ZAMM 59, T 10 – T 15 (1979)

D. J. OLBERS

Weak Coupling in Statistical Geophysical Systems

1. Introduction

The development of the geophysical sciences in the recent years is inconceivable without the use of digital computers. Aside from the analysis and interpretation of the large data sets which usually are obtained in geophysical experiments, the computer is used for simulating physical systems to understand their behaviour in space and time. Prominent examples are the general circulation in the ocean and atmosphere or the dynamics of the synoptic turbulent eddy field in the ocean. As a common feature these systems are governed by strongly nonlinear dynamics. However, nature provides a variety of problems where the evolution of the system can be studied to a large extent by analytical means. Generally, these problems can be formulated as *interaction of weakly coupled systems*. In this note I will briefly review some results from weakly coupled *statistical* systems and present some recent applications.

The concept of treating weakly coupled systems is very simple. Consider a system described by a *state vector* $\varphi(t) = (\varphi_1, \varphi_2, \dots)$ which is in interactive contact with a set of *external* (i.e. considered as given) *fields* $\psi(t) = (\psi_1, \psi_2, \dots)$. The vectors φ and ψ may be field functions by dependence on a *spatial coordinate* x. Suitable scaling of the equations of motion governing $\varphi(t)$ will reveal a small parameter $\varepsilon \ll 1$ characterising the *weakness* of the coupling. The *evolution of the system* will be governed by equations of motion

$$\partial_t \varphi + H\varphi = \varepsilon A^{(1)}(\varphi, \psi) + \varepsilon^2 A^{(2)}(\varphi, \psi) + \dots \tag{1}$$

which may be assumed to be of first order without loss of generality. The operators H and $A^{(n)}$ are of order unity. The use of only one coupling parameter is a simplification, in general there is one for each type of coupling.

Suppose now that equation (1) can be solved for $\varepsilon = 0$, the solution $\varphi^{(0)}$ will be called the *free state* of the system. A solution to the full equation of motion can be attempted in the form of a *perturbation expansion*

$$\varphi = \varphi^{(0)} + \varepsilon\varphi^{(1)} + \varepsilon^2\varphi^{(2)} + \dots \tag{2}$$

with the initial condition $\varphi(t = 0) = \varphi^{(0)}$. In general the solutions $\varphi^{(n)}$, $n \geqq 1$, contain *secular contributions* which grow indefinitely in time. Since the perturbations of the solution φ about the free state $\varphi^{(0)}$ are required to be small, the occurrence of secularities leads to an integration time limit for the validity of the expansion (2). However, the nonuniform behaviour of the perturbation series is not a disaster because the secular contributions are just those which reflect the effect of the coupling. The higher oder solutions can be interpreted as being forced by interactions of the free state and the external fields. *Secular terms can then be reinterpreted as a slow change of the free state as the consequence of the weak coupling.* This heuristic procedure can be put in a more rigorous form by using a multiple-scale representation of the time variable (BOGOLJUBOW and MITROPOLSKI [1], SANDRI [17]).

Many geophysicl systems appear to be erratic in space and time. Only mean values are resolvable in experiments. Then the state vector φ and possibly also the external fields ψ must be regarded as *random variables*. In this case it is desirable to construct equations describing the evolution of the probability distribution $p(\varphi, t)$ of φ at time t or even the multivariate probability distributions $p(\varphi_{i_1}, t_1, \dots, \varphi_{i_n}, t_n)$. Equivalently, equations of motion for the (infinite) set of correlations

$$\langle \delta\varphi_{i_1}(t) \dots \delta\varphi_{i_n}(t)\rangle = \int \mathrm{d}\varphi\, p(\varphi, t)\, \delta\varphi_{i_1} \dots \delta\varphi_{i_n}\,, \qquad \delta\varphi_i = \varphi_i - \langle\varphi_i\rangle \tag{3}$$

and the multiple time correlations $\langle \delta\varphi_{i_1}(t_1) \dots \delta\varphi_{i_n}(t_n)\rangle$ may be considered. These are easily derived from the equations of motion (1). In most problems one would be satisfied to predict the variances $\langle \delta\varphi_i(t)\, \delta\varphi_i(t)\rangle$ which characterize the statistical state of the system to the lowest order. If the equations of motion contain non-linearities the evolution equations of the variances are not closed. One is faced then with an *infinite hierarchy of evolution equations* which generally can be solved only if a closure hypothesis is imposed. In contrast to strongly coupled statistical systems (turbulence) this problem is settled for weakly coupled systems occurring in geophysics.

