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Inequivalent coherent state representations in group field theory

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Abstract

In this paper we propose an algebraic formulation of group field theory and consider non-Fock representations based on coherent states. We show that we can construct representations with an infinite number of degrees of freedom on compact manifolds. We also show that these representations break translation symmetry. Since such representations can be regarded as quantum gravitational systems with an infinite number of fundamental pre-geometric building blocks, they may be more suitable for the description of effective geometrical phases of the theory.

Keywords: group field theory, quantum gravity, quantum field theory, spin foam models

Introduction

Many contemporary approaches to quantum gravity see spacetime and geometry as collective phenomena of more fundamental degrees of freedom. In such theories, a transition from fundamental and non-geometric to the effective geometric level is often associated with a phase transition and requires control over many degrees of freedom. A key goal is then to provide a consistent description of this phase transition. In algebraic quantum field theory different phases are associated with inequivalent representations of the operator algebra of observables; the study of phase transitions becomes the study of the operator algebra and its



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inequivalent representations. In this paper we will investigate the operator algebra for group field theory (GFT) and provide examples of inequivalent representations for GFT on a compact base manifold.

Group field theory [1–3] is one candidate theory that aims at the description of emergence of geometry. It is a statistical quantum field theory in which space-time geometry and dynamics of general relativity suppose to arise as an effective field theory. It is closely related to canonical loop quantum gravity (LQG) [4–7] and its covariant formulation in terms of spin foams [8, 9]; for details on this relation, see [10]. On the other hand, it can also be seen as a group-theoretic enrichment of random tensor models [11–13], in which tensor indices over finite sets are replaced by field arguments [13–17].

The quanta of GFT models formally describe point particles labeled by a (generally non-abelian) Lie group in the same way that quanta of ordinary field theories are formally labeled by points of spacetime. However, canonical quantization and the resulting Hamiltonian dynamics or evolution which entirely relies on a time variable cannot be applied here since time does not (yet) exist. Still, a Hilbert space for ‘particles on the group’ can be defined guided by a discrete geometric intuition; in particular, the GFT quanta can be understood as quantized simplices (tetrahedra in 4 dimension), whose quantum algebra and single particle Hilbert space are obtained by geometric quantization of a classical discrete geometry (see for example [18–20]). Applying second quantization techniques, one can construct a Fock space of quantum simplices that serves as the Hilbert space for GFT. The simplicial building blocks that are populating the Fock space admit a dual interpretation in terms of spin network vertices [19–21].

Nevertheless, the Fock vacuum provides trivial topology and geometry and therefore, can be intuitively considered ‘far away’ from any state that carries information about non-trivial smooth spacetime geometry. On the other hand, finitely many-particle states in GFT have a discrete geometric interpretation, shared with loop quantum gravity and simplicial quantum gravity, and provide a notion of generalized piecewise-flat geometries [10]. However, for a description of smooth geometries the number of degrees of freedom, or GFT quanta, should be very large and states with an infinite particle number are likely to be needed.

In turn, the interactions among large numbers of GFT quanta, i.e. their collective behavior, may give rise to phase transitions, as in any other non-trivial quantum field theory (see for example [22]). New questions, then, arise: which phase of a given GFT model, if any, admits a geometric interpretation and a description in terms of effective field theory and general relativity? Which quantum representation of the fundamental GFT is appropriate to the description of such geometric physics?

This prompts us to study the definition of new representations in GFT, taking full advantage of its field-theoretic structures, and complementing parallel work on GFT renormalization [23–29]. Our approach provides a GFT counterpart of similar studies, with identical motivations, carried out in the context of canonical loop quantum gravity, spin foam models, tensor models and dynamical triangulations [30–39].

Our work is motivated by the use of GFT coherent states in the extraction of an effective continuum dynamics [40–46], and the requirement of an infinite number of degrees of freedom that is needed for description of smooth geometries. To study these two requirements we construct coherent state representations with an infinite number of GFT quanta and study their relation with the Fock representation. The idea is to avoid the limiting procedures of the particle number for thermodynamical potentials but instead define directly representations that correspond to an infinite system.

Using this approach we can explicitly formulate the theory on Hilbert spaces with infinite particle number. Such Hilbert spaces could be better suitable for a description of geometrical

states. The structure of the constructed representations is however still very simple and more realistic representations with richer structure have to be understood in future work.

In the first part of this paper (section 1) we set up the algebraic formulation of GFT. Using this formulation in the second part (section 2) we show how one can construct inequivalent representations for GFT and provide simple examples of representations associated to infinite systems with breaking of translation symmetry.

Notation

In this paper we will use the following notation and conventions. The base manifold of GFT is considered to be $G^{\times n}$ with $G = SU(2)$ and some fixed $n \in \mathbb{N}$; it will be denoted, $M \doteq G^{\times n}$. A generalization of statements from this paper to compact Lie groups other than $SU(2)$ is straightforward, but a treatment of non-compact base manifold requires more care. Throughout the whole paper the letter h is reserved as an element of G , and dh refers to the Haar measure on G , the Haar integral on G is denoted by $\int_G (\cdot) dh$. The Haar measure is normalized to 1, $\int_G dh = 1$, and is invariant under left and right multiplication and inversion on G , that is for an integrable function f and $h_1, h_2 \in G$

$$\int_G f(h_1 h h_2) dh = \int_G f(h) dh. \quad (1)$$

$$\int_G f(h^{-1}) dh = \int_G f(h) dh. \quad (2)$$

It is a unique measure on G with this properties.

The letters x and y are reserved for elements of M , and dx refers to the Haar measure on M , the Haar integral on M is denoted by $\int_M (\cdot) dx$. The Haar measure dx is, as above, normalized to 1 and invariant under left and right multiplication as well as inversion on M . Whenever necessary, we will use subscripts for the components of x and write $x = (x_1, x_2, \dots, x_n) \in M$. We denote the Lie algebra of M by \mathfrak{m} and by convention choose it to be isomorphic to the space of right invariant vector fields on M .

We denote the space of square integrable functions on M by $L^2(M, dx)$ and define the bracket $(\cdot, \cdot)_{L^2}$, such that for any $f, g \in L^2(M, dx)$,

$$(f, g)_{L^2} = \int_M \bar{f}(x) g(x) dx. \quad (3)$$

The real and imaginary part of expressions are referred to as $\text{Re}(\cdot)$ and $\text{Im}(\cdot)$, respectively. The Dirac-delta distribution on M is denoted $\delta(\cdot)$ and satisfies

$$f(y) = \int_M \delta(yx^{-1}) f(x) dx, \quad (4)$$

where yx^{-1} denotes the group product between y and x^{-1} .

Throughout the paper we will use different norms on several different spaces. We will introduce them in the text whenever we use them, but here we summarize the notation for better overview:

$\|\cdot\|_{L^2} = \sqrt{(\cdot, \cdot)_{L^2}}$ is the L^2 norm, $\|\cdot\|_{k, \infty}$ is the family of semi norms with respect to which the space of smooth functions is complete, in particular $\|\cdot\|_{\infty}$ is the supremums norm for smooth functions, $\|f\|_{\infty} = \sup_{x \in M} |f(x)|$, $\|\cdot\|_{\star}$ refers to the C^{\star} -norm, $\|\cdot\|_{\mathcal{H}}$ refers to the

Hilbert space norm for whatever Hilbert space is in question, and $\|\cdot\|_{\text{op}} = \sup_{x \in \mathcal{H}} \|\cdot x\|_{\mathcal{H}}$ is the operator norm for bounded linear operators on the Hilbert space \mathcal{H} .

