Inequivalent coherent state representations in Group Field Theory

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In this paper we study an algebraic formulation of group field theory and consider non-Fock (condensate) representations based on coherent states. We show that we can construct representations with infinite numbers of 'particles' on compact base (group) manifolds, similar to the case of non-compact manifolds, and show that they break translation symmetry. Since such representations can be regarded as quantum gravitational systems with an infinite number of fundamental pre-geometric building blocks, they may be more suitable for the description of effective spatio-temporal phases of the theory.

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Introduction

Many contemporary approaches to quantum gravity see spacetime and geometry as collective phenomena of more fundamental degrees of freedom which are not themselves geometric in the usual sense of general relativity. The transition from fundamental and non-geometric to the effective geometric level, then, requires the control of many fundamental degrees of freedom and it is often associated with a phase change. Therefore, a key goal of such theories is to provide a consistent description of this transition and of the emergence of spacetime and geometry.

Group field theory (GFT) [1–3] is one such candidate formalism for emergent quantum gravity that is formulated in the language of (quantum) field theory. It is closely related to canonical loop quantum gravity [4–7] and its covariant formulation in terms of spin foams [8, 9]; for details on this relation, see [10]. On the other hand, it can also be seen as a group-theoretic enrichment of random tensor models [11–13], in which tensor indices over finite sets are replaced by field arguments [13–17].

The quanta of GFT models formally describe point particles labeled by a (generally non-abelian) Lie group in the same way that quanta of ordinary field theories are formally labeled by points of spacetime. The main conceptual difference though, between GFT and usual many-body particle systems, is that the former does not have any time variable that can be used to define a Hamiltonian or an evolution of the system, whereas in the latter, the Hamiltonian is a fundamental feature that offers a guideline for quantization. Still, a Hilbert space for "particles on the group" can be defined guided by a discrete geometric intuition; in particular, for quantum gravity models, the GFT quanta can be understood as quantized simplices (tetrahedra in 4d for example), whose quantum algebra and single particle Hilbert space are obtained by geometric quantization of a classical discrete geometry (see for example [18–20]). Applying second quantization techniques, one can construct a Fock space of quantum simplices that serves as the Hilbert space for GFT. The simplicial building blocks that are populating the Fock space admit a dual interpretation in terms of spin network vertices [19–21].

Nevertheless, the Fock vacuum lacks any topological or discrete geometric structure, missing even purely combinatorial information. Therefore, it can be intuitively considered "far away" from any state that carries information about smooth spacetime or geometry. On the other hand, many-particle states in GFT have a discrete geometric interpretation, shared with loop quantum gravity and simplicial quantum gravity, and broadly speaking provide a notion of generalized piecewise-flat geometries.

In certain limited circumstances and under additional assumptions, piecewise-flat geometries can be effectively used to approximate spacetime physics. Nevertheless, in cases when the background curvature is large or quantum fluctuations over a background geometry become important, the number of fundamental building blocks required for a sensible approximation should be large as well. In other words, the piling up of GFT quanta is likely to be needed to study the non-perturbative sector of the theory and to define states that can be well-approximated by smooth geometries, such that the dynamics of GFT can be approximated by usual effective field theory on smooth spacetimes.

In turn, the interactions among large numbers of GFT quanta, i.e. their collective behavior, may give rise to phase transitions, as in any other non-trivial quantum field theory. In statistical mechanics and field theory, in fact, different phases are associated with different, unitarily inequivalent, representations of the algebra of observables. The Fock space is just one example of such representation but in general a quantum field theory has infinitely many unitarily-inequivalent irreducible representations [22].

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New questions, then, arise: which phase of a given GFT model, if any, admits a geometric interpretation and a description in terms of effective field theory (and GR)? Which quantum representation of the fundamental GFT is appropriate to the description of such geometric physics? In fact, if our experience from ordinary quantum field theory is of any use we do not expect the Fock representation to be the interesting one for the description of macroscopic physics.

This prompts us to study the definition of new representations in GFT, taking full advantage of its field-theoretic structures, and complementing parallel work on GFT renormalization [23–29], aimed at the unraveling of the macroscopic phase diagram of different models. While GFT renormalization is most conveniently carried out in the functional integral formulation of the theory, the analysis of representations calls for an operator framework. This is the GFT counterpart of similar studies, with identical motivations, being carried out in the context of canonical LQG, spin foam models, tensor models and dynamical triangulations [30–39].

Further, recent work aimed at the extraction of effective continuum physics from GFT models focused on the possibility of a GFT condensate phase [40–42]. Effective, Friedmann-like dynamical equations have been extracted from the simplest GFT condensate hydrodynamics, leading to a number of interesting results [43].

In this work we study coherent state representations of the GFT algebra and their relationship with the Fock representation. The suitable choice from many inequivalent representations (or different phases of a given quantum system) is usually guided by experimental input. Lacking this, we can only rely on insights coming from concurrent attempts to extract continuum physics from the theory.

We present an algebraic formulation of GFT and provide a partial characterization of irreducible representations in terms of symmetry breaking. The algebraic approach is beneficial for several reasons. First, it does not rely on a special choice of a Hilbert space and therefore allows one to address the question of phase transitions in a consistent way. Second, it provides insight into the structural, model-independent aspects of the theory, and could guide our understanding and model development on the basis of more general principles. Third, the mathematical basis of statistical systems is very well established in this language, which can then become an appropriate basis for a definition of important physical concepts in GFT, such as symmetry breaking, phase transitions, dynamical evolution or equilibrium states.

Our approach is strongly motivated by the analysis of Araki and Woods [44] of condensate representations in bosonic many-body systems. The idea is to avoid the limiting procedures of the particle number for thermodynamical potentials but instead define directly representations that correspond to an infinite system. We also rely heavily on the Gel'fand-Naimark-Segal (GNS) theorem, which establishes the relation between the representations of the algebra and the space of algebraic states (for an earlier application of the GNS theorem to GFT n-point functions, see [45]).

In order to obtain inequivalent representations that are believed to be "close" to the geometrical ones, we consider two heuristically motivated conditions. The first is a large number of particles, and the second is the assumption on the form of the state, which we require to be coherent.

Using this approach we can explicitly formulate the theory on Hilbert spaces with infinite particle number (over the Fock vacuum) and as such could be better suitable for a description of geometrical states. The structure of the constructed representations is however still very simple and more realistic representations with richer structure have to be understood in future work.

We also leave a more complete analysis of the symmetry properties of GFT representations and symmetry breaking in GFT models to future work, and limit ourselves to the construction of inequivalent representations, not delving into the physics they may encode in any detail. We do not attempt a full characterization of representations in terms of the quantum dynamics of specific GFT models, due to the present poor understanding of dynamics in the operator formulation of GFT.

In the first part of this paper (I) we set up the algebraic formulation of GFT. After a brief introduction of the operator formulation of GFT, we introduce the basics of the algebraic formulation of statistical physics. We then motivate the algebra of observables and the role of algebraic states in GFT.

In the second part (II) we show how one can construct inequivalent representations for GFTs formulated on compact base manifolds and provide simple examples of representations associated to infinite systems.

I. Group Field Theory

A. Operator formulation of GFT

Group field theory is a field theoretical description of spin networks and simplicial geometry. It can be formulated in terms of functional integrals [1, 3, 46, 47] or in operator language [48]. In the latter, the natural starting point is a Fock space spanned by creation and annihilation operators $\varphi^\dagger(g_1, g_2, \cdots), \varphi(g_1, g_2, \cdots)$, acting on the Fock vacuum of zero quanta $|o\rangle$. The arguments of the field operators are d group elements of some Lie group G (usually SU(2) or SL(2,$\mathbb{C}$), which can be formally seen as labels for a basis of the single particle Hilbert space[1],

[1] In the same generalized, and rather formal, sense in which one assumes that delta distributions peaked on given points provide a basis for the usual one-body Hilbert space for point particles.

taken then to be the space $L^2(G^d)$,

$$\varphi^{\dagger}(g)|o) = |g), \qquad \varphi(g)|o) = 0, \tag{1}$$

where $g = (g_1, g_2, \cdots, g_d)$ denotes a finite collection of group elements. We take the valence d to be fixed, determining the simplicial interpretation of the formalism. In models with a simplicial or more general topological interpretation, like the ones related to loop quantum gravity and simplicial quantum gravity, one requires the operators to be invariant under the right multiplication by an element of the group, such that for all $h \in G$ we have

$$\phi(g_1, g_2, \cdots) = \phi(g_1 h, g_2 h, \cdots). \tag{2}$$

This symmetry requirement is called the closure constraint, since it encodes the closure of the simplices or cells dual to the GFT quanta [18–21, 49]. The commutation relations between the fields generate the usual CCR algebra,

$$\left[\varphi(g_1, \cdots), \varphi^{\dagger}(\tilde{g}_1, \cdots)\right] = \prod_{i=1} \delta\left(g_i \tilde{g}_i^{-1}\right) \cdots, \tag{3}$$

or the simple generalization of it encoding the closure condition, i.e. involving the identity on the space of invariant functions. The Fock space created by these operators can be understood as a kinematical Hilbert space $\mathcal{H}_{\text{kin}}$ for the theory, formed by generic quantum states on which no dynamics has yet been imposed. As we mentioned, this is where one can find all spin network states, which also form the kinematical Hilbert space of loop quantum gravity. As in any background-independent formulation of quantum gravity, one expects the quantum dynamics to be encoded in a *constraint operator* $C : \mathcal{H}_{\text{kin}} \to \mathcal{H}_{\text{kin}}$. Following the idea of Dirac quantization the role of C is twofold: first to select the space of physical states formally as

$$\mathcal{H}_{\text{phys}} = \{|\psi) \in \mathcal{H}_{\text{kin}} \,|\, C|\psi) = 0\}, \tag{4}$$

and second, select the relevant observables $\mathcal{O}$ by

$$[C, \mathcal{O}] = 0. \tag{5}$$

In canonical quantum gravity, the constraint operator would be a combination of diffeomorphism constraint operators, including the Hamiltonian constraint encoding invariance under temporal diffeomorphisms as well as the Einstein's equations. More generally, but in a similar 'time-independent'manner, the operator C is interpreted as implementing relational evolution of some internal degrees of freedom with respect to others, and in this way encodes the quantum dynamical equations of the theory. Any concrete choice of operator C defines a different GFT model.

