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Extremum Seeking for Time-Varying Functions using Lie Bracket Approximations ⋆

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Abstract: The paper presents a control algorithm that steers a system to an extremum point of a time-varying function. The proposed extremum seeking law depends on values of the cost function only and can be implemented without knowing analytical expression of this function. By extending the Lie brackets approximation method, we prove the local and semi-global practical uniform asymptotic stability for time-varying extremum seeking problems. For this purpose, we consider an auxiliary non-autonomous system of differential equations and propose asymptotic stability conditions for a family of invariant sets. The obtained control algorithm ensures the motion of a system in a neighborhood of the curve where the cost function takes its minimal values. The dependence of the radius of this neighborhood on the bounds of the derivative of a time-varying function is shown.

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1. INTRODUCTION

This paper is devoted to the development of extremum seeking control algorithms, i.e. algorithms that steer a system to the point of minimum (or maximum) of a function whose values can be measured but the analytical expression is partially or completely unknown. Unlike many other publications in this area (see, e.g., Tan et al. (2010)), we consider the problem of extremum seeking with a *time-varying* cost. Namely, let us consider mappings $J{:}\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$, $\gamma{:}\mathbb{R}^+\to\mathbb{R}^m$, $x^*{:}\mathbb{R}^m\to\mathbb{R}^n$, and assume that the values of the cost function $J(x,\gamma(t))$ may be measured for each $x=(x_1,\dots,x_n)^T\in\mathbb{R}^n$ and $t\in\mathbb{R}^+=[0,+\infty)$. We assume that the function $x^*(\gamma(t))$ gives a unique (possibly time-varying) point of minimum of J at time t:

$$J(x^*(\gamma(t)),\gamma(t)) = J^*(\gamma(t)) < J(x,\gamma(t)),$$
$$\text{for each } x \neq x^*(\gamma(t)) \text{ and each } t\in\mathbb{R}^+.$$

The purpose is *to construct a control system whose trajectories $x(t)$ tend asymptotically to an arbitrary small neighborhood of $x^*(\gamma(t))$, assuming that only the values of the cost function $J(x(t),\gamma(t))$ are available for control design.* We will propose a solution to this problem in Section 3 by exploiting the concept of practical asymptotic stability of a family of sets. Such study is motivated by many practical problems, e.g. when a vehicle should follow a moving target measuring the distance to the target only. The problem of time-varying extremum seeking is studied in several papers, however, the most of them consider particular classes of cost functions or use more complicated techniques such as gradient estimation. In particular, a number of papers (e.g., Yu and Özgüner (2005); Scheinker and Krstić (2012); Mandi and Mišković (2015); Vweza et al. (2015)) is devoted to applications of extremum seeking techniques for tracking a curve with known magnitude and variation bounds, i.e. the cost function is assumed to be the distance between the state of a system and a target. In the paper Zhu et al. (2013), a cooperative control strategy for a multi-agent system was proposed for seeking a signal source moving with constant velocity. A cost function defined as the sum of a quadratic function and a certain time-varying term was considered in Hua et al. (2013). A gradient-based control algorithm solving the optimization problem for cost functions with unknown time-varying parameters was constructed in DeHaan and Guay (2005); Moshksar et al. (2015). The extremum-seeking problem for a more general class of time-varying cost functions was studied in Sahneh et al. (2012) under several assumptions. In particular, it is assumed that a time-varying point of extremum has bounded time derivatives up to the third order. The control algorithm proposed

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in Sahneh et al. (2012) exploits a delay-based strategy for the gradient estimation and a gradient-search method. A similar approach was applied in Ye and Hu (2013) for time-varying extremum seeking with constrained inputs.

In this paper, we consider the time-varying cost function of general form. An idea of our construction relies on the method of a Lie bracket approximation for extremum seeking systems proposed in Dürr et al. (2013). By using a class of highly oscillating inputs, we approximate the trajectories of an extremum seeking system by the trajectories of the non-autonomous system representing the gradient flow of the cost function. Asymptotic stability conditions of a neighborhood of the curve $x^*(\gamma(t))$ are obtained by exploiting methods for studying systems with constantly acting perturbations. Thus, the main contribution of this paper is twofold. First, we present an extremum seeking solution for a broad class of time-varying functions. Second, we provide a rigorous analysis of an auxiliary non-autonomous system and obtain novel local and global asymptotic stability conditions for a family of sets representing the level sets of a time-varying Lyapunov function. Although we consider a special class of systems, the proposed approach can also be applied for deriving asymptotic stability conditions for families of invariant sets for general non-autonomous systems.

The rest of the paper is organized as follows. Section 2 contains some auxiliary results and definitions. In subsection 3.1, we propose asymptotic stability conditions for the system representing the gradient flow of the function $J(x,\gamma(t))$, and also obtain several corollaries applicable for particular classes of $J(x,\gamma(t))$. The extremum tracking control strategy is proposed in subsection 3.2. The results obtained are illustrated with several examples in Section 4. The appendices contain technical details for the proofs.

2. PRELIMINARIES

This section presents some definitions and statements which will be used throughout the paper.

Consider the control-affine system

$$\dot{x} = f_0(t,x) + \sum_{j=1}^{l} f_j(t,x)\sqrt{\omega}u_j(t,\omega t), \tag{1}$$

where $x\in\mathbb{R}^n$, $u_j:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$, $x(t_0)=x^0\in\mathbb{R}^n$, $\omega>0$, and the so-called *Lie bracket system* for (1):

$$\dot{\bar{x}}=f_0(t,\bar{x})+\frac{1}{T}\sum_{i<j}[f_i,f_j](t,\bar{x})\int_0^T\int_0^\theta u_j(t,\theta)u_i(t,\tau)d\tau d\theta, \tag{2}$$

where $T>0$, $[f_i,f_j](t,\bar{x})=\frac{\partial f_j(t,\bar{x})}{\partial x}f_i(t,\bar{x})-\frac{\partial f_i(t,\bar{x})}{\partial x}f_j(t,\bar{x})$. Assume that the solutions of (1), (2) are defined for all $t\geq0$. Let $\mathcal{L}_t\subset\mathbb{R}^n$, $t\in\mathbb{R}^+$, be a one-parameter family of non-empty sets. For a $\delta>0$, we denote the δ-neighborhood of the set $\mathcal{L}_t$ at time t as $B_\delta(\mathcal{L}_t)=\bigcup_{y\in\mathcal{L}_t}\{x\in\mathbb{R}^n : \|x-y\|<\delta\}$ and its closure by $\bar{B}_\delta(\mathcal{L}_t)$.

Definition 1. A family of sets $\mathcal{L}_t$ is said to be *locally practically uniformly asymptotically stable* for (1) if

– it is *practically uniformly stable*, i.e. for each $\varepsilon>0$ there are $\delta>0$ and $\omega_0>0$ such that, for all $t_0\in\mathbb{R}^+$ and $\omega>\omega_0$, if $x^0\in B_\delta(\mathcal{L}_{t_0})$ then the corresponding solution of (1) satisfies $x(t)\in\mathcal{B}_\varepsilon(\mathcal{L}_t)$ for all $t\geq t_0$;

– $\hat{\delta}$-*practically uniformly attractive* with some $\hat{\delta}>0$, i.e. for each $\varepsilon>0$ there are $t_1\in[0,\infty)$ and $\omega_0>0$ such that, for all $t_0\in\mathbb{R}^+$ and $\omega>\omega_0$, if $x^0\in B_{\hat{\delta}}(\mathcal{L}_{t_0})$ then the corresponding solution of (1) satisfies $x(t)\in B_\varepsilon(\mathcal{L}_t)$ for all $t\geq t_0+t_1$;

– the solutions of system (1) are *practically uniformly bounded*, i.e. for each $\delta>0$ there are $\varepsilon>0$ and $\omega_0>0$ such that, for all $t_0\in\mathbb{R}^+$ and $\omega>\omega_0$, if $x^0\in B_\delta(\mathcal{L}_{t_0})$ then $x(t)\in B_\varepsilon(\mathcal{L}_t)$ for all $t\geq t_0$.

