

**PAPER**

L lines, C points and Chern numbers: understanding band structure topology using polarization fields

OPEN ACCESS**RECEIVED**

27 March 2017

REVISED

22 July 2017

ACCEPTED FOR PUBLICATION

6 September 2017

PUBLISHED

27 November 2017

Original content from this work may be used under the terms of the [Creative Commons Attribution 3.0 licence](#).

Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.

Thomas Fösel^{1,2,4}, Vittorio Peano^{1,3}  and Florian Marquardt^{1,2} ¹ Friedrich-Alexander University Erlangen-Nürnberg (FAU), Department of Physics, Staudtstr. 7, D-91058 Erlangen, Germany² Max Planck Institute for the Science of Light, Staudtstr. 2, D-91058 Erlangen, Germany³ Department of Physics, University of Malta, Msida MSD 2080, Malta⁴ Author to whom any correspondence should be addressed.E-mail: thomas.foesel@mpl.mpg.de**Keywords:** Chern number, polarization singularities, Chern insulators

Abstract

Topology has appeared in different physical contexts. The most prominent application is topologically protected edge transport in condensed matter physics. The Chern number, the topological invariant of gapped Bloch Hamiltonians, is an important quantity in this field. Another example of topology, in polarization physics, are polarization singularities, called L lines and C points. By establishing a connection between these two theories, we develop a novel technique to visualize and potentially measure the Chern number: it can be expressed either as the winding of the polarization azimuth along L lines in reciprocal space, or in terms of the handedness and the index of the C points. For mechanical systems, this is directly connected to the visible motion patterns.

1. Introduction

Ever since the discovery of the transverse wave nature of electromagnetic waves, the study of the polarization properties of such vector waves has attracted a great deal of attention and led to the introduction of novel mathematical concepts. For a generic plane wave, the tip of the electric field vector traces an ellipse that defines the polarization state. In just the same way, we can describe the mechanical motion of a single pendulum that is free to move along two orthogonal directions.

For an arbitrary (monochromatic) field, its polarization becomes position-dependent, and the resulting polarization field can display complex spatial patterns. Again, there is a mechanical analogy, in the form of 2D arrays of coupled mechanical oscillators. Such coupled oscillator arrays have already been used as a platform to implement topologically protected transport of sound waves, using coupled pendula [1] and coupled gyroscopes [2]. Eventually, they could also be realized on the nanoscale, e. g. using nanopillar arrays [3, 4]. The time-evolution of such an array under monochromatic driving also exhibits elliptical motion that has the same mathematical description as the electromagnetic polarization fields.

The study of the complex spatial polarization patterns in random electromagnetic waves has led to interesting topological concepts. The central objects of interest are lines where the polarization gets linear, and points with circular polarization. These ‘L lines’ [5, 6] and ‘C points’ [6, 7] have been studied thoroughly in random optical fields, but are still not widely known. However, they have been found in many different physical contexts, for example the sunlight in the sky [8] and speckle fields [9]. In recent years, unconventional C points in optical fields where the rotation of the electric field occurs in a plane containing the propagation direction of light have been both predicted theoretically and verified experimentally [10–12].

There is, of course, another branch of physics where topology has become very prominent recently: the analysis of band structures. According both to their phenomenology and their theoretical description, two categories can be identified. On the one hand, there are the Chern insulators associated with the quantum Hall effect [13, 14] and the anomalous quantum Hall effect [15]; they exhibit chiral edge transport as a result of a non-trivial topological invariant, the Chern number [16–18]. On the other hand, topological insulators have been

established by the discovery of the quantum spin Hall effect [19–23]; their edge channels are helical, and their topological properties are encoded in a binary (\mathbb{Z}_2) topological invariant [24]. The importance of those topological features on transport is by now well-documented [25].

In the present work, it is our aim to connect these two strands of topology, in a general way that is particularly useful for mechanical systems. Our approach helps to visualize (and, in principle, measure) the Chern numbers based on the polarization fields of bulk excitations. In fact, we offer two different approaches to extract the Chern numbers of the band structure, one based on L lines and the other based on C points.

