# Lorentz signature and twisted spectral triples

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**Abstract**

We show how twisting the spectral triple of the Standard Model of elementary particles naturally yields the Krein space associated with the Lorentzian signature of spacetime. We discuss the associated spectral action, both for fermions and bosons. What emerges is a tight link between twist and Wick rotation.

## 1 Introduction

Noncommutative differential geometry (NCG) provides a unified framework from which to describe both Einstein-Hilbert gravity (in Euclidean signature) and classical gauge theories [1]. In particular, it gives an elegant description of the full Standard Model of particle physics in all of its detail, including the Higgs mechanism and neutrino mixing, as gravity on a certain "almost commutative manifold" [2, 3]. A recent and comprehensive review can be found in [4].

The main benefit of the NCG approach to physics is that it offers a more constrained description of gauge theories than the usual effective field theory approach. Indeed, the added geometric constraints impose a range of successful and phenomenologically accurate restrictions on the allowed particle content of the Standard Model of particle physics [5–8]. Despite this success, an early estimate for the Higgs mass was also furnished at $m_H \simeq$ 170 Gev. This prediction was disfavored by the Tevatron data, and has since been ruled out by the LHC [9, 10]. While falling short of an accurate comparison with experiment, this prediction depended on a number of assumptions including the big desert hypothesis, as well as the presence of a scale at which the coupling constants of the three gauge interactions unify.

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In light of the many successes of the NCG construction, one is led to question the various assumptions that went into the incorrect Higgs mass calculation, and above all the validity of the big desert assumption. This concern is particularly pressing as the "low" experimental value detected for the Higgs mass causes an instability (or meta-stability) in the electroweak vacuum at intermediate energies (see [11] for a quite recent update), that may be cured by the addition of a new scalar field suitably coupled to the Higgs - usually denoted $\sigma$ (e.g. [12, 13]). If the addition of such a scalar field were admissible within the NCG construction, it would not only stabilize the electroweak vacuum, but also allow compatibility with the experimentally observed Higgs mass [14]. Such an outcome is difficult to achieve however as the extra geometric constraints imposed by the NCG formalism severely restrict any allowed addition to the Standard Model. Early attempts in [15, 16] to generate an extra scalar field within the framework required the adjunction of new fermions. More recently, there have been a number of phenomenologically viable Standard Model extensions singled out by the NCG framework which preserve the fermionic sector. This has been achieved for example in [17, 18] by relaxing some of the geometrical constraints, and in [19] by taking full account of the outer symmetries of the model (let us also mention some proposals to modify the grading, based on Morita equivalence, developed in [20, 21], as well as other modification of the grading in [22]).

In this paper we are mostly concerned with the outcome of an extension of the Standard Model known as *grand symmetry*, proposed in [23] (see [24] for a shorter non-technical presentation). It relies on an enlargement of the Standard Model algebra, and ultimately allows one to obtain the field $\sigma$ in agreement with the NCG principles, namely as an internal part of a connection. As shown in [25] however, in order to make the grand symmetry extension work, one is required to twist the noncommutative geometry of the Standard Model, in the sense of Connes-Moscovici [26]. Besides solving some technical difficulties, the twist also permits one to understand the breaking of the grand symmetry down to the Standard Model symmetries in a dynamical process induced by the spectral action. Our first result establishes the fact that *the required twist corresponds to a Wick rotation.* More precisely, we show in section 3 that the twist turns the inner product of the Hilbert space of (Euclidean) spinors into a Krein product. The latter is precisely the product associated with spinors on a Lorentzian manifold. In a sense made precise in §3.2, the twist is actually the square of the Wick rotation.

Our second result concerns the spectral action in the twisted context. While the behavior of gauge transformations for twisted spectral triples has already been worked out in [27], the corresponding gauge invariance of the spectral action has not yet been addressed. We investigate this question in section 4:

- We begin with the fermionic action $S^F$ in §4.1, showing that the straightforward adaptation to the twisted case of the usual formula is indeed invariant under a twisted gauge transformation. However, it is not antisymmetric when restricted to the (positive) eigenvectors of the chirality operator, unlike the non-twisted case. This leads us to propose two possible definitions of $S^F$ in the twisted context:

either by restricting to the eigenvectors of the unitary implementing the twist, or by considering a Dirac operator Krein adjoint instead of selfadjoint.

- The bosonic action $S^B$ is adressed in §4.2. We show that there is an easy way to rewrite it so that it becomes invariant under a twisted gauge transformation. We also investigate the possibility of a Krein adjoint Dirac operator. Our formula then gives back the Euclidean bosonic action.

The picture that emerges is that twisted geometries may provide an appropriate framework from which to facilitate the description of non-Euclidean signatures in NCG. Before that, we begin in the following section by recalling the main feature of the twisted spectral triple of the Standard Model.

# 2 Twisted spectral geometry for the standard model

This section deals with the twisted spectral triple of the Standard Model of [25]. We do not discuss the usual non-twisted version, which can be found in [3]; neither do we motivate the importance of twists in noncommutative geometry. Let us just recall that twisted spectral triples were introduced in [26] in order to build spectral triples from algebras which do not exhibit a trace. Quite unexpectedly, they also provide the correct mathematical framework to write the "beyond SM" model of [23].

A *twisted spectral triple* $(\mathcal{A}, \mathcal{H}, D; \rho)$ consists of a *-algebra $\mathcal{A}$ of bounded operators in a Hilbert space* $\mathcal{H}$, together with a non-necessarily bounded self-adjoint operator $D$ on $\mathcal{H}$ with compact resolvent, and an automorphism $\rho$ of $\mathcal{A}$ such that the twisted commutator

$$[D, a]_\rho := Da - \rho(a)D \tag{2.1}$$

is bounded for any $a$ in $\mathcal{A}$. The twisted spectral triple is *even* if there is a $\mathbb{Z}_2$ grading, i.e. an operator $\Gamma$ on $\mathcal{H}$, $\Gamma = \Gamma^\dagger$, $\Gamma^2 = 1$, such that $\Gamma D + D\Gamma = 0$ and $\Gamma a - a\Gamma = 0$ for any $a \in \mathcal{A}$. It is *real* if there is an antilinear isometry $J$ (called real structure) which satisfies

$$J^2 = \epsilon \mathbb{I}, \quad JD = \epsilon' DJ, \quad J\Gamma = \epsilon'' \Gamma J \tag{2.2}$$

where $\epsilon, \epsilon', \epsilon'' \in \{1, -1\}$ define the $KO$-dimension (see e.g. [28] for details).