If the attractivity property holds for every $\hat{\delta}>0$, then the family of sets $\mathcal{L}_t$ is called *semi-globally practically uniformly asymptotically stable* for (1). For systems independent of ω we omit the terms *practically* and *semi*.

Note that the concept of practical stability of an equilibrium was studied by LaSalle and Lefschetz (1961) for systems without parameters and successfully applied for parametrized systems in extremum seeking problem with time-invariant costs (see, e.g., Dürr et al. (2013) and references therein). Definition 1 extends the notion of stability of a family of invariant sets (cf. Langa et al. (2002)) for the case of practical stability.

Definition 2. The family of sets $\mathcal{L}_t$ is called to be *uniformly positively invariant* for system (1) if, for all $t_0\in\mathbb{R}^+$, the following property holds: if $x^0\in\mathcal{L}_{t_0}(\gamma)$ then the corresponding solution of (1) satisfies $x(t)\in\mathcal{L}_t(\gamma)$ for all $t\geq t_0$.

Definition 3. A function $w\in C(\mathbb{R}^+;\mathcal{E}_w)$, $\mathcal{E}_w\subseteq\mathbb{R}^+$, *belongs to the class* $\mathcal{K}$ if it is strictly increasing and $w(0)=0$. If, in addition, w is unbounded then it *belongs to the class* $\mathcal{K}_\infty$.

Remark 1. Note that if $w\in C(\mathbb{R}^+;\mathcal{E}_w)$ is of the class $\mathcal{K}$ and surjective, then there exists the unique inverse function $w^{-1}(x)\in\mathcal{K}:\mathcal{E}_w\to\mathbb{R}^+$.

Assume that the vector fields of system (1) and control functions $u_i(t,\omega t)$ satisfy the following assumptions:

A1 $f_i\in C^2(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)$, $i=0,1,\ldots,l$.

A2 The functions $\|f_i(t,x)\|$, $\|\frac{\partial f_i(t,x)}{\partial t}\|$, $\|\frac{\partial f_i(t,x)}{\partial x}\|$, $\|\frac{\partial^2 f_j(t,x)}{\partial t\partial x}\|$, $\left\|\frac{\partial[f_j,f_k](t,x)}{\partial t}\right\|$, $\left\|\frac{\partial[f_j,f_k](t,x)}{\partial x}\right\|$ are bounded on each compact set $x\in\mathcal{X}\subset\mathbb{R}^n$, uniformly in $t\geq0$, for $i=\overline{0,l}$, $j=\overline{1,l}$, $k=\overline{j,l}$.

A3 The functions $u_i\in L^\infty(\mathbb{R}^+\times\mathbb{R}^+;\mathbb{R})$ are bounded and Lipschitz with respect to the first argument.

A4 The functions $u_i(t,\cdot)$ are T-periodic with $T>0$, and $\int_0^T u_i(t,\tau)d\tau=0$ for all $t\in\mathbb{R}^+$, $i=1,\ldots,l$.

Theorem 1. (Dürr et al. (2013)). Let A1–A4 be satisfied, and let $I\subseteq\mathbb{R}^n$ be the set of initial conditions such that system (2) has unique, uniformly bounded solutions. Then, for every bounded set $I_1\subseteq I$, for every $\xi>0, t_1>0$, there exists an $\omega_0>0$ such that, for all $\omega>\omega_0$, $t_0\in\mathbb{R}^+$, and $x^0=x(t_0)\in I_1$, there exist unique solutions x and $\bar{x}$ of systems (1) and (2), respectively, through $x(t_0)=\bar{x}(t_0)=x^0$ which satisfy $\|x(t)-\bar{x}(t)\|<\xi$ for $t\in[t_0,t_0+t_1]$.

3. MAIN RESULTS

The following result states the practical asymptotic stability of the family of sets $\mathcal{L}_t$ for system (2).

Theorem 2. Let A1–A4 be satisfied. Suppose that a family of sets $\mathcal{L}_t$ is locally (globally) uniformly asymptotically stable for (2), and there exists a compact set $S\subset\mathbb{R}^n$ such that $\mathcal{L}_t\subseteq S$ for all $t\in\mathbb{R}^+$. Then $\mathcal{L}_t$ is locally (semi-globally) practically uniformly asymptotically stable for (1).

The proof of this theorem is given in Appendix A. Further, we apply Theorem 2 for seeking an extremum point of a time-varying cost function. Consider the system

$$\dot{x} = u, \quad (3)$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^n$ is the control, $x(t_0) = x^0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}^+$. It was shown in Dürr et al. (2013) that the controls

$$u_s = c_{1,s} J(x, \gamma(t)) \sqrt{\omega_s} \cos(\omega_s t) + c_{2,s} \sqrt{\omega_s} \sin(\omega_s t) \quad (4)$$

ensure the practical uniform asymptotic stability of the point x^* for constant γ, provided that $\omega_s = k_s \omega$, $\omega > 0$, $k_{s_1} \neq k_{s_2}$ for all $s_1 \neq s_2$, $k_s \in \mathbb{Q}_{++}$, $c_{1,s}, c_{2,s} > 0$, $s = \overline{1,n}$, where $\mathbb{Q}_{++}$ denotes the set of positive rational numbers. In this section, we will prove the practical uniform asymptotic stability of the family of sets

$$\mathcal{L}_{\lambda,t} = \{x \in \mathbb{R}^n : J(x, \gamma(t)) - J^*(\gamma(t)) \leq \lambda\},\ t \in \mathbb{R}^+, \quad (5)$$

with a time-varying function $\gamma(t)$ and some $\lambda > 0$ which can be chosen arbitrary small under a suitable choice of the control parameters. First, we will propose asymptotic stability conditions for a system representing the gradient flow of the function J. Then we will show that, under certain assumptions, the trajectories of system (3) with controls (4) approximate the gradient flow dynamics, and prove the practical uniform asymptotic stability.

3.1 Stability of the gradient flow

We start with the consideration of a system representing the gradient flow of J:

$$\dot{x}_s = -c_s \frac{\partial J(x, \gamma(t))}{\partial x_s}, \quad s = \overline{1,n}, \quad (6)$$

where c_s are some positive constants. In this section, we will propose several results concerning stability properties of system (6) under the assumption that $\|\dot{\gamma}(t)\|$ is bounded. In Theorem 3, we will prove local asymptotic stability conditions for an arbitrary small neighborhood of the curve where the function J takes its minimal values, provided that c_s in (6) are large enough. Global asymptotic stability conditions are given in Theorem 4.

Let us mention that there are several approaches for establishing stability properties of non-autonomous systems. In particular, system (6) under a suitable change of variables can be considered as a special case of *systems with constantly acting perturbations* which were studied, e.g., in Malkin (1952); Khalil (1996); Savchenko and Ignatyev (1989). It was proved that the existence of a Lyapunov function with certain properties for a "non-perturbed" system ensures the stability under constantly acting perturbations. Furthermore, stability of the curve $x^*(\gamma(t))$ holds if (6) has a manifold of exponentially stable constant equilibria (see Kelemen (1986); Lawrence and Rugh (1990); Khalil (1996)). The asymptotic stability property was proved for the case of vanishing inputs, i.e. for $\|\dot{\gamma}(t)\| \to 0$. In this paper, we exploit some methods used in the abovementioned studies for obtaining *asymptotic* stability conditions for system (6). Note that, in contrast to the abovemention papers, we do not assume that the velocity of $\gamma(t)$ tends to zero, and prove the asymptotic stability of the family of sets (5).

