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# Obstacle Avoidance Problem for Driftless Nonlinear Systems with Oscillating Controls

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**Abstract:** The paper focuses on the development of the navigation function approach for nonlinear systems with fast oscillating controls. This approach allows to solve the obstacle avoidance problem and generate reference trajectories on the state space with obstacles by using the gradient flow of a navigation function. In general, such gradient flow cannot be implemented for underactuated control systems, and an approximation of non-admissible velocities is needed for the control design. We present here an approximation result under low-order controllability assumptions. Our control design scheme is illustrated by an example of a nonholonomic system.



*Keywords:* Lyapunov methods, asymptotic stabilization, application of nonlinear analysis and design, analytic design, control of constrained systems.

## 1. INTRODUCTION

The motion planning problem in the presence of obstacles is of high practical and theoretical importance. From the practical point of view, there are many examples where it is necessary to navigate a robot avoiding collisions with static objects, e.g., car parking and various manipulating tasks with state constraints. From the theoretical viewpoint, the development of controllers ensuring the collision-free motion for *general* classes of systems remains a challenging issue in spite of numerous publications in this area. Let us briefly overview some approaches for solving this problem. One of the basic ideas is to exploit special functions whose gradient flows produce trajectories converging to the destination point and remaining in the *free space* (i.e. the domain that remains after removing all the obstacles from the configuration space). In particular, Koditschek and Rimon (1990) introduced a class of potential functions called *navigation functions*, and proposed an explicit formula for their construction. The use of such functions for the collision-free motion of a point-mass robot was shown by Rimon and Koditschek (1992). An extension of these results for the case of an arbitrary convex target function was proposed by Paternain et al. (2016). In Tanner et al. (2001, 2003), navigation functions were used for constructing discontinuous controls that ensure collision-free paths for a unicycle and mobile manipulator systems. The concept of *dipolar inverse Lyapunov functions* was introduced to improve the convergence property. Thus, the method of navigation functions was extended by Loizu et al. (2004) for the control of a multi-agent system with unicycle dynamics, and then modified with dipolar functions by Dimarogonas et al. (2006) Another method based on the concept of artificial potential fields is originated from the work by Khatib (1986). Its main idea is to use a combination of attractive and repulsive potential fields. Such concept has been exploited in a number of further studies. In particular, by using Lyapunov's direct method, control schemes for the obstacle avoidance problem have been developed by Vanualailai et al. (2008) for car-like and manipulator systems, see also Sharma et al. (2012). The potential field method for the case of moving obstacles and target has been proposed by Ge and Cui (2002).

There are many other publications dealing with the obstacle avoidance problem for nonholonomic systems, however, the most of them is devoted to particular cases, and a few works deals with more general classes of systems. In particular, a path-iteration algorithm for the motion planning with obstacles avoidance has been proposed by Popa and Wen (1996) with the use of so-called penalty functions for systems that can be converted to a chained form. The main result in that area has been proved for an $N$-trailer system. The stabilization problem for nonholonomic systems in the presence of obstacles is considered in the paper by Lizarralde and Wen (1996). The method proposed in that paper allows to reduce the stabilization problem to finding the root by an iterative approach, however, no explicit formulas for control functions have been given in the cited publication.

The present paper is devoted to the construction of a general control algorithms generating a collision-free path

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for driftless nonlinear systems whose vector fields together with the first-order Lie brackets satisfy the controllability rank condition. We prove the local solvability of this problem in the class of trigonometrical control functions and propose explicit formulas for their coefficients. The novel contribution of this paper relies on the control design scheme based on navigation functions for general step-2 bracket generating systems. Up to our knowledge, this problem has not been treated with such generality for underactuated dynamical systems. Besides, it should be mentioned that the main result of this paper is applicable for an arbitrary navigation function, so that rather general shapes of the obstacles are possible within our approach.

The remaining part of this paper is organized as follows. In Section II, we formulate the obstacle avoidance problem and recall some facts concerning the navigation functions approach and the Volterra series. The main result of this paper is stated in Section III. As an example, we consider a unicycle system in Section IV.

## 2. PROBLEM STATEMENT AND SOME PRELIMINARIES

Consider a control system

$$\dot{x} = \sum_{i=1}^{m} u_i f_i(x), \quad x \in D \subset \mathbb{R}^n,\ u \in \mathbb{R}^m, \tag{1}$$

where $x = (x_1, ..., x_n)^T$ is the state and $u = (u_1, ..., u_m)^T$ is the control, $m < n$, $f_i \in C^2(D)$. We assume that the closed domain $D$ is represented as $D = \mathcal{W} \backslash \bigcup_{j=1}^{N} \mathcal{O}_j$, where $\mathcal{W} \subset \mathbb{R}^n$ is a closed bounded domain, and $\mathcal{O}_j \subset \mathcal{W}$ are open domains, $j = \overline{1, N}$. We will refer to $\mathcal{W}$, $D$, and $\mathcal{O}_j$ as *the workspace, free space, and obstacles*, respectively. The free space $D$ is assumed to be *valid*, i.e. $\overline{\mathcal{O}_i} \subset \operatorname{int} \mathcal{W}$, $\overline{\mathcal{O}_i} \cap \overline{\mathcal{O}_j} = \emptyset$ if $\neq j$, for all $i, j = \overline{1, N}$, where $\operatorname{int} \mathcal{W}$ is the interior of $\mathcal{W}$ and $\overline{\mathcal{O}_i}$ is the closure of $\mathcal{O}_i$.

