

PAPER • OPEN ACCESS

On curvature squared invariants in 6D supergravity

To cite this article: J Novak and G Tartaglino-Mazzucchelli 2018 *J. Phys.: Conf. Ser.* **965** 012029

View the [article online](#) for updates and enhancements.

Related content

- [KNEADING INVARIANTS FOR MAPPINGS OF A SEGMENT WITH AN ABSOLUTELY CONTINUOUS INVARIANT MEASURE](#)
N I Chernov
- [Normal invariants of manifold pairs and assembly maps](#)
A Bak and Yu V Muranov
- [Exact invariants and adiabatic invariants of the singular Lagrange system](#)
Chen Xiang-Wei and Li Yan-Min

On curvature squared invariants in 6D supergravity¹

J Novak¹ and G Tartaglino-Mazzucchelli²

¹Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut,
Am Mühlenberg 1, D-14476 Golm, Germany

²Instituut voor Theoretische Fysica, KU Leuven,
Celestijnenlaan 200D, B-3001 Leuven, Belgium

E-mail: joseph.novak@aei.mpg.de, gabriele.tartaglino-mazzucchelli@kuleuven.be

Abstract.

We review recent developments in the construction of curvature squared invariants in off-shell $\mathcal{N} = (1, 0)$ supergravity in six dimensions.

1. Introduction and Motivations

It is well known that the low-energy effective action of string theory is described by supergravity modified by an infinite series of higher-derivative quantum corrections, which can schematically be represented by the Lagrangian

$$\mathcal{L}_{\text{string}}^{\text{low}} = \mathcal{L}_{\text{SG}} + \sum \mathcal{D}^p \mathcal{R}^q + \text{forms contributions} + \text{susy completion} . \quad (1.1)$$

Besides the purely gravitational higher-derivative terms, there are contributions involving the extra fields of the low-energy spectrum of string theory, including the tensor hierarchy of forms, and the fermions needed for local supersymmetry invariance of the action. The supersymmetric higher-derivative terms are in general poorly understood but are of crucial importance in various contexts. For instance, the contributions depending on the p -forms play an important role in the dynamics of the moduli in compactified string theory and the low-energy description of string dualities; see, e. g., [1, 2, 3]. Moreover, higher-derivative terms control the stringy corrections to the entropy of black-holes (see for example [4, 5, 6, 7]) and they are needed for precision tests within the context of the AdS/CFT correspondence. Locally supersymmetric higher-derivative terms also describe counterterms for UV divergencies in supergravity and their classification is intimately connected to the understanding of surprising quantum properties of extended supergravities such as in the case of 4D $\mathcal{N} = 8$ supergravity (see [8] for a short review and a list of references).

Even the supersymmetric extension of curvature squared invariants, which would be the next to leading order in the expansion (1.1), is in general not fully understood. However, non-supersymmetric curvature squared gravity has attracted much attention for over 50 years, irrespective of string theory motivations, for various reasons. For instance, the renormalization of quantum field theories in curved spacetime requires counterterms containing curvature squared terms [9]. In four dimensions, these govern the structure of conformal anomalies that are relevant

¹ Based on the talk presented by GT-M at ISQS25 (Prague, Czech Republic, 6 – 10 June, 2017).



in studying renormalization group flows, see for example the a -theorem of Komargodski and Schwimmer [10]. Forty years ago it was also realized that Lagrangians defined by a linear combination of the Riemann tensor squared, the Ricci tensor squared and the scalar curvature squared, $\alpha(\mathcal{R}_{abcd})^2 + \beta(\mathcal{R}_{ab})^2 + \gamma\mathcal{R}^2$, lead to renormalizable but not unitary theories of gravity [11]. Recently, it was also realized that the Starobinsky model of inflation [12], based on the addition of a \mathcal{R}^2 term to the Einstein-Hilbert gravity, was proposed to be a promising inflationary candidate consistent with the Cosmic-Microwave-Background (CMB) data.

An interesting curvature squared invariant is described by the Gauss-Bonnet (GB) combination

$$\mathcal{L}_{\text{GB}} = \mathcal{R}_{abcd}\mathcal{R}^{abcd} - 4\mathcal{R}_{ab}\mathcal{R}^{ab} + \mathcal{R}^2 = 6\mathcal{R}_{[ab}{}^{ab}\mathcal{R}_{cd]}{}^{cd} . \quad (1.2)$$

In 4D it is a topological term, the Euler characteristic, associated with the type A conformal anomaly [13]. In D-dimensions the GB combination, which is ghost free, is actually expected to govern the first order α' corrections in string theory [14, 15]. In general its structure for any spacetime dimensions and number of supersymmetry charges is not known. In particular, the dependence upon the extra supergravity fields, i.e. the dilaton and the various p -forms, is still not in general understood. In 4D and 5D the Gauss-Bonnet was constructed off-shell and completely classified, see [16, 17, 18, 19, 20, 21, 22, 23] and [24, 25, 26], respectively. In the 6D case only partial results were obtained 30 years ago [27, 28, 29, 30] and a classification of curvature squared invariants has been absent until new insights this year. In particular, the GB invariant in six dimensions has been constructed in our recent paper [31]. One of the main aims of this note is to review this result.

An obvious question that one may ask is: how should one efficiently construct the higher-derivative supergravity invariants? A useful approach would be a formalism that guarantees manifest supersymmetry in a model independent way. Off-shell approaches to supergravity, when available, can indeed be used to construct general supergravity-matter couplings possessing model independent supersymmetry. Two possibilities have been largely developed in the literature: (i) a component field approach (superconformal tensor calculus), see for example the recent ‘‘Supergravity’’ book by Freedman and Van Proeyen [32] for an exhaustive review and references; and (ii) superspace approaches, see the classic books [33, 34, 35, 36].

It turns out that the two approaches can be linked and powerfully used together through the so called ‘‘conformal superspace.’’ In this formalism the entire superconformal algebra is gauged in superspace combining the main advantages of both approaches. Conformal superspace was first introduced by D. Butter for 4D $\mathcal{N} = 1$ [37] and $\mathcal{N} = 2$ [38] supergravity (see also the seminal work by Kugo and Uehara [39]) and it was developed and extended to 3D \mathcal{N} -extended supergravity [40], 5D $\mathcal{N} = 1$ supergravity [26], and recently to 6D $\mathcal{N} = (1, 0)$ supergravity [41], see also [42].

