 0 & 1 & 0 \\ \delta^i{}_j & 0 & 0_{d\times d} & 0 & 0_{d\times n} \\ 0 & 1 & 0 & 0 & 0 \\ 0_{n\times d} & 0 & 0_{n\times d} & 0 & \mathbb{I}_{n\times n} \end{pmatrix}\ . \tag{3.29}$$

It is then straightforward to see that restricting all fields to depend exclusively on $d$ coordinates $y^i$ defined as

$$\left\{y^i \equiv Y^{i0}\right\}\ , \qquad \Phi(x^\mu, Y^{MN}) = \Phi(x^\mu, y^i)\ , \tag{3.30}$$

constitutes a solution to (2.6).[4] Upon evaluating the above constructed theory for this solution of the section constraint, it reproduces the field equations of the bosonic string in $d + 3$ dimensions, coupled to $n$ abelian vectors, i.e., for $n = 16$ the field equations of the heterotic string truncated to the Cartan subalgebra of the full gauge group.

An alternative solution to the section constraints (2.6) is found by starting from the theory based on $\mathrm{O}(3 + n_\mathrm{L}, 3 + n_\mathrm{R})$ and decomposing it under a $\mathrm{GL}(3)$ embedded as

$$\mathrm{O}(3 + n_\mathrm{L}, 3 + n_\mathrm{R}) \supset \mathrm{O}(3, 3) \supset \mathrm{GL}(3)\ , \tag{3.31}$$

with fundamental vectors breaking into

$$\begin{aligned} \{V^M\} \longrightarrow \left\{V^i, V_i, \tilde{V}^p, \bar{V}^q\right\}\ ,& \\ i = 1, \dots, 3\ , \quad p = 1, \dots, n_\mathrm{L}\ , \quad q = 1, \dots, n_\mathrm{R}\ ,& \end{aligned} \tag{3.32}$$

and a Cartan-Killing form

$$\eta_{MN} = \begin{pmatrix} 0_{d\times d} & \delta_i{}^j & 0_{d\times n_\mathrm{L}} & 0_{d\times n_\mathrm{R}} \\ \delta^i{}_j & 0_{d\times d} & 0_{d\times n_\mathrm{L}} & 0_{d\times n_\mathrm{R}} \\ 0_{n\times d} & 0_{n\times d} & \mathbb{I}_{n_\mathrm{L}\times n_\mathrm{L}} & 0_{n_\mathrm{L}\times n_\mathrm{R}} \\ 0_{n\times d} & 0_{n\times d} & 0_{n_\mathrm{R}\times n_\mathrm{L}} & -\mathbb{I}_{n_\mathrm{R}\times n_\mathrm{R}} \end{pmatrix}\ . \tag{3.33}$$

[4] More elaborately, we could in a first step have broken down $\mathrm{O}(d{+}1, d{+}1{+}n)$ to $\mathrm{O}(d, d)$ and selected coordinates $\{Y^I\} \equiv \{Y^{i0}, Y_{i0}\}$, such that the section constraints (2.6) reduce to $\eta^{IJ}\,\partial_I \otimes \partial_J = 0$ and reproduce the structures of standard double field theory. In a second step, this remaining section constraint is then solved by (3.30).

Restricting all fields to depend exclusively on coordinates $\tilde{y}_i$ defined as

$$\left\{\tilde{y}_i \equiv \varepsilon_{ijk} Y^{jk}\right\} , \qquad \Phi(x^\mu, Y^{MN}) = \Phi(x^\mu, \tilde{y}_i) , \tag{3.34}$$

again constitutes a solution to (2.6). In this case, the above constructed theory reproduces the field equations of $D = 6$ gravity, coupled to $n_{\rm L}$ selfdual and $n_{\rm R}$ anti-selfdual antisymmetric two-form tensors, as well as to $n_{\rm L} \cdot n_{\rm R}$ scalar fields.[5] Indeed, it follows from inspection that fields depending on the full set of coordinates $\{\tilde{y}^i\}$ cannot depend on any further internal coordinate without violating the section constraints (2.6). The resulting theory cannot be lifted beyond six dimensions which is the case for the chiral theories coupling (anti-)selfdual tensor fields.[6]

Comparing the two solutions (3.30), (3.34) it is obvious that for $d \leq 2$ the coordinates (3.30) can be considered as a subset of (3.34). Indeed, in this case the $D \leq 5$ theories described by (3.30) are obtained by dimensional reduction (and possible truncation) from the $D = 6$ theories described by (3.34). The two solutions thus are not independent. For $d > 3$ on the other hand, the different choices of coordinates are inequivalent (as discussed, the set of coordinates (3.34) cannot be extended without violating the section constraints (2.6), thus never be equivalent to the $d > 3$ coordinates (3.30)) — and so are the resulting higher-dimensional theories. An interesting case is the theory with $d = 3$, $n = 0$ (i.e. $n_{\rm L} = n_{\rm R} = 1$), built on the group ${\rm O}(4,4)$. In this case, the two choices of coordinates (3.30) and (3.34) can be shown to be related by an outer automorphism (a triality rotation) of ${\rm SO}(4,4)$, they hence describe equivalent theories. Indeed, the $D = 6$ theory from (3.34) coupling gravity to one selfdual tensor, one anti-selfdual tensor, and a scalar field, is precisely the bosonic string described by (3.30). We will come back to this equivalence later when discussing Scherk-Schwarz reductions and consistent truncations.

Let us finally discuss two important series of theories, based on the groups ${\rm O}(4,n)$ and ${\rm O}(8,n)$, respectively. These theories can be supersymmetrized upon adding fermionic fields into half-maximal and quarter-maximal field theories, respectively. According to the above discussion, the ${\rm O}(4,4)$ theory has a unique solution of the section constraint which describes the embedding of the $D = 6$, $\mathcal{N} = (1,0)$ supergravity coupled to one tensor multiplet, such that its full field content and couplings are non-chiral. The theories built from ${\rm O}(4,4+n)$, $n > 0$, on the other hand admit two inequivalent solutions (3.30), (3.34) of the section constraint, describing the coupling of $\mathcal{N} = (1,0)$ vector multiplets and chiral tensor multiplets, respectively, to this $D = 6$ supergravity. The ${\rm O}(8,n)$ theories can be supersymmetrized into half-maximal field theories. For these theories, the solution (3.30) of the section constraint, describes the embedding of $D = (2+n)$ half-maximal supergravity for $n \leq 8$ and of $D = 10$, $\mathcal{N} = 1$ supergravity with $n - 8$ vector multiplets for $n \geq 8$,

[5] In particular, the special case $n_{\rm L} = n_{\rm R} = 0$ corresponds to pure $D = 6$ gravity with ${\rm SO}(3,3) \sim {\rm SL}(4)$ encoding the Ehlers symmetry group upon reduction to three dimensions. The gauge structure and section constraints in this case have also been considered in [17, 27].

[6] Similarly, the section constraints in exceptional field theory in general admit two inequivalent solutions corresponding to a higher-dimensional IIA and IIB origin [18]. Specifically, the two solutions (3.30), (3.34) are based on different embeddings of ${\rm GL}(3)$ into ${\rm SO}(3,3)$, in analogy to the two inequivalent solutions in ${\rm SL}(5)$ exceptional field theory [29].