We consider the following obstacle avoidance problem: *given an initial point $x^0 \in \operatorname{int} D$ and a destination point $x^* \in \operatorname{int} D$, the goal is to construct an admissible control such that the corresponding solution of system* (1) *with the initial data $x(0) = x^0$ satisfies the following conditions:*

$$x(t) \in \operatorname{int} D \ \text{ for all } \ t \geq 0, \quad x(t) \to x^* \ \text{ as } \ t \to +\infty.$$

We will solve this problem for the class of 2-step bracket generating systems by constructing a time-varying feedback law $u = h(t, x)$ and defining the solutions of system (1) in the sense of sampling. The precise formulation of this result will be given in the next section.

### 2.1 Navigation functions

As it has been mentioned in the introduction, there are several classes of potential function that can be used for generating collision-free paths. In this work we refer to the navigation functions introduced in the work by Koditschek and Rimon (1990). Such functions are bounded, approach their maximum value on the boundary of the free space, and vanish at the destination point only.

*Definition 1.* (Koditschek and Rimon (1990)). Let the free space $D \subset \mathbb{R}^n$ be a compact connected analytic manifold with boundary. A map $\phi : D \to [0, 1]$ is a *navigation function* in $D$, if it is:

- analytic on $D$;
- polar on $D$, i.e. it has a unique minimum $x^*$ which belongs to the interior of the free space $D$;
- Morse on $D$, i.e. all the critical points on $D$ are non-degenerate;
- admissible on $D$, i.e. all boundary components have the same maximal height, namely $\partial D = \phi^{-1}(1)$.

Let us recall the construction of a navigation function for the case of "a spherical world".

*Lemma 1.* (Koditschek and Rimon (1990)). Assume that the workspace $\mathcal{W}$ and the obstacles $\mathcal{O}_j$ are spheres:

$$\mathcal{W} = \{x \in \mathbb{R}^n : \|x\|^2 \leq r_0^2\},$$

$$\mathcal{O}_j = \{x \in \mathbb{R}^n : \|x - x_o^j\|^2 < r_j^2\},\ j = \overline{1, N},$$

where $r_0, r_j > 0$ are the radii of the corresponding spheres, and $x_o^j \in \mathbb{R}^n$ is the center of the $j$-th obstacle. Assume, moreover, that the free space $D$ is valid, i.e.

$$\|x_o^j\| + r_j < r_0,\ \|x_o^j - x_o^l\| > r_j + r_l,\ j \neq l, \quad j, l = \overline{1, N}.$$

Then there exists a $\nu \in \mathbb{N}$ such that, for every $\lambda \geq \nu$ and any destination point $x^*$ in the interior of $D$, the function

$$\phi(x) = \frac{\|x - x^*\|^2}{\left(\|x - x^*\|^{2\lambda} + \prod_{j=0}^{N} \beta_j(x)\right)^{1/\lambda}} \tag{2}$$

is a navigation function on $D$. Here $\beta_0(x) = r_0^2 - \|x\|^2$, $\beta_j(x) = \|x - x_o\|^2 - r_j^2$ for $j = \overline{1, N}$, and $\|\cdot\|$ stands for the Euclidean norm on $\mathbb{R}^n$.

If $\phi(x)$ is a navigation function in $D$ with the destination point $x^* \in \operatorname{int} D$, then a collision-free path from any initial point $x^0 \in \operatorname{int} D$ to $x^*$ can be generated by using the solution $x(t)$ of the following Cauchy problem

$$\dot{x}(t) = -\nabla\phi(x(t)), \quad 0 \leq t < t^* \leq +\infty,\ x(0) = x^0, \tag{3}$$

where $\nabla\phi(x)$ is the gradient of $\phi(x)$. It is clear that the above function $x(t)$ is not a solution of system (1) in general, as the control system (1) is underactuated ($m < n$). However, the curve $\gamma = \{x(t) : 0 \leq t < t^*\}$ can be approximated by admissible trajectories of (1) with high-frequency high-amplitude open-loop controls (see, e.g., Liu (1997); Jean (2014); Gauthier et al. (2010)).

We will use another approach, based on a sampling strategy, to produce a family of time-varying feedback controls with bounded frequencies (and small amplitudes near the destination point $x^*$). Our control design scheme exploits the Volterra series expansion of solutions of system (1) and extends the approach of Zuyev (2016); Zuyev et al. (2016) for a class of navigation functions. Note that these publications address the exponential stabilization problem by using quadratic forms (and perturbations of quadratic forms) as a Lyapunov function, so that the convergence proof, described there, is not applicable for the case of navigation functions.