The above is very much in the spirit of many-body physics, where the constraint operator replaces the role of the Hamiltonian, or of the Schrödinger equation for many-body wave functions. It can be considered an intuitive guideline, but its concrete and rigorous implementation needs much more technical care. This is just as in canonical quantum gravity, where the implementation of the Hamiltonian constraint is a key open issue [50], but also as in ordinary quantum field theory, where the interacting Hamiltonian cannot be represented on the same Fock space as the free part.

For example, if the zero eigenvalue lies in the continuous part of the spectrum of C the states $|\psi)$ that satisfy $C|\psi) = 0$ are not contained in the kinematical Hilbert space, but sometimes may be regarded as distributions over it. This is already the case for finite-dimensional systems in the presence of gauge symmetries like reparametrization invariance, and it is an even more severe issue in continuum quantum gravity. There it can be partially tackled by the method of refined algebraic quantization [51, 52], but experience with quantum field theories tells us that the need for going beyond the Fock representation can be expected to arise simply because one needs to take into account the collective physics of many (ideally, infinite) interacting degrees of freedom. In particular, as we discussed in the introduction, this is of direct interest in attempts to extract effective continuum physics from GFT models. GFTs, then, combine both types of difficulties: a constraint operator (rather than a Hamiltonian) encoding a background independent quantum dynamics, and the need to study the limit of an infinite number of degrees of freedom.

To this end, we first want to establish a more rigorous operator formulation of GFTs, and then provide simple examples of representations with an infinite number of particles, by using tools form algebraic QFT.

B. Weyl algebra in GFT

The algebra of observables is defined along the lines of many-body systems, motivated by the single-body Hilbert space, which corresponds, in most interesting GFT models, with the space of a single quantized simplex.

Starting from the heuristic CCR relations of group field theory, equation (3), we define the corresponding Weyl algebra in the usual way, first by formal smearing the field operators with some test functions (see next section for more details on the space of suitable smearing functions)

$$\psi(f) = \int \psi(g)\, \bar{f}(g)\, \mathrm{d}\mu_{\text{H}} \quad \psi^{\dagger}(f) = \int \psi^{\dagger}(g)\, f(g)\, \mathrm{d}\mu_{\text{H}}$$

and then exponentiating the field operators to obtain the bounded operators

$$W^F_{(f)} := e^{\frac{\imath}{\sqrt{2}}\left[\psi(f) + \psi^{\dagger}(f)\right]}. \tag{6}$$

By the Baker-Hausdorff formula and equation (3), these satisfy the multiplication rule

$$e^{\imath\left[\psi(f)+\psi^{\dagger}(f)\right]} e^{\imath\left[\psi(g)+\psi^{\dagger}(g)\right]} = e^{\imath\left[\psi(f+g)+\psi^{\dagger}(f+g)\right]} e^{-\frac{\imath}{2}\,\mathrm{Im}(f,g)}.$$

These steps serve as a motivation for the definition of an abstract Weyl algebra [53, 54]:

Definition 1. The Weyl algebra $\mathfrak{A}$ is a $C^\star$-algebra of polynomials generated by the Weyl operators

$$\begin{aligned} W_{(f)}W_{(g)} &= W_{(f+g)}e^{-\frac{i}{2}\operatorname{Im}(f,g)} \\ W^\star_{(f)} &= W_{(-f)} \ , \end{aligned} \tag{7}$$

which is closed in the l_1 norm; that is, in the norm such that a polynomial $P\left[W((f)\right] = \sum_n^N a_n W\left(f\right)^n$ has the norm $\|P\| = \sum_n^N |a_n|$, and subsequently in the $C^\star$-norm

$$\|W_{(f)}\|_\star = \sup_{\pi\in\mathrm{Rep}} \|\pi\left(W_{(f)}\right)\| \ . \tag{8}$$

The supremum is taken over all irreducible representations of the algebra.

Alternatively the norm can be understood in terms of the spectral radius, such that for any $A \in \mathfrak{A}$

$$\|A\|^2_\star = \sup_{\lambda\in\sigma(A^\star A)} |\lambda| = r\left(A^\star A\right) \ . \tag{9}$$

In particular, for the Weyl operators,

$$\|W_{(f)}\| = 1 \ , \tag{10}$$

for any $f \in \mathcal{S}$.

Remark 2. At this point we have already made the choice to regularize the unbounded smeared fields $\psi\left(f\right)$ by exponentiation. It should be noticed, however, that this is not the only possibility, and in most QFT systems the algebra obtained in this way is rather poor, as it does not admit automorphisms that implement dynamical evolution [55]. Also, such an algebra admits many irregular representations that are not suitable for most physical applications (at least in ordinary QFT). In [56] the resolvent algebra was introduced, which instead of the exponentiation of $\psi + \psi^\dagger$, uses its resolvent $R\left(\lambda\right) = \left[\lambda\mathbb{1} - \left(\psi + \psi^\dagger\right)\right]^{-1}$ to generate the $C^\star$-algebra. This algebra cures many of the problems of the Weyl algebra and may be a more suitable candidate for the description of systems with non-trivial dynamics. In our paper we focus on the Weyl algebra as the first step to provide an algebraic formulation of GFT, but further research is required for more conclusive statements.

1. *The space of smearing functions for compact base manifold M*

In order to fully define the Weyl algebra we need to define the space of smearing functions $\mathcal{S}$. We have certain freedoms in the specification of $\mathcal{S}$, and an explicit choice is often made to best reflect the physical properties of the system. In this section we consider the Lie group G on which the fields are defined to be compact. The case of the non-compact G is treated in the next section.

Firstly, we require $\mathcal{S}$ to be dense in the space of square integrable functions $L^2\left(M, \mu_\mathrm{H}\right)$ with respect to the Hilbert space norm (here and in the rest of the paper μ_H denotes the Haar measure on G). This ensures that a cyclic representation of the Weyl algebra will be large enough to contain the one-body Hilbert space of the theory. Secondly, it seems to be appropriate to require stability of $\mathcal{S}$ under translation invariance. The base manifold $M = G^{\times n}$ is a Lie group and hence admits a natural action of the group $G^{\times n}$. Explicitly, for $g \in G^{\times n}$ let $L_g : M \to M$ be left group multiplication. Then it is feasible to require the existence of an automorphism group $\alpha_g : \mathfrak{A} \to \mathfrak{A}$, such that for the Weyl operators we get

$$\alpha_g\left(W_{(f)}\right) = W_{\left(L_g^\star f\right)}, \tag{11}$$

extended to the full algebra by linearity and the product property. This automorphism is the GFT counterpart of translations in ordinary field theory and we will refer to it as the translation automorphism in the rest of the paper[2]. Thirdly, in the case $G = \mathrm{SU}\left(2\right)$ we require the Wigner matrices $D^J_{\alpha,\beta}\left(g\right)$ to be inside of $\mathcal{S}$, which allows us to use the standard spin representation of GFT states, and the spin network basis [48], directly on the level of the algebra, and to provide a closer relation of GFT to loop quantum gravity and tensor models. For these reasons it seems most natural to set $\mathcal{S} = \mathcal{C}^\infty\left(M\right)$, the space of smooth functions on the compact manifold M[3].

To formulate the closure constraint equation (2) algebraically, let $R_g : M \to M$ denote right multiplication on M and the corresponding automorphism by β_g as above. Let $D : G \to G^{\times n}$ be the diagonal map defined by $D_g \equiv D\left(g\right) = \left(g, \cdots, g\right) \in G^{\times n}$. Then the gauge invariant $C^\star$-algebra becomes a $C^\star$-sub-algebra $\mathfrak{A}_G \subset \mathfrak{A}$ such that for any $A \in \mathfrak{A}_G$, we have

$$\beta_{D_g}\left(A\right) = A \ . \tag{12}$$

That is, each element of the algebra is invariant under diagonal right translations. There are two ways to implement this condition:

1. Start with the Weyl algebra introduced above and construct a smaller algebra of observables following the formalism for constrained systems developed in [54], or

2. Implement the constraints at the level of the smearing functions by restricting $\mathcal{S}$ such that for any $f \in \mathcal{S}$ we have

$$f = f \circ R_{D_g}, \tag{13}$$

[2] In contrast to ordinary QFT we do not require here the existence of a unitary operator implementing this transformation.

