THE FINGERPRINTS OF BLACK HOLES - SHADOWS AND THEIR DEGENERACIES

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Abstract. We show that, away from the axis of symmetry, no continuous degeneration exists between the shadows of observers at any point in the exterior region of any Kerr-Newman black hole spacetime of unit mass. Therefore, except possibly for discrete changes, an observer can, by measuring the black holes shadow, determine the angular momentum and the charge of the black hole under observation, as well as the observer's radial position and angle of elevation above the equatorial plane. Furthermore, his/her relative velocity compared to a standard observer can also be measured. On the other hand, the black hole shadow does not allow for a full parameter resolution in the case of a Kerr-Newman-Taub-NUT black hole, as a continuous degeneration relating specific angular momentum, electric charge, NUT charge and elevation angle exists in this case.

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1. Introduction

The shadow of the black hole is defined as the set of trajectories on which no light from a background source, passing a black hole, can reach the observer. There is hope for the Event Horizon Telescope to be able to resolve the black hole at the center of the Milky Way (Sgr A*) well enough that one can compare it to the predictions from theoretical calculations, see for example [5]. Therefore, analyzing the shadows of black holes is of direct astronomical interest. The first discussion of the shadow in Schwarzschild spacetimes can be found in [19], and, for extremal Kerr at infinity, it was later calculated in [1]. The perspective of an actual observation has led to a number of advancements in the theoretical treatment of black hole shadows in recent years [2, 4, 7, 8, 18, 20, 21]. In [10, 12] possible ways to extract the black hole parameters from the observation of the shadow have been explored. However, a strict treatment of the question "How much information about the black hole is there in the shape of the shadow?" has, to our knowledge, not been carried out. The only work we are aware of, that takes into account the fact that an observer cannot a priori know what his detailed motion with respect to the manifold is, can be found in [6]. However, the focus in that work is more on the explicit deformation due to different velocities rather than a systematic study on how the freedom of picking any observer at a point influences the possibility of extracting information about the black hole from the shape of the shadow.

The most important conceptual idea introduced in the present work is the notion of what it means for the shadows at two points to be degenerate. In the case of degeneracy there exist two distinct observers for which the shadow is absolutely identical. Consequently, an observer cannot distinguish - from shape and size of the shadow alone - between the two situations.

We will put the concept of degeneracy to work in this paper by proving the existence of two continuous degeneracies, one parameter curves in the parameter space of observers in the exterior region of Kerr-Newman-Taub-NUT spacetimes. Beyond that, we show that there are no further continuous degeneracies.

Note that even though the shadow parametrization in [8] is given for the entire Plebański-Demiański class of black hole spacetimes, we will focus on the subclass of Kerr-Newman-Taub-NUT black holes in the present work. This has two reasons, first for observations within our galaxy the cosmological constant should be negligible, second including the cosmological constant increases the complexity of the arguments without providing additional insight. The case with cosmological constant will be treated in a separate paper.

Overview of this paper. In section 2 we collect some background on the Kerr-Newman-Taub-NUT spacetime. We discuss its properties and discuss the geodesic equation in these spacetimes in section 2.1. In section 3 we introduce the framework for the discussion of the black hole shadows, in particular the notion of the celestial sphere. In section 3.2 we discuss the shadow for observers at points of symmetry. We use this context to introduce the formal definition of degeneracies and how they arise. In section 3.3 we recall the explicit form of the shadow in the spacetimes we consider. In section 4 we introduce the recipe of the search for continuous degeneracies. Finally, in section 4.3 we present the proof for the main result of our paper. Appendix A is devoted to deriving several results on Möbius transformations needed in the main text and a list of somewhat long, explicit expressions have been shifted to Appendix B.

2. The Kerr-Newman-Taub-NUT Spacetime

In the following introduction of the spacetimes discussed in this paper, we follow the work of Grenzebach et al. [7]. However, we are setting the cosmological constant to zero. The Kerr-Newman-Taub-NUT black holes are stationary, axially symmetric, type D spacetimes. In Boyer-Lindquist coordinates, (t, r, θ, ϕ), the metric is given by [9, p. 314]:

$$\begin{aligned} \mathrm{d}s^2 =& \Sigma\left(\frac{1}{\Delta}\mathrm{d}r^2 + \mathrm{d}\theta^2\right) + \frac{1}{\Sigma}\left((\Sigma + a\chi)^2 \sin^2\theta - \Delta\chi^2\right)\mathrm{d}\phi^2 + \\ & \frac{2}{\Sigma}\left(\Delta\chi - a(\Sigma + a\chi)\sin^2\theta\right)\mathrm{d}t\mathrm{d}\phi - \frac{1}{\Sigma}\left(\Delta - a^2\sin^2\theta\right)\mathrm{d}t^2, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \Sigma &= r^2 + (l + a\cos\theta)^2, \\ \chi &= a\sin^2\theta - 2l(\cos\theta + C), \\ \Delta &= r^2 - 2Mr + a^2 - l^2 + Q^2. \end{aligned}$$

The coordinate t takes values in $(-\infty, \infty)$, r in (r_+, ∞), where r_+ is the largest root of $\Delta = 0$, while θ and ϕ are standard coordinates on the two-sphere. The relevant physical parameters in the metric are the mass M and charge Q, the former will be assumed to be positive based on physical reasons. Further, we will assume w.l.o.g. that the spin parameter satisfies $a \geq 0$. The NUT parameter l which can be interpreted as a gravitomagnetic charge can in principle take any value in $\mathbb{R}$.
In addition, there is a parameter C, first introduced by Manko and Ruiz [13], that induces pathologies on both parts of the rotation axis, unless $C = \pm 1$. In the present work we will simply ignore these pathologies and take $C = \pm 1$ and only consider the regular part of the rotation axis in these cases (the case $C = -1$ corresponds to the original definition of the NUT metric [15, 13]). For a detailed discussion of the rotation axis in the case $l \neq 0$ and $a \neq 0$ see [14].
We will frequently make use of the following orthonormal tetrad at point p:

$$\begin{aligned} e_0 &= \left.\frac{(\Sigma + a\chi)\partial_t + a\partial_\phi}{\sqrt{\Sigma\Delta}}\right|_p, & e_1 &= \left.\sqrt{\frac{1}{\Sigma}}\partial_\theta\right|_p, \\ e_2 &= \left.\frac{-(\partial_\phi + \chi\partial_t)}{\sqrt{\Sigma}\sin\theta}\right|_p, & e_3 &= -\left.\sqrt{\frac{\Delta}{\Sigma}}\partial_r\right|_p. \end{aligned} \tag{2.2}$$

The metric does not depend on the coordinates t and ϕ and therefore features at least two independent Killing vector fields independent of the choice of the parameters.
The Kerr-Newman-Taub-NUT family of metrics contains the Schwarzschild ($a = Q = l = 0$), Kerr ($Q = l = 0$), Reissner-Nordström ($a = l = 0$), Kerr-Newman ($l = 0$), and Taub-NUT ($a = Q = 0$) metrics as special cases.
Note that we are only interested in the exterior region of the black hole, hence the region at $r > r_+$, where the two horizons $r_\pm$ are defined by:

$$r_\pm = M \pm \sqrt{M^2 - a^2 + l^2 - Q^2}, \tag{2.3}$$

and we are assuming that the inequality $M^2 - a^2 + l^2 + Q^2 \geq 0$ holds. Note that $\Delta > 0$ is satisfied in the exterior region.

2.1. **Geodesic Equation.** We now focus our attention on null geodesics. For all members of the Kerr-Newman-Taub-NUT family of spacetimes there exist four linearly independent constants of motions for the geodesic equation. The norm of the tangent vector is directly related to the mass of the test body:

$$m = g_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu \tag{2.4}$$

which we will assume to be equal to zero from here on. The dot denotes differentiation with respect to the affine parameter λ. The two quantities arising from the

Killing vector fields ∂_t, ∂_ϕ:

$$E = -(\partial_t)_\mu \dot{\gamma}^\mu, \qquad L_z = (\partial_\phi)_\mu \dot{\gamma}^\mu \tag{2.5}$$

are, for appropriate choice of the parameter along the geodesic, the test body's energy and angular momentum in the direction of the axis of symmetry. The fourth constant of motion is called Carter's constant, K, and it originates from the existence of a Killing tensor, given by:

$$\sigma_{\mu\nu} = \Sigma((e_1)_\mu (e_1)_\nu + (e_2)_\mu (e_2)_\nu) - (l + a\cos\theta)^2 g_{\mu\nu}, \qquad K := \sigma_{\mu\nu}\dot{\gamma}^\mu \dot{\gamma}^\nu. \tag{2.6}$$

This tensor can be obtained from the general expression of the conformal Killing-Yano tensor for the Plebański-Demiański family of solutions, as presented in [11]. Carter's constant corresponds somewhat loosely to the total angular momentum of the test body.

The constants of motion can be used to write the geodesic equation as a system of first order ODEs, e.g. [3, p. 242]:

$$\dot{t} = \frac{\chi(L_z - E\chi)}{\Sigma \sin^2\theta} + \frac{(\Sigma + a\chi)((\Sigma + a\chi)E - aL_z)}{\Sigma\Delta}, \tag{2.7a}$$

$$\dot{\phi} = \frac{L_z - E\chi}{\Sigma \sin^2\theta} + \frac{a((\Sigma + a\chi)E - aL_z)}{\Sigma\Delta}, \tag{2.7b}$$

$$\Sigma^2 \dot{r}^2 = R(r, E, L_z, K) := ((\Sigma + a\chi)E - aL_z)^2 - \Delta K, \tag{2.7c}$$

$$\Sigma^2 \dot{\theta}^2 = \Theta(\theta, E, L_z, Q) := K - \frac{(\chi E - L_z)^2}{\sin^2\theta}. \tag{2.7d}$$

Note that $\Sigma + a\chi$ depends only on r, so the radial and the θ equations are separated. Moreover, they are homogeneous in E and thus for $E \neq 0$ we have:

$$R(r, E, L_z, Q) = E^2 R(r, 1, L_E, K_E), \tag{2.8}$$

$$\Theta(\theta, E, L_z, Q) = E^2 \Theta(r, 1, L_E, K_E), \tag{2.9}$$

where $L_E = L_z/E$ and $K_E = K/E^2$. These constants are invariant under affine reparametrization of the geodesic.