We will explain how this generalizes the well-known Skyrmion configuration method [26, 27]. Furthermore, we will discuss that our method is an alternative to other recently explored techniques to obtain Chern numbers for bosonic systems, both from bulk features [28, 29], from dynamics at the boundary [30] and in more general settings [31, 32].

Besides the mechanical arrays of pendula and nanopillars mentioned above, we will also explain how our method can be applied in general to arbitrary mechanical structures (e. g. phononic crystals), and also to the electromagnetic fields in photonic structures. It is, thus, applicable in principle to a large class of the recently proposed or implemented topological devices for sound waves [1, 2, 33–44], light waves [45–57], ultracold atoms [58, 59], or magnons [60–64]. These include Chern insulators, as well as time-reversal preserving topological insulators whose Hamiltonian can be decomposed into a pair of Chern insulator Hamiltonians with opposite Chern numbers.

2. Connecting topological bands to polarization fields

2.1. Polarization fields

In this section, we will very briefly provide a qualitative description of polarization fields, L lines [5, 6] and C points [6, 7]. The prototypical example is a monochromatic electromagnetic wave. We observe a two-dimensional cut through this field, and restrict to the projection on its in-plane components. For purely transverse polarization, this is naturally given for a plane perpendicular to the propagation direction. Using the common complex representation of the oscillating electric field, we are hence dealing with a (continuous) two-component complex vector field in two spatial dimensions.

This situation can be depicted as follows: for each point in space, we draw the orbit that the electric field vector traces during one cycle of oscillation. In general, this will be an ellipse (traversed with a certain handedness). A typical example for a random polarization pattern is shown in figure 1. Two particular features stand out: first, all the points with a linear polarization pattern (where the ellipse degenerates into a line) are arranged on closed curves; these are called the L lines. Second, there are isolated points of circular polarization, the so called C points.

Both L lines and C points are topologically robust in the sense that they are stable under smooth deformations of the polarization field. That is, they will only deform and move around, but they cannot suddenly appear or disappear. Destruction and creation of C points is only possible pairwise, i. e. two C points of the same handedness (same circular polarization) have to annihilate, and always two C points of the same handedness are created together. The only way for an L line to disappear is to contract down to a point, and new L lines can only emerge in the inverse process; in addition, L lines can also merge and split.