[3] Notice that the commutative algebra of smooth functions over a manifold can be used to reconstruct the underlying differential structure of the manifold and in this sense $\mathcal{S}$ simply replaces the geometrical information of M (as n copies of a sphere) by the algebraic one [57, Chapter 7].

which makes the Weyl operators gauge invariant.

In general, these two procedures may lead to different algebras. For example, in the first approach it is conceivable to have Weyl operators that are not gauge invariant but lead to gauge invariant combinations or, even weaker, gauge invariant expectation values. Instead, the second procedure would introduce gauge invariance already on the level of the Weyl operators. Both possibilities seem viable at this stage.

In this paper we choose to implement the invariance by restricting the space of smearing functions, since from our perspective it better reflects the intuition of the closure constraint in equation(2). Then, for Weyl operators the closure constraint reads, for $W_{(f)} \in \mathfrak{A}_G$

$$\beta_{D_g}\left(W_{(f)}\right) = W_{\left(R^\star_{D_g} f\right)} = W_{(f)} \ . \tag{14}$$

To specify the space of gauge invariant functions we define a projection operator P as follows:

Lemma 3. *Let G be a compact group and $M = G^{\times n}$. Let $D_h = (h, h, \cdots, h)$ be a diagonal element of M and $R^\star_{D_h} f = f \circ R_{D_h}$ denote the pullback of $f \in \mathcal{C}^\infty(M)$. Let*

$$(Pf)(g) = \int_G \left(R^\star_{D_h} f\right)(g)\ dh, \tag{15}$$

with dh the Haar measure on G. Then $P\mathcal{C}^\infty(M) \subseteq \mathcal{C}^\infty(M)$. Furthermore P is an orthogonal projection on $L^2(M, \mu_{\mathrm{H}})$.

Proof. P is a continuous operator on $\mathcal{C}(M)$ since for any $\eta \in C(M)$ we have

$$\|P\eta\|_\infty = \| \int_G R^\star_{D_h}\eta \,\mathrm{d}h\|_\infty \leq \int_G \|R^\star_{D_h}\eta\|_\infty \mathrm{d}h = \|\eta\|_\infty .$$

Hence $P\eta \in \mathcal{C}(M)$ whenever $\eta \in \mathcal{C}(M)$. Further, let $\xi \in \mathcal{C}^\infty(M)$ and X be a vector field on M; then the Lie derivative $\mathcal{L}_X\xi$ of ξ along X is given by

$$\lim_{t\to 0} \frac{\xi\left(g \cdot e^{tX}\right) - \xi(g)}{t} = X(\xi)\,|_g. \tag{16}$$

and P commutes with X since

$$\begin{aligned} X(P\xi)(g) &= \lim_{t\to 0} \frac{(P\xi)\left(g \cdot e^{tX}\right)(g) - (P\xi)(g)}{t} \\ &= \lim_{t\to 0} \int_G R^\star_h \left(\frac{R^\star_{e^{tX}}\xi - \xi}{t}\right)(g) \\ &= \int_G R^\star_h \left(\lim_{t\to 0} \frac{R^\star_{e^{tX}}\xi - \xi}{t}\right)(g) \\ &= P(X\xi)(g) . \end{aligned}$$

In the third equality we used the continuity of R_h and the dominated convergence theorem estimating the integrand by $\sup_{g\in M}|X\xi|$. Hence P commutes with derivatives on $\mathcal{C}^\infty(M)$ and for any $\xi \in \mathcal{C}^n(M)$ we get $\mathcal{L}_{X_1} \cdots \mathcal{L}_{X_n} P\xi = P\left(\mathcal{L}_{X_1} \cdots \mathcal{L}_{X_n}\xi\right) \in \mathcal{C}(M)$. Therefore $P\mathcal{C}^\infty(M) \subset \mathcal{C}^\infty(M)$.

Let $\chi, \xi \in \mathcal{C}^\infty(M)$. Then by Fubini and the invariance of the Haar measure under right multiplication and inversion, we have

$$\begin{aligned} (\chi, P\xi) &= \int_M \bar{\chi} \cdot \left(\int_G R^\star_h \xi \,\mathrm{d}h\right) \mathrm{d}\mu_{\mathrm{H}} \\ &= \int_M \left(\int_G R^\star_h \bar{\chi} \mathrm{d}h\right) \cdot \xi \,\mathrm{d}\mu_{\mathrm{H}} \\ &= (P\chi, \xi) \ , \end{aligned} \tag{17}$$

and

$$\begin{aligned} (PP\xi)(g) &= \int_G \int_G \left(R^\star_{D_k} R^\star_{D_h}\xi\right)(g)\ \mathrm{d}h\,\mathrm{d}k \\ &= \int_G \int_G \left(\left(R_{D_h} R_{D_k}\right)^\star \xi\right)(g)\ \mathrm{d}h\,\mathrm{d}k \\ &= \int_G \int_G \left(\left(R_{D_{kh}}\right)^\star \xi\right)(g)\ \mathrm{d}h\,\mathrm{d}k \\ &= \int_G \left(\left(R_{D_h}\right)^\star \xi\right)(g)\ \mathrm{d}h = (P\xi)(g) \ . \end{aligned} \tag{18}$$

Therefore, P is an orthogonal projection on the dense domain of $L^2(M, \mu_{\mathrm{H}})$ and extends uniquely to $L^2(M, \mu_{\mathrm{H}})$ by continuity. □

Theorem 4. *The space $P\mathcal{C}^\infty(M)$ is dense in $PL^2(M, \mu_{\mathrm{H}})$ - the image of the orthogonal projection P on $L^2(M, \mu_{\mathrm{H}})$.*

Proof. Since PL^2 is given by the projection P, the space PL^2 is a closed subspace of $L^2(M, \mu_{\mathrm{H}})$, so the set $PL^2 \cap \mathcal{C}^\infty(M)$ is dense in PL^2 since $\mathcal{C}^\infty(M)$ is dense in $L^2(M, \mu_{\mathrm{H}})$. Further, any $f \in PL^2 \cap \mathcal{C}^\infty(M)$ is an almost-everywhere gauge invariant function that is smooth. Define $g = f - Pf$. Then g vanishes almost everywhere and is smooth due to above lemma. Hence g has to be zero everywhere, so we get $f \in P\mathcal{C}^\infty(M)$ and $PL^2 \cap \mathcal{C}^\infty(M) \subseteq P\mathcal{C}^\infty(M)$. The opposite inclusion is obvious since any $f \in P\mathcal{C}^\infty(M)$ is square integrable for compact M and $P\mathcal{C}^\infty(M) \subseteq \mathcal{C}^\infty(M)$ by the above lemma. □

Thus, if we want to consider GFT models with a closure constraint, we restrict ourselves to the space of smearing functions $\mathcal{S}_G = P\mathcal{C}^\infty(M)$.

2. Space of smearing functions for non-compact M

If the base manifold is non-compact, say $G = SL(2, \mathbb{C})$, the space of smearing functions is most conveniently chosen to be the space of smooth compactly supported functions on M, i.e. $\mathcal{S} = \mathcal{C}^\infty_c(M)$. The main motivation for this choice is mere mathematical convenience and the algebraic relation of the commutative algebra between smooth functions and smooth manifolds

[57]. Having specified the space of smearing functions, we could proceed to construct representations of the Weyl algebra, in both compact and non-compact cases.

Before we do so, we mention one other possibility for the base manifold, that somehow relates the two. This is the case in which one interprets the use of a compact manifold like $SU(2)$ as a sort of regularization (analogous to putting fields in a finite box to cure IR divergences); then the issue arises, what happens if the regularization is removed and the radius of the $SU(2)$ sphere is sent to infinity as part of a 'group-theoretic thermodynamic limit'. The 'decompactification'of $SU(2)$ can be performed in many different ways. However most of them single out at least one point of M that is then associated with infinity. If we denote this point with $p \in M$ than we need to choose the space of smearing functions to be the space of compactly supported, smooth functions on $M_p \equiv M/\{p\}$. In this case $\mathcal{S}_p \equiv \mathcal{C}_c^\infty(M/\{p\})$ which now corresponds to the non-compact manifold M_p. The definition of the Weyl algebra in this case does not differ from the more general non-compact case, so we can treat both cases in the same way.

Notice, however, that this construction presents several drawbacks from the standard GFT point of view, which make this route unappealing. First, in this case we no longer have a simple group structure of the base manifold and it becomes unclear, for example, how to implement the closure constraint. Second, we also lose the possibility of considering Wigner matrices as smearing functions, since there is no point p on which all of the Wigner functions vanish. As a consequence, we can not use the spin representation at the algebraic level, and thus a procedure like the above clearly assigns a different status to the group and spin representations of GFT models, which has no obvious physical motivation.

