Double-Copy Structure of One-Loop Open-String Amplitudes

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In this Letter, we provide evidence for a new double-copy structure in one-loop amplitudes of the open superstring. Their integrands with respect to the moduli space of genus-one surfaces are cast into a form where gauge-invariant kinematic factors and certain functions of the punctures—so-called generalized elliptic integrands—enter on completely symmetric footing. In particular, replacing the generalized elliptic integrands by a second copy of kinematic factors maps one-loop open-string correlators to gravitational matrix elements of the higher-curvature operator R^4.

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Introduction.—Recent investigations of scattering amplitudes revealed a variety of hidden relations between field theories of seemingly unrelated particle content. The oldest and possibly most prominent example of such connections is the double-copy structure of gravity [1–3] whose scattering amplitudes can be reduced to squares of gauge-theory building blocks. This kind of double copy is geometrically intuitive from the realization of gravitons and gauge bosons as vibration modes of closed and open strings, respectively. Its first explicit realization at the level of scattering amplitudes in string theory was pinpointed by Kawai, Lewellen, and Tye (KLT) in 1985 [1].

The first loop-level generalization of the gravitational double copy was found by Bern, Carrasco, and Johansson (BCJ) [3]: Gauge-theory ingredients in a suitable gauge can be conjecturally squared to gravitational loop integrands at the level of cubic diagrams. The gauge dependence of the BCJ construction has been recently bypassed through a generalized double copy [4]—see Ref. [5] for an impressive five-loop application—and a one-loop KLT formula in field theory [6].

It has been recently discovered that tree-level amplitudes of the *open* superstring admit a double-copy representation [7] which mimics the field-theory version of the KLT formula [8]: Gauge-theory trees are double copied with moduli-space integrals whose expansion in the inverse string tension α' suggests an interpretation as scattering amplitudes in effective scalar field theories [9].

One-loop open-string amplitudes exhibit two sorts of invariances that are intertwined through a similar double-copy structure: While gauge invariance is also required for field-theory amplitudes, string-theory correlators defined over a Riemann surface of genus one must be additionally invariant under monodromy variations, i.e., transporting their punctures around the homology cycles.

In this Letter, we introduce a one-to-one map between gauge-invariant kinematic factors of the external states and doubly periodic functions on genus-one Riemann surfaces, and the latter will be traced back to so-called generalized elliptic integrands. The examples given up to six points provide evidence for a double-copy structure in one-loop open-string amplitudes. In particular, when the generalized elliptic integrands are double copied to their gauge-invariant kinematic counterparts, we obtain gravitational tree-level matrix elements: those with a single insertion of the higher-curvature operator R^4 from an effective Lagrangian $\sim R + R^4$ along with its supersymmetrization.

The results of this Letter yield the first manifestly supersymmetric representations of seven-point integrands for open- and closed-string one-loop amplitudes, and we will report on cross-checks and higher-multiplicity results in Ref. [10].

Open-string correlators.—Color-stripped one-loop amplitudes of n open-string states are given by the moduli-space integral

$$A^{1-\text{loop}}_{\text{open}}(\lambda) = \int d^D\ell \int_{D(\lambda)} d\tau \prod_{j=2}^{n} dz_j |\mathcal{I}_n| \mathcal{K}_n. \qquad (1)$$

Following the *chiral-splitting* techniques of Ref. [11], the integrations of Eq. (1) involve D-dimensional loop momenta ℓ. The integration domain $D(\lambda)$ for the moduli z_j, τ depends on the topology of the genus-one world sheet—the cylinder and the Möbius strip—represented by λ. Both of these topologies can be derived from a torus via suitable

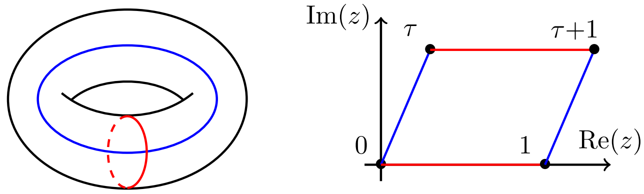

FIG. 1. Parameterization of a torus as a lattice $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ with discrete identifications $z \cong z+1 \cong z+\tau$ of the punctures and modular parameter τ in the upper half plane.

involutions [12], and its usual parametrization depicted in Fig. 1 requires the quantities $|\mathcal{I}_n|\mathcal{K}_n$ to be doubly periodic functions as $z \to z+1$ and $z \to z+\tau$—at least after integration over ℓ.