1. Group field theory

1.1. Operator formulation of GFT

Group field theory is a field theoretical description of spin networks and simplicial geometry. It can be formulated in terms of functional integrals [1, 3, 47, 48] or in operator language [49]. In the latter, the natural starting point is a Fock space spanned by creation and annihilation operators $\varphi^\dagger(x)$, $\varphi(x)$, acting on the Fock vacuum of zero quanta $|o\rangle$, such that

$$\varphi^\dagger(x)|o\rangle = |x\rangle, \quad \varphi(x)|o\rangle = 0. \quad (5)$$

In models with a simplicial or more general topological interpretation, like the ones related to loop quantum gravity and simplicial quantum gravity, one requires the operators to be invariant under the right multiplication by an arbitrary element of the group G , such that for all $h \in G$ we have

$$\varphi(x_1, x_2, \dots, x_n) = \varphi(x_1 h, x_2 h, \dots, x_n h). \quad (6)$$

This symmetry requirement is called the closure constraint [18–21, 50]. The functions that satisfy the closure constraint are called gauge invariant functions. The canonical commutation relation (CCR) between the fields without closure constraint is given by,

$$[\varphi(x), \varphi^\dagger(y)] = \delta(xy^{-1}). \quad (7)$$

And for gauge invariant fields the CCR read

$$[\varphi(x), \varphi^\dagger(y)] = \int_G \prod_{j=1}^n \delta(x_j h y_j^{-1}) dh. \quad (8)$$

The Fock space created by these operators can be understood as a kinematical Hilbert space \mathcal{H}_{kin} , formed by generic quantum states on which no dynamics has yet been imposed. As in any background-independent formulation of quantum gravity, one expects the quantum dynamics to be encoded in a finite set of constraint operators $C_i : \mathcal{H}_{\text{kin}} \rightarrow \mathcal{H}_{\text{kin}}$ for $i \in \{1, \dots, N\}$. Following the idea of Dirac quantization the role of C_i is twofold: first to select the space of physical states formally as

$$\mathcal{H}_{\text{phys}} = \{|\psi\rangle \in \mathcal{H}_{\text{kin}} \mid C_i |\psi\rangle = 0 \forall i \in \{1, \dots, N\}\}, \quad (9)$$

and second, select the relevant observables \mathcal{O} by

$$[C_i, \mathcal{O}] = 0 \quad \forall i \in \{1, \dots, N\}. \quad (10)$$

Any concrete choice of such operators C_i defines a different GFT model. In analogy to this, the constraint operators in LQG would be the diffeomorphism constraints and the Hamiltonian constraint, but the GFT constraints cannot be directly interpreted as diffeomorphisms or Hamiltonian constraints, since the degrees of freedom of the GFT theory do not live on a continuous spacetime manifold where diffeomorphisms would be defined and act as symmetry transformations.

A treatment of constraint systems can be technically challenging [3, 51, 52]. In particular, if the zero eigenvalue lies in the continuous part of the spectrum of C_i the states $|\psi\rangle$, that satisfy

$C_i|\psi\rangle = 0$, are not contained in the kinematical Hilbert space, and one needs to generalize the construction using the notion of rigged Hilbert spaces [53]. This is already the case for finite-dimensional systems in the presence of gauge symmetries like reparametrization invariance, and it is an even more severe issue in continuum quantum gravity. There, it can be partially tackled by the method of refined algebraic quantization [3, 54], but experience with quantum field theories tells us that we need Hilbert spaces other than Fock to describe an infinite number of interacting degrees of freedom [22]. Hence, GFT combines both types of difficulties: a constrained system without explicit Hamiltonian evolution, and the need to study an infinite number of degrees of freedom.

To approach this problem and establish its rigorous operator formulation, we use the algebraic formalism for quantum statistical mechanics in GFT. In the following we will put the above formulation of GFT in algebraic terms and construct Hilbert spaces with infinite particle number as representations of the GFT algebra of observables. This will require the definition of a Weyl algebra for GFT.

1.2. Algebraic formulation of GFT

The first step in the construction of an algebraic formulation is the construction of the algebra of observables. In GFT, by convenience, we choose this algebra to be the Weyl algebra. The later is a C^* -algebra that is constructed over a symplectic space of the classical theory. For that reason we start our discussion of the algebraic construction with a definition of the suitable symplectic space in GFT.

1.2.1. Symplectic space of GFT. We begin with the space of smooth, complex valued functions on M that we denote by $\mathcal{S} = C^\infty(M)$. Let $L_x : M \rightarrow M$ denote the left and $R_x : M \rightarrow M$ the right multiplication on M by $x \in M$ and denote the pull-back of $f \in \mathcal{S}$ by L_x (respectively R_x) as

$$L_x^*f = f \circ L_x \quad (\text{respectively } R_x^*f = f \circ R_x). \quad (11)$$

Lemma 1. \mathcal{S} is closed under translations; that is for any $y \in M$ and $f \in \mathcal{S}$ the functions L_y^*f and R_y^*f are again in \mathcal{S} . Moreover, L_y^* and R_y^* leave the L^2 -bracket, $(\cdot, \cdot)_{L^2}$, invariant.

Proof. The first statement follows from smoothness of the maps L_x and R_x . The second statement is a direct consequence of the left (respectively right) invariance of the Haar measure dx . That is for $f, g \in \mathcal{S}$ and $y \in M$,

$$\begin{aligned} (L_y^*f, L_y^*g)_{L^2} &= \int_M \bar{f}(yx) g(yx) dx \\ &= \int_M \bar{f}(x) g(x) dx \\ &= (f, g)_{L^2}. \end{aligned}$$

And similar for R_y^*f . □

Let $X_i \in \mathfrak{m}$ be a Lie algebra element of M , then X_i acts as a derivation on smooth functions such that for $f \in \mathcal{S}$, $I \subset \mathbb{R}$ an interval containing zero and $t \in I$,

$$X_i f(x) \doteq \partial_i f(e^{tX_i} x) |_{t=0}, \quad (12)$$

where e^{tX_i} denotes the exponential map on M [55].

Lemma 2. \mathcal{S} equipped with topology induced by the family of semi-norms

$$\{\|f\|_{k,\infty} = \|X_1 \cdots X_k f(g)\|_\infty : X_1, \dots, X_k \in \mathfrak{m}; \forall k \in \mathbb{N}\},$$

is a complete, topological, locally convex, vector space.

Proof. See⁴ [55]. □

When the topology of \mathcal{S} will be important in our discussion we will denote this topological space by \mathcal{S}_∞ .

Since M is compact, every smooth function on it is finite integrable and we can equip \mathcal{S} with the norm-topology induced by the norm,

$$\|f\|_{L^2}^2 = \int_M \bar{f}(x) f(x) dx. \quad (13)$$

Lemma 3. \mathcal{S} equipped with the norm topology is not complete and its completion is the space of square integrable functions on M .

Proof. See [55]. □

To distinguish this topological space from the above, we will denote it \mathcal{S}_{L^2} whenever this will be necessary.

Let $h \in G$ and $D : G \rightarrow M$ be a diagonal map such that $D_h \equiv D(h) = (h, \dots, h)$. We say f satisfies the closure constraint (or f is gauge invariant) if

$$R_{D_h}^* f = f \quad \forall h \in G. \quad (14)$$

We denote the space of functions that satisfy the closure constraint by \mathcal{S}_G .

Proposition 4. \mathcal{S} can be decomposed in complementary subspaces \mathcal{S}_G and \mathcal{S}_{NG} such that

$$\mathcal{S}_\infty = \mathcal{S}_G + \mathcal{S}_{NG}, \quad (15)$$

and $\mathcal{S}_G \cap \mathcal{S}_{NG} = \{0\}$. Where \mathcal{S}_G is a space of gauge invariant functions and \mathcal{S}_{NG} is a space of functions that do not satisfy the closure constraint.

Proof. Let P define an operator on \mathcal{S} and pointwise acting as

$$(Pf)(x) = \int_G (R_{D_h}^* f)(x) dh.$$

P is linear since it is a composition of linear operators, $R_{D_h}^*$ and $\int_G (\cdot) dh$. We show that the image of P is in \mathcal{S}_∞ . By [55, lemma 2.1] it is enough to show that $\|Pf\|_{k,\infty} < \infty$ for any $k \in \mathbb{N}$. For an arbitrary fixed k we get

$$\begin{aligned} \|Pf\|_{k,\infty} &= \sup_{x \in M} |X_1 \cdots X_k (Pf)(x)| \\ &= \sup_{x \in M} \left| X_1 \cdots X_k \int_G (R_{D_h}^* f)(x) dh \right|. \end{aligned}$$

⁴In this reference the authors define the lie algebra by left invariant vector fields as opposed to our definition as right invariant vector fields. For that reason in the original paper equation (12) is defined by right multiplication with the exponential map. This small change, however, does not change the results of the paper.

