Lattice oscillator model on noncommutative space, eigenvalues problem for the perturbation theory

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Abstract

Harmonic oscillator in noncommutative two dimensional lattice are investigated. Using the properties of non-differential calculus and its applications to quantum mechanics, we provide the eigenvalues and eigenfunctions of the corresponding Hamiltonian. First we consider the case of ordinary quantum mechanics, and we point out the thermodynamic properties of the model. Then we consider the same question when both coordinates and momentums are noncommutative.

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1 Introduction

Several experiments and theoretical results show that the continuous space time in the description of modern physics should need revision at the scale where quantum theory and gravitation can be concile [1]-[3]. The discrete spacetime has become a tool of choice for the investigation of physics around this scale. It maybe consider as an alternative way to prove the existence of a minimum length (for example $l_p \approx 1.6 \cdot 10^{35}$meters required by string theory). The idea of a discrete structure of spacetime was first suggested by Wheeler [4] and well after by Wilson [5], which is considered as an attempt to understand the confinement of subatomic particles. The numerical simulations of quantum field theories on Euclidean lattices have proven to be a very successful tool for studying nonperturbative phenomena. Consequently, a lot of efforts has been put into the lattice formulation of quantum and field theories, see [6]-[21] and references therein. Hence, the discrete structure of spacetime is inherent in many models of quantum gravity, such as loop quantum gravity, noncommutative (NC) field theory, spin foam, black hole physics, random tensors models.

Recent results obtained in the framework of nonperturbative string theory and quantum Hall effect, have boosted interest in a deeper understanding of the role played by NC geometry in different

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sectors of theoretical physics [7]-[9]. In physics, the most important achievement of NC geometry is to overcome the distinction between continuous and discrete spaces, in the same way as quantum mechanics unified the concepts of waves and particles. However a NC space is an intriguing and revolutionary possibility that could have important consequences in our conception of the quantum structure of nature. The description of noncommutativity in quantum and field theory can be achieved by replacing the ordinary product of functions in classical theory by the so called Moyal star product. It can also be realized by defining the field theory on a coordinates operators space that are intrinsically NC, which satisfy the commutation relation $[\hat{X}^\rho, \hat{X}^\sigma] = i\theta^{\rho\sigma}$. The simplest case corresponds to $\theta^{\rho\sigma}$ being constant skew-symmetric matrix. In our current investigation we wish to define the noncommutativity of quantum theory, in which both coordinates and momentums are NC, i.e.

$$[\hat{X}^\rho, \hat{X}^\sigma] = i\theta^{\rho\sigma}, \quad [\hat{X}^\rho, \hat{P}^\sigma] = i\hbar_{eff}\delta^{\rho\sigma}, \quad [\hat{P}^\rho, \hat{P}^\sigma] = i\bar{\theta}^{\rho\sigma}. \tag{1}$$

The case $\theta^{\rho\sigma} = 0 = \bar{\theta}^{\rho\sigma}$ corresponds to ordinary quantum mechanics, for which $\hbar_{eff} = \hbar$ (the Planck constant). In the general possible representations, one obtained from standard Bopp-shifts in the conventional canonical variables $\hat{x}^\rho$, $\hat{p}^\rho$, with nonvanishing commutators $[\hat{x}^\rho, \hat{p}^\sigma] = i\hbar\delta^{\rho\sigma}$, is $\hat{X}^\rho = a\,\hat{x}^\rho + b\,\hat{p}^\rho$, $\hat{P}^\rho = c\,\hat{x}^\rho + d\,\hat{p}^\rho$, where a, b, c, d are constants. With these transformations, all Hamiltonians dynamics in NC space correspond to others problems in ordinary quantum space. As an example the harmonic oscillator in NC space corresponds to the Landau problem in ordinary quantum space. It would therefore be interesting to investigate the harmonic oscillator in the NC discrete space in which the continuous variables $\hat{X}^\rho$ and $\hat{P}^\rho$ become discrete with a spacing ε. It turns out that this question is not trivial, but may be solved in the perturbation to ε.

Let us recall very briefly what we know about the lattice oscillator in classical and quantum mechanics. Lattice oscillator systems are the standard model for the vibrational degrees of freedom, known as phonons, in crystal lattices [18]-[21]. These phonons interact with the other degrees of freedom, such as spins and electrons, in ways that often significantly modify their behavior. The lattice quantum theory is based on the non-differential calculus with discrete derivatives and integrals, which is studied by several mathematicians and physicists [12]-[16], and continues to interest scientists up to nowadays. As an example, it is applied to study non-local or time-dependent processes, as well as to model phenomena involving coarsegrained, fractal spaces and fractional systems as well as more simple systems such as harmonic oscillator [15]-[16]. Most of the models of interacting quantum oscillators are related with solids such as ionic crystals containing localized light particles oscillating in the field created by heavy ionic complexes. The energy spectrum is obtained by the ladder operators method, similar to the quantum harmonic oscillator problem. A lattice at a nonzero temperature has an energy that is not constant, but fluctuates randomly about some mean value. The thermodynamics properties and the quantum radiation maybe also examinated closely. That makes the study of the oscillator in the lattice, as a very interesting ingredient to understand physics beyond continuous limit. We note that, several points of view are developed and are represented as the generalizations of the Heisenberg algebra to a discrete space. There are many lattice models which is reduced, classically to the same continuum theory in the zero lattice spacing, and this includes the q-deformations and those extensions [22]-[30].

Our aim in the following work is to solve the quantum dynamics in the general NC discrete space and determine the eigenvalue problem of the corresponding oscillator Hamiltonian. Thus the paper is organized as follows: In section (2) we briefly review some definitions and properties concerning the discrete differential calculus and it application to quantum mechanics. We also introduce the noncommutativity in this discrete space and show how the Heisenberg uncertainly relations are modified. Section (3) is devoted to the study of $2d$ lattice harmonic oscillator in both commutative and NC quantum space. The eigenvalue problem is solved in these two different cases. The thermodynamic behavior is also point out in this section. In section (4) we provide some remarks and conclusion of our work. The direct computation of the states and energies of the oscillator in ordinary quantum space performed using the Ladder operator method in [29] is also discussed.

2 Discrete differential calculus and lattice quantum mechanics

In this section, we review some basis properties of the differential calculus on a $2d$ lattice (particularly we consider the case where $d = 1$). It is based on the work in [3]. For more details see also [1]-[2] and reference therein. The lattice is the subset $\Gamma = ([0, \ell] \times [0, \ell]) \cap \mathbb{Z}^2)$ of the plane $\mathbb{R}^2$ endowed with the discrete points $M_{n,m} := M(x_n, y_m)$ such that the coordinates $\{x_n\}_n$ and $\{y_m\}_m$, $n, m \in \mathbb{N}$ are spacing by $\varepsilon << 1$ and ε have the dimension of Planck lenght: $[[\varepsilon]] \equiv [[l_p]]$. We write $x_n = n\varepsilon$ and $y_m = m\varepsilon$, (see figure (1)). Without all confusion we use the simple notation $x_n := x$ and $y_m := y$.

The most obvious choice for discretize the continuous derivatives is to use the discrete symmetric derivative: Naturally, the derivative $\partial_x := \frac{\partial}{\partial x}$ and $\partial_y := \frac{\partial}{\partial y}$ are replaced by the forward and backward difference operators $\mathbb{d}_j^+$ and $\mathbb{d}_j^-$, $j = x, y$ so called the left and right non-differential operators acting on the two variables dependent function $f(x, y)$ as

$$\mathbb{d}_x^+ f(x,y) = \frac{1}{\varepsilon}\Big[f(x+\varepsilon, y) - f(x,y)\Big], \quad \mathbb{d}_x^- f(x,y) = \frac{1}{\varepsilon}\Big[f(x,y) - f(x-\varepsilon, y)\Big], \tag{2}$$

$$\mathbb{d}_y^+ f(x,y) = \frac{1}{\varepsilon}\Big[f(x, y+\varepsilon) - f(x,y)\Big], \quad \mathbb{d}_y^- f(x,y) = \frac{1}{\varepsilon}\Big[f(x,y) - f(x, y-\varepsilon)\Big]. \tag{3}$$

Remark that, in the limit $\varepsilon \to 0$, $\mathbb{d}_x^+ = \mathbb{d}_x^- = \partial_x$ and $\mathbb{d}_y^+ = \mathbb{d}_y^- = \partial_y$. All the computations performed here can be generalized to arbitrary dimensions $d > 1$. The operators $\mathbb{d}_x^+$, $\mathbb{d}_x^-$, $\mathbb{d}_y^+$, $\mathbb{d}_y^-$ are related to the translation operators in x and y directions denoted by τ_x^ε and τ_y^ε with group parameter ε as:

$$\tau_x^\varepsilon = e^{\varepsilon \partial_x}, \quad \tau_y^\varepsilon = e^{\varepsilon \partial_y}, \tag{4}$$

and such that $\tau_x^\varepsilon f(x,y) = f(x+\varepsilon, y)$ and $\tau_y^\varepsilon f(x,y) = f(x, y+\varepsilon)$. We get the followings identities:

$$\mathbb{d}_j^+ = \frac{1}{\varepsilon}\big(\tau_j^\varepsilon - \mathbb{1}\big), \quad \mathbb{d}_j^- = -\frac{1}{\varepsilon}\big(\tau_j^{-\varepsilon} - \mathbb{1}\big), \quad j = x, y. \tag{5}$$