Koba-Nielsen factor and the correlators.—A universal contribution to genus-one integrands in Eq. (1) is furnished by the Koba-Nielsen factor $|\mathcal{I}_n|$ with

$$\mathcal{I}_n \equiv \exp\left(\sum_{i<j}^{n} s_{ij} \ln\theta(z_{ij},\tau) + \sum_{j=1}^{n} z_j(\ell\cdot k_j) + \frac{\tau\ell^2}{4\pi i}\right) \tag{2}$$

and $z_{ij} \equiv z_i - z_j$. Here, $s_{ij} \equiv k_i \cdot k_j$ are the Mandelstam invariants in units $2\alpha' = 1$ built from lightlike external momenta k_j, and θ is the odd Jacobi theta function:

$$\theta(z,\tau) \equiv \sin(\pi z) \prod_{n=1}^{\infty}(1-e^{2\pi i(n\tau+z)})(1-e^{2\pi i(n\tau-z)}). \tag{3}$$

Finally, the correlators $\mathcal{K}_n$ in Eq. (1) are the main subject of this Letter's investigations: They comprise kinematic factors for the external states written in pure-spinor superspace as well as meromorphic functions of the moduli to be introduced as generalized elliptic integrands. We will provide evidence via explicit examples at $n \leq 6$ points that the kinematic factors and generalized elliptic integrands satisfy identical relations and that their composition can be viewed as a double copy.

By virtue of chiral splitting, the moduli-space integrands of closed-string one-loop amplitudes follow as the holomorphic square $\mathcal{K}_n \to |\mathcal{K}_n|^2$ along with $|\mathcal{I}_n| \to |\mathcal{I}_n|^2$ [11]. Hence, the double-copy structure to be described for $\mathcal{K}_n$ immediately propagates to the closed string.

Kinematic factors from pure spinors.—In the pure-spinor formulation of the superstring [13], the gauge invariance and supersymmetry of the amplitudes are unified to an invariance under the Becchi-Rouet-Stora-Tyutin (BRST) operator Q. A classification of BRST-invariant kinematic factors of various tensor ranks that can arise from the one-loop amplitude prescription has been given in Ref. [14]. The simplest scalar BRST invariants can be expressed in terms of gauge-theory trees [15], e.g.,

$$C_{1|2,3,4} = s_{12}s_{23}A^{\rm tree}_{\rm YM}(1,2,3,4),$$
$$C_{1|23,4,5} = s_{45}[s_{34}A^{\rm tree}_{\rm YM}(1,2,3,4,5) - (2\leftrightarrow 3)], \tag{4}$$

and further examples of various tensor ranks are available for download in Ref. [16]. For instance, the bosonic components of a vector invariant $C^m_{1|2,3,4,5}$ (with Lorentz indices $m,n = 0,1,\ldots,D-1$ in D spacetime dimensions) involve tensor structures such as $e^m_1 t_8(2,3,4,5)$ and $(k^m_2/s_{12})t_8(12,3,4,5)$. The t_8 tensor with multiparticle insertions is defined in Ref. [6], and e_i denotes the polarization vector of the ith gluon. Here and in the following, groups of external-state labels in a subscript that are separated by a comma (rather than a vertical bar) can be freely interchanged; e.g., $C_{1|23,4,5} = C_{1|23,5,4} = C_{1|4,23,5}$.