By lemma 1 the integrand is a smooth function and can be upper bounded by $\sup_{x \in M} |(R_{D_h}^* f)(x)|$. Hence, by dominant convergence theorem

$$\|Pf\|_{k,\infty} \leq \int_G \sup_{x \in M} |X_1 \cdots X_k (R_{D_h}^* f)(x)| dh$$

For any fixed $h \in G$ we have

$$X_1 \cdots X_k (R_{D_h}^* f)(x) = \partial_{t_1} \cdots \partial_{t_k} f(e^{t_1 X_1} \cdots e^{t_k X_k} x D_h),$$

where all derivatives are taken at zero. Since $x D_h \in M$ it follows that

$$\sup_{x \in M} |X_1 \cdots X_k (R_{D_h}^* f)(x)| = \sup_{x \in M} |X_1 \cdots X_k f(x)|$$

and we obtain

$$\|Pf\|_{k,\infty} \leq \|f\|_{k,\infty}.$$

Therefore, $P : \mathcal{S}_\infty \rightarrow \mathcal{S}_\infty$, is a continuous linear operator on \mathcal{S} .

Further, by right invariance of the Haar measure it follows that $P^2 f = Pf$. By [56, theorem 1.1.8] it follows that \mathcal{S}_∞ can be decomposed as

$$\mathcal{S}_\infty = \mathcal{S}_G + \mathcal{S}_{NG},$$

where $\mathcal{S}_G = P\mathcal{S}_\infty$ and $\mathcal{S}_{NG} = (1 - P)\mathcal{S}_\infty$ and $\mathcal{S}_G \cap \mathcal{S}_{NG} = \{0\}$. □

Lemma 5. *P is an orthogonal projector on $L^2(M, dx)$.*

Proof. *P is bounded on \mathcal{S}_{L^2} since for any $f \in \mathcal{S}$ we have by right invariance of the Haar measure*

$$\|Pf\|_{L^2}^2 = \int_M \int_G |f(x D_h)| dh dx = \int_M |f(x)| dx = \|f\|_{L^2}^2.$$

Let $f, g \in \mathcal{S}$. Then by Fubini and the invariance of the Haar measure under right multiplication and inversion, we have

$$\begin{aligned} (f, Pg)_{L^2} &= \int_M \bar{f}(x) \left(\int_G (R_{D_h}^* g)(x) dh \right) dx \\ &= \int_M \left(\int_G \overline{(R_{D_h}^* f)}(x) dh \right) g(x) dx \\ &= (Pf, g)_{L^2}. \end{aligned}$$

And for $h_1, h_2 \in G$ we have

$$\begin{aligned} (PPf)(x) &= \int_G \int_G (R_{D(h_1)}^* R_{D(h_2)}^* f)(x) dh_1 dh_2 \\ &= \int_G \int_G (R_{D(h_1)}^* R_{D(h_2)}^* f)(x) dh_1 dh_2 \\ &= \int_G \int_G ((R_{D(h_1 h_2)})^* f)(x) dh_1 dh_2 \\ &= \int_G ((R_{D_h})^* f)(x) dh = (Pf)(x). \end{aligned}$$

Therefore, P is an orthogonal projection on the dense domain of $L^2(M, dx)$ and extends uniquely to the whole $L^2(M, dx)$ by continuity. \square

Theorem 6. *The space $\mathcal{S}_G = P\mathcal{S}$ is dense in $PL^2(M, dx)$ —the image of the orthogonal projection P on $L^2(M, dx)$.*

Proof. Since $PL^2(M, dx)$ is given by the projection P , it is a closed subspace of $L^2(M, dx)$. By lemma 3 the set $PL^2(M, dx) \cap \mathcal{S}$ is dense in $PL^2(M, dx)$. Further, any $f \in PL^2(M, dx) \cap \mathcal{S}$ is an almost-everywhere gauge invariant function that is smooth. Define $g = f - Pf$. Then g vanishes almost everywhere and is smooth. Hence g is zero everywhere, and we get $f \in \mathcal{S}_G$ and $PL^2(M, dx) \cap \mathcal{S} \subseteq \mathcal{S}_G$. The opposite inclusion, $\mathcal{S}_G \subseteq PL^2(M, dx) \cap \mathcal{S}$, is obvious since any $f \in \mathcal{S}_G$ is square integrable and $\mathcal{S}_G \subseteq \mathcal{S}$ by lemma 4. \square

To proceed with the construction of the symplectic space we equip \mathcal{S}_∞ with a symplectic form $\mathfrak{s} : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}$ defined for any $f, g \in \mathcal{S}$ by

$$\mathfrak{s}(f, g) = \text{Im}[(f, g)_{L^2}]. \tag{16}$$

Restricting \mathcal{S} to the subspace \mathcal{S}_G we obtain the symplectic form on \mathcal{S}_G that we denote by the same symbol \mathfrak{s} .

The above theorem ensures that after quantization, the one particle Hilbert space, that is given by the L^2 closure of the \mathcal{S}_G will be that of a quantized polygon [50]. However, the symplectic structure of our space is different from the symplectic structure of a single polygon.

Remark 7. The space \mathcal{S}_G is not closed under right multiplication, meaning that in general for $f \in \mathcal{S}_G$ and $y \in M$ not of the diagonal form (that is $y \neq D_h$ for any $h \in G$), the function R_y^*f will not be gauge invariant. To see this we observe

$$(R_y^*f)(x) = f(xy),$$

which is in general not equal to

$$f(xD_hy) = (R_y^*f)(xD_h). \tag{17}$$

For this reason we choose the definition of the Lie algebra to be given by right invariant vector fields on M (generated by left translation) to ensure that for any $f \in \mathcal{S}_G$, the function Xf stays in \mathcal{S}_G .

1.2.2. The Weyl algebra of GFT. To define the Weyl algebra from the space \mathcal{S} we follow the standard procedure presented for example in [51, 52] and that we recall below for convenience.

First we define a \star -algebra $\mathcal{A}(\mathcal{S})$ such that:

1. The elements of $\mathcal{A}(\mathcal{S})$ are complex valued functions on \mathcal{S} with support consisting of a finite subset of \mathcal{S} . It follows that $\mathcal{A}(\mathcal{S})$ is a vector space.
2. Then we define a ℓ^1 norm on $\mathcal{A}(\mathcal{S})$ by

$$\|A\|_1 = \sum_{f \in \mathcal{S}} |A(f)|.$$

The sum on the right hand side is well defined since each element in $\mathcal{A}(\mathcal{S})$ is supported on a finite subset of \mathcal{S} .

3. For $f \in \mathcal{S}$ we define functionals $W_{(f)} \in \mathcal{A}(\mathcal{S})$ such that for any $g \in \mathcal{S}$

$$W_{(f)}(g) = \begin{cases} 1 & \text{if } f = g \text{ pointwise} \\ 0 & \text{otherwise} \end{cases}.$$

These functionals form a dense linear subspace in $\mathcal{A}(\mathcal{S})$.

4. We then define the multiplication on that subspace by

$$W_{(f)} \cdot W_{(g)} = e^{-\frac{i}{2}\mathfrak{S}(f,g)} W_{(f+g)}.$$

and extend it to the full $\mathcal{A}(\mathcal{S})$ by linearity.

5. Finally, we define the involution $W_{(f)}^* = W_{(-f)}$.

Closing $\mathcal{A}(\mathcal{S})$ in the ℓ^1 norm provides a Banach \star -algebra $\mathbb{A}(\mathcal{S})$. This algebra can be represented by bounded linear operators on some Hilbert space. Denoting the space of all non-degenerate, irreducible representations of $\mathbb{A}(\mathcal{S})$ by *Irreps*, we define the Weyl algebra.

Definition 8. The Weyl algebra is a C^* -algebra over \mathcal{S} obtained by completion of $\mathbb{A}(\mathcal{S})$ in the C^* -norm

$$\|W_{(f)}\|_{\star} := \sup_{\pi \in \text{Irreps}} \|\pi(W_{(f)})\|_{\mathcal{H}}. \quad (18)$$

We denote it $\mathfrak{A}(\mathcal{S})$.