The followings generalized Leibnitz rule are well satisfied:

$$\mathbb{d}_j^+(fg) = \frac{1}{\varepsilon}\big(\tau_j^\varepsilon f \tau_j^\varepsilon g - fg\big) = g\mathbb{d}_j^+ f + f\mathbb{d}_j^+ g + \varepsilon \mathbb{d}_j^+ f \mathbb{d}_j^+ g \tag{6}$$

$$\mathbb{d}_j^-(fg) = \frac{1}{\varepsilon}\Big(fg - \tau_j^{-\varepsilon} f \tau_j^{-\varepsilon} g\Big) = g\mathbb{d}_j^- f + f\mathbb{d}_j^- g - \varepsilon \mathbb{d}_j^- f \mathbb{d}_j^- g, \tag{7}$$

which are reduced to usualy Leibnitz rule in the limit where $\varepsilon \to 0$. One can also define the discrete Laplacian as

$$\mathbb{d}^2 = \mathbb{d}_x^+ \mathbb{d}_x^- + \mathbb{d}_y^+ \mathbb{d}_y^- = \frac{2}{\varepsilon^2}\Big[\cosh(\varepsilon\partial_x) + \cosh(\varepsilon\partial_y) - 2\Big]. \tag{8}$$

This quantity plays an important role to defined the kinetic part for the Hamiltonian in classical or quantum description of dynamic systems on the $2d$ lattice i.e. $\hat{H} = \Omega \mathbb{d}^2 + \hat{V}(x,y)$, $\Omega \in \mathbb{R}$ and $\hat{V}(x,y)$ is the interaction potential.

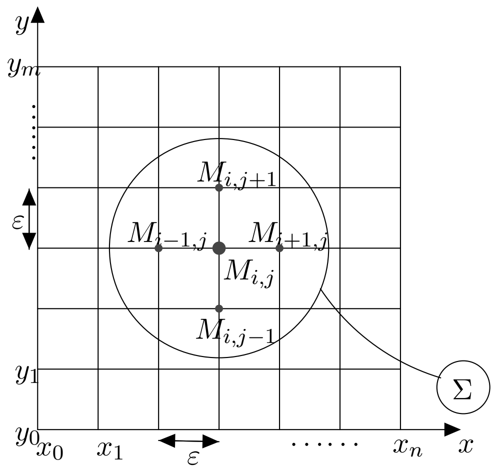


Figure 1: Two dimensions lattice and the representation of the subset Σ with gives the interactions of the point $M_{i,j}$ with his neighbours

We are ready to defined the relation between non differential geometry and quantum mechanics on the lattice (for more detail see [1]-[3]). Let us consider the Hilbert space $\mathcal{H}$ endowed with the Hermitian structure

$$\langle f|g\rangle = \varepsilon^2 \sum_{x,y} f^*(x,y)g(x,y). \tag{9}$$

Consider the subset $\mathcal{H}_\otimes$ of $\mathcal{H}$ in which the states $f(x,y)$ can be decomposed into $f_1(x)\otimes f_2(y)$. Then $\mathcal{H}_\otimes = \mathcal{H}_x \otimes \mathcal{H}_y \equiv L^2(\mathbb{R}, dx) \bigotimes L^2(\mathbb{R}, dx)$ and the tensors product operators $\mathbb{d}^+ = \mathbb{d}^+_x \otimes \mathbb{d}^+_y$ and $\mathbb{d}^- = \mathbb{d}^-_x \otimes \mathbb{d}^-_y$ acting on $\mathcal{H}_\otimes$ are not Hermitian. We find that $\left(i\mathbb{d}^+\right)^\dagger = i\mathbb{d}^-$. However the operators $\mathbb{d}^-\mathbb{d}^+ = \mathbb{d}^+\mathbb{d}^-$, which corresponds to the laplacian (8) is Hermitian. Now let us consider the positions and momentums operators $\hat{x}^\varepsilon$ and $\hat{y}^\varepsilon$, defined by the eigen-equations

$$\left(\hat{x}^\varepsilon f\right)(x,y) = xf(x,y), \quad \left(\hat{y}^\varepsilon f\right)(x,y) = yf(x,y) \tag{10}$$

and the momentum $\hat{p}^\varepsilon_x$ and $\hat{p}^\varepsilon_y$ as

$$\left(\hat{p}^\varepsilon_x f\right)(x,y) = -\frac{i\hbar}{2}\left(\mathbb{d}^+_x + \mathbb{d}^-_x\right)f(x,y) = -\frac{i\hbar}{\varepsilon}\sinh(\varepsilon\partial_x)f(x,y) \tag{11}$$

$$\left(\hat{p}^\varepsilon_y f\right)(x,y) = -\frac{i\hbar}{2}\left(\mathbb{d}^+_y + \mathbb{d}^-_y\right)f(x,y) = -\frac{i\hbar}{\varepsilon}\sinh(\varepsilon\partial_y)f(x,y). \tag{12}$$

The operators (11) and (12) are Hermitian and have the nonvanish commutation relations:

$$\left[\hat{x}^\varepsilon, \hat{p}^\varepsilon_x\right] = i\hbar\cosh\left(\varepsilon\partial_x\right), \quad \left[\hat{y}^\varepsilon, \hat{p}^\varepsilon_y\right] = i\hbar\cosh\left(\varepsilon\partial_y\right). \tag{13}$$

As a quantum system the space of states of a physical model defined with the Hamiltonian $H = \frac{|\hat{p}^\varepsilon|^2}{2m} + V(|\hat{x}^\varepsilon|)$ should thus provide a linear representation space of the generalized Heisenberg algebra, equiped with the Hermitean inner product (9) for with these two operators be self-adjoint. The uncertainly relation is now generalized as

$$\Delta\hat{x}^\varepsilon\Delta\hat{p}^\varepsilon_x \geq \frac{\hbar}{2}\Big|\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\Big(\frac{\varepsilon}{\hbar}\Big)^{2n}\langle\hat{p}^{2n}_x\rangle\Big|, \quad \Delta\hat{y}^\varepsilon\Delta\hat{p}^\varepsilon_y \geq \frac{\hbar}{2}\Big|\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\Big(\frac{\varepsilon}{\hbar}\Big)^{2n}\langle\hat{p}^{2n}_y\rangle\Big| \tag{14}$$

where $\hat{p}_x = -i\hbar\partial_x$, $\hat{p}_y = -i\hbar\partial_y$ corresponding to the momentum operators in ordinary quantum mechanics, the first order expansion of the relations (14) gives

$$\Delta\hat{x}^\varepsilon\Delta\hat{p}^\varepsilon_x \geq= \frac{\hbar}{2}\Big(1+\frac{\varepsilon^2}{2\hbar^2}|\langle\hat{p}_x^2\rangle| + \mathcal{O}(\varepsilon^2)\Big), \quad \Delta\hat{y}^\varepsilon\Delta\hat{p}^\varepsilon_y \geq= \frac{\hbar}{2}\Big(1+\frac{\varepsilon^2}{2\hbar^2}|\langle\hat{p}_y^2\rangle| + \mathcal{O}(\varepsilon^2)\Big). \tag{15}$$

Recall that we are interested in investigating the behavior of the oscillator model in noncommutative space. For this purpose, we must to determine the energy spectrum of the Hamiltonian for the small value of the spacing ε. We use the "*capital*" notation to specify the NC quantum operators, such as coordinates and momentums. Suppose that the NC variables are related to the commutative coordinates operators by the relations:

$$\hat{X}^\varepsilon = \hat{x}^\varepsilon - \frac{\theta}{2\hbar}\hat{p}^\varepsilon_y, \quad \hat{Y}^\varepsilon = \hat{y}^\varepsilon + \frac{\theta}{2\hbar}\hat{p}^\varepsilon_x \tag{16}$$

$$\hat{P}^\varepsilon_x = \hat{p}^\varepsilon_x + \frac{\bar{\theta}}{2\hbar}\hat{y}^\varepsilon, \quad \hat{P}^\varepsilon_y = \hat{p}^\varepsilon_y - \frac{\bar{\theta}}{2\hbar}\hat{x}^\varepsilon \tag{17}$$

The commutation relations between coordinates and momentums are then taking to be:

$$\Big[\hat{X}^\varepsilon, \hat{Y}^\varepsilon\Big] = i\theta\hat{I}^\varepsilon_0, \ \Big[\hat{P}^\varepsilon_x, \hat{P}^\varepsilon_y\Big] = i\bar{\theta}\hat{I}^\varepsilon_0, \ \Big[\hat{X}^\varepsilon, \hat{P}^\varepsilon_x\Big] = i\hbar\hat{I}^\varepsilon_1, \ \Big[\hat{Y}^\varepsilon, \hat{P}^\varepsilon_y\Big] = i\hbar\hat{I}^\varepsilon_2, \tag{18}$$

where the operators $\hat{I}^\varepsilon_0, \hat{I}^\varepsilon_1, \hat{I}^\varepsilon_2$ are given by

$$\hat{I}^\varepsilon_0 = \frac{1}{2}\Big[\cosh\big(\varepsilon\partial_x\big) + \cosh\big(\varepsilon\partial_y\big)\Big], \tag{19}$$

$$\hat{I}^\varepsilon_1 = \cosh\big(\varepsilon\partial_x\big) + \frac{\theta\bar{\theta}}{4\hbar^2}\cosh\big(\varepsilon\partial_y\big), \tag{20}$$

$$\hat{I}^\varepsilon_2 = \cosh\big(\varepsilon\partial_y\big) + \frac{\theta\bar{\theta}}{4\hbar^2}\cosh\big(\varepsilon\partial_x\big). \tag{21}$$