In addition to BRST-invariant kinematic factors, the six-point correlator [17] gives rise to *pseudoinvariants* with nonvanishing BRST variations,

$$QC^{mn}_{1|2,3,4,5,6} = -\eta^{mn}V_1 Y_{2,3,4,5,6},$$
$$QP_{1|2|3,4,5,6} = -V_1 Y_{2,3,4,5,6}, \tag{5}$$

where V_1 denotes an unintegrated vertex operator and $Y_{2,3,4,5,6}$ is related to the anomaly kinematic factor $\sim\varepsilon_{10}F^5$ of the gluon field strength [18]. The BRST variation of the correlator localizes on the boundary of moduli space, and the cancellation of the hexagon anomaly [19] thus follows as usual in the integrated amplitude Eq. (1).

The construction of (pseudo)invariants from Berends-Giele currents [14] gives rise to the shuffle symmetries within the individual groups of labels; e.g.,

$$C^{\cdots}_{1|23,\ldots} = -C^{\cdots}_{1|32,\ldots}, \quad C^{\cdots}_{1|234,\ldots} + {\rm cyc}(2,3,4) = 0. \tag{6}$$

Double-copy representations.—In order to exemplify the main result of this work, the correlators for open-string amplitudes Eq. (1) up to multiplicity six can be written as [10]

$$\mathcal{K}_4 = C_{1|2,3,4}E_{1|2,3,4},$$
$$\mathcal{K}_5 = C^m_{1|2,3,4,5}E^m_{1|2,3,4,5} + [s_{23}C_{1|23,4,5}E_{1|23,4,5} + (2,3|2,3,4,5)],$$
$$\mathcal{K}_6 = \frac{1}{2}C^{mn}_{1|2,\ldots,6}E^{mn}_{1|2,\ldots,6} - [P_{1|2|3,4,5,6}E_{1|2|3,4,5,6} + (2\leftrightarrow 3,\ldots,6)] + [s_{23}C^m_{1|23,4,5,6}E^m_{1|23,4,5,6} + (2,3|2,3,\ldots,6)]$$
$$+ \{[s_{23}s_{45}C_{1|23,45,6}E_{1|23,45,6} + {\rm cyc}(3,4,5)] + (6\leftrightarrow 5,4,3,2)\}$$
$$+ \{[s_{23}s_{34}C_{1|234,5,6}E_{1|234,5,6} + {\rm cyc}(2,3,4)] + (2,3,4|2,\ldots,6)\}. \tag{7}$$

Throughout this Letter, $(i_1, \ldots, i_p | i_1, \ldots, i_q)$ denotes a sum over the $\binom{q}{p}$ choices of p indices $i_1, \ldots, i_p$ out of $i_1, \ldots, i_q$. The entire dependence of the correlators Eq. (7) on the external polarizations is captured by the above (pseudo) invariants, and they are accompanied by generalized elliptic integrands $E^{\cdots}_{1|\ldots}$ to be spelled out in the next section. In particular, the two kinds of ingredients in Eq. (7) will be shown to enter on completely symmetric footing and to be freely interchangeable. This symmetry is at the heart of the double-copy structure of one-loop open-string amplitudes.

Generalized elliptic integrands.—At tree level, the double-copy structure of the open superstring arises from a relation between kinematic factors and world sheet functions defined on a disk [8]. The same Kleiss-Kuijf and BCJ relations among gauge-theory amplitudes $A^{\rm tree}_{\rm YM}(1,2,\ldots,n)$ [2,20] are satisfied by the disk integrals of the so-called Parke-Taylor factors $(z_{12}z_{23}\ldots z_{n-1,n}z_{n,1})^{-1}$, where z_j represent the locations of the punctures on the disk boundary.

In this section, we will introduce the notion of generalized elliptic integrands (GEIs) to specify the $E^{\cdots}_{1|\ldots}$ in the correlators Eq. (7). They refer to functions on genus-one Riemann surfaces that play a similar role in one-loop open-string amplitudes as the Parke-Taylor factors at tree level.