### 2.2 The Volterra series

Any solution $x(t)$ of system (1) with the initial data $x(0) = x^0$ and controls $u_i = u_i(t)$, $u_i \in C[0, \tau]$, can be represented by means of the Volterra type series (cf. Nijmeijer and van der Schaft (1990); Zuyev (2016)):

$$
\begin{aligned}
x(t) = x^0 &+ \sum_{i=1}^{m} f_i(x^0) \int_0^t u_i(s)ds \\
&+ \frac{1}{2}\sum_{i<j}[f_i, f_j](x^0)\int_0^t \int_0^v \{u_j(v)u_i(s) - u_i(v)u_j(s)\}\, ds\, dv \\
&+ \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial f_j(x^0)}{\partial x} f_i(x^0) \int_0^t u_i(s)ds \int_0^t u_j(s)ds + r_t(x^0),
\end{aligned}
\tag{4}
$$

$t \le \tau$, where $[f_j, f_l](x) = \frac{\partial f_l(x)}{\partial x} f_j(x) - \frac{\partial f_j(x)}{\partial x} f_l(x)$ is the Lie bracket, and $\frac{\partial f_j(x^0)}{\partial x}$ is the Jacobian matrix evaluated at $x = x^0$. In order to estimate the remainder $r_t(x^0)$ of the Volterra expansion (4) and to prove the main result, we need two auxiliary lemmas from Zuyev (2016).

*Lemma 2.* Let $\tilde{D} \subset \mathbb{R}^n$ be a convex domain, and let $x(t) \in \tilde{D}$, $0 \le t \le \tau$, be a solution of (1) corresponding to the initial value $x(0) = x^0 \in \tilde{D}$ and control $u \in C[0, \tau]$. If the vector fields $f_1(x), f_2(x), ..., f_m(x)$ satisfy assumptions

$$
\left\| \frac{\partial f_i(x)}{\partial x} \right\| \le L,\ \left\| \frac{\partial^2 f_{ij}(x)}{\partial x^2} \right\| \le H,\ i = \overline{1, m},\ j = \overline{1, n}, \tag{5}
$$

in $\tilde{D}$ with some constants $H$, $L > 0$, then the remainder $r_\tau(x^0)$ of the Volterra expansion (4) satisfies the estimate

$$
\begin{aligned}
\|r_\tau(x^0)\| &\le \frac{M}{L}\left\{ e^{LU\tau} - \frac{1}{2}\left((LU\tau + 1)^2 + 1\right)\right\} \\
&+ \frac{HM^2\sqrt{n}}{4L^3}\left\{ \left(e^{LU\tau} - 2\right)^2 + 2LU\tau - 1\right\} \\
&= \frac{M(L^2 + HM\sqrt{n})}{6}U^3\tau^3 + O(U^4\tau^4).
\end{aligned}
\tag{6}
$$

Here $M = \max_{1\le i\le m} \|f_i(x^0)\|$, $U = \max_{0\le t\le\tau} \sum_{i=1}^m |u_i(t)|$.

*Lemma 3.* Let $x(t) \in \tilde{D} \subset \mathbb{R}^n$, $0 \le t \le \tau$, be a solution of system (1) with a control $u \in C[0, \tau]$, and let

$$
\|f_i(x') - f_i(x'')\| \le L\|x' - x''\|, \quad \forall x', x'' \in \tilde{D},\ i = 1, ..., m.
$$

Then

$$
\|x(t) - x(0)\| \le \frac{M}{L}(e^{LUt} - 1), \quad t \in [0, \tau], \tag{7}
$$

where $M = \max_{1\le i\le m} \|f_i(x(0))\|$, $U = \max_{0\le t\le\tau} \sum_{i=1}^m |u_i(t)|$.

## 3. MAIN RESULT

Assume that the control system (1) is step-2 bracket generating, i.e. the vector fields $f_1(x)$, $f_2(x)$, ..., $f_m(x)$ together with a fixed set of their first order Lie brackets satisfy the Hörmander condition:

$$
\operatorname{span}\{f_i(x), [f_j, f_l](x) : i = \overline{1, m},\ (j, l) \in S\} = \mathbb{R}^n, \tag{8}
$$

for each $x \in D$, where $S \subseteq \{1, 2, ..., m\}^2$. Without loss of generality, we assume that each pair of indices $(j, l) \in S$ is ordered with $j < l$.

For a given $\varepsilon > 0$, we denote by $\pi_\varepsilon$ the *partition* of $\mathbb{R}^+ = [0, +\infty)$ into intervals

$$
I_j = [t_j, t_{j+1}),\ t_j = \varepsilon j, \quad j = 0, 1, 2, \dots .
$$

We will define solutions of system (1) corresponding to a time-varying feedback law in the sense of sampling as follows (cf. Zuyev (2016)).