In this paper, we review the description of curvature squared invariants for 6D $\mathcal{N} = (1, 0)$ off-shell supergravity and show how using conformal superspace it is possible to efficiently: describe off-shell supermultiplets through simple differential constraints; provide manifestly supersymmetric off-shell action principles by using powerful cohomological ‘‘superform’’ techniques; and reduce the results to component fields and the superconformal tensor calculus. With these techniques at hand, one has a systematic approach to construct higher-derivative off-shell invariants.

This paper is organized as follows. In section 2 we review how Poincaré gravity can be obtained by gauge fixing conformal gravity. By using the same logic, we review basic aspects of 6D conformal superspace and show how to construct 6D $\mathcal{N} = (1, 0)$ Poincaré supergravity from superspace in section 3. Section 4 is devoted to the construction of curvature squared invariants, while in section 5 we analyze Einstein-Gauss-Bonnet supergravity, which is related

to the low-energy description of 6D compactified string theory. Conclusions are presented in section 6.

2. 6D conformal and Poincaré gravity

Before turning to supergravity, it is useful to remind the reader about conformal gravity and how Poincaré gravity can be viewed as a gauge fixed limit of it. As we will see in the next section, the same principle generalizes to the supersymmetric case.

Conformal gravity in six dimensions may be constructed by gauging the entire conformal group $SO(6,2)$, which possesses the generators $X_{\bar{a}} = \{P_a, M_{ab}, \mathbb{D}, K_a\}$. To each generator we then associate a connection. The vielbein e_m^a and its inverse e_a^m are associated with the momentum P_a generating local-translations/diffeomorphism. Together with the other connections for the Lorentz (ω_m^{bc}), dilatation (b_m) and special conformal (f_m^b) generators, one can introduce the covariant derivatives

$$\nabla_a = e_a^m \left(\partial_m - \frac{1}{2} \omega_m^{bc} M_{bc} - b_m \mathbb{D} - f_m^b K_b \right). \quad (2.1)$$

The covariant derivative algebra is constrained to be

$$[\nabla_a, \nabla_b] = -\frac{1}{2} C_{ab}^{cd} M_{cd} - \frac{1}{6} \nabla^d C_{abcd} K^c. \quad (2.2)$$

The algebra is expressed entirely in terms of a single primary dimension two tensor C_{ab}^{cd} ,

$$K_f C_{ab}^{cd} = 0, \quad \mathbb{D} C_{ab}^{cd} = 2 C_{ab}^{cd}, \quad (2.3)$$

satisfying the following algebraic constraints

$$C_{abcd} = C_{[ab][cd]}, \quad \eta^{ac} C_{abcd} = 0, \quad C_{[abc]d} = 0. \quad (2.4)$$

The solution of the previous constraints determine $\omega_m^{bc} = \omega_m^{bc}(e)$ and $f_m^b = f_m^b(e)$ as composite functions of the vielbein e_m^a while b_m proves to be a pure gauge degree of freedom. The tensor C_{ab}^{cd} coincides with the 6D Weyl tensor: $C_{ab}^{cd} = \mathcal{R}_{ab}^{cd} - \delta_{[a}^{[c} \mathcal{R}_{b]}^{d]} + \frac{1}{10} \delta_{[a}^{[c} \delta_{b]}^{d]} \mathcal{R}$ with $\mathcal{R}_{ab}^{cd} = \mathcal{R}_{ab}^{cd}(e)$ being the standard Riemann tensor, $\mathcal{R}_a^b := \mathcal{R}_{ad}^{bd}$ the Ricci tensor and $\mathcal{R} := \mathcal{R}_a^a$ the scalar curvature.

So far we have considered the gauging of the full conformal algebra. Standard Poincaré gravity can be described by coupling conformal gravity to a primary dimension two scalar field ϕ . If one chooses a gauge for dilatation and special conformal transformations in which $b_m = 0$ and $\phi = 1$ then what is left is standard gravity, gauging only diffeomorphisms and Lorentz transformations. Along this line, the Einstein-Hilbert action of general relativity is described by the action of the dilaton field ϕ in a conformal gravity background

$$S_{\text{EH}} = -\frac{5}{2} \int d^6x e \phi \nabla^a \nabla_a \phi. \quad (2.5)$$

In a gauge where $b_m = 0$ and $\phi = 1$ the previous action reduces to the standard Einstein-Hilbert term

$$S_{\text{EH}} = -\frac{1}{2} \int d^6x e \mathcal{R}. \quad (2.6)$$

Although the previous approach might seem cumbersome in the case of gravity, its supersymmetric analogue proves to be a natural way to describe in a uniform framework different multiplets of off-shell Poincaré supergravity, and general supergravity-matter couplings. The strategy is to first describe off-shell conformal supergravity and then break unnecessary local symmetries, as dilatations and local superconformal transformations, by coupling to compensating multiplets. Let us show how this works in the case of 6D $\mathcal{N} = (1, 0)$ supergravity.

3. 6D $\mathcal{N} = (1, 0)$ conformal and Poincaré supergravity

The standard Weyl multiplet of $\mathcal{N} = (1, 0)$ conformal supergravity is associated with the local off-shell gauging of $\text{OSp}(6, 2|1)$, the $\mathcal{N} = (1, 0)$ superconformal group in 6D [43]. The multiplet contains $40 + 40$ physical components described by the following independent gauge fields: the vielbein e_m^a and a dilatation connection b_m (as in conformal gravity); the gravitino $\psi_{m_i}^\alpha$, associated to the gauging of Q -supersymmetry; and $\text{SU}(2)_R$ gauge fields \mathcal{V}_m^{ij} . The other gauge fields associated with the remaining generators of $\text{OSp}(6, 2|1)$ are composite fields. In addition to the independent gauge connections, the standard Weyl multiplet comprises a set of covariant matter fields given by an anti-self-dual tensor T_{abc}^- , a real scalar field D and a chiral fermion χ^i . These extra independent degrees of freedom are necessary to close the local $\text{OSp}(6, 2|1)$ algebra of transformations without imposing any equation of motions.