The followings uncertainly relations are well satisfied:

$$\Delta\hat{X}^\varepsilon\Delta\hat{Y}^\varepsilon \geq \frac{\theta}{4}\Big|\sum_{n=0}^{\infty}\frac{\varepsilon^{2n}}{(2n)!}\Big(\langle\partial_x^{2n}\rangle + \langle\partial_y^{2n}\rangle\Big)\Big|, \quad \Delta\hat{P}^\varepsilon_x\Delta\hat{P}^\varepsilon_y \geq \frac{\bar{\theta}}{4}\Big|\sum_{n=0}^{\infty}\frac{\varepsilon^{2n}}{(2n)!}\Big(\langle\partial_x^{2n}\rangle + \langle\partial_y^{2n}\rangle\Big)\Big| \tag{22}$$

$$\Delta\hat{X}^\varepsilon\Delta\hat{P}^\varepsilon_x \geq \frac{\hbar}{2}\Big|\sum_{n=0}^{\infty}\frac{\varepsilon^{2n}}{(2n)!}\Big(\langle\partial_x^{2n}\rangle + \frac{\theta\bar{\theta}}{4\hbar^2}\langle\partial_y^{2n}\rangle\Big)\Big|, \quad \Delta\hat{Y}^\varepsilon\Delta\hat{P}^\varepsilon_y \geq \frac{\hbar}{2}\Big|\sum_{n=0}^{\infty}\frac{\varepsilon^{2n}}{(2n)!}\Big(\frac{\theta\bar{\theta}}{4\hbar^2}\langle\partial_x^{2n}\rangle + \langle\partial_y^{2n}\rangle\Big)\Big| \tag{23}$$

The first order perturbation gives:

$$\Delta\hat{X}^\varepsilon\Delta\hat{Y}^\varepsilon \geq \frac{\theta}{4}\Big[1 - \frac{\varepsilon^2}{2\hbar^2}(p_x^2 + p_y^2)\Big], \quad \Delta\hat{P}^\varepsilon_x\Delta\hat{P}^\varepsilon_y \geq \frac{\bar{\theta}}{4}\Big[1 - \frac{\varepsilon^2}{2\hbar^2}(p_x^2 + p_y^2)\Big] \tag{24}$$

$$\Delta\hat{X}^\varepsilon\Delta\hat{P}^\varepsilon_x \geq \frac{\hbar}{2}\Big[1 + \frac{\theta\bar{\theta}}{4\hbar^2} - \frac{\varepsilon^2}{2\hbar^2}\Big(p_x^2 + \frac{\theta\bar{\theta}}{4\hbar^2}p_y^2\Big)\Big], \quad \Delta\hat{Y}^\varepsilon\Delta\hat{P}^\varepsilon_y \geq \frac{\hbar}{2}\Big[1 + \frac{\theta\bar{\theta}}{4\hbar^2} - \frac{\varepsilon^2}{2\hbar^2}\Big(p_y^2 + \frac{\theta\bar{\theta}}{4\hbar^2}p_x^2\Big)\Big] \tag{25}$$

which corresponds to the uncertainty relation for physics at the Planck scale predicted by the string theory, and as is readily checked it implies a minimal uncertainly in the positions $\Delta\hat{X}^\varepsilon$ and $\Delta\hat{Y}^\varepsilon$ namely $\Delta\hat{X}^\varepsilon_0$ and $\Delta\hat{Y}^\varepsilon_0$ given by $\Delta\hat{X}^\varepsilon_0 = \Delta\hat{Y}^\varepsilon_0 \propto \varepsilon = l_p$. However, we have a maximal dispersion for the momentums $\Delta\hat{P}^\varepsilon_{x0} = \Delta\hat{P}^\varepsilon_{y0} \propto \frac{1}{\varepsilon\sqrt{\varepsilon}}$, which is infinite in the continuum limit: see [31]-[33] for more details.

3 Perturbation method for the harmonic oscillator on a lattice

In this section, the low energy approximation is given for the harmonic. First we examinate the case of ordinary quantum space defined with the commutation relation (13). The next part is devoted to the same computation where we have to consider noncommutativity in general case given in (18).

3.1 Harmonic oscillator in the ordinary quantum space lattice

Consider the subset Σ of the lattice given in figure (1), in which the point $M_{k,j}$ interact with the four neighbours $M_{k+1,j}$, $M_{k-1,j}$, $M_{k,j+1}$, $M_{k,j-1}$. These interactions are suppose to be harmonic and the Hamiltonian becomes

$$\hat{H}_{\varepsilon,kj} = \frac{1}{2m}\Big[\big(\hat{p}^{\varepsilon}_{x_k}\big)^2 + \big(\hat{p}^{\varepsilon}_{y_j}\big)^2\Big] + \frac{m\omega^2}{2}\Big[\big(\hat{x}^{\varepsilon}_k\big)^2 + \big(\hat{y}^{\varepsilon}_j\big)^2\Big]. \tag{26}$$

The total Hamiltonian that describes the oscillation of all points of the lattice is

$$\hat{H}_{\varepsilon} = \sum_{\{k,j\}} \hat{H}_{\varepsilon,kj} \in L(\mathcal{H}_{\otimes}), \tag{27}$$

where $L(\mathcal{H}_{\otimes})$ is the set of linear operators on the Hilbert space $\mathcal{H}_{\otimes}$. For simplicity, the sum in expression (27) will not be write. Then, using (11) and (12), expression (27) becomes

$$\hat{H}_{\varepsilon} \;=\; -\frac{\hbar^2}{2m\varepsilon^2}\Big[\sinh^2(\varepsilon\partial_x) + \sinh^2(\varepsilon\partial_y)\Big] + \frac{m\omega^2}{2}\Big[(\hat{x}^{\varepsilon})^2 + (\hat{y}^{\varepsilon})^2\Big] \tag{28}$$

Let us turn now to the solution of the eigen-values problem by using the corresponding partial differential equation, which is widely given in [29]:

$$\hat{H}_{\varepsilon}\phi_n(x,y) = E_n\phi_n(x,y). \tag{29}$$

We consider now the wave function in the Fourier space. The coordinates and momentums operators are given by

$$\hat{p}^{\varepsilon}_x = \frac{\hbar}{\varepsilon}\sin(\varepsilon k_x), \quad \hat{p}^{\varepsilon}_y = \frac{\hbar}{\varepsilon}\sin(\varepsilon k_y), \quad \text{and} \quad \hat{x}^{\varepsilon} = i\frac{\partial}{\partial k_x}, \quad \hat{y}^{\varepsilon} = i\frac{\partial}{\partial k_y} \tag{30}$$

where k_x, k_y, taken on the Brillouin zone $]-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]$ are called the quasi-momenta. Now defining the annihilation and creation operators as

$$\hat{a}_x = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x}^{\varepsilon} + i\hat{p}^{\varepsilon}_x), \qquad \hat{a}^{\dagger}_x = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x}^{\varepsilon} - i\hat{p}^{\varepsilon}_x) \tag{31}$$

$$\hat{a}_y = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{y}^{\varepsilon} + i\hat{p}^{\varepsilon}_y), \qquad \hat{a}^{\dagger}_y = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{y}^{\varepsilon} - i\hat{p}^{\varepsilon}_y), \tag{32}$$

which are suppose to diagonalized the Hamiltonian $H_{\varepsilon} = H_{x,\varepsilon} + H_{y,\varepsilon}$, with

$$H_{x,\varepsilon} = \frac{(\hat{p}^{\varepsilon}_x)^2}{2m} + \frac{m\omega^2(\hat{x}^{\varepsilon})^2}{2}, \quad H_{y,\varepsilon} = \frac{(\hat{p}^{\varepsilon}_y)^2}{2m} + \frac{m\omega^2(\hat{y}^{\varepsilon})^2}{2}. \tag{33}$$