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| | | | | | |
|---|---|---|---|---|---|
| $\mathrm{O}(d{+}1,d{+}1{+}n)$<br>$Y^{MN}$ | $\hookleftarrow$<br>$\longrightarrow$ | $\mathrm{O}(d,d)$<br>$(Y^{i0},Y_{i0})$ | $\hookleftarrow$<br>$\longrightarrow$ | $\mathrm{GL}(d)$<br>$y^i{=}Y^{i0}$ | $D=d+3$ bosonic string,<br>with (3.30) and $n_{\mathrm{V}}=n$ |
| $\mathrm{O}(3{+}n_L,3{+}n_R)$<br>$Y^{MN}$ | $\hookleftarrow$<br>$\longrightarrow$ | $\mathrm{O}(3,3)$<br>$(Y^{i0},Y_{i0})$ | $\hookleftarrow$<br>$\longrightarrow$ | $\mathrm{GL}(3)$<br>$y^i{=}Y^{i0}$ | $D=6$ gravity,<br>with (3.34) and $n_\pm=n_{\mathrm{R,L}}$ |
| $\mathrm{O}(4,4+n)$ | $\hookleftarrow$ | $\mathrm{O}(3,3)$ | $\hookleftarrow$ | $\mathrm{GL}(3)$ | $D=6$ bosonic string,<br>for (3.30): with $n_{\mathrm{V}}=n$<br>for (3.34): with $n_-{=}n$<br>upon adding fermions: $\frac{1}{4}$ SUSY |
| $\mathrm{O}(8,n+1)$ | $\hookleftarrow$ | $\mathrm{O}(7,7)$ | $\hookleftarrow$ | $\mathrm{GL}(7)$ | for (3.30): $D=10$ bosonic string,<br>with $n_{\mathrm{V}}=n-7$ for $n\geq 7$<br>upon adding fermions: $\frac{1}{2}$ SUSY |
| $\mathrm{O}(8,n+1)$ | $\hookleftarrow$ | $\mathrm{O}(n,n)$ | $\hookleftarrow$ | $\mathrm{GL}(n)$ | for (3.30): bosonic sector of<br>$D=n+3$ sugra, for $n\leq 7$<br>upon adding fermions: $\frac{1}{2}$ SUSY |
| $\mathrm{O}(8,n+1)$ | $\hookleftarrow$ | $\mathrm{O}(3,3)$ | $\hookleftarrow$ | $\mathrm{GL}(3)$ | for (3.34): $D=6$,<br>bosonic sector of $\mathcal{N}=(2,0)$ sugra,<br>$n-2$ tensor multiplets |

**Table 1**. Table of gravitational theories which can be embedded into the present construction together with the corresponding solutions of the section constraint. Notations are the following: $n_{\mathrm{s}}$ — number of scalar multiplets, $n_{\mathrm{V}}$ — number of abelian vector multiplets, $n_\pm$ — number of (anti)self-dual 2-forms.

respectively. The solution (3.34) on the other hand describes the embedding of $D=6$, $\mathcal{N}=(2,0)$ chiral supergravity coupled to $n-3$ tensor multiplets. In accordance with the above counting, every one of these multiplets combines a selfdual tensor with five scalar fields while the $\mathcal{N}=(2,0)$ supergravity multiplet carries five anti-selfdual tensors.

Table 1 summarizes the embedding of the various higher-dimensional theories. For completeness, let us mention that the theory based on the group $\mathrm{O}(2,1)$ constructed in [14] which describes pure $D=4$ gravity with the Ehlers group made manifest, does not seem to fit in the present construction. This is seen from the fact the section constraints (2.6) for $\mathrm{O}(2,1)$ do not admit any solution whereas the construction of [14] is based on a weaker version of the section constraints (suppressing only the $\mathbf{1}\oplus\mathbf{3}$ in $\mathbf{3}\otimes\mathbf{3}$) which allows for a one-dimensional solution.

## 4 Generalized Scherk-Schwarz reduction

In this section, we study reductions of the $\mathrm{O}(p,q)$ enhanced double field theory via a generalized Scherk-Schwarz ansatz [22, 23, 30–36]. We derive the consistency conditions on the Scherk-Schwarz twist matrices and rephrase them as a generalized parallelizability condition. The particular structure of generalized diffeomorphisms (2.9) and in particular the presence of constrained rotations in the diffeomorphism algebra requires a modification of the standard constructions. We discuss in some detail the structure of three-dimensional



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gauge theories obtained by these generalized Scherk-Schwarz reductions. We finally decompose the system of compatibility equations according to the solution (3.30) of the section constraints and reproduce as a particular case the structures known from $\mathrm{SL}(d+1)$ generalized geometry. In turn, this allows to employ known solutions of this system in order to describe consistent truncations to three dimensions.

### 4.1 Reduction ansatz and consistency equations

The generalised Scherk-Schwarz reduction ansatz is encoded in an $\mathrm{O}(p,q)$ matrix $U_M{}^{\bar N}(Y)$ and a weight factor $\rho(Y)$. As in exceptional field theory [23], we impose the following reduction ansatz on the fields

$$\begin{aligned} g_{\mu\nu}(x,Y) &= \rho(Y)^{-2}\, \mathrm{g}_{\mu\nu}(x) \,, \\ \mathcal{M}_{MN}(x,Y) &= U_M{}^{\bar M}(Y)\, U_N{}^{\bar N}(Y)\, \mathcal{M}_{\bar M\bar N}(x) \,, \\ \mathcal{A}_\mu{}^{MN}(x,Y) &= \rho(Y)^{-1}\, U^M{}_{\bar M}(Y)\, U^N{}_{\bar N}(Y)\, A_\mu{}^{\bar M\bar N}(x) \,, \\ \mathcal{B}_{\mu\, KL}(x,Y) &= -\frac{1}{4}\, \rho(Y)^{-1}\, U^M{}_{\bar N}(Y)\, \partial_{KL} U_{M\bar M}(Y)\, A_\mu{}^{\bar M\bar N}(x) \,. \end{aligned} \tag{4.1}$$

Fundamental indices on the twist matrix are raised and lowered with the invariant tensor $\eta_{MN}$, such that in particular $U_M{}^{\bar M} U^{M\bar N} = \eta^{\bar M\bar N}$ . Note that the ansatz for the constrained gauge connection $\mathcal{B}_{\mu\, KL}$ is manifestly compatible with the constraints (2.45). The gauge parameters $\Lambda^{MN}$, $\Sigma_{MN}$ associated with $\mathcal{A}_\mu{}^{MN}$, $\mathcal{B}_{\mu\, MN}$ factor accordingly

$$\begin{aligned} \Lambda^{MN}(x,Y) &= \rho(Y)^{-1} U^M{}_{\bar M}(Y) U^N{}_{\bar N}(Y) \Lambda^{\bar M\bar N}(x) \,, \\ \Sigma_{KL}(x,Y) &= -\frac{1}{4}\, \rho(Y)^{-1}\, U_{M\bar N}(Y)\, \partial_{KL} U^M{}_{\bar M}(Y)\, \Lambda^{\bar M\bar N}(x) \,. \end{aligned} \tag{4.2}$$

The consistency constraints on the twist matrix are straightforwardly obtained by working out the gauge transformations of these objects. E.g. we find that

$$\begin{aligned} \mathcal{L}_{(\Lambda,\Sigma)}\, g_{\mu\nu} &= 2\,\rho^{-2} \left(\Lambda^{\bar K\bar L}\, \theta_{\bar K\bar L}\, \mathrm{g}_{\mu\nu}\right) \,, \\ \mathcal{L}_{(\Lambda,\Sigma)}\, \mathcal{M}_{MN} &= -2\, U_M{}^{\bar M} U_N{}^{\bar N} \left(\Lambda^{\bar K\bar L}\, X_{\bar K\bar L,(\bar M}{}^{\bar Q}\, \mathcal{M}_{\bar N)\bar Q}\right) \,, \end{aligned} \tag{4.3}$$

where the embedding tensor $X_{\bar K\bar L,\bar M}{}^{\bar N}$ captures the gauge structure of the three-dimensional theory, and is given by

$$X_{\bar K\bar L,\bar P\bar Q} = \theta_{\bar K\bar L\bar P\bar Q} + \frac{1}{2}\left(\eta_{\bar P[\bar K}\theta_{\bar L]\bar Q} - \eta_{\bar Q[\bar K}\theta_{\bar L]\bar P}\right) + \theta\, \eta_{\bar P[\bar K}\eta_{\bar L]\bar Q} \,, \tag{4.4}$$

with the various components defined in terms of the twist matrix as

$$\begin{aligned} \theta_{\bar K\bar L\bar P\bar Q} &= 6\,\rho^{-1}\, \partial_{LP} U_{N\,[\bar K} U^N{}_{\bar L} U^L{}_{\bar P} U^P{}_{\bar Q]} \,, \\ \theta_{\bar P\bar Q} &= 4\,\rho^{-1}\, U^K{}_{\bar P}\, \partial_{KL} U^L{}_{\bar Q} - \frac{4\,\rho^{-1}}{p+q}\, \eta_{\bar P\bar Q} U^{K\bar L} \partial_{KL} U^L{}_{\bar L} - 4\rho^{-2}\, \partial_{\bar P\bar Q}\rho \,, \\ \theta &= \frac{4\,\rho^{-1}}{p+q}\, U^{K\bar L} \partial_{KL} U^L{}_{\bar L} \,. \end{aligned} \tag{4.5}$$