2.2. Constructing an auxiliary polarization field in momentum space for Bloch bands

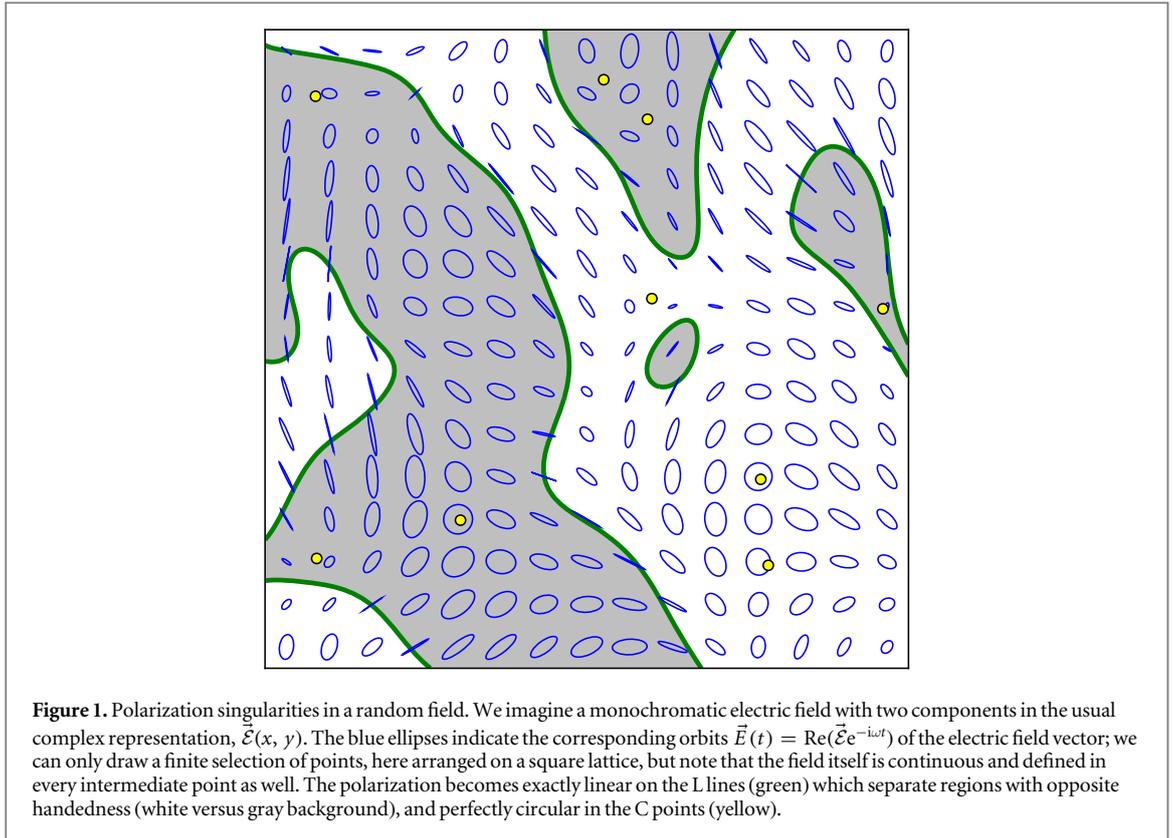
Now, we will look at a different situation and relate it to the polarization picture subsequently. We consider a particle or excitation hopping around on a lattice in two dimensions. We will assume that there is some internal degree of freedom associated with the particle, or, equivalently, a certain number of basis states situated at each lattice site. The physical origin may vary: the spin of a particle, the atomic orbital for atoms in a crystal (s, p, d, ...), the different oscillation directions of a mechanical resonator in a phononic lattice, the sense of circulation of the whispering gallery modes in a lattice of coupled optical disk resonators [65], or the polarization of photon-exciton polaritons in micropillars [66–68]. In addition, one unit cell might consist of multiple sites, depending on the lattice geometry. For all these cases, we will speak of the ‘polarization degree of freedom’.

The total number N of basis states associated with a unit cell of the Bravais lattice determines the number of bands. In the following, we will denote the real-space basis states as $|j, s\rangle$ where $j \in \mathbb{Z}^2$ labels the unit cell and $s \in \{1, \dots, N\}$ refers to the polarization degree of freedom.

The eigenmodes of every Hamiltonian \hat{H} respecting the translational invariance are the Bloch states

$$|\Phi_n(\vec{k})\rangle = \sum_{j,s} \Psi_s^{(n)}(\vec{k}) e^{i\vec{k}\cdot\vec{r}_j} |j, s\rangle, \quad (1)$$

where n is the index of the band (with $1 \leq n \leq N$) and \vec{r}_j the position of the unit cell.



In order to make a connection with the polarization fields introduced before, we need to identify a complex two-component vector field which encodes the Bloch state. The projections of any Bloch wave function along two arbitrary, linearly independent polarization directions define such a complex vector field in momentum space:

$$\vec{\psi}^{(n)}(\vec{k}) = \begin{pmatrix} \Psi_{s_1}^{(n)}(\vec{k}) \\ \Psi_{s_2}^{(n)}(\vec{k}) \end{pmatrix} \in \mathbb{C}^2. \quad (2)$$

Without loss of generality, the two projections can always be chosen along orthogonal directions, which we will assume in the following.

For particles with spin-1/2 or equivalent systems, like mechanical oscillators with two degrees of freedom (oscillation directions), s_1 and s_2 can refer to the two basis states of the internal degree of freedom. If the lattice is a non-Bravais lattice, the projection onto one sublattice is a natural choice. In the mechanical case, $\vec{\psi}^{(n)}(\vec{k})$ corresponds to the directly visible motion pattern $\vec{r}_{\vec{k}}(t) = \text{Re}(\vec{\psi}^{(n)}(\vec{k})e^{-i\omega t})$ on the selected site. We note that, in the most general case, the orbit will be elliptical, relating directly to the polarization patterns observed in electromagnetic waves.