The real structure implements an action of the opposite algebra† $\mathcal{A}^\circ$, obtained by identifying $Jb^*J^{-1}$ with $b^\circ \in \mathcal{A}^\circ$ (for any $b \in \mathcal{A}$), which is asked to commute with $\mathcal{A}$:

$$[a, JbJ^{-1}] = 0 \quad \forall\, a, b \in \mathcal{A}. \tag{2.3}$$

*We denote $T^\dagger$ the adjoint of an operator $T$ on $\mathcal{H}$. As usual, we omit the symbol of representation for the algebra and identifies $\pi(a^*) = \pi(a)^\dagger$ with $a^*$.

†Identical to $\mathcal{A}$ as a vector space, but with reversed product: $a^\circ b^\circ = (ba)^\circ$.

This is called the *order zero condition* and it permits to define a right action of $\mathcal{A}$ on $\mathcal{H}$

$$\psi a := a^\circ \psi = Ja^* J^{-1}\psi \quad \forall \psi \in \mathcal{H}. \tag{2.4}$$

Another condition that plays an important role is the *order one condition* [29] which, for twisted spectral triples, writes [25, 30]

$$[[D, a]_\rho, JbJ^{-1}]_{\rho_0} = 0 \quad \forall a, b \in \mathcal{A} \tag{2.5}$$

where‡

$$\rho_0(JbJ^{-1}) := J\rho(b)J^{-1}. \tag{2.6}$$

Usual spectral triples are retrieved taking for $\rho$ the identity automorphism $\rho(a) = a$.

A gauge theory is described - in its non twisted version - by an *almost commutative geometry*, that is the product

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_F,\ \mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_F,\ D = \partial\!\!\!/ \otimes \mathbb{I}_F + \gamma_E \otimes D_F \tag{2.7}$$

of the canonical spectral triple $\left(C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), \partial\!\!\!/\right)$ associated to an (oriented closed) Riemannian spin manifold $\mathcal{M}$ of even dimension§ $m$, with a finite dimensional spectral triple $(\mathcal{A}_F, \mathcal{H}_F, D_F)$. Recall that $L^2(\mathcal{M}, S)$ denotes the Hilbert space of square integrable spinors on $\mathcal{M}$, on which $C^\infty(\mathcal{M})$ acts as

$$\pi(f) = f \begin{pmatrix} \mathbb{I}_{\frac{n}{2}} & 0 \\ 0 & \mathbb{I}_{\frac{n}{2}} \end{pmatrix} \qquad f \in C^\infty(\mathcal{M}), \tag{2.8}$$

with $n = 2^{\frac{m}{2}}$ the dimension of the spin representation. The Dirac operator is

$$\partial\!\!\!/ = -i \sum_{\mu=1}^{m} \gamma_E^\mu \nabla_\mu^S \quad \text{where} \quad \nabla_\mu^S = \partial_\mu + \omega_\mu^S \tag{2.9}$$

with $\gamma_E^\mu = \gamma_E^{\mu\,\dagger}$ the selfadjoint Euclidean-signature Dirac matrices and $\omega_\mu^S$ the spin connection. The grading

$$\gamma_E = \mathrm{diag}(\mathbb{I}_{\frac{n}{2}}, -\mathbb{I}_{\frac{n}{2}}) \tag{2.10}$$

is the product of the Dirac matrices (it is usually called $\gamma^5$ in the physics literature).

To describe the standard model, under natural assumptions on the representation of the algebra, it is shown in [5] that the finite dimensional algebra in (2.7) has to be

$$\mathcal{A}_F := \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \tag{2.11}$$

‡ $\rho_0$ is the "natural" image of $\rho$ in the automorphism group of $\mathcal{A}^\circ$: $\rho_\circ(b^\circ) = (\rho(b))^\circ$.

§In this paper we will consider only manifolds of even dimension. The odd case has technical issue which we prefer to ignore. For a full discussion of product geometries see [31] and the references therein.

acting on the finite dimensional Hilbert space.

$$\mathcal{H}_F = \mathcal{H}_R \oplus \mathcal{H}_L \oplus \mathcal{H}_R^c \oplus \mathcal{H}_L^c = \mathbb{C}^{96} \tag{2.12}$$

where $\mathcal{H}_R = \mathbb{C}^8 \times \mathbb{C}^3$ is spanned by the $N = 3$ generations of 8 right-handed fermions (electron, neutrino, up and down quarks with three colors each), $\mathcal{H}_L$ stands for left fermions, and the exponent $^c$ is for the antiparticles. The finite dimensional Dirac operator $D_F$ is a $96 \times 96$ matrix whose entries are the Yukawa couplings of fermions, the Dirac and Majorana masses of neutrinos, the Cabibbo matrix and the mixing matrix for neutrinos.

The twisted spectral triple of the Standard Model is obtained by making

$$C^\infty(\mathcal{M}) \otimes \mathbb{C}^2 \simeq C^\infty(\mathcal{M}) \oplus C^\infty(\mathcal{M}) \tag{2.13}$$

act on $L^2(\mathcal{M}, S)$ as

$$\pi((f,g)) := \begin{pmatrix} f\mathbb{I}_{\frac{n}{2}} & 0 \\ 0 & g\mathbb{I}_{\frac{n}{2}} \end{pmatrix} \qquad \forall (f,g) \in C^\infty(\mathcal{M}) \oplus C^\infty(\mathcal{M}), \tag{2.14}$$

choosing as automorphism

$$\rho((f,g)) = (g,f) \qquad \forall (f,g) \in C^\infty(\mathcal{M}) \oplus C^\infty(\mathcal{M}). \tag{2.15}$$

The complete twisted spectral triple thus consists in

$$\left((C^\infty(\mathcal{M}) \otimes \mathcal{A}_F) \otimes \mathbb{C}^2, L^2(\mathcal{M}, S) \otimes \mathbb{C}^{96}, D = \partial\!\!\!/ \otimes \mathbb{I}_{96} + \gamma_E \otimes D_F;\, \rho\right), \tag{2.16}$$

where the "doubled" algebra $(C^\infty(\mathcal{M}) \otimes \mathcal{A}_F) \otimes \mathbb{C}^2$ acts on $\mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathbb{C}^{96}$ as in the Standard Model, except that the representation (2.8) of $C^\infty(\mathcal{M})$ is substituted with the representation (2.14) of $C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$. The Dirac operator is unchanged.

The grading $\Gamma$ and the real structure $J$ are as in the non twisted case, namely

$$\Gamma = \gamma_E \otimes \gamma_F \quad \text{ where } \quad \gamma_F := \text{diag}\,(\mathbb{I}_{8N}, -\mathbb{I}_{8N}, -\mathbb{I}_{8N}, \mathbb{I}_{8N}), \tag{2.17}$$

and

$$J = \mathcal{J} \otimes J_F \quad \text{ where } \quad J_F := \begin{pmatrix} 0 & \mathbb{I}_{16N} \\ \mathbb{I}_{16N} & 0 \end{pmatrix} cc \tag{2.18}$$

with $\mathcal{J} = i\gamma^0\gamma^2 cc$ the charge conjugation on $L^2(\mathcal{M}, S)$(with $cc$ the complex conjugation).