The truncation (4.1) thus is consistent, if all the components (4.5) of the embedding tensor are constant, i.e.

$$\partial_{\bar{M}}\theta_{\bar{K}\bar{L}\bar{P}\bar{Q}} = 0 = \partial_{\bar{M}}\theta_{\bar{K}\bar{L}} = \partial_{\bar{M}}\theta \,. \tag{4.6}$$

This provides a set of differential equations on the twist matrix and the weight factor which encodes the consistency of the truncation. In terms of $\mathrm{O}(p,q)$ representations, the components (4.5) of the embedding tensor transform as

$$\ydiagram{1,1} \otimes \ydiagram{1,1} \longrightarrow \bullet \oplus \ydiagram{1,1} \oplus \ydiagram{2} \oplus \ydiagram{1,1,1,1} \;, \tag{4.7}$$

in a subrepresentation of the full tensor product (2.79).

For those theories admitting a supersymmetric embedding (i.e. the $\mathrm{O}(p,q)$ enhanced double field theories with $p = 2,4,8$), the structure (4.3), (4.4) precisely reproduces the gauge structure of the associated three-dimensional gauged supergravities [37]. Here, that same structure appears more generally for an arbitrary group $\mathrm{O}(p,q)$. The anti-symmetric tensor $\theta_{[\bar{P}\bar{Q}]}$ triggers three-dimensional gaugings in which the trombone scaling symmetry is part of the gauge group [38]. This follows directly from the first line of (4.3): a non-vanishining $\theta_{[\bar{P}\bar{Q}]}$ implies that the three-dimensional metric $g_{\mu\nu}$ is charged under part of the gauge group. The resulting theories do not admit a three-dimensional action and are defined only on the level of the field equations. For most of the following discussions we will thus require that $\theta_{[\bar{P}\bar{Q}]} = 0$.

In a generic three-dimensional gauge theory, the embedding tensor (4.4) is subject to the quadratic constraints

$$X_{\bar{K}\bar{L}\bar{P}}{}^{\bar{R}}X_{\bar{M}\bar{N}\bar{R}}{}^{\bar{Q}} - X_{\bar{M}\bar{N}\bar{P}}{}^{\bar{R}}X_{\bar{K}\bar{L}\bar{R}}{}^{\bar{Q}} = 2\, X_{\bar{K}\bar{L}[\bar{M}}{}^{\bar{R}}X_{\bar{N}]\bar{R}\bar{P}}{}^{\bar{Q}} \,, \tag{4.8}$$

which guarantees closure of the gauge algebra. With the embedding tensor defined by a twist matrix as (4.5), these constraints follow directly from the section constraint (2.6). Note that the section constraint combined with (4.5) furthermore implies that

$$\theta_{[\bar{N}_1\ldots\bar{N}_4}\theta_{\bar{N}_5\ldots\bar{N}_8]} = 0 \,. \tag{4.9}$$

I.e. the generalized Scherk-Schwarz ansatz with twist matrices that obey the section condition can only reproduce gaugings whose embedding tensor satisfies the additional quadratic condition (4.9). This is consistent with the fact, that the general potential of $D = 3$ half-maximal supergravity carries a term proportional to $\theta_{\bar{N}_1\ldots\bar{N}_4}\theta_{\bar{N}_5\ldots\bar{N}_8}\,\mathcal{M}^{\bar{N}_1\ldots\bar{N}_8}$, with a scalar dependent totally antisymmetric tensor $\mathcal{M}^{\bar{N}_1\ldots\bar{N}_8}$ [39], that is not reproduced by the Scherk-Schwarz ansatz from the scalar potential given in (3.6).

### 4.2 Generalized parallelizability

Here we discuss the notion of *generalized parallelizability* outlined in the introduction, which gives a more 'geometric' perspective on the consistency conditions on the twist matrices discussed above. We claim that for the doubled tensor (in the sense of (2.25))

$$\mathfrak{U}_{\bar{M}\bar{N}} \equiv \left(\rho^{-1}U^K{}_{[\bar{M}}U^L{}_{\bar{N}]} \,,\; -\frac{1}{4}\,\rho^{-1}(\partial_{KL}U^P{}_{\bar{M}})U_{P\bar{N}}\right) , \tag{4.10}$$

which is manifestly compatible with the constraints on the second component by having the indices $KL$ be carried by a derivative, the consistency conditions can be stated simply in terms of the (generalized) Dorfman product (2.26) as

$$\mathfrak{U}_{\bar{M}\bar{N}} \circ \mathfrak{U}_{\bar{K}\bar{L}} = -X_{\bar{M}\bar{N},\bar{K}\bar{L}}{}^{\bar{P}\bar{Q}}\, \mathfrak{U}_{\bar{P}\bar{Q}}\ . \tag{4.11}$$

Here $X$ is the constant embedding tensor.

We will now show that for the gauge vectors and its associated gauge symmetries the consistency of the Scherk-Schwarz ansatz is an immediate consequence of the fact that all relations are governed by the same Dorfman product '$\circ$' satisfying the Leibniz identity (2.43). We make the following Scherk-Schwarz ansatz for gauge fields and parameters:

$$\begin{aligned} \mathfrak{A}_\mu(x,Y) &= \mathfrak{U}_{\bar{M}\bar{N}}(Y)\, A_\mu{}^{\bar{M}\bar{N}}(x)\ , \\ \Upsilon(x,Y) &= \mathfrak{U}_{\bar{M}\bar{N}}(Y)\, \Lambda^{\bar{M}\bar{N}}(x)\ . \end{aligned} \tag{4.12}$$

It immediately follows with (2.44) and (4.1) that this is equivalent to the Scherk-Schwarz ansatz given above for the vector components. Let us now consider the gauge transformation of the Scherk-Schwarz ansatz:

$$\begin{aligned} \delta_\Upsilon \mathfrak{A}_\mu(x,Y) &= \partial_\mu \Upsilon - \mathfrak{A}_\mu \circ \Upsilon \\ &= \mathfrak{U}_{\bar{M}\bar{N}}\, \partial_\mu \Lambda^{\bar{M}\bar{N}} - \mathfrak{U}_{\bar{K}\bar{L}} \circ \mathfrak{U}_{\bar{P}\bar{Q}}\, A_\mu{}^{\bar{K}\bar{L}} \Lambda^{\bar{P}\bar{Q}} \\ &= \mathfrak{U}_{\bar{M}\bar{N}} \big(\partial_\mu \Lambda^{\bar{M}\bar{N}} + X_{\bar{K}\bar{L},\bar{P}\bar{Q}}{}^{\bar{M}\bar{N}} A_\mu{}^{\bar{K}\bar{L}} \Lambda^{\bar{P}\bar{Q}}\big) \\ &= \mathfrak{U}_{\bar{M}\bar{N}}\, \delta_\Lambda A_\mu{}^{\bar{M}\bar{N}}\ , \end{aligned} \tag{4.13}$$

where we used (4.11) and defined in the last line

$$\delta_\Lambda A_\mu{}^{\bar{M}\bar{N}} = \partial_\mu \Lambda^{\bar{M}\bar{N}} + X_{\bar{K}\bar{L},\bar{P}\bar{Q}}{}^{\bar{M}\bar{N}} A_\mu{}^{\bar{K}\bar{L}} \Lambda^{\bar{P}\bar{Q}}\ . \tag{4.14}$$

In here the $Y$-dependence encoded in $\mathfrak{U}(Y)$ has factored out, and this is precisely the expected gauge transformation in gauged supergravity. Thus, the gauge transformations reduce consistently under Scherk-Schwarz. Similarly, one may show for all objects defined in terms of the Dorfman product, such as the non-abelian field strengths (2.54), that they reduce consistently under Scherk-Schwarz. In general, the consistency conditions on the twist matrix are fully encoded in the algebra property (4.11).