Let $\phi_0 = \phi_{0x} \otimes \phi_{0y}$ is the fundamental eigen-state such that $\hat{a}_x \otimes \hat{a}_y(\phi_{0x} \otimes \phi_{0y}) = \hat{a}_x \phi_{0x} \otimes \hat{a}_y \phi_{0y} = 0$. It leads to the solution

$$\phi_{0x} = C_x \exp\Big[\frac{\hbar}{m\omega\varepsilon^2}\cos(\varepsilon k_x)\Big], \quad \phi_{0y} = C_y \exp\Big[\frac{\hbar}{m\omega\varepsilon^2}\cos(\varepsilon k_y)\Big], \tag{34}$$

where the constants C_x and C_y, are given, using the normalization condition

$$\int_{-\pi/\varepsilon}^{+\pi/\varepsilon} dk_x\, \phi_{0x}^2 = \int_{-\pi/\varepsilon}^{+\pi/\varepsilon} dk_y\, \phi_{0y}^2 = 1, \quad C_x = C_y = \Big[2\pi J_0\Big(\frac{2i\hbar}{m\omega\varepsilon^2}\Big)\Big]^{-\frac{1}{2}}, \tag{35}$$

such that

$$\phi_0(k_x, k_y) = \Big[2\pi J_0\Big(\frac{2i\hbar}{m\omega\varepsilon^2}\Big)\Big]^{-1} \exp\Big[\frac{\hbar}{m\omega\varepsilon^2}\cos(\varepsilon k_x)\Big] \otimes \exp\Big[\frac{\hbar}{m\omega\varepsilon^2}\cos(\varepsilon k_y)\Big] \tag{36}$$

where J_0 stands for the first kind Bessel function. Replacing the solution (36) in the eigen-value equation (29), the fundamental energy becomes

$$E_0 = \frac{\hbar\omega}{2}(\cos(\varepsilon k_x) + \cos(\varepsilon k_y)). \tag{37}$$

The others states maybe determine order by order using the creation operators $\hat{a}_x \otimes \hat{a}_y$. Let us now comment this result. The limit $k_x, k_y \to 0$ is not well defined using (36). Also, the energy $E_0(k_x, k_y)$ depends on k_x and k_y, which means that after inverse Fourier transformation we get

$$\begin{aligned} E_0(x, y) &= \frac{1}{(2\pi)^2}\Big|\int_{-\frac{\pi}{\varepsilon}}^{\frac{\pi}{\varepsilon}} e^{i(k_x x + k_y y)} E_0(k_x, k_y) dk_x dk_y\Big| \\ &= \frac{\hbar\omega}{2\pi^2}\Big|\frac{x\sin(\pi x/\varepsilon)\sin(\pi y/\varepsilon)}{y(\varepsilon^2 - x^2)} + \frac{y\sin(\pi y/\varepsilon)\sin(\pi x/\varepsilon)}{x(\varepsilon^2 - y^2)}\Big|, \end{aligned} \tag{38}$$

which depends on the coordinates functions x and y. It should also be noted that the continuous limit is given by

$$\lim_{\varepsilon\to 0} E_0(x, y) = \hbar\omega\delta(x)\delta(y), \tag{39}$$

which is also not well defined as ground state energy of the harmonic oscillator, owing to the presence of the Dirac delta function. Thereby, to fill these gaps, we'll consider the perturbation method to derived the eigen-value equation (29). Considering the following Taylor expansion

$$\sinh^2(\varepsilon\partial_x) + \sinh^2(\varepsilon\partial_y) \approx \varepsilon^2(\partial_x^2 + \partial_y^2) + \frac{\varepsilon^4}{3}(\partial_x^4 + \partial_y^4) + \mathcal{O}(\varepsilon^5). \tag{40}$$

The first order expansion to ε^2 of the Hamiltonian H_ε takes the form

$$\begin{aligned} \hat{H}_\varepsilon &= -\frac{\hbar^2}{2m}\Big(\partial_x^2 + \partial_y^2\Big) + \frac{m\omega^2}{2}\Big[\big(\hat{x}^\varepsilon\big)^2 + \big(\hat{y}^\varepsilon\big)^2\Big] - \frac{\varepsilon^2\hbar^2}{6m}\Big(\partial_x^4 + \partial_y^4\Big) + \mathcal{O}(\varepsilon^2) \\ &= \hat{H}_0 + \varepsilon^2\hat{W} + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{41}$$

$\hat{H}_0$ corresponds to the harmonic oscillator Hamiltonian in ordinary space and $\hat{W}$ is considered to be the perturbation term. Thus, we can introduce the new o annihilation and creation operators as:

$$\hat{b}_x = \frac{1}{\sqrt{2\hbar m\omega}}\Big(i\hat{p}_x + m\omega\hat{x}^\varepsilon\Big), \quad \hat{b}_x^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}\Big(-i\hat{p}_x + m\omega\hat{x}^\varepsilon\Big) \tag{42}$$

$$\hat{b}_y = \frac{1}{\sqrt{2\hbar m\omega}}\Big(i\hat{p}_y + m\omega\hat{y}^\varepsilon\Big), \quad \hat{b}_y^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}\Big(-i\hat{p}_y + m\omega\hat{y}^\varepsilon\Big), \tag{43}$$

such that the canonical commutation relation $\left[\hat{b}_i, \hat{b}_j^\dagger\right] = \mathbb{1}\delta_{ij}$ is satisfied, and $\hat{H}_0$ and $\hat{W}$ take the form

$$\hat{H}_0 = \hbar\omega\Big(\hat{b}_x\hat{b}_x^\dagger + \hat{b}_y\hat{b}_y^\dagger - 1\Big), \quad \hat{W} = -\frac{m\omega^2}{24}\Big((\hat{b}_x - \hat{b}_x^\dagger)^4 + (\hat{b}_y - \hat{b}_y^\dagger)^4\Big). \tag{44}$$

One constructs the Fock states as $\{|n_x, n_y; 0\rangle = |n_x\rangle \otimes |n_y\rangle \in \mathfrak{H},\ n_x, n_y \in \mathbb{N}\}$ such that the followings relations hold:

$$\hat{b}_x \otimes \hat{b}_y|n_x, n_y; 0\rangle = \sqrt{n_x n_y}|n_x - 1, n_y - 1; 0\rangle, \quad \hat{b}_x \otimes \hat{b}_y|0, 0; 0\rangle = 0 \tag{45}$$

$$\hat{b}_x^\dagger \otimes \hat{b}_y^\dagger|n_x, n_y; 0\rangle = \sqrt{(n_x + 1)(n_y + 1)}|n_x + 1, n_y + 1; 0\rangle. \tag{46}$$

The eigen-equation $\hat{H}_0|n_x, n_y; 0\rangle = E^0_{n_x,n_y}|n_x, n_y; 0\rangle$, gives $n_x + n_y + 1$ degenerate states such that $E^0_{n_x,n_y} = \hbar\omega(n_x + n_y + 1)$. We denote these degenerate states $|n_x, n_y; 0\rangle^j$, such that

$$|n_x, n_y; 0\rangle^j = \Big\{|n_x + n_y - j, j; 0\rangle;\ j = 0, 1, 2, \cdots, n_x + n_y\Big\}. \tag{47}$$

Consider now the vector $|N\rangle = \sum_{j=0}^n c_j|n_x, n_y; 0\rangle^j + \varepsilon^2|n_x, n_y; 1\rangle + \mathcal{O}(\varepsilon^2)$ and the energy $E_{\varepsilon,n_x,n_y} = E^0_{n_x,n_y} + \varepsilon^2 E^1_{n_x,n_y} + \mathcal{O}(\varepsilon^2)$, which solve the eigen-problem $\hat{H}_\varepsilon|N\rangle = E_{\varepsilon,n_x,n_y}|N\rangle$, with $n = n_x + n_y$ and $c_j \in \mathbb{C}, j = 0, 1, \cdots, n+1$. We have the orthogonality relation $\sum_{j=0}^n c_j\ {}^k\langle 0; n_x, n_y|n_x, n_y; 0\rangle^j = \delta^{jk}$ and the first order correction of the energy i.e. $E^1_{n_x,n_y}$ is determined by the following linear homogeneous system

$$\sum_{j=0}^n c_j\ {}^k\langle 0; n_x, n_y|\hat{W}|n_x, n_y; 0\rangle^j = E^1_{n_x,n_y} c_k. \tag{48}$$

While the above system is completely determine by the matrix G such that $\det G = 0$

$$G : \begin{cases} G_{kk} = {}^k\langle 0; n_x, n_y|\hat{W}|n_x, n_y; 0\rangle^k - E^1_{n_x,n_y} \\ G_{kj} = {}^k\langle 0; n_x, n_y|\hat{W}|n_x, n_y; 0\rangle^j \quad \text{for} \quad k \neq j \end{cases} \tag{49}$$

A few computation shows that G is diagonal matrix and the diagonal elements G_{kk} are given by

$$G_{kk} = -\frac{m\omega^2}{4}\Big(n(n+1) - 2kn + 2k^2 + 1\Big) - E^1_n \tag{50}$$

where $E^1_n = E^1_{n_x,n_y}$, satisfies the following equation

$$\prod_{k=0}^n \Big[-\frac{m\omega^2}{4}\Big(n^2 + n(1-2k) + 2k^2 + 1\Big) - E^{1k}_n\Big] = 0. \tag{51}$$

The index "k" in E^{1k}_n is used to specify the degeneracy, such that

$$E^k_{\varepsilon,n} = \hbar\omega\Big(n+1\Big) - \frac{m\omega^2\varepsilon^2}{4}\Big[n^2 + n(1-2k) + 2k^2 + 1\Big] + \mathcal{O}(\varepsilon^2). \tag{52}$$