In the applications considered in this paper the operator H is *linear*. The free state is a *linear wave system* (or a steady state system if H is zero). Problems with a non-linear H are conceivable but hardly accessible by analytical methods. The concept of solitons can be viewed as a first step in this direction. *Solitons* are a class of exact solutions to nonlinear dispersive systems and can be superposed in a similar way as linear waves (see e.g. SCOTT et al. [18]). Attempts have been made to apply the soliton concept to geophysical systems but to my knowledge weak interactions among solitons and external fields have not yet been studied, not to mention the statistical part of such a problem.

There are two prominent cases to which many weak coupling problems in geophysics can be reduced. The first case treats the weak coupling of two systems with *widely different intrinsic time scales*. This case turns out to be identical to the problem of the diffusion of particles in a turbulent fluid already treated by TAYLOR [19]. Since this problem has recently been reviewed by HASSELMANN [7] the emphasis of this note will be put on the second case which treats the weak coupling of linear systems with *comparable intrinsic time scales*. This problem can be solved by the theory of resonant interactions between random wave fields. The basics of this theory have been formulated by PEIERLS [15] studying non-linearly interacting phonons in solids.

It will be convenient for the formal analysis to remove the linear term in the equations of motion by transforming to interaction variables

$$\chi = e^{\Pi t} \varphi \tag{4}$$

which yields

$$\partial_t \chi = \varepsilon B^{(1)}(\chi, \psi, t) + \varepsilon^2 B^{(2)}(\chi, \psi, t) + \dots, \qquad B^{(n)}(\chi, \psi, t) = e^{\Pi t} A^{(n)}(e^{-\Pi t} \chi, \psi) . \tag{5}$$

2. The diffusion problem

Assume that $\psi(t)$ is rapidly fluctuating and can be regarded as a statistically stationary random variable for times small compared with the intrinsic time scale of the state $\varphi(t)$. *Statistical stationarity* means that correlations are invariant against translation in time. The *first order problem*

$$\partial_t \chi^{(1)} = B^{(1)}(\chi^{(0)}, \psi, t) = \beta(t) \tag{6}$$

is then identical to the diffusion of a particle in a turbulent fluid where $\chi^{(1)}$ represents the position vector of the particle and $\beta(t)$ the turbulent LAGRANGian fluid velocity. The forcing function $\beta(t)$ is also statistically stationary with expectation value

$$\langle \beta(t) \rangle = \text{constant} \tag{7}$$

and covariance function

$$R_{ij}(\tau) = \langle \beta_i(t)\, \beta_j(t + \tau) \rangle . \tag{8}$$

It is a well-known result (TAYLOR [19]) that *the response to stationary forcing is non-stationary.* Straightforeward integration of (6) yields a *linear increase of expectation values and covariances*

$$\langle \chi_i^{(1)}(t) \rangle = t \langle \beta_i \rangle , \qquad \langle \delta\chi_i^{(1)}(t)\, \delta\chi_j^{(1)}(t) \rangle \sim t \int_{+\infty} d\tau\, R_{ij}(\tau) . \tag{9}$$

The second relation is an asymptotic time limit with respect to the rapidly fluctuating field $\beta(t)$ (or $\psi(t)$ respectively). The time t is large compared to the integral correlation scale $\max_{i,j} \left| \int_{-\infty}^{+\infty} d\tau\, R_{ij}(\tau) \right| \left(R_{ii}(0)\, R_{jj}(0)\right)^{-1/2}$ of $\beta(t)$. At the same time the analysis applies only to small deviations $\varepsilon\chi^{(1)}$ from the free state $\chi^{(0)}$ which requires $t \ll \varepsilon^{-1}$. This limit on the validity of (9) can be removed by interpreting the infinitesimal increments as rate of change of a slowly varying state described by a probability distribution $p(\chi, t)$. It has been shown (CHANDRASEKHAR [3]) that $p(\chi, t)$ *is governed by a* FOKKER-PLANCK *equation* (heat conduction equation)