### 4.3 GL($d+1$) twist equations

In the following, we will be interested in constructing explicit solutions to the consistency equations (4.6). Obviously, the precise content of these equations will depend on the solution of the section constraints (2.6), i.e. on the choice of physical coordinates among the $\{Y^M\}$. We have discussed the different choices in subsection 3.3 above. Let us stress that in this paper we will only be interested in constructing twist matrices that satisfy the section conditions (2.6), i.e. in constructing consistent truncations from actual higher-dimensional supergravities. It is known [23, 33, 35] that the match with lower-dimensional gauged supergravity formally holds even in the case the section constraint is replaced by the weaker quadratic constraint on the resulting embedding tensor (provided the initial scalar potential is written in an appropriate form). On the other hand the higher-dimensional origin of the resulting gaugings within a well-defined theory remains mysterious.

As an ansatz for the solutions constructed in this section, we consider Scherk-Schwarz twist matrices $U_M{}^{\bar{N}}(Y)$ living in the maximal $\mathrm{GL}(d+1)$ subgroup of $\mathrm{O}(d+1,d+1)$, i.e. of the explicit type

$$U_M{}^{\bar{M}} = \left[\begin{array}{c:c} \varphi\, V_A{}^{\bar{A}} & 0 \\ \hdashline 0 & \varphi^{-1}\,(V^{-1})_{\bar{A}}{}^{A} \end{array}\right] , \tag{4.15}$$

with an $\mathrm{SL}(d+1)$ matrix $V_A{}^{\bar{A}}$ and a scalar function $\varphi$. Under this subgroup, the extended coordinates decompose as

$$\{Y^{MN}\} \longrightarrow \{Y^{AB},\, Y_{AB},\, Y_A{}^B\} , \tag{4.16}$$

with the indices $A, B = 1, \ldots, d+1$ labelling the fundamental representation of $\mathrm{SL}(d+1)$. We moreover restrict the physical coordinates to $\{Y^{AB}\}$, suppressing all dependence on $\{Y_{AB}, Y_A{}^B\}$. This restriction is compatible with the choice (3.30) of physical coordinates. What we will show in the following is that with this ansatz the consistency equations (4.5)–(4.6) can be reduced to the $\mathrm{SL}(d+1)$ system of equations that has ben solved in [23] with solutions corresponding to sphere and hyperboloid geometries.

Rather than directly plugging the ansatz (4.15) into the consistency equations (4.5)–(4.6), it is useful to first analyze the representation content of the latter. With the ansatz (4.15), the consistency equations (4.5) turn into equations linear in the currents

$$\begin{aligned} J_{\bar{A}\bar{B},\bar{C}}{}^{\bar{D}} &\equiv (V^{-1})_{\bar{A}}{}^{A}(V^{-1})_{\bar{B}}{}^{B}\,(V^{-1})_{\bar{C}}{}^{C}\,\partial_{AB}V_C{}^{\bar{D}} , \\ j_{\bar{A}\bar{B}} &\equiv \varphi^{-1}\,(V^{-1})_{\bar{A}}{}^{A}(V^{-1})_{\bar{B}}{}^{B}\,\partial_{AB}\varphi , \end{aligned} \tag{4.17}$$

which under $\mathrm{SL}(d+1)$ transform in the representations

$$\begin{aligned} J_{\bar{A}\bar{B},\bar{C}}{}^{\bar{D}} &: [0,1,0,\ldots,0] \,\oplus\, [2,0,0,\ldots,0] \,\oplus\, [0,0,1,\ldots,0,1] \,\oplus\, [1,1,0,\ldots,0,1] , \\ j_{\bar{A}\bar{B}} &: [0,1,0,\ldots,0] , \end{aligned} \tag{4.18}$$

denoted by their standard Dynkin labels. We may trace back the appearance of the various components of these currents within the various components of the consistency equations by decomposing the $\mathrm{O}(d+1,d+1)$ representations (4.7) of the latter under $\mathrm{SL}(d+1)$. Specifically, we find that the different components of the consistency equations (4.5)–(4.6) accommodate the following components of the currents (4.18)

$$\begin{aligned} \bullet &\longrightarrow - \\ \yng(1,1) &\longrightarrow [0,1,0,\ldots] \\ \yng(2) &\longrightarrow [2,0,0,\ldots] \\ \yng(1,1,1,1) &\longrightarrow [0,1,0,\ldots] + [0,0,1,0,\ldots,0,1] \quad . \end{aligned} \tag{4.19}$$



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We thus conclude that the consistency equations (4.5)–(4.6) translate into

$$
J_{\bar{A}\bar{B},\bar{C}}{}^{\bar{D}}\Big|_{[2,0,0,\dots,0]\,\oplus\,[0,0,1,\dots,0,1]} = \text{const.}\,, \tag{4.20}
$$

together with two equations combining $j_{\bar{A}\bar{B}}$ with the projection $J_{\bar{A}\bar{B},\bar{C}}{}^{\bar{D}}|_{[0,1,0,\dots,0]}$ which take the explicit form

$$
\begin{aligned}
-\rho^{-1}\varphi^{-2}\Big(\partial_{AB}(V^{-1})_{\bar{A}\bar{B}}{}^{AB} + (d-1)\,(V^{-1})_{\bar{A}\bar{B}}{}^{AB}\,\partial_{AB}\ln\varphi\Big) &= \theta_{\bar{A}\bar{B}\bar{C}}{}^{\bar{C}} \overset{!}{=} \text{const}\,,\\
2\,\rho^{-1}\varphi^{-2}\left(\partial_{AB}(V^{-1})_{\bar{A}\bar{B}}{}^{AB} - 2\,(V^{-1})_{\bar{A}\bar{B}}{}^{AB}\,\partial_{AB}\ln(\varphi\rho)\right) &= \theta_{[\bar{A}\bar{B}]} \overset{!}{=} \text{const}\,.
\end{aligned} \tag{4.21}
$$

It follows that with the ansatz

$$
\rho = \varphi^{-(d+1)/2}\,, \tag{4.22}
$$

for the weight factor $\rho$, these two equations coincide and the full system (4.20)–(4.21) of consistency equations reproduces the $\mathrm{SL}(d+1)$ consistency equations solved in [23] for sphere and hyperboloid compactifications. In particular, for these solutions $\theta_{[\bar{A}\bar{B}]} = 0 = \theta_{\bar{A}\bar{B}\bar{C}}{}^{\bar{C}}$. Translating the solutions of [23] into our conventions here, we identify physical coordinates $\{y^i\}$, $i = 1,\dots,d$, as (3.30) among the $Y^{AB}$ and accordingly split the upper left block of (4.15) as

$$
\begin{aligned}
\varphi V_A{}^{\bar{A}} &= \begin{pmatrix} \varphi V_0{}^0 & \varphi V_0{}^j \\ \varphi V_i{}^0 & \varphi V_i{}^j \end{pmatrix}\\
&= \left(\begin{array}{c|c} (1-u)^{-1}\,(1+u\,k(u)) & -y^j\,(1-u)^{-1/2}\,k(u) \\ \hline -y^i\,(1-u)^{-1/2} & \delta_i{}^j \end{array}\right)\,,
\end{aligned} \tag{4.23}
$$

with $u \equiv y^i y^i$, and with a scalar function $k(u)$ found as a solution of the differential equation

$$
2\,u\,(1-u)\,k'(u) = ((d-1)\,u - d)\,k(u) - 1\,. \tag{4.24}
$$

The weight factor $\rho$ is given by[7]

$$
\rho = (1-u)^{1/2}\,. \tag{4.25}
$$

The resulting $U$-matrix (4.15) induces an embedding tensor $\theta_{(AB)} \propto \delta_{AB}$ in the $\square\!\square$ within (4.7). When evaluated in (4.3), (4.4) it describes a gauge group

$$
\mathrm{G}_{\text{gauge}} = \mathrm{SO}(d+1) \ltimes \mathbb{T}^{d(d+1)/2}\,, \tag{4.26}
$$

which is the semi-direct product of $\mathrm{SO}(d+1)$ with $\frac{1}{2}\,d(d+1)$ nilpotent generators transforming in the adjoint representation of $\mathrm{SO}(d+1)$. It is important to note that the gauge sector

[7]To avoid confusion let us point out that $\rho$ in (4.25) denotes the weight factor of the $\mathrm{O}(d+1,d+1)$ consistency equations (4.5) and *not* the weight factor of the $\mathrm{SL}(d+1)$ equations in [23] from which it differs by a power of $\frac{d+1}{d-3}$. In particular, in the present context, the construction applies to any values of $d$ without an analogue of the relation (4.28) in [23]. This is due to the fact that the additional factor $\varphi$ in (4.15) has been fixed such as to compensate for the missing powers of $(1-u)$.