So far we have described the Weyl multiplet in components. Let us review the basic aspects of its description in the 6D $\mathcal{N} = (1, 0)$ conformal superspace developed in [41]. In this paper, we adopt the notation and conventions of [41, 42]. Take a $\mathcal{N} = (1, 0)$ curved superspace $\mathcal{M}^{6|8}$ parametrised by local coordinates

$$z^M = (x^m, \theta_i^\mu), \quad m = 0, 1, 2, 3, 4, 5, \quad \mu = 1, 2, 3, 4, \quad i = 1, 2. \quad (3.1)$$

The structure group of conformal superspace, denoted by X , contains the generators for $\text{SO}(5, 1)$ Lorentz, $\text{SU}(2)_R$ R -symmetry, dilatation, S -supersymmetry and conformal boosts transformations. The superspace covariant derivatives are

$$\nabla_A = E_A^M \partial_M - \frac{1}{2} \Omega_A^{ab} M_{ab} - \Phi_A^{ij} J_{ij} - B_A \mathbb{D} - \mathfrak{F}_{AB} K^B, \quad (3.2)$$

where $E_A^M(z)$ is the supervielbein associated with translations in superspace, $P_A = (P_a, Q_\alpha^i)$, and $\partial_M = \partial/\partial z^M$; $\Omega_A^{cd}(z)$ is the Lorentz connection; $\Phi_A^{ij}(z)$ is the $\text{SU}(2)_R$ connection; B_A is the dilatation connection; and \mathfrak{F}_{AB} is the special superconformal connection, with $K^A = (K^a, S_i^\alpha)$. The supergravity gauge group, which is associated with the gauging of the $\text{OSp}(6, 2|1)$ transformations, is generated by the following transformations

$$\delta_{\mathcal{K}} \nabla_A = [\mathcal{K}, \nabla_A], \quad \mathcal{K} := \xi^A \nabla_A + \frac{1}{2} \Lambda^{bc} M_{bc} + \Lambda^{ij} J_{ij} + \tau \mathbb{D} + \Lambda_A K^A. \quad (3.3)$$

Similarly to conformal gravity, to describe the 6D $(1, 0)$ Weyl multiplet, one constrains the algebra of covariant derivatives

$$\begin{aligned} [\nabla_A, \nabla_B] &= -T_{AB}^C \nabla_C - \frac{1}{2} R(M)_{AB}{}^{cd} M_{cd} - R(J)_{AB}{}^{kl} J_{kl} \\ &\quad - R(\mathbb{D})_{AB} \mathbb{D} - R(S)_{AB\gamma}{}^k S_k^\gamma - R(K)_{ABc} K^c \end{aligned} \quad (3.4)$$

to be completely determined in terms of the super-Weyl tensor $W^{\alpha\beta} = (\tilde{\gamma}^{abc})^{\alpha\beta} W_{abc}$ (see, e.g. [44, 41] for its definition) describing in superspace the 6D $\mathcal{N} = (1, 0)$ Weyl multiplet of conformal supergravity. The details of the algebra are more intricate than in the non-supersymmetric case and we refer the reader to [41] for details. It is only relevant to mention that $W^{\alpha\beta}$ is a dimension-1 primary superfield

$$K^A W^{\beta\gamma} = 0, \quad \mathbb{D} W^{\alpha\beta} = W^{\alpha\beta}, \quad (3.5)$$

and satisfies the Bianchi identities

$$\nabla_\alpha^{(i} \nabla_\beta^{j)} W^{\gamma\delta} = -\delta_{[\alpha}^{(\gamma} \nabla_{\beta]}^{(i} \nabla_{\rho]}^{j)} W^{\delta)\rho}, \quad \nabla_\alpha^k \nabla_{\gamma k} W^{\beta\gamma} - \frac{1}{4} \delta_\alpha^\beta \nabla_\gamma^k \nabla_{\delta k} W^{\gamma\delta} = 8i \nabla_{\alpha\gamma} W^{\gamma\beta}. \quad (3.6)$$

It can be shown that the component fields of the standard Weyl multiplet can be identified as $\theta = 0$ projections of the superspace one-forms and descendants of $W^{\alpha\beta}$, see [42].

So far we have considered only the standard Weyl multiplet possessing the covariant component fields: T_{abc}^- , $\chi^{\alpha i}$ and D . However, there exists a variant multiplet for conformal supergravity called the dilaton-Weyl multiplet [43], which plays an important role in the formulation of 6D Poincaré supergravity. The dilaton-Weyl multiplet is obtained by coupling the standard Weyl multiplet to a (on-shell) tensor multiplet. In superspace, the tensor multiplet can be described by a scalar superfield Φ embedded in a gauge super two-form B_2 . Its field strength $H_3 = dB_2$ is constrained to have the following component superfields

$$H_{\alpha\beta\gamma}^{ijk} = 0, \quad H_{a\alpha\beta}^{ij} = 2i\varepsilon^{ij}(\gamma_a)_{\alpha\beta}\Phi, \quad H_{ab\alpha}^i = (\gamma_{ab})_{\alpha}{}^{\beta}\nabla_{\beta}^i\Phi, \quad (3.7a)$$

$$H_{abc} = -\frac{i}{8}(\tilde{\gamma}_{abc})^{\gamma\delta}\nabla_{\gamma}^k\nabla_{\delta k}\Phi - 4W_{abc}\Phi. \quad (3.7b)$$

Here Φ is a primary superfield of dimension 2 satisfying the differential constraint $\nabla_{\alpha}^{(i}\nabla_{\beta}^{j)}\Phi = 0$. Equation (3.7b) shows that one can redefine the super-Weyl tensor as a composite of the tensor multiplet

$$W_{abc} = -\frac{1}{4}H_{abc} - \frac{i}{32}(\tilde{\gamma}_{abc})^{\gamma\delta}\nabla_{\gamma}^k\nabla_{\delta k}\Phi, \quad (3.8)$$

which ends up being the dynamical multiplet for conformal supergravity in the dilaton-Weyl formulation. At the component level, the previous result is equivalent to showing that the covariant fields of the standard Weyl multiplet, T_{abc}^- , $\chi^{\alpha i}$ and D , are exchanged with the component fields of the tensor multiplet, σ , ψ_{α}^i and b_{mn} in the following way²

$$T_{abc}^- = \frac{1}{2\sigma}H_{abc}^-, \quad D = \frac{15}{4\sigma}\left(\nabla^a\nabla_a\sigma + \frac{1}{3}T^{-abc}H_{abc}\right) + \text{fermions}. \quad (3.9)$$

So far we have discussed kinematical properties of 6D supergravity. Let us turn to constructing dynamical systems. To do so, we will use the superform approach to engineer invariant actions [45, 46, 47, 48]. This approach has been rediscovered a number of times in the literature and it has been developed and used in, e. g., the study of the properties of UV counterterms in supersymmetric Yang-Mills and supergravity theories, see [49, 50, 51, 52, 53]. Since 2009 it has been employed and developed also by the two of us together with our collaborators to construct off-shell higher-derivative invariants in various dimensions, see e. g. [54, 55, 56, 57].