$$\partial_t p + \frac{\partial}{\partial \chi_i}(v_i p) - \frac{\partial}{\partial \chi_i} D_{ij} \frac{\partial}{\partial \chi_j} p = 0 \tag{10}$$

f the expectation values $\langle \chi_i \rangle$ *and the covariances* $\langle \delta\chi_i\, \delta\chi_j \rangle$ *grow linearly for infinitesimal times.* The advection of probability in the phase space χ_i is given by the velocity

$$v_i(\chi) = \langle \beta_i \rangle - \frac{\partial}{\partial \chi_j} D_{ij} \tag{11}$$

and the diffusion of probability by the tensor

$$D_{ij}(\chi) = \tfrac{1}{2} \int_{-\infty}^{+\infty} d\tau\, R_{ij}(\tau) \tag{12}$$

which is equal to the spectral density of the forcing at zero frequency. Thus *the long-term behaviour of the system is governed by the low frequency components of the forcing.* For a more detailed review of the diffusion problem and applications in geophysics the reader is referred to HASSELMANN [7].

3. The wave interaction problem

The ocean, the atmosphere and the solid earth support a variety of wave motions. Surface gravity waves on top of the ocean are a familiar phenomenon. There are other wave types which fill the interior of ocean and atmosphere, such as internal gravity waves, acoustic waves and planetary waves. The solid earth allows propagation of seismic waves. It should be emphasized that waves by definition can only exist if interactions are weak.

3.1 The radiative transfer equation

Linear waves are characterized by an amplitude a, a wavevector k and a frequency ω which are related by a dispersion relation $\omega = \Omega(k)$. Large scale inhomogeneities (compared to period and wavelength) of the wave carrying background can be treated by WKB methods. *Waves appear then in form of slowly varying wave trains which may locally be represented by wave groups characterized by a local dispersion relation* $\omega = \Omega(k, x, t)$. A wave group propagates with the group velocity

$$\dot{x}_i = \frac{\partial \Omega}{\partial k_i} . \tag{13}$$

On the trajectory wavevector and frequency change according to

$$\dot{k}_i = -\frac{\partial \Omega}{\partial x_i}; \qquad \dot{\omega} = \frac{\partial \Omega}{\partial t} \tag{14}$$

while changes in amplitude are conveniently expressed in the form of action conservation (WHITHAM [20], BRETHERTON [2])

$$\partial_t \left(\frac{E}{\omega}\right) + \frac{\partial}{\partial x_i}\left(\dot{x}_i \frac{E}{\omega}\right) = 0 \,. \tag{15}$$

The wave energy $E = |a(k)|^2$ is a quadratic functional of the amplitude. Equation (15) states that *wave action* $\int (E/\omega)\mathrm{d}x$ *is an adiabatic invariant* (see e.g. LANDAU and LIFSHITZ [8]) *for slowly varying linear wave groups.*

It will soon become apparent that the function appropriate to describe the state of a weakly interacting random wave field is the action spectrum

$$N(k, x, t) = \langle |a(k)|^2 \rangle / \omega \,. \tag{16}$$

This second order wave correlation can loosely be interpreted as *number density of waves in the phase space* (k, x). *Its evolution is governed by a radiative transfer equation*

$$\left\{\partial_t + \dot{x}_i \frac{\partial}{\partial x_i} + \dot{k}_i \frac{\partial}{\partial k_i}\right\} N(k, x, t) = S(k, x, t) \tag{17}$$

which is the generalization of action conservation (15) *of linear wave groups.* Interaction processes have been accounted for by the source function $S(k, x, t)$ which determines the local change of action of the wave groups due to weak coupling between them and with external fields.

Before proceeding with the derivation of S I will briefly focus on the obvious resemblance between an ensemble of interacting wave groups and an ensemble of interacting particles. Indeed, equations (13) and (14) are HAMILTONian equations with a HAMILTONian $\Omega(k, x, t)$ for a particle with generalized coordinate x and momentum k and energy ω. Action conservation is then conservation of particle number and the radiative transfer equation is analogous to transport equations (such as BOLTZMANN's) governing the evolution of the particle distribution function in phase space. We will see that the analogy becomes even closer when considering weak interaction processes in the wave field.