$|n_x, n_y; 1\rangle$ can be computed in the same manner. Let $\mathcal{D} = \{n - j;\ 0 \leq j \leq n\}$ and $\mathcal{D}' = \{j;\ 0 \leq j \leq n\}$, we get

$$|n_x, n_y; 1\rangle = -\sum_{j=0}^n \sum_{\substack{m_x,m_y \\ m_x \notin \mathcal{D} \\ m_y \notin \mathcal{D}'}} c_j \frac{\langle 0; m_x, m_y|\hat{W}|n_x, n_y; 0\rangle^j}{E^0_{m_x,m_y} - E^0_{n_x,n_y}}|m_x, m_y; 0\rangle = 0. \tag{53}$$

Quantities	Formulas
Partition function	$Z(T,\varepsilon) = \frac{1}{4\sinh\left(\frac{\beta\hbar\omega}{2}\right)}\left(1+\frac{m\omega^2\varepsilon^2\beta}{4}\coth^2\left(\frac{\beta\hbar\omega}{2}\right)\right)+\mathcal{O}(\varepsilon^2)$
Free energy	$F(T,\varepsilon) = \frac{2}{\beta}\log\left[2\sinh\left(\frac{\beta\hbar\omega}{2}\right)\right] - \frac{m\omega^2\varepsilon^2}{4}\coth^2\left(\frac{\beta\hbar\omega}{2}\right)+\mathcal{O}(\varepsilon^2)$
Entropy	$S(T,\varepsilon) = -2k_B\log\left(2\sinh(\frac{\beta\hbar\omega}{2})\right)+k_B\hbar\omega\beta\coth(\frac{\beta\hbar\omega}{2})+\frac{1}{4}mk_B\hbar\omega^3\beta^2\varepsilon^2\frac{\cosh(\frac{\beta\hbar\omega}{2})}{\sinh^3(\frac{\beta\hbar\omega}{2})}+\mathcal{O}(\varepsilon^2)$
Internal energy	$U(T,\varepsilon) = \hbar\omega\coth(\frac{\beta\hbar\omega}{2})\left[1-\frac{m\omega^2\varepsilon^2}{4}\left(\frac{1}{\hbar\omega}\coth(\frac{\beta\hbar\omega}{2})+\frac{\beta}{\sinh^2(\frac{\beta\hbar\omega}{2})}\right)\right]+\mathcal{O}(\varepsilon^2)$
Heat capacity	$C_v(T,\varepsilon) = -k_B\left(\frac{\hbar\omega\beta/2}{\sinh^2(\frac{\hbar\omega\beta}{2})}\right)^2\left[1+m\omega^2\varepsilon^2\beta-(1-\frac{1}{2}m\omega^2\varepsilon^2\beta)\cosh(\hbar\omega\beta)\right]+\mathcal{O}(\varepsilon^2)$

Figure 2: The thermodynamic quantities of the lattice oscillator.

Let us now deal with the thermodynamic behavior of the oscillator model. The useful ingredient for this study in the partition function $Z^k(T,\varepsilon)$ depending with the degeneracy index k as:

$$Z^k(T,\varepsilon) = \sum_{n=0}^{\infty}(n+1)e^{-\beta E^k_{\varepsilon,n}}, \quad \beta = \frac{1}{k_BT}, \quad k_B \text{ is the Boltzmann constant.} \tag{54}$$

The full partition function is the sum under all degeneracies terms as $Z(T,\varepsilon) = \sum_{k=0}^{n} Z^k(T,\varepsilon)$. From statistical mechanics point of view, the probability $p(i,j)$ of finding a system in a state $|i,j\rangle$ is given by the Boltzmann formula

$$p(i,j) = \frac{e^{-\beta E^k_{\varepsilon,n}(i,j)}}{Z(T,\varepsilon)}. \tag{55}$$

Thereby the others thermodynamic quantities such as the free enery $F = -\frac{1}{\beta}\log Z(T,\varepsilon)$, the entropy $S = -\frac{\partial F}{\partial T}$, the internal energy $U = -\frac{\partial \ln Z}{\partial \beta}$, the heat capacity $C_v = \frac{\partial U}{\partial T}$ are given in the table built in the figure (2). We give the asymptotic behavior of the various functions at low temperatures $T << \hbar\omega$. The free energy and the internal energy is reduced to

$$F(T << \hbar\omega,\varepsilon) = U(T << \hbar\omega,\varepsilon) \approx \hbar\omega - \frac{m\omega^2\varepsilon^2}{4}, \tag{56}$$

which corresponds to the ground state energy. The continuum limit $\varepsilon = 0$ is also well satisfied and the result is very close to what is obtained in thermodynamic quantum mechanics of the harmonic oscillator. On the other hand a divergence appears at high temperatures $T >> \hbar\omega$. This behavior is illustrated in the figures (3) and (4). The same analysis can be performed for the entropy S and the heat capacity C_v.

3.2 Harmonic oscillator in a noncommutative lattice

In this subsection we consider the case where both momentums and coordinates are noncommutative, as presented in the equations (18). We will show that in particular case $\bar{\theta} = -m^2\omega^2\theta$ the eigen-value problem can be determined as in the previous section. Some of the most ambiguous aspect i.e. the case where $\bar{\theta} \neq -m^2\omega^2\theta$ is also considered and studied. The Hamiltonian of the oscillator in the first order of ε^2 is splited into:

$$\hat{\mathfrak{H}}_\varepsilon = \hat{\mathfrak{H}}_0 + \varepsilon^2\hat{\mathfrak{W}} + \mathcal{O}(\varepsilon^2) \tag{57}$$

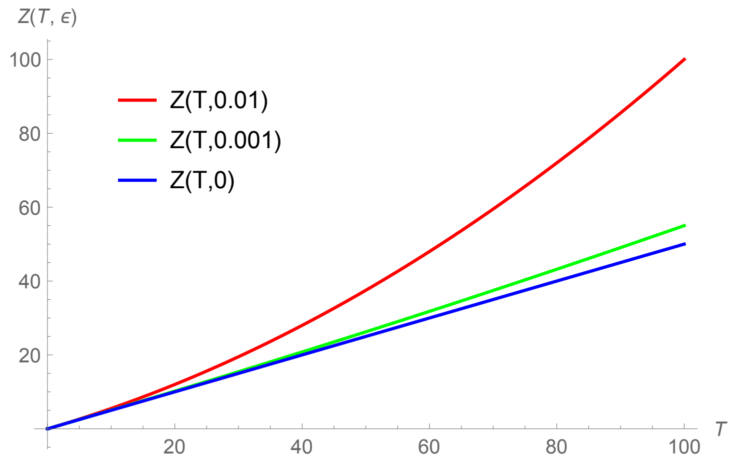
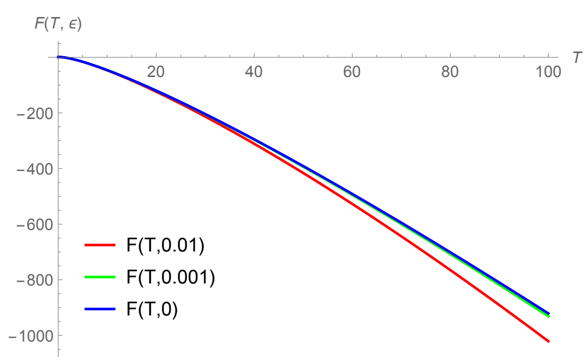


Figure 3: Plot of the partition function for different values of the lattice spacing ε.



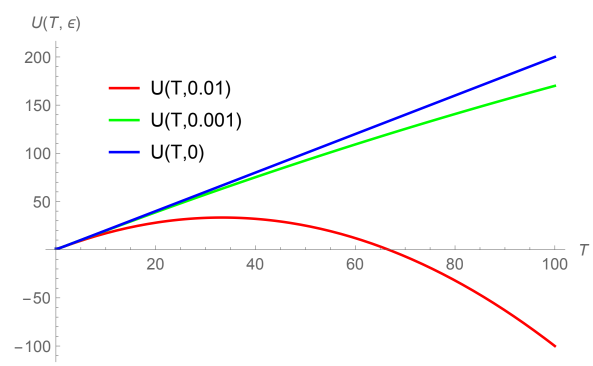


Figure 4: Plot of the free energy and the internal energy for different values of the lattice spacing ε. We find that for low temperatures, $\forall \varepsilon$ small the thermodynamic quantities coincide with what it was supposed to be when $\varepsilon = 0$ (the continuum limit). The divergence appears for the high temperatures as we can remark in these figures.

where $\hat{\mathfrak{H}}_0$ is assume to be the nonperturbative Hamiltonian and $\hat{\mathfrak{W}}$ stands for the perturbation:

$$\begin{aligned} \hat{\mathfrak{H}}_0 &= \frac{\Omega^2}{2m}(\hat{p}_x^2+\hat{p}_y^2)+\frac{m\omega^2\bar{\Omega}^2}{2}\Big((\hat{x}^\varepsilon)^2+(\hat{y}^\varepsilon)^2\Big)+\frac{\widetilde{\Omega}}{m}(\hat{y}^\varepsilon\hat{p}_x-\hat{x}^\varepsilon\hat{p}_y), & (58)\\ \hat{\mathfrak{W}} &= -\frac{1}{6m\hbar^2}\Big[\Omega^2(\hat{p}_x^4+\hat{p}_y^4)+\tilde{\Omega}(\hat{y}^\varepsilon\hat{p}_x^3-\hat{x}^\varepsilon\hat{p}_y^3)\Big]. & (59)\end{aligned}$$