In six dimensions, the construction of a supersymmetric invariant from a superform goes as follows. Consider a closed super 6-form $J = \frac{1}{6!}dz^{M_6} \wedge \dots \wedge dz^{M_1} J_{M_1\dots M_6}$,

$$dJ = \frac{1}{6!}dz^{M_6} \wedge \dots \wedge dz^{M_0} \partial_{M_0} J_{M_1\dots M_6} = 0. \quad (3.10)$$

By using J one can define the following action principle

$$S = \int d^6x {}^*J|_{\theta=0}, \quad {}^*J = \frac{1}{6!}\varepsilon^{mnpqrs} J_{mnpqrs}. \quad (3.11)$$

Under superdiffeomorphisms, with $\xi = \xi^A E_A = \xi^M \partial_M$, a closed 6-form satisfies

$$\delta_{\xi}J = \mathcal{L}_{\xi}J \equiv i_{\xi}dJ + di_{\xi}J = di_{\xi}J, \quad (3.12)$$

² Here we focus on just the bosonic fields.

which implies that (up to boundary terms) the action (3.11) is invariant under local supersymmetry transformations. Expressing the action in terms of the tangent frame gives

$$\begin{aligned} S &= \int d^6x \frac{1}{6!} \varepsilon^{m_1 \dots m_6} E_{m_6}{}^{A_6} \dots E_{m_1}{}^{A_1} J_{A_1 \dots A_6} |_{\theta=0} , \\ &\propto \int d^6x e \varepsilon^{a_1 \dots a_6} \left[J_{a_1 \dots a_6} + 3\psi_{a_1 i}{}^\alpha J_{\alpha a_2 \dots a_6}^i + \frac{15}{4} \psi_{a_2 j}{}^\beta \psi_{a_1 i}{}^\alpha J_{\alpha \beta a_3 \dots a_6}^{ij} \right. \\ &\quad \left. + \frac{5}{2} \psi_{a_3 k}{}^\gamma \psi_{a_2 j}{}^\beta \psi_{a_1 i}{}^\alpha J_{\alpha \beta \gamma a_4 a_5 a_6}^{ijk} + \mathcal{O}(\psi^4) \right] |_{\theta=0} , \end{aligned} \quad (3.13)$$

which makes evident how to obtain from (3.11) the component expression of supergravity actions as a power series expansion in the gravitini. To impose invariance under the entire local supergravity gauge group, it is necessary to also require the action to be invariant under the structure group X and any possible additional gauge transformations. This means that J should transform by (at most) an exact 6-form under these transformations

$$\delta_X J = d\Xi , \quad \text{for some } 5\text{-form } \Xi . \quad (3.14)$$

The classification of X -exact closed super 6-forms J is equivalent to the classification of supersymmetric invariants.

Now that we have identified a powerful way to construct supersymmetric invariants, it is natural to ask: how can we describe the $\mathcal{N} = (1, 0)$ extension of the Einstein-Hilbert term of [43] in superspace? To describe this invariant one has to make use of the dilaton-Weyl multiplet coupled to a linear-multiplet conformal compensator. This is based on a scalar isotriplet superfield $G^{ij}(z)$, constrained by $\nabla_\alpha^{(i} G^{jk)} = 0$. Remarkably, this multiplet can be engineered as a closed super 5-form such that $H_5 = dB_4$ for some gauge super 4-form and whose lowest dimension nonvanishing component is

$$H_{abc\alpha\beta}{}^{ij} = -2i(\gamma_{abc})_{\alpha\beta} G^{ij} . \quad (3.15)$$

By using the linear multiplet 5-form, one can construct an invariant action in the following way. First, in a conformal supergravity background, consider a $(1, 0)$ vector multiplet described by a spinor superfield $W^{\alpha i}$ obeying the Bianchi identities $\nabla_\gamma^k W_k^\gamma = 0$, $\nabla_\alpha^{(i} W^{\beta j)} = \frac{1}{4} \delta_\alpha^\beta \nabla_\gamma^{(i} W^{\gamma j)}$ and appropriately embedded in a closed super two-form F_2 . Next, consider the following closed 6-form $J_{B_4 \wedge F_2}$

$$J_{B_4 \wedge F_2} = B_4 \wedge F_2 - \Sigma_{B_4 \wedge F_2} , \quad d\Sigma_{B_4 \wedge F_2} = -F_2 \wedge H_5 , \quad (3.16)$$

with $\Sigma_{B_4 \wedge F_2}$ a covariant 6-form constructed entirely in terms of the superfields $W^{\alpha i}$ and G^{ij} in F_2 and H_5 .³ By using the closed 6-form $J_{B_4 \wedge F_2}$, together with the superform action principle (3.11), one obtains the following $B_4 \wedge F_2$ action principle⁴

$$S_{B_4 \wedge F_2} = \frac{1}{2} \int d^6x e \left(\frac{1}{4!} \varepsilon^{abcdef} f_{ab} b_{cdef} + X^{ij} G_{ij} + \text{fermions} \right) , \quad (3.17)$$

with f_{ab} the component field strength and X^{ij} the scalar component of the vector multiplet, while b_{abcd} is the component gauge 4-form of the linear multiplet. Note that f_{ab} and b_{abcd} are directly related to the $\theta = 0$ component of the field strength $F_{mn}(z)$ and the gauge superfield $B_{mnpq}(z)$ respectively. The previous action reproduces the one first constructed in [43].