The fermionic fields are elements of the Hilbert space $\mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_F$. The bosonic fields are obtained by the so-called *twisted fluctuations of $D$ by $\mathcal{A}$*, which amount to substituting $D$ with [25]

$$D_{A_\rho} := D + A_\rho + \epsilon'\, J A_\rho J^{-1} \tag{2.19}$$

where $A_\rho$ is an element of the set of twisted generalized one forms [26]

$$\Omega^1_D(\mathcal{A},\rho) := \left\{ \sum_i a_i[D,b_i]_\rho,\ a_i,b_i \in \mathcal{A} \right\}. \tag{2.20}$$

When $D_{A_\rho}$ is selfadjoint, we call it a *twisted-covariant Dirac operator.* One then shows [25] that $(\mathcal{A},\mathcal{H},D_{A_\rho};\rho)$ is a real twisted spectral triple, with the same real structure and $KO$-dimension as $(\mathcal{A},\mathcal{H},D;\rho)$.

A gauge transformation for a twisted spectral triple [27] is implemented by the simultaneous action on $\mathcal{H}$ and $\mathcal{L}(\mathcal{H})$ (the space of linear operators on $\mathcal{H}$) of the group of unitaries of $\mathcal{A}$,

$$\mathcal{U}(\mathcal{A}) := \{u \in \mathcal{A}, u^*u = uu^* = \mathbb{I}\}\,. \tag{2.21}$$

The action on $\mathcal{H}$ follows from the adjoint action of $\mathcal{A}$ (on the left via its representation, on the right by (2.4)), that is

$$\mathrm{Ad}(u)\psi = u\psi u^* = uJuJ^{-1}\psi \quad \forall\psi\in\mathcal{H},\, u\in\mathcal{U}(\mathcal{A}). \tag{2.22}$$

The action on $\mathcal{L}(\mathcal{H})$ is defined as

$$T \mapsto \mathrm{Ad}(\rho(u))\,T\,u \qquad \forall T\in\mathcal{L}(\mathcal{H}), \tag{2.23}$$

where

$$\mathrm{Ad}(\rho(u)) = \rho(u)J\rho(u)J^{-1}. \tag{2.24}$$

In particular, for $T = D_{A_\rho}$ a twisted covariant Dirac operator (2.19), one has (see details in appendice, and also [27])

$$\mathrm{Ad}(\rho(u))\,D_{A_\rho}\mathrm{Ad}(u^*) = D_{A^u_\rho} \tag{2.25}$$

where

$$A^u_\rho := \rho(u)A_\rho u^* + \rho(u)\,[D,u^*]_\rho\,. \tag{2.26}$$

The map $A_\rho \mapsto A^u_\rho$ is a twisted version of the usual law of transformation of the gauge potential in noncommutative geometry [29].

The interest of twisting the spectral triple of the Standard Model is that whereas the part $D_R$ of the operator $D_F$ that contains the Yukawa coupling $k_R$ of the right handed neutrino is transparent to usual inner fluctuations,

$$[D_R,a] = 0 \quad \forall a\in\mathcal{A}_F, \tag{2.27}$$

it is not transparent to twisted inner fluctuations (2.19),

$$[D_R,a]_\rho \neq 0 \quad \text{for some } a\in\mathcal{A}_F\otimes\mathbb{C}^2. \tag{2.28}$$

That $D_R$ did not fluctuate remained almost unnoticed until the observation in [14] that turning the (constant) $k_R$ into a field $k_R\sigma$ provides precisely the extra scalar field required to stabilize the electroweak vacuum, and also provides a way of naturally accommodating the mass of the Higgs boson. The non-fluctuation of $D_R$ by internal symmetries can be traced back to the first-order condition (as already noticed in [32]). To justify the substitution $k_R \to k_R\sigma$, various solutions have been proposed:

- Make the first-order condition more flexible, as investigated in [17, 18], with phenomenological consequences in [33] (see also [34]).

- Attempt to fluctuate the $\sigma$ field using the outer symmetries of the theory, as initiated in [19], leading to a minimal and phenomenologically viable Standard Model extension.

- Double the algebra and twist the first order condition, as explained above.

# 3 Twist and Lorentz Structure

We show that when the automorphism $\rho$ in a twisted spectral triple $(\mathcal{A}, \mathcal{H}, D; \rho)$ is inner, then there exists a natural $\rho$-twisted inner product on $\mathcal{H}$. Furthermore, for the twisted geometry of the Standard Model (2.7), this inner-product is a Krein product of Lorentzian spinors.

## 3.1 Twisted inner product

Let $\mathcal{H}$ be an Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and $\rho$ be an automorphism of $\mathcal{B}(\mathcal{H})$.

**Definition 3.1.** *A $\rho$-twisted inner product $\langle\cdot,\cdot\rangle_\rho$ is an inner product on $\mathcal{H}$ such that*

$$\langle\Psi, \mathcal{O}\Phi\rangle_\rho = \langle\rho(\mathcal{O})^\dagger\Psi, \Phi\rangle_\rho \qquad \forall\mathcal{O}\in\mathcal{B}(\mathcal{H}),\ \Psi,\ \Phi\in\mathcal{H}, \tag{3.1}$$

*where $\rho(\mathcal{O})^\dagger$ is the adjoint of $\rho(\mathcal{O})$ with respect to the initial Hilbert inner product $\langle\cdot,\cdot\rangle$.*

We denote

$$\mathcal{O}^+ := \rho(\mathcal{O})^\dagger \tag{3.2}$$

the adjoint of $\mathcal{O}$ with respect to the $\rho$-twisted inner product. For short we call the later the $\rho$-product, and $\mathcal{O}^+$ the $\rho$-adjoint of $\mathcal{O}$. An operator $\mathcal{O}$ (resp. $U$) on $\mathcal{H}$ is said *$\rho-$hermitian* (*$\rho-$unitary*) if it is hermitian (unitary) with respect to the $\rho$-product: $\mathcal{O}^+ = \mathcal{O}$, $U^+U = UU^+ = \mathbb{I}$. In terms of the initial Hilbert product on $\mathcal{H}$, this reads

$$\mathcal{O} = \rho(\mathcal{O})^\dagger, \quad \rho(U)^\dagger U = U\rho(U)^\dagger = \mathbb{I}. \tag{3.3}$$

If $\rho$ is an inner automorphism of $\mathcal{B}(\mathcal{H})$, such that there exists a unitary operator $R$ on $\mathcal{H}$ satisfying

$$\rho(\mathcal{O}) = R\mathcal{O}R^\dagger \qquad \forall\mathcal{O}\in\mathcal{B}(\mathcal{H}), \tag{3.4}$$

then a natural $\rho$-product is

$$\langle\Psi,\Phi\rangle_\rho = \langle\Psi, R\Phi\rangle = \langle R^\dagger\Psi,\Phi\rangle. \tag{3.5}$$