3.2 The evolution of wave correlations

The evolution of a weakly interacting wave system proceeds on two widely different time scales, the period of the waves and a much larger interaction time scale. The perturbation parameter ε in (1) is the ratio of these time scales. Usually, it is a measure a wave slope so that the theory applies to infinitesimal amplitudes. The operators $A^{(n)}$ or $B^{(n)}$ in equations (1) and (5) are then polynomial operators of degree $n + 1$. To evaluate the source function S it is convenient to project the state vector χ and the equations of motion (5) into a normal mode base which is defined as the base in which H is diagonal. Since H describes free linear waves it has purely imaginary eigenvalues

$$H u_s = i\omega_s u_s \,. \tag{18}$$

The *normal mode expansion* of the state vector is given by

$$\chi(x, t) = \sum_s a_s(t)\, u_s(x) \,. \tag{19}$$

Normalization of the eigenvectors $u_s(x)$ is conveniently applied such that the wave energy is given by $\sum |a_s|^2$. The projection operator onto normal mode amplitudes follows from the solution of the adjoint eigenvalue problem

$$H_{ad} u^s = i\omega_s u^s \tag{20}$$

since u_s and u^s can be chosen orthonormalized

$$\int \mathrm{d}x\, u^s u_{s'} = \delta_{ss'} \,. \tag{21}$$

In general the eigenvalue problems (18) and (20) have a *threefold structure.* The *physical space* (with coordinate x) can be decomposed into a *propagation space* (with coordinate y) and a *cross-space* (with coordinate z). In the propagation space the system is homogeneous and infinite (or periodic), eigenfunctions are of sinusoidal form. In the cross-space the waves are trapped and form standing modes. If the state vector consists of more than one field component, an additional algebraic eigenvalue problem remains to be solved for the different wave branches. The eigenvector $u_s(x)$ then takes the form

$$u_s(x) = u^l_{k,\nu}(y, z) = U^l_{k,\nu} q^\nu_k(z)\, \mathrm{e}^{iky} \,, \tag{22}$$

where ν labels for each wavevector k the cross-space modes and l labels the wave branches characterized by the polarization vectors $U^l_{k,\nu}$. Thus $s = (l, k, \nu)$ and $\omega^l_{k,\nu}$ is the dispersion function $\Omega^l_\nu(k)$ of the wave modes ν and branches l. If the state vector χ is real, the sum in the representation (19) includes summation over the complex eigenvectors $u_{-s} = u^*_s$ with eigenvalue $\omega_{-s} = -\omega_s$.

Projection of the equations of motion (5) yields the *rate of change of the normal mode amplitudes*

$$\dot{a}_s(t) = \mathrm{e}^{i\omega_s t}\, \{\varepsilon G^{(0)}_s(b) + \varepsilon G^{(1)}_s(a, b) + \varepsilon^2 G^{(2)}_s(a, b) + \dots\} \,, \tag{23}$$

where b_r are the amplitudes of the external fields in a suitable representation

$$\psi(x, t) = \sum_r b_r v_r(x)\, e^{-i\omega_r t} \tag{24}$$

and $G_s^{(n)}$ are polynomials of degree $n + 1$, thus e.g.

$$G_s^{(1)}(a, b) = \sum C_{ss's''} a_{s'} a_{s''}\, e^{-i(\omega_{s'} + \omega_{s''})t} + \sum C_{ss'}^{r} a_{s'} b_r\, e^{-i(\omega_{s'} + \omega_r)t} + \sum C_s^{rr'} b_r b_{r'}\, e^{-i(\omega_r + \omega_{r'})t} . \tag{25}$$