2.3. Geometrical interpretation of the polarization

The vector field $\vec{\psi}^{(n)}(\vec{k})$ defined in equation (2) can be rewritten in the useful parameterization

$$\vec{\psi}^{(n)}(\vec{k}) = \sqrt{S_0} e^{i\varphi} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \chi \\ i \sin \chi \end{pmatrix}. \quad (3)$$

In a mechanical setting, the angle $\theta(\vec{k})$ gives the direction of the major axis of the elliptical orbit and is therefore called the polarization azimuth, $\chi(\vec{k})$ represents a measure of its ellipticity and handedness, and S_0 (the total intensity along the two projections) is the squared diagonal of the axis-aligned bounding box, see figures 2(a) and (b). In addition, φ is the oscillation phase which is a gauge freedom for the eigenstate.

One can characterize the polarization of the field $\vec{\psi}^{(n)}(\vec{k})$ by the Stokes parameters [69, 70]

$$\begin{aligned} S_0 &= |\psi_1|^2 + |\psi_2|^2, \\ S_1 &= |\psi_1|^2 - |\psi_2|^2 = S_0 \cos(2\chi) \cos(2\theta), \\ S_2 &= 2 \text{Re}(\psi_1^* \psi_2) = S_0 \cos(2\chi) \sin(2\theta), \\ S_3 &= 2 \text{Im}(\psi_1^* \psi_2) = S_0 \sin(2\chi). \end{aligned} \quad (4)$$