*Definition 2.* Assume given a time-varying feedback law $u = h(t, x)$, $h : \mathbb{R}^+ \times D \to \mathbb{R}^m$, $x^0 \in D$, and $\varepsilon > 0$. A $\pi_\varepsilon$-solution of system (1) corresponding to $x^0$ and $h(t, x)$ is an absolutely continuous function $x(t) \in D$, defined for $0 \le t < t^* \le +\infty$, which satisfies the initial condition $x(0) = x^0$ and the following differential equations

$$
\dot{x}(t) = f(x(t), h(t, x(t_j))), \quad t_j \le t < \min\{t_{j+1}, t^*\},
$$

for each $j \in \mathbb{N} \cup \{0\}$ such that $t_j < t^*$.

The above definition extends the notion of "$\pi$-trajectories", introduced by Clarke et al. (1997), for the case of time-varying feedback laws.

For solving the obstacle avoidance problem for system (1), we will use a time-varying feedback control of the form

$$
\begin{aligned}
u^\varepsilon(t, x) = &a(x) + \sum_{(i,l)\in S} a_{il}(x)\Big\{\cos\left(\frac{2\pi k_{il}(x)}{\varepsilon}t\right) e_i \\
&+ \sin\left(\frac{2\pi k_{il}(x)}{\varepsilon}t\right) e_l\Big\}
\end{aligned}
\tag{9}
$$

on each interval $I_j$ of length $\varepsilon$, where $e_i$ denotes the $i$-th unit vector in $\mathbb{R}^m$, and the functions

$$
a(x) = \left(a_1(x), ..., a_m(x), a_{il}(x)_{(i,l)\in S}\right)^T \in \mathbb{R}^n,
$$

$K(x) = \left(k_{il}(x)_{(i,l)\in S}\right)^T \in (\mathbb{Z} \setminus \{0\})^{n-m}$ are defined below.

Our main idea is to choose the feedback control (9) in order to approximate the direction of $-\nabla\phi(x)$ by trajectories of system (1), where $\phi(x)$ is a navigation function. For this purpose, we fix $x \in D$ and $\varepsilon > 0$, and consider the following system of second order algebraic equations:

$$
\begin{aligned}
&\sum_{i=1}^{m} a_i f_i(x) + \frac{\varepsilon}{4\pi}\sum_{(i,j)\in S}\frac{a_{ij}^2}{k_{ij}}[f_i, f_j](x) \\
&+ \frac{\varepsilon}{2}\sum_{i,j=1}^{m} a_i a_j \frac{\partial f_j(x)}{\partial x} f_i(x) + \frac{\varepsilon}{2\pi}\sum_{i<j}\left(a_j \sum_{(q,i)\in S}\frac{a_{qi}}{k_{qi}}\right. \\
&\left. - a_i \sum_{(q,j)\in S}\frac{a_{qj}}{k_{qj}}\right)[f_i, f_j](x) = -\nabla\phi(x),
\end{aligned}
\tag{10}
$$

with respect to the variables $a_i$, $a_{ql}$, $i \in \{1, 2, ..., m\}$, $(q, l) \in S$, assuming that the numbers $k_{ql} \in \mathbb{Z} \setminus \{0\}$ are chosen without resonances, i.e.

$$
|k_{ql}| \ne |k_{jr}| \quad \text{if} \quad S \ni (q, l) \ne (j, r) \in S. \tag{11}
$$

For $x \in \mathbb{R}$, $X \subset \mathbb{R}^n$, and $Y \subset \mathbb{R}^n$, we define the distances

$$
\rho(x, Y) = \inf_{y\in Y}\|x - y\|,\ \rho(X, Y) = \inf_{x\in X, y\in Y}\|x - y\|.
$$

An $\varepsilon$-neighborhood of a set $X \subset \mathbb{R}^n$ is denoted by $B_\varepsilon(X)$. We use the notation $\mathcal{L}_c = \{x \in D : \phi(x) \le c\}$ for the level set of a function $\phi : D \to \mathbb{R}$, and

$$
\|a(x)\|_1 = \sum_{j=1}^{m}|a_j(x)| + \sqrt{2}\sum_{(i,l)\in S}|a_{il}(x)|
$$

denotes the modified 1-norm of a vector $a(x) \in \mathbb{R}^n$. Our main result is as follows.

*Theorem 1.* Let a function $\phi \in C^2(D)$, $x^0 \in \operatorname{int} D$, $\epsilon_0 = \rho\left(\mathcal{L}_{\phi(x^0)}, \partial D\right) > 0$, $\epsilon_1 \in (0, \epsilon_0)$, and let

$$
\left\|\frac{\partial^2\phi(x)}{\partial x^2}\right\| \le \mu, \quad \forall x \in \overline{B_{\epsilon_1}}\left(\mathcal{L}_{\phi(x^0)}\right) \subset D.
$$