Theorem 3. Given functions $J \in C^2(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R})$, $\gamma \in C^1(\mathbb{R}^+; \mathbb{R}^m)$, and a closed bounded domain $\overline{\Gamma} \subset \mathbb{R}^m$, let there exist a function $x^* \in C^1(\overline{\Gamma}; \mathbb{R}^n)$, a constant $\Delta > 0$, and functions $w_{11}, w_{12}, w_2 \in \mathcal{K} : [0, \Delta] \to \mathbb{R}^+$ such that the following conditions are satisfied:

C1.1 the curve $\gamma(t) \in \Gamma$ for all $t \geq 0$, and $\max\limits_{t \in \mathbb{R}^+} \|\dot{\gamma}(t)\| \leq \nu$ with some $\nu \geq 0$;

C1.2 for all $x \in B_\Delta(x^*(g))$, $g \in \Gamma$,

$$w_{11}(\|x - x^*(g)\|) \leq J(x - x^*(g), g) \leq w_{12}(\|x - x^*(g)\|),$$
$$\left\| \frac{\partial J(x - x^*(g), g)}{\partial x} \right\|^2 \geq w_2(\|x - x^*(g)\|),$$

C1.3 there are $L, H > 0$, $M \geq 0$ such that $\left\| \frac{\partial V(x,g)}{\partial x} \right\| \leq L$, $\left\| \frac{V(x,g)}{\partial g} \right\| \leq M$, $\left\| \frac{\partial^2 V(x,g)}{\partial x^2} \right\| \leq H$, for all $x \in B_\Delta(x^*(g))$ and for all $g \in \Gamma$, where $V(x,g) = J(x + x^*(g)) - J^*(g)$.

Then, for every $\lambda \in (0, w_{11}(\Delta))$, there exists a $c > 0$ such that the family of sets $\mathcal{L}_{\lambda,t}$ (5) is locally uniformly asymptotically stable for system (6) with $\min\{c_1, \ldots, c_n\} > c$.

The proof of this theorem is given in Appendix B.

For fixed values of parameters c_s in system (6), the above result can be formulated in the following way.

Corollary 1. Let the conditions of Theorem 3 be satisfied, and suppose that there exists a $\lambda \in (0, w_{11}(\Delta))$ such that $w_2\left(w_{12}^{-1}(\lambda)\right) > \frac{\nu M}{\min\{c_1, \ldots, c_n\}}$. Then the family of sets $\mathcal{L}_{\lambda,t}$ is locally uniformly asymptotically stable for system (6).

Requiring additional properties for the cost function, it is possible to establish global asymptotic stability conditions.

Theorem 4. Given functions $J \in C^2(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R})$, $\gamma \in C^1(\mathbb{R}^+; \mathbb{R}^m)$, and a closed bounded domain $\overline{\Gamma} \subset \mathbb{R}^m$, let there exist a function $x^* \in C^1(\overline{\Gamma}; \mathbb{R}^n)$ such that the following conditions hold:

C2.1 the curve $\gamma(t) \in \Gamma$ for all $t \geq 0$, and $\max\limits_{t \in \mathbb{R}^+} \|\dot{\gamma}(t)\| \leq \nu$ with some $\nu \geq 0$;

C2.2 there exist $\mu_1 > 0$, $\mu_2 \in [0, \infty)$, $p \in [0, 1)$, and functions $w_{11}, w_{12} \in \mathcal{K}_\infty : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all $x \in \mathbb{R}^n$, $g \in \Gamma$, $w_{11}(\|x - x^*(g)\|) \leq J(x - x^*, g) \leq w_{12}(\|x - x^*(g)\|)$, and $\frac{1}{\mu_2^{1/p}} \left\| \frac{\partial V(x,g)}{\partial g} \right\|^{1/p} \leq V(x,g) \leq \frac{1}{\mu_1} \left\| \frac{\partial V(x,g)}{\partial x} \right\|^2$, where $V(x,g) = J(x + x^*(g)) - J^*(g)$.

C2.3 for every compact $\mathcal{X} \subset \mathbb{R}^n$, the functions $\left\| \frac{\partial V(x,g)}{\partial x} \right\|$, $\left\| \frac{\partial^2 V(x,g)}{\partial x^2} \right\|$ are bounded for all $x \in \mathcal{X} \subset \mathbb{R}^n$, $g \in \Gamma$.

Then the family of sets $\mathcal{L}_{\lambda,t}$ with $\lambda = \left(\frac{\nu \mu_2}{\mu_1 \min\{c_1, \ldots, c_n\}} \right)^{\frac{1}{(1-p)}}$ is globally uniformly asymptotically stable for system (6).

The proof is given in Appendix C.

3.2 Stability of the extremum seeking system

Let $x^*(\gamma(t))$ be a time-varying extremum point of the function $J(x, \gamma(t))$. By introducing the variables $\tilde{x} = x - x^*(\gamma(t))$ in equations (3) with controls (4), we obtain the following system:

$$\dot{\tilde{x}}_s = c_{1,s} J(\tilde{x} + x^*(\gamma(t)), \gamma(t)) \sqrt{\omega_s} \cos(\omega_s t) + c_{2,s} \sqrt{\omega_s} \sin(\omega_s t) - \dot{x}_s^*(\gamma(t)). \quad (7)$$

It is easy to see that system (7) is of the type (1) with $f_0(t, \tilde{x}) = -\dot{x}_s^*(\gamma(t))$,

$$f_j(t, \tilde{x}) = \begin{cases} c_{1,j} k_j^{-1/2} J(\tilde{x} + x^*(\gamma(t)) e_j, & \text{for } j = \overline{1,n}, \\ c_{2,j} k_j^{-1/2} e_{j-n}, & \text{for } j = \overline{n+1,l}, \end{cases}$$

where $l = 2n$, and $e_m \in \mathbb{R}^n$ denotes the standard unit vector with non-zero m-th component; $u_j(t, \omega t) = \cos\left(\omega t k_j^{-1}\right)$ for

$j=\overline{1,n}$, and $u_j(t,\omega t)=\sin\left(\omega t k_j^{-1}\right)$ for $j=\overline{n+1,l}$.
To satisfy A1–A4, we need the following requirement:
B1 the functions $J(\tilde{x}+x^*(\gamma(t)),\gamma(t))$, $\left\|\frac{\partial J(\tilde{x}+x^*(\gamma(t)),\gamma(t))}{\partial \tilde{x}}\right\|$, $\left\|\frac{\partial^2 J(\tilde{x}+x^*(\gamma(t)),\gamma(t))}{\partial \tilde{x}^2}\right\|$, $\left\|\frac{\partial^2 J(\tilde{x}+x^*(\gamma(t)),\gamma(t))}{\partial \tilde{x}\partial t}\right\|$, $\|\dot{x}^*(\gamma(t))\|$, $\|\ddot{x}^*(\gamma(t))\|$ are bounded on each compact set $x\in\mathcal{X}\subset\mathbb{R}^n$, uniformly in $t\geq 0$.
Then observe that the Lie bracket system (2) for (7) coincides with (B.2) in variables $\bar{x}$ with $c_s=\frac{1}{2}c_{1,s}c_{2,s}$, $s=\overline{1,n}$. Besides, if there exists a function $w\in\mathcal{K}$ such that $J(\tilde{x}+x^*(g))-J^*(g)\geq w(\|\tilde{x}\|)$ then, for any $\lambda\in\mathcal{E}_w$, $\widetilde{\mathcal{L}}_{\lambda,t}\subseteq\bar{B}_{w^{-1}(\lambda)}(0)$. Thus, Theorems 3,4 together with Theorem 2 imply practical uniform asymptotic stability (local or semi-global, respectively) of the family of sets $\widetilde{\mathcal{L}}_{\lambda,t}$ for system (7) and, consequently, of the family of sets $\mathcal{L}_{\lambda,t}$ for system (3) with controls (4). This result can be formulated as follows.