³ Its existence is guaranteed by the fact that $F_2 \wedge H_5$ is Weil trivial.

⁴ The procedure here very much follows the one adopted in [58] in 4D.

We are now ready to introduce the (1,0) Poincaré supergravity EH invariant. Consider the following composite vector multiplet built by using the linear multiplet

$$\begin{aligned} \mathbb{W}^{\alpha i} &= \frac{1}{G} \nabla^{\alpha\beta} \chi_{\beta}^i + \frac{4}{G} (W^{\alpha\beta} \chi_{\beta}^i + 10iX_j^{\alpha} G^{ji}) - \frac{1}{2G^3} G_{jk} (\nabla^{\alpha\beta} G^{ij}) \chi_{\beta}^k \\ &+ \frac{1}{2G^3} G^{ij} E^{\alpha\beta} \chi_{\beta j} + \frac{i}{16G^5} \varepsilon^{\alpha\beta\gamma\delta} \chi_{\beta j} \chi_{\gamma k} \chi_{\delta l} G^{ij} G^{kl} , \end{aligned} \quad (3.18)$$

where

$$\nabla_{\alpha}^{(i} G^{jk)} = 0 , \quad \chi_{\alpha}^i := \frac{2}{3} \nabla_{\alpha j} G^{ij} , \quad E^{\alpha\beta} := \frac{i}{8} \varepsilon^{\alpha\beta\gamma\delta} \nabla_{[\gamma}^k \chi_{\delta]k} . \quad (3.19)$$

By plugging back this composite vector multiplet in the $B_4 \wedge F_2$ action, and gauge fixing dilatation, conformal-boosts and $SU(2)_R$ symmetry, one ends up with the (1,0) Poincaré supergravity action

$$e^{-1} \mathcal{L}_{\text{EH}} = -\frac{1}{2} e^{-2\varphi} [\mathcal{R} - 4\partial_m \varphi \partial^m \varphi + \frac{1}{12} H_{abc} H^{abc}] + \text{fermions} , \quad (3.20)$$

where φ is the only component of G^{ij} surviving the $SU(2)_R \rightarrow U(1)_R$ gauge fixing condition. The results for the EH terms obtained from superspace coincide with the original results of [43] but the path we have followed will prove more useful for higher-derivative actions.

4. Curvature squared invariants

Let us show how it is possible to construct supersymmetric extensions of all curvature squared Lagrangians in superspace. We are searching for extensions of a Lagrangian that is schematically the following linear combination of Weyl squared, Riemann squared and the scalar curvature squared

$$\mathcal{L}_{\text{curvature-squared}} \propto a C^{abcd} C_{abcd} + b \mathcal{R}^{abcd} \mathcal{R}_{abcd} + c \mathcal{R}^2 + \dots . \quad (4.1)$$

In constructing the supersymmetric extension of the EH term a crucial role was played by the $B_4 \wedge F_2$ action principle. It turns out that we can construct the two known invariants plus one new invariant by using a new action principle corresponding to the supersymmetrization of a $B_2 \wedge H_4$ terms where B_2 is the gauge 2-form of a tensor multiplet with field strength $H_3 = dB_2$ and H_4 is a closed 4-form, $dH_4 = 0$, possessing the following components

$$H_{\alpha\beta\gamma\delta}^{ijkl} = H_{a\beta\gamma\delta}^{jkl} = 0 , \quad H_{ab\gamma\delta}^{kl} = i(\gamma_{abc})_{\gamma\delta} B^{ckl} , \quad (4.2a)$$

$$H_{abc\delta}^l = -\frac{1}{12} \varepsilon_{abcdef} (\gamma^{de})_{\delta}{}^{\rho} \nabla_{\rho p} B^{flp} , \quad H_{abcd} = \frac{i}{48} \varepsilon_{abcdef} (\tilde{\gamma}^e)^{\alpha\beta} \nabla_{\alpha k} \nabla_{\beta l} B^{fkl} . \quad (4.2b)$$

The superfield $B^{\alpha\beta ij} = (\tilde{\gamma}^a)^{\alpha\beta} B_a^{ij}$, is a dimension three primary superfield satisfying the following differential constraints induced by the constraint $dH_4 = 0$

$$\nabla_{\alpha}^{(i} B^{\beta\gamma jk)} = -\frac{2}{3} \delta_{\alpha}^{[\beta} \nabla_{\delta}^{(i} B^{\gamma]jk)} , \quad [\nabla_{\alpha}^{(i}, \nabla_{\beta k}] B^{\alpha\beta jk)} = -8i \nabla_{\alpha\beta} B^{\alpha\beta ij} . \quad (4.3)$$

We can now engineer the new action principle by using the superform approach. We start with the following closed 6-form $J_{B_2 \wedge H_4}$

$$J_{B_2 \wedge H_4} = B_2 \wedge H_4 - \Sigma_{B_2 \wedge H_4} , \quad d\Sigma_{B_2 \wedge H_4} = H_3 \wedge H_4 \quad (4.4)$$

with $\Sigma_{B_2 \wedge H_4}$ a covariant super 6-form constructed only in terms of the covariant superfields Φ and B_a^{ij} , respectively associated with the field strengths H_3 and H_4 . Once one explicitly constructs $J_{B_2 \wedge H_4}$ and plugs the result into (3.11) we obtain the new $B_2 \wedge H_4$ action principle

$$S_{B_2 \wedge H_4} = \int d^6x e \left\{ \frac{1}{4} (b_{ab} - \eta_{ab} \sigma) C^{ab} + \text{fermions} \right\}, \quad (4.5)$$

which proves to be locally superconformal invariant. Here $\sigma := \Phi|_{\theta=0}$ and b_{ab} is related to the $\theta = 0$ component of the gauge superfield $B_{mn}(z)$, while C_{ab} is

$$C_{ab} := \frac{i}{12} (\tilde{\gamma}_a)^{\alpha\beta} \nabla_{\alpha k} \nabla_{\beta l} B_b^{kl} |_{\theta=0}. \quad (4.6)$$

Given a B_a^{ij} superfield satisfying (4.3), which plays the role of a Lagrangian for the $B_2 \wedge H_4$ action principle, we will automatically have a (1,0) locally superconformal invariant. Let us describe relevant examples.