Lemma 9. For any $x \in M$ the maps α_x and β_x from $\mathfrak{A}(\mathcal{S})$ to $\mathfrak{A}(\mathcal{S})$ defined such that for any $f \in \mathcal{S}$

$$\alpha_x(W_{(f)}) = W_{(L_x^* f)}, \quad \beta_x(W_{(f)}) = W_{(R_x^* f)}, \quad (19)$$

and extended to the whole $\mathfrak{A}(\mathcal{S})$ by linearity are \star -automorphisms.

Proof. By definition α_x and β_x are linear. Further let $f, g \in \mathcal{S}$, then by lemma 1

$$\begin{aligned} \alpha_x(W_{(f)}W_{(g)}) &= \alpha_x\left(W_{(f+g)}e^{-\frac{i}{2}\text{Im}(f,g)_{L^2}}\right) \\ &= e^{-\frac{i}{2}\text{Im}(f,g)_{L^2}} W_{(L_x^* f + L_x^* g)} \\ &= e^{-\frac{i}{2}\text{Im}(L_x^* f, L_x^* g)_{L^2}} W_{(L_x^* f + L_x^* g)} \\ &= W_{(L_x^* f)} W_{(L_x^* g)} \\ &= \alpha_x(W_{(f)}) \alpha_x(W_{(g)}). \end{aligned}$$

Also

$$\alpha_x(W_{(f)}^*) = \alpha_x(W_{(-f)}) = W_{(-L_x^* f)} = [\alpha_x(W_{(f)})]^*.$$

We can similarly address β_x . □

Restricting \mathcal{S} to \mathcal{S}_G we obtain a subset \mathfrak{A}_G defined as

$$\mathfrak{A}_G = \overline{\text{span}\{W_{(f)} \in \mathfrak{A}(\mathcal{S}) \mid f \in \mathcal{S}_G\}}^{\|\cdot\|_{\mathfrak{A}(\mathcal{S})}}, \quad (20)$$

where $\overline{\cdot}^{\|\cdot\|_{\mathfrak{A}(\mathcal{S})}}$ denotes the closure in the $\mathfrak{A}(\mathcal{S})$ - C^* -algebra norm.

Theorem 10. \mathfrak{A}_G is a maximal C^* -sub-algebra of $\mathfrak{A}(\mathcal{S})$ that satisfies $\forall A \in \mathfrak{A}_G$, $\beta_{D_h}(A) = A$ for any $h \in G$.

Proof. \mathfrak{A}_G is spanned by Weyl elements of the form $W_{(f)}$ with $f \in \mathcal{S}_G \subset \mathcal{S}$, hence, $\mathfrak{A}_G \subset \mathfrak{A}(\mathcal{S})$. Since \mathcal{S}_G is closed under addition, and multiplication by real numbers, \mathfrak{A}_G is closed under multiplication and involution,

$$W_{(f)}W_{(g)} = W_{(f+g)}e^{-\frac{i}{2}\text{Im}(f,g)} \in \mathfrak{A}_G, W_{(f)}^* = W_{(-f)} \in \mathfrak{A}_G.$$

To show that \mathfrak{A}_G is invariant under β_{D_h} for any $h \in G$ let $(A_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in \mathfrak{A}_G such that

$$A_n = \sum_{i=0}^n c_i W_{(f_i)} \quad \text{with } c_i \in \mathbb{C}, f_i \in \mathcal{S}_G$$

and that converges to $A \in \mathfrak{A}_G$. Choose $h \in G$. Then by lemma 9 β_{D_h} is a \star -automorphism on $\mathfrak{A}(\mathcal{S})$ and the sequence $(\beta_{D_h}(A_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathfrak{A}(\mathcal{S})$ that converges to $\beta_{D_h}(A) \in \mathfrak{A}(\mathcal{S})$. However, if $f_i \in \mathcal{S}_G$ then $\beta_{D_h}(W_{(f_i)}) = W_{(R_{D_h}^* f_i)} = W_{(f_i)}$ and the two sequences are identical in \mathfrak{A}_G . Thus, the limit points have to be equal and we get, $\beta_{D_h}(A) = A$. The fact that \mathfrak{A}_G is maximal follows from proposition 4 and the fact that we can decompose, $\mathcal{S} = \mathcal{S}_G + \mathcal{S}_{NG}$ with $\mathcal{S}_G \cap \mathcal{S}_{NG} = \{0\}$. \square

Corollary 11. The Weyl algebra over \mathcal{S}_G , denoted $\mathfrak{A}(\mathcal{S}_G)$, is a maximal C^* -sub-algebra of $\mathfrak{A}(\mathcal{S})$ whose elements are invariant under β_{D_h} for any $h \in G$.

Proof. This follows from the fact that $\eta : \mathfrak{A}(\mathcal{S}) \rightarrow \mathfrak{A}(\mathcal{S}_G)$ defined on Weyl elements by

$$\eta(W_{(f)}) = W_{(Pf)}, \tag{21}$$

and extended to $\mathfrak{A}(\mathcal{S})$ by linearity is an invertible \star -homomorphism from \mathfrak{A}_G to $\mathfrak{A}(\mathcal{S}_G)$. The later is obvious since on \mathfrak{A}_G , η acts as an identity. \square

This concludes our construction of the Weyl algebra for GFT. In the following we will not distinguish between the algebra $\mathfrak{A}(\mathcal{S})$ and $\mathfrak{A}(\mathcal{S}_G)$ since all the following statements equally apply to both cases. For that reason we will use \mathfrak{A} to refer to the Weyl algebra (gauge invariant or not) and use \mathcal{S} for the space of smooth function (gauge invariant or not). \mathcal{S}_∞ and \mathcal{S}_{L^2} then refer to the corresponding topological spaces (gauge invariant or not). In section 2.4, however, we will use the gauge invariant algebra $\mathfrak{A}(\mathcal{S}_G)$ since it is more relevant for GFT's with simplicial interpretation.

1.3. Algebraic states

In order to deal with states directly at the level of the algebra, we briefly introduce the concept of algebraic states. An algebraic state is a linear, positive, normalized functional on the algebra \mathfrak{A} ,

$$\omega : \mathfrak{A} \rightarrow \mathbb{C}$$

such that for any $A \in \mathfrak{A}$ we get

$$\omega(A^*A) \geq 0 \quad \omega(\mathbb{1}) = 1. \tag{22}$$

The first inequality is the condition of positivity and the second is the normalization. Specifically for the Weyl algebra the positivity condition reads as follows:

Definition 12. The functional $\omega : \mathfrak{A} \rightarrow \mathbb{C}$ is positive if, for any finite $N \in \mathbb{N}$ and any set of complex coefficients $\{c_n\}_{n \in \{0, \dots, N\}}$ and test functions $\{f_n \in \mathcal{S}\}_{n \in \{0, \dots, N\}}$, the following holds

$$\sum_{n,m}^N c_n \bar{c}_m \omega(W_{(f_n - f_m)}) e^{-i \frac{\text{Im}(f_n f_m)}{2}} \geq 0.$$

By the GNS construction, every algebraic state provides a triple $(\mathcal{H}_\omega, \pi_\omega, |\Omega\rangle)$, where \mathcal{H}_ω is a Hilbert space, $\pi_\omega : \mathfrak{A} \rightarrow \mathcal{L}(\mathcal{H}_\omega)$ is a representation of \mathfrak{A} in terms of bounded linear operators on \mathcal{H}_ω , and the state vector $|\Omega\rangle \in \mathcal{H}_\omega$, such that $\forall A \in \mathfrak{A}$

$$\omega(A) = (\Omega | \pi_\omega[A] | \Omega). \tag{23}$$

This representation is unique, up to unitarily equivalence [57].

The algebra of observables $\pi_\omega(\mathfrak{A}(\mathcal{S}))$ on the GNS Hilbert space \mathcal{H}_ω is a sub-algebra of bounded linear operators on \mathcal{H}_ω , that we denote \mathfrak{M} . The commutant of \mathfrak{M} is a subset of bounded linear operators of $\mathcal{L}(\mathcal{H}_\omega)$ on \mathcal{H}_ω such that

$$\mathfrak{M}' = \{A \in \mathcal{L}(\mathcal{H}_\omega) \mid \forall B \in \mathfrak{M} \ AB = BA\}. \tag{24}$$

Usually, \mathfrak{M} is not closed in the strong operator topology on \mathcal{H}_ω . This is because the C^* -norm (equation (18)) is stronger than the operator norm. The closure of \mathfrak{M} in the strong (or equivalently, weak) operator topology is called the von Neumann algebra and is equal to the bicommutant of \mathfrak{M} by the von Neumann theorem (see for example [58]). We denote the von Neumann algebra of the ω -GNS representation \mathfrak{M}'' .