Indeed, for any bounded operator $\mathcal{O}$ on $\mathcal{H}$ one checks that

$$\langle\Psi,\mathcal{O}\Phi\rangle_\rho = \langle\Psi, R\mathcal{O}\Phi\rangle = \langle\mathcal{O}^\dagger R^\dagger\Psi,\Phi\rangle = \langle\mathcal{O}^\dagger R^\dagger\Psi, R^\dagger R\Phi\rangle \tag{3.6}$$

$$= \langle R\mathcal{O}^\dagger R^\dagger\Psi, R\Phi\rangle = \langle\rho(\mathcal{O}^\dagger)\Psi,\Phi\rangle_\rho = \langle\rho(\mathcal{O})^\dagger\Psi,\Phi\rangle_\rho = \langle\mathcal{O}^+\Psi,\Phi\rangle_\rho \tag{3.7}$$

where we used that an inner automorphism is necessarily a $^*$-automorphism, that is

$$\rho(O)^\dagger = (R\mathcal{O}R^\dagger)^\dagger = R\mathcal{O}^\dagger R^\dagger = \rho(\mathcal{O}^\dagger). \tag{3.8}$$

Notice that $R$ is both unitary (by definition) and $\rho$-unitary: one has $\rho(R) = RRR^\dagger = R$ so that $\rho(R)^\dagger R = R^\dagger R = \mathbb{I} = R\,\rho(R)^\dagger$.

All the definitions above extend to any (i.e. non necessarily bounded) linear operator $T$ on $\mathcal{H}$ whose domain contains $R^\dagger\mathcal{H}$. One defines the action of $\rho$ on $T$ as $\rho(T)\Psi := R\,T\,R^\dagger\Psi$. The $\rho$-adjoint of $T$ is given as in (3.2) by $T^+ := \rho(T)^\dagger$. An unbounded operator is $\rho$-selfadjoint when $T^+ = T$ on the domain of $T$ and both operators have the same domain.

The extension of an inner automorphism $a \to uau^*$ of $\mathcal{A}$ to an automorphism of $\mathcal{B}(\mathcal{H})$ is not unique (just consider two distinct unitaries $R_1$, $R_2$ in $\mathcal{B}(\mathcal{H})$ such that $R_1 a R_1^\dagger = R_2 a R_2^\dagger$ for any $a \in \mathcal{A}$). Any such extension defines an automorphism of $\mathcal{A}_\circ$:

$$\rho(Ja^*J^{-1}) = R\,Ja^*J^{-1}R^\dagger \quad \forall a \in \mathcal{A}. \tag{3.9}$$

We say that an inner automorphism is compatible with $J$ if it admits an extension such that (3.9) agrees with $\rho_\circ \in \mathrm{Aut}(\mathcal{A}_\circ)$ defined in (2.6). More precisely:

**Definition 3.2.** *Given a real spectral triple $(\mathcal{A},\mathcal{H},D)$, an inner automorphism $\rho$ of $\mathcal{A}$ is compatible with the real structure $J$ if there exists a unitary $R \in \mathcal{B}(\mathcal{H})$ such that*

$$\rho(a) = RaR^\dagger \quad \text{and} \quad J\,R\,a^*\,R^\dagger J^{-1} = RJa^*J^{-1}R^\dagger \qquad \forall a \in \mathcal{A}. \tag{3.10}$$

This condition is verified in particular when the inner automorphism can be implemented by a a unitary $R$ such that

$$JR = \pm RJ. \tag{3.11}$$

## 3.2 Lorentzian signature and Krein space

In definition 3.1 we do not require the $\rho$-product to be positive definite. Since $R$ is unitary, one has that $\langle\cdot,\cdot\rangle_\rho$ is non-degenerate. If in addition we impose that $R$ is selfadjoint and different from the identity (i.e. the automorphism $\rho$ is not the trivial one), then $R$ has eigenvalues $\pm1$.The two corresponding eigenspaces $\mathcal{H}_+, \mathcal{H}_-$ are such that $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$

and the $\rho$-product is positive definite on $\mathcal{H}_+$, negative definite on $\mathcal{H}_-$. In other terms, a space $\mathcal{H}$ equipped with the product $\langle\cdot,\cdot\rangle_\rho$ is a Krein space. Furthermore, the operator $R$ is a fundamental symmetry, that is it satisfies $R^2 = \mathbb{I}$ and the inner product $\langle\cdot, R\cdot\rangle_\rho$ is positive definite on $\mathcal{H}$ (in our case, this is simply the Hilbert product one started with).

In the twisted spectral triple of the Standard Model, the automorphism $\rho$ is the flip (2.15). It is implemented on $L^2(\mathcal{M}, S)$ by the adjoint action of the selfadjoint unitary operator

$$R = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & \mathbb{0} \end{pmatrix}. \tag{3.12}$$

This matrix has eingenvalues $\pm 1$, hence $\mathcal{H}$ equipped with the $\rho$-product is a Krein space. The Euclidean Dirac matrices in the chiral basis are, for $\mu = 0, j$ with $j = 1, 2, 3$,

$$\gamma_E^\mu = \left( \begin{array}{cc} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{array} \right) \qquad \text{where } \sigma^\mu = \left\{\mathbb{I}_2, -i\sigma^i\right\},\ \tilde{\sigma}^\mu = \left\{\mathbb{I}_2, i\sigma^i\right\} \tag{3.13}$$

where $\sigma_j$, $j = 1, 2, 3$ are the Pauli matrices. Thus $R$ is nothing but $\gamma_E^0$ and the $\rho$-product (3.1) is now the usual inner product of quantum field theory in Lorentz signature, where instead of $\psi^\dagger$ it appears $\bar{\psi} = \psi^\dagger\gamma_E^0$:

$$\langle\psi, \phi\rangle_\rho = \langle\psi, \gamma_E^0\phi\rangle = \int d^4x\psi^\dagger\gamma_E^0\phi := \int d^4x\bar{\psi}\phi. \tag{3.14}$$

Furthermore, one has

$$\rho(\gamma_E^0) = (\gamma_E^0)^3 = \gamma_E^0, \quad \rho(\gamma_E^j) = \gamma_E^0\gamma_E^j\gamma_E^0 = -\gamma_E^j. \tag{3.15}$$

The twist therefore performs some some sort of Wick rotation whereby the sign of the time-component Dirac matrix is changed with respect to the spatial directions. The matrix $R$ is of course expressed in a particular basis, and its action on the Clifford algebra generated by the $\gamma$'s singles out one particular direction, which we identify with time. Then it makes sense to define the integral on a time slice and have fields normalized only for the space integral, which is what is commonly done. However, the $\rho(\gamma_E^i)$'s are not the Lorentzian signature (i.e. Minkowskian) gamma matrices,

$$\gamma_M^0 = \gamma_E^0\ , \quad \gamma_M^j = i\gamma_E^j \quad j = 1, 2, 3. \tag{3.16}$$