C. Algebraic states

In order to deal with states directly at the level of the algebra, we briefly introduce the concept of *algebraic states*.

An algebraic state is a linear, positive, normalized functional on the algebra $\mathfrak{A}$,

$$\omega : \mathfrak{A} \to \mathbb{C}$$

such that for any $A \in \mathfrak{A}$ we get

$$\begin{aligned} \omega(A^\star A) &\geq 0 \\ \omega(\mathbb{1}) &= 1 \,. \end{aligned} \tag{19}$$

The first inequality is the condition of positivity and the second is the normalization. Specifically for the Weyl algebra the positivity condition reads as follows:

Definition 5. Let $\mathfrak{A}$ be the Weyl algebra. The functional $\omega : \mathfrak{A} \to \mathbb{C}$ is positive if, for any finite $N \in \mathbb{N}$ and any set of complex coefficients $\{c_n\}_{n\in(0,\cdots,N)}$ and test functions $\{f_n \in \mathcal{S}\}_{n\in(0,\cdots,N)}$, the following holds

$$\sum_{n,m}^{N} c_n \bar{c}_m\, \omega\left(W_{(f_n - f_m)}\right) e^{-\imath \frac{\mathrm{Im}(f_n, f_m)}{2}} \geq 0 \,.$$

The set of states is a convex subset of the Banach space of continuous, linear functionals on $\mathfrak{A}$. We denote the space by $\mathfrak{S}(\mathfrak{A})$, or simply by $\mathfrak{S}$ when there is no confusion possible.

By the GNS construction, every algebraic state provides a triple $(\mathcal{H}_\omega, \pi_\omega, \Omega)$, where $\mathcal{H}_\omega$ is a Hilbert space, $\pi_\omega : \mathfrak{A} \to \mathcal{L}(\mathcal{H}_\omega)$ is a representation of $\mathfrak{A}$ in terms of bounded linear operators on $\mathcal{H}_\omega$, and the state vector $\Omega \in \mathcal{H}_\omega$, such that $\forall A \in \mathfrak{A}$

$$\omega(A) = (\Omega, \pi_\omega(A)\, \Omega) \,. \tag{20}$$

This representation is unique, in the sense that it is unitarily equivalent to any other representation that produces the same expectation values of observables [58].

The algebra of observables $\pi(\mathfrak{A})$ on the GNS Hilbert space $\mathcal{H}_\omega$ is usually not closed in the strong operator topology. This is because the $C^\star$- norm (see equation (9)) is stronger than the operator norm. The closure of the representation of the algebra of observables in the strong (or equivalently, weak) operator topology is called the von Neumann algebra and is equal to the bicommutant of the represented algebra of observables by the von Neumann theorem. See for example [59].

A state ω is called pure if it can not be written as a convex combination of two or more states

$$\omega = \lambda\omega_1 + (1-\lambda)\,\omega_2 \qquad 0 < \lambda < 1 \qquad \omega_1 \neq \omega_2 \neq \omega \,,$$

otherwise it is called mixed.

The GNS representation of a state is irreducible if and only if the state is pure [59, Theorem 2.3.19].

On the other hand any normalized vector, or more generally every density matrix operator ρ on the GNS Hilbert space provides a notion of an algebraic state by

$$\omega_\rho(A) := \mathrm{tr}\left[\rho\, \pi_\omega(A)\right] \,. \tag{21}$$

The space of all algebraic states constructed in this way is called the folium of the representation π_ω, or the folium of the state ω. Each state of the same folium provides an equivalent representation of the algebra.

The folium of any representation is $w^\star$-dense in the space of all algebraic states [60], meaning that we can obtain any algebraic state as a limit point of a sequence of states within one folium. Generally, most of the algebraic states are mere mathematical artifacts, and one needs a prescription for selecting interesting specific states that can be considered of physical relevance.

As we discussed above, one strategy is to rely on the quantum dynamics, encoded in a constraint operator. The constraint operator may indeed select a Hilbert space which is considered physical. From the algebraic point of view the constraint operator is therefore related to the choice of the folium, or conversely, information about the constraint operator is partly encoded in the algebraic state.

In this paper, however, we do not discuss the constraint operator explicitly, since little is known at present about the constraint operators underlying specific GFT models, and even less about general properties of such constraint operators for GFT models of quantum gravity. Instead, our procedure will be the following. Starting from the Fock representation of GFT, we consider sequences that satisfy two conditions:

1. All states in the sequence are coherent states.

   This is mainly motivated by the use of GFT coherent states in the extraction of an effective continuum dynamics in the series of works [40, 41, 43, 61]. Of course, coherent states are also key for the classical approximation of any QFT, and routinely used in particle physics, many-body systems and condensed matter theory, which provides further motivation.

2. The particle number of the limit state diverges.

   As described above, the interpretation of CCRs in GFT as creation and annihilation of quanta of space, themselves with a discrete geometric interpretation, leads to the expectation that quantum states of direct use for approximating smooth spacetime and geometry will have infinitely many particles. This is only possible if the particle number operator in the corresponding representation is not defined (in the sense that it is formally divergent); that is, if the corresponding representation is non-Fock.

In the next section we provide simple explicit examples for GFT representations that satisfy these two properties. Let us also remark that symmetries would be another criterion for selecting and interpreting algebraic states and the corresponding GFT representations, as is customary in usual quantum field theories. We partially use their symmetric properties in the following, for a partial characterization of the representations we will construct, but a better understanding of symmetries in GFTs (building on [62–64]) is needed to exploit them more fully.

II. States and representations

A. Fock state and the Fock representation

By construction, the Weyl algebra admits the Fock representation, which is given by the GNS representation of the algebraic state

$$\omega\left(W_{(f)}\right)=e^{-\frac{\|f\|^2}{4}} . \tag{22}$$

Since the state (22) is regular, i.e. the function $\Omega\left(t\right):=\omega_c\left(W_{(tf)}\right)$ for $t\in I\subset\mathbb{R}$ and any fixed $f\in\mathcal{S}$ is smooth, the generator of the Weyl operator exists by Stone's theorem [65]. Denoting the corresponding GNS triple by $(\mathcal{H},\pi_F,o)$, we can write

$$(o|\pi_F\left[W_{(f)}\right]|o)=(o|e^{\imath\Phi_F(f)}|o) , \tag{23}$$

where $\Phi_F\left(f\right)$ is an essentially self-adjoint generator of $\pi_F\left[W_{(f)}\right]$ in the Fock representation, defined by differentiation on the dense domain $D\left(\Phi\right)=\left\{\mathrm{Poly}_n\left(W_{(f)}\right)|\Omega)|\forall n\in\mathbb{N}\right\}\subset\mathcal{H}$, where Poly_n denotes the space of all polynomials of degree n. We can also define the creation and annihilation operators by

$$\psi_F\left(f\right)=\frac{1}{\sqrt{2}}\left[\Phi_F\left(f\right)+\imath\Phi_F\left(\imath f\right)\right] \tag{24}$$

$$\psi_F^\dagger\left(f\right)=\frac{1}{\sqrt{2}}\left[\Phi_F\left(f\right)-\imath\Phi_F\left(\imath f\right)\right], \tag{25}$$

with $\psi_F\left(f\right)^\dagger=\psi_F^\dagger(f)$, such that $\psi\left(f\right)$ is anti-linear in f, $\psi^\dagger\left(f\right)$ is linear in f, both are closed on $D\left(\phi\right)$ and fulfill the canonical commutation relations

$$\left[\psi_F\left(f\right),\psi_F\left(g\right)\right]=\left[\psi_F^\dagger\left(f\right),\psi_F^\dagger\left(g\right)\right]=0 \tag{26}$$

and

$$\left[\psi_F\left(f\right),\psi_F^\dagger\left(g\right)\right]=\left(f,g\right)\mathbb{1} . \tag{27}$$

In the following we will consider only regular representations and denote the generators of the representations π_ω by

$$\psi_\omega\left(f\right),\psi_\omega^\dagger\left(f\right). \tag{28}$$

Without providing the proof, we present here some well-known properties of the Fock representation:

Particle Number: The Fock representation is the unique representation (up to unitary equivalence) in which the particle number operator N exists in the sense of a strong limit of the operator

$$s-\lim_{n\to\infty}e^{\imath\sum_i^n\psi_\omega^\dagger(f_i)\psi_\omega(f_i)}=e^{\imath N},$$

for some complete orthonormal basis $\{f_i\}_{i\in\mathbb{N}}$ of $L^2\left(M,\mu_{\mathrm{H}}\right)$.

Purity: The Fock state is pure and hence the GNS representation of ω_F is irreducible.

Folium: As a consequence of the general theorem stated above, the folium of the Fock representation is dense in the space of algebraic states $\mathfrak{S}$.

B. Coherent states and non-Fock representations

Usually coherent states are characterized by the condition that they are eigenstates of the annihilation operators in the Fock representation, and hence require a notion of the Hilbert space for their very definition. In the algebraic approach to many-body physics, this characterization is avoided by introducing a generalized notion of coherent states directly at the level of the algebra. This is described in [66, 67], where the authors also provide a classification of such generalized Fock and non-Fock coherent states, that lie in the folium of the Fock representation and those which do not, respectively.