A linear forcing term $\varepsilon G_s^{(0)}(b)$ has been included in (23). This must be small since the interaction concept is based on a free state for $\varepsilon = 0$. The solution to (23) can be obtained by a perturbation expansion

$$a_s = a_s^{(0)} + \varepsilon a_s^{(1)} + \varepsilon^2 a_s^{(2)} + \dots \tag{26}$$

with initially $a_s(t = 0) = a_s^{(0)}$. This expansion contains secular terms which arise from resonances. Thus e.g. $a_s^{(1)}$ contains the term

$$\Delta a_s^{(1)} = t \sum C_{ss's''} a_{s'}^{(0)} a_{s''}^{(0)} \tag{27}$$

where the sum is restricted to components satisfying the *resonance conditions*

$$\omega_s - \omega_{s'} - \omega_{s''} = 0 , \qquad k - k' - k'' = 0 . \tag{28}$$

The latter condition follows from the homogeneity of the system in the propagation space. *The linear increase of $a_s^{(1)}$ can be interpreted as a slow change of the free wave amplitudes due to interactions with other free wave components.*

It is worth mentioning that visualization of the secular terms in the perturbation expansion in terms of interaction diagrams may be helpful (Hasselmann [6]). Thus the resonance (28) can be represented by two wave components with "energies" $\omega_{s'}$ and $\omega_{s''}$ and "momenta" k' and k'', respectively, entering a vertex associated with a coupling coefficient $C_{ss's''}$ and one outgoing wave component with "energy" ω_s and "momentum" k leaving the vertex.

The analogy to colliding particles is hindered somewhat by the fact that the number of particles is not conserved and the "energy" ω_s is allowed to have negative values. Further the true energy $|a_s|^2$ and momentum $k|a_s|^2/\omega_s$ are conserved only if the coupling coefficients have a certain symmetry. For the interacting triplett (28) one finds

$$\frac{d}{dt}\{\alpha_s|a_s|^2 + \alpha_{s'}|a_{s'}|^2 + \alpha_{s''}|a_{s''}|^2\} = a_s a_{s'}^* a_{s''}^* \{\alpha_s C_{-s\ s'\ s''} + \alpha_{s'} C_{s's-s''} + \alpha_{s''} C_{s''s\ s'}\} \tag{29}$$

for arbitrary α_s. *Energy* ($\alpha_s = 1$) *and momentum* ($\alpha_s = k/\omega_s$) *are* thus *conserved if $C_{-ss's''}/\omega_s$ is symmetric in the three indices s, s' and s'' and the corresponding wave components satisfy the resonance conditions* (28). Indeed it is observed in most applications that the coupling coefficients $C_{ss_1\dots s_n}$ describing internal interactions in the wave field show this kind of symmetry whereas the external coupling coefficients $C_{ss_1\dots s_m}^{r_1\dots r_n}$ have only the trivial symmetry associated with the polynomial structure of (23). In general, the energy conserving internal interactions can be derived from a variational principle (Hasselmann [6]) where symmetry of coupling coefficients is a trivial by-product.

Equations of motion for wave correlations $\langle a_{s_1} \dots a_{s_n}\rangle$ are readily obtained from (23). The evolution of the second order correlation is governed by

$$\partial_t\langle a_s a_{s'}\rangle = e^{i\omega_s t}\{\varepsilon\langle G_s^{(0)} a_{s'}\rangle + \varepsilon\langle G_s^{(1)} a_{s'}\rangle + \dots\} + (s \leftrightarrow s') . \tag{30}$$

But since $G_s^{(1)}$ is a quadratic polynomial of the wave amplitudes this equation already involves triple correlations $\langle a_s a_{s'} a_{s''}\rangle$ even in the lowest non-trivial order. Thus one has to consider an infinite hierarchy of equations. As in turbulence theory a closure hypothesis is needed. There is no generally accepted scheme by which hierarchies describing strongly non-linear systems can be cut short. A variety of hypotheses exist each of which seems to explain only limited aspects of experimental results. This dilemma is much reduced in the theory of statistical wave fields. Because of the dispersive nature of wave propagation linear random wave fields are in a Gaussian state, that is a state in which wave amplitudes are mutually independent. Weakly non-linear wave fields never depart much from a Gaussian state. It has been shown by Prigogine [16] that *in the limit of infinitely weak non-linear coupling the forcing correlations $\langle G_s^{(n)} a_{s'}\rangle$ can be determined under the assumption that the lowest order amplitudes $a_s^{(0)}$ are elements of a Gaussian ensemble.* This implies that $\langle a_{s_1}^{(0)} \dots a_{s_n}^{(0)}\rangle$ vanishes for odd n and becomes a sum if all possible products $\langle a_{s_i}^{(0)} a_{s_j}^{(0)}\rangle \langle a_{s_k}^{(0)} a_{s_l}^{(0)}\rangle$ for even n (we assume initially $\langle a_s\rangle = \langle a_s^{(0)}\rangle = 0$ which remains so for homogeneous systems due to a general property of the coupling coefficients).