The parameters Ω, $\bar{\Omega}$, $\widetilde{\Omega}$ are given by

$$\Omega^2=1+\frac{m^2\omega^2\theta^2}{4\hbar^2},\quad \bar{\Omega}^2=1+\frac{\bar{\theta}^2}{4m^2\hbar^2\omega^2},\quad \widetilde{\Omega}=\frac{\bar{\theta}}{2\hbar}+\frac{m^2\omega^2\theta}{2\hbar}. \tag{60}$$

The more general result concerning the eigenvalues probem of the Hamiltonian (58) can be obtained by essentially the same method as last section. We construct the annihilation and creation operators ($\hat{\mathfrak{b}}$ and $\hat{\mathfrak{b}}^\dagger$) as follows:

$$\begin{aligned} \hat{\mathfrak{b}}_x &= \frac{1}{\sqrt{2\Omega\bar{\Omega}\hbar m\omega}}\Big(i\Omega\hat{p}_x+m\omega\bar{\Omega}\hat{x}^\varepsilon\Big),\quad \hat{\mathfrak{b}}_x^\dagger=\frac{1}{\sqrt{2\Omega\bar{\Omega}\hbar m\omega}}\Big(-i\Omega\hat{p}_x+m\omega\bar{\Omega}\hat{x}^\varepsilon\Big) & (61)\\ \hat{\mathfrak{b}}_y &= \frac{1}{\sqrt{2\Omega\bar{\Omega}\hbar m\omega}}\Big(i\Omega\hat{p}_y+m\omega\bar{\Omega}\hat{y}^\varepsilon\Big),\quad \hat{\mathfrak{b}}_y^\dagger=\frac{1}{\sqrt{2\Omega\bar{\Omega}\hbar m\omega}}\Big(-i\Omega\hat{p}_y+m\omega\bar{\Omega}\hat{y}^\varepsilon\Big). & (62)\end{aligned}$$

They satisfy the canonical commutation relation $\Big[\hat{\mathfrak{b}}_x,\hat{\mathfrak{b}}_x^\dagger\Big]=\mathbb{1}=\Big[\hat{\mathfrak{b}}_y,\hat{\mathfrak{b}}_y^\dagger\Big]$, and according to which the Hamiltonian $\hat{\mathfrak{H}}_0$ can be factorized as follows:

$$\hat{\mathfrak{H}}_0 = \Omega\bar{\Omega}\hbar\omega\big(\hat{\mathfrak{b}}_x^\dagger\hat{\mathfrak{b}}_x+\hat{\mathfrak{b}}_y^\dagger\hat{\mathfrak{b}}_y+\mathbb{1}\big)-i\frac{\widetilde{\Omega}\hbar}{m}\big(\hat{\mathfrak{b}}_x\hat{\mathfrak{b}}_y^\dagger-\hat{\mathfrak{b}}_x^\dagger\hat{\mathfrak{b}}_y\big). \tag{63}$$

which corresponds to the Hamiltonian of two dimensional Landau problem in the symmetric gauge on NC space. Equivalently the presence of magnetic fields in this relation also suggest a NC structure for the spacetime. The perturbation term is

$$\hat{\mathfrak{W}}=-\frac{\bar{\Omega}\omega}{24\Omega}\Big[\Omega\bar{\Omega}m\omega\Big((\hat{\mathfrak{b}}_x-\hat{\mathfrak{b}}_x^\dagger)^4+(\hat{\mathfrak{b}}_y-\hat{\mathfrak{b}}_y^\dagger)^4\Big)+i\widetilde{\Omega}(\hat{\mathfrak{b}}_y+\hat{\mathfrak{b}}_y^\dagger)(\hat{\mathfrak{b}}_x-\hat{\mathfrak{b}}_x^\dagger)^3-i\widetilde{\Omega}(\hat{\mathfrak{b}}_x+\hat{\mathfrak{b}}_x^\dagger)(\hat{\mathfrak{b}}_y-\hat{\mathfrak{b}}_y^\dagger)^3\Big]. \tag{64}$$

For some purposes, it is useful to remark that the states of the form $|n_x,n_y;0\rangle$ defined in (47) may diagonalyzed the Hamiltonian (63). In the hope to include the perturbation term $\hat{\mathcal{W}}$ in our diagonalization procedure let us consider the new operators:

$$\begin{aligned} \hat{\mathfrak{b}}_+ &= \frac{1}{\sqrt{2}}\big(\hat{\mathfrak{b}}_x+i\hat{\mathfrak{b}}_y\big),\quad \hat{\mathfrak{b}}_+^\dagger=\frac{1}{\sqrt{2}}\big(\hat{\mathfrak{b}}_x^\dagger-i\hat{\mathfrak{b}}_y^\dagger\big), & (65)\\ \hat{\mathfrak{b}}_- &= \frac{1}{\sqrt{2}}\big(\hat{\mathfrak{b}}_x-i\hat{\mathfrak{b}}_y\big),\quad \hat{\mathfrak{b}}_-^\dagger=\frac{1}{\sqrt{2}}\big(\hat{\mathfrak{b}}_x^\dagger+i\hat{\mathfrak{b}}_y^\dagger\big). & (66)\end{aligned}$$

Here also, the canonical commutation relation is well satisfied i.e.: $\Big[\hat{\mathfrak{b}}_+,\hat{\mathfrak{b}}_+^\dagger\Big]=\mathbb{1}=\Big[\hat{\mathfrak{b}}_-,\hat{\mathfrak{b}}_-^\dagger\Big]$. Then the Hamiltonian $\hat{\mathfrak{H}}_0$ and $\hat{\mathfrak{W}}$ becomes

$$\begin{aligned} \hat{\mathfrak{H}}_0 &= \Omega\bar{\Omega}\hbar\omega\Big(\hat{\mathfrak{N}}_++\hat{\mathfrak{N}}_-+\mathbb{1}\Big)-\frac{\widetilde{\Omega}\hbar}{m}\big(\hat{\mathfrak{N}}_--\hat{\mathfrak{N}}_+\big) & (67)\\ \hat{\mathfrak{W}} &= -\frac{\bar{\Omega}\omega}{96\Omega}\Big[\Omega\bar{\Omega}m\omega\Big(A^4+B^4\Big)+\widetilde{\Omega}\Big(AC^3+BD^3\Big)\Big], & (68)\end{aligned}$$

where

$$A = \hat{\mathfrak{b}}_+ - \hat{\mathfrak{b}}_+^\dagger + \hat{\mathfrak{b}}_- - \hat{\mathfrak{b}}_-^\dagger, \quad B = \hat{\mathfrak{b}}_+ + \hat{\mathfrak{b}}_+^\dagger - \hat{\mathfrak{b}}_- - \hat{\mathfrak{b}}_-^\dagger \tag{69}$$

$$C = \hat{\mathfrak{b}}_+ - \hat{\mathfrak{b}}_+^\dagger - \hat{\mathfrak{b}}_- + \hat{\mathfrak{b}}_-^\dagger, \quad D = \hat{\mathfrak{b}}_+ + \hat{\mathfrak{b}}_+^\dagger + \hat{\mathfrak{b}}_- + \hat{\mathfrak{b}}_-^\dagger \tag{70}$$

and $\hat{\mathfrak{N}}_+ = \hat{\mathfrak{b}}_+^\dagger \hat{\mathfrak{b}}_+$ and $\hat{\mathfrak{N}}_- = \hat{\mathfrak{b}}_-^\dagger \hat{\mathfrak{b}}_-$ are the number operators. Let $\{|n_+, n_-; 0\rangle = |n_+\rangle \otimes |n_-\rangle, n_-, n_+ \in \mathbb{N}\}$ be a set of Fock vectors such that

$$\hat{\mathfrak{b}}_+^\dagger |n_+, n_-; 0\rangle = \sqrt{n_+ + 1}|n_+ + 1, n_-; 0\rangle, \quad \hat{\mathfrak{b}}_-^\dagger |n_+, n_-; 0\rangle = \sqrt{n_- + 1}|n_+, n_- + 1; 0\rangle \tag{71}$$