Viewing the Wick rotation as the operator $W(\gamma_E^0) = \gamma_E^0$, $W(\gamma_E^j) = i\gamma_E^j$, one has that the twist (3.15) is the square of the Wick rotation

$$\rho(\gamma_E^0) = W(W(\gamma_E^0)), \quad \rho(\gamma_E^j) = W(W(\gamma_E^j)). \tag{3.17}$$

The Euclidean Dirac matrices are selfadjoint for the Hilbert product of $L^2(\mathcal{M}, S)$, but (except for $\gamma_E^0$) not $\rho$-hermitian since from (3.2) and (3.15) one has

$$(\gamma_E^j)^+ = \rho(\gamma_E^j)^\dagger = -\gamma_E^{j\,\dagger}. \tag{3.18}$$

On the contrary, the Minkowskian gamma matrices (except $\gamma_M^0$) are not selfadjoint for the Hilbert product since (3.16) yields $(\gamma_M^j)^\dagger = -\gamma_M^j$; but they are $\rho$-hermitian since

$$\rho(\gamma_M^j) = i\rho(\gamma_E^j) = -i\gamma_E^j = -\gamma_M^j, \tag{3.19}$$

so that

$$(\gamma_M^j)^+ = \rho(\gamma_M^j)^\dagger = (\gamma_0\gamma_M^j\gamma_0)^\dagger = \gamma_0(\gamma_M^j)^\dagger\gamma_0 = -\gamma_0\gamma_M^j\gamma_0 = -\rho(\gamma_M^j) = \gamma_M^j. \tag{3.20}$$

The "temporal" gamma matrix $\gamma^0 := \gamma_E^0 = \gamma_M^0$ is both selfadjoint and $\rho$-hermitian.

The twist naturally defines a Krein structure, maintaining in the background the Euclidean structure. Applications of Krein spaces to noncommutative geometry framework have been recently studied in [35, 36] as well as in [38, 39, 39] (see reference therein for earlier attempts of adapting Connes noncommutative geometry to the Minkowskian signature).

# 4 Actions

For an almost commutative geometry (2.7), the fermionic action is [3]

$$S^F(D) := \langle J\tilde{\psi}, D\tilde{\psi}\rangle \tag{4.1}$$

where $\psi$ is a vector in

$$\mathcal{H}_+ := \{\psi \in \mathcal{H}, \gamma\psi = \psi\} \tag{4.2}$$

which, seen as an operator on the Fock space, is a Grassmanian variable $\tilde{\psi}$. The bosonic action is [2]

$$S^B(D) := \mathrm{Tr}\, f\left(\frac{D^2}{\Lambda^2}\right) \tag{4.3}$$

where $\Lambda$ is an energy cutoff and $f$ a smooth approximation of the characteristic function of the interval $[0, 1]$. Both actions are invariant under a gauge transformation [3], that is the simultaneous transformation

$$D \mapsto \mathrm{Ad}(u)\, D\, \mathrm{Ad}(u^*) \quad \text{and} \quad \psi \longmapsto \mathrm{Ad}(u)\psi. \tag{4.4}$$

For the almost commutative geometry (2.7) with $\mathcal{M}$ a Riemannian spin manifold and $(\mathcal{A}_F, \mathcal{H}_F, D_F)$ as described in §2, $S^F$ yields the fermionic action of the Standard Model, while the asymptotic expansion of $S^B$ yields the bosonic part, including the Higgs, together with the Einstein-Hilbert action (in Euclidean signature) and an extra Weyl term.

As explained at the end of section 2, a gauge transformation for a twisted spectral triple is the simultaneous transformation

$$D \mapsto \mathrm{Ad}(\rho(u))\, D\, \mathrm{Ad}(u^*) \quad \text{and} \quad \psi \longmapsto \mathrm{Ad}(u)\psi. \tag{4.5}$$

None of the actions (4.1) and (4.3) are invariant under (4.5), unless $u$ is invariant under the twist, that is $u = \rho(u^*)$. We show below how the twisted inner product (3.5) allows to modify $S^F$ and $S^B$, so that to define an action invariant under (4.5) for any unitary $u$.

## 4.1 Fermionic Action

To build a fermionic action for a twisted spectral triple invariant under (4.5), it suffices to substitute the Hilbert inner product in (4.1) with the $\rho$-product. In order to do so we first prove some technical results, and then discuss possible fermionic actions.

The first proposition shows that the immediate generalization of the usual fermionic action obtained by twisting the inner product is still invariant under the (twisted) gauge transformations.

**Proposition 4.1.** *Let $(\mathcal{A}, \mathcal{H}, D; \rho)$ be a real twisted spectral triple with $\rho$ an inner automorphism of $\mathcal{B}(\mathcal{H})$ compatible with the real structure in the sense of Def. 3.2. Then*

$$\mathfrak{A}^\rho_D(\psi, \phi) = \langle J\psi, D\phi\rangle_\rho \quad \forall \psi, \phi \in \mathcal{H} \tag{4.6}$$

*is a bilinear form invariant under the simultaneous transformations of* (4.5)*. That is*

$$\mathfrak{A}^\rho_D(\psi, \phi) = \mathfrak{A}^\rho_{Ad(\rho(u))\, D\, Ad(u^*)}\, (Ad(u)\psi,\ Ad(u)\phi) \quad \forall \psi, \phi \in \mathcal{H},\ u \in \mathcal{U}(\mathcal{A}). \tag{4.7}$$

*Proof.* One simply adapts to the twisted case, the proof of [28, Prop. 1.213]. The Hilbert product on $\mathcal{H}$ is antilinear on the first variable, and the same is true for the $\rho$-product. Since $J$ is antilinear, one has that $\mathfrak{A}^\rho_D(\cdot, \cdot)$ is linear in both variables.