Key definition.—By the quasiperiodicity $\theta(z+\tau,\tau) = -e^{-i\pi\tau-2\pi i z}\theta(z,\tau)$ of the theta function Eq. (3), the Koba-Nielsen factor Eq. (2) by itself is not a doubly periodic function of the punctures. However, its monodromies as $z_j \to z_j + \tau$ can be compensated by a shift in the loop momentum $\ell \to \ell - 2\pi i k_j$:

$$\mathcal{I}_n\big|^{\ell\to\ell-2\pi i k_j}_{z_j\to z_j+\tau} = \mathcal{I}_n. \tag{8}$$

We refer to meromorphic functions of z_j, ℓ, τ invariant under $(z_j,\ell)\to(z_j+\tau,\ell-2\pi i k_j)$ and $(z_j,\ell)\to(z_j+1,\ell)$ as GEIs. After integrating the loop momentum in Eq. (1), GEIs give rise to doubly periodic but generically non-meromorphic functions of z_j and τ. Since $\mathcal{I}_n$ transforms by a complex phase under $z_j \to z_j+1$, the quantity $|\mathcal{I}_n|$ in Eq. (1) is a GEI.

Scalar GEIs.—A variety of GEIs can be generated from the Kronecker-Eisenstein series [21],

$$F(z,\alpha,\tau) \equiv \frac{\theta'(0,\tau)\theta(z+\alpha,\tau)}{\theta(\alpha,\tau)\theta(z,\tau)} \equiv \sum_{n=0}^{\infty} \alpha^{n-1} g^{(n)}(z,\tau), \tag{9}$$

whose expansion in α defines meromorphic functions such as $g^{(0)}(z,\tau)=1$ and $g^{(1)}(z,\tau)=\partial_z \ln\theta(z,\tau)$ as well as

$$2g^{(2)}(z,\tau) = [\partial_z \ln\theta(z,\tau)]^2 + \partial_z^2 \ln\theta(z,\tau) - \frac{\theta'''(0,\tau)}{3\theta'(0,\tau)}. \tag{10}$$

The importance of the Kronecker-Eisenstein series to the description of one-loop open-string integrands has been recently emphasized in Ref. [22], where it was shown to reproduce the spin-sum identities of Ref. [23].

The quasiperiodicity $F(z+\tau,\alpha,\tau) = e^{-2\pi i\alpha}F(z,\alpha,\tau)$ implies that the functions $g^{(n)}(z,\tau)$ are not elliptic,

$$g^{(n)}(z+\tau,\tau) = \sum_{k=0}^{n} \frac{(-2\pi i)^k}{k!} g^{(n-k)}(z,\tau); \tag{11}$$

for example, $g^{(1)}(z+\tau,\tau) = g^{(1)}(z,\tau) - 2\pi i$. However, these monodromies cancel in cyclic products,

$$\begin{aligned} &F(z_{12},\alpha,\tau)F(z_{23},\alpha,\tau)\ldots F(z_{k-1,k},\alpha,\tau)F(z_{k,1},\alpha,\tau) \\ &\quad\equiv \alpha^{-k}\sum_{w=0}^{\infty}\alpha^w V_w(1,2,\ldots,k), \end{aligned} \tag{12}$$

which define elliptic functions V_w in k variables with w simultaneous poles as $z_j \to z_{j+1}$ such as $V_0(1,2,\ldots,k)=1$ and $V_1(1,2,\ldots,k) = \sum_{j=1}^{k} g^{(1)}_{j,j+1}$, as well as

$$V_2(1,2,\ldots,k) = \sum_{j=1}^{k} g^{(2)}_{j,j+1} + \sum_{i<j}^{k} g^{(1)}_{i,i+1} g^{(1)}_{j,j+1}, \tag{13}$$

with $g^{(n)}_{ij} \equiv g^{(n)}(z_i - z_j,\tau)$ and $z_{k+1}\equiv z_1$. Therefore, the following functions are elliptic:

$$\begin{aligned} &E_{1|23,4,5} \equiv V_1(1,2,3), \qquad E_{1|234,5,6} \equiv V_2(1,2,3,4), \\ &\qquad E_{1|23,45,6} \equiv V_1(1,2,3)V_1(1,4,5). \end{aligned} \tag{14}$$

In addition to the above scalar elliptic functions, we also introduce the formal definition $E_{1|2,3,4}\equiv 1$ and

$$E_{1|2|3,4,5,6} \equiv \partial_{z_1} g^{(1)}_{12} + s_{12}(g^{(1)}_{12})^2 - 2s_{12}g^{(2)}_{12}, \tag{15}$$

which exhaust the scalar GEIs in the correlators Eq. (7).