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of the resulting three-dimensional theory, obtained by evaluating the action (3.26) under the Scherk-Schwarz ansatz, is governed by a Chern-Simons action rather than a Yang-Mills action for the vector fields. With the gauge group (4.26) and the particular structure of the embedding tensor (4.4), this theory may be rewritten as an $\mathrm{SO}(d+1)$ Yang-Mills gauge theory [40] upon furthermore eliminating $\frac{1}{2}\,d(d+1)$ scalar fields from the action. The three-dimensional scalar coset space then reduces from $\mathrm{SO}(d+1,d+1)/(\mathrm{SO}(d+1)\times\mathrm{SO}(d+1))$ to $\mathrm{GL}(d+1)/\mathrm{SO}(d+1)$. The generalized Scherk-Schwarz reduction in this case reproduces the consistent truncation of the $(d+3)$-dimensional bosonic string on the sphere $\mathbb{S}^d$ which has been explicitly constructed in [41]. In particular, it describes the $\mathbb{S}^7$ reduction of the NS-NS sector of ten-dimensional supergravity to an $\mathcal{N}=8$ half-maximal supergravity in three dimensions. The theory does not admit an $\mathrm{AdS}_3$ solution but a domain-wall solution that preserves half of the supersymmetry.

Note that here we have only given the explicit twist matrix for the case of compact gauge groups underlying sphere compactifications. It is straightforward to also employ the solutions from [23] with non-compact gauge groups to describe consistent truncations on (warped) hyperboloid backgrounds.

Let us finally stress that our construction of explicit twist matrices here has been based on restricting the coordinates to the antisymmetric bifundamental $\{Y^{AB}\}$ in the decomposition (4.16) under $\mathrm{GL}(d+1)\subset\mathrm{O}(d+1,d+1)$. In principle, one may also explore other choices of physical coordinates, e.g. within the adjoint representation $\{Y_A{}^B\}$ of $\mathrm{SL}(d+1)$, which together with an ansatz (4.15) for a $\mathrm{GL}(d+1)$ twist matrix will give rise to yet other solutions.

## 5 Consistent truncations from $D=6$ dimensions

In this section, we evaluate the generic reductions from the previous section for the particular case of an $\mathbb{S}^3$ reduction from $D=6$ dimensions. As it turns out, in this case inequivalent reductions can be constructed based on the alternative solution (3.34) of the section constraint. Moreover, the above constructed twist matrices admit a one-parameter deformation corresponding to turning on an internal flux for the three-form field strength. The resulting three-dimensional theories capture the compactification of six-dimensional supergravities around the supersymmetric $\mathrm{AdS}_3\times\mathbb{S}^3$ vacuum.

### 5.1 Generic $\mathbb{S}^3$ reduction

For $d=3$, the $\mathrm{GL}(4)$ twist matrix (4.15), (4.23) describes the generic $\mathbb{S}^3$ reduction [41] from the minimal $D=6$, $\mathcal{N}=(1,0)$ supergravity coupled to a tensor multiplet. The total $D=6$ field content thus combines the metric, a (non-chiral) two-form and a scalar field. After $\mathbb{T}^3$ reduction, this theory gives rise to a $D=3$ theory with scalar coset space $\mathrm{SO}(4,4)/(\mathrm{SO}(4)\times\mathrm{SO}(4))$. It induces an embedding tensor of the form

$$\theta_{\bar{A}\bar{B}} = 4\,\delta_{\bar{A}\bar{B}} \qquad \Longrightarrow \qquad \Theta_{\bar{A}\bar{B},\bar{C}\bar{D}} = 2\,\delta_{\bar{C}[\bar{A}}\delta_{\bar{B}]\bar{D}}\,. \tag{5.1}$$

As described above, the resulting three-dimensional theory is a Chern-Simons gauge theory with gauge group $\mathrm{SO}(4)\ltimes\mathbb{T}^6$, which may be rewritten as a more standard $\mathrm{SO}(4)$ Yang-Mills gauge theory upon eliminating the six nilpotent gauge fields together with six of the scalar fields [40]. The scalar coset space then reduces to GL(4)/SO(4) and can be parametrized in terms of a symmetric GL(4) matrix $T_{AB}$. The theory has a runaway potential given by [41]

$$V \,=\, 4\left(\mathrm{Tr}\,(T^2) - \frac{1}{2}(\mathrm{Tr}\,T)^2)\right) \,, \tag{5.2}$$

and no ground state.

Interestingly, this solution allows for an alternative presentation upon using the dual coordinates (3.34). Switching from (3.30) to these coordinates and changing the twist matrix (4.15) into

$$U_M{}^{\bar M}(\tilde y) = \left(\begin{array}{c|c} \varphi\,(V^{-1})_{\bar A}{}^{A} & 0 \\ \hline 0 & \varphi^{-1}\,V_A{}^{\bar A} \end{array}\right) \,, \tag{5.3}$$

with $V$ still given by (4.23), produces another solution to the consistency equations (4.5)–(4.6), with an embedding tensor given by

$$\theta_{\bar A\bar B\bar C}{}^{\bar D} \,=\, \varepsilon_{\bar A\bar B\bar C\bar E}\,\delta^{\bar E\bar D}\,. \tag{5.4}$$

This is a DFT analogue of the construction used in [42] to relate consistent ExFT truncations from IIA and IB supergravity by accompanying the change of coordinates by the action of an outer automorphism $V\to(V^T)^{-1}$ on the SL(4) twist matrix. Here, the resulting gaugings are equivalent as can be seen by comparing the representations of the embedding tensors (5.1), (5.4) within SO(4, 4)

$$\theta_{\bar A\bar B} \subset \Box\!\Box = \mathbf{35}_v\,, \qquad \theta_{\bar A\bar B\bar C}{}^{\bar D} \;\subset\; \begin{smallmatrix}\Box\\ \Box\\ \Box\\ \Box\end{smallmatrix} = \mathbf{35}_s\oplus\mathbf{35}_c\,. \tag{5.5}$$

The two embedding tensors (5.1), (5.4) then are related by a triality flip $\mathbf{35}_v\leftrightarrow\mathbf{35}_c$, the two gaugings hence equivalent. They both describe the $\mathbb{S}^3$ reduction of minimal $D=6$, $\mathcal{N}=(1,0)$ supergravity coupled to a tensor multiplet.

### 5.2 $D=6$, $\mathcal{N}=(1,0)$ on $\mathrm{AdS}_3\times\mathbb{S}^3$

In the three-dimensional case, the generic $\mathbb{S}^3$ reduction constructed in [41] can be modified by integrating out the two-form from the resulting three-dimensional theory which gives rise to an additional contribution to the scalar potential. In turn, the new potential then supports a stable supersymmetric $\mathrm{AdS}_3$ solution [43], corresponding to the supersymmetric $\mathrm{AdS}_3\times\mathbb{S}^3$ solution of minimal $D=6$ supergravity. For the description in terms of a Scherk-Schwarz twist matrix this corresponds to a deformation of the above construction by an extra matrix factor

$$\mathcal{U}(y) \,=\, U(y)\,\mathring{U}_\alpha(y)\,, \tag{5.6}$$

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with the GL(4) matrix $U(y)$ from (4.15), (4.23), and the matrix $\mathring{U}(y)$ obtained by exponentiating some nilpotent generators of SO(4,4) according to

$$\mathring{U}_\alpha = \exp\left(\alpha\,(1+k(u))(1-u)^{-1/2}N_0\right)\,,$$
$$N_0 \equiv \begin{pmatrix} 0_{4\times4} & n_0 \\ 0_{4\times4} & 0_{4\times4} \end{pmatrix}\,,\qquad n_0 \equiv \begin{pmatrix} 0 & y^3 & -y^2 & 0 \\ -y^3 & 0 & y^1 & 0 \\ y^2 & -y^1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,, \tag{5.7}$$

with the function $k(u)$ from (4.24) and a constant $\alpha$. It is straightforward to check that the matrices $U$ and $\mathring{U}_\alpha$ commute and that their product (5.6) remains a solution of the Scherk-Schwarz consistency equations. It results in an embedding tensor that in addition to (5.1) has the further non-vanishing component

$$\theta_{\bar{A}\bar{B}\bar{C}\bar{D}} = -2\,\alpha\,\varepsilon_{\bar{A}\bar{B}\bar{C}\bar{D}}\,. \tag{5.8}$$