Assume that, for some $\varepsilon > 0$, the system of algebraic equations (10) has a solution $a = a(x)$ for each $x \in D_0 = \mathcal{L}_{\phi(x^0)}$

with $K{=}K(x)$ such that the non-resonance assumption (11) holds, and

$$\|a(x)\|_1 \le \frac{1}{L\varepsilon}\log\left(1+\frac{L\epsilon_1}{M}\right), \tag{12}$$

$$\|a(x)\|_1^3\left\{(1+\varepsilon\mu)\|\nabla\phi(x)\| + \frac{\mu C}{2}\|a\|_1^3\varepsilon^3\right\} \le \frac{1}{\varepsilon^2 C}\left(1-\frac{\varepsilon\mu}{2}\right)\|\nabla\phi(x)\|^2,\ \forall x\in D_0, \tag{13}$$

where $C{>}M(L^2{+}HM\sqrt{n})/6$, $M=\sup_{i=\overline{1,m},\,x\in D}\|f_i(x)\|$, $L = \sup_{i=\overline{1,m},\,x\in D}\left\|\frac{\partial f_i(x)}{\partial x}\right\|$, $H{=}\sup_{i=\overline{1,m},\,j=\overline{1,n},\,x\in D}\left\|\frac{\partial^2 f_{ij}(x)}{\partial x^2}\right\|$. Then the $\pi_\varepsilon$-solution of system (1) with the control (9) and the initial data $x(0)=x^0$ is well-defined on $t\in[0,+\infty)$ and has the following properties:

$$x(t)\in \operatorname{int} D,\quad \forall t\ge 0, \tag{14}$$

$$x(t)\to Z_0=\{x\in D_0\,:\,\nabla\phi(x)=0\}\quad\text{as}\quad t\to+\infty. \tag{15}$$

**Proof.** For a given point $x^0{\in}\operatorname{int} D$, positive numbers $\epsilon_0{=}\rho\left(\mathcal{L}_{\phi(x^0)},\partial D\right){>}0$, $\epsilon_1{\in}(0,\epsilon_0)$, $\varepsilon{>}0$, and the functions $a(x)$, $K(x)$ satisfying the system of algebraic equations (10) and assumption (11) for all $x{\in}D_0{=}\mathcal{L}_{\phi(x^0)}$, we introduce the map $F\,:\,\xi\in D_0\mapsto F(\xi)=x(\varepsilon;\xi,u^\varepsilon_\xi)$, where $x(t;\xi,u^\varepsilon_\xi)$, $t{\in}[0,\varepsilon]$, is the solution of (1) with the initial condition $x|_{t=0}{=}\xi$ and control $u^\varepsilon_\xi(t){=}u^\varepsilon(t,\xi)$ of the form (9) with the coefficients $a(\xi)$ and integer parameters $K(\xi)$. Lemma 3 implies that $x(t;\xi,u^\varepsilon_\xi)$ is well-defined on $t\in[0,\varepsilon]$ provided that condition (12) holds. We assume

$$F(\xi)=\xi\quad\text{if}\ \ \nabla\phi(\xi)=0, \tag{16}$$

as $a(\xi)=0$ satisfies system (10) with $\nabla\phi(\xi)=0$, and $x(t;\xi,u^\varepsilon_\xi)\equiv\xi$ in this case.

By exploiting the Volterra expansion (4) and computing the integrals under assumption (11), we conclude that

$$F(\xi)=\xi+\Delta x,\ \Delta x=-\varepsilon\nabla\phi(\xi)+r_\varepsilon, \tag{17}$$

where $r_\varepsilon=r_\varepsilon(\xi)$ is the remainder of the Volterra expansion (4). By applying Taylor's formula with the Lagrange form of the remainder to $\phi(\xi+\Delta x)$, we get:

$$\begin{aligned}\phi(F(\xi)){=}&\phi(\xi){-}\left.\frac{\partial\phi}{\partial x}\right|_\xi(\varepsilon\nabla\phi(\xi){-}r_\varepsilon){+}\frac{1}{2}\left\langle\left.\frac{\partial^2\phi}{\partial x^2}\right|_\theta\Delta x,\Delta x\right\rangle\\ \le&\phi(\xi){-}\varepsilon\left(1{-}\frac{\mu\varepsilon}{2}\right)\|\nabla\phi(\xi)\|^2{+}(1{+}\mu\varepsilon)\|\nabla\phi(\xi)\|\cdot\|r_\varepsilon\|\\ &+\frac{\mu}{2}\|r_\varepsilon\|^2,\quad \|\theta-\xi\|\le\|\Delta x\|.\end{aligned} \tag{18}$$

The above inequality implies that

$$\phi(F(\xi))<\phi(\xi) \tag{19}$$

provided that

$$\left\{(1+\mu\varepsilon)\|\nabla\phi(\xi)\|+\frac{\mu}{2}\|r_\varepsilon\|\right\}\|r_\varepsilon\| < \varepsilon\left(1-\frac{\mu\varepsilon}{2}\right)\|\nabla\phi(\xi)\|^2,\ \nabla\phi(\xi)\ne 0. \tag{20}$$