Definition 6. Let $L : \mathcal{S} \to \mathbb{C}$ be a continuous linear form on the space of test functions $\mathcal{S}$. A state ω of the form

$$\omega_L \left(W_{(f)}\right) = \omega_F \left(W_{(f)}\right) e^{\imath \sqrt{2}\,\mathrm{Re}[L(f)]}, \tag{29}$$

is called a coherent state. It is pure and regular [66].

Notice that the Fock state is the special case of the above family of coherent states for $L = 0$.

Any linear functional L corresponds to a well defined state [66]. It should be noticed that there exist even more general definitions of a coherent state, but this is the one that most closely reflects the condition of being an eigenfunction of the annihilation operator.

Proposition 7 ([67, Proposition 2.5])**.** *The state ω of the above form is equivalent to the Fock representation, if and only if L is continuous on $L^2(M, \mu_{\mathrm{H}})$.*

The detailed proof of this proposition is presented in [67], but we provide a brief sketch.

Assume that L is a continuous functional on $L^2(M, \mu_{\mathrm{H}})$. Then by the Riesz lemma there exists an $l \in L^2(M, \mu_{\mathrm{H}})$ such that for any $f \in L^2(M, \mu_{\mathrm{H}})$

$$L(f) = \int_M f \cdot \bar{l}\, \mathrm{d}\mu_{\mathrm{H}}, \tag{30}$$

and

$$\|L\|_{op} = \|l\|_{L^2}. \tag{31}$$

It is not difficult to see that the algebraic state (29) provides a Fock-equivalent representation and that $\overline{L(f)}$ is the eigenvalue of the GNS cyclic state vector $|\Omega)$ [66], i.e.

$$\psi_F(f)\,|\Omega) = \overline{L(f)}|\Omega) = (f, l)_{L^2}\,|\Omega)\,. \tag{32}$$

Since the representation is Fock, the particle number operator exists such that $N = \sum_{i\in\mathbb{N}} \psi^\dagger(f_i)\,\psi(f_i)$ for some orthonormal basis $\{f_i\}_{i\in\mathbb{N}}$ of $L^2(M, \mu_{\mathrm{H}})$ is defined on the dense domain of $\mathcal{H}_\omega$, and its expectation value is given by

$$(\Omega|N|\Omega) = \sum_{i\in\mathbb{N}} |L(f_i)|^2 = \|l\|^2_{L^2} = \|L\|_{op}\,. \tag{33}$$

That is, the particle number is given by the L^2 norm of l or equivalently the operator norm of L. When L is unbounded on $L^2(M, \mu_{\mathrm{H}})$ the global particle number is ill-defined and the representation can not be Fock.

The non-Fock coherent states are hence classified by functionals L which are continuous on $\mathcal{S}$ but unbounded as operators on $L^2(M, \mu_{\mathrm{H}})$. By the Riesz-Markov theorem every functional L on $\mathcal{S}$ is of the form

$$L(f) = \int_M f\, \mathrm{d}\mu, \tag{34}$$

for some Baire measure μ.

From this we can easily state

Corollary 8. *Let $G = SU(2)$ and $M = G^{\times n}$, if L is invariant under left multiplication i.e. $L\left(L^\star_g f\right) = L(f)$ for any $f \in \mathcal{S}$ and $g \in SU(2)^{\times n}$, then the coherent state ω_L is Fock.*

Proof. Let L be invariant under left translations. Then for any $f \in \mathcal{S}$ we have

$$L\left(L^\star_g f\right) = \int_M L^\star_g f\, \mathrm{d}\mu = L(f) = \int_M f\, \mathrm{d}\mu, \tag{35}$$

hence the measure μ is a left-invariant measure on $SU(2)$, which has to be identified with the Haar measure up to rescaling,

$$\mu = c \cdot \mu_{\mathrm{H}}, \tag{36}$$

for some $c \in \mathbb{R}$. Since $|L(f)| \leq c\|f\|$, L is continuous on $L^2(M, \mu_{\mathrm{H}})$. ☐

1. UV and IR divergences of L

From the above discussion it follows that we need the integrand in equation(34) to diverge on $L^2(M, \mu_{\mathrm{H}})$ in order to have inequivalent coherent state representations. There are two different reasons for which the functional (34) can become unbounded on $L^2(M, \mu_{\mathrm{H}})$, which are related to the long (IR) and short (UV) scale behavior of the measure[4].

The IR divergences appear when the integral becomes infinite due to regions with arbitrary large measure. This is what happens in ordinary many-body physics.

For example, the broken $U(1)$ phase of a free massive bosonic theory is described by the coherent state with $L(f) = \sqrt{\rho} \int f\, \mathrm{d}^3x$. This implies that the Baire measure in equation(34) is given by $\mu = \sqrt{\rho}\, \mathrm{d}^3 x$, where the

[4] Notice that here, as customary in the GFT renormalization literature, we label IR and UV scales with respect to the metric on the group manifold, without necessarily invoking any matching with the notion of UV and IR on any continuum spacetime emerging from the same GFT models.

Radon–Nikodym derivative is a constant function and thus not integrable over an infinite domain.

In case when M is not compact (for example, when the Lie group defining the base manifold of a given GFT model is $SL(2,\mathbb{C})$), we can construct such representations simply by choosing a non-integrable Radon-Nikodym derivative for the Baire measure, that is by choosing $\mu = g\,\mu_h$ with $\int_M |g| \to \infty$.

On a compact manifold, there are no regions with an arbitrary large measure and hence IR divergences can not occur. Nevertheless, the integral (34) can diverge due to short scale behavior of the measure, called the UV divergence. This can happen when the Radon-Nikodym derivative diverges too quickly[5] in local regions or even develops a pure point component. For simplicity we want to exclude cases when the measure μ develops a singular component with respect to the Haar measure.[6]

Physically, an IR divergent state can be understood as a state with an infinite number of quanta but with a finite density. Locally - restricted to a region of the base manifold - the notion of particle as well as the particle number are well-defined. This is the typical situation in condensed matter physics [60]. A UV divergence on the other hand corresponds to a state in which an infinite number of fundamental quanta are concentrated at a single point on the base manifold and, correspondingly, the density at this point blows up. The notion of particle as well as the particle number operator are defined globally except for such a local region with infinite density. From the point of view of ordinary field theory on spacetime, this situation is clearly not physical: an infinite number of particles in a finite region corresponds to a situation in which an infinite energy is similarly localized. Accordingly, ordinary QFT a compact base manifold requires a finite particle number and hence forces us to stay in the Fock representation. This requirement is usually captured in the statement that no phase transition can occur in field theories in a finite volume.

In GFTs, however, the notion of energy is not present and the base (group) manifold does not relate to local regions of space-time. Thus, even in the compact case, the restriction to the Fock representation would not be well-motivated. In fact, UV divergences in the above sense, i.e., concentrations of the functional measure on specific regions of the underlying group manifold could even be desirable, or at least reasonable, from the point of view of the interpretation of GFT quanta as "building blocks of spacetime and geometry." Heuristically, these types of coherent states would correspond to condensates with a collective wave-function sharply peaked on a given value of the underlying discrete connection, and thus on a given curvature of the fundamental tetrahedra. This is not unreasonable from the point of view of approximating smooth geometries via large numbers of such tetrahedra. Notice also that wave functions of this type have been used for condensate states more general than coherent states, in [68, 69], while hints of similar divergences of the GFT particle number were found in the GFT condensate cosmology context in [42]. Hence, in this case, even if our base manifold M is compact we can not and should not exclude non-Fock inequivalent representations.

To summarize, GFT models on the compact manifold can exhibit inequivalent representations due to UV divergences, whereas models on non-compact manifolds can exhibit UV as well as IR divergences.

Remark 9. A fundamental difference between UV and IR divergences is their behavior under translations. Whereas the IR divergence can be generated by translation invariant measures as in the example of the Bose-Einstein condensation, the UV divergences on the compact manifold cannot, by corollary (8).

C. Examples

1. Compact case

Our procedure to construct inequivalent representations following the idea described in section (I C) is fairly straightforward. By the above discussion, we simply need to construct a sequence of continuous functionals L_n on $\mathcal{S}$ that converge pointwise to a functional L_∞ unbounded on $L^2(M,\mu_{\mathrm{H}})$. Here we provide a very simple example in which the sequence of regular measures converges to a pure point measure. It should be clear, however, that any measure that satisfies the property of being unbounded on L^2 leads to a new inequivalent representation.

Let us first define the Dirac measure μ_D, such that for any open $U \subset M$

$$\mu_D(U) = \begin{cases} 1 & \text{if } \mathbb{1} \in U \\ 0 & \text{otherwise} \end{cases}, \tag{37}$$

or, equivalently, on smooth functions $f \in \mathcal{S}$

$$\mu_D(f) = f(\mathbb{1}) \ . \tag{38}$$

Obviously, such a Riesz functional is continuous on $\mathcal{S}$ but unbounded on $L^2(M,\mu_{\mathrm{H}})$ due to the possible singular behavior of functions at single points.