This gives rise to a three-dimensional gauging with the same gauge group $\mathrm{SO}(4)\ltimes T^6$ but a modified scalar potential

$$V = 4\left(\mathrm{Tr}\,(T^2) - \frac{1}{2}\,(\mathrm{Tr}\,T)^2 + 2\,\alpha^2\det T\right)\,, \tag{5.9}$$

which (for $\alpha = 1$) exhibits a critical point at the scalar origin which corresponds to a supersymmetric $\mathrm{AdS}_3$ solution [43]. The product of twist matrices (5.6) thus describes the consistent truncation of $D=6$, $\mathcal{N}=(1,0)$ supergravity on $\mathrm{AdS}_3\times\mathbb{S}^3$.

Similar to the discussion in the previous subsection, also the deformed twist matrix (5.6) can be expressed in terms of the dual coordinates (3.34). In dual coordinates, the twist matrix $U$ in (5.6) is replaced by (5.3) whereas the factor $\mathring{U}_\alpha$ now is given by

$$\mathring{U}_\alpha(\tilde{y}) = \exp\left(\alpha\,(1+k(\tilde{u}))(1-\tilde{u})^{-1/2}N_0\right)\,,$$
$$N_0 \equiv \begin{pmatrix} 0_{4\times4} & n_0 \\ 0_{4\times4} & 0_{4\times4} \end{pmatrix}\,,\qquad n_0 \equiv \begin{pmatrix} 0 & 0 & 0 & \tilde{y}^1 \\ 0 & 0 & 0 & \tilde{y}^2 \\ 0 & 0 & 0 & \tilde{y}^3 \\ 0 & -\tilde{y}^1 & -\tilde{y}^2 & -\tilde{y}^3 \end{pmatrix}\,. \tag{5.10}$$

Again, the two matrices $U$ and $\mathring{U}_\alpha$ commute with their product solving the Scherk-Schwarz conistency equations (4.5)–(4.6). The resulting embedding tensor turns out to be given by the sum of (5.4) and (5.8):

$$\theta_{\bar{A}\bar{B}\bar{C}}{}^{\bar{D}} = \varepsilon_{\bar{A}\bar{B}\bar{C}\bar{E}}\,\delta^{\bar{E}\bar{D}}\,,\qquad \theta_{\bar{A}\bar{B}\bar{C}\bar{D}} = -2\,\alpha\,\varepsilon_{\bar{A}\bar{B}\bar{C}\bar{D}}\,, \tag{5.11}$$

inducing the same scalar potential (5.9). I.e. with respect to the decomposition (5.5) of the embedding tensor, its new component $\theta_{\bar{A}\bar{B}\bar{C}\bar{D}}$ lives in the $\mathbf{35}_s$ and is not affected by the SO(4,4) triality flip $\mathbf{35}_v \leftrightarrow \mathbf{35}_c$.



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### 5.3 $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,0)$ on $\mathrm{AdS}_3 \times \mathbb{S}^3$

We have presented the consistent truncations of $D = 6$ $\mathcal{N} = (1,0)$ supergravity on $\mathbb{S}^3$ described by an $\mathrm{SO}(4,4)$ twist matrix $\mathcal{U}$. Upon embedding $\mathrm{SO}(4,4)$ into $\mathrm{SO}(4+m, 4+n)$, the same twist matrix can be employed to describe consistent truncation of $D = 6$ supergravity coupled to vector or tensor multiplets.

E.g. choosing in the $\mathrm{SO}(8,4)$ theory physical coordinates according to (3.30) together with a twist matrix (5.6) describes the consistent truncation of half-maximal $D = 6$, $\mathcal{N} = (1,1)$ non-chiral supergravity on $\mathrm{AdS}_3 \times \mathbb{S}^3$. The embedding tensor of this theory is given by the sum of (5.1) and (5.8) as

$$\theta_{\bar{A}\bar{B}} = 4\,\delta_{\bar{A}\bar{B}}\,, \qquad \theta_{\bar{A}\bar{B}\bar{C}\bar{D}} = -2\,\alpha\,\varepsilon_{\bar{A}\bar{B}\bar{C}\bar{D}}\,, \tag{5.12}$$

for indices $\bar{A}, \bar{B}, \bar{C}, \bar{D} \in \{1,2,3,4\}$ and zero otherwise. On the other hand, choosing for the $\mathrm{SO}(8,4)$ theory the dual physical coordinates according to (3.34), together with a twist matrix (5.3), (5.7) describes the consistent truncation of half-maximal $D = 6$, $\mathcal{N} = (2,0)$ chiral supergravity (coupled to a tensor multiplet) on $\mathrm{AdS}_3 \times \mathbb{S}^3$. The embedding tensor of this theory is given by (5.11) as

$$\theta_{\bar{A}\bar{B}\bar{C}}{}^{\bar{D}} = \varepsilon_{\bar{A}\bar{B}\bar{C}\bar{E}}\,\delta^{\bar{E}\bar{D}}\,, \qquad \theta_{\bar{A}\bar{B}\bar{C}\bar{D}} = -2\,\alpha\,\varepsilon_{\bar{A}\bar{B}\bar{C}\bar{D}}\,, \tag{5.13}$$

for indices $\bar{A}, \bar{B}, \bar{C}, \bar{D} \in \{1,2,3,4\}$ and zero otherwise. This is precisely the embedding tensor derived in [44] for the gauging associated with the $\mathcal{N} = (2,0)$ compactification (given in a different basis). The present construction provides the full non-linear embedding of the three-dimensional theory in six dimensions. In this case, the gaugings induced by (5.12) and by (5.13) are no longer equivalent since the different representations (5.5) of the embedding tensor are no longer related by triality within $\mathrm{SO}(8,4)$. Accordingly, the higher-dimensional theories are strictly in-equivalent. It is straightforward to extend the construction such as to include the couplings to further $\mathcal{N} = (1,1)$ vector or $\mathcal{N} = (2,0)$ tensor multiplets. The resulting three-dimensional gaugings in particular reproduce the mass spectra computed in [45, 46].

## 6 Conclusions and outlook

We have constructed enhanced double field theories in which the usual $\mathrm{O}(d,d)$ is enlarged to at least $\mathrm{O}(d+1,d+1)$ due to the inclusion of 'dual graviton' graviton degrees of freedom. In this we have employed the 'split formulation' common for exceptional field theory, in which one has external and internal coordinates. The structure of the resulting theory parallels maximal $\mathrm{E}_{8(8)}$ ExFT [13] and minimal $\mathrm{SL}(2)$ ExFT [14]. It can certainly be further generalized for other choices of groups together with coordinates in the adjoint representation, cf. the classifications in [12, 17, 37, 47]. For three external dimensions the dual graviton components arise among the 'scalar' fields. One may also introduce the dual graviton in the more familiar 'non-split' double field theory, for which they take the form of higher-rank $\mathrm{O}(d,d)$ representations, but so far this has only been achieved at the linearized



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level [48, 49]. It remains as an open problem to find a non-split formulation for the dual graviton at the full non-linear level.

The theories we have constructed for the groups $\mathrm{SO}(8,n)$ and $\mathrm{SO}(4,n)$ reproduce the bosonic sectors of half-maximal and quarter-maximal supergravities, respectively. Depending on the solution of the section constraint, these theories describe chiral or non-chiral theories in six dimensions. It should be straightforward and parallel to the maximal case [26] to introduce the fermion fields directly in the ExFT formulation given in this paper. This will require to identify the proper $\mathrm{SO}(p)\times\mathrm{SO}(q)$ spin connections, determine their relevant components via the torsionlessness condition (2.83) and work out the supersymmetric field equations.