The center of the von Neumann algebra is then defined as $Z = \mathfrak{M}' \cap \mathfrak{M}''$. A state is called factor if the center of its von Neumann algebra contains only multiples of identity.

A state ω is called pure if it can not be written as a convex combination of two or more states

$$\omega = \lambda \omega_1 + (1 - \lambda) \omega_2 \quad 0 < \lambda < 1$$

where $\omega_1, \omega_2, \omega$ are pairwise distinct. Otherwise it is called mixed. The GNS representation of a state is irreducible if and only if the state is pure [58, theorem 2.3.19]. The GNS representation of a state is irreducible if the state is factor.

Most algebraic states are mere mathematical artifacts, and one needs a prescription for selecting interesting specific states that can be considered of physical relevance. One strategy is to rely on the quantum dynamics, encoded in a constraint operator. From the algebraic point of view the constraint operator is therefore related to the choice of the folium, or conversely, information about the constraint operator is partly encoded in the algebraic state.

We will not discuss the constraint operator explicitly, since little is known at present about the constraint operators underlying specific GFT models. Instead and reasonably, using the following criteria starting from the Fock representation of GFT, we consider state sequences that satisfy two conditions:

1. All states in the sequence are coherent states.

This is mainly motivated by the use of GFT coherent states in the extraction of an effective continuum dynamics in the series of works [40, 43, 44, 46]. Of course, coherent states are also key for the classical approximation of any QFT, and routinely used in particle physics, many-body systems and condensed matter theory, which provides further motivation.

2. The particle number of the limit state diverges.

As described above, it is reasonable to expect that quantum states that describe smooth geometries contain infinitely many particles. This is only possible if the particle number operator in the corresponding representation is formally divergent and by consequence, if the corresponding representation is non-Fock.

In the next section we provide simple explicit examples for GFT representations that satisfy these two properties.

2. States and representations

2.1. Fock states and the Fock representation

The Weyl algebra \mathfrak{A} admits the Fock representation, which is given by the GNS representation of the algebraic state

$$\omega_{\mathbb{F}}(W_{(f)}) = e^{-\frac{\|f\|_t^2}{4}}. \quad (25)$$

Since the above state is regular, i.e. the function $\Omega(t) := \omega_{\mathbb{F}}(W_{(f)})$ for $t \in I \subset \mathbb{R}$ and any fixed $f \in \mathcal{S}$ is smooth, the generator of the Weyl operator exists by Stone's theorem [59]. Denoting the corresponding GNS triple by $(\mathcal{H}_{\mathbb{F}}, \pi_{\mathbb{F}}, |o\rangle)$, we can write

$$(o|\pi_{\mathbb{F}}[W_{(f)}]|o) = (o|e^{i\Phi_{\mathbb{F}}(f)}|o), \quad (26)$$

where $\Phi_{\mathbb{F}}(f)$ is an essentially self-adjoint generator of $\pi_{\mathbb{F}}[W_{(f)}]$ in the Fock representation, defined on the dense domain

$$D(\Phi_{\mathbb{F}}) = \left\{ \sum_i^N c_i \pi_{\mathbb{F}}[W_{(f_i)}]|o\rangle \mid c_i \in \mathbb{C}, f_i \in \mathcal{S}, N \in \mathbb{N} \right\}.$$

We can obtain the action of $\Phi_{\mathbb{F}}(f)$ on $D(\Phi_{\mathbb{F}})$ by differentiation. For any $|\psi\rangle \in D(\Phi_{\mathbb{F}})$ and appropriate set of complex coefficients $\{c_i\}_{i \in \{0, \dots, N\}}$ and test functions $\{f_i\}_{i \in \{0, \dots, N\}}$ such that

$$|\psi\rangle = \sum_{i=0}^N c_i \pi_{\mathbb{F}}[W_{(f_i)}]|o\rangle, \quad (27)$$

we get

$$(o|\Phi_{\mathbb{F}}(f)|\psi) = -i\partial_t \omega_{\mathbb{F}}\left(W_{(f)} \sum_{i=0}^N c_i W_{(f_i)}\right) \Big|_{t=0}. \quad (28)$$

In particular we obtain for any $f \in \mathcal{S}$

$$(o|\Phi_{\mathbb{F}}(f)|o) = 0, \quad (29)$$

and

$$\|\Phi_{\mathbb{F}}(f)|o\rangle\|_{\mathcal{H}}^2 = \langle o|\Phi_{\mathbb{F}}(f)\Phi_{\mathbb{F}}(f)|o\rangle = \frac{1}{2}\|f\|_{L^2}^2. \quad (30)$$

By similar calculations it follows that the operators $\Phi_{\mathbb{F}}(f)$ satisfy the commutation relation, for any $f, g \in \mathcal{S}$

$$[\Phi_{\mathbb{F}}(f), \Phi_{\mathbb{F}}(g)] = i\text{Im}[(f, g)_{L^2}]. \quad (31)$$

We call $\Phi_{\mathbb{F}}(f)$ the field operator of GFT.

We can also define the creation and annihilation operators by

$$\psi_{\mathbb{F}}(f) = \frac{1}{\sqrt{2}}[\Phi_{\mathbb{F}}(f) + i\Phi_{\mathbb{F}}(if)] \quad (32)$$

$$\psi_{\mathbb{F}}^\dagger(f) = \frac{1}{\sqrt{2}}[\Phi_{\mathbb{F}}(f) - i\Phi_{\mathbb{F}}(if)], \quad (33)$$

with $\psi_{\mathbb{F}}(f)^\dagger = \psi_{\mathbb{F}}^\dagger(f)$, such that $\psi_{\mathbb{F}}(f)$ is anti-linear in f , $\psi_{\mathbb{F}}^\dagger(f)$ is linear in f , both are closed on $D(\Phi_{\mathbb{F}})$ and fulfill the canonical commutation relations

$$[\psi_{\mathbb{F}}(f), \psi_{\mathbb{F}}(g)] = [\psi_{\mathbb{F}}^\dagger(f), \psi_{\mathbb{F}}^\dagger(g)] = 0 \quad (34)$$

and

$$[\psi_{\mathbb{F}}(f), \psi_{\mathbb{F}}^\dagger(g)] = (f, g). \quad (35)$$

From equations (32), (28) and (30) it follows that

$$\|\psi_{\mathbb{F}}(f)|o\rangle\|_{\mathcal{H}}^2 = \langle o|\psi_{\mathbb{F}}^\dagger(f)\psi_{\mathbb{F}}(f)|o\rangle = 0,$$

and therefore

$$\psi_{\mathbb{F}}(f)|o\rangle = 0, \quad (36)$$

for all $f \in \mathcal{S}$. Hence, $|o\rangle$ is the Fock vacuum with respect to the annihilation operator $\psi_{\mathbb{F}}(f)$ and the space $\mathcal{H}_{\mathbb{F}}$ is spanned by polynomials of creation operators $\psi_{\mathbb{F}}^\dagger(f)$ applied on $|o\rangle$.

The Fock state is pure and hence the GNS representation of $\omega_{\mathbb{F}}$ is irreducible [60]. Also the Fock representation is the unique representation (up to unitary equivalence) in which the particle number operator N exists, formally given by

$$N = \sum_{i \in \mathbb{N}} \psi_{\mathbb{F}}^\dagger(f_i)\psi_{\mathbb{F}}(f_i) \quad (37)$$

for some complete orthonormal basis $\{f_i\}_{i \in \mathbb{N}}$ of $L^2(M, dx)$.

2.2. Coherent states and non-Fock representations

Usually coherent states are characterized as eigenstates of the annihilation operators in the Fock representation, and hence require a notion of the Hilbert space for their very definition. In the algebraic approach, this characterization is avoided by introducing a generalized notion of coherent states directly at the level of the algebra. This is described in [61, 62]. Below we briefly summarize some of the results of that work that will be important for our discussion.