Assume further a contracting sequence of open sets $\{U_n\}_{n\in\mathbb{N}}$ around $\mathbb{1} \in SU(2)$, such that $U_{n+1} \subset U_n$ and $\cap_n U_n = \{\mathbb{1}\}$, and consider a sequence of measures defined as

$$\mu_n = \frac{1}{\mu_{\mathrm{H}}(U_n)} \chi_{U_n} \mu_{\mathrm{H}}, \tag{39}$$

where χ_{U_n} is the characteristic function on U_n,

$$\chi_{U_n} = \begin{cases} 1 & \text{if } g \in U_n \\ 0 & \text{otherwise} \end{cases}, \tag{40}$$

[5] By 'too quickly'we mean that the function is integrable but not square integrable around a local region.

[6] This is for mathematical convenience only, at this point, since we do not have a strong physical motivation to exclude such representations.

and μ_{H} is the Haar measure.

Lemma 10. *On $\mathcal{S}$ the sequence of functionals defined by (39) converges to the Dirac measure in the distributional sense. That is for any $f \in \mathcal{S}$*

$$\lim_{n\to\infty} \mu_n(f) = \mu_D(f) = f(\mathbb{1}). \tag{41}$$

Proof. Since f is continuous, we can find $N_\epsilon(\mathbb{1})$ - a neighborhood of $\mathbb{1}$ - such that $\forall g \in N_\epsilon(\mathbb{1})$ we have $f(g) \in B_\epsilon(f(\mathbb{1}))$ - an ϵ-ball around $f(\mathbb{1})$. Since the sequence is contracting $\exists N \in \mathbb{N}$ such that $\forall n > N$, $U_n \subset N_\epsilon(\mathbb{1})$ then

$$\begin{aligned}
|\mu_n(f) - \mu_D(f)| &= \left| \frac{1}{\mu_{\mathrm{H}}(U_n)} \int f\chi_{U_n} \mathrm{d}\mu_{\mathrm{H}} - f(\mathbb{1}) \right| \\
&= \frac{1}{\mu_{\mathrm{H}}(U_n)} \left| \int \chi_{U_n} (f - f(\mathbb{1})) \, \mathrm{d}\mu_{\mathrm{H}} \right| \\
&\le \frac{1}{\mu_{\mathrm{H}}(U_n)} \int \chi_{U_n} |f - f(\mathbb{1})| \, \mathrm{d}\mu_{\mathrm{H}} \\
&\le \epsilon.
\end{aligned}$$

□

At every finite n the measure μ_n is absolutely continuous with respect to the Haar measure and by the above proposition every state

$$\omega_n\left(W_{(f)}\right) := \omega_F\left(W_{(f)}\right) \cdot e^{\imath\sqrt{2}\,\mathrm{Re}\int f\,\mathrm{d}\mu_n}, \tag{42}$$

is equivalent to the Fock representation. From the convergence of the measure, the convergence of the algebraic sequence is obvious.

Lemma 11. *The sequence of states ω_n converges in the $w^\star$-topology to*

$$\omega_\infty^{\mathbb{1}}\left(W_{(f)}\right) = \omega_F\left(W_{(f)}\right) \cdot e^{\imath\sqrt{2}\,\mathrm{Re}(f(\mathbb{1}))}. \tag{43}$$

Proof. For any $W_{(f)} \in \mathfrak{A}$ we have

$$\begin{aligned}
&\left|\omega_n\left(W_{(f)}\right) - \omega_\infty^{\mathbb{1}}\left(W_{(f)}\right)\right| \\
=&\left|\omega_F\left(W_{(f)}\right)\right| \left| e^{\imath\sqrt{2}\,\mathrm{Re}\left[\int f\,\mathrm{d}\mu_n\right]} - e^{\imath\sqrt{2}\,\mathrm{Re}\left[\int f\,\mathrm{d}\mu_D\right]} \right| \\
=&\left|\omega_F\left(W_{(f)}\right)\right| \left| e^{\imath\sqrt{2}\,\mathrm{Re}\left[\int f\,\mathrm{d}\mu_n - \int f\,\mathrm{d}\mu_D\right]} - 1 \right| \\
\to& 0.
\end{aligned}$$

By linearity of the state and the product property of the Weyl algebra, this extends to the whole algebra $\mathfrak{A}$. □

So, at finite n the representation is Fock, the particle number operator exists and the particle number of the nth member of the sequence is given by

$$\|L\|_{op} = \frac{1}{\mu_{\mathrm{H}}(U_n)}, \tag{44}$$

where $|\Omega_n)$ is the GNS vector of the corresponding algebraic state ω_n. Hence the particle number increases with increasing n as the volume of U_n becomes smaller. At the limit point the total particle number is not defined (it diverges) and the corresponding representation becomes inequivalent to the Fock one.

Notice that we can define states ω_∞^g using the automorphisms $\alpha_{g^{-1}}$ introduced in the previous section such that

$$\omega_\infty^g = \omega_\infty^{\mathbb{1}} \circ \alpha_{g^{-1}}. \tag{45}$$

Each of the states ω_∞^g leads to an inequivalent representation.

We will show in the next section that ω_∞^g is inequivalent to $\omega_\infty^{g'}$ whenever $g \neq g'$.

2. Non-compact case

If the base manifold M is non-compact the above construction can still be used, which leads to the UV divergence of the integral (34). However, in addition to this we can construct inequivalent representations that steam from the IR divergences of equation (34). In this case any non-integrable function l such that

$$L(f) = \int \bar{l} \cdot f \, \mathrm{d}\mu_{\mathrm{H}}, \tag{46}$$

provides an inequivalent representation. At least in some cases, for example when the function l is a real-valued constant function, we can associate the physical interpretation of a constant global particle density to l^2. This case was extensively discussed in the fundamental work of Araki and Woods in [44] and we refer the reader to their work for the explicit analysis and construction of the corresponding representations. As mentioned above such representations do not break translation invariance and still allow for a local definition of particles, since locally (confined to a small region of the base manifold) the Hilbert space is equivalent to Fock because the particle number in such a region is bounded by $V \cdot \rho$, the volume of the region and the particle density therein respectively.

D. Explicit representation for ω_∞^g

We now construct an explicit representation that is generated by the above algebraic state following the construction in [44].

Take $L^2(M, \mu_D^g)$ to be the space of L^2 functions with respect to the Dirac measure concentrated at $g \in M$. Note that this space is one-dimensional. Define the commuting operators for $L^2(M, \mu_D^g)$ such that for any $\varphi \in L^2(M, \mu_D^g)$

$$[A(f)\varphi](h) = f(h)\varphi(h) \quad [B(f)\varphi](h) = \bar{f}(h)\varphi(h). \tag{47}$$

With the Fock creation and annihilation operators $\psi_F(f), \psi_F^\dagger(f)$, the unitary operators

$$W^g_{(f)} = e^{\frac{\imath}{\sqrt{2}}\left[\psi_F(f)+\psi_F^\dagger(f)\right]} \otimes e^{\frac{\imath}{\sqrt{2}}[A(f)+B(f)]}, \qquad (48)$$

together with the state vector

$$|\Omega^g_\infty) \equiv |o) \otimes 1 \qquad (49)$$

where 1 is the constant function on M and $|o)$ is the Fock vacuum, define the representation of ω^g_∞, since

$$(\Omega^g_\infty | W^g_{(f)} | \Omega^g_\infty) = e^{-\frac{\|f\|^2}{4}} \cdot e^{\imath\sqrt{2}\,\mathrm{Re}[f(g)]}. \qquad (50)$$

Irreducibility and cyclicity of this representation are inherited from the Fock representation since $L^2(M, \mu_D)$ is one-dimensional.

1. *Breaking of translation symmetry*

The state above breaks the translation symmetry in the sense that α_g can not be represented by a unitary operator on $\mathcal{H}_\omega$. To see this, we first note that

$$\omega^{\mathbb{1}}_\infty\left(W_{(f)}\right) \neq \omega^g_\infty\left(W_{(f)}\right) = \omega^{\mathbb{1}}_\infty\left(\alpha_g^{-1}\left[W_{(f)}\right]\right) \qquad (51)$$

for generic $f \in \mathcal{S}$ and α_g as defined in (11). From this, it follows that:

Corollary 12. *Let $g, g' \in G^{\times n}$ and $g \neq g'$ then the states ω^g_∞ and $\omega^{g'}_\infty$ are inequivalent.*

Proof. The states ω^g_∞ and $\omega^{g'}_\infty$ are pure, and hence their representations are factor. Define the state $\omega = \frac{1}{2}\left(\omega^g_\infty + \omega^{g'}_\infty\right)$. It is easy to see that the operator $(\mathbb{1} \oplus c\mathbb{1})$ for $c \in \mathbb{C}$ are in the center of the representation π_ω and hence the representation generated by ω is not factor. By theorem [59, Proposition 2.4.27] the states ω^g and $\omega^{g'}$ are not equivalent. □

Since ω^g_∞ and $\omega^{g'}_\infty$ are inequivalent, the translation automorphism α_g^{-1} can not be implemented by a unitary operator, i.e. it is broken.