If, in addition, the *free wave amplitudes are uncorrelated with external fields* we can evaluate the wave correlations to any order in ε in terms of the initial second order correlation $\langle a_s^{(0)} a_{s'}^{(0)}\rangle$. As with the amplitude equations, the integration of the correlation equations yield secular contributions which arise from resonances. *The second order correlation asymptotically takes the form*

$$\langle a_s a_{s'}\rangle \sim \langle a_s^{(0)} a_{s'}^{(0)}\rangle + \varepsilon t S_1(\langle a^{(0)} a^{(0)}\rangle, \langle b\rangle) + \varepsilon^2 t\{S_2(\langle a^{(0)} a^{(0)}\rangle, \langle a^{(0)} a^{(0)}\rangle) + S_3(\langle a^{(0)} a^{(0)}\rangle, \langle bb\rangle) + S_4(\langle bb\rangle) + S_5(\langle bbb\rangle) + S_6(\langle bbbb\rangle)\} + O(\varepsilon^3 t) . \tag{31}$$

The S_j are operators which have the form of scattering integrals involving the resonance conditions as δ-functions. The explicit structure will be illustrated in applications. The limitation in time ($\varepsilon t \ll 1$ or $\varepsilon^2 t \ll 1$) for the validity of the expanison can be removed by interpreting the linear increase as a rate of change of a slowly varying correlation $\langle a_s a_{s'}\rangle$ on a time scale of orders ε^{-1} or ε^{-2}. This procedure yields a *closed evolution equation for the second order correlations*

$$\partial_t\langle a_s a_{s'}\rangle = S_1 + \dots + S_6 + O(\varepsilon^3) . \tag{32}$$

The somewhat heuristic approach to this equation presented here can be put on a mathematically firmer ground by using multipletime-scale methods (DAVIDSON [4]).

It should be mentioned that equation (32) is irreversible with respect to time and thus allows the description of a system which evolves towards an equilibrium. This is not the case for the exact infinite hierarchy. This is a common feature in statistical mechanics: by smoothing the correlations (neglection of cumulants implied by the GAUSSian hypothesis) irreversability is introduced and equations are obtained which describe the macroscopic behaviour of the system.

The statistical description of a weakly interacting wave field can be simplified further *if the free state can be regarded as statistically homogeneous* (in the propagation space) *and stationary.* This means that correlations are nvariant against translations in the propagation space and in time. *Then the wave field is completely described by he action spectra* $N_s = N_\nu^l(k)$ of wave branch l and mode ν defined by

$$\tfrac{1}{2}\langle a_s a_{s'}\rangle = \delta_{ss'} N_s \tag{33}$$

Equation (32) then reduces to a local evolution equation of the spectra $N_\nu^l(k)$ which constitute the source function S introduced in the radiative transfer equation (17).

3.3. Applications

The weak interaction theory has found many applications in the recent years in oceanography because advances in measuring techniques have revealed that considerable amount of energy of the oceanic motions in the deep-sea is stored in the form of waves. For illustration I present some applications concerning the oceanic internal wave field. Internal waves exchange energy and momentum with low-frequency motions by means of wave-induced REYNOLDS-stresses. This is of considerable importance for the understanding and modelling of the large scale motion, i.e. the general circulation pattern and meso-scale synoptic eddies. Interaction processes with the wave field must be accounted for in numerical models of the circulation in form of carefully defined parametrizations. A detailed understanding of the energy balance of the wave field is required, i.e. the generation, transfer and dissipation processes affecting the wave field must be studied. In this task the weak interaction theory turned out to be a powerful tool (MÜLLER and OLBERS [11]).