We estimate the remainder $r_\varepsilon$ by Lemma 2 with the use of condition (13) to show that (20) holds under our assumptions if $\nabla\phi(\xi)\ne0$. Conditions (16) and (19) imply

$$\phi(F(\xi))\le\phi(\xi)\quad\text{for all}\ \ \xi\in D_0. \tag{21}$$

Therefore, $F(\xi)\in D_0$ because of the construction of $D_0$. It means that

$$x^j=F(x^{j-1}),\quad F:D_0\to D_0,\ j=1,2,\dots, \tag{22}$$

is a discrete-time dynamical system. It is easy to see that $x^j=x(j\varepsilon)$, $j=0,1,2,\dots$, where $x(t)$ is the $\pi_\varepsilon$-solution of system (1) with the initial data $x(0)=x^0$ and the control $u^\varepsilon(t,x)$ given by formula (9). As it has been already mentioned, the restriction of $x(t)$ on each $I_j=[\varepsilon j,\varepsilon(j{+}1))$ is well-defined, and $x(t)\in\operatorname{int} D$ for all $t\ge0$ because of Lemma 3 and condition (12) (see Fig. 1).

It remains to prove assertion (15). The invariance principle (cf. (LaSalle, 2012, Proposition 2.6, p. 9), (Sundarapandian, 2003, Corollary 1)) together with conditions (16) and (21) implies that

$$x(j\varepsilon)\to Z_1\quad\text{as}\ \ j\to+\infty, \tag{23}$$

where $Z_1$ is the largest invariant subset of $Z_0=\{x\in D_0\,:\,\nabla\phi(x)=0\}$ for the dynamical system (22) ($D_0$ is compact under our assumptions). Condition (13) shows that $a(x)\to0$ as $\nabla\phi(x)\to0$, therefore, the continuity of $\nabla\phi(x)$ and property (23) yields $a(x^j)\to0$ as $j\to\infty$. Then we conclude that

$$\lim_{j\to+\infty}\left(\sup_{s\in[0,\varepsilon]}\|x(j\varepsilon+s)-x^j\|\right)=0.$$

Therefore, property (23) also implies that $x(t){\to}Z_1$ as $t{\to}+\infty$, which completes the proof.

Thus, Theorem 1 states that the control functions (9) with coefficients satisfying the algebraic equations (10), where $\phi(x)$ is a given potential function, steer system (1) to the set where the gradient of $\phi(x)$ vanishes. In order to apply Theorem 1, we have to choose an $\varepsilon>0$ and solve the system of $n$ quadratic equations (10) with respect to the components of $a(x)\in\mathbb{R}^n$ for some integer parameters $K(x)$ depending on $x\in D_0$. A local solvability results for this algebraic system is presented below.

*Lemma 4.* Assume that the vector fields $f_1(x)$, $f_2(x)$, ..., $f_m(x)$ satisfy the rank condition (8) in $D$, and let $\phi\in C^2(D)$. Then, for any small enough $\varepsilon>0$, there exists a $\Delta>0$ such that the system of algebraic equations (10) has a solution $a=a(x)\in\mathbb{R}^n$ for each

$$x\in\hat{D}=B_\Delta(\{x\in D:\nabla\phi(x)=0\})\cap D$$

with some $K=K(x)\in(\mathbb{Z}\setminus\{0\})^{n-m}$ such that condition (11) holds. This solution satisfies the estimate

$$\|a(x)\|_1=O\left(\max\left\{\|\nabla\phi(x)\|,\sqrt{\frac{\|\nabla\phi(x)\|}{\varepsilon}}\right\}\right),\quad x\in\hat{D}.$$

The idea of the proof, based on the topological degree theory, is similar to Theorem 3.2 in Zuyev and Grushkovskaya (2017). We omit the proof of Lemma 4 due to lack of place.

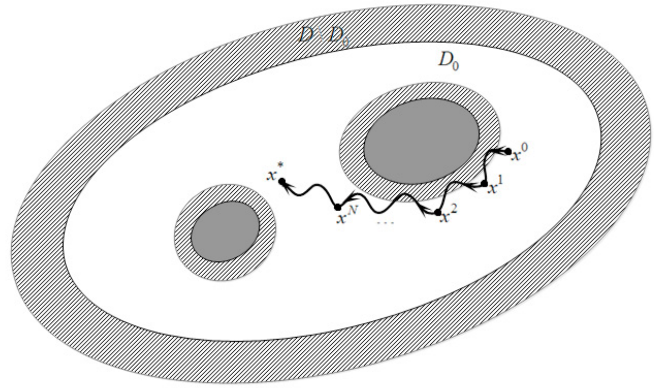


Fig. 1. $\pi_\varepsilon$-solution of system (1) with $x^0\in D_0$.