Theorem 5. Given $J\in C^2(\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R})$, $\gamma\in C^3(\mathbb{R}^+;\mathbb{R}^m)$, and a closed bounded domain $\overline{\Gamma}\subset\mathbb{R}^m$, let there exist a function $x^*\in C^3(\overline{\Gamma};\mathbb{R}^n)$ such that B1, C1.1–C1.3 hold. Then, for every $\lambda\in(0,w_{11}(\Delta))$, there exist c_s such that the family of sets $\mathcal{L}_{\lambda,t}$ is locally practically uniformly asymptotically stable for system (3) with controls (4).

Theorem 6. Given $J\in C^2(\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R})$, $\gamma\in C^3(\mathbb{R}^+;\mathbb{R}^m)$, and a closed bounded domain $\overline{\Gamma}\subset\mathbb{R}^m$, let there exist a function $x^*\in C^3(\overline{\Gamma};\mathbb{R}^n)$ such that B1, C2.1–C2.3 hold. Then the family of sets $\mathcal{L}_{\lambda,t}$ is semi-globally practically asymptotically stable for system (3) with controls (4), where λ is given by Theorem 4.

Remark 2. The conditions of the above theorems can be relaxed for certain classes of costs functions, in particular, for functions of the form

$$J(x-\varphi(t),t)=\sum_{s=1}^{n}\gamma_s(t)J_s(x_s-\varphi_s(t)),$$

under the assumption that J satisfy B1, C1.1-C1.3 (or C2.1-C2.3) for all $\tilde{x}\in\mathbb{R}^n$, $t\geq 0$. In this case, $\varphi_s(t)$ are allowed to be unbounded: indeed, in the variables $\tilde{x}_s=x_s-\varphi_s(t)$, the function $J(\tilde{x},t)$ has the unique constant extremum point, and system (7) does not depend on $\dot{\varphi}_i(t)$. Besides, the corresponding Lie bracket system has the form

$$\dot{\bar{x}}_s=-c_s\gamma_s(t)\frac{\partial J_s(\bar{x}_s)}{\partial \bar{x}_s}-\dot{\varphi}_s(t),\quad s=\overline{1,n}.$$

Using the Lyapunov function $V(\bar{x})=\sum_{s=1}^{n}(J_s(\bar{x}_s)-J_s^*)$ and taking into account the boundedness of $\gamma_s(t)$, we may obtain asymptotic stability conditions for a time-invariant compact set $\widetilde{\mathcal{L}}_\lambda=\{\bar{x}\in\mathbb{R}^n:V(\bar{x})\leq\lambda\}$ with some $\lambda>0$. Applying Theorem 2 and returning to the x variable, we achieve the practical asymptotic stability of a neighborhood of the curve $x^*+\varphi(t)$ for an extremum seeking system.

Remark 3. As it is common in the extremum seeking literature (see, e.g, Dürr et al. (2013); Grushkovskaya and Ebenbauer (2016)), we may introduce a washout filter $G(s)=s/(s+h)$, $h>0$, in order to improve the behavior of an extremum seeking system.

4. EXAMPLES

In this section, we consider illustrative examples with time-varying cost functions which do not satisfy the assumptions made in the publications mentioned in the introduction, and apply our results to a consensus problem.

Example A. The time-varying extremum point of the function $J(x,\gamma(t))=(x-\ln(1+t^2))^4$ is not exponentially stable for system (6).That is why the asymptotic stability results from Kelemen (1986); Lawrence and Rugh (1990) are not applicable. Note also that the papers mentioned in Section 3.1 require the boundedness of $\gamma(t)$ for all $t\in\mathbb{R}^+$. In this example, $\gamma(t)=\ln(1+t^2)$ is unbounded; however, our asymptotic stability results are applicable, as it is mentioned in the Remark 2. Fig. 1 illustrates the behavior of the trajectories of the corresponding extremum system (3) with $c_{1,1}=5, c_{2,1}=1, h=1, \omega=50, x(0)=0.5$.

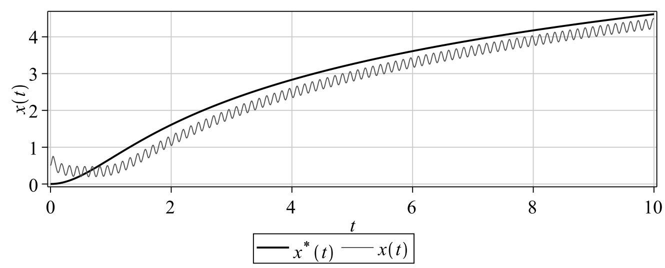

Fig. 1. Trajectory of system (3)-(4) for Example A.

Example B. Consider $J_2(x,\gamma(t))=x_1^2+(2+\sin(0.01t^2))x_2+e^{-\frac{2}{1+0.01t^2}x_1-x_2}$. It does not satisfy assumptions of the papers mentioned in the introduction. In particular, $J_2(x,\gamma(t))$ is not the distance function; besides, the third derivative of the extremum point is unbounded, while the paper Sahneh et al. (2012) requires boundedness of the first three derivatives of $x^*(t)$. Fig. 2 shows the corresponding time-plots for solutions of (3) with $c_{1,i}=2, c_{2,i}=1, h=1, \omega_1=100, \omega_2=50, x(0)=(-0.5,0.5)$.

Example C: Consensus problem. In this example, we show that the proposed control algorithm can be also applied for solving the consensus problem for multi-agent systems. Consider a system consisting of 3 two-dimensional single integrators (3). Assume that the velocities of agents A and B can be controlled, while agent C moves with an unknown velocity $v(t)=v_1^2(t)+v_2^2(t)\in[0,\nu]$, $\nu\in[0,\infty)$:

$$\dot{x}_i^A=u_{1,i},\ \dot{x}_i^B=u_{2,i},\ \dot{x}_i^C=v_i(t),\ i=1,2. \quad (8)$$

Moreover, assume that agent A can measure the squared distance to agent C, and agent B – to agent A. Let us introduce the cost functions $J_1(x^A,x^C)=\|x^A-x^C\|^2$,

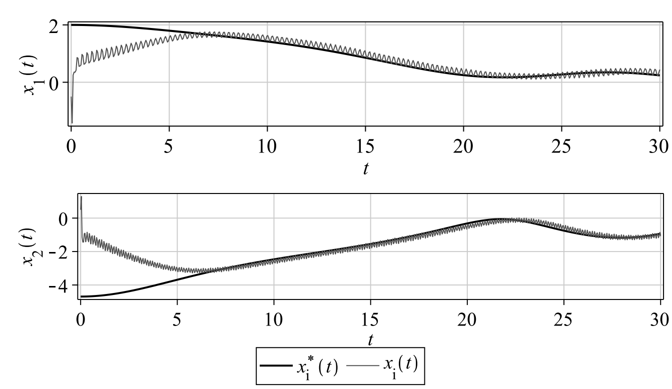

Fig. 2. Trajectories of system (3)-(4) for Example B.

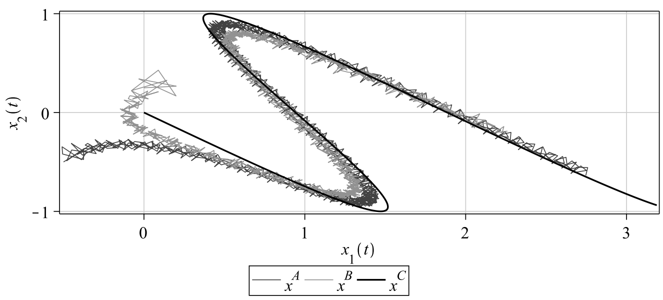

Fig. 3. Trajectories of the agents for Example C.