Let $U = \mathrm{Ad}(u) = uJuJ^{-1}$. By (3.10) one has

$$\mathrm{Ad}(\rho(u)) = \rho(u)J\rho(u)J^{-1} = \rho(u)\rho(JuJ^{-1}) = \rho(uJuJ^{-1}) = \rho(U). \tag{4.8}$$

Therefore

$$\mathfrak{A}^\rho_{\rho(U)DU^*}(U\psi, U\phi) = \langle JU\psi, \rho(U)DU^*U\phi\rangle_\rho = \langle UJ\psi, \rho(U)D\phi\rangle_\rho \tag{4.9}$$

$$= \langle J\psi, U^+\rho(U)D\phi\rangle_\rho = \langle J\psi, D\phi\rangle_\rho \tag{4.10}$$

where in the first line we have used the fact that $J$ commutes with $U$,

$$JU = J(uJuJ^{-1}) = JJuJ^{-1}u = \epsilon'' u(JJ^{-1})J^{-1}u = \epsilon'' uJ(J^{-1})^2 u = uJu = UJ, \tag{4.11}$$

and the last line comes from (3.2):

$$U^+\rho(U) = \rho(U)^*\rho(U) = \mathbb{I}. \tag{4.12}$$

This proves the result. ■

In the non twisted case, on an almost commutative geometry of $KO$-dimension 2 (which is the case of the Standard Model), the bilinear form $\langle J\phi, D\psi\rangle$ is antisymmetric, $\langle J\phi, D\psi\rangle = -\langle J\psi, D\phi\rangle$, and does not vanish when restricted to $\mathcal{H}_+$. This makes $S^F$ in (4.1) vanishing if computed with usual spinors, but gives the expected fermionic action when computed with Grassman fermionic fields. One restricts to $\mathcal{H}_+$ in order to solve the fermion doubling problem (see [28, I.§16.2] for details). In the twisted case, the bilinear form (4.6) is not necessarily antisymmetric. It is however, as we show in the next proposition, if one restricts to

$$\mathcal{H}_R := \{\psi \in \mathcal{H}, R\psi = \psi\}. \tag{4.13}$$

**Proposition 4.2.** *Let $(\mathcal{A}, \mathcal{H}, D; \rho)$ be a twisted spectral triple for which $\rho$ is compatible with the real structure in the sense of* (3.11)*. Then the bilinear form $\mathfrak{A}_D^\rho$ is such that*

$$\mathfrak{A}_D^\rho(\psi, \phi) = \epsilon'\epsilon''\, \mathfrak{A}_D^\rho(\phi, \psi) \quad \forall \psi, \phi \in \mathcal{H}_R. \tag{4.14}$$

*Proof.* By the definition of an antiunitary operator, one has $\langle J\phi, J\psi\rangle = \langle \phi, \psi\rangle$ for any $\psi, \phi \in \mathcal{H}$. Thus

$$\mathfrak{A}_D^\rho(\psi, \phi) = \langle J\psi, RD\phi\rangle = \epsilon''\langle J\psi, J^2RD\phi\rangle = \epsilon''\langle JRD\phi, \psi\rangle. \tag{4.15}$$

Let $\epsilon'' = 1$ or $-1$ given by $JR = \epsilon'''RJ$. This yields

$$\mathfrak{A}_D^\rho(\psi, \phi) = \epsilon''\epsilon'''\langle RJD\phi, \psi\rangle = \epsilon'\epsilon''\epsilon'''\langle RDJ\phi, \psi\rangle = \epsilon'\epsilon''\epsilon'''\langle J\phi, DR^\dagger\psi\rangle. \tag{4.16}$$

Finally, using the fact that $\psi$ and $\phi$ are in $\mathcal{H}_R$, one obtains

$$\mathfrak{A}_D^\rho(\psi, \phi) = \epsilon'\epsilon''\epsilon'''\langle JR^\dagger R\phi, D\psi\rangle = \epsilon'\epsilon''\langle R^\dagger JR\phi, D\psi\rangle = \epsilon'\epsilon''\langle J\phi, RD\psi\rangle = \epsilon'\epsilon''\mathfrak{A}_D^\rho(\phi, \psi). \blacksquare$$

In $KO$-dimension 2, one has $\epsilon' = 1$, $\epsilon'' = -1$, hence $\mathfrak{A}_D^\rho$ is antisymmetric as expected.

An alternative to imposing $\psi \in \mathcal{H}_R$ is to instead assume that $D$ is $\rho$-hermitian instead of selfadjoint:

**Proposition 4.3.** *Let $(\mathcal{A}, \mathcal{H}, D; \rho)$ be as in Prop. 4.2 except that $R$ is now selfadjoint and $D$ (non necessarily selfadjoint) is such that $\rho(D^\dagger) = D$. Then the bilinear form $\mathfrak{A}_D^\rho$ satisfies*

$$\mathfrak{A}_D^\rho(\psi, \phi) = \epsilon'\epsilon''\epsilon'''\, \mathfrak{A}_D^\rho(\phi, \psi) \quad \forall \psi, \phi \in \mathcal{H}. \tag{4.17}$$

*Proof.* By a similar calculation to that of Prop. 4.2, one arrives at

$$\mathfrak{A}_D^\rho(\psi,\phi) = \epsilon'\epsilon''\epsilon'''\langle J\phi, D^\dagger R^\dagger\psi\rangle. \tag{4.18}$$

The $\rho$-hermicity of $D$ together with $R = R^\dagger$ implies $D^\dagger R = RD$. Hence

$$\mathfrak{A}_D^\rho(\psi,\phi) = \epsilon'\epsilon''\epsilon'''\langle J\phi, RD\psi\rangle = \epsilon'\epsilon''\epsilon'''\,\mathfrak{A}_D^\rho(\phi,\psi). \qquad \blacksquare$$

For the Standard Model, one has $\epsilon''' = 1$, $\epsilon' = 1$ and $\epsilon'' = -1$, so $\epsilon'\epsilon''\epsilon''' = -1$ and $\mathfrak{A}_D^\rho$ is antisymmetric as expected.

Understanding whether all the tools of twisted spectral triples still make sense with an operator $D$ that is $\rho$-hermitian instead of selfadjoint would require some mathematical studies that will be done elsewhere. Here, we simply check that $\rho$-hermicity is preserved by twisted-fluctuation of the metric, and is well behaved under gauge transformation.

**Proposition 4.4.** *Let $(\mathcal{A}, \mathcal{H}, D; \rho)$ satisfy the requirements of a twisted spectral triple except that $D$ is $\rho$-hermitian. Assuming $\rho$ is compatible with the real structure in the sense of* (3.11)*, one has that a twisted fluctuation*

$$D_{A_\rho} = D + A_\rho + JA_\rho J^{-1} \tag{4.19}$$

*of $D$ is $\rho$-adjoint as long as $A_\rho$ is $\rho$-adjoint, i.e.*

$$A_\rho = A^+ = \rho(A_\rho^\dagger). \tag{4.20}$$

*In addition, when $\rho$ is inner with $R = R^\dagger$, then a gauge transformation*

$$D'_{A_\rho} := \rho(U)D_{A_\rho}U^\dagger \quad \text{with } U := Ad(u) \text{ for } u \in \mathcal{U}(\mathcal{A}) \tag{4.21}$$

*of a $\rho$-adjoint operator $D_{A_\rho}$ is still $\rho$-adjoint.*

*Proof.* One has

$$\left(JA_\rho J^{-1}\right)^+ = R\left(JA_\rho J^{-1}\right)^\dagger R^\dagger = R\,JA_\rho^\dagger J^{-1}\,R^\dagger \tag{4.22}$$

$$= J\,RA_\rho^\dagger R^\dagger\,J^{-1} = JA_\rho^+ J^{-1} = JA_\rho J^{-1}. \tag{4.23}$$