Tensorial GEIs.—Open-string integrands at $(n\geq 5)$ points also involve loop momenta from the zero modes of certain world sheet fields. Appearances of ℓ will be combined with the coefficients $g^{(n)}$ of the Kronecker-Eisenstein series Eq. (9) to form GEIs such as

$$\begin{aligned} E^m_{1|2,3,4,5} &\equiv \ell^m + k_2^m g^{(1)}_{12} + k_3^m g^{(1)}_{13} + k_4^m g^{(1)}_{14} + k_5^m g^{(1)}_{15}, \\ E^m_{1|23,4,5,6} &\equiv (\ell^m + k_4^m g^{(1)}_{14} + k_5^m g^{(1)}_{15} + k_6^m g^{(1)}_{16}) V_1(1,2,3) \\ &\quad + [k_2^m (g^{(1)}_{13} g^{(1)}_{23} + g^{(2)}_{12} - g^{(2)}_{13} - g^{(2)}_{23}) - (2\leftrightarrow 3)], \\ E^{mn}_{1|2,3,4,5,6} &\equiv \ell^m \ell^n + [k_2^{(m} k_3^{n)} g^{(1)}_{12} g^{(1)}_{13} + (2,3|2,\ldots,6)] \\ &\quad + [\ell^{(m} k_2^{n)} g^{(1)}_{12} + 2k_2^m k_2^n g^{(2)}_{12} + (2\leftrightarrow 3,4,5,6)]. \end{aligned} \tag{16}$$

Vector indices are symmetrized according to $\ell^{(m}k_2^{n)} = \ell^m k_2^n + \ell^n k_2^m$, and the notation for the permutations is explained below Eq. (7).

One can explicitly check that the above $E^{\cdots}_{1|\ldots}$ constitute GEIs after using Eq. (11) and momentum conservation. These GEIs suffice to describe open-string correlators Eq. (7) up to six points, and higher multiplicities or tensor ranks will be addressed in Ref. [10].

Shuffle symmetries from Fay identities.—Similar to the kinematic factors, GEIs obey shuffle symmetry within the individual groups of labels; e.g.,

$$E^{\cdots}_{1|23,\ldots} = -E^{\cdots}_{1|32,\ldots}, \quad E^{\cdots}_{1|234,\ldots} + \mathrm{cyc}(2,3,4) = 0. \tag{17}$$

These shuffle symmetries can be traced back to the symmetry $g^{(n)}_{ij} = (-1)^n g^{(n)}_{ji}$ and the components of the Fay relations [21],

$$F(z_1,\alpha_1)F(z_2,\alpha_2) = F(z_1,\alpha_1+\alpha_2)F(z_2-z_1,\alpha_2) + (1\leftrightarrow 2), \tag{18}$$

such as [22] $g^{(1)}_{12}g^{(1)}_{23} + g^{(2)}_{13} + \mathrm{cyc}(1,2,3) = 0$.

Double-copy structure.—In this section we will show surprising relations between the BRST-invariant kinematic factors and GEIs that underpin the double-copy structure of the open superstring at one loop. When trading the GEIs in the correlators Eq. (7) for another copy of kinematic factors, gravitational matrix elements of R^4 will be seen to emerge.

BRST-invariant kinematic factors versus GEIs.—Given the GEIs defined above, one can show that

$$k_2^m E^m_{1|2,3,4,5} + [s_{23}E_{1|23,4,5} + (3\leftrightarrow 4,5)] = 0 \tag{19}$$

up to a total world sheet derivative $[(\partial \ln \mathcal{I}_5)/(\partial z_2)]$ that vanishes under the integrals of Eq. (1). Rather surprisingly, in 2014 the following kinematic identity of identical structure was proven in the cohomology of the BRST operator [14],

$$k_2^m C^m_{1|2,3,4,5} + [s_{23}C_{1|23,4,5} + (3\leftrightarrow 4,5)] = 0, \tag{20}$$

as can be explicitly verified with the data provided on the website [16]. Note that Eq. (20) enters the field-theory amplitudes of Ref. [24] as a kinematic Jacobi identity [3].