Definition 13. Let $\Gamma : \mathcal{S}_\infty \rightarrow \mathbb{C}$ be a continuous linear form on \mathcal{S}_∞ . A state ω defined on the Weyl elements as

$$\omega_\Gamma (W_{(f)}) = \omega_F (W_{(f)}) e^{i\sqrt{2}\text{Re}[\Gamma(f)]}, \tag{38}$$

and extended to $\mathfrak{A}(\mathcal{S})$ by linearity, is called a coherent state. It is pure and regular [61].

With this definition the Fock state is the special case of the above family of coherent states for $\Gamma = 0$.

Any linear functional Γ corresponds to a well defined state [61]. It should be noticed that there exist even more general definitions of coherent states, but this is the one that most closely reflects the condition of being an eigenfunction of the annihilation operator.

Proposition 14 ([62, proposition 2.5]). *The state ω of the above form is equivalent to the Fock state, if and only if Γ is continuous on \mathcal{S}_{L^2} .*

The detailed proof of this proposition is presented in [62], but we provide an intuitive sketch.

Assume that Γ is a continuous functional on \mathcal{S}_{L^2} , and hence, it extends by continuity to $L^2(M, dx)$. Then by the Riesz lemma there exists an $\gamma \in L^2(M, dx)$ such that for any $f \in \mathcal{S}_{L^2}$

$$\Gamma(f) = \int_M f(x) \cdot \bar{\gamma}(x) dx, \tag{39}$$

and

$$\|\Gamma\|_{\text{op}} = \|\gamma\|_{L^2}. \tag{40}$$

The state ω_Γ provides a GNS triple $(\mathcal{H}_\Gamma, \pi_\Gamma, |\Gamma\rangle)$. It is not difficult to see that in this case the GNS Hilbert space is Fock and that $\overline{L(f)}$ is the eigenvalue of the state vector $|\Gamma\rangle$ [61], i.e.

$$\psi_\Gamma(f) |\Gamma\rangle = \overline{\Gamma(f)} |\Gamma\rangle = (f, \gamma)_{L^2} |\Gamma\rangle. \tag{41}$$

Since the representation is Fock, the particle number operator, equation (37), exists and its expectation value is given by

$$(\Gamma|N|\Gamma) = \sum_{i \in \mathbb{N}} |\Gamma(f_i)|^2 = \|\gamma\|^2 = \|\Gamma\|_{\text{op}}^2. \tag{42}$$

That is, the particle number is given by the L^2 norm of γ or equivalently the operator norm of Γ . When Γ is discontinuous on \mathcal{S}_{L^2} and, hence, unbounded on $L^2(M, dx)$ the global particle number is ill-defined and the representation can not be Fock.

The non-Fock coherent states are hence classified by functionals Γ which are continuous on \mathcal{S}_∞ but discontinuous on \mathcal{S}_{L^2} , sometimes called the space of tempered microfunctions.

By the Riesz–Markov theorem every functional Γ on \mathcal{S}_∞ is of the form

$$\Gamma(f) = \int_M f(x) d\nu(x), \tag{43}$$

for some Baire measure ν .

From this we can easily state

Corollary 15. *If Γ is invariant under left multiplication i.e. for any $y \in M$, $\Gamma(L_y^* f) = \Gamma(f)$ for any $f \in \mathcal{S}$, then the coherent state ω_Γ is Fock.*

Proof. Let Γ be invariant under left translations. Then for any $f \in \mathcal{S}$ we have

$$\Gamma(L_y^* f) = \int_M L_y^* f(x) d\nu(x) = \Gamma(f) = \int_M f(x) d\nu(x), \tag{44}$$

hence the measure ν is a left-invariant measure on M . By uniqueness of the Haar measure, $d\nu = c \cdot dx$, for some $c \in \mathbb{R}$. Then by Hölder's inequality $|\Gamma(f)| \leq c\|f\|_{L^2}$, and hence Γ is continuous on $L^2(M, dx)$. \square

2.2.1. Remarks on the discontinuity of Γ . From the above discussion it follows that in order to have inequivalent coherent state representations we need the integrand in equation (43) to diverge on some square integrable functions on M . There are two reasons for which the functional in equation (43) can become unbounded on $L^2(M, dx)$, which are related to the long (IR) and short (UV) scale behavior of the measure $d\nu$.

The IR divergences appear when the integral becomes infinite due to regions with arbitrary large measure. This is what happens in ordinary many-body physics. On a compact manifold, however, IR divergences can not occur. But the UV divergence can.

Physically, an IR divergent state can be understood as a state with an infinite number of quanta but with a finite density. On finite regions of the base manifold the particle number is, however, finite. This is the typical situation in condensed matter physics [63]. A UV divergence, on the other hand, corresponds to a state in which infinitely many particles are concentrated at a single point on the base manifold and, correspondingly, the density at this point blows up. The particle number operator is defined globally except for such a local region with infinite density. From the point of view of field theory on spacetime, this situation is clearly not physical: an infinite number of particles in a finite region corresponds to an infinite energy density. Accordingly, quantum field theories on compact spacetimes require a finite particle number and hence forces us to stay in the Fock representation. This requirement is usually captured in the statement that no phase transition can occur in field theories in a finite volume (for example [22]).

In GFTs, however, the notion of energy is not present and the base manifold does not relate to local regions of space-time. Thus, even in the compact case, the restriction to the Fock representation would not be well-motivated. In fact, UV divergences in the above sense could even be desirable from the point of view of the interpretation of GFT quanta as 'building blocks of spacetime and geometry'. Heuristically, these types of coherent states would correspond to condensates with a collective wave-function sharply peaked on a given value of the underlying discrete connection. Wave functions of this type have been used for condensate states more general than coherent states, in [41, 42], while hints of similar divergences of the GFT particle number were found in the GFT condensate cosmology context in [45].

To summarize: GFT models on the compact manifold can exhibit inequivalent representations due to UV divergences, even though the IR divergences can not occur.

Remark 16. A fundamental difference between UV and IR divergences is their behavior under translations. Whereas the IR divergence can be generated by translation invariant measures as in the example of the Bose–Einstein condensation, the UV divergences on the compact manifold cannot, by corollary 15.

2.3. Example

Our procedure to construct inequivalent representations is fairly straightforward. By the above discussion, we simply need to construct a sequence of continuous functionals Γ_n on \mathcal{S}_∞ that converge pointwise to a functional Γ_∞ unbounded on $L^2(M, dx)$. Here we provide a very simple example in which the sequence of regular measures converges to a pure point measure. It should be clear, however, that any measure that satisfies the property of being unbounded on $L^2(M, dx)$ leads to a new inequivalent representation.

Let us first define the Dirac measure ν_D , such that for any open $U \subset M$ and $\mathbb{1} \in M$ denoting the identity on M ,

$$\nu_D(U) = \begin{cases} 1 & \text{if } \mathbb{1} \in U \\ 0 & \text{otherwise} \end{cases}. \tag{45}$$

It follows that on smooth functions $f \in \mathcal{S}$ we have,

$$\nu_D(f) = f(\mathbb{1}). \tag{46}$$

Such a Riesz functional is continuous on \mathcal{S}_∞ , since

$$|\nu_D(f)| = |f(\mathbb{1})| \leq \|f\|_\infty. \tag{47}$$

However, it is unbounded on $L^2(M, dx)$ due to the possible singular behavior of functions at sets of Haar measure zero.

Assume further a contracting sequence of open sets $\{U_n\}_{n \in \mathbb{N}}$ around the identity $\mathbb{1} \in M$, such that $U_{n+1} \subset U_n$ and $\bigcap_{n \in \mathbb{N}} U_n = \{\mathbb{1}\}$, and consider a sequence of measures defined as

$$d\nu_n = \frac{\chi_{U_n}}{|U_n|} dx, \tag{48}$$

where χ_{U_n} is the characteristic function on U_n ,

$$\chi_{U_n}(x) = \begin{cases} 1 & \text{if } x \in U_n \\ 0 & \text{otherwise} \end{cases}, \tag{49}$$

and $|U_n| = \int_{U_n} dx$.