As an application of these theories we have worked out a number of consistent truncations via the generalized Scherk-Schwarz ansatz with suitably chosen twist matrices. In particular, the truncations from six-dimensional supergravity on $\mathrm{AdS}_3\times\mathbb{S}^3$ are constructed from a new class of twist matrices that give rise to three-dimensional supergravities with supersymmetric ground states. The consistent truncations of $D=6, \mathcal{N}=(1,1)$ and $D=6$, $\mathcal{N}=(2,0)$ supergravity on $\mathrm{AdS}_3\times\mathbb{S}^3$ should be important in the context of the associated AdS/CFT dualities. It is interesting, that the reduction of the chiral $\mathcal{N}=(2,0)$ supergravity appears consistent only in presence of an additional tensor multiplet which vanishes in the background. It would be interesting to explore if similar consistent truncations can be constructed upon including massive vector multiplets, leading to the three-dimensional gaugings constructed in [44]. The techniques recently developed in [50, 51] for generalized consistent truncations in exceptional field theory may be very useful here.

We have found that the generalized Scherk-Schwarz ansatz cannot produce arbitrary three-dimensional gaugings but only theories whose embedding tensor satisfies the additional condition (4.9) — at least as long as the twist matrices satisfy the section constraints. A geometrical higher-dimensional origin of gaugings violating (4.9) thus remains unclear. Similar no-go theorems have been found in [42, 52, 53] for higher-dimensional theories. Interestingly, most three-dimensional theories that seem to describe parts of the spectrum on $\mathrm{AdS}_3\times\mathbb{S}^3\times\mathbb{S}^3\times\mathbb{S}^1$ appear to violate the condition (4.9) [54]. The quest for consistent truncations around this background thus remains elusive. This may be related to recent surprises in the BPS spectrum on this background [55, 56]. A notable exception is the lowest massive spin-3/2 multiplet in the BPS spectrum which fits into a maximal three-dimensional supergravity [54] whose ten-dimensional uplift may be constructible within maximal ExFT.

Another interesting generalization would be the explicit inclusion of Ramond-Ramond (RR) fields to the presented formulation in order to enhance supersymmetry from half-maximal to maximal. In the standard $\mathrm{O}(d,d)$ DFT the RR fields fit into spinor representations [5, 6] and it would be interesting the work out the generalization to the enhanced DFT discussed here. Since the extended section constraint allows for solutions corresponding to chiral supergravity in six dimensions it is tempting to speculate that the maximally supersymmetric and enhanced DFT may shed a light on Hull's conjectured six-dimensional $(4,0)$ theory [57].

## Acknowledgments

This work is partially supported by a PHC PROCOPE, projet No 37667ZL. The work of O.H. is supported by a DFG Heisenberg fellowship. The work of E.T.M. is supported by the Alexander von Humboldt Foundation and in part by the Russian Government programme of competitive growth of Kazan Federal University.


## A O$(p, q)$ tensors and identities

In this section, we present our O$(p,q)$ conventions, define a number of relevant tensors and collect some useful identities. Generators $T_{MN} = T_{[MN]}$ of O$(p,q)$ are labelled by antisymmetric pairs of fundamental indices $M,N = 1,\ldots,p+q$. Their structure constants are given as

$$f_{PQ,MN}{}^{KL} = 8\,\delta_{[P}{}^{[K}\eta_{Q][M}\delta_{N]}{}^{L]}\,, \tag{A.1}$$

with the O$(p,q)$ invariant tensor $\eta_{MN}$, which we use to raise and lower indices. The Cartan-Killing form is given by

$$\eta_{KL,MN} \equiv -\eta_{M[K}\eta_{L]N}\,. \tag{A.2}$$

The projector of a product of two adjoint representations onto the adjoint representation reads

$$\begin{aligned}\mathbb{P}^{PQ}{}_{RS}{}^{MN}{}_{KL} &= \frac{1}{16\,(p+q-2)}\,f^{UV,PQ}{}_{RS}f_{UV}{}^{MN}{}_{KL}\\ &= \frac{1}{p+q-2}\Big(\delta_{[R}{}^{P}\delta_{S]}{}^{[M}\delta^{N]}{}_{[K}\delta_{L]}{}^{Q} - \delta_{R}{}^{[P}\eta^{Q][M}\delta^{N]}{}_{[K}\eta_{L]S}\\ &\quad - \delta_{[R}{}^{Q}\delta_{S]}{}^{[M}\delta^{N]}{}_{[K}\delta_{L]}{}^{P} + \delta_{S}{}^{[P}\eta^{Q][M}\delta^{N]}{}_{[K}\eta_{L]R}\Big)\,.\end{aligned} \tag{A.3}$$

We also define the tensor

$$s^{PQ,MN}{}_{KL} = 8\,\delta_{(K}{}^{[P}\eta^{Q][M}\delta_{L)}{}^{N]}\,, \tag{A.4}$$

symmetric under exchange of $[PQ]$ with $[MN]$, as well as the projector

$$\mathbb{A}^{PQMN}{}_{KLMN} \equiv \delta_{KLMN}{}^{PQMN}\,. \tag{A.5}$$

In terms of these tensors, the O$(p,q)$ section constraints (2.6) can then be written as

$$\begin{aligned}\mathbb{A}^{PQMN}{}_{KLMN}\,\partial_{PQ}\otimes\partial_{MN} &= 0 = \eta^{PM}\eta^{QN}\,\partial_{PQ}\otimes\partial_{MN}\,,\\ s^{PQ,MN}{}_{UV}\,\partial_{PQ}\otimes\partial_{MN} &= 0 = f^{PQ,MN}{}_{UV}\,\partial_{PQ}\otimes\partial_{MN}\,.\end{aligned} \tag{A.6}$$

A useful identity for the projection tensor (A.3) (the analogue of the $E_{8(8)}$ identity (2.3) in [13]) is the following

$$\begin{aligned}\frac{p+q-2}{2}\,\mathbb{P}^{PQ}{}_{RS}{}^{MN}{}_{KL} = &-\frac{1}{4}\left(\delta_{RS}{}^{PQ}\,\delta_{KL}{}^{MN} + \delta_{KL}{}^{PQ}\,\delta_{RS}{}^{MN}\right) + \frac{3}{2}\,\mathbb{A}^{PQMN}{}_{RSKL}\\ &+\frac{1}{64}\,s^{PQ,MN}{}_{UV}s_{RS,KL}{}^{UV} + \frac{1}{64}\,f^{PQ,MN}{}_{UV}f_{RS,KL}{}^{UV}\,,\end{aligned} \tag{A.7}$$

which together with (A.6) shows in particular that

$$2\,(p+q-2)\,\mathbb{P}^{PQ}{}_{RS}{}^{MN}{}_{KL}\,\partial_{PQ}\otimes\partial_{MN} \;=\; -\left(\partial_{RS}\otimes\partial_{KL}+\partial_{KL}\otimes\partial_{RS}\right)\,. \tag{A.8}$$

Another useful identity is given by

$$2\,(K-2)\,\mathbb{P}^{PQ}{}_{RS}{}^{MN}{}_{KL} \;=\; -f^{PQ,U[M}{}_{RS}\,\eta_{U[K}\delta^{N]}{}_{L]}\,. \tag{A.9}$$