The striking resemblance between the identities Eq. (19) on a genus-one Riemann surface and Eq. (20) in the cohomology of the kinematic BRST operator motivates us to search for further instances. Indeed, the symmetry properties [14]

$$\begin{aligned} C_{2|34,1,5} &= C_{1|34,2,5} + C_{1|23,4,5} - C_{1|24,3,5},\\ C_{2|13,4,5} &= -C_{1|23,4,5},\\ C^m_{2|1,3,4,5} &= C^m_{1|2,3,4,5} + [k_3^m C_{1|23,4,5} + (3\leftrightarrow 4,5)] \end{aligned} \tag{21}$$

hold for their dual GEIs in identical form,

$$\begin{aligned} E_{2|34,1,5} &= E_{1|34,2,5} + E_{1|23,4,5} - E_{1|24,3,5},\\ E_{2|13,4,5} &= -E_{1|23,4,5},\\ E^m_{2|1,3,4,5} &= E^m_{1|2,3,4,5} + [k_3^m E_{1|23,4,5} + (3\leftrightarrow 4,5)], \end{aligned} \tag{22}$$

as can be verified from their explicit expressions above. Similarly, the kinematic identities at six points [14],

$$k_{23}^m C^m_{1|23,4,5,6} = P_{1|2|3,4,5,6} - P_{1|3|2,4,5,6} + [s_{24}C_{1|324,5,6} - s_{34}C_{1|234,5,6} + (4\leftrightarrow 5,6)], \tag{23}$$

$$k_1^m C^{mn}_{1|2,3,4,5,6} = -[k_2^n P_{1|2|3,4,5,6} + (2\leftrightarrow 3,4,5,6)], \tag{24}$$

$$\eta_{mn}C^{mn}_{1|2,3,4,5,6} = 2[P_{1|2|3,4,5,6} + (2\leftrightarrow 3,4,5,6)], \tag{25}$$

all have a direct counterpart in terms of GEIs up to boundary terms in moduli space. Under $C^{\cdots}_{1|\ldots} \to E^{\cdots}_{1|\ldots}$, Eqs. (23) and (24) translate into total derivatives with respect to the punctures z_j that can be immediately discarded. The GEI analogue of Eq. (25) additionally involves a τ derivative,

$$\eta_{mn}E^{mn}_{1|2,\ldots,6} = 2[E_{1|2|3,4,5,6} + (2\leftrightarrow 3,\ldots,6)] + 4\pi i\frac{\partial \ln \mathcal{I}_6}{\partial \tau}, \tag{26}$$

resulting in the expected BRST anomaly $Q(\mathcal{K}_6\mathcal{I}_6) \sim [(\partial \mathcal{I}_6)/(\partial\tau)]$.

The above identities among GEIs can be derived from the antisymmetry $g^{(1)}_{ij} = -g^{(1)}_{ji}$, momentum conservation, and the Fay identity Eq. (18). Together with the shuffle symmetries Eq. (6), these relations signal a fascinating duality between BRST invariants and GEIs which will be shown on a case-by-case basis to persist at higher points [10].

The duality even extends to anomalies: The BRST variations Eq. (5) can be mapped to a modular anomaly in the ℓ integral over $E^{mn}_{1|2,\ldots,6}$ and $E_{1|2|3,4,5,6}$ [10], which cancels by the kinematic identity Eq. (25) dual to Eq. (26).