$J_2(x^A, x^B){=}\|x^A{-}x^B\|^2$, and the controls of type (4). For the corresponding Lie bracket system, consider the Lyapunov function $V(\bar{x}^A, \bar{x}^B, \bar{x}^C){=}J_1(\bar{x}^A, \bar{x}^C){+}J_2(\bar{x}^A, \bar{x}^B)$. Its time-derivative along the trajectories of (8) can be estimated as $\dot{V}{\le}{-}\mu V{+}\nu V^{1/2}$, where μ depends on the coefficients of (4). Similarly to the proof of Theorem 4, we conclude that the family of sets $\mathcal{L}_{\lambda,t}{=}\{\|x^A - x^B\|^2{+}\|x^A{-}x^C\|^2{\le}\lambda\}$ with $\lambda{=}\nu\mu^{-2}$ is semi-globally practically asymptotically stable for system (8). For the simulation (see Fig. 3), we use $v_1(t){=}0.15t{+}\sin(0.5t)$, $v_2(t){=}{-}\sin(0.5t)$, $\omega_1{=}100$, $\omega_2{=}150$, $\omega_2{=}120$, $\omega_4{=}180$, $c_{1,i}{=}2$, $c_{2,i}{=}0.5$, $i{=}\overline{1,4}$.

5. CONCLUSIONS

In this paper, we have studied the extremum seeking problem for a class of time-varying cost functions $J(x(t), \gamma(t))$. The proposed control depends only on the values of the cost function, so that it can be used in situations when the analytical expressions of the functions J and γ are unknown, and gradient-based algorithms are not applicable. We have extended the Lie bracket approximations method and have shown that it can be applied to a family of one-parameter sets. It has been proved that, under certain assumptions, the asymptotic stability of a family of sets $\mathcal{L}_t$ for the gradient system implies practical asymptotic stability of $\mathcal{L}_t$ for the extremum seeking system. Moreover, we have shown that the trajectories of the system converge to a neighborhood of $x^*(\gamma(t))$ and estimated the radius of such a neighborhood. Note that $x^*(\gamma(t))$ is not a solution of a system representing the gradient flow, in general, however, it is the equilibrium for each fixed value of γ. Let us emphasize that, to the best of our knowledge, most of the existing results on stability of such non-autonomous systems either give conditions for the stability (but not the asymptotic stability), or require that $\lim_{t\to\infty} \|\dot{\gamma}(t)\| \to 0$ for the local attractivity property. In contrast to these approaches, we introduce *constructive local and global asymptotic stability* conditions for *families of invariant sets* given by the level sets of a Lyapunov function under the assumption that $\|\dot{\gamma}(t)\|$ is bounded, but not necessary tends to zero. Although these results are proved for gradient systems, they can also be extended for obtaining asymptotic stability conditions for families of invariant sets of general non-autonomous systems.

REFERENCES

DeHaan, D. and Guay, M. (2005). Extremum-seeking control of state-constrained nonlinear systems. *Automatica*, 41(9), 1567–1574.

Dürr, H.B., Stanković, M.S., Ebenbauer, C., and Johansson, K. (2013). Lie bracket approximation of extremum seeking systems. *Automatica*, 49, 1538–1552.

Grushkovskaya, V. and Ebenbauer, C. (2016). Multi-agent coordination with Lagrangian measurements. In *IFAC-PapersOnLine*, volume 49, 115–120.

Hua, J., Qing-Hao, M., Yang, W., Ming, Z., Su-Lin, Z., and Wei, L. (2013). Extremum seeking control for time-varying scalar signal. In *Proc. 32nd Chinese Control Conference*, 5710–5715.

Kelemen, M. (1986). A stability property. *IEEE Trans. Autom. Control*, 31(8), 766–768.

Khalil, H.K. (1996). *Nonlinear Systems [2nd Edition]*. Prentice Hall.

Langa, J.A., Robinson, J.C., and Suárez, A. (2002). Stability, instability, and bifurcation phenomena in non-autonomous differential equations. *Nonlinearity*, 15, 887–903.

LaSalle, J.P. and Lefschetz, S. (1961). *Stability by Lyapunov's second method with applications*. Academic Press.

Lawrence, D.A. and Rugh, W.J. (1990). On a stability theorem for nonlinear systems with slowly varying inputs. *IEEE Trans. Autom. Control*, 35(7), 860–864.

Malkin, I.G. (1952). *Theory of Stability of Motion*. United States Atomic Energy Commission.

Mandi, F. and Mišković, N. (2015). Tracking underwater target using extremum seeking. *IFAC-PapersOnLine*, 48(2), 149 – 154.

Moshksar, E., Dougherty, S., and Guay, M. (2015). Model-based extremum-seeking control for unstable systems with time-varying extremum. In *Proc. 54th IEEE Conference on Decision and Control*, 6960–6965.

Sahneh, F.D., Hu, G., and Xie, L. (2012). Extremum seeking control for systems with time-varying extremum. In *Proc. 31st Chinese Control Conference*, 225–231.

Savchenko, A.Y. and Ignatyev, A.O. (1989). *Some Problems of Stability of Nonautonomous Dynamical Systems (in Russian)*. Naukova Dumka.

Scheinker, A. and Krstić, M. (2012). Extremum seeking-based tracking for unknown systems with unknown control directions. In *Proc. 51st IEEE Conference on Decision and Control*, 6065–6070.

Tan, Y., Moase, W.H., Manzie, C., Nešić, D., and Mareels, I.M.Y. (2010). Extremum seeking from 1922 to 2010. In *Proc. 29th Chinese Control Conference*, 14–26.

Vweza, A.O., Chong, K.T., and Lee, D.J. (2015). Gradient-free numerical optimization-based extremum seeking control for multiagent systems. *Int. J. of Control, Automation and Systems*, 13(4), 877–886.

Ye, M. and Hu, G. (2013). Extremum seeking under input constraint for systems with a time-varying extremum. In *Proc. 52nd IEEE Conference on Decision and Control*, 1708–1713.

Yu, H. and Özgüner, U. (2005). Adaptive tracking control via extremum-seeking method. *IFAC Proceedings Volumes*, 38(1), 259 – 264. 16th IFAC World Congress.

Zhu, S., Wang, D., and Low, C.B. (2013). Cooperative control of multiple UAVs for moving source seeking. In *Proc. International Conference on Unmanned Aircraft Systems International Conference on*, 193–202.

Zuyev, A. (2016). Exponential stabilization of nonholonomic systems by means of oscillating controls. *SIAM*

Journal on Control and Optimization, 54(3), 1678–1696.
Zuyev, A., Grushkovskaya, V., and Benner, P. (2016). Time-varying stabilization of a class of driftless systems satisfying second-order controllability conditions. In *Proc. 15th European Control Conference*, 1678–1696.

Appendix A. PROOF OF THEOREM 2

The proof is similar to the proof of (Dürr et al., 2013, Theorem 2), (Grushkovskaya and Ebenbauer, 2016, Theorem 7). Due to the lack of space, we present here the proof of semi-global practical uniform asymptotic stability only.