3.3.1 Generation of internal waves in the seasonal thermocline

Two generation mechanisms have been proposed to explain the occurrence of high-frequency internal waves which are trapped in the seasonal thermocline of the ocean: resonant generations by atmospheric turbulence and the weak coupling of surface and internal waves.

The *source function of atmospheric generation* is of the form S_4 in equation (31)

$$S_\nu^{\text{atm}}(k) = \int dk' \int d\omega'\, T_\nu^{\text{atm}}(k', \omega')\, \delta(k - k')\, \delta(\omega_{k,\nu} - \omega')\, F_{\text{atm}}(k', \omega')\,, \tag{34}$$

where $F_{\text{atm}}(k, \omega)$ is the spectrum of the turbulent atmospheric field, such as pressure, windstress or buoyancy flux at the sea surface. Insufficient knowledge of the spectral structure of F_{atm} in the wavenumber-frequency region of internal waves presently prevents a detailed theoretical investigation of this mechanism.

Generation of internal waves by resonant interaction of two surface wave components leads to a source function emerging from the term S_6 *in* (31) *by evaluating the quadruple correlation of the external field for a* GAUSSian *wave field.* This yields (OLBERS and HERTERICH [14])

$$S_\nu^g(k) = \int dk' \int dk''\, T_\nu^g(k, k', k'')\, \delta(k' - k'' - k)\, \delta(\omega' - \omega'' - \omega_{k,\nu})\, N_g(k')\, N_g(k'')\,, \tag{35}$$

where $N_g(k)$ is the (two-dimensional) surface wave spectrum. This quantity is well-known from observations and parametrical models have been found. Evaluation of the scattering integral (35) yields a parametrization of the transfer rate in terms of spectral parameters of the surface wave field which in case of wind sea may further be related to the surface wind speed. Some features of observations can be explained by the mechanism, such as the burstlike occurrence of internal waves in the seasonal thermocline which might be due to the very strong dependence of the transfer rate on the local sea state: the rate is proportional to the fourth power of the significant wave height and the seventh power of the local wind speed (in case of wind sea). The process will be investigated in the JASIN experiment.

3.3.2 Energy balance of the deep-sea internal wave field

Observations of the internal wave field in the main thermocline of the ocean lead to the conjection of a universal shape and level of the internal wave spectrum (GARRETT and MUNK [5], MÜLLER et al. [12]). This is a surprising result in view of the large number of the interaction processes which may affect the state of the wave field. However, even in the vicinity of possible generation regions, such as the sea surface, rough bottom topography, continental slopes or strong shear currents, the shape and the level of the spectrum do not show significant differences from a universal form (WUNSCH [21]).

A first conception of a balance of the universal spectrum evolved from the investigations of the resonant interactions within the internal wave field (OLBERS [13]). The *source function for triad interactions*

$$\begin{aligned} S(k) = \int dk' \int dk''\, \{T^+(k, k', k'')\, \delta(k - k' - k'')\, \delta(\omega - \omega' - \omega'')\, [N'N'' - NN' - NN''] + \\ + 2T^-(k, k', k'')\, \delta(k - k' + k'')\, \delta(\omega - \omega' + \omega'')\, [N'N'' + NN' - NN'']\} \end{aligned} \tag{36}$$

is derived from the term S_2 in equation (31). (The propagation space is here — in contrast to the last example — the three-dimensional space. This is an adequate simplification since observations showed that the vertical scales of the waves are small compared to the vertical scale of the stratification). Notice that *internal resonant interactions conserve the total energy* (not action!), i.e.

$$\int dk\, \omega\, S(k) = 0 \qquad (37)$$

but for a given spectrum $N(k)$ there will generally be a transfer of energy within the spectral region, i.e. $S(k)$ will show sources and sinks. In a state of equilibrium these must be balanced by external generation and dissipation processes. Following these ideas a rough balance could be proposed by which internal waves draw energy from the low-frequency oceanic shear current and loose this energy after non-linear transfer by wave breaking (OLBERS [13], MÜLLER and OLBERS [11]).