Comparison with R^4.—In the low-energy limit, one-loop amplitudes of the closed string are known to yield matrix elements of higher-curvature operators R^4 [25]. Up to and including six points, they have been expressed in terms of the above BRST (pseudo)invariants [17]:

$$\begin{aligned}
\mathcal{M}_4^{R^4} &= C_{1|2,3,4}\tilde{C}_{1|2,3,4},\\
\mathcal{M}_5^{R^4} &= C^m_{1|2,3,4,5}\tilde{C}^m_{1|2,3,4,5} + [s_{23}C_{1|23,4,5}\tilde{C}_{1|23,4,5} + (2,3|2,3,4,5)],\\
\mathcal{M}_6^{R^4} &= \frac{1}{2}C^{mn}_{1|2,\ldots,6}\tilde{C}^{mn}_{1|2,\ldots,6} - [P_{1|2|3,4,5,6}\tilde{P}_{1|2|3,4,5,6} + (2\leftrightarrow 3,\ldots,6)] + [s_{23}C^m_{1|23,4,5,6}\tilde{C}^m_{1|23,4,5,6} + (2,3|2,3,\ldots,6)]\\
&\quad + \{[s_{23}s_{45}C_{1|23,45,6}\tilde{C}_{1|23,45,6} + \mathrm{cyc}(3,4,5)] + (6\leftrightarrow 5,4,3,2)\}\\
&\quad + \{[s_{23}s_{34}C_{1|234,5,6}\tilde{C}_{1|234,5,6} + \mathrm{cyc}(2,3,4)] + (2,3,4|2,\ldots,6)\}.
\end{aligned} \tag{27}$$

The tilde refers to a second copy of the superspace kinematic factors, where the gravitational polarizations can be reconstructed from the tensor product of the gauge-theory polarizations. The double-copy structure of the above $\mathcal{M}_n^{R^4}$ is shared by the open-string correlators Eq. (7) which are converted to Eq. (27) by trading the GEIs for another copy of their kinematical correspondents: $E \leftrightarrow \{\tilde{C}, \tilde{P}\}$. This motivates us to conjecture that

$$\mathcal{K}_n = \mathcal{M}_n^{R^4}|_{\tilde{C},\tilde{P}\to E} \tag{28}$$

for arbitrary multiplicities n, where all the vector indices and external-particle labels in the subscripts are understood to be inert under the replacements. At multiplicity $n = 7$, Eq. (28) leads to a new supersymmetric expression for $\mathcal{K}_7$; i.e., the results of this Letter allow us to probe uncharted terrain of multiparticle string amplitudes (see Ref. [10] for details and consistency checks).

Conclusions and outlook.—In this Letter, we have presented evidence for a duality between GEIs and BRST-invariant kinematic factors: identities among GEIs that vanish up to boundary terms in moduli space are mapped to identities among kinematic factors that vanish up to BRST-exact terms. This duality has been exploited to reveal a double-copy structure in the one-loop amplitude of the open superstring. Trading GEIs in *open-string* correlators by another copy of BRST-invariant kinematic factors leads to *gravitational* matrix elements of supersymmetrized R^4 operators.

The duality between elliptic functions and BRST invariants presented here turns out to be even richer. Alternative double-copy representations of the above open-string correlators will be given in Ref. [10], which manifest their locality instead of gauge invariance. These representations will illustrate further aspects of the duality between kinematic factors and world sheet functions, in closer contact with conformal-field-theory techniques.

It is a fascinating possibility that the duality between kinematic invariants and (generalized) elliptic functions is a generic feature of string-theory correlators. At genus $g = 2$, 3, the low-energy limits of closed-string amplitudes have been recently computed with the pure-spinor formalism, resulting in matrix elements of $D^{2g}R^4$ [26]. It is conceivable that their double-copy structure applies to open-string correlators at the respective loop order.

The new double-copy structures unraveled in this Letter should lead to great simplification of higher-order calculations in string theory, deducing the structure of the integrands from effective-field-theory quantities. Moreover, the study of GEIs is expected to trigger conceptual advances in the mathematics of string theory related to the interplay of higher-genus geometry and algebra. Finally, the $\alpha' \to 0$ limit of our string-theory results yields new representations of field-theory amplitudes and will shed further light on the BCJ double copy at loop level [27].

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