Notice that the automorphism $\alpha_g\beta_g$ implements the isometry of the base manifold and hence the above representations break the isometry transformation. This is rather different from ordinary field theory, in which Poincaré symmetry is not allowed to be broken [22, 44, 58, 70]. Again, this is possible because no spacetime interpretation is attached to the GFT base manifold.

III. Interpretation of new representations

Let pontificate on the interpretation of the newly found non-Fock representations, expanding on some of the points above.

The vacua ω^g of the new representations can be interpreted as states in which infinitely many GFT quanta carry a label (or equivalently have the property) g. It is instructive to think about the label g as one of the "continuous modes" of the theory. Let us call this mode the 'basic mode'. In this case the representation described above is very similar to the usual case of Bose-Einstein condensation [44]. The creation and annihilation operators of particles in the basic mode g are given by A and B operators respectively. They commute since the number of particles in this mode is infinite, which is the manifestation of the usual Bogoliubov argument. States of the Hilbert space are then created by excitations of other "modes" on top of the basic one and hence can be considered quantum fluctuations over a background field with infinitely many particles in the basic mode.

If we relate the group elements of GFT with the basic notion of holonomy/curvature, which is well-justified at the discrete, simplicial level but requires more work at the continuum, effective level, we could think about the ground state of new representations as a truly infinite gas of particles that all carry the same geometrical information. The resulting continuum geometry would be then reconstructed from such infinite particle states. This could be a generic geometry, since approximately equal curvature building blocks can be used, if they also have progressively vanishing size, to approximate any geometry, as in Regge calculus [71]. Another possibility is that they could generate a homogeneous background with the constant holonomy (curvature) g. Choosing $g = \mathbb{1}$ we would obtain a flat background on top of which excitations are created by ψ_F and $\psi_F^\dagger$. The type of states created/annihilated on top of such a condensate background would be formally analogous to the fundamental spin network states or cylindrical functions that are also found in the fundamental (Fock) Hilbert space of the theory. Importantly, though, in these representations the role of the Fock creation and annihilation operators is that of collective excitations and not of single building blocks of quantum geometry. This is because the generators of the Weyl operators in this representation differ from the Fock operators by a non-normalizable "function." The origin of inequivalent representations for different gs stems from the fact that the corresponding Hilbert spaces are created by excitations over backgrounds with different geometry than the fully degenerate one corresponding to the Fock vacuum. Being a specific case of the condensate state with $L = 0$, the Fock representation corresponds to the case in which the background consists of no GFT quanta at all.

The above paragraph should be read of course with a grain of salt. It should provide a nice intuition, but it does not amount yet to a compelling, let alone complete, physical interpretation of the newly constructed representations. In fact:

1. The mode g in our case is not selected by any physical principle such as energy minimization, entropy maximization or the enforcement of a specific phys-

ical symmetry. It is rather postulated by hand, which makes the construction non-unique. In contrast to this, we recall, the ground mode in condensed matter physics is selected as the minimum of the (free) Hamiltonian. A detailed analysis of the constraint operator underlying interesting GFT models is necessary, before assigning any physical interpretation to the above representations.

2. The states $|\Omega^g_\infty\rangle$ are quantum states, whose physical properties should be ascertained by computing expectation values of observables with a clear macroscopic, geometric meaning. Partial results in this direction have been obtained in the mentioned context of GFT condensate cosmology [41, 43, 72, 73], and in the first steps towards the GFT definition of quantum (black hole) horizons [68]. But it is clear that also in this direction, much further work is needed before making any conclusive statement from the physical point of view.

3. The form of the constraint operator at this moment is not fully understood, however if the constraint operator is symmetric under the described translation automorphisms the inequivalent states for different gs should be physically indistinguishable and any association of geometrical properties to the inequivalent vacua ω^g would be incorrect.

Conclusions

We have described an algebraic formulation for GFTs. We believe that this formulation has potential, not only allowing us to formulate problems in a rigorous way, but also to efficiently tackle some conceptual and technical issues related to the problem of phase transitions and continuum limits in this class of quantum gravity models. We have used the algebraic formulation to more rigorously define the Fock representation for GFT states, and then to construct inequivalent, non-Fock representations of the same quantum system. In particular, we focused on coherent representations, which are the GFT analog of Bose condensate representations in usual quantum field theories. We have treated both the case of compact and non-compact group manifolds, showing that inequivalent condensate representations may be considered also in the first case, in GFT, in contrast to usual spacetime-based QFT. Finally, we have given a partial symmetry characterization of the non-Fock representations, and attempted a preliminary geometric interpretation of them, leaving a more complete analysis to future work, to be conducted in parallel with current attempts at extracting effective continuum spacetime physics from GFT models for quantum gravity.

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[1] D. Oriti, "The Group field theory approach to quantum gravity", *Approaches to Quantum Gravity, Editor D. Oriti, Cambridge University Press, Cambridge*, 2009 310–331, `arXiv:gr-qc/0607032`.
[2] D. Oriti, "The microscopic dynamics of quantum space as a group field theory", in "Proceedings, Foundations of Space and Time: Reflections on Quantum Gravity: Cape Town, South Africa", pp. 257–320. 2011. `arXiv:1110.5606`.
[3] T. Krajewski, "Group field theories", *PoS* **QGQGS2011** (2011) 005, `arXiv:1210.6257`.
[4] T. Thiemann, "Lectures on loop quantum gravity", *Lect. Notes Phys.* **631** (2003) 41–135, `arXiv:gr-qc/0210094`, [,41(2002)].
[5] C. Rovelli, "Loop quantum gravity", *Living Reviews in Relativity* **11** (2008), no. 1, 5.
[6] A. Ashtekar and J. Pullin, "The Overview Chapter in Loop Quantum Gravity: The First 30 Years", `arXiv:1703.07396`.
[7] N. Bodendorfer, "An elementary introduction to loop quantum gravity", `arXiv:1607.05129`.
[8] J. C. Baez, "An Introduction to spin foam models of quantum gravity and BF theory", *Lect. Notes Phys.* **543** (2000) 25–94, `arXiv:gr-qc/9905087`.
[9] A. Perez, "The Spin Foam Approach to Quantum Gravity", *Living Rev. Rel.* **16** (2013) 3, `arXiv:1205.2019`.
[10] D. Oriti, "Group Field Theory and Loop Quantum Gravity", 2014. `arXiv:1408.7112`.
[11] R. Gurau and J. P. Ryan, "Colored Tensor Models - a review", *SIGMA* **8** (2012) 020, `arXiv:1109.4812`.
[12] R. Gurau, "Invitation to Random Tensors", *SIGMA* **12** (2016) 094, `arXiv:1609.06439`.
[13] V. Rivasseau, "The Tensor Track, IV", in "Proceedings, 15th Hellenic School and Workshops on Elementary Particle Physics and Gravity (CORFU2015): Corfu, Greece, September 1-25, 2015". 2016. `arXiv:1604.07860`.
[14] V. Rivasseau, "Quantum Gravity and Renormalization: The Tensor Track", *AIP Conf. Proc.* **1444** (2011) 18–29, `arXiv:1112.5104`.
[15] V. Rivasseau, "The Tensor Track: an Update", in "29th International Colloquium on Group-Theoretical Methods in Physics (GROUP 29) Tianjin, China, August 20-26, 2012". 2012. `arXiv:1209.5284`.

[16] V. Rivasseau, "The Tensor Track, III", *Fortsch. Phys.* **62** (2014) 81–107, arXiv:1311.1461.

[17] D. Oriti, "Non-commutative quantum geometric data in group field theories", *Fortsch. Phys.* **62** (2014) 841–854, arXiv:1405.1830.

[18] J. C. Baez and J. W. Barrett, "The Quantum tetrahedron in three-dimensions and four-dimensions", *Adv. Theor. Math. Phys.* **3** (1999) 815–850, arXiv:gr-qc/9903060.

[19] F. Conrady and L. Freidel, "Quantum geometry from phase space reduction", *J. Math. Phys.* **50** (2009) 123510, arXiv:0902.0351.

[20] E. Bianchi, P. Dona, and S. Speziale, "Polyhedra in loop quantum gravity", *Phys. Rev.* **D83** (2011) 044035, arXiv:1009.3402.

[21] A. Baratin, B. Dittrich, D. Oriti, and J. Tambornino, "Non-commutative flux representation for loop quantum gravity", *Class. Quant. Grav.* **28** (2011) 175011, arXiv:1004.3450.

[22] R. Haag and D. Kastler, "An algebraic approach to quantum field theory", *Journal of Mathematical Physics* **5** (1964), no. 7, 848–861.

[23] D. Benedetti, J. Ben Geloun, and D. Oriti, "Functional Renormalisation Group Approach for Tensorial Group Field Theory: a Rank-3 Model", *JHEP* **03** (2015) 084, arXiv:1411.3180.

[24] J. Ben Geloun, R. Martini, and D. Oriti, "Functional Renormalisation Group analysis of Tensorial Group Field Theories on $\mathbb{R}^d$", *Phys. Rev. D* **94** (2016), no. 2, 024017, arXiv:1601.08211.

[25] J. Ben Geloun, "Renormalizable Tensor Field Theories", in "18th International Congress on Mathematical Physics (ICMP2015) Santiago de Chile, Chile, July 27-August 1, 2015". 2016. arXiv:1601.08213.