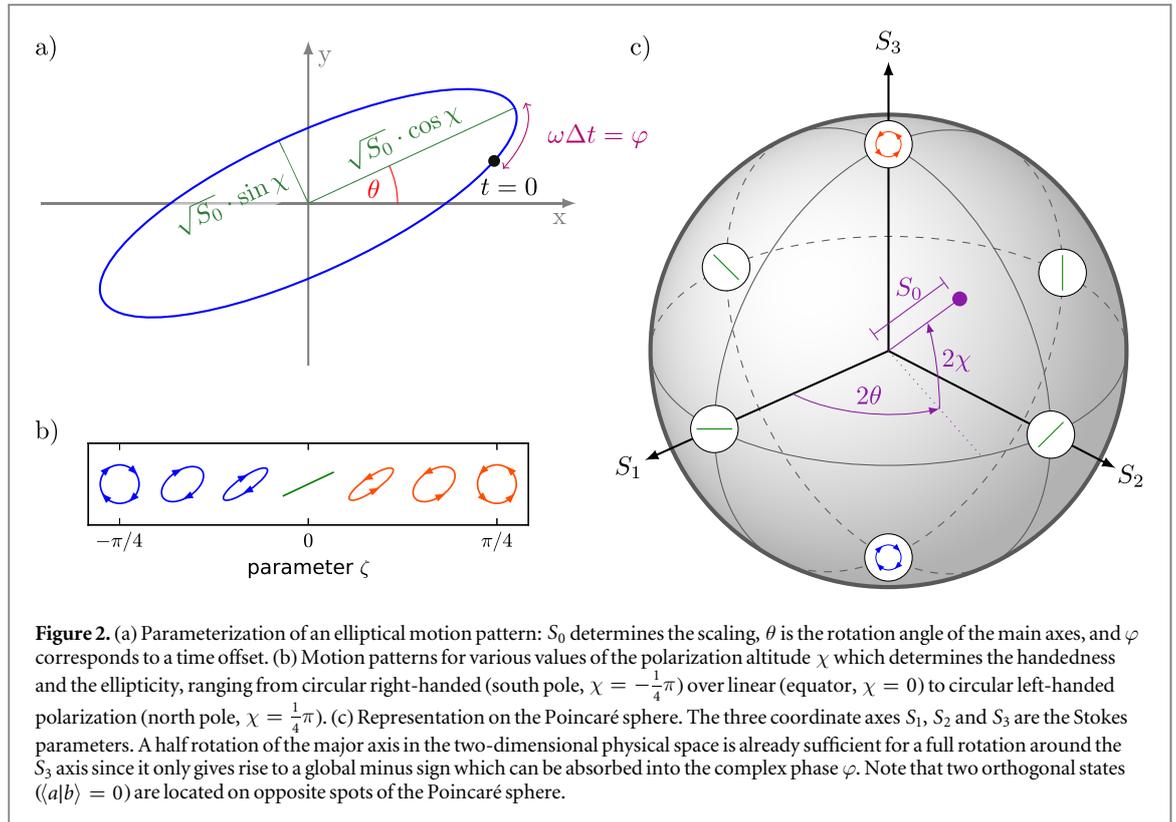


Figure 2. (a) Parameterization of an elliptical motion pattern: S_0 determines the scaling, θ is the rotation angle of the main axes, and φ corresponds to a time offset. (b) Motion patterns for various values of the polarization altitude χ which determines the handedness and the ellipticity, ranging from circular right-handed (south pole, $\chi = -\frac{1}{4}\pi$) over linear (equator, $\chi = 0$) to circular left-handed polarization (north pole, $\chi = \frac{1}{4}\pi$). (c) Representation on the Poincaré sphere. The three coordinate axes S_1 , S_2 and S_3 are the Stokes parameters. A half rotation of the major axis in the two-dimensional physical space is already sufficient for a full rotation around the S_3 axis since it only gives rise to a global minus sign which can be absorbed into the complex phase φ . Note that two orthogonal states ($\langle a|b \rangle = 0$) are located on opposite spots of the Poincaré sphere.

The parameters S_1 , S_2 , and S_3 can be identified with the coordinates on the Poincaré sphere, while S_0 is the corresponding radius, see figure 2(c).

2.4. Chern number

By now, it is well-established [25, 27] that Bloch wave functions defined on a two-dimensional momentum space can be viewed as a mapping from a torus (the Brillouin zone) to a complex vector space, and that close inspection of this mapping can be used to define topological invariants. We will now consider the situation of an arbitrary periodic Hamiltonian where time-reversal symmetry is broken (e. g. due to application of a magnetic field, in the case of electrons). Then, the topological invariant in question is the so-called Chern number (an integer). This quantity is defined for band insulators where the bands remain distinct, i. e. there is a gap between those bands. The Chern numbers of individual bands cannot change under smooth transformations of the Hamiltonian as long as the bands do not touch.

Chern numbers derive their importance from the so-called bulk-boundary correspondence: they can be used to predict whether there will be chiral, topologically protected edge channels located at the boundary of any finite sample (and the net number of clockwise versus counter-clockwise channels). In this sense, a bulk property (the Chern number associated to the infinitely extended periodic system) is connected to the boundary properties of a finite sample.