Hence

$$(D_{A_\rho})^+ = D^+ + A_\rho^+ + \left(JA_\rho J^{-1}\right)^+ = D + A_\rho + JA_\rho J^{-1} = D_{A_\rho}. \tag{4.24}$$

For the second claim, one has

$$\left(D'_{A_\rho}\right)^+ = \rho\left(D'^{\dagger}_{A_\rho}\right) = \rho\left(UD^\dagger_{A_\rho}\rho(U)^\dagger\right) = \rho(U)\rho\left(D^\dagger_{A_\rho}\right)U^\dagger \tag{4.25}$$

$$= \rho(U)D^+_{A_\rho}U^\dagger = \rho(U)D_{A_\rho}U^\dagger = D'_{A_\rho} \tag{4.26}$$

where we use

$$\rho(\rho(U)^\dagger) = \rho\left((RUR)^\dagger\right) = \rho(RU^\dagger R) = R^2U^\dagger R^2 = U^\dagger. \qquad \blacksquare$$

Condition (4.20) is the twisted version of the usual requirement that the gauge potential $A$ in a fluctuation of the metric should be selfadjoint.

In the Standard Model, the twist $\rho$ is given by the selfadjoint unitary $R = \gamma^0$ in (3.12), thus it satisfies all the requirements of propositions 4.1, 4.2, 4.3. Moreover, the Minkowskian Dirac operator

$$D^M := \partial\!\!\!/_M \otimes \mathbb{I}_{96} + \gamma_M \otimes D_F \tag{4.27}$$

with

$$\partial\!\!\!/_M := -i\gamma_M^\mu \partial_\mu \,, \quad \gamma_M = \gamma^0\gamma_M^1\gamma_M^2\gamma_M^3 = i^3\gamma^0\gamma_E^1\gamma_E^2\gamma_E^3 = -i\gamma_E \tag{4.28}$$

is $\rho$-hermitian for the Krein structure induced by $\gamma^0$. Indeed, from (2.10) one gets

$$\gamma^0\gamma_M\gamma^0 = -i\gamma^0\gamma_E\gamma^0 = i\gamma_E = -\gamma_M, \tag{4.29}$$

hence, noticing that $\gamma_M^\dagger = -\gamma_M$, one obtains

$$(D^M)^+ = \rho((D^M)^\dagger) = \gamma^0(\partial\!\!\!/_M)^\dagger\gamma^0 \otimes \mathbb{I}_{96} + \gamma^0\gamma_M^\dagger\gamma^0 \otimes D_F, \tag{4.30}$$

$$= -i\gamma^0(\gamma_M^\mu)^\dagger\gamma^0\partial_\mu + \gamma_M \otimes D_F = -i\gamma_M^\mu\partial_\mu + \gamma_M \otimes D_F = D^M, \tag{4.31}$$

where, to get the second line, we used $\partial\!\!\!/_M^\dagger = -i(\gamma_M^\mu)^\dagger\partial_\mu$, then (3.20). By Prop. 4.4, one has that any twisted-fluctuation $D^M_{A_\rho}$ with $A_\rho = A_\rho^+$ is $\rho$-Hermitian as well. Consequently, there are two candidates for the fermionic action:

- A Lorentzian one: $\langle J\psi, D^M_{A_\rho}\psi\rangle_\rho$, where $D^M_{A_\rho}$ is a $\rho$-hermitian twisted fluctuation of the Minkowskian operator $D^M$.
- A Euclidean one: $\langle J\psi, D^E_{A_\rho}\psi\rangle_\rho$ with $\psi \in \mathcal{H}_R$ and $D^E_{A_\rho}$ is a selfadjoint twisted fluctuation of the Euclidean operator

$$D_E := \partial\!\!\!/ \otimes \mathbb{I}_{96} + \gamma_E \otimes D_F \quad \text{with } \partial\!\!\!/ := -i\gamma_E^\mu\partial_\mu. \tag{4.32}$$

The Lorentzian action has been considered in [35, 40]. The Euclidean action is similar to the one of the Standard Model [3], except that $\psi$ is in $\mathcal{H}_R$ instead of $\mathcal{H}_+$. The consequence of this latter choice on the fermion doubling will be studied elsewhere.

## 4.2 Bosonic action

The easiest way to make the bosonic action (4.3) invariant under a twisted gauge transformation is to rewrite it as

$$\mathrm{Tr}\, f\left(\frac{D^\dagger D}{\Lambda^2}\right), \tag{4.33}$$

since under the map

$$D \to \rho(U)DU^\dagger, \tag{4.34}$$

one gets that $D^\dagger D$ is mapped to $UD^\dagger DU^\dagger$ which has the same trace as $D^\dagger D$. The map (4.34) does not necessarily preserve selfadjointness. However, the gauge invariance guarantees that (4.33) still makes sense for non selfadjoint $\rho(U)DU^\dagger$, as soon as $D$ is selfadjoint. This is another advantage with respect to formula (4.3), which no longer makes sense when the argument of $f$ is non selfadjoint (or at least not a normal operator), by the lack of the spectral theorem. Obviously, in case $D = D^\dagger$ both formulas (4.33) and (4.3) coincide. This action has been computed in [25] for a selfadjoint twisted fluctuation $D_{A_\rho}$ of the Dirac operator of the Standard Model.

In case $D$ is $\rho$-hermitian, that is

$$D = D^+ = \rho(D)^\dagger, \tag{4.35}$$

it is well known that substituting the Hilbert adjoint with the Krein adjoint in (4.33) is problematic for $D^+D$ is an hyperbolic operator, whereas the heat kernel technique used for the asymptotic expansion of $S^B$ are well defined only for elliptic operators. Nevertheless, since (4.35) implies that $D^\dagger = \rho(D)$, one can still write (4.33) in a twisted form (that is, without reference to the Hilbert adjoint) as

$$\mathrm{Tr}\, f\left(\frac{\rho(D)D}{\Lambda^2}\right). \tag{4.36}$$

This action is well defined and gauge invariant. Actually, taking for $D$ the $\rho$-Hermitian Minkowskian Dirac operator $-i\gamma_M^\mu\partial_\mu$, it returns the Euclidean action: by cyclicity of the trace, one can substitute $\rho(D)D$ with $\frac{1}{2}\left(\rho(D)D + D\rho(D)\right)$, which is nothing but the Euclidean Laplacian (up to a sign):

$$\begin{aligned}
\frac{1}{2}\left(\rho(D)D + D\rho(D)\right) &= \frac{1}{2}\left(i\gamma_M^{\mu\dagger}\partial_\mu i\gamma_M^\nu\partial_\nu + i\gamma_M^\mu\partial_\mu i\gamma_M^{\nu\dagger}\partial_\nu\right) && (4.37)\\
&= -\frac{1}{2}\left(\gamma_M^{\mu\dagger}\gamma_M^\nu\partial_\mu\partial_\nu + \gamma_M^\mu\gamma_M^{\nu\dagger}\partial_\mu\partial_\nu\right) && (4.38)\\
&= -\frac{1}{2}\left(\gamma_M^{\mu\dagger}\gamma_M^\nu + \gamma_M^\mu\gamma_M^{\nu\dagger}\right)\partial_\mu\partial_\nu && (4.39)\\
&= -g_E^{\mu\nu}\partial_\mu\partial_\nu && (4.40)
\end{aligned}$$

where $g_E$ is the Euclidean metric.