Practical uniform stability

Let us show that the family of sets $\mathcal{L}_t$ is practically uniformly stable for system (1). For an arbitrary $\epsilon > 0$, let $c_1 \in (0, \epsilon)$. From the assumptions of the theorem, the family of sets $\mathcal{L}_t$ is uniformly stable for system (2). It means that there exists a $\delta > 0$ such that, for all $t_0 \in \mathbb{R}^+$, if $\bar{x}^0 \in B_\delta(\mathcal{L}_{t_0})$ then $\bar{x}(t) \in \mathcal{B}_{c_1}(\mathcal{L}_t), t \in [t_0, \infty)$. Besides, from the global uniform attractiveness, for every $c_2 \in (0, \delta)$, there exists $t_f > 0$ such that, for all $t_0 \in \mathbb{R}^+$:

$$\bar{x}^0 \in B_\delta(\mathcal{L}_{t_0}) \Rightarrow \bar{x}(t) \in B_{c_2}(\mathcal{L}_t), t \in [t_0 + t_f, \infty). \quad (A.1)$$

Let $\xi = \min\{\epsilon - c_1, \delta - c_2\}$, and $I_1 = B_\delta(S) \supseteq B_\delta(\mathcal{L}_{t_0})$ for all $t_0 \in \mathbb{R}^+$. Since S is compact and the family of sets $\mathcal{L}_t$ is globally uniformly asymptotically stable, the solutions of system (2) with initial conditions from I_1 are uniformly bounded, so we may apply Theorem 1 which yields the existence of such an ω_0 that, for all $\omega > \omega_0$ and all $x^0 \in B_\delta(\mathcal{L}_{t_0})$, $\|x(t) - \bar{x}(t)\| < \xi$, $t \in [t_0, t_0 + t_f]$. Then, for each $t \in [t_0, t_0 + t_f]$, for all $\omega > \omega_0$, $x^0 \in B_\delta(\mathcal{L}_{t_0}) \Rightarrow x(t) \in B_\varepsilon(\mathcal{L}_t)$, $t \in [t_0, t_0 + t_f]$ and, moreover, estimate (A.1) implies that $x(t_0 + t_f) \in B_\delta(\mathcal{L}_{t_0+t_f})$. Therefore, repeating the above procedure with the other solution of (2) through $\bar{x}(t_0 + t_f)$ with the same choice of parameters, we obtain the stability property: for all $t_0 \in \mathbb{R}^+$, for all $\omega > \omega_0$,

$$x^0 \in B_\delta(\mathcal{L}_{t_0}) \Rightarrow x(t) \in B_\varepsilon(\mathcal{L}_t), \ t \in [t_0, \infty).$$

Practical uniform boundedness

Similarly, we prove the practical uniform boundedness of the solutions of system (1). Let $\delta>0$ and $c_1 \in (0, \delta)$. From the global uniform asymptotic stability of the family of sets $\mathcal{L}_t$ for (2), there exist $c_2>0$, $t_f>0$ such that, for all $t_0 \in \mathbb{R}^+$:

$$\bar{x}^0 \in B_\delta(\mathcal{L}_{t_0}) \Rightarrow \bar{x}(t) \in \mathcal{B}_{c_2}(\mathcal{L}_t), t \in [t_0, \infty),$$
$$\text{and } \bar{x}(t) \in \mathcal{B}_{c_1}(\mathcal{L}_t), t \in [t_0 + t_f, \infty)$$

Put $\epsilon>c_2$, $\xi=\min\{\delta-c_1, \epsilon-c_2\}$, $I_1=B_\delta(S)$. From Theorem 1, there exists an $\omega_0>0$ such that, for all $\omega>\omega_0$, $x^0 \in B_\delta(\mathcal{L}_{t_0})$, we have $\|x(t)-\bar{x}(t)\|<\xi$, $t \in [t_0, t_0+t_f]$. Thus, if $x^0 \in B_\delta(\mathcal{L}_{t_0})$ then $x(t) \in \mathcal{B}_\epsilon(\mathcal{L}_t), t \in [t_0, t_f]$, and $x(t_0+t_f) \in B_\delta(\mathcal{L}_{t_0+t_f})$. Repeating the same procedure for the other solution of (2) through $\bar{x}(t_0+t_f)$, we obtain the required property.

Practical uniform attractivity

Let $\delta, \varepsilon > 0$. Since $\mathcal{L}_t$ is practically uniformly stable, there exist $c_1 > 0$, $\omega_{0,1} > 0$, such that, for all $t_0 \in \mathbb{R}^+$, $\omega > \omega_{0,1}$,

$$x^0 \in B_{c_1}(\mathcal{L}_{t_0}) \Rightarrow x(t) \in B_\varepsilon(\mathcal{L}_t), \ t \in [t_0, \infty). \quad (A.2)$$

Let $\varepsilon_1 \in (0, c_1)$. Due to the uniform attractiveness of $\mathcal{L}_t$ for system (2), there exists a $t_f > 0$ such that, for all $t_0 \in \mathbb{R}^+$:

$$\bar{x}^0 \in B_\delta(\mathcal{L}_{t_0}) \Rightarrow \bar{x}(t) \in B_{\varepsilon_1}(\mathcal{L}_t), \ t \in [t_0 + t_f, \infty). \quad (A.3)$$

Besides, the uniform boundedness yields the existence of an $A>0$ such that, for every $t_0 \in \mathbb{R}^+$, if $\bar{x}^0 \in B_\delta(\mathcal{L}_{t_0})$ then $\bar{x}(t) \in B_A(\mathcal{L}_t)$ for all $t \in [t_0, \infty)$. Again, applying Theorem 1 with $I_1 = B_\delta(S)$, $\xi = c_1 - \varepsilon_1$, we conclude that there is an $\omega_{0,2}>0$ such that for all $\omega>\omega_{0,2}$, $x^0 \in B_\delta(\mathcal{L}_{t_0})$, we have $\|x(t)-\bar{x}(t)\|<\xi$, $t \in [t_0, t_0+t_f]$. Taking into account (A.3) we conclude that, for all $t_0 \in \mathbb{R}^+$, $\omega>\omega_{0,2}$, if $x^0 \in B_\delta(\mathcal{L}_{t_0})$ then $x(t_0+t_f) \in B_{c_1}(\mathcal{L}_{t_0+t_f})$. Combining the last estimate with (A.2) and choosing $\omega_0 = \max\{\omega_{0,1}, \omega_{0,2}\}$, we complete the proof of the practical uniform attractivity: for all $t_0 \in \mathbb{R}^+$, $\omega>\omega_0$, if $x^0 \in B_\delta(\mathcal{L}_{t_0})$ then $x(t) \in B_\varepsilon(\mathcal{L}_t)$, for all $t \in [t_0 + t_f, \infty)$.

Appendix B. PROOF OF THEOREM 3

Preliminary constructions

Condition C1.2 implies that, for each $g \in \Gamma$, $x^*(g)$ is the unique solution of the equations $\frac{\partial J(x,g)}{\partial x_s}=0$, $s=\overline{1,n}$, continuously depending on g. With the new variables

$$\tilde{x}_s = x_s - x_s^*(\gamma(t)), \quad s = \overline{1,n}, \quad (B.1)$$

system (6) can be represented in the following form:

$$\dot{\tilde{x}}_s = -c_s \frac{\partial J(\tilde{x} + x^*(\gamma(t)), \gamma(t))}{\partial \tilde{x}_s} - \dot{x}_s^*, \quad s = \overline{1,n}, \quad (B.2)$$

where $\dot{x}_s^* = \sum_{j=1}^m \frac{\partial x_s^*(\gamma(t))}{\partial \gamma_j} \dot{\gamma}_j(t)$.

The equation $\frac{\partial J(\tilde{x}+x^*(\gamma(t)),\gamma(t))}{\partial \tilde{x}}=0$ has the unique solution $\tilde{x}^*=0$ for all $t \in \mathbb{R}^+$, which is, in general, not the solution of (B.2). For the further study, we will use the argumentation similar to the proof of the stability under constantly acting perturbations in Malkin (1952).

Consider the function

$$V(\tilde{x}, \gamma(t)) = J(\tilde{x} + x^*(\gamma(t)), \gamma(t)) - J^*(\gamma(t)), \quad (B.3)$$

which is positive definite and strictly radially increasing because of properties of J and J^*. Moreover, for each constant value of $\gamma(t) = g \in \Gamma$, the time-derivative V' of the function (B.3) along the trajectories of system

$$\dot{\tilde{x}}_s = -\frac{\partial J(\tilde{x} + x^*(g), g)}{\partial \tilde{x}}$$

is negative-definite since $V' = -\sum_{j=1}^n \left(\frac{\partial J(\tilde{x}+x^*(g),g)}{\partial \tilde{x}_s}\right)^2$.