## 4. EXAMPLE: A UNICYCLE

In this section, we apply controls of the form (9) to solve the obstacle avoidance problem for a unicycle example. The equations of motion are as follows:

$$\begin{aligned}&\dot{x}_1 = u_1 \cos x_3,\ \dot{x}_2 = u_1 \sin x_3,\\ &\dot{x}_3 = u_2,\end{aligned} \tag{24}$$

where $(x_1, x_2)^T \in \mathbb{R}^2$ are the coordinates of the contact point of the unicycle, $x_3$ is the angle between the wheel and the $x_1$ axis, $u_1$ and $u_2$ control the forward and the angular velocity, respectively. We identify the angles $x_3'$ and $x_3''$ if $x_3' \equiv x_3'' (\mathrm{mod}\ 2\pi)$, so the configuration space for system (24) is $X = \mathbb{R}^2 \times S^1$. Let the workspace be defined as

$$\mathcal{W} = \{(x_1, x_2, x_3)^T \in X : \beta_0(x_1, x_2) \geq 0\},$$

where $\beta_0(x) = R^2 - x_1^2 - x_2^2$. We formulate the following goal: to reach a point $x^* \in \mathrm{int}\,\mathcal{W}$ avoiding $N$ obstacles $\mathcal{O}_i = \{x \in X : \beta_i(x_1, x_2) < 0\}$ $(x^* \notin \overline{\mathcal{O}_i})$,

$$\beta_i(x) = (x_1 - x^i_{o,1})^2 + (x_2 - x^i_{o,2})^2 - r_i^2, \quad i = \overline{1, N}.$$

For this purpose, we use the construction of a navigation function $\phi : D \to [0, 1]$, $D = \mathcal{W} \setminus \bigcup_{i=1}^N \mathcal{O}_i$ from Lemma 1:

$$\phi(x) = \frac{\|x - x^*\|^2}{\left(\|x - x^*\|^{2\lambda} + \prod_{i=0}^{N} \beta_i(x)\right)^{1/\lambda}}, \quad \lambda \in \mathbb{N},\ \lambda > 1.$$

Control system (24) satisfies the Hörmander condition of the type (8) with $S = \{(1, 2)\}$:

$$\mathrm{span}\{\mathrm{f}_1(\mathrm{x}), \mathrm{f}_2(\mathrm{x}), [\mathrm{f}_1, \mathrm{f}_2](\mathrm{x})\} = \mathbb{R}^3 \quad \text{for all } \mathrm{x} \in \mathrm{X},$$

where $f_1(x) = (\cos x_3, \sin x_3, 0)^T$, $f_2(x) = (0, 0, 1)^T$. Following the control design methodology, described in the previous section, we define the control functions as

$$\begin{aligned}u_1^\varepsilon(t, x) &= a_1(x) + a_{12}(x) \cos\left(2\pi k(x) t \varepsilon^{-1}\right),\\ u_2^\varepsilon(t, x) &= a_2(x) + a_{12}(x) \sin\left(2\pi k(x) t \varepsilon^{-1}\right),\end{aligned} \tag{25}$$

where $k \in \mathbb{Z} \setminus \{0\}$, $a_1$, $a_2$, and $a_{12}$ are defined from the algebraic equations (10):

$$a_1(x) = -\frac{1}{\varepsilon}\left(\frac{\partial \phi}{\partial x_1} \cos x_3 + \frac{\partial \phi}{\partial x_2} \sin x_3\right),\ a_2(x) = -\frac{1}{\varepsilon}\frac{\partial \phi}{\partial x_3},$$

$$a_{12}(x) = \begin{cases} a_1(x) \pm \sqrt{a_1(x) + 2\pi k(x) A(x)}, & A(x) \neq 0,\\ 0, & A(x) = 0,\end{cases}$$

$$k(x) = \begin{cases} \mathrm{sign}\, A(x), & A(x) \neq 0,\\ 1, & A(x) = 0,\end{cases}$$

$$A(x) = a_1(x) a_2(x) + \frac{2}{\varepsilon}\Big(\frac{\partial \phi}{\partial x_2} \cos x_3 - \frac{\partial \phi}{\partial x_1} \sin x_3\Big).$$

It can be shown that the conditions of Theorem 1 are satisfied; thus, controls (25) solve the obstacle avoidance problem for system (24) with any $x^* \in \mathrm{int}\, D$, provided that $\varepsilon > 0$ is small enough.

We illustrate the proposed control design scheme by numerical simulations. Let us consider two cases. In the first case, Fig. 2 shows the trajectory of system (24) with the initial condition $x^0 = (0, -3, \pi/2)^T$ and the destination point $x^* = (0, 2.5, 0)^T$. The parameters of functions $\beta_i$ are:

$x^1_{o,1} = 2$, $x^1_{o,2} = 1$, $r_1 = 1$, $x^2_{o,1} = 0$, $x^2_{o,2} = -1$, $r_2 = 1$, $x^3_{o,1} = -1.5$, $x^3_{o,2} = 2$, $r_3 = 0.75$, $R = 3.5$.