[26] D. Benedetti and V. Lahoche, "Functional Renormalization Group Approach for Tensorial Group Field Theory: A Rank-6 Model with Closure Constraint", *Class. Quant. Grav.* **33** (2016), no. 9, 095003, arXiv:1508.06384.

[27] S. Carrozza and V. Lahoche, "Asymptotic safety in three-dimensional SU(2) Group Field Theory: evidence in the local potential approximation", *Class. Quant. Grav.* **34** (2017), no. 11, 115004, arXiv:1612.02452.

[28] S. Carrozza, "Flowing in Group Field Theory Space: a Review", *SIGMA* **12** (2016) 070, arXiv:1603.01902.

[29] S. Carrozza, V. Lahoche, and D. Oriti, "Renormalizable Group Field Theory beyond melons: an example in rank four", arXiv:1703.06729.

[30] T. Koslowski and H. Sahlmann, "Loop quantum gravity vacuum with nondegenerate geometry", *SIGMA* **8** (2012) 026, arXiv:1109.4688.

[31] B. Dittrich and M. Geiller, "A new vacuum for Loop Quantum Gravity", *Class. Quant. Grav.* **32** (2015), no. 11, 112001, arXiv:1401.6441.

[32] B. Bahr, B. Dittrich, and M. Geiller, "A new realization of quantum geometry", arXiv:1506.08571.

[33] C. Delcamp and B. Dittrich, "Towards a phase diagram for spin foams", arXiv:1612.04506.

[34] B. Dittrich, S. Mizera, and S. Steinhaus, "Decorated tensor network renormalization for lattice gauge theories and spin foam models", *New J. Phys.* **18** (2016), no. 5, 053009, arXiv:1409.2407.

[35] B. Dittrich, "The continuum limit of loop quantum gravity - a framework for solving the theory", arXiv:1409.1450.

[36] J. Ambjorn, A. Görlich, J. Jurkiewicz, A. Kreienbuehl, and R. Loll, "Renormalization Group Flow in CDT", *Class. Quant. Grav.* **31** (2014) 165003, arXiv:1405.4585.

[37] T. Delepouve and R. Gurau, "Phase Transition in Tensor Models", *JHEP* **06** (2015) 178, arXiv:1504.05745.

[38] V. Bonzom, "Large N Limits in Tensor Models: Towards More Universality Classes of Colored Triangulations in Dimension $d \geq 2$", *SIGMA* **12** (2016) 073, arXiv:1603.03570.

[39] J. Ambjorn, D. Coumbe, J. Gizbert-Studnicki, A. Gorlich, and J. Jurkiewicz, "New higher-order transition in causal dynamical triangulations", *Phys. Rev.* **D95** (2017), no. 12, 124029, arXiv:1704.04373.

[40] S. Gielen, D. Oriti, and L. Sindoni, "Cosmology from Group Field Theory Formalism for Quantum Gravity", *Phys. Rev. Lett.* **111** (2013), no. 3, 031301, arXiv:1303.3576.

[41] D. Oriti, L. Sindoni, and E. Wilson-Ewing, "Emergent Friedmann dynamics with a quantum bounce from quantum gravity condensates", *Class. Quant. Grav.* **33** (2016), no. 22, 224001, arXiv:1602.05881.

[42] A. G. A. Pithis, M. Sakellariadou, and P. Tomov, "Impact of nonlinear effective interactions on group field theory quantum gravity condensates", *Phys. Rev.* **D94** (2016), no. 6, 064056, arXiv:1607.06662.

[43] S. Gielen and L. Sindoni, "Quantum Cosmology from Group Field Theory Condensates: a Review", *SIGMA* **12** (2016) 082, arXiv:1602.08104.

[44] H. Araki and E. Woods, "Representations of the canonical commutation relations describing a nonrelativistic infinite free bose gas", *Journal of Mathematical Physics* **4** (1963), no. 5, 637–662.

[45] A. Perez and C. Rovelli, "Observables in quantum gravity", *Clay Math. Proc.* **11** (2010) 501–518, arXiv:gr-qc/0104034.

[46] L. Freidel, "Group field theory: An Overview", *Int. J. Theor. Phys.* **44** (2005) 1769–1783, arXiv:hep-th/0505016.

[47] D. Oriti, "The Group field theory approach to quantum gravity: Some recent results", *AIP Conf. Proc.* **1196** (2009) 209–218, arXiv:0912.2441.

[48] D. Oriti, "Group field theory as the 2nd quantization of Loop Quantum Gravity", *Class. Quant. Grav.* **33** (2016), no. 8, 085005, arXiv:1310.7786.

[49] M. Kapovich, J. Millson, *et al.*, "The symplectic geometry of polygons in euclidean space", *J. Differential Geom* **44** (1996), no. 3, 479–513.

[50] T. Thiemann, "Loop Quantum Gravity: An Inside View", *Lect. Notes Phys.* **721** (2007) 185–263, arXiv:hep-th/0608210.

[51] D. Giulini and D. Marolf, "On the generality of refined algebraic quantization", *Classical and Quantum Gravity* **16** (1999), no. 7, 2479.

[52] T. Thiemann, "Lectures on loop quantum gravity", *Quantum Gravity*, 2003 219–229.

[53] A. Carey, J. Gaffney, and C. Hurst, "A C*-algebra formulation of the quantization of the electromagnetic field", *Journal of Mathematical Physics* **18** (1977), no. 4, 629–640.

[54] H. B. Grundling and C. Hurst, "Algebraic quantization of systems with a gauge degeneracy", *Communications*

in mathematical physics **98** (1985), no. 3, 369–390.

[55] M. Fannes and A. Verbeure, "On the time evolution automorphisms of the ccr-algebra for quantum mechanics", *Communications in Mathematical Physics* **35** (1974), no. 3, 257–264.

[56] D. Buchholz and H. Grundling, "The resolvent algebra: A new approach to canonical quantum systems", *Journal of Functional Analysis* **254** (2008), no. 11, 2725–2779.

[57] J. Nestruev, "Smooth manifolds and observables", Springer Science & Business Media, 2006.

[58] F. Strocchi, "Spontaneous Symmetry Breaking in Quantum Systems. A review for Scholarpedia", *Scholarpedia* **7** (2012), no. 1, 11196, `arXiv:1201.5459`.

[59] O. Bratteli and D. Robinson, "Operator algebras and quantum statistical mechanics 1: C*- and w*-algebras. symmetry groups. decomposition of states", Springer-Verlag, 1997.

[60] G. G. Emch, "Algebraic methods in statistical mechanics and quantum field theory", Courier Corporation, 2009.

[61] D. Oriti, L. Sindoni, and E. Wilson-Ewing, "Bouncing cosmologies from quantum gravity condensates", *Class. Quant. Grav.* **34** (2017) 04, `arXiv:1602.08271`.

[62] A. Kegeles and D. Oriti, "Generalized conservation laws in non-local field theories", *J. Phys.* **A49** (2016), no. 13, 135401, `arXiv:1506.03320`.

[63] A. Kegeles and D. Oriti, "Continuous point symmetries in Group Field Theories", *J. Phys.* **A50** (2017), no. 12, 125402, `arXiv:1608.00296`.

[64] J. Ben Geloun, "Classical Group Field Theory", *J. Math. Phys.* **53** (2012) 022901, `arXiv:1107.3122`.

[65] M. Reed and B. Simon, "Methods of modern mathematical physics. vol. 1. functional analysis", Academic New York, 1980.

[66] R. Honegger and A. Rapp, "General Glauber coherent states on the Weyl algebra and their phase integrals", *Physica A: Statistical Mechanics and its Applications* **167** (1990), no. 3, 945–961.

[67] R. Honegger and A. Rieckers, "The general form of non-fock coherent boson states", *Publications of the Research Institute for Mathematical Sciences* **26** (1990), no. 2, 397–417.

[68] D. Oriti, D. Pranzetti, and L. Sindoni, "Horizon entropy from quantum gravity condensates", *Phys. Rev. Lett.* **116** (2016), no. 21, 211301, `arXiv:1510.06991`.

[69] D. Oriti, D. Pranzetti, J. P. Ryan, and L. Sindoni, "Generalized quantum gravity condensates for homogeneous geometries and cosmology", *Class. Quant. Grav.* **32** (2015), no. 23, 235016, `arXiv:1501.00936`.

[70] F. Strocchi, "Symmetry breaking", Springer Science & Business Media, 2005.

[71] R. Friedberg and T. D. Lee, "Derivation of Regge's Action From Einstein's Theory of General Relativity", *Nucl. Phys.* **B242** (1984) 145, [,213(1984)].

[72] M. de Cesare, A. G. A. Pithis, and M. Sakellariadou, "Cosmological implications of interacting Group Field Theory models: cyclic Universe and accelerated expansion", *Phys. Rev. D* **94** (2016), no. 6, 064051, `arXiv:1606.00352`.

[73] A. G. A. Pithis and M. Sakellariadou, "Relational evolution of effectively interacting group field theory quantum gravity condensates", *Phys. Rev.* **D95** (2017), no. 6, 064004, `arXiv:1612.02456`.