From C1.2, $V(\tilde{x}, \gamma(t)) \geq w_{11}(\|\tilde{x}\|)$ for all $t \in \mathbb{R}^+$, and $V' \leq -w_2(\|\tilde{x}\|)$ for all $t \in \mathbb{R}^+$. Note that due to the properties of the functions V, w_{11}, w_{12}, $\mathcal{E}_{w_{11}} \subseteq \mathcal{E}_V \subseteq \mathcal{E}_{w_{12}}$. For Δ defined from the conditions of the theorem, and an arbitrary positive $\lambda \in (0, w_{11}(\Delta))$, consider

$$\widetilde{\mathcal{L}}_{\lambda,t} = \{\tilde{x} \in \mathbb{R}^n : V(\tilde{x}, \gamma(t)) \leq \lambda\},$$

and its boundary $\partial\widetilde{\mathcal{L}}_{\lambda,t}=\{\tilde{x} \in \mathbb{R}^n : V(\tilde{x}, \gamma(t))=\lambda\}$. Defining the positive constants $\rho_1=w_{12}^{-1}(\lambda)$, $\rho_2=w_{11}^{-1}(\lambda)<\Delta$, we get

$$\rho_1 \leq \|\tilde{x}\| \leq \rho_2 \text{ for all } \tilde{x} \in \partial\widetilde{\mathcal{L}}_{\lambda,t}.$$

Let $\alpha_1 = \inf_{\|\tilde{x}\| \in [\rho_1, \rho_2]} w_2(\|\tilde{x}\|) = w_2(\rho_1)$. Since $w_2 \in \mathcal{K}$ is positive definite and $\rho_1 > 0$, we have $\alpha_1 > 0$. Thus,

$$V' \leq -c\alpha_1 \text{ for all } \tilde{x} \in \partial\widetilde{\mathcal{L}}_{\lambda,t}, \ t \in \mathbb{R}^+,$$

where $c = \min\{c_1, \ldots, c_n\}$. Furthermore, the time derivative of the function (B.3) along the trajectories of system (B.2) can be estimated in the following way:

$$\begin{aligned}\dot{V} &= \sum_{s=1}^n \frac{\partial J}{\partial \tilde{x}_s}\left(\dot{\tilde{x}}_s + \dot{x}_s^*\right) + \frac{\partial J}{\partial \gamma}\dot{\gamma} - \frac{\partial J^*}{\partial \gamma}\dot{\gamma} \\ &= V' + \frac{\partial (J - J^*)}{\partial \gamma}\dot{\gamma} \\ &\leq -c\alpha_1 + \nu M \text{ for all } \tilde{x} \in \partial\widetilde{\mathcal{L}}_{\lambda,t}, \ t \in \mathbb{R}^+,\end{aligned} \quad (B.4)$$

where M is defined from C1.3. Taking $c > c^{(1)} = M\nu\alpha^{-1}$, we deduce from (B.4) that

$$\dot V \le -c^{(1)}\alpha_1 + \nu M < 0 \text{ for all } \tilde x \in \partial\widetilde{\mathcal{L}}_{\lambda,t},\ t \in \mathbb{R}^+. \quad \text{(B.5)}$$

The next step is to prove local uniform asymptotic stability of the family of sets $\widetilde{\mathcal{L}}_{\lambda,t}$ for system (B.2) with such parameters that $\min\{c_1,\dots,c_n\} > c^{(1)}$. Let $t_0 \in \mathbb{R}^+$ be an arbitrary fixed number, and denote $\tilde x^0 = x^0 + x^*(\gamma(t_0))$.

Invariance

Let us prove by contradiction the invariance of the family of sets $\widetilde{\mathcal{L}}_{\lambda,t}$ for system (B.2). For $\tilde x^0$ such that $V(\tilde x^0,\gamma(t_0)){\le}\lambda$, suppose that there exists a t_1 such that $\tilde x(t_1)$ is not in the set $\widetilde{\mathcal{L}}_{\lambda,t_1}$, i.e. $V(\tilde x(t_1),\gamma(t_1)){>}\lambda$. By the continuity of V, there exist $\tau_1{\in}[t_0,t_1)$, $\tau_2{\in}(\tau_1,t_1]$ such that $V(\tilde x(\tau_1),\gamma(\tau_1)){=}\lambda$, and $V(\tilde x(t),\gamma(t)){>}\lambda$ for all $t{\in}(\tau_1,\tau_2]$, which implies $\dot V \ge 0$ for all $\tilde x \in \partial\mathcal{L}_{\lambda,\tau}$. Comparing this inequality with (B.5), we get the contradiction.

Uniform stability

For an arbitrary $\varepsilon > 0$, let $\varepsilon^*{\in}\left(0,\min\left\{\varepsilon,\frac{2\hat\alpha}{H},\right\}\right)$, and

$$\delta{\in}\Big(0,\min\Big\{\varepsilon^*,\tfrac{1}{L}\Big(\varepsilon^*\hat\alpha-\tfrac12\varepsilon^{*2}H\Big),\ \Delta - w_{11}^{-1}(\lambda), \tfrac{1}{L}(w_{11}(\Delta)-\lambda)\Big\}\Big),$$

where $\hat\alpha{=}w_2(w_{12}^{-1}(\lambda))$, and the constants L, H are given by C1.3. Consider the solution of (B.2) with the initial condition $\tilde x^0{\in}B_\delta(\widetilde{\mathcal{L}}_{\lambda,t_0})$, and define $\hat\lambda{=}\sup_{\tilde x\in\bar B_\delta(\widetilde{\mathcal{L}}_{\lambda,t_0})} V(\tilde x,t_0)$. The case $\hat\lambda{\in}(0,\lambda]$ has been considered in the previous step. Assume now that $\hat\lambda{\in}(\lambda,\infty)$. To estimate the value of $\hat\lambda$, consider $\xi^1{\in}\bar B_\delta(\widetilde{\mathcal{L}}_{\lambda,t_0})$ such that $V(\xi^1,t_0){=}\hat\lambda$, and ξ^2 such that $V(\xi^2,t_0){=}\lambda$. From the mean value theorem,

$$V(\xi^1,t_0) - V(\xi^2,t_0) = \frac{\partial V}{\partial\tilde x}(\xi^2+\theta(\xi^1-\xi^2))(\xi^1-\xi^2)$$

with some $\theta\in(0,1)$. Note that $\|\xi^2+\theta(\xi^1-\xi^2)\| \le \|\xi^2\| + \|\xi^1-\xi^2\| \le w_{11}^{-1}(\lambda)+\delta < \Delta$, therefore, $\hat\lambda-\lambda < L\delta$. Due to the choice of δ, $w_{11}^{-1}(\hat\lambda)\le\Delta$, and

$$\hat\lambda < \lambda + \varepsilon^*\hat\alpha - \tfrac12\varepsilon^{*2}H. \quad \text{(B.6)}$$

Observe that, for each $l{\in}[\lambda,\hat\lambda]$, $\|\tilde x\|{\ge}w_{12}^{-1}(l){\ge}\rho_1$, for all $\tilde x{\in}\partial\widetilde{\mathcal{L}}_{l,t}$, $t{\in}\mathbb{R}^+$, and $\hat\rho_2{=}w_{11}^{-1}(\hat\lambda){\le}\Delta$. Therefore, we can conclude that $w_2(\|\tilde x\|){\ge}\alpha_1$, for all $\tilde x{\in}\widetilde{\mathcal{L}}^{\hat\lambda}_{\lambda,t}$, $t{\in}\mathbb{R}^+$, where $\widetilde{\mathcal{L}}^{\hat\lambda}_{\lambda,t}{=}\{\tilde x{\in}\mathbb{R}^n{:}V(\tilde x,\gamma(t)){\in}[\lambda,\hat\lambda]\}$, and inequality (B.5) holds for all $\tilde x{\in}\widetilde{\mathcal{L}}^{\hat\lambda}_{\lambda,t}$, $t{\in}\mathbb{R}^+$. Arguing similarly to the prove of the invariance, we receive: if $\tilde x^0{\in}B_\delta(\widetilde{\mathcal{L}}_{\lambda,t_0})$, then $\tilde x(t){\in}\widetilde{\mathcal{L}}_{\hat\lambda,t}$, for all $t{\ge}t_0$, and $\|\tilde x(t)\|{\le}w_{11}^{-1}(\hat\lambda)$, for all $\tilde x{\in}\widetilde{\mathcal{L}}_{\hat\lambda,t}$, $t{\ge}t_0$.