Then we get $\hat{\mathfrak{N}}_+|n_+, n_-; 0\rangle = n_+|n_+, n_-; 0\rangle$, $\hat{\mathfrak{N}}_-|n_+, n_-; 0\rangle = n_-|n_+, n_-; 0\rangle$. The states $|n_+, n_-; 0\rangle$ solve the eigenvalue problem

$$\hat{\mathfrak{H}}_0 |n_+, n_-; 0\rangle = \mathcal{E}_{0,n_+,n_-}|n_+, n_-; 0\rangle \tag{72}$$

with the corresponding energies

$$\mathcal{E}_{0,n_+,n_-} = \Omega\bar{\Omega}\hbar\omega\big(n_+ + n_- + 1\big) - \frac{\widetilde{\Omega}\hbar}{m}\big(n_- - n_+\big) = \Omega\bar{\Omega}\hbar\omega\big(n + 1\big) - \frac{\widetilde{\Omega}\hbar}{m}\jmath. \tag{73}$$

where $n = n_+ + n_-$ and $\jmath = n_- - n_+$. Concerning the perturbation $\hat{\mathfrak{W}}$, it seems that the states $|n_+, n_-; 0\rangle$ are a diagonalyzed basis, in the case where $\bar{\theta} = -m^2\omega^2\theta$, which refer to the solvable condition of the harmonic oscillator in noncommutative space when both momentum and coordinates are suppose to satisfy the nonvanish commutation relations. The Hamiltonian $\hat{\mathfrak{H}}_0$ and $\hat{\mathfrak{W}}$ become

$$\hat{\mathfrak{H}}_0 = \Omega^2\Big[\frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{m\omega^2}{2}\Big((x^\varepsilon)^2 + (y^\varepsilon)^2\Big)\Big] = \Omega^2 \hat{H}_0 \tag{74}$$

$$\hat{\mathfrak{W}} = -\frac{\Omega^2}{6m\hbar^2}(\hat{p}_x^4 + \hat{p}_y^4) = \Omega^2 \hat{W} \tag{75}$$

where $\hat{H}$ and $\hat{W}$ are defined in (41). Then, we find that the eigen-energy of $\hat{\mathfrak{H}}$ is

$$\mathcal{E}_{\varepsilon,n}^k = \Omega^2\Big[\hbar\omega(n+1) - \frac{m\omega^2\varepsilon^2}{4}\Big(n^2 + n(1-2k) + 2k^2 + 1\Big)\Big]. \tag{76}$$

with $k = 0, 1, ..., n$ are the order of degeneracy. Let remark that all the thermodynamic properties derived in the last subsection can also be performed using the spectrum (76) and the same conclusion can be made.

Now we will focus on the case of arbitrary positive values of the parameters θ and $\bar{\theta}$. The first order correction energy $\mathcal{E}_{n_+,n_-}^{1k}$ is obtained by the following system

$$\sum_{\substack{j=0\\ j\neq k}}^{n} c_j \, {}^k\langle\hat{\mathfrak{W}}\rangle^j + c_k\big({}^k\langle\hat{\mathfrak{W}}\rangle^k - \mathcal{E}_{n_+,n_-}^{1k}\big) = 0, \tag{77}$$

where the matrix elements ${}^k\langle\hat{\mathfrak{W}}\rangle^j := {}^k\langle 0; n_+, n_-|\hat{\mathfrak{W}}|n_+, n_-; 0\rangle^j$ is explicitly given by

$${}^k\langle\hat{\mathfrak{W}}\rangle^j = \begin{cases} -\frac{\bar{\Omega}^2 m\omega^2}{8}\Big((n+1)(n+2) + 2kn - 2k^2\Big) - \frac{\widetilde{\Omega}\bar{\Omega}\omega}{8\Omega}(n+1)(n-2k), & j = k, \\ -\frac{\bar{\Omega}^2 m\omega^2}{8}\sqrt{k(k-1)(n-k+1)(n-k+2)}, & j = k-2, \\ -\frac{\bar{\Omega}^2 m\omega^2}{8}\sqrt{(k+1)(k+2)(n-k-1)(n-k)}, & j = k+2, \\ 0 \quad \text{otherwise} \end{cases} \tag{78}$$

What we are interested in is the determinant of the matrix $(\mathfrak{G})$ similar to expression (49):

$$(\mathfrak{G}):\begin{cases}\mathfrak{G}_{k,k}=-\frac{\bar{\Omega}^2m\omega^2}{8}\Big((n+1)(n+2)+2kn-2k^2\Big)-\frac{\widetilde{\Omega}\bar{\Omega}\omega}{8\Omega}(n+1)(n-2k)-\mathcal{E}_n^{1k},\\ \mathfrak{G}_{k,k-2}=-\frac{\bar{\Omega}^2m\omega^2}{8}\sqrt{k(k-1)(n-k+1)(n-k+2)},\\ \mathfrak{G}_{k,k+2}=-\frac{\bar{\Omega}^2m\omega^2}{8}\sqrt{(k+1)(k+2)(n-k-1)(n-k)},\\ 0\quad\text{otherwise}\end{cases}\tag{79}$$

The matrix $\mathfrak{G}$ is not diagonal but it is symmetric, i.e. $\mathfrak{G}_{k,k+2}=\mathfrak{G}_{k+2,k}$. The solution of equation $\det\mathfrak{G}=0$ can no longer be done by direct calculation for arbitrary value of the integer n. We provide here this solution order by order to this quantum number n. Thus the first order corrections of the energy spectrum become

$$n=0:\qquad \mathcal{E}_0^{10}=-\frac{\bar{\Omega}^2m\omega^2}{4}\tag{80}$$

$$n=1:\begin{cases}\mathcal{E}_1^{11}=-\frac{3\bar{\Omega}^2m\omega^2}{4}+\frac{\widetilde{\Omega}\bar{\Omega}\omega}{4\Omega}\\ \mathcal{E}_1^{10}=-\frac{3\bar{\Omega}^2m\omega^2}{4}-\frac{\widetilde{\Omega}\bar{\Omega}\omega}{4\Omega}\end{cases}\tag{81}$$

$$n=2:\begin{cases}\mathcal{E}_2^{12}=-\frac{3\bar{\Omega}^2m\omega^2}{2}+\frac{\omega\bar{\Omega}}{4\Omega}\sqrt{9\widetilde{\Omega}^2+\Omega^2\bar{\Omega}^2m^2\omega^2}\\ \mathcal{E}_2^{11}=-\frac{7\bar{\Omega}^2m\omega^2}{4}\\ \mathcal{E}_2^{10}=-\frac{3\bar{\Omega}^2m\omega^2}{2}-\frac{\omega\bar{\Omega}}{4\Omega}\sqrt{9\widetilde{\Omega}^2+\Omega^2\bar{\Omega}^2m^2\omega^2}\end{cases}\tag{82}$$

$$n=3:\begin{cases}\mathcal{E}_3^{13}=\frac{2\bar{\Omega}\widetilde{\Omega}\omega-11\Omega\bar{\Omega}^2m\omega^2}{4\Omega}+\frac{\bar{\Omega}\omega}{2\Omega}\sqrt{4\widetilde{\Omega}^2+2\Omega\bar{\Omega}\widetilde{\Omega}m\omega+\Omega^2\bar{\Omega}^2m^2\omega^2}\\ \mathcal{E}_3^{12}=-\frac{2\bar{\Omega}\widetilde{\Omega}\omega+11\Omega\bar{\Omega}^2m\omega^2}{4\Omega}+\frac{\bar{\Omega}\omega}{2\Omega}\sqrt{4\widetilde{\Omega}^2-2\Omega\bar{\Omega}\widetilde{\Omega}m\omega+\Omega^2\bar{\Omega}^2m^2\omega^2}\\ \mathcal{E}_3^{11}=\frac{2\bar{\Omega}\widetilde{\Omega}\omega-11\Omega\bar{\Omega}^2m\omega^2}{4\Omega}-\frac{\bar{\Omega}\omega}{2\Omega}\sqrt{4\widetilde{\Omega}^2+2\Omega\bar{\Omega}\widetilde{\Omega}m\omega+\Omega^2\bar{\Omega}^2m^2\omega^2}\\ \mathcal{E}_3^{10}=-\frac{2\bar{\Omega}\widetilde{\Omega}\omega+11\Omega\bar{\Omega}^2m\omega^2}{4\Omega}-\frac{\bar{\Omega}\omega}{2\Omega}\sqrt{4\widetilde{\Omega}^2-2\Omega\bar{\Omega}\widetilde{\Omega}m\omega+\Omega^2\bar{\Omega}^2m^2\omega^2}\end{cases}\tag{83}$$