A locally supersymmetric extension of the Riemann squared term was first constructed in 1987 in [30]. We can reproduce it by using the $B_2 \wedge H_4$ action principle and the composite superfield

$$B^{\alpha\beta ij} = -\frac{i}{2} \Lambda^{\alpha(i} \gamma^{\delta} \Lambda^{\beta j)} \delta^{\gamma} \quad (4.7)$$

with $\Lambda^{\alpha i} \gamma^{\delta}$ given by the following dimension-3/2 primary superfield

$$\begin{aligned} \Lambda^{\alpha i} \gamma^{\delta} &= X_{\beta}^{i\alpha\gamma} - \frac{1}{3} \delta_{\beta}^{\alpha} X^{\gamma i} + \frac{1}{12} \delta_{\beta}^{\gamma} X^{\alpha i} + \frac{i}{4} \Phi^{-1} \psi_{\beta}^i W^{\alpha\gamma} + \frac{i}{12} \Phi^{-1} \delta_{\beta}^{\alpha} W^{\gamma\delta} \psi_{\delta}^i \\ &\quad - \frac{i}{12} \Phi^{-1} \delta_{\beta}^{\gamma} W^{\alpha\delta} \psi_{\delta}^i + \frac{i}{12} \varepsilon^{\alpha\gamma\delta\rho} \Phi^{-1} \nabla_{\delta(\rho} \psi_{\beta)}^i - \frac{i}{8} \varepsilon^{\alpha\gamma\delta\rho} \Phi^{-2} (\nabla_{\delta(\rho} \Phi) \psi_{\beta)}^i \\ &\quad + \frac{i}{32} \varepsilon^{\alpha\gamma\delta\rho} \Phi^{-2} H_{\rho\beta} \psi_{\delta}^i - \frac{1}{16} \varepsilon^{\alpha\gamma\delta\rho} \Phi^{-3} \psi_{\delta}^i \psi_{(\rho}^k \psi_{\beta)k}, \end{aligned} \quad (4.8)$$

where

$$\psi_{\alpha}^i = \nabla_{\alpha}^i \Phi, \quad \nabla_{\alpha}^i \psi_{\beta}^j = -\frac{i}{2} \varepsilon^{ij} (\gamma^{abc})_{\alpha\beta} H_{abc}^+ - i \varepsilon^{ij} (\gamma^a)_{\alpha\beta} \nabla_a \Phi. \quad (4.9)$$

In the gauge $\Phi = 1$ and $B_A = 0$, the composite (4.7) together with the action principle (4.5) reproduces the Riemann squared invariant of [30] whose leading term is

$$e^{-1} \mathcal{L}_{\text{Riem}^2} = \mathcal{R}^{abcd} \mathcal{R}_{abcd} + \dots \quad (4.10)$$

A scalar curvature squared invariant, with leading term \mathcal{R}^2 , was constructed in components in [59] by using results of the superconformal tensor calculus of [43]. We can reproduce it from superspace by using the $B_2 \wedge H_4$ action principle where

$$B^{\alpha\beta ij} = -\frac{i}{2} \mathbb{W}^{\alpha(i} \mathbb{W}^{\beta j)}, \quad (4.11)$$

and $\mathbb{W}^{\alpha i}$ is the composite vector multiplet (3.18) used to construct the EH term.

Remarkably, by using the $B_2 \wedge H_4$ action principle we can construct a new curvature squared invariant. A composite built entirely out of the super-Weyl tensor can be constructed [41]

$$B^{\alpha\beta ij} = -4W^{\gamma[\alpha} Y_{\gamma}^{\beta]ij} - 32i X_{\gamma}^{\alpha\delta(i} X_{\delta}^{\beta\gamma j)} + 10i X^{\alpha(i} X^{\beta j)}, \quad (4.12)$$

where $X^{\alpha i}$, $X_{\alpha}^{\beta\gamma i}$ and $Y_{\alpha}^{\beta ij}$ are descendant superfields constructed in terms of spinor derivatives of the super-Weyl tensor $W^{\alpha\beta}$. The previous composite superfield was originally constructed to describe one of the two (1,0) conformal supergravity actions [41, 42]. On the other hand, by plugging B_a^{ij} of (4.12) in the $B_2 \wedge H_4$ action we obtain the following new curvature squared invariant

$$\begin{aligned}
e^{-1}\mathcal{L}_{\text{new}} = & \frac{1}{32} \left\{ \sigma C_{ab}{}^{cd} C_{cd}{}^{ab} - 3\sigma \mathcal{R}_{ab}{}^{ij} \mathcal{R}^{ab}{}_{ij} + \frac{4}{15} \sigma D^2 - 8\sigma T^{-dab} \nabla_d \nabla^c T_{abc}^- \right. \\
& + 4\sigma (\nabla_c T^{-abc}) \nabla^d T_{abd}^- + 4\sigma T^{-abc} T_{ab}{}^{-d} T^{-ef}{}_c T_{efd}^- - \frac{8}{45} H_{abc} T^{-abc} D \\
& - 2H_{abc} C^{ab}{}_{de} T^{-cde} + 4H_{abc} T_d{}^{-ab} \nabla_e T^{-cde} - \frac{4}{3} H_{abc} T^{-dea} T^{-bcf} T_{def}^- \\
& \left. - \frac{1}{4} \epsilon^{abcdef} b_{ab} (C_{cd}{}^{gh} C_{efgh} - \mathcal{R}_{cd}{}^{ij} \mathcal{R}_{efij}) \right\} + \text{fermions} , \quad (4.13)
\end{aligned}$$

where $\mathcal{R}_{ab}{}^{ij}$ is the $SU(2)_R$ field strength. In the gauge $\sigma = 1$ and $b_a = 0$, equation (4.13) reduces to

$$e^{-1}\mathcal{L}_{\text{new}} = \frac{1}{32} \mathcal{R}_{ab}{}^{cd} \mathcal{R}_{cd}{}^{ab} - \frac{1}{32} \mathcal{R}_b{}^d \mathcal{R}_d{}^b + \frac{1}{128} \mathcal{R}^2 + \dots \quad (4.14)$$