The next step is to prove that $\tilde x(t)\in B_\varepsilon(\widetilde{\mathcal{L}}_{\lambda,t})$ for all $t\ge t_0$. Fix an arbitrary $t_f\in(t_0,\infty)$. One can see that it is enough to show that for all $\tilde x{\in}\partial\widetilde{\mathcal{L}}_{\hat\lambda,t_f}$ there exists $\tilde x^1{\in}\partial\widetilde{\mathcal{L}}_{\lambda,t_f}$ such that $\|\tilde x{-}\tilde x^1\|{<}\varepsilon$. For $\tilde x^f$ such that $V(\tilde x^f,t_f){=}\hat\lambda$, let

$$\tilde x^1 = \tilde x^f - \varepsilon^*\frac{\nabla V(\tilde x^f,t_f)}{\|\nabla V(\tilde x^f,t_f)\|},$$

where $\nabla V(\tilde x^f,t_f)$ denotes the gradient vector of the function $V(\tilde x,t_f)$ calculated at x^f. Obviously, $\|\tilde x^1{-}\tilde x^f\|{<}\varepsilon$. Let us show that $V(\tilde x^1,t_f){\le}\lambda$. By using the Taylor formula with the Lagrange form of the reminder, we receive

$$\begin{aligned}V(\tilde x^1,t_f) &= V(\tilde x^f,t_f) + \frac{\partial V}{\partial\tilde x}(\tilde x^f)(\tilde x^1-\tilde x^f) + R(\tilde x^1,\tilde x^f)\\ &= \hat\lambda - \varepsilon^*\|\nabla V(\tilde x^f,t_f)\| + R(\tilde x^1,\tilde x^f),\end{aligned}$$

where $R(\tilde x^1,\tilde x^f){=}\frac12\sum_{i,j=1}^n\frac{\partial^2V}{\partial\tilde x_i\partial\tilde x_j}(\tilde x^f)(\tilde x_i^1{-}\tilde x_i^f)(\tilde x_j^1{-}\tilde x_j^f)$. Note that a similar technique based on the second-order expansion of a Lyapunov function was used, e.g., in Zuyev (2016); Zuyev et al. (2016). From the conditions of the theorem and from estimate (B.6),

$$V(\tilde x^1,t_f) \le \hat\lambda - \varepsilon^*\hat\alpha + \tfrac12\varepsilon^{*2}H < \lambda,$$

that proves the stability property.

Uniform attraction

Let δ be defined as in the previous step, and $\tilde x^0{\in}B_\delta(\mathcal{L}_{\lambda,t_0})$. Then there is a $\mu>0$ such that

$$\dot V \le -\mu^2 < 0 \text{ for } \tilde x \in \widetilde{\mathcal{L}}^{\lambda_1}_{\lambda,t},\ t\ge t_0, \quad \text{(B.7)}$$

where $\widetilde{\mathcal{L}}^{\lambda_1}_{\lambda,t}{=}\{\tilde x{\in}\mathbb{R}^n : V(\tilde x,\gamma(t)){\in}[\lambda,\lambda_1]\}$, and $\lambda_1{=}V(x^0,t_0)$. Assume that $\tilde x(t){\in}\widetilde{\mathcal{L}}^{\lambda_1}_{\lambda,t}$ for all $t{\ge}t_0$. Then, from (B.7):

$$\begin{aligned}V(\tilde x(t),\gamma(t)) &\equiv V(\tilde x^0,\gamma(t_0)) + \int_{t_0}^t\frac{dV(\tilde x(s),\gamma(s))}{ds}ds\\ &\le V(\tilde x^0,\gamma(t_0)) - \mu_2^2(t-t_0).\end{aligned}$$

It is easy to see that the left-hand side of the above inequality is positive for all $t \ge t_0$ while its right-hand side becomes negative for t large enough, so we get a contradiction. Therefore, there exists a $t_1{>}t_0$ such that $\tilde x(t_1){\in}\mathcal{L}_{\lambda,t_1}$. Since $\dot V\Big|_{\tilde x\in\partial\mathcal{L}_{\lambda,t}}{<}0$, we conclude that $\tilde x(t){\in}\mathcal{L}_{\lambda,t}$ for all $t{\in}[t_1,\infty)$. Taking into account (B.1), we obtain the statement of the theorem.

Appendix C. PROOF OF THEOREM 4

From the conditions C2.1 and C2.2, the time-derivative of function (B.3) along the trajectories of system (B.2) can be estimated as $\dot V{\le}{-}c\mu_1V{+}\nu\mu_2V^p$ for all $t{\in}\mathbb{R}^+$, where $c = \min\{c_1,\dots,c_n\}$. Thus, $\dot V < 0$ for all $\tilde x$, t such that $V(\tilde x,t) > \lambda$. Arguing similarly to the proof of Theorem 3 and taking into account condition C2.3, we conclude that the family of sets $\widetilde{\mathcal{L}}_{\lambda,t}$ is uniformly stable for system (6). Furthermore, the solution of the corresponding comparison equation $\dot V = -c\mu_1V + \nu\mu_2V^p$ with $V(t_0) = V^0 = J(x^0,\gamma(t_0)) - J^*(\gamma(t_0))$ is the function

$$\widehat V(t) = \left(\lambda^{1-p}\left(1-e^{-\kappa(t-t_0)}\right) + V_0^{1-p}e^{-\kappa(t-t_0)}\right)^{1/(1-p)},$$

with $\kappa{=}c\mu_1(1{-}p)$, so it can be shown that the family of sets $\widetilde{\mathcal{L}}_{\lambda,t}$ is invariant. Furthermore, C2.3 yields that for all $t_0{\in}\mathbb{R}^+$, the solutions of (B.2) are uniformly bounded:

$$\|\tilde x(t)\| \le w^{-1}(\hat V) \le w^{-1}\left(\lambda^{1-p}+V_0^{1-p}\right),\ t\ge t_0. \quad \text{(C.1)}$$

Putting for every $\varepsilon{>}0$ and for $t_0{\in}\mathbb{R}^+$ such that $V^0 = V(\tilde x(t_0),t_0) > \lambda$,

$$t_1 \ge \frac{1}{\kappa}\ln\left(\frac{w_{12}(\|x^0\|)^{1-p}-\lambda^{1-p}}{(\lambda+\varepsilon)^{1-p}-\lambda^{1-p}}\right),$$

we conclude that the corresponding solution $x(t)\in\widetilde{\mathcal{L}}_{\lambda+\varepsilon,t}$ for all $t\in[t_0+t_1,\infty)$, so the solutions of system (B.2) with arbitrary initial conditions reach an arbitrary small neighborhood of the the family of sets $\widetilde{\mathcal{L}}_{\lambda,t}$ in a finite time.