The arrows at $x^0$ and $x^*$ illustrate the behavior of $x_3$. In the second case (see Fig. 3), we choose $x^0 = (1.5, -1, 0)^T$, $x^* = (-3, 0, \pi)^T$, and

$x^1_{o,1}=2$, $x^1_{o,2}=1$, $r_1=1$, $x^2_{o,1}=0$, $x^2_{o,2}=-2$, $r_2=1$, $x^3_{o,1}=-1.5$, $x^3_{o,2}=2$, $r_3=0.75$, $x^4_{o,1}=2$, $x^4_{o,2}=-2$, $r_4=0.5$, $x^5_{o,1}=-1.5$, $x^5_{o,2}=0$, $r_5=0.5$, $R=3.5$.

For both cases, $\varepsilon = 0.01$ and $\lambda = 3$.

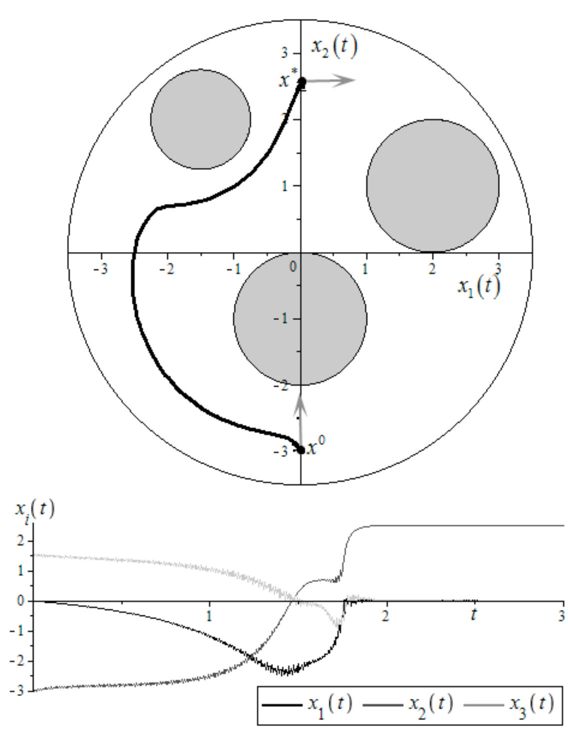


Fig. 2. Trajectory of system (24) with controls (25) in the presence of three obstacles.

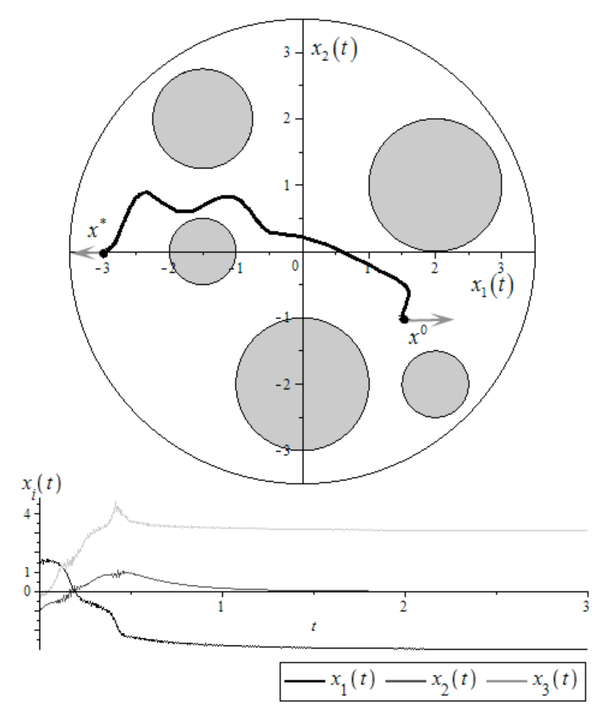


Fig. 3. Trajectory of system (24) with controls (25) in the presence of five obstacles.

## 5. CONCLUSIONS

In this paper, we have considered the motion planning problem in the presence of static obstacles. In contrast to known publications in this area, we developed a feedback control algorithm ensuring a collision-free motion for a general class of nonholonomic systems satisfying the controllability rank condition with the first-order Lie brackets. The main contribution of this paper is an explicit control design scheme and the proof of the local solvability of the problem under consideration. The proposed control strategy not only allows to steer the system from an initial to a destination point avoiding the obstacles, but also stabilizes the system in a neighborhood of the final point. This result is obtained by expanding solutions of the system into the Volterra series and approximating the gradient flow corresponding to a navigation function by trajectories of the original underactuated system. One of the key features of our approach is the reduction of the control problem under consideration to solving the system of second-order algebraic equations (10) with the absence of high-order terms, that leads to an attractive control design scheme. The procedure of omitting the high-order terms is possible because of subtle estimates of the remainder $r_\varepsilon$ in formula (4) and Lemma 2. In order to simplify the presentation, this paper uses a spherical workspace and spherical obstacles. However, other types of obstacles can also be considered (for example, the construction of navigation functions for "star worlds" is shown by Rimon and Koditschek (1991), and "trees and forests of stars" by Rimon and Koditschek (1992)). Moreover, the approach proposed is universal in the sense that, although the main result of the paper is formulated in terms of navigation functions, it can also be extended for other classes of potential functions generating collision-free paths, e.g. artificial potential fields proposed by Khatib (1986).

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