Lemma 17. *On \mathcal{S} the sequence of functionals defined by (48) converges to the Dirac measure in the distributional sense. That is for any $f \in \mathcal{S}$*

$$\lim_{n \rightarrow \infty} \nu_n(f) = \nu_D(f) = f(\mathbb{1}). \tag{50}$$

Proof. Since f is continuous, we can find for some $\epsilon > 0$ a neighborhood $N_\epsilon(\mathbb{1})$ around $\mathbb{1}$ on M such that $\forall x \in N_\epsilon(\mathbb{1})$ $f(x)$ is in an ϵ -ball around $f(\mathbb{1})$ in \mathbb{C} . Since the sequence is contracting $\exists N \in \mathbb{N}$ such that $\forall n > N$, $U_n \subset N_\epsilon(\mathbb{1})$ then

$$\begin{aligned} |\nu_n(f) - \nu_D(f)| &= \left| \frac{1}{|U_n|} \int_M f(x) \chi_{U_n}(x) dx - f(\mathbb{1}) \right| \\ &= \frac{1}{|U_n|} \left| \int_M \chi_{U_n}(x) (f(x) - f(\mathbb{1})) dx \right| \\ &\leq \frac{1}{|U_n|} \int_M \chi_{U_n}(x) |f(x) - f(\mathbb{1})| dx \\ &\leq \epsilon. \end{aligned} \tag{51}$$

At every finite n the measure ν_n is absolutely continuous with respect to the Haar measure and by the above proposition 14 every state

$$\omega_n(W_{(f)}) := \omega_F(W_{(f)}) \cdot e^{z\sqrt{2}\text{Re}[\Gamma_n(f)]}, \tag{51}$$

is equivalent to the Fock one. Where $\Gamma_n(f) \doteq \int_M f(x) d\nu_n$. From the convergence of the measure, the convergence of the algebraic sequence is obvious.

Lemma 18. *The sequence of states ω_n converges in the w^* -topology to $\omega_D^{\mathbb{1}}$, defined on Weyl*

elements such that for each $f \in \mathcal{S}$

$$\omega_D^1(W_{(f)}) \doteq \omega_F(W_{(f)}) \cdot e^{i\sqrt{2}\text{Re}(\Gamma[f](1))}, \tag{52}$$

and extended by linearity to the whole \mathfrak{A} .

Proof. For any $W_{(f)} \in \mathfrak{A}$ we have

$$\begin{aligned} |\omega_n(W_{(f)}) - \omega_D^1(W_{(f)})| &= |\omega_F(W_{(f)})| \left| e^{i\sqrt{2}\Re[f f d\nu_n]} - e^{i\sqrt{2}\Re[f f d\nu_D]} \right| \\ &= |\omega_F(W_{(f)})| \left| e^{i\sqrt{2}\Re[f f d\nu_n - f f d\nu_D]} - 1 \right| \\ &\rightarrow 0. \end{aligned}$$

By linearity of the state and the product property of the Weyl algebra, this extends to the whole algebra \mathfrak{A} . \square

At finite n the representation is Fock, the particle number operator exists and the particle number of the n th member of the sequence is given by

$$\|\Gamma_n\|_{\text{op}} = \frac{1}{|U_n|}. \tag{53}$$

But with increasing n the particle number grows since the volume of U_n shrinks. At the limit point the total particle number diverges and the corresponding representation becomes inequivalent to the Fock one.

We can define states ω_D^x peaked at points $x \in M$ using the automorphisms $\alpha_{x^{-1}}$ introduced in the previous section such that

$$\omega_D^x = \omega_D^1 \circ \alpha_{x^{-1}}. \tag{54}$$

We will show in the next section that each of the states ω_D^x leads to an inequivalent representation and breaks translation invariance.

2.4. Explicit representations

In this section we will focus on the algebra $\mathfrak{A}(\mathcal{S}_G)$, since it is more relevant for GFT's with simplicial interpretation, however, all the constructions can be directly applied to $\mathfrak{A}(\mathcal{S})$ leading to similar results.

We now construct an explicit representation that is generated by the above algebraic state following the construction in [64].

Take $L^2(M, d\nu_D^x)$ to be the space of L^2 functions with respect to the Dirac measure concentrated at $x \in M$, i.e. for any $f \in \mathcal{S}_G$

$$\nu_D^x(f) = f(x). \tag{55}$$

The space $L^2(M, d\nu_D^x)$ is one-dimensional. For any $f \in \mathcal{S}_G$ define commuting operators $A(f)$ and $B(f)$ on $L^2(M, d\nu_D^x)$ such that for any $\varphi \in L^2(M, d\nu_D^x)$ and $f \in \mathcal{S}_G$

$$[A_x(f)\varphi](x) = f(x)\varphi(x) \quad [B_x(f)\varphi](x) = \bar{f}(x)\varphi(x).$$

We define the state vector

$$|\Omega_D^x\rangle \equiv |o\rangle \otimes 1 \tag{56}$$

where 1 is the constant function on M and $|o\rangle$ is the Fock vacuum. Further we define unitary operators

$$W_{(f)}^x = e^{\frac{i}{\sqrt{2}}[\psi_F(f) + \psi_F^\dagger(f)]} \otimes e^{\frac{i}{\sqrt{2}}[A(f) + B(f)]}, \tag{57}$$

where $\psi_F(f), \psi_F^\dagger(f)$ are the Fock operators. We denote the closure of the space generated by polynomials of operators $W_{(f)}^x$ acting on $|\Omega_D^x\rangle$ by \mathcal{H}_x . It follows that the operator algebra spanned by $W_{(f)}^x$ for $f \in \mathcal{S}_G$ is equivalent to \mathfrak{M} (the ω_D^x -GNS representation of the Weyl algebra $\mathfrak{A}(\mathcal{S}_G)$) since the expectation values coincide,

$$(\Omega_D^x | W_{(f)}^x | \Omega_D^x) = e^{-\frac{\|f\|_{L^2}^2}{4}} \cdot e^{i\sqrt{2}\text{Re}[f(x)]}. \tag{58}$$

Irreducibility and cyclicity of this representation are inherited from the Fock representation since $PL^2(M, \nu_D)$ is one-dimensional.

Let f_y be a real valued function on M defined such that for some fixed $a \in \mathbb{R}$

$$f_y(x) = \begin{cases} a & \text{if } \exists h \in G \text{ such that } x = yD_h, \\ 0 & \text{else} \end{cases}, \tag{59}$$

clearly $f_y \in PL^2(M, dx)$ and is zero almost everywhere with respect to the Haar measure.

Lemma 19. *Let $\{f_n | f_n \in \mathcal{S}_G\}_{n \in \mathbb{N}}$ be a sequence that converges to f_y in the L^2 -norm. Then the limit $\lim_{n \rightarrow \infty} W_{(f_n)}^x$ exists in \mathfrak{M} . We call this element $W_{(f_y)}^x$. Moreover, $W_{(f_y)}^x$ is in the center and there exists a complex number $c \in \mathbb{C}$ such that $W_{(f_y)}^x = c\mathbb{1}$.*

Proof. Since $f_y \in PL^2(M, dx)$ and \mathcal{S}_G is dense in $PL^2(M, dx)$ there exists a Cauchy sequence $\{f_n | f_n \in \mathcal{S}_G\}$ that converges to f_y . Then for n and m large enough and any $g \in \mathcal{S}_G$ we get by direct calculation

$$\begin{aligned} \left\| \left(W_{(f_n)}^x - W_{(f_m)}^x \right) W_{(g)}^x | \Omega_D^x \right\|_{\mathcal{H}} &= 2(\Omega_D^x | \Omega_D^x) - 2\text{Re} \left[e^{-\frac{\|f_n - f_m\|_{L^2}^2}{4}} e^{i\sqrt{2}\text{Re}[f_n(x) - f_m(x)]} \right] \\ &\quad \times \text{Re} \left[e^{-\frac{i}{2}\text{Im}[(f_n - f_m, g)_{L^2} + (-f_m - g, f_n + g)_{L^2}]} \right] \leq \epsilon. \end{aligned}$$