5. Applications: Einstein-Gauss-Bonnet supergravity

It turns out that the supersymmetric GB invariant can be constructed by taking a linear combination of the new curvature squared invariant (4.13) and the Riemann squared invariant. By comparing (1.2) with (4.10) and (4.14) it is simple to see that the following combination

$$S_{\text{GB}} = -3S_{\text{Riem}^2} + 128S_{\text{new}} \quad (5.1)$$

describes a 6D (1,0) off-shell supersymmetric extension of the Gauss-Bonnet term. In the gauge $\sigma = 1$ and $b_a = 0$, the resulting Lagrangian becomes

$$\begin{aligned}
e^{-1}\mathcal{L}_{\text{GB}} = & \mathcal{R}_{abcd} \mathcal{R}^{abcd} - 4\mathcal{R}_{ab} \mathcal{R}^{ab} + \mathcal{R}^2 \\
& - \frac{1}{2} \mathcal{R}_{abcd} H^{abe} H^{cd}{}_e + \mathcal{R}^{ab} H_{ab}^2 - \frac{1}{6} \mathcal{R} H^2 + \frac{1}{144} (H^2)^2 - \frac{1}{8} (H_{ab}^2)^2 + \frac{5}{24} H^4 \\
& + \epsilon^{abcdef} b_{ab} \left(-\frac{1}{4} \mathcal{R}_{cd}{}^{gh} (\tilde{\omega}) \mathcal{R}_{efgh} (\tilde{\omega}) + \mathcal{R}_{cd}{}^{ij} \mathcal{R}_{efij} \right) + \text{fermions} ,
\end{aligned}$$

where we have introduced the torsionfull connection $\tilde{\omega}$ and various contractions as

$$\tilde{\omega}_m{}^{cd} := \omega_m{}^{cd} - \frac{1}{2} e_m{}^a H_a{}^{cd} , \quad (5.2a)$$

$$H^2 := H_{abc} H^{abc} , \quad H_{ab}^2 := H_a{}^{cd} H_{bcd} , \quad H^4 := H_{abe} H_{cd}{}^e H^{acf} H^{bd}{}_f . \quad (5.2b)$$

There are various advantages in having an off-shell version of the (1,0) Gauss-Bonnet invariant. First of all, it is possible to couple it to general supergravity-matter couplings without having to modify the supersymmetry transformations. Secondly, we have a complete off-shell descriptions of the dependance on the NSNS 2-form gauge potential. Let us show how these advantages can be used in a relevant case.

Consider the combination

$$\mathcal{L} = \mathcal{L}_{\text{EH}} + \frac{1}{16} \alpha' \mathcal{L}_{\text{GB}} . \quad (5.3)$$

This turns out to be an off-shell extension of the α' corrected low-energy effective action of string theory compactified to six dimensions. More precisely, once one properly integrated out

the $SU(2)_R$ gauge connection and an auxiliary field of the linear multiplet compensator, it can be shown that the previous Lagrangian reduces to

$$\begin{aligned}
2\kappa^2 e^{-1} \mathcal{L} = & e^{-2\varphi} \left[-\mathcal{R} + 4\partial_m \varphi \partial^m \varphi - \frac{1}{12} H_{abc} H^{abc} \right] \\
& + \frac{1}{16} \alpha' \left[6\mathcal{R}_{[ab}{}^{ab} \mathcal{R}_{cd]}{}^{cd} - \frac{1}{6} \mathcal{R} H^2 + \mathcal{R}^{ab} H_{ab}^2 + \frac{5}{24} H^4 + \frac{1}{144} (H^2)^2 - \frac{1}{8} (H_{ab}^2)^2 \right. \\
& \left. - \frac{1}{2} \mathcal{R}_{abcd} H^{abe} H^{cd}{}_e + \frac{1}{4} \epsilon^{abcdef} b_{ab} R_{cd}{}^g{}_h(\tilde{\omega}) R_{ef}{}^h{}_g(\tilde{\omega}) \right] + \text{fermions} \quad (5.4)
\end{aligned}$$

which precisely matches the on-shell Einstein-Gauss-Bonnet supergravity action that was first described in [3]. In [3], Liu and Minasian constructed the Lagrangian (5.4) by fixing the one-loop four-derivative corrections in six dimensions by means of a K3 reduction of the R^4 corrected type IIA strings and requiring that the dyonic string remains a solution, as well as the duality of this model to heterotic strings compactified on T^4 . Our result relies only on supersymmetry and can be considered as an important consistency check of the results and assumptions of [3].

6. Conclusion

The examples reviewed in this paper demonstrate that there exist powerful “top-down” approaches based on superspace to construct off-shell higher-derivative supergravity invariants. By using these techniques we can potentially improve the classification of higher-derivative invariants in various dimensions. For instance, the new 6D curvature squared invariant completes a piece that has been missing for three decades. The supersymmetric extension of the Gauss-Bonnet invariant is of importance in the study of the low-energy limit of string theory and for α' -corrected AdS/CFT tests, and might find various interesting applications. For instance, the Einstein-Gauss-Bonnet action (5.1) possesses a supersymmetric $AdS_3 \times S^3$ solution analogue of the famous $AdS_5 \times S^5$ solution in IIB string theory. Since we know the full off-shell description of (5.1), in [31] we were able to compute the α' -corrected Kaluza-Klein spectrum of fluctuations around $AdS_3 \times S^3$ and show that it organizes in short and long multiplets of its super-isometry group, $SU(1,1|2) \times SL(2, \mathbb{R}) \times SU(2)$. Our results could be used to better understand the dynamics of strings in $AdS_3 \times S^3 \times K3(T^4)$ backgrounds and for applications to AdS/CFT.

Acknowledgments

We are grateful to D. Butter, M. Ozkan and Y. Pang for collaboration. JN is supported by a Humboldt research fellowship of the Alexander von Humboldt Foundation. GT-M was supported in part by the Interuniversity Attraction Poles Programme initiated by the Belgian Science Policy (P7/37) and in part by the KU Leuven C1 grant ZKD1118 C16/16/005.