2.1.1. *The Trapped Set.* Trapping for null geodesics in black hole spacetimes describes the phenomenon that there exist null geodesics which never leave a spatially compact region of the exterior region. Consequently, these null geodesics never cross the future or the past event horizons, $\mathcal{H}^\pm$, and they also neither go to future or past null infinity $\mathscr{I}^\pm$. From the properties of the function $R(r, E, L_z, K)$, it follows (see [7] and [16]) that the trapped null geodesics are those which stay at a fix value of r and hence satisfy $\dot{r} = \ddot{r} = 0$, which corresponds to:

$$R(r, L_E, K_E) = \frac{d}{dr} R(r, L_E, K_E) = 0. \tag{2.10}$$

These equations can be solved for the constants of motion in terms of the constant value $r = r_{trapp}$ as [7]:

$$K_E = \left.\frac{16 r^2 \Delta}{(\Delta')^2}\right|_{r=r_{trapp}}, \qquad aL_E = (\Sigma + a\chi) - \left.\frac{4r\Delta}{\Delta'}\right|_{r=r_{trapp}}, \tag{2.11}$$

where Δ' denotes the derivative of Δ with respect to r.

The allowed values of r_{trapp} are obtained from the condition $\Theta \geq 0$ and read [7]:

$$(4r\Delta - \Sigma\Delta')^2 \leq 16 a^2 r^2 \Delta \sin^2\theta. \tag{2.12}$$

3. Trapping as a Set of Directions

For the following, instead of the set of trapped null geodesics we will be investigating future trapped or past trapped null geodesics. This kind of trapping exists at every point in the exterior region.

3.1. **Framework.** First we have to introduce the basic framework and notations. Let $\mathcal{M}$ be a smooth manifold with Lorentzian metric g. At any point p in $\mathcal{M}$ choose an orthonormal basis (e_0, e_1, e_2, e_3) for the tangent space, with e_0 time-like and future directed. It is sufficient to treat only past directed null geodesics, as the future directed ones can be obtained by a sign flip in the parametrization. The tangent vector to any past pointing null geodesic at p can be written as:

$$\dot{\gamma}(k|_p)|_p = \alpha(-e_0 + k_1 e_1 + k_2 e_2 + k_3 e_3), \tag{3.1}$$

where $\alpha = g(\dot{\gamma}, e_0) > 0$ and $k = (k_1, k_2, k_3)$ satisfies $|k|^2 = 1$, hence $k \in \mathbb{S}^2$. The geodesic is independent of the scaling of the tangent vector as this corresponds to an affine reparametrization for the null geodesic, so the specific value of α is irrelevant. The $\mathbb{S}^2$ is often referred to as the celestial sphere of a time-like observer at p, whose tangent vector is given by e_0, e.g. [17, p.8].
For the further discussion we fix the tetrad. We can make the following definition:

Definition 1. *Let $\gamma(k|_p)$ denote a null geodesic through p for which the tangent vector at p is given by equation* (3.1)*.*

It is clear that $\gamma(k_a|_p)$ and $\gamma(k_b|_p)$ are equivalent up to parametrization if $k_a = k_b$. Suppose now that $\mathcal{M}$ is the exterior region of a black hole spacetime with a complete $\mathcal{I}^\pm$ and boundary $\mathcal{H}^+ \cup \mathcal{H}^-$. As in [16] we can then define the following sets on $\mathbb{S}^2$ at every point p:

Definition 2. *The future infalling set:* $\Omega_{\mathcal{H}^+}(p) := \{k \in \mathbb{S}^2 | \gamma(k|_p) \cap \mathcal{H}^+ \neq \emptyset\}$.
The future escaping set: $\Omega_{\mathcal{I}^+}(p) := \{k \in \mathbb{S}^2 | \gamma(k|_p) \cap \mathcal{I}^+ \neq \emptyset\}$.
The future trapped set: $\mathbb{T}_+(p) := \{k \in \mathbb{S}^2 | \gamma(k|_p) \cap (\mathcal{H}^+ \cup \mathcal{I}^+) = \emptyset\}$.
The past infalling set: $\Omega_{\mathcal{H}^-}(p) := \{k \in \mathbb{S}^2 | \gamma(k|_p) \cap \mathcal{H}^- \neq \emptyset\}$.
The past escaping set: $\Omega_{\mathcal{I}^-}(p) := \{k \in \mathbb{S}^2 | \gamma(k|_p) \cap \mathcal{I}^- \neq \emptyset\}$.
The past trapped set: $\mathbb{T}_-(p) := \{k \in \mathbb{S}^2 | \gamma(k|_p) \cap (\mathcal{H}^- \cup \mathcal{I}^-) = \emptyset\}$

Definition 3. *We refer to the set $\Omega_{\mathcal{H}^-}(p) \cup \mathbb{T}_-(p)$ as the shadow of the black hole.*

Note that light from a background source, i.e. not in between the black hole and the observer and sufficiently far away, can only reach the observer in the set $\Omega_{\mathcal{I}^-}(p)$ and hence the shadow will be black. For any practical purposes one can only extract information about the boundary of the shadow from an observation. In [16] it was shown that for the Kerr-Newman-Taub-NUT black hole the boundary of the shadow is given by the set $\mathbb{T}_-(p)$ and that this set consists of those directions that asymptote to the trapped null geodesics in the past.

3.2. **Degeneration for observers located on an axis of symmetry.** The following discussions applies for observers located on an axis of symmetry, i.e. an observer located at any regular point p in the exterior region of a black hole spacetime, for which there exists a one parameter family of diffeomorphisms with closed orbits that leave p invariant. This includes in particular any point in the exterior region of a spherically symmetric black hole spacetime, as well as observers located on the rotation axis of e.g. Kerr. The discussion for points of symmetry, is here treated in a separate section to introduce several important concepts needed for our main theorem. Points of symmetry are special with respect to degeneracies as it was shown in [16] that the shadow for observers at regular points of symmetry in the exterior of Kerr-Newman-Taub-NUT black holes is circular.
It is well-known (see e.g. [17, p.14]) that a change of observer (i.e. an ortochronus Lorentz transformation of the tetrad) corresponds to a conformal transformation on the celestial sphere, and vice versa. Restricting oneself to orientation preserving transformations, they are isomorphic to Möbius transformations. A fundamental property of conformal transformations on $\mathbb{S}^2$ is that they map circles into circles. As a consequence if p_1 and p_2 are points in (possibly different) spacetimes in the

family under consideration, and both lie on an axis of symmetry then, upon identification of the two celestial spheres by a respective choice of time oriented orthonormal basis, there exists a Lorentz transformation[1] (**LT**) of the observer such that $\mathbb{T}_-(p_1) = \mathbf{LT}[\mathbb{T}_-(p_2)]$. This concept is central to our argument.

Definition 4. *The shadows at two points p_1, p_2 are called degenerate if, upon identification of the two celestial spheres by the orthonormal basis, there exists an element of the conformal group on $\mathbb{S}^2$ that transforms $\mathbb{T}_-(p_1)$ into $\mathbb{T}_-(p_2)$.*

Remark 5. *The shadow at two points p_1, p_2 being degenerate implies that for every observer at p_1 there exists an observer at p_2 for which the shadow on $\mathbb{S}^2$ is identical. Because this notion compares structures on $\mathbb{S}^2$, the two points need not be in the same manifold for their shadows to be degenerate. Just from the shadow alone an observer can not distinguish between these two configurations.*

Combining the discussion above with the shape of the shadow at points off the axis when $a \neq 0$ described in the next subsection, we concluide that the only reliable information that an observer knowing to live in the exterior of a Kerr-Newman-Taub-NUT black hole can extract from observing a circular shadow is that he/she is on an axis of the symmetry of the black hole.

3.3. **Parametrization of the Shadow for generic observers.** The shadow at any point in the exterior region of a Kerr-Newman-Taub-NUT spacetime has been explicitly obtained in [7] and its smoothness properties discussed in [16]. Here we only summarize the results that we need later.

From here on for the rest of the paper we will always assume that $a \neq 0$ as $a = 0$ has been treated in the previous section. Fixing the orthonormal tetrad (2.2) to which we will refer as "standard observer", the celestial sphere can be coordinated by standard spherical coordinates $\rho \in [0, \pi]$ and $\psi \in [0, 2\pi)$ so that (3.1) can be written as:

$$\dot{\gamma}(\rho, \psi)|_p = \alpha(-e_0 + e_1 \sin\rho\cos\psi + e_2 \sin\rho\sin\psi + e_3\cos\rho). \tag{3.2}$$

At any point p in the exterior region of a Kerr-Newman-Taub-NUT black hole the curve $\mathbb{T}_-(p)$ that defines the shadow is given by the parametric expression:

$$\begin{aligned} \sin\psi &= \frac{\Delta'(x)\{x^2 + (l + a\cos[\theta(p)])^2\} - 4x\Delta(x)}{4ax\sqrt{\Delta(x)}\sin[\theta(p)]} && \text{(3.3a)} \\ &:= f(x, \theta, M, a, Q, l), \\ \sin\rho &= \frac{4x\sqrt{\Delta(r(p))\Delta(x)}}{\Delta'(x)(r(p)^2 - x^2) + 4x\Delta(x)} && \text{(3.3b)} \\ &:= h(x, r, M, a, Q, l), \end{aligned}$$

where the parameter x takes values in the compact interval $[r_{min}(\theta(p)), r_{max}(\theta(p))]$ and $r_{min}(\theta)$ and $r_{max}(\theta)$ are obtained by solving the equality case in (2.12). Geometrically they correspond to the smallest and largest values that r can take at the intersection of a cone of constant θ with the area of trapping. The parameter x corresponds to the asymptotic value of r along the past null geodesic with initial tangent vector along the direction defined by $\{\rho(x), \psi(x)\}$. Note that the shadow curve is independent of the Manko-Ruiz parameter C.

In [7] the parametrization of the shadow curve was in fact obtained for the more general case of the Kerr-Newman-Taub-NUT-(anti-)de Sitter spacetime family. It was further extended in [8] to the full Plebański-Demiański class. A rigorous proof of the fact that the sets $\mathbb{T}_-(p)$ and $\mathbb{T}_+(p)$ given by this parametrization are smooth

[1]By this we always mean an ortochronus Lorentz transformation.

curves on the celestial sphere in the exterior region of any observer in a Kerr-Newman-Taub-NUT spacetime was given in [16].