# 5 Conclusions and Outlook

The twist of the spectral triple corresponding to the Standard Model makes the Lorentzian signature naturally emerge as a twisted-inner product. The spectral action can be modified accordingly in order to make sense either of a selfadjoint Dirac operator, or of a $\rho$-hermitian one. The first choice permits one to maintain the usual definition for the twisted spectral triple, but restricts the gauge group to those unitaries whose twisted adjoint action preserves selfadjointness (see [27]). On the other hand, choosing a $\rho$-hermitian Dirac operator does not restrict the gauge group ($\rho$-hermicity is preserved by twisted fluctuations), but requires one to modify the definition of twisted spectral triples in order to accommodate an operator $D$ that is twisted selfadjoint rather than selfadjoint.

The modifications of the spectral action that we propose here do not yield the bosonic action in a Lorentzian signature, which is a well-known and difficult problem. Nevertheless, twists shed a new light on the problem, if one considers that the question is not so much to obtain directly from a spectral formula the Einstein-Hilbert action in the Lorentzian signature, than to be able to implement Wick rotation in a coherent way. Traditionally in quantum field theory, one begins in a given Lorentz signature, Wick rotates to perform some calculation, then Wick rotates back to obtain physical predictions. So far in noncommutative geometry, one starts with a bosonic action in Euclidean signature, expands with heat kernel techniques and then Wick rotates. The results of this paper suggest to start with a Lorentzian signature, for which a twist is adapted (for twisted fluctuations preserve Krein hermicity, whereas usual fluctuations do not). The spectral action (4.33) then yields the Einstein-Hilbert action in Euclidean signature, and physical predictions are obtained by Wick rotating back to the Lorentzian model one has started with. The added value of the twist is thus to prescribe a geometry upon which to "Wick rotate back" [41].

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# A Appendix

Given a twisted spectral $(\mathcal{A}, \mathcal{H}, D), \rho$, we check that the twisted adjoint action (2.25) of the unitaries of $\mathcal{A}$ yields the twisted gauge transformation (2.26). In agreement with (2.21) and (2.24)), for $u$ a unitary of $\mathcal{A}$, we write

$$U := \mathrm{Ad}(u) = uJuJ^{-1}, \quad \rho(U) = \mathrm{Ad}(\rho(u)) = \rho(u)J\rho(u)J^{-1}. \tag{A.1}$$

Assuming $\rho$ is a $^*$-automorphism (which is the case of the flip in the Standard Model), one first checks that

$$\begin{aligned}
\rho(U)DU^* &= \rho(u)J\rho(u)J^*DJu^*J^*u^* \\
&= \epsilon'\rho(u)J\rho(u)Du^*J^*u^* \quad \text{using } DJ=\epsilon' JD \\
&= \epsilon'\rho(u)J\rho(u)\left(\rho(u^*)D+[D,u^*]_\rho\right)J^*u^* \\
&= \epsilon'\rho(u)JDJ^*u^* + \epsilon'\rho(u)J\rho(u)\left[D,u^*\right]_\rho J^*u^* \\
&= \rho(u)Du^* + \epsilon' JJ^*\rho(u)J\rho(u)\left[D,u^*\right]_\rho J^*u^* \\
&= \rho(u)Du^* + \epsilon' J\rho(u)J^*\rho(u)J\left[D,u^*\right]_\rho J^*u^* \;\text{ using } [J^*\rho(u)J,\rho(u)]=0 \\
&= \rho(u)Du^* + \epsilon' J\rho(u)\left[D,u^*\right]_\rho J^*uJJ^*u^* \;\text{ using } \Big[\left[D,u^*\right]_\rho, J^*\rho(u)J\Big]_\rho=0 \\
&= \rho(u)Du^* + \epsilon' J\rho(u)\left[D,u^*\right]_\rho J^* \\
&= \rho(u)\left(\rho(u^*)D+[D,u^*]_\rho\right) + \epsilon' J\rho(u)\left[D,u^*\right]_\rho J^* \\
&= D+\rho(u)\left[D,u^*\right]_\rho + \epsilon' J\rho(u)\left[D,u^*\right]_\rho J^*.
\end{aligned} \tag{A.2}$$

Then, noticing that by the order zero and the twisted first order condition one has

$$[A, J\rho(a)J^*]_\rho = 0 \tag{A.3}$$

for any twisted 1-form $A=a^i[D,b_i]_\rho$, one has

$$\begin{aligned}
\rho(U)AU^* &= \rho(u)J\rho(u)J^*AJu^*J^*u^* \\
&= \rho(u)J\rho(u)J^*J\rho(u^*)J^*Au^* \;\text{ using } \left[A,J\rho(u^*)J^*\right]_\rho=0 \\
&= \rho(u)Au^*.
\end{aligned} \tag{A.4}$$

As well,

$$\begin{aligned}
\rho(U)JAJ^*U^* &= \rho(u)J\rho(u)J^*JAJ^*Ju^*J^*u^* \\
&= \rho(u)J\rho(u)Au^*J^*u^*JJ^* \\
&= \rho(u)J\rho(u)AJ^*u^*Ju^*J^* \;\text{ using } [J^*u^*J,u^*]=0 \\
&= \rho(u)J\rho(u)J^*\rho(u^*)JAu^*J^* \;\text{ using } \left[A,J^*u^*J\right]_\rho=0 \\
&= J\rho(u)J^*\rho(u)\rho(u^*)JAu^*J^* \;\text{ using } [J\rho(u)J^*,\rho(u)]=0 \\
&= J\rho(u)Au^*J^*.
\end{aligned} \tag{A.5}$$

Therefore, collecting (A.2), (A.4) and (A.5), one finds that a twisted covariant operator $D_A = D+A+\epsilon' JAJ^*$ is mapped under a twisted gauge trasformation to

$$\rho(U)D_AU^* = D+\rho(u)Au^*+\rho(u)\left[D,u^*\right]_\rho + \epsilon' J\left(\rho(u)Au^*+\rho(u)\left[D,u^*\right]_\rho\right)J^* \tag{A.6}$$

which is nothing but $D+A^u+JA^uJ^*$ for $A^u$ the twisted gauge transform (2.26).

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