References

- [1] Antoniadis I, Ferrara S, Minasian R and Narain K S 1997 *Nucl. Phys. B* **507** 571
- [2] Antoniadis I, Minasian R, Theisen S and Vanhove P 2003 *Class. Quant. Grav.* **20** 5079
- [3] Liu J T and Minasian R 2013 *Nucl. Phys. B* **874** 413
- [4] Maldacena J M, Strominger A and Witten E 1997 *JHEP* **9712** 002
- [5] Vafa C 1998 *Adv. Theor. Math. Phys.* **2** 207
- [6] Lopes Cardoso G, de Wit B and Mohaupt T 1999 *Phys. Lett. B* **451** 309
- [7] Lopes Cardoso G, de Wit B, Kappeli J and Mohaupt T 2000 *JHEP* **0012**, 019
- [8] Kallosh R arXiv:1412.7117 [hep-th]
- [9] Utiyama R and DeWitt B S 1962 *J. Math. Phys.* **3** 608
- [10] Komargodski Z and Schwimmer A 2011 *JHEP* **1112** 099
- [11] Stelle K S 1977 *Phys. Rev. D* **16**, 953
- [12] Starobinsky A A 1980 *Phys. Lett. B* **91** 99
- [13] Deser S and Schwimmer A 1993 *Phys. Lett. B* **309** 279
- [14] B. Zwiebach 1985 *Phys. Lett. B* **156** 315
- [15] S. Deser and A. N. Redlich 1986 *Phys. Lett. B* **176** 350 Erratum: [1987 *Phys. Lett. B* **186** 461]

- [16] Cecotti S, Ferrara S, Girardello L and Porrati M 1985 *Phys. Lett. B* **164** 46
- [17] Theisen S 1986 *Nucl. Phys. B* **263**, 687
- [18] Buchbinder I L and Kuzenko S M, 1988 *Nucl. Phys. B* **308**, 162
- [19] Cecotti S, Ferrara S, Girardello L, Porrati M and Pasquinucci A 1986 *Phys. Rev. D* **33** 2504
- [20] Ferrara S, Sabharwal S and Villasante M 1988 *Phys. Lett. B* **205**, 302
- [21] Ferrara S and Villasante M 1989 *J. Math. Phys.* **30**, 104
- [22] Le Du R 1998 *Eur. Phys. J. C* **5**, 181
- [23] Butter D, de Wit B, Kuzenko S M and Lodato I 2013 *JHEP* **1312** 062
- [24] Ozkan M and Pang Y 2013 *JHEP* **1303** 158 Erratum: [2013 *JHEP* **1307** 152]
- [25] Ozkan M and Pang Y 2013 *JHEP* **1308**, 042
- [26] Butter D, Kuzenko S M, Novak J and Tartaglino-Mazzucchelli G 2015 *JHEP* **1502** 111
- [27] Bergshoeff E, Salam A and Sezgin E 1986 *Phys. Lett. B* **173**, 73
- [28] Bergshoeff E, Salam A and Sezgin E 1987 *Nucl. Phys. B* **279**, 659
- [29] Nishino H and Gates S J Jr. 1986 *Phys. Lett. B* **173** 417
- [30] Bergshoeff E and Rakowski M 1987 *Phys. Lett. B* **191**, 399
- [31] Novak J, Ozkan M, Pang Y and Tartaglino-Mazzucchelli G 2017 *Phys. Rev. Lett.* **119** no.11, 111602
- [32] Freedman D Z and Van Proeyen A 2012 Cambridge Univ. Press
- [33] Gates S J Jr., Grisaru M T, Rocek M and Siegel W 1983 *Front. Phys.* **58** 1
- [34] Wess J and Bagger J 1992 *Princeton, USA: Univ. Pr.* 259 p
- [35] Buchbinder I L and Kuzenko S M 1998 *Bristol, UK: IOP* 656 p
- [36] Galperin A S, Ivanov E A, Ogievetsky V I and Sokatchev E S 2001 *Cambridge, UK: Univ. Pr.* 306 p
- [37] Butter D 2010 *Annals Phys.* **325**, 1026
- [38] Butter D 2011 *JHEP* **1110**, 030
- [39] Kugo T and Uehara S 1985 *Prog. Theor. Phys.* **73** 235
- [40] Butter D, Kuzenko S M, Novak J and Tartaglino-Mazzucchelli G 2013 *JHEP* **1309** 072
- [41] Butter D, Kuzenko S M, Novak J and Theisen S 2016 *JHEP* **1612**, 072
- [42] Butter D, Novak J and Tartaglino-Mazzucchelli G 2017 *JHEP* **1705**, 133
- [43] Bergshoeff E, Sezgin E and Van Proeyen A 1986 *Nucl. Phys. B* **264**, 653, Erratum: [2001 *Nucl. Phys. B* **598**, 667]
- [44] Linch W D III and Tartaglino-Mazzucchelli G 2012 *JHEP* **1208**, 075
- [45] Castellani L, D'Auria R and Fre P 1991 *World Scientific, Singapore* pp. 680–684
- [46] Hasler M F 1998 *Eur. Phys. J. C* **1**, 729
- [47] Gates S J Jr. 1999 *Nucl. Phys. B* **541**, 615
- [48] Gates S J Jr., Grisaru M T, Knutt-Wehlau M E and Siegel W 1998 *Phys. Lett. B* **421**, 203
- [49] Bossard G, Howe P S and Stelle K S 2009 *Gen. Rel. Grav.* **41**, 919
- [50] Bossard G, Howe P S and Stelle K S 2009 *Phys. Lett. B* **682**, 137
- [51] Bossard G, Howe P S and Stelle K S 2011 *JHEP* **1101**, 020
- [52] Bossard G, Howe P S, Lindstrom U, Stelle K S and Wulff L, 2011 *JHEP* **1105** 021
- [53] Bossard G, Howe P S and Stelle K S 2013 *JHEP* **1307**, 117
- [54] Butter D, Kuzenko S M, Novak J and Tartaglino-Mazzucchelli G 2015 *JHEP* **1502**, 111
- [55] Butter D, Kuzenko S M, Novak J and Tartaglino-Mazzucchelli G 2013 *JHEP* **1310**, 073
- [56] Kuzenko S M, Novak J and Tartaglino-Mazzucchelli G 2014 *JHEP* **1401**, 121
- [57] Kuzenko S M, Novak J and Tartaglino-Mazzucchelli G 2015 *JHEP* **1509**, 081
- [58] Butter D, Kuzenko S M, Novak J 2012 *JHEP* **1209**, 131
- [59] Ozkan M 2013 *PhD Thesis, Texas A&M University*