For $n\geq 4$ the computation of the determinant of $\mathfrak{G}$ leads to the mixing of real and complex values as solution of equation (77). The complex energies cannot be taking into account in our analysis. Moreover for the moment we have no method to classify these solutions. Then we consider only the quantum numbers $n=0,1,2,3$ as given above. The correction of the states $|N\rangle$ namely $|n_+,n_-;1\rangle$ are given using the first order perturbation equation:

$$|n_+,n_-;1\rangle=-\sum_{\substack{\ell_+,\ell_-\\ \ell_+\notin\mathcal{D}\\ \ell_-\notin\mathcal{D}'}}\sum_{j=0}^{n}c_j\frac{\langle 0;\ell_+,\ell_-|\hat{\mathfrak{W}}|n_+,n_-;0\rangle^j}{\mathcal{E}^0_{\ell_+,\ell_-}-\mathcal{E}^0_{n_+,n_-}}|\ell_+,\ell_-;0\rangle,\tag{84}$$

where $\mathcal{D}=\{n_++n_--j;\ 0\leq j\leq n_++n_-\}$, $\mathcal{D}'=\{j;\ 0\leq j\leq n_++n_-\}$. Let $\langle 0;\ell_+,\ell_-|\hat{\mathfrak{W}}|n_+,n_-;0\rangle^j:=\langle\hat{\mathfrak{W}}\rangle^j_{\ell n}$ we can determine $|n_+,n_-;1\rangle$ by replacing in (84) the following relation

$$\langle\hat{\mathfrak{W}}\rangle^j_{\ell n}=\begin{cases}-\frac{\bar{\Omega}^2m\omega^2}{8}\sqrt{j(j-1)(n-j)(n-j-1)} & \ell_-=j-2;\ell_+=n-j-2\\ -\frac{\bar{\Omega}^2m\omega^2}{8}\sqrt{(j+1)(j+2)(n-j+1)(n-j+2)} & \ell_-=j+2;\ell_+=n-j+2\\ (\frac{\widetilde{\Omega}\bar{\Omega}\omega}{8\Omega}(n-2j)+\frac{\bar{\Omega}^2m\omega^2}{4}(n+2))\sqrt{(j+1)(n-j+1)} & \ell_-=j+1;\ell_+=n-j+1\\ (\frac{\bar{\Omega}^2m\omega^2}{12}-\frac{\widetilde{\Omega}\bar{\Omega}\omega}{24\Omega})\sqrt{(j+1)(j+2)(j+3)(n-j)} & \ell_-=j+3;\ell_+=n-j-1\\ (-\frac{\widetilde{\Omega}\bar{\Omega}\omega}{4\Omega}j+(\frac{\bar{\Omega}^2m\omega^2}{4}+\frac{\widetilde{\Omega}\bar{\Omega}\omega}{8\Omega})n)\sqrt{j(n-j)} & \ell_-=j-1;\ell_+=n-j-1\\ (\frac{\bar{\Omega}^2m\omega^2}{12}-\frac{\widetilde{\Omega}\bar{\Omega}\omega}{24\Omega})\sqrt{j(j-2)(j-1)(n-j+1)} & \ell_-=j-3;\ell_+=n-j+1\\ (\frac{\bar{\Omega}^2m\omega^2}{12}+\frac{\widetilde{\Omega}\bar{\Omega}\omega}{24\Omega})\sqrt{(j+1)(n-j)(n-j-1)(n-j-2)} & \ell_-=j+1;\ell_+=n-j-3\\ (\frac{\bar{\Omega}^2m\omega^2}{12}+\frac{\widetilde{\Omega}\bar{\Omega}\omega}{24\Omega})\sqrt{j(n-j+1)(n-j+2)(n-j+3)} & \ell_-=j-1;\ell_+=n-j+3\end{cases}\tag{85}$$

For $n=0$ we get:

$$\text{for } n=0, \quad |0,0;1\rangle = \frac{1}{\sqrt{17}}|2,2;0\rangle - \frac{4}{\sqrt{17}}|1,1;0\rangle. \tag{86}$$

For $n=1$, we get

$$\begin{aligned}|n_+,n_-;1\rangle &= c_0\Big[\frac{\bar{\Omega}^2m^2\omega^2}{8\hbar(2\Omega\bar{\Omega}m\omega+\tilde{\Omega}n_-)}\sqrt{3}\,|3,2;0\rangle - \frac{\tilde{\Omega}\bar{\Omega}m\omega+6\Omega\bar{\Omega}^2m^2\omega^2}{16\Omega\hbar(\Omega\bar{\Omega}m\omega+\tilde{\Omega}n_-)}\sqrt{2}\,|2,1;0\rangle \\ &- \frac{2\Omega\bar{\Omega}^2m^2\omega^2+\tilde{\Omega}\bar{\Omega}m\omega}{48\Omega\hbar(\Omega\bar{\Omega}m\omega+\tilde{\Omega}(n_--2))}\sqrt{6}\,|0,3;0\rangle\Big] + c_1\Big[\frac{\bar{\Omega}^2m^2\omega^2}{8\hbar(2\Omega\bar{\Omega}m\omega+\tilde{\Omega}(n_--1))}\sqrt{3}\,|2,3;0\rangle \\ &- \frac{-\tilde{\Omega}\bar{\Omega}m\omega+6\Omega\bar{\Omega}^2m^2\omega^2}{16\Omega\hbar(\Omega\bar{\Omega}m\omega+\tilde{\Omega}(n_--1))}\sqrt{2}\,|1,2;0\rangle - \frac{2\Omega\bar{\Omega}^2m^2\omega^2+\tilde{\Omega}\bar{\Omega}m\omega}{48\Omega\hbar(\Omega\bar{\Omega}m\omega+\tilde{\Omega}(n_-+1))}\sqrt{6}\,|3,0;0\rangle\Big]\end{aligned} \tag{87}$$

where $n_+=1$ and $n_-=0$ or $n_+=0$ and $n_-=1$. The constants c_0 and c_1 are determined using the normalization conditions $\langle 1;1,0|1,0;1\rangle = 1$ and $\langle 1;0,1|0,1;1\rangle = 1$.
For $n=2$, we get

$$\begin{aligned}|n_+,n_-;1\rangle &= c_0\Big[\frac{\bar{\Omega}^2m^2\omega^2\sqrt{6}}{8\hbar(2\Omega\bar{\Omega}m\omega+\tilde{\Omega}n_-)}\,|4,2;0\rangle - \frac{(\tilde{\Omega}\bar{\Omega}m\omega+4\Omega\bar{\Omega}^2m^2\omega^2)\sqrt{3}}{8\Omega\hbar(\Omega\bar{\Omega}m\omega+\tilde{\Omega}n_-)}\,|3,1;0\rangle \\ &\quad - \frac{(2\Omega\bar{\Omega}^2m^2\omega^2+\tilde{\Omega}\bar{\Omega}m\omega)\sqrt{3}}{24\Omega\hbar(\Omega\bar{\Omega}m\omega+\tilde{\Omega}(n_--2))}\,|1,3;0\rangle\Big] \\ &+ c_1\Big[\frac{3\bar{\Omega}^2m^2\omega^2}{8\hbar(2\Omega\bar{\Omega}m\omega+\tilde{\Omega}(n_--1))}\,|3,3;0\rangle - \frac{\bar{\Omega}^2m^2\omega^2}{\hbar(\Omega\bar{\Omega}m\omega+\tilde{\Omega}(n_--1))}\,|2,2;0\rangle \\ &- \frac{(2\Omega\bar{\Omega}^2m^2\omega^2+\tilde{\Omega}\bar{\Omega}m\omega)\sqrt{6}}{24\Omega\hbar(\Omega\bar{\Omega}m\omega+\tilde{\Omega}(n_--3))}\,|0,4;0\rangle + \frac{\bar{\Omega}^2m^2\omega^2}{4\hbar(\Omega\bar{\Omega}m\omega-\tilde{\Omega}(n_--1))}\,|0,0;0\rangle \\ &- \frac{(2\Omega\bar{\Omega}^2m^2\omega^2+\tilde{\Omega}\bar{\Omega}m\omega)\sqrt{6}}{24\Omega\hbar(\Omega\bar{\Omega}m\omega+\tilde{\Omega}(n_-1))}\,|4,0;0\rangle\Big] + c_2\Big[\frac{\bar{\Omega}^2m^2\omega^2\sqrt{6}}{8\hbar(2\Omega\bar{\Omega}m\omega+\tilde{\Omega}(n_--2))}\,|2,4;0\rangle \\ &- \frac{(-\tilde{\Omega}\bar{\Omega}m\omega+4\Omega\bar{\Omega}^2m^2\omega^2)\sqrt{3}}{8\Omega\hbar(\Omega\bar{\Omega}m\omega+\tilde{\Omega}(n_--2))}\,|1,3;0\rangle - \frac{(2\Omega\bar{\Omega}^2m^2\omega^2+\tilde{\Omega}\bar{\Omega}m\omega)\sqrt{3}}{24\Omega\hbar(\Omega\bar{\Omega}m\omega+\tilde{\Omega}n_-)}\,|3,1;0\rangle\Big],\end{aligned} \tag{88}$$

where $n_+=2$ and $n_-=0$ or $n_+=1$ and $n_-=1$ or $n_+=0$ and $n_-=2$. The constants c_0, c_1 and c_2 are determined using the normalization conditions $\langle 1;2,0|2,0;1\rangle = 1$, $\langle 1;1,1|1,1;1\rangle = 1$ and $\langle 1;0,2|0,2;1\rangle = 1$.

4 Conclusion and remarks

In this paper we have solved the harmonic oscillator in the $2d$ lattice. First we have considered the case of ordinary quantum mechanics. We showed that the direct computation of the eigenvalues by using the Ladder operator does not give a satisfactory as to the physical relevance of the result, due to the appearance of coordinates dependency in the energies. Also the continuous limit i.e. $\varepsilon = 0$ is not well satisfied. The first order approximation of the lattice spacing ε have been considered and the perturbation computation of the energy spectrum is given. The statistical thermodynamic properties of the model are also given. In the other hand the same question is addressed for general noncommutativity between coordinates and momentums. We come to the conclusion that the eigenvalue problem maybe solved in the case where $\bar{\theta} = -m^2\omega^2\theta$. The more general case where this relation is not satisfied is also examinated. We hope that it will be possible to construct a new Fock states in which the matrix $\hat{\mathfrak{W}}$ maybe diagonalizable. This question deserve to be addressed and will be considered in forthcoming work.

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