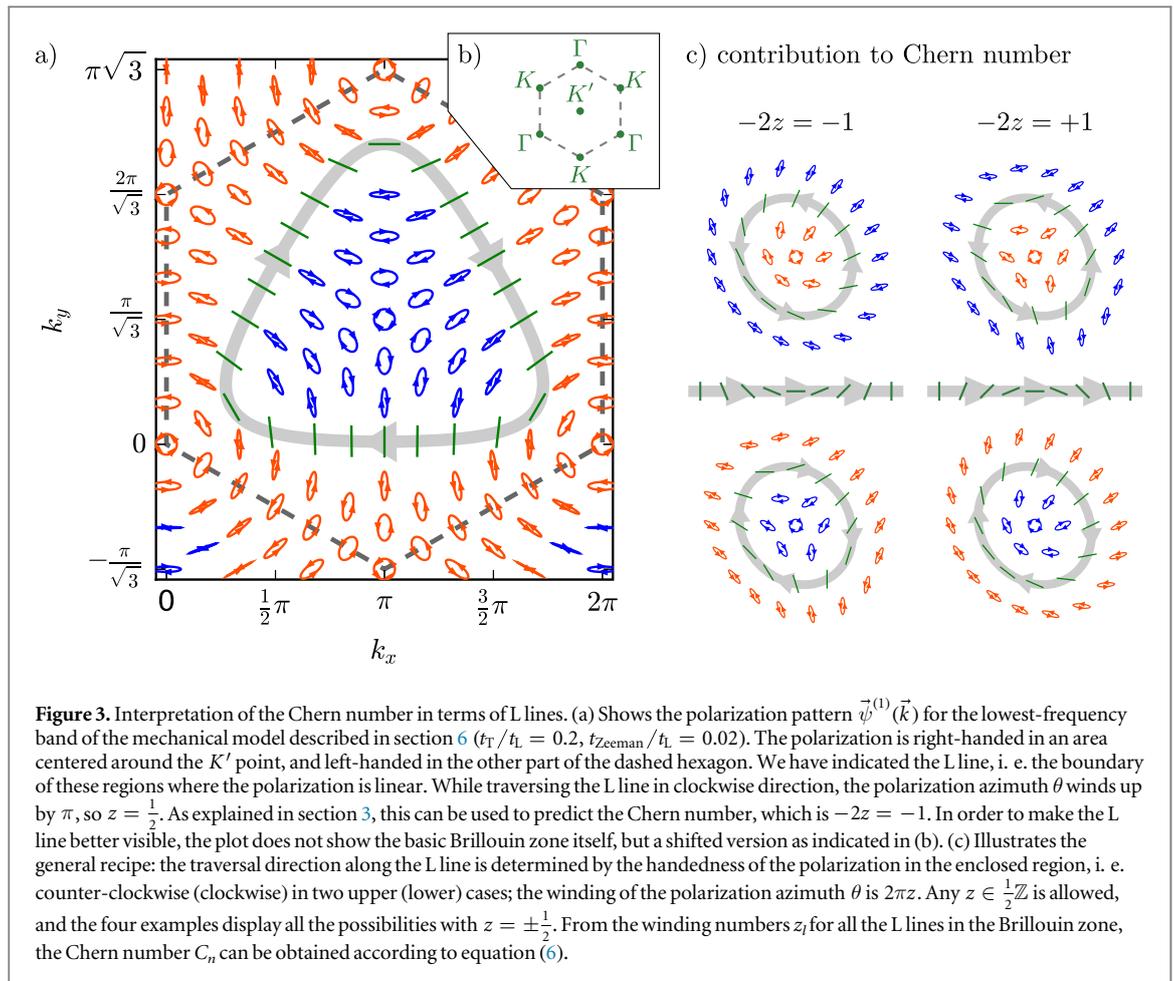
The Chern number of the band n can be calculated as

$$C_n = \frac{1}{2\pi} \iint_{\text{BZ}} [\vec{\nabla}_{\vec{k}} \times \vec{A}_n(\vec{k})] \cdot \vec{e}_z d^2k \in \mathbb{Z}, \quad (5)$$

where $\vec{A}_n(\vec{k}) = -i \langle \Phi_n(\vec{k}) | \vec{\nabla}_{\vec{k}} | \Phi_n(\vec{k}) \rangle$ is the Berry connection and \vec{e}_z is the unit vector in the direction of the z -axis.

The complex amplitudes $\Psi_s^{(n)}(\vec{k})$ from equation (1) completely characterize the Bloch states in the band n . They can already be deduced by inspecting a Bloch state in a single unit cell. Observation of the polarization pattern within one unit cell as a function of \vec{k} is therefore sufficient to determine the Chern number of the n th band. This may even be of experimental relevance, for platforms like mechanical systems where direct measurement of the motion pattern in a given Bloch state is feasible.