## B $\mathrm{E}_{8(8)}$ generalized Dorfman structure

For completeness we present in this appendix the generalized Dorfman product for $\mathrm{E}_{8(8)}$, which allows one to formulate the gauge sector of the $\mathrm{E}_{8(8)}$ ExFT constructed in [13] in the same way as in section 2.2. We use the same notation and conventions as in [13], to which we refer the reader for further details. In particular, $M,N,\ldots=1,\ldots,248$ denote the adjoint $\mathrm{E}_{8(8)}$ index. We group the two gauge parameters, as in the main text, into the 'doubled' object

$$\Upsilon=\left(\Lambda^M,\Sigma_M\right)\,, \tag{B.1}$$

and assume that the second component is a covariantly constrained object. The generalized Lie derivative of an adjoint vector with density weight $\lambda$ can then be written as

$$\mathbb{L}^{[\lambda]}_{\Upsilon}V^M=\Lambda^N\partial_N V^M+f^M{}_{NK}R^N(\Upsilon)V^K+\lambda\,\partial_N\Lambda^N V^M\,, \tag{B.2}$$

where we defined

$$R^M(\Upsilon)\equiv f^{MN}{}_K\,\partial_N\Lambda^K+\Sigma^M\,. \tag{B.3}$$

We recall from [13] that $\Lambda^M$ has weight one, $\Sigma_M$ has weight zero, and $\partial_M$ lowers the weight by one, $[\partial_M]=-1$, so that $R^M$ has weight zero. The above Lie derivatives close according to the 'E-bracket', $[\mathbb{L}_{\Upsilon_1},\mathbb{L}_{\Upsilon_2}]=\mathbb{L}_{[\Upsilon_1,\Upsilon_2]}$, whose explicit form we will give momentarily. A useful intermediate relation for proving closure, in terms of (B.3), is

$$R_M([\Upsilon_1,\Upsilon_2])=2\,\Lambda_{[1}{}^N\partial_N R_M(\Upsilon_{2]})+f_{MNK}R^N(\Upsilon_1)R^K(\Upsilon_2)\,. \tag{B.4}$$

We also recall that there are trivial gauge parameters with respect to which the generalized Lie derivatives act trivially on fields as a consequence of the section constraints. They take the form

$$\begin{aligned}
\Lambda^M &= \eta^{MN}\Omega_N\,, &&\text{with } \Omega_M \text{ covariantly constrained}\,,\\
\Lambda^M &= (\mathbb{P}_{3875})^{MK}{}_{NL}\,\partial_K\chi^{NL} &&\\
\Lambda^M &= f^{MN}{}_K\,\Omega_N{}^K\,, &&\Sigma_M=\partial_M\Omega_N{}^N+\partial_N\Omega_M{}^N\,,
\end{aligned} \tag{B.5}$$

where $\Omega_M{}^N$ is covariantly constrained in the first index.

Let us now turn to the definition of the generalized Dorfman product in terms of the doubled vectors (B.1):

$$\Upsilon_1\circ\Upsilon_2\equiv\Big(\mathbb{L}^{[1]}_{\Upsilon_1}\Lambda_2{}^M\,,\;\mathbb{L}^{[0]}_{\Upsilon_1}\Sigma_{2M}+\Lambda_2{}^N\partial_M R_N(\Upsilon_1)\Big)\,. \tag{B.6}$$

This definition is such that the E-bracket is given by

$$[\Upsilon_1, \Upsilon_2] = \frac{1}{2}\big(\Upsilon_1 \circ \Upsilon_2 - \Upsilon_2 \circ \Upsilon_1\big) \, . \tag{B.7}$$

More precisely, this agrees with the bracket given in [13] upon adding a trivial parameter of the last form in (B.5), with $\Omega_N{}^K = \Sigma_{[1N}\Lambda_{2]}{}^K$, which is manifestly compatible with the constraint. On the other hand, the symmetric part of the product is trivial: one finds by an explicit computation

$$\begin{aligned}\frac{1}{2}\big(\Upsilon_1 \circ \Upsilon_2 + \Upsilon_2 \circ \Upsilon_1\big) = \Big(& 7\,(\mathbb{P}_{3875})^{MK}{}_{NL}\,\partial_K\big(\Lambda_1^N\Lambda_2^L\big) + \frac{1}{8}\,\partial^M\big(\Lambda_1^N\Lambda_{2N}\big) + f^{MN}{}_K\,\Omega_N{}^K \, , \\ & \partial_M\Omega_N{}^N + \partial_N\Omega_M{}^N\Big) \, ,\end{aligned} \tag{B.8}$$

where

$$\Omega_M{}^N \equiv \Lambda_{(1}{}^N\Sigma_{2)M} - \frac{1}{2}\,f^N{}_{KL}\,\Lambda_{(1}{}^K\,\partial_M\Lambda_{2)}{}^L \, . \tag{B.9}$$

Since $\Omega_M{}^N$ so defined is manifestly covariantly constrained in the first index, this is indeed a trivial parameters of the last form in (B.5).

We next prove that the Dorfman product satisfies the Leibniz algebra relation discussed in the main text. To this end we define again an extended generalized Lie derivative on doubled vectors $\mathfrak{A} = (A^M, B_M)$ according to

$$\mathbb{L}_\Upsilon\mathfrak{A} \equiv \Upsilon \circ \mathfrak{A} \, , \tag{B.10}$$

and verify that they satisfy the same algebra w.r.t. (B.7):

$$\big[\mathbb{L}_{\Upsilon_1}, \mathbb{L}_{\Upsilon_2}\big]\mathfrak{A} = \mathbb{L}_{[\Upsilon_1,\Upsilon_2]}\mathfrak{A} \, . \tag{B.11}$$

This relation only needs to be proved when acting on the second, covariantly constrained component of $\mathfrak{A}$, for which closure can be quickly seen to be equivalent to

$$\partial_M R_N([\Upsilon_1, \Upsilon_2]) = \mathbb{L}^{[-1]}_{\Upsilon_1}\big(\partial_M R_N(\Upsilon_2)\big) - \mathbb{L}^{[-1]}_{\Upsilon_2}\big(\partial_M R_N(\Upsilon_1)\big) \, . \tag{B.12}$$

This in turn can be proved by taking the derivative of (B.4) and using the Lemma (2.13) of [13]. The proof of the Leibniz identity (2.43) finally follows precisely as in the main text.

Let us now turn to the definition of an invariant inner product on the space of doubled vectors in order to construct a Chern-Simons action. The following symmetric pairing transforms covariantly (i.e. as a scalar density of weight one in the sense of (2.62))

$$\langle\langle\mathfrak{A}_1, \mathfrak{A}_2\rangle\rangle = 2\,A_{(1}{}^M B_{2)M} - f^K{}_{MN}A_{(1}{}^M\partial_K A_{2)}{}^N \, . \tag{B.13}$$

In order to prove this covariance property one has to compute the non-covariant variation of the second term, which in turn cancels the effect of the 'anomalous' term in the definition of the Dorfman product (B.6). Specifically, we have to establish

$$\begin{aligned}\Delta_\Upsilon\big(f^K{}_{MN}A_{(1}{}^M\partial_K A_{2)}{}^N\big) &= f^M{}_{NK}f^K{}_{PQ}A_{(1}{}^N\partial_M R^P(\Upsilon)A_{2)}{}^Q \\ &= 2\,A_{(2}{}^M A_{1)}{}^N\partial_M R_N(\Upsilon) \, ,\end{aligned} \tag{B.14}$$

which follows by a somewhat tedious computation, writing out $R$ and using Lemma (2.13) and (A.1) in [13] in order to reduce the number of $f$'s. Given the covariance property, it follows that under an integral we have an invariant inner product:

$$\langle \mathfrak{A}_1, \mathfrak{A}_2 \rangle \equiv \int \mathrm{d}^{248}Y \big( A_1{}^M B_{2M} + A_2{}^M B_{1M} - f^M{}_{NK} A_1{}^N \partial_M A_2{}^K \big) \,, \tag{B.15}$$

where the second term was simplified by integration by parts. We can rewrite this as

$$\langle \mathfrak{A}_1, \mathfrak{A}_2 \rangle \equiv \int \mathrm{d}^{248}Y \big( A_1{}^M R_M(\mathfrak{A}_2) + A_2{}^M B_{1M} \big) \,. \tag{B.16}$$

This form makes it manifest, as in the main text, that if one argument is trivial the inner product is zero, cf. (2.65).

With the above we have established that the analogues of all Dorfman-type identities used in the main text also hold for the $\mathrm{E}_{8(8)}$ case. This implies that the discussion of covariant derivatives, gauge fields and the tensor hierarchy proceeds in complete parallel. In particular, there is a (generalized) Chern-Simons formulation for the (doubled) gauge vector $\mathfrak{A}_\mu$ for the $\mathrm{E}_{8(8)}$ ExFT that takes precisely the same form as (2.67).



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