One important observation, already made in [7], is that the shadow for the standard observer is symmetric on the celestial sphere with respect to the $k_1 = 0$ plane (i.e. the great circle in the celestial sphere defined by the meridians $\psi = \pi/2$ and $\psi = -\pi/2$) This is simply due to the form of equation (2.7d) that gives two solution $\pm\sqrt{\Theta(\theta, L_E, K_E)/\Sigma}$ for any combination of conserved quantities L_E and K_E. Therefore if $(k_1, k_2, k_3) \in \mathbb{T}_-(p)$ then we always have that $(-k_1, k_2, k_3) \in \mathbb{T}_-(p)$. Further note that from the radial equation (2.7c) we get immediately that if $k = (k_1, k_2, k_3) \in \mathbb{T}_+(p)$ then $k = (k_1, k_2, -k_3) \in \mathbb{T}_-(p)$. Hence the properties of the past and the future trapped sets are equivalent. In particular this implies that if there exists a conformal map Ψ from $\mathbb{T}_-(p_1)$ to $\mathbb{T}_-(p_2)$ then there exists another conformal map, related to Ψ by conjugation with a reflection about the $k_3 = 0$ plane, that maps $\mathbb{T}_+(p_1)$ to $\mathbb{T}_+(p_2)$.

An observer can only see the past, hence $\mathbb{T}_-(p)$, so we will concentrate on this curve in the search of degeneracies. However the fact that the properties $\mathbb{T}_-(p)$ and $\mathbb{T}_+(p)$ are equivalent tells us that our results will hold true for both.

4. Which degeneracies exist?

The question one would like to answer is, which observers can be fully distinguished based on the shape of the shadow they observe. A quick inspection of the equations (3.3) shows that the shadow is invariant up to a reparametrization $x \to x/M$ as long as the following quantities are constant:

$$\theta,\ \frac{r}{M},\ \frac{a}{M},\ \frac{Q}{M},\ \frac{l}{M}. \tag{4.1}$$

With this degeneracy we see that the shape of the shadow can only be affected by the change of dimensionless parameters. There is a discrete degeneracy for two observers with:

$$M_1 = M_2 \quad l_1 = -l_2 \quad r_1 = r_2 \quad a_1 = a_2 \quad Q_1 = Q_2 \quad \theta_1 = \pi - \theta_2 \tag{4.2}$$

In the case $l = 0$ this corresponds to a reflection of the observers position with respect to the equatorial plane, while when $l \neq 0$ the spacetime itself changes. In either case, two observers related by this transformation are fully indistinguishable from the observation of the shadow.

The comparison for the shadows of two arbitrary observers is a difficult problem and it is unclear to the authors how to determine all possible degeneracies. We will therefore restrict ourselves to continuous degeneracies. Hence a family of observers who form a C^1 curve in the space of parameters for which the shadows are indistiguishable.

In the following we will introduce a method to systematically search for continuous degeneracies and to prove when such degeneracies do not exist. We will heavily rely on the fact that we have an explicit parametrization $c(x; r, \theta, M, a, Q, l)$ for the curve defining the boundary of the shadow at a point p with coordinates r, θ in the exterior region of a Kerr-Newman-Taub-NUT spacetime with parameters (M, a, Q, l) (and curve parameter x).

To studying continuous degenerations we impose that the first variation of the curve is zero. From here on we will look at the shadow as a curve in the complex plane which is obtained from the parametrization (3.3) by stereographic projection of the celestial sphere [17, p.10]:

$$c(x) = \frac{X(x) + iY(x)}{1 - Z(x)}, \tag{4.3a}$$

$$X(x) = \sin(\rho)\sin(\psi) = h(x) \cdot f(x), \tag{4.3b}$$

$$Y(x) = \sin(\rho)\cos(\psi) = \pm h(x) \cdot \sqrt{1 - f^2(x)}, \tag{4.3c}$$

$$Z(x) = \cos(\rho) = -sgn\left(\frac{\partial h}{\partial x}\right)\sqrt{1 - h^2(x)}. \tag{4.3d}$$

The freedom of sign choice in (4.3c) comes from that fact that upon stereographic projection the symmetry with respect to the $k_1 = 0$ plane on the celestial sphere becomes a reflection symmetry with respect to the real axis for the curve in the complex plane. The sign in Z makes the curve C^1 and is the right choice to describe $\mathbb{T}_-$ [16]. If we were to describe $\mathbb{T}_+$ instead, the global minus sign in front would have to be dropped. Outside the area of trapping $sgn\left(\frac{\partial h}{\partial x}\right)$ has a fixed sign. Inside the area of trapping it changes sign when $x = r$, where $h = 1$ and thus $Z = 0$. From the definition of degeneracies for black hole shadows it follows that any degeneracy is characterized by a change in parameters together with a Möbius transformation on the shadow (as the Möbius transformation on the complex plane are equivalent to the orientation preserving conformal transformations on the Riemann sphere). Therefore when searching for continuous degeneracies we have to take the Möbius transformation into consideration. The limitation of our result to continuous degeneracies arises from the fact that we analyze small perturbations, hence we linearize the problem.

The first order of the action of any member of the conformal group on $\mathbb{S}^2$ on a curve is given by:

$$\Psi_\epsilon(c) = c(x) + \epsilon\vec{\xi}|_{c(x)} + \mathcal{O}(\epsilon^2), \tag{4.4}$$

where ϵ is a small parameter and where ξ is a conformal Killing vector field on $\mathbb{S}^2$. The first variation of the curve with respect to a parameter p is given by:

$$c(x; p + dp) = c(x, p) + \vec{V}_p dp + \mathcal{O}(dp^2), \tag{4.5}$$

where dp is an infinitesimal change of the parameter and V_p is given by $\partial_p c(x, p)$. The most generic variation vector for a curve is then:

$$\vec{V} = \sum_{p \in \mathcal{P} = \{r,\theta,M,a,Q,l\}} \vec{V}_p dp + \sum_{\xi \in Lie(Mb)} \vec{\xi}|_{c(x)}\epsilon_\xi. \tag{4.6}$$

We can now formulate a necessary and sufficient condition for the curve to be invariant under a continuous deformation. This is the case if there exists a nontrivial combination of dp and ϵ_ξ such that V is tangential to the curve. Letting n be the normal to the curve $c(x, p)$, the condition is that:

$$\vec{V} \cdot \vec{n} \equiv 0 \tag{4.7}$$

has a nontrivial solution in terms of dp and ϵ_ξ. Here we did not yet restrict the vector field ξ, however as we will discuss next, there are a priori restrictions on the most general conformal Killing vector capable of compensating the deformations induced by the change in parameters.

4.1. **Vector Fields from Möbius Transformations.** By the definition of degeneracies for every observer at point p_1 there exists an observer at point p_2 who observes the exact same shadow. For our purpose we can reformulate this statement the following way: If the shadows at two points are degenerate, then there exists a Möbius transformation that maps the stereographic projection of the shadow of a standard observer at point p_1 to the stereographic projection of the shadow of the shadow of the standard observer at point p_2.

As we have observed in section 3.3 the stereographic projection of the shadow of

any standard observer is reflection symmetric with respect to the real line. A rather involved argument (which may be of independent interest) is needed to show that only those conformal transformation that preserve the reflection symmetry can be used to "counter" the deformation from the change in parameters (as those correspond to a change between standard observers). The detailed proof is given in Appendix A.
On finds that the most general such conformal Killing vector is an arbitrary linear combination of the three linearly independent vector fields given by:

$$\vec{\xi}_1 = \partial_x, \qquad \vec{\xi}_2 = x\partial_y + y\partial_x, \qquad \vec{\xi}_3 = (x^2 - y^2)\partial_x + 2xy\partial_y, \tag{4.8}$$

in terms of Cartesian coordinates $\{x, y\}$ on the complex plane, i.e. $z = x + iy$.

4.2. **Conditions for Continuous Degenerations.** We now start with the explicit calculations. Most of them ae by no means difficult, but they are lengthy and have thus been performed mostly in Mathematica. Here we will describe the essential steps involved. From here on we will restrict to a domain of x such that $sgn\left(\frac{\partial h}{\partial x}\right)$ does not change. This does not restrict our argument, as our aim is to prove that a certain quantity is zero independent of x. So it is equivalent to consider the problem in an open and dense interval. With:

$$\vec{V}_p = \begin{pmatrix} \frac{d(Re(c(x,p)))}{dp} \\ \frac{d(Im(c(x,p)))}{dp} \end{pmatrix}, \tag{4.9}$$

and with the normal vector to a curve parametrized by x in two dimensions given by:

$$\vec{n} = \pm \begin{pmatrix} \frac{d(Im(c(x)))}{dx} \\ -\frac{d(Re(c(x)))}{dx} \end{pmatrix}, \tag{4.10}$$

we can calculate the various terms that show up in (4.7). Note here that the sign choice in the definition of the normal vector corresponds to the choice between the inward and the outward pointing normal to the curve. Because we want to find curves with $V \cdot n = 0$ it doesn't matter which orientation or normalization we choose for n as long as we choose it consistently, hence we pick the plus. From equation (4.3a) we directly get:

$$Re(c(x)) = \frac{X(f(x), h(x))}{1 - Z(h(x))}, \tag{4.11}$$

$$Im(c(x)) = \frac{Y(f(x), h(x))}{1 - Z(h(x))}. \tag{4.12}$$

Plugging everything in, we obtain the following result in terms of $f(x)$ and $h(x)$:

$$\vec{V}_p \cdot \vec{n} = \frac{h(x)\left(\frac{\partial f(x,p)}{\partial x}\frac{\partial h(x,p)}{\partial p} - \frac{\partial f(x,p)}{\partial p}\frac{\partial h(x,p)}{\partial x}\right)}{\sqrt{1-f^2(x)}\sqrt{1-h^2(x)}(1-\sqrt{1-h^2(x)})^2}, \tag{4.13}$$

$$\vec{\xi}_1 \cdot \vec{n} = \frac{\sqrt{1-h^2(x)}f(x)h(x)\frac{\partial f(x)}{\partial x} + (1-f^2(x))\frac{\partial h(x)}{\partial x}}{\sqrt{1-f^2(x)}\sqrt{1-h^2(x)}(1-\sqrt{1-h^2(x)})^2}, \tag{4.14}$$

$$\vec{\xi}_2 \cdot \vec{n} = \frac{h^2(x)\frac{\partial f(x)}{\partial x}}{\sqrt{1-f^2(x)}(1-\sqrt{1-h^2(x)})^2}, \tag{4.15}$$

$$\begin{aligned}\vec{\xi}_3 \cdot \vec{n} = & \frac{h^2(x)(1-\sqrt{1-h^2(x)})\left(f(x)h(x)\frac{\partial f(x)}{\partial x} + \frac{\partial h(x)}{\partial x} - f^2(x)\frac{\partial h(x)}{\partial x}\right)}{\sqrt{1-f^2(x)}\sqrt{1-h^2(x)}(1-\sqrt{1-h^2(x)})^4} \\ & - \frac{f(x)h^4(x)\frac{\partial f(x)}{\partial x}}{\sqrt{1-f^2(x)}\sqrt{1-h^2(x)}(1-\sqrt{1-h^2(x)})^4}.\end{aligned} \tag{4.16}$$

At this point it is important to note that $f(x, \theta, M, a, Q, l)$, $h(x, r, M, a, Q, l)$ and all their partial derivatives are rational functions in x after multiplication with $\sqrt{\Delta(x)}$. For a list of all partial derivatives of $f(x, \theta, M, a, Q, l)$ and $h(x, r, M, a, Q, l)$ see Appendix B. Hence any product of f, h and their derivatives which contain an even number of factors is a rational function in x, while any such product with an odd number of factors is a rational function in x after multiplication with $\Delta_r(x)$. (i.e. $h(x)f(x)\frac{\partial f(x,p)}{\partial x}\frac{\partial h(x,p)}{\partial p}$ and $h^3(x)\left(\frac{\partial f(x,p)}{\partial x}\right)^2\sqrt{\Delta(x)}$ are both rational functions in x).
Further we notice that away from the real axis we have $f^2(x) < 1$ and outside the area of trapping we always have $h^2(x) < 1$.

Definition 6. *An degeneration is called intrinsic when there is no need to act with a Möbius transformation to counter the deformation in the shadow due to the change in parameters.*

The condition for an intrinsic degeneracy of the shadow is then the existence of a non-trivial value of dp such that the following linear combination vanishes:

$$\sum_{p\in\mathcal{P}}\left(\frac{\partial f(x,p)}{\partial x}\frac{\partial h(x,p)}{\partial p}-\frac{\partial f(x,p)}{\partial p}\frac{\partial h(x,p)}{\partial x}\right)dp \equiv 0, \tag{4.17}$$

where $\mathcal{P}$ is the set of parameters within which we are searching for degeneracies of the shadow. If we now write down the general linear combination that we required to be zero in condition (4.7):

$$\beta\vec{\xi}_1\cdot\vec{n}+\alpha\vec{\xi}_2\cdot\vec{n}+\gamma\vec{\xi}_3\cdot\vec{n}+\sum_{p\in\mathcal{P}}\vec{V}_p\cdot\vec{n}dp \equiv 0, \tag{4.18}$$

we get one set of terms which are products of f, h and their derivatives with an odd number of total powers and another set of terms with an odd number of total powers but an additional factor of $\sqrt{1-h^2(x)}$. Now if $\sqrt{1-h^2(x)}$ is not a rational function (showing this will be part of our program), then for the above condition to be true, both sets of terms have to be equal to zero on their own, as adding a rational and an irrational function can never be equal to zero unless both functions themselves are equal to zero on their own. This gives us a system of two equations that we can solve for β and γ:

$$\begin{aligned}\beta =&\frac{\sum_{p\in\mathcal{P}}h(x)\left(\frac{\partial f(x,p)}{\partial x}\frac{\partial h(x,p)}{\partial p}-\frac{\partial f(x,p)}{\partial p}\frac{\partial h(x,p)}{\partial x}\right)dp}{2\left((1-h^2)f(x)h(x)\frac{\partial f(x)}{\partial x}-(1-f^2(x))\frac{\partial h(x)}{\partial x}\right)} \\ &+\alpha\frac{h^2(x)\frac{\partial f(x)}{\partial x}}{2\left(f(x)h(x)\frac{\partial f(x)}{\partial x}-(1-f^2(x))\frac{\partial h(x)}{\partial x}\right)},\end{aligned} \tag{4.19}$$

$$\begin{aligned}\gamma =&\frac{\sum_{p\in\mathcal{P}}h(x)\left(\frac{\partial f(x,p)}{\partial x}\frac{\partial h(x,p)}{\partial p}-\frac{\partial f(x,p)}{\partial p}\frac{\partial h(x,p)}{\partial x}\right)dp}{2\left((1-h^2)f(x)h(x)\frac{\partial f(x)}{\partial x}-(1-f^2(x))\frac{\partial h(x)}{\partial x}\right)} \\ &-\alpha\frac{h^2(x)\frac{\partial f(x)}{\partial x}}{2\left(f(x)h(x)\frac{\partial f(x)}{\partial x}-(1-f^2(x))\frac{\partial h(x)}{\partial x}\right)}.\end{aligned} \tag{4.20}$$

Now we know that β and γ both are constants. With the above result this is only possible if both terms are independent of x individually. Addding them and noticing that $\alpha = 0$ is always a possibility, we conclude that the condition for the existence of a degeneracy of the shadow within a certain set of parameters $\mathcal{P}$ is the existence of a non-trivial dp satisfying:

$$\frac{\partial}{\partial x}\left(\frac{\sum_{p\in\mathcal{P}} h(x)\left(\frac{\partial f(x,p)}{\partial x}\frac{\partial h(x,p)}{\partial p}-\frac{\partial f(x,p)}{\partial p}\frac{\partial h(x,p)}{\partial x}\right)dp}{2\left((1-h^2)f(x)h(x)\frac{\partial f(x)}{\partial x}-(1-f^2(x))\frac{\partial h(x)}{\partial x}\right)}\right)\equiv 0 \tag{4.21}$$

has a non-trivial solution in terms of the dp. If only the trivial solution exists than there exists no continuous degeneracy within the parameter set $\mathcal{P}$. We can ellaborate further on the role of α as follows: either

$$\frac{\partial}{\partial x}\left(\frac{h^2(x)\frac{\partial f(x)}{\partial x}}{2\left(f(x)h(x)\frac{\partial f(x)}{\partial x}-(1-f^2(x))\frac{\partial h(x)}{\partial x}\right)}\right)\equiv 0, \tag{4.22}$$

and α can take any value but with the the only effect of modiying both β and γ, or (4.22) does not hold, and we must take $\alpha = 0$. In no case the validity of (4.22) affects the existence of a degeneracy. In fact, one can show that the above condition can never be satisfied but since this is of no relevance to our argument, we will omit the proof here and just assume α to be zero. For the actual proof this leaves us with the following strategy:

(1) Check whether or not intrinsic degeneracies exist using condition (4.17).
(2) Check whether eventual intrinsic degeneracies can be used to eliminate parameters from the set within which one has to search for degeneracies.
(3) Check that $\sqrt{1-h^2(x)}$ is an irrational function for all possible combinations of the remaining parameters in $\mathcal{P}$.
(4) Check that the denominator of the first term in (4.19) is not equivalent to zero for all possible combinations of parameters.
(5) Check whether there exist any non-trivial solutions to (4.21) for all possible combinations of the remaining parameters.

Note that wherever in these steps we have to show that something is either equivalent to zero or not equivalent to zero the expressions we have to check are polynomials. Hence the condition is that the coefficients for every order of x have to be equal to zero simultaneously, which leaves us with a system of equations that has to be satisfied. These system of equations in the steps above are of different complexities, however for most steps too involved to be solved by hand. Note that the derivation until here is independent of the detailed form of $f(x)$ and $h(x)$ and hence in principle valid for any black hole spacetime where the parametrization (3.3) exists, hence given the results in [8] the following analysis can in principle be carried out for the entire Plebański-Demiański class of black hole spacetimes.

4.3. **Continuous Degeneracies.** We now apply the above recipe to the Kerr-Newman-Taub-NUT family, which we are interested in the present work, hence our set of parameters is given by $\mathcal{P} = \{M, a, Q, l, r, \theta\}$ for this section. We will here rederive the degeneracy already mentioned in (4.1) to illustrate the way the method works. We start with the first point in the list, the search for intrinsic degeneracies.

Lemma 7. *There are two intrinsic degeneracies given by:*

$$\frac{a}{M} = C_1, \qquad \frac{r}{M} = C_2, \qquad \frac{Q}{M} = C_3, \qquad \frac{l}{M} = C_4, \qquad \theta = C_5. \tag{4.23}$$

and

$$a\sin\theta = C_1, \qquad l + a\cos\theta = C_2, \qquad Q + 2a\cos\theta(l + a\cos\theta) = C_3,$$
$$r = C_4, \qquad M = C_5.$$

Proof. The derivatives in Appendix B are written such that every term in (4.17):

$$\left(\frac{\partial f}{\partial x}\frac{\partial h}{\partial a} - \frac{\partial f}{\partial a}\frac{\partial h}{\partial x}\right) da + \left(\frac{\partial f}{\partial x}\frac{\partial h}{\partial M} - \frac{\partial f}{\partial M}\frac{\partial h}{\partial x}\right) dM + \left(\frac{\partial f}{\partial x}\frac{\partial h}{\partial r}\right) dr$$
$$+ \left(\frac{\partial f}{\partial x}\frac{\partial h}{\partial Q} - \frac{\partial f}{\partial Q}\frac{\partial h}{\partial x}\right) dQ + \left(\frac{\partial f}{\partial x}\frac{\partial h}{\partial l} - \frac{\partial f}{\partial l}\frac{\partial h}{\partial x}\right) dl - \left(\frac{\partial f}{\partial \theta}\frac{\partial h}{\partial x}\right) d\theta \equiv 0 \tag{4.24}$$

has the same denominator. The numerator in the above equation is a polynomial in x of order 11. Hence this condition gives us a system of 11 equation. Solving this system leaves us with two degrees of freedom. One of the solution is given by $dl = l dM/M$. Insewrting this yields the following set of ODEs:

$$\frac{da}{a} = \frac{dM}{M}, \qquad \frac{dr}{r} = \frac{dM}{M}, \qquad \frac{dQ}{Q} = \frac{dM}{M}, \qquad \frac{dl}{l} = \frac{dM}{M}, \qquad d\theta = 0, \tag{4.25}$$

which can be integrated to give:

$$\frac{a}{M} = C_1, \qquad \frac{r}{M} = C_2, \qquad \frac{Q}{M} = C_3, \qquad \frac{l}{M} = C_4, \qquad \theta = C_5. \tag{4.26}$$

where C_1, C_2, C_3, C_4 and C_5 are integration constants. We now explain a method that will be used several times below. A degeneration involving four integration constants means that (locally) the parameter space is threaded by a congruence of curves, with any two points along the same curve having identical shadows. Consider now another degeneracy, independent of the previous one. This means that the vector field tangent to the new congruence of curves is linearly independent of the previous one. Consider a point p in parameter space where the two vectors are linearly independent. At that point, and in fact in an open neighbourhood thereof, the two congruences of curves are nowhere tangent to each other. It follows that the shadow at any point in this open set is now invariant under a two parameter family of transformations, i.e. a two-dimensional surface in parameter space. Consider a hypersurface passing through p and transverse to the first congruence. The intersection of this hypersurface with the invariant two-dimensional surface is necessarily a non-trivial degeneration curve, which obviously does not belong to the first congruence. This means that we can look for linearly independent degenerations by restricting the problem to a hypersurface transverse to the original one. This greatly simplifies the computations. Geometrically, the procedure is analogous to performing a gauge fixing. In summary, the idea is to use the existing degenerations to reduce the order of the problem. Point (2) in the strategy outlined above refers precisely to this "gauge fixing" procedure.

Applying this strategy, the second degeneracy condition can be found without loss of generality by setting $dM = 0$ (the foliation by hypersurfaces is given by $M =$ const, which indeed is transverse to the congruence of curves defined by (4.26)). Solving the set of equations obtained from (4.24) with $dM = 0$ yields:

$$d\theta = \frac{\sin\theta}{a} dl, \quad da = -\cos\theta dl, \quad dQ = 2(l + a\cos\theta) dl, \quad dr = 0, \quad dM = 0, \tag{4.27}$$

and can be integrated to yield:

$$a\sin\theta = C_1,\ l + a\cos\theta = C_2,\ Q + 2a\cos\theta(l + a\cos\theta) = C_3,\ r = C_4,\ M = C_5, \tag{4.28}$$

where C_1, C_2, C_3, C_5 and C_5 are again integration constants. □

The first degeneracy can be "gauge fixed" immediately and globally by fixing $M =$ const and restricting the whole problem to this lower dimensional parameter space. We want to exploit in a similar way the second degeneracy and reduce the problem further. The vector field along the second degeneration can be read off directly from (4.27) and it always has a non-zero component along the l direction. Thus, a suitable family of transverse hypersurfaces is $l = \text{const}$.

Now we want to prove that if we set $M = \text{const}$ and $l = \text{const}$ then there is no further degeneracy in $\mathcal{P} = \{a, r, Q, \theta\}$. We start with point (3) of the recipe in the previous section.

Lemma 8. *The function* $\sqrt{1-h^2(x)}$ *is irrational.*

Proof. We need to prove that

$$[\Delta'(x)(r^2-x^2)+4x\Delta(x)]^2 - 16x^2\Delta(r)\Delta(x) = P(x)^2 \tag{4.29}$$

admits no solution where $P(x)$ is a polynomial on x. The leading term in the right-hand side is $4x^6$, which combined with the fact that a global sign in $P(x)$ is irrelevant, shows that $P(x)$ must be of the form $P(x) = 2x^3 + K_2x^2 + K_1x + K_0$. The zero, first and fifth order coefficients in (4.29) are immediately solved to give:

$$K_2 = -6M, \qquad K_1 = -2\epsilon(r^2+2\beta), \qquad K_0 = 2\epsilon M r^2,$$

where $\epsilon = \pm 1$ and $\beta := a^2 - l^2 + q^2$. The choice $\epsilon = -1$ makes $P(x) \equiv \Delta'(x)(r^2 - x^2) + 4x\Delta(x)$ and equation (4.29) becomes $16x^2\Delta(r)\Delta(x) = 0$, which is impossible for r in the exterior region. For the choice $\epsilon = 1$ the coefficients in x^4 and x^3 in (4.29) impose, respectively:

$$\begin{aligned} 2Mr + \beta &= 0, \\ -2(2Mr+\beta) - \Delta(r) + \beta &= 0. \end{aligned}$$

Since in the exterior region $r > r_+ > 0$, the first requires $\beta < 0$ and the second $\Delta(r) = \beta < 0$, which is impossible. We conclude that $\sqrt{1-h^2(x)}$ is an irrational function. □

Next we check that the denominator in (4.21) is non-trivial for all allowed parameter combinations.

Lemma 9.

$$\left((1-h^2)f(x)h(x)\frac{\partial f(x)}{\partial x} - (1-f^2(x))\frac{\partial h(x)}{\partial x}\right) \not\equiv 0 \tag{4.30}$$

Proof. Plugging in the parametrization (3.3) we get:

$$\frac{\sqrt{\Delta(r)}\{-x(\Delta'(x))^2 - 2\Delta(x)(\Delta'(x) - x\Delta''(x))\}g_1(x)}{16a^2x\sqrt{\Delta(x)}\sin^2\theta(4x\Delta(x)+(r^2-x^2)\Delta'(x))^3} \not\equiv 0, \tag{4.31}$$

where $g_1(x)$ is given by the following polynomial of order six:

$$\begin{aligned} g_1(x) =& 16x\left[x^2+(l+a\cos\theta)^2\right]\Delta(r)\left(2\left[x^2+(l+a\cos\theta)^2\right]\Delta'(x) - 8x\Delta(x)\right) - \\ &\left(4x\Delta(x)+(r^2-x^2)\Delta'(x)\right)\left(32a^2x(r^2-x^2)\sin^2\theta\right. \\ &\left.+(r^2+(l+a\cos\theta)^2)\left(-32x\Delta(x)+8(x^2+(l+a\cos\theta)^2)\Delta'(x)\right)\right). \end{aligned} \tag{4.32}$$

The first factor in the numerator of (4.31) is clearly non-zero for an observer in the exterior region. The second factor is a poynomial in x with leading term $-4x^3$, hence non-indetically zero. The zeroth order coefficient for $g_1(x)$ is:

$$-32M^2r^2(l+a\cos\theta)^2(r^2+(l+a\cos\theta)^2), \tag{4.33}$$

thus the only way this can be zero for an observer in the exterior region is if $l = -a\cos\theta$. Plugging that in for the other coefficient we get that the first order coefficient is given by $-64MQ^2r^4$ and the fifth order coefficient is given by $-192M(-Q^2+2Mr)$. Those can never be equal to zero at the same time which finishes the proof for this lemma. □

When we plug the parametrization into the numerator inside the parenthsis in (4.21) we get this is equal to:

$$\frac{\{-x(\Delta'(x))^2 - 2\Delta(x)(\Delta'(x) - x\Delta''(x))\}g_2(x)}{2a^2\sqrt{\Delta(x)}\sin\theta(4x\Delta(x) + (r^2 - x^2)\Delta'(x))^3}, \tag{4.34}$$

where $g_2(x)$ is given by the following polynomial of order five:

$$\begin{aligned} g_2(x) =& 2a\left[x^2 + (l + a\cos\theta)^2\right](4x\Delta(x) + (r^2 - x^2)\Delta'(x))\cdot \\ & (2QdQ + 2ada + (2r - 2M)dr) - \\ & \Big\{16x(x^2 - r^2)(da + a\cot\theta d\theta)\Delta(x) + 16aQx\left[r^2 + (l + a\cos\theta)^2\right]dQ + \\ & 16a^2x\left[r^2 + (l + a\cos\theta)^2\right]da + (2x - 2M)\cdot \\ & \Big[8ar\left[x^2 + (l + a\cos\theta)^2\right]dr + 4(r^2 - x^2)(x^2 + l^2 - a^2\cos^2\theta)da \\ & + \frac{4a(r^2 - x^2)(2al + \cos\theta(x^2 + l^2 + a^2 + a^2\sin^2\theta))}{\sin\theta}d\theta\Big]\Big\}\Delta(r). \end{aligned} \tag{4.35}$$

We can now plug (4.31) and (4.34) into condition (4.21) to obtain:

$$\frac{\partial}{\partial x}\left(\frac{8x\sin\theta g_2(x)}{\sqrt{\Delta(r)}g_1(x)}\right) \equiv 0. \tag{4.36}$$

At this point we introduce the notion of a restricted degeneracy.

Definition 10. *A restricted degeneracy is one where a combination of parameters has to be zero instead of just being constant.*

Since a degeneracy is defined by a curve in parameter space, two things may happen. Either the curve is tangent to the submanifold of parameter space defined by the restrited degeneracy, or it is transverse to it. Im the latter case, the curve leaves immediately the submanifold, and hence the degeneration curve must exist away from the submanifold. It follows that the only degeneracies that one could be missing by the general analysis are those satisfying not only that the parameters are zero, but also that their variation is zero, so that the curve is tangent to the restricted submanifold.

An example of a restricted degeneracy is $\sin\theta = 0$, under which condition (4.36) is obviously satisfied. By the argument above, the corresponding degeneration curves must satsify $d\theta = 0$. The other parameters can vary arbitrarily in this case. Thus, with a slight abuse, we recover the degeneracy on the rotation axis. Of course, the argument in this case is not fully sound since it ignores the fact that the coordinate system and the shadow parametrization breaks down on the axis. This argument just serves the purpose to illustrate the concept of a restricted degeneracy. The situation on the rotation axis was treated properly in section 3.2. In the following we will always assume that $\sin\theta \neq 0$.

Lemma 11.

$$\frac{\partial}{\partial x}\left(\frac{8x\sin\theta g_2(x)}{\sqrt{\Delta(r)}g_1(x)}\right) \equiv 0 \qquad \Longrightarrow \qquad g_2(x) \equiv 0. \tag{4.37}$$

Proof. First note that (4.36) can only be true if either $g_1(x) = Bxg_2(x)$ for some non-zero constant B, or if $g_2(x) \equiv 0$. We will now exclude the first possibility. Note that the zeroth order coefficient of $xg_2(x)$ is zero and with the zeroth order coefficient for $g_1(x)$ given in (4.33). Thus, the only chance for the two to be proportional is if:

$$l = -a\cos\theta. \tag{4.38}$$

We fixed l this requires for $d\theta = \cot\theta a^{-1} da$ to hold. However plugging these two condition into $g_2(x)$ we get that now its zeroth order coefficient also vanishes. Thus not only the zeroth, but also the first term in $g_1(x)$ must be zero. Plugging (4.38) into $g_1(x)$ it follows that its first order coefficient is given by $-64MQ^2r^4$, which vanishes only if $Q = 0$. Setting $Q = 0$ and $dQ = 0$ everywhere, the first order coefficient in $g_2(x)$ zero, while the second order coefficient of $g_1(x)$ is $96M^2r^4$. This is manifestly non.zero and we reach a contradiction. Thus, the only possibility is $g_2(x) = 0$ and the Lemma is proved. □

From this lemma, the remaining task is to show that there exists no non-trivial solution for $g_2(x) \equiv 0$ which is equivalent to condition (4.17). We emphasize, in particular, that the previous lemma already implies that all degenerations of the shadow must be intrinsic.

The next Theorem, which is the main result of this paper, proves that there are no more degeneracies than those already found.

Theorem 12. *The only continuous degenerations of the black hole shadow for observers located at coordinate position r, θ in the exterior region of Kerr-Newman-Taub-NUT black holes with parameters M, a, Q and l are given for observers such that their parameters have the same value for all the following functions:*

$$\frac{a}{M} = C_1, \qquad \frac{r}{M} = C_2, \qquad \frac{Q}{M} = C_3, \qquad \frac{l}{M} = C_4, \qquad \theta = C_5. \tag{4.39}$$

or

$$a\sin\theta = C_1,\ l + a\cos\theta = C_2,\ Q + 2a\cos\theta(l + a\cos\theta) = C_3,\ r = C_4\ M = C_5. \tag{4.40}$$

Proof. The two degeneracies have already been derived in Lemma 7. Given Lemma 11 we know that the condition for degeneracies to exist is given by (4.36). The only thing remaining to show is that $g_2(x) \equiv 0$ has no non-trivial solutions. The highest order coefficient is given by:

$$aQdQ + a(r - M)dr + (a^2 - \Delta(r))da - \frac{a\cos\theta\Delta(r)}{\sin\theta} d\theta = 0. \tag{4.41}$$

We solve this for dr and substitute back into g_2. This leads to a a third order polynomial in x, i.e. $g_2 = \sum_{i=0}^{3} w_i x^i$, and each coefficient w_i must vanish. The combination $Mw_3 + w_2$ is very simple:

$$Mw_3 + w_2 = -\frac{16\Delta(r)M(r^2 + (l + a\cos\theta)^2)}{\sin\theta} (a\cos\theta d\theta + \sin\theta da) = 0.$$

The first factor is nowhere zero in the exterior region, so we can solve for da:

$$da = -\frac{a\cos\theta}{\sin\theta} d\theta, \tag{4.42}$$

and substitute back into $g_2(x)$, which factorizes as:

$$g_2(x) = \frac{16a\Delta(r)(r - x)}{r - M} g_3(x),$$

where $g_3(x)$ is a quadratic polynomial in x. Obviously, g_2 is identically zero only if $g_3 \equiv 0$. The highest order term of g_3 is:

$$-rQdQ + \frac{a}{\sin\theta} \left(l(M - r) + aM\cos\theta\right) d\theta = 0. \tag{4.43}$$

At this point we need to split the treatment in two cases depending on whether $Q \equiv 0$ or not.

For the case with $Q \not\equiv 0$, we solve (4.43) for dQ and substitute back into g_3 to obtain:

$$g_3(x) = \frac{aM(r - M)(r^2 + (l + a\cos\theta)^2)}{r\sin\theta} (l + a\cos\theta)\, d\theta.$$

Thus $g_3(x) \equiv 0$ can can only happen if $l + a\cos\theta = 0$. Taking its differential and inserting da from (4.42) yields $-a(\sin\theta)^{-1}d\theta = 0$, hence $d\theta = da = dr = dQ = 0$ and we have no continuous degeneration.

The remaining case is when $Q = 0$ and $dQ = 0$. We want to impose $g_3(x) \equiv 0$, so that in particular it must be that $g_3(x = M) = 0$. Evaluating:

$$g_3(x = M) = \frac{Ma^2\cos\theta d\theta\left(r^2 + (l + a\cos\theta)^2\right)}{\sin\theta},$$

which implies $\cos\theta d\theta = 0$, and hence $d\theta = 0$ (if $d\theta \neq 0$ it must be $\theta = \pi/2$ so that $d\theta = 0$ anyway). Consequently $d\theta = da = dr = dQ = 0$, which finishes the proof. □

5. Conclusion

In the present work we showed that there exist only two continuous degeneracies for the shadow of any observer in the exterior region of a Kerr-Newman-Taub-NUT spacetime. In particular when one focuses on the physically relevant case of Kerr-Newman (hence $l = 0$) the only continuous degeneracy is given by scaling of all parameters with the mass. If one assumes that, apart from the discrete spacetime isometries, no discrete degeneracies exist, then the result presented in this paper suggests that in principle an observer in the exterior region of a Kerr-Newman spacetime could extract the relative angular momentum a/M of the black hole, as well as the relative charge Q/M, the relative distance r/M, and the angle of observation relative to the rotation axis of the black hole. Additionally, one could extract how fast one is moving in comparison to a standard observer at that point in the manifold. Preliminary calculations suggest that the same result should hold true for the Kerr-Newman-de Sitter case. The proof for this case is work in progress. It is interesting however, that from an observation of the shadow alone an observer can never conclude that the Taub-NUT charge must vanish.

Note that if one looks at the projection of the shadows of the standard observers on the complex plane and chooses a parabolic and a hyperbolic Möbius transformation (see Appendix A) for each standard observer such that all shadows intersect the real axis at $+1$ and -1, it turns out that the changes of the shape due to variations of r/M and Q/M are extremely small. Hence reading off these parameters from the shadow would require a very precise measurement of the shadow curve. Adding in the fact that the light sources can be rather messy, the observational task is certainly formidable, so that at least in the foreseeable future there is little hope that from the shape of the shadows alone one can extract in practice more than a rough estimate on a/M. However, from a theoretical point of view it seems plausible (and our results are a strong indication in this direction) that one can extract very detailed information about a black hole just by looking at it.

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Appendix A. Möbius transformation

The Riemann sphere $\mathbb{S}^2$ can be blobally parametrized by stereographic projection by means of $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. A Möbius transformation is a map

$$\chi : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}},$$
$$c \longrightarrow \chi(z) := \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbb{C}, \qquad ad - bc = 1.$$

The set of all Möbius transformations define a group, denoted by Mb.

This group is isomorphic to the set of positively oriented conformal maps of $\mathbb{S}^2$ endowed with the standard round metrid.

In this appendix we prove the following theorem.

Theorem 13. *Let $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and Mb the set of Möbieus transformations. Let $c : \mathbb{S}^1 \longrightarrow \overline{\mathbb{C}}$ be an embedding. If there exists $\chi \neq Id_{\overline{\mathbb{C}}}$ that leaves c invariant as a set, then c is a generalized circle (i.e. a circle or a straight line with the point at infinity attached), or there exists $n \in \mathbb{N}$ such that $\chi^n = Id_{\overline{\mathbb{C}}}$ and c is conjugate to a closed curve invariant under rotations of angle $\frac{2\pi m}{n}$, $m \in \mathbb{Z}$ around the origin of $\mathbb{C}$.*

By "invariant as a set" we mean that there is a diffeomorphism $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ such that $\chi \circ c = c \circ f$ (the image of c and $\chi \circ c$ are obviously the same). A closed embedded curve c is *conjugate* to another closed embedded curve c_1 if there exists $\chi_1 \in \mathrm{Mb}$ such that $\chi_1 \circ c = c_1$.

Proof: We first note that the problem is invariant under conjugation: for any $\xi \in \mathrm{Mb}$ the conjugate curve $c_\xi := \xi \circ c$ is invariant (as a set) under the conjugate transformation $\chi_\xi := \xi \cdot \chi \cdot \xi^{-1}$, as it is obvious from:

$$\chi_\xi \circ c_\xi = (\xi \circ \chi \circ \xi^{-1}) \circ (\xi \circ c) = \xi \circ (\chi \circ c) = \xi \circ c \circ f = c_\xi \circ f.$$

It is well-known that all Möbius transformations (different from the identity) can be classified by conjugation into four disjoint classes: parabolic, elliptic, hyperbolic or loxodromic. Each class admits a canonical representative, in the sense that any element in the class is conjugate to this representative. The representatives can be chosen as follows:

$$\begin{aligned}
&\text{Parabolic:} && \chi_P(z) = \frac{z}{1+z} && && \text{(A.1)}\\
&\text{Elliptic:} && \chi_E(z) = e^{i\theta} z, && 0 \neq \theta \in \mathbb{R}_{\text{mod } 2\pi} \\
&\text{Hyperbolic:} && \chi_H(z) = e^{\lambda} z && \lambda \in \mathbb{R} \setminus \{0\} \\
&\text{Loxodromic:} && \chi_L(z) = kz && k \in \mathbb{C} \setminus \{\mathbb{R}\} \quad \text{and} \quad |k| \neq 1
\end{aligned}$$

Thus, we may assume without loss of generality that the transformation χ leaving c invariant is one of these canonical transformations. Obviously $\chi^m, m \in \mathbb{Z}$ also leaves c invariant. The action of χ^m is immediate to write down in the elliptic, hyperbolic and loxodromic canonical cases. In the parabolic case, a simple inductive argument shows that:

$$\chi_P^m(z) = \frac{z}{mz+1}, \qquad m \in \mathbb{Z}.$$

Thus, it follows that the cyclic group $\{\chi^n; n \in \mathbb{Z}\}$ is finite (i.e. $\chi^m = \mathrm{Id}$ for some $m \in \mathbb{Z}$) if and only if χ is elliptic and $\frac{\theta}{2\pi} \in \mathbb{Q}_{\text{mod } 1}$.

Let us consider first the loxodromic, hyperbolic and parabolic cases. We start by showing that the embedded loop c must pass through the origin $z = 0$ of the complex plane. Let $0 \neq z_0 \in \mathbb{C}$ be any point on the curve, i.e. $z_0 \in \mathrm{Im}(c)$ and define, for each $m \in \mathbb{Z}$, $z_m := \chi^m(z_0) \in \overline{\mathbb{C}}$. From invariance of the curve under χ, all points in the sequence $\{z_m\}$ lie on the image of the curve. From compactness of $\mathrm{Im}(c) \subset \overline{\mathbb{C}}$ it follows that the set of accumulation points of $\{z_m\}$ is non-empty and a subset of $\mathrm{Im}(c)$.

When χ is hyperbolic or loxodromic, the canonical form is $\chi_k(z) := kz$ with $|k| \neq 1$. The sequences are now $z_m := \chi_k^m(z_0) = k^m z_0$. If $|k| > 1$, the sequence converges to $z = 0$ as $m \to -\infty$. If $|k| < 1$, the sequence converges to $z = 0$ as $m \to \infty$. In either case $z = 0$ is an accumulation point, so the loop c passes throught $z = 0$. When χ is parabolic, the sequence is $z_m = \frac{z_0}{mz_0+1}$ which converges to $z = 0$ as $|m| \to \infty$, and we reach the same conclusion.

We can now show that a loxodromic Möbius transformation does not leave any closed embedded loop invariant. Let us take differentials in the invariance equation $\chi \circ c = c \circ f$ and evaluate at the invariant point $p := \{z = 0\}$:

$$d\chi|_p(\dot{c}) = \dot{f}|_{c^{-1}(p)}\dot{c},$$

which simply states the fact that the differential map of χ_p must preserve the direction of $\dot{c}|_p$ (it may change its scale, but not the direction). The differential of $\chi(z) = kz$ at $z = 0$ is $d\chi|_{z=0} = k$. Thus, this differential acts on a vector v by scaling with $|k|$ and rotating by $\arg(k)$. When k is not real, all vectors $v \neq 0$ change direction and we reach a contradiction. Thus, no embedded loop is invariant under a loxodromic Möbius transformation.

We next consider the hyperbolic case. The canonical representative is now $\chi = \chi_H$. Let ξ be a rotation of the form $\xi(z) = e^{i\alpha}z$ $\alpha \in \mathbb{R}$. Upon conjugation with ξ, the map χ_H remains unchanged. The conjugate curve $\xi \circ c$ passes though $z = 0$, and the parameter α can be adjusted so that its tangent vector there points along the real axis x. Since c is an embedded curve, there is a neighbourhood U of $z = 0$ such that $U \cap c$ is connected and in fact a graph over the real axis. After restricting U if necessary we may assume that U is an open disk centered at $z = 0$. We consider the curve $c_U := c \cap U$ from now on. This curve can be parametrized by x, i.e. $c(x) = x + iy(x)$ where $y(x)$ is a smooth function of $x \in (-\epsilon, \epsilon)$. The parameter λ in the definition of χ_H can be assumed to be negative (if it were positive simply replace χ_H by χ_H^{-1}). Then χ_H maps U into itself, and leaves the curve c_U invariant. So, it must be the case that, for all $x \in (-\epsilon, \epsilon)$:

$$e^{\lambda}(x + iy(x)) = x'(x) + iy(x'(x)) \quad \Longleftrightarrow \quad y(e^{\lambda}x) = e^{\lambda}y(x). \tag{A.2}$$

where $x'(x)$ indicates the reparametrization of the curve induced by the Möbius transformation χ_H. Define the function $P(u) := e^{-\lambda u}y(e^{\lambda u})$. By construction, $P(u)$ is smooth on $(-\infty, \lambda^{-1}\ln\epsilon)$. In terms of P, the function $y(x)$ restricted to $x > 0$ takes the form $y(x) = xP(\lambda^{-1}\ln x)$. The invariance property (A.2) becomes, when applied at the point $x = e^{\lambda u}$:

$$P(u + 1) = e^{-\lambda u}e^{-\lambda}y(e^{\lambda u}e^{\lambda}) = e^{-\lambda u}y(e^{\lambda u}) = P(u).$$

So $P(u)$ is a periodic function of period one. We can now compute the derivative of $y(x)$ (prime denotes derivative with respect to u):

$$\frac{dy(x)}{dx} = P(\lambda^{-1}\ln x) + \lambda^{-1}P'|_{\lambda^{-1}\ln x}.$$

If $P(u)$ is not a constant function the combination $P(u) + \lambda^{-1}P'(u)$ does not converge as $u \to -\infty$. To show this, take the sequence $u_n = u_0 - n$ with $u_0 \in [-1, 0)$ defined by the condition that $P(u_0)$ attains the supremum of $P(u)$ and another sequence $u'_n = u_1 - n$ where $u_1 \in [-1, 0)$ is the value where $P(u)$ attains the infimum. By periodicity, the sequences $P(u_n)$ and $P(u'_n)$ are both constant. Moreover, P' vanishes on all points u_n and u'_n. Thus, the sequences $\{P(u_n) + \lambda^{-1}P'(u_n)\}$ and $\{P(u'_n) + \lambda^{-1}P'(u'_n)\}$ converge to the same limit if and only if $P(u_0) = P(u_1)$, i.e. if the function $P(u)$ is constant, as claimed. As a consequence, $\frac{dy}{dx}$ converges as $x \to 0^+$ if and only if $P(u) = a$ for some constant a, or equivalently iff $y(x) = ax$. Since, in our setup, $\frac{dy}{dx} = 0$ at $x = 0$ we conclude that $y(x) = 0$. We have proved this fact in a neighbourhood U of 0, but this extends to the whole loop c by applying repeatedly the transformation χ_H. In summary, we have shown that the only embedded loops invariant under the canonical representative χ_H of hyperbolic Möbius transformations is the line $(x, y = 0)$, and arbitrary rotations thereof around the origin. We now use the property that Möbius transformations map generalized circles into generalized circles, and conclude that an embedded loop which is not a generalized circle can never be invariant under a hyperbolic Möbius transformation.

We want to use a similar argument for the parabolic case. To that aim, it is preferable to use a different representative. More precisely, recall that for $\chi = \chi_P$ given in (A.1) the invariant embedded loop c necessarily passes through $z = 0$. Let us apply a conjugation with the inversion map $\hat{\xi}(z) = -1/z$. The conjugate $\widehat{\chi}_P = \hat{\xi} \circ \chi_P \circ \hat{\xi}^{-1}$ is given by $\widehat{\chi}_P(z) = z - 1$ and the conjugate loop $\hat{c} := \hat{\xi} \circ c$ passes through the point at infinity. Consider the vector field:

$$\zeta = z^2 \partial_z + \overline{z}^2 \partial_{\overline{z}}.$$

This field is smooth in a neighbourhood of the point at infinity. Indeed, the vector field $\partial_{x'} = \partial_{z'} + \partial_{\overline{z}'}$ is clearly smooth in a neighbourhood of zero. The inversion map $z' = -\frac{1}{z}$ transforms this neighbourhood of zero into a neighbourhood of infinity and transforms the vector field $\partial_{x'}$ into ζ, from which smoothness follows. In the coordinates $\{x, y\}$ defined by $z = x + iy$ this vector field takes the form:

$$\zeta = \left(x^2 - y^2\right) \partial_x + 2xy \partial_y.$$

The property of invariance of an embedded loop under a Möbius transformation is preserved by reparametrizations of the curve, so we are free to choose the parametrization of $\hat{c}$. However, we must make sure that the parameter is smooth everywhere, including a neighbourhood of infinity. To that aim we choose to parametrize $\hat{c}$ with arc length s with respect to the round sphere metric:

$$ds^2 = \frac{1}{\left(1 + \frac{1}{4}(x^2 + y^2)\right)^2}(dx^2 + dy^2), \tag{A.3}$$

which extends smoothly to the point at infinity. As before, let $0 \neq z_0 = \hat{c}(s_0) = (x_0, y_0) \in \mathbb{C}$ be a point on the curve. From the condition that the tangent vector $T|_p$ of the curve is unit with respect to (A.3), there exists $\alpha \in [0, 2\pi)$ such that:

$$T|_p = F|_p \left(\cos\alpha \partial_x + \sin\alpha \partial_y\right),$$

with $F|_p$ determined by:

$$F|_p = 1 + \frac{1}{4}\left(x^2 + y^2\right)\bigg|_{(x_0, y_0)}.$$

We compute the scalar product with the vector ζ to find:

$$\langle T|_p, \zeta|_p \rangle = \frac{\cos\alpha(x^2 - y^2) + 2\sin\alpha xy}{1 + \frac{1}{4}\left(x^2 + y^2\right)}\bigg|_{(x_0, y_0)}.$$

Consider now the sequence of points $\{z_m = (x_0 - m, y_0)\}$. From invariance under $\widehat{\chi}_P$, they also also lie on the curve $\hat{c}$. In fact, the set $\mathrm{Im}(\hat{c})$ defines a periodic submanifold, in the sense that a unit translation along the x axis leaves it invariant. As a consequence, all the the tangent vectors T_{p_m} of the curve at each point z_m must be parallel to each other (in the natural euclidean sense of the term). Hence α is the same for all z_m. Let us compute the limit along the sequence of the scalar product $\langle T|_{p_m}, \zeta|_{p_m} \rangle$:

$$\lim_{m \longrightarrow \infty} \langle T|_{p_m}, \zeta|_{p_m} \rangle = \lim_{m \longrightarrow \infty} \frac{\cos\alpha((x_0 - m)^2 - y_0^2) + 2\sin\alpha(x_0 - m)y_0}{1 + \frac{1}{4}\left((x_0 - m)^2 + y_0^2\right)} = 4\cos\alpha.$$

Given that the curve is smooth evergywhere, including infinity, and that the sequence $\{z_m\}$ converges to the point at infinity, it follows that all the tangent vectors $T|_{p_m}$ must converge, namely to the unit tangent vector T_∞ to the curve there. The scalar products above must then converge to a single finite value, and this must happen independently of the initial point z_0. Since the limit depends on α we conclude that α must be the same for all points along the curve. If $\alpha = \frac{\pi}{2}$ or $\alpha = \frac{3\pi}{2}$ then the curve would be an infinite collection of vertical lines in the $\{x, y\}$ plane, all of them passing though the point at infinity and the curve $\hat{c}$ would not be embedded. Thus the tangent vector T_p must have a non-zero component along the x axis everywhere along the curve. This implies that it can be described as a graph $y(x)$ on the x axis. Since $y(x)$ must reach a local maximum and α vanishes there we conclude that $\alpha = 0$ at all points, and hence that $y = y_0 =$ const. So, the embedded loop $\hat{c}$ must be the straight line $y = y_0$. This claim is for embedded curves invariant under the parabolic transformation $z \to z - 1$. Upon conjugation, and using again that Möebius transformations map generalized circles into generalized circes, we conclude that the only embedded closed loops invariant under a parabolic transformation are generalized circles.

It only remains to consider the elliptic case, i.e. $\chi = \chi_E$. Since χ_E is a rotation of angle θ of the complex plane around its origin, the invariant embedded loop c defines a figure invariant under a rotation of angle $\theta \neq 2\pi k$, $k \in \mathbb{Z}$. Consider the set of all angles $\beta \in (0, 2\pi)$ under which this figure is invariant and let β_0 be its infimum. If $\beta_0 = 0$, the curve must be a circle. If β_0 is different from zero, then there must exist $n \in \mathbb{N}$ such that $\beta_0 = \frac{2\pi}{n}$ (if such n did not exist, define $n \in \mathbb{N}$ by $n\beta_0 < 2\pi < (n+1)\beta_0$, the angle $(n+1)\beta_0 - 2\pi$ is positive, smaller that β_0 and belongs to the set of rotation angles that leave the figure invariant, which is a contradiction.) Thus $\beta_0 = \frac{2\pi}{n}$ and in fact all other symmetry angles must be a multiple of this (by a similar argument as before). The number n is called the *order of symmetry* of the figure. In summary, the closed embedded loop c is invariant under χ_E if and only if it is a circle centered at zero, or a figure with a discrete rotational symmetry of order n. The statement of the theorem then follows once again from the fact that the collection of generalized circles is preserved under Möbius transformations. □

As discussed in the main text, the shadow curve for suitable chosen observers at any point in the class of black hole spacetimes under consideration here has the property of being reflection symetric. In precise terms, let the map $r : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ be defined by reflection with respect to the real axis $y = 0$, i.e. $r(z) = \overline{z}$. A closed embedded loop $c : \mathbb{S}^1 \longrightarrow \overline{\mathbb{C}}$ is **reflection symmetric** if there exists a smooth map $f_1 : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ such that $r \circ c = c \circ f_1$. One checks immediately that f_1 is a diffeomorphism of $\mathbb{S}^1$ (in fact an orientation reversing diffeomorphism). Our aim is to determine which elements $\chi \in \mathrm{Mb}$ have the property that the conjugate curve $\chi \circ c$ is also reflection symmetric. Thus, we want to impose the condition that there exists a diffemorphism $f_2 : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ such that $r \circ \chi \circ c = \chi \circ c \circ f_2$, which in turn is equivalent to $\chi^{-1} \circ r \circ \chi \circ r^{-1} \circ c \circ f_1 = c \circ f_2$, i.e. to:

$$\chi^{-1} \circ r \circ \chi \circ r^{-1} \circ c = c \circ f,$$

where $f := f_2 \circ f_1^{-1}$ is an orientation preserving diffeomorphism of $\mathbb{S}^1$. The map $\widetilde{\chi} := \chi^{-1} \circ r \circ \chi \circ r^{-1}$ is by construction an element of the Möbius group, and leaves the loop defined by c invariant (as a submanifold). From Theorem 13 it follows that $\widetilde{\chi}$ is the indentity map, unless either $\mathrm{Im}(c)$ is conjugate to a figure with discrete rotational symmetry of order n and, in addition, $\widetilde{\chi}$ is conjugate to $\chi_{m,n} := z \to e^{i\frac{2\pi m}{n}} z$ for some integer m between $-n$ and n, or else c is a generalized circle.

In this paper we are interested in Möbius transformations sufficiently close to the identity that map reflection symmetric curves into reflection symmetric curves. Since, for fixed n $\{\xi_{m,n}; -n < m < n\}$ is discrete, it is disjoint to a sufficiently small neighbourhood of the identity map $\mathrm{Id}_{\mathbb{C}}$, and we can ignore the case of discrete rotational symmetry of order n. Also, we restrict ourselves to non-degenerate spacetimes points, where the shadow curve is not a generalized circle (for simplicity we call such curves "non–circular"). So, we conclude that $\widetilde{\chi}$ must be the identity map, i.e.:

$$\chi^{-1} \circ r \circ \chi \circ r^{-1} = \mathrm{Id}_{\overline{\mathbb{C}}} \qquad \Longleftrightarrow \qquad r \circ \chi \circ r^{-1} = \chi.$$

Letting χ correspond to the SL$(2,\mathbb{C})$ matrix:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

it is immediate to compute that $r \circ \chi \circ r^{-1}$ corresponds to the SL$(2,\mathbb{C})$ matrix:

$$\begin{pmatrix} \overline{\alpha} & \overline{\beta} \\ \overline{\gamma} & \overline{\delta} \end{pmatrix}.$$

Thus, is χ is sufficiently close to the identity map and the reflection symmetric curve c is non-circular, it must be the case that $\chi \in SL(2,\mathbb{R})$, i.e. all $\alpha, \beta, \gamma, \delta$ are real parameters.

Our second aim is to identify the infinitessimal transformations with generate this subgroup of Möbius transformations. Consider a one parameter subgroup $\tau : \mathbb{R} \longrightarrow \mathrm{SL}(2,\mathbb{C})$ of SL$(2,\mathbb{C})$ and denote by $\chi_{\tau(s)}$, $s \in \mathbb{R}$ the corresponding curve in the Möbius group. A straightforward computation gives, for each $z \in \mathbb{C}$:

$$\frac{d\chi_{\tau(s)}(z)}{ds} = \beta_0 + (\alpha_0 - \delta_0) - \gamma_0 z^2,$$

where $\alpha_0 = \left.\frac{d\alpha(s)}{ds}\right|_{s=0}$, $\beta_0 = \left.\frac{d\beta(s)}{ds}\right|_{s=0}$, $\gamma_0 = \left.\frac{d\gamma(s)}{ds}\right|_{s=0}$, $\delta_0 = \left.\frac{d\delta(s)}{ds}\right|_{s=0}$. The condition that the curve $\tau(s)$ takes values in SL$(2,\mathbb{C})$ requires that $\delta_0 = -\alpha_0$. Thus, the infinitessimal generator of this one-parameter subgroup is:

$$\xi = \left(\beta_0 + 2\alpha_0 z - \gamma_0 z^2\right)\partial_z + \left(\overline{\beta_0} + 2\overline{\alpha_0}\,\overline{z} - \overline{\gamma_0}\,\overline{z}^2\right)\partial_{\overline{z}}.$$

Thus if we restrict ourselves to the subgroup of transformation preserving the reflection symmetry of a non-circular curve c, the generators are:

$$\xi = \beta_0\left(\partial_z + \partial_{\overline{z}}\right) + 2\alpha_0\left(z\partial_z + \overline{z}\partial_{\overline{z}}\right) - \gamma_0\left(z^2\partial_z + \overline{z}^2\partial_{\overline{z}}\right), \qquad \alpha_0, \beta_0, \gamma_0 \in \mathbb{R}.$$

In terms of Cartesian coordinates $\{x, y\}$ on the stereograhic plane, i.e. $z = x + iy$, this vector field becomes:

$$\xi = \beta_0\partial_x + 2\alpha_0\left(x\partial_x + y\partial_y\right) - \gamma_0\left(\left(x^2 - y^2\right)\partial_x + 2xy\partial_y\right).$$

So, the three generators of Möbius transformations preserving reflection symmetry turn out to be the translations along the x axis $\xi_1 = \partial_x$, the dilations about the origin $\xi_2 = x\partial_y + y\partial_x$ and a third conformal Killing vector given by $\xi_3 = (x^2 - y^2)\partial_x + 2xy\partial_y$. These vector fields generate a Lie algebra with structure constants:

$$[\xi_1, \xi_2] = \xi_1, \qquad [\xi_1, \xi_3] = 2\xi_2, \qquad [\xi_2, \xi_3] = \xi_3.$$

Note that the subset of reflection symmetric transformations that leave the origin $\{x = 0, y = 0\}$ invariant is generated by $\{\xi_2, \xi_3\}$, which is, naturally, a two-dimensional subalgebra. Another observation is that the only element in $\{\xi_1, \xi_2, \xi_3\}$ which is a Killing vector of $\mathbb{C} \cup \{\infty\}$ endowed with the spherical metric $ds^2 = \left(1 + \frac{1}{4}(x^2 + y^2)\right)^{-2}(dx^2 + dx^2)$, is $4\xi_1 + \xi_3$ (and its constant multiples). This Killing field corresponds to rotations of the sphere leaving invariant the antipodal points for which the corresponding equator maps onto the real axis by stereographic projection.

Appendix B. Partial derivatives of f and h

$$\frac{\partial f}{\partial a} = \frac{4x\Delta^2 - a^2\{x^2+(l+a\cos\theta)^2\})\Delta' - \Delta(4a^2x+(x^2+l^2-a^2\cos^2\theta)\Delta')}{4a^2x\Delta^{3/2}\sin\theta} \tag{B.1a}$$

$$\frac{\partial f}{\partial l} = \frac{l\{x^2+(l+a\cos\theta)^2\}\Delta' - 2\Delta(2lx+(l+a\cos\theta)\Delta')}{4ax\Delta^{3/2}\sin\theta} \tag{B.1b}$$

$$\frac{\partial f}{\partial M} = -\frac{\{x^2+(l+a\cos\theta)^2\}(2\Delta+x\Delta')+4x^2\Delta}{4ax\Delta^{3/2}\sin\theta} \tag{B.1c}$$

$$\frac{\partial f}{\partial Q} = -\frac{Q(\Delta'\{x^2+(l+a\cos\theta)^2\}+4x\Delta)}{4ax\Delta^{3/2}\sin\theta} \tag{B.1d}$$

$$\frac{\partial f}{\partial \theta} = -\frac{2a(l+a\cos\theta)\Delta'\sin^2\theta+\cos\theta(\Delta'\{x^2+(l+a\cos\theta)^2\}-4x\Delta)}{4ax\Delta^{1/2}\sin^2\theta} \tag{B.1e}$$

$$\frac{\partial f}{\partial x} = \frac{\{x^2+(l+a\cos\theta)^2\}((M-x)^3-M(M^2-a^2-Q^2+l^2))}{2ax^2\Delta^{3/2}\sin\theta} \tag{B.1f}$$

$$\frac{\partial h}{\partial a} = \frac{4ax(-8x\Delta(r)\Delta(x)+(\Delta(r)+\Delta(x))((r^2-x^2)\Delta'(x)+4x\Delta(x))}{\sqrt{\Delta(x)\Delta(r)}((r^2-x^2)\Delta'(x)+4x\Delta(x))^2} \tag{B.2a}$$

$$\frac{\partial h}{\partial Q} = \frac{4Qx(-8x\Delta(r)\Delta(x)+(\Delta(r)+\Delta(x))((r^2-x^2)\Delta'(x)+4x\Delta(x))}{\sqrt{\Delta(x)\Delta(r)}((r^2-x^2)\Delta'(x)+4x\Delta(x))^2} \tag{B.2b}$$

$$\frac{\partial h}{\partial l} = \frac{4lx(8x\Delta(r)\Delta(x)-(\Delta(r)+\Delta(x))((r^2-x^2)\Delta'(x)+4x\Delta(x))}{\sqrt{\Delta(x)\Delta(r)}((r^2-x^2)\Delta'(x)+4x\Delta(x))^2} \tag{B.2c}$$

$$\frac{\partial h}{\partial M} = \frac{4x\left(r\Delta(x)(-4x\Delta(x)+(x^2-r^2)\Delta'(x))+\Delta(r)(2(r^2+x^2)\Delta(x)+x(x^2-r^2)\Delta'(x)\right)}{\sqrt{\Delta(x)\Delta(r)}((r^2-x^2)\Delta'(x)+4x\Delta(x))^2} \tag{B.2d}$$

$$\frac{\partial h}{\partial r} = \frac{2x\Delta(x)(4x\Delta(x)\Delta'(r)+(\Delta'(r)(r^2-x^2)-4r\Delta(r))\Delta'(x))}{\sqrt{\Delta(x)\Delta(r)}((r^2-x^2)\Delta'(x)+4x\Delta(x))^2} \tag{B.2e}$$

$$\frac{\partial h}{\partial x} = \frac{2(r^2-x^2)\Delta(r)((x-M)^3+M(M^2-a^2-Q^2+l^2))}{\sqrt{\Delta(x)\Delta(r)}((r^2-x^2)\frac{\Delta'(x)}{2}+2x\Delta(x))^2}. \tag{B.2f}$$

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