Since $|\Omega_D^x\rangle$ is cyclic we can reach every element of \mathcal{H}_x acting on it by polynomials of Weyl operators. Hence, the sequence $\{W_{(f_n)}^x | f_n \in \mathcal{S}_G\}$ is a Cauchy sequence in the strong operator topology and therefore converges to an element in the von Neumann algebra \mathfrak{M}'' . We call this element $W_{(f_y)}^x$. For any $f \in \mathcal{S}_G$ we have

$$W_{(f)}^x W_{(f_y)}^x = \lim_{n \rightarrow \infty} W_{(f+f_n)}^x e^{-\frac{i}{2}\text{Im}[(f, f_n)]} = W_{(f+f_y)}^x = W_{(f_y)}^x W_{(f)}^x,$$

where the second equality follows from the fact that $W_{(f+f_n)}^x$ is a Cauchy sequence and f_y is zero almost everywhere. Hence, the element $W_{(f_y)}^x$ is in the center of the von Neumann algebra \mathfrak{M}'' . Since the state ω_D^x is pure the center contains only multiples of identity, thus there exists a $c \in \mathbb{C}$ such that $W_{(f_y)}^x = c\mathbb{1}$. \square

2.4.1. Breaking of translation symmetry. We show that for any $x \in M$ the state ω_D^x breaks translation symmetry in the sense that for any non-trivial $y \in M$ the translation automorphism α_y can not be represented by a unitary operator on \mathcal{H}_x .

Corollary 20. *Let $x, y \in M$ with $y \neq \mathbb{1}$, then the states ω_D^x and ω_D^{yx} are inequivalent.*

Proof. ω_D^x and ω_D^{yx} are pure states and therefore factor. By [58, proposition 2.4.27] factor states ω_D^x and ω_D^{yx} are (quasi)-equivalent if and only if the state $\omega = \frac{1}{2}(\omega_D^x + \omega_D^{yx})$ is factor as well. Therefore it is enough to show that the center of the von Neumann algebra of ω is non-trivial. For some fixed $a \neq 0 \in \mathbb{R}$ define the function

$$f_{yx}(z) = \begin{cases} a & \text{if } \exists h \in G \quad M \ni z = yx D_h \\ 0 & \text{else} \end{cases}. \quad (60)$$

The GNS triple of ω is given by

$$\left(\mathcal{H} = \mathcal{H}_x \oplus \mathcal{H}_{yx}, \pi_\omega = \pi_{\omega_D^x} \oplus \pi_{\omega_D^{yx}}, |\Omega_D\rangle = |\Omega_D^x\rangle \oplus |\Omega_D^{yx}\rangle \right),$$

By lemma 19 there exists an element $W_{(f_{yx})}$ in the von Neumann algebra of ω as a limit of an appropriate sequence of operators $W_{(f_n)}$, and $W_{(f_{yx})}$ is in the center of the von Neumann algebra. However, a direct calculation shows that

$$(\Omega_D | W_{(f_{yx})} | \Omega_D) = \frac{1}{2} \left[(\Omega_D^x | W_{(f_{yx})}^x | \Omega_D^x) + (\Omega_D^{yx} | W_{(f_{yx})}^{yx} | \Omega_D^{yx}) \right] = \frac{1}{2} (1 + e^{i\sqrt{2}a}).$$

Hence, $W_{(f_{yx})} \neq \mathbb{1}$ and the GNS representation of ω is not factor. By theorem [58, proposition 2.4.27] the states ω_D^x and ω_D^{yx} are inequivalent. \square

Since ω_D^x and ω_D^{yx} are inequivalent, the translation automorphism α_y can not be implemented by a unitary operator for any non-trivial $y \in M$. Hence, the translation symmetry is broken and moreover for $x, z \in M$ not of the diagonal form the states ω^x and ω^z , lead to inequivalent representations since they are related by translation $y = zx^{-1} \in M$.

Notice that the automorphism α_x implements the isometry of the base manifold and hence the above representations break the isometry transformation. This is rather different from ordinary field theory, in which Poincaré symmetry is not allowed to be broken [57, 60, 64, 65]. Again, this is possible because no spacetime interpretation is attached to the GFT base manifold.

3. Interpretation of new representations

Let us pontificate on the interpretation of the newly found non-Fock representations, expanding on some of the points above.

The state ω_D^x contain infinitely many GFT quanta carrying a label (or equivalently have the property) x . It is instructive to think about the label x as one of the ‘continuous modes’ of the theory. Let us call this mode the ‘basic mode’. In this case the representation described above is very similar to the usual case of Bose–Einstein condensation [64]. The creation and annihilation operators of particles in the basic mode x are given by A and B operators respectively.

They commute since the number of particles in this mode is infinite, which is the manifestation of the usual Bogoliubov argument (for example [60, 64]). States of the Hilbert space are then created by excitations of other ‘modes’ on top of the basic one and hence can be considered quantum fluctuations over a background that is created by infinitely many particles in the basic mode.

We can now have the following interpretation. If we relate the group elements of GFT with the basic notion of holonomy/curvature, which is well-justified at the discrete level we could think about the ground state of new representations as a truly infinite gas of particles that all carry the same geometrical information. The resulting continuum geometry would be then reconstructed from such an infinite particle state. This could be a generic geometry, since approximately equal curvature building blocks can be used, if they also have progressively vanishing size, to approximate any geometry, as in Regge calculus [66]. Another possibility is that they could generate a homogeneous background with the constant holonomy (curvature) x . Choosing $x = \mathbb{1}$ we would obtain a flat background on top of which excitations are created by $\psi_F(f)$ and $\psi_F^\dagger(f)$. The type of states created/annihilated on top of such a condensate background would be formally analogous to the fundamental spin network states or cylindrical functions that are also found in the Fock Hilbert space of the theory. Importantly, though, in these representations the role of the Fock creation and annihilation operators is that of collective excitations and not of single building blocks of quantum geometry. The origin of inequivalent representations for different x 's stems from the fact that the corresponding Hilbert spaces are created by excitations over backgrounds with different geometry than the fully degenerate one corresponding to the Fock vacuum. Being a specific case of the condensate state with $\Gamma = 0$, the Fock representation corresponds to the case in which the background consists of no GFT quanta at all.

The above description provides a useful intuition, but it does not amount yet to a compelling nor complete, physical interpretation. In fact:

1. The basic mode x in our case is not selected by any physical principle such as energy minimization, entropy maximization or the enforcement of a specific physical symmetry. It is rather postulated by hand, which makes the construction non-unique. In contrast to this, we recall, the ground mode in condensed matter physics is selected as the minimum of the Hamiltonian. A detailed analysis of the constraint operator underlying interesting GFT models is necessary, before assigning any physical interpretation to the above representations.
2. The states $|\Omega_D^x\rangle$ are quantum states, whose physical properties should be ascertained by computing expectation values of observables with a clear macroscopic, geometric meaning. This obscures the interpretation of the elements $x \in M$ in terms of holonomy/curvature of the reconstructed geometry.
3. The form of the constraint operator at this moment is not fully understood, however if it is symmetric under the described translation automorphisms the inequivalent states for different x 's should be physically indistinguishable and any association of geometrical properties to the points x in the inequivalent states ω^x would be incorrect.

4. Conclusions

We have constructed an algebraic formulation for GFT. We believe that this formulation has potential, not only allowing us to formulate problems in a rigorous way, but also to efficiently tackle some conceptual and technical issues related to the problem of phase transitions and

continuum limits in this class of quantum gravity models. We have used the algebraic formulation to construct inequivalent, non-Fock representations of the GFT algebra of observables and studied its operator algebras in absence of dynamics in the case when the base manifold of the GFT is compact. In particular, we focused on coherent state representations. We have given a partial symmetry characterization of the non-Fock representations, and attempted a preliminary geometric interpretation of them, leaving a more complete analysis to future work.

For the non-compact base manifolds the analysis requires different techniques since the closure constraint can not be imposed in the same way as we did in this paper, since the Haar measure for non-compact groups is not normalized. Nevertheless, we believe that for GFT's without the closure constraint similar results regarding the construction of the operator algebra and definition of its inequivalent representations will hold true even for non-compact base manifolds. We leave a careful and rigorous discussion of the non-compact case for future work.

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