It turns out that the Chern number can even be calculated by knowing the projections (in polarization space) of the Bloch state $|\Phi_n(\vec{k})\rangle$ along just two linearly independent directions, as long as these projections do not both vanish simultaneously for the same quasimomentum \vec{k} (for a proof, see section 5). This fact is the basis for what



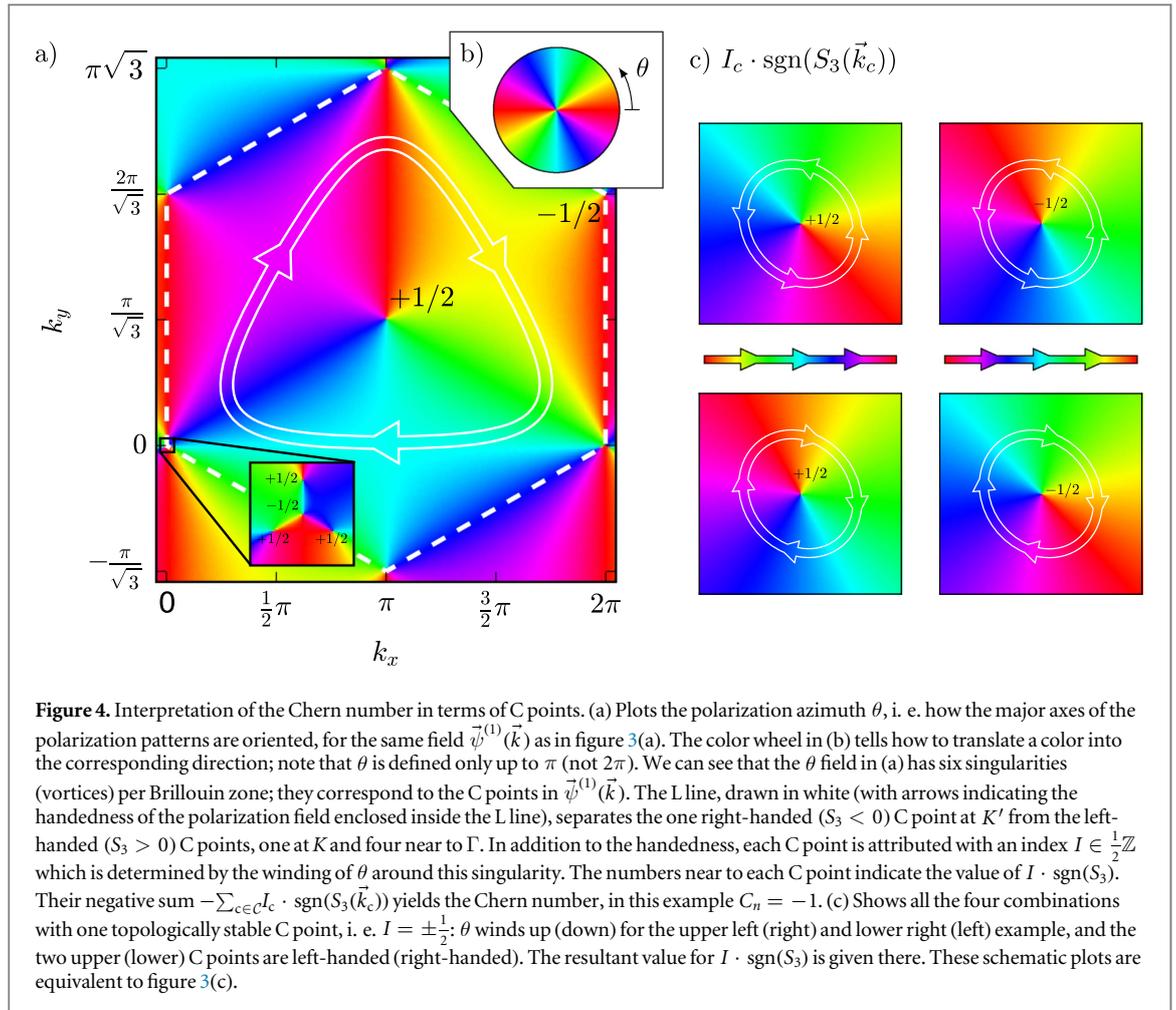
we will discuss in the remainder of this article: the Chern number is directly related to the polarization pattern encoded in the two-component complex vector field $\vec{\psi}^{(n)}(\vec{k})$ introduced in equation (2).

3. L lines and Chern numbers

In the previous section, we have shown how to assign to each band n an auxiliary polarization field $\vec{