At present other generation processes cannot be excluded. The general picture which has emerged from a detailed investigation of the scattering integral (MCCOMAS and BRETHERTON [9]; MCCOMAS [10]) can be expressed as follows. Any deviation from the equilibrium state results in a large transfer of energy to the spectral dissipation region, i.e. wherever energy is supplied from external fields it is very efficiently transferred to a region where it can be dissipated by wave breaking. Time scales of the non-linear transfer frequently lie below the wave period and the weak interaction condition becomes questionable. A simple illustration of this conception is a system of connected tubes filled to the top with water: wherever water is supplied an equivalent portion is immediately spilled at the lowest out-flow and the equilibrium shape is restored.

References

1 BOGOLJUBOW, N. N.; MITROPOLSKI, J. A., Asymptotische Methoden in der Theorie der nichtlinearen Schwingungen, Akademie-Verlag, Berlin 1965.
2 BRETHERTON, F. P., Propagation in slowly varying waveguides. Proc. Roy. Soc. Ser. A **302**, 555 (1968).
3 CHANDRASEKHAR, S., Stochastic problems in physics and astronomy, Rev. Modern Phys. **15**, 1—89 (1943).
4 DAVIDSON, R. C., The evolution of wave correlations in uniformly turbulent, weakly non-linear systems, J. Plasma Physics **1**, 341—359 (1967).
5 GARRETT, C.; MUNK, W., Space-time scales of internal waves: a progress report, J. Geophys. Res. **80**, 291—298 (1975).
6 HASSELMANN, K., FEYNMAN diagrams and interactions rules of wave-wave scattering processes, Rev. Geophys. **4**, 1, 1—32 (1966).
7 HASSELMANN, K., Application of two-timing methods in statistical geophysics, J. Geophys. **43**, 351—358 (1977).
8 LANDAU, L. D.; LIFSCHITZ, E. M., Mechanik, Akademie-Verlag, Berlin 1970.
9 MCCOMAS, C. H.; BRETHERTON, F. P., Resonant interactions of oceanic internal waves, J. Geophys. Res. **82**, 1397—1412 (1977).
10 MCCOMAS, C. H., Equilibrium mechanics within the oceanc internal wave field, J. Phys. Oceanogr. J· Phys. Oceanogr. **7**, 836—845 (1978).
11 MÜLLER, P.; OLBERS, D. J., On the dynamics of internal waves in the deep ocean, J. Geophys. Res. **80**, 3848—3860 (1975).
12 MÜLLER, P.; OLBERS, D. J.; WILLEBRAND, J., The IWEX spectrum, J. Geophys. Res. **83**, 479—500 (1978).
13 OLBERS, D. J., Non-linear energy transfer and the energy balance of the internal wave field in the deep ocean, J. Fluid Mech. **74**, 375—399 (1976).
14 OLBERS, D. J.; HERTERICH, K., The spectral energy transfer from surface waves to internal waves in the ocean, J. Fluid Mech. (to appear in 1978).
15 PEIERLS, R., Zur kinetischen Theorie der Wärmeleitungen in Kristallen, Ann. der Phys. **3**, 1055—1101 (1929).
16 PRIGOGINE, I., Non Equilibrium Statistical Mechanics, Interscience, New York 1962.
17 SANDRI, G., Uniformization of asymptotic expansions, in: Non-linear partial differential equations, Academic Press, New York 1977.
18 SCOTT, A. C.; CHU, F. Y. F.; MCLAUGHLIN, D. W., The soliton: a new concept in applied science, Proc. IEEE **61**, 1443—1483 (1973).
19 TAYLOR, G. I., Diffusion by continuous movements, Proc. London Math. Soc. **20**, 196 (1921).
20 WHITHAM, G. B., Variational methods and applications to water waves, Proc. Roy. Soc. Ser. A **299**, 6—25 (1967).
21 WUNSCH, C., Geographical variability of the internal wave field: a search for sources and sinks, J. Phys. Oceanogr. **6**, 471—485 (1976).

Address: Dr. DIRK J. OLBERS, Universität Hamburg, Institut für Geophysik und Max-Planck-Institut für Meteorologie, Bundesstraße 55, D-2 Hamburg 13, BRD
Present affiliation: Institut für Meereskunde, Universität Kiel, Düsterbrooker Weg 20, D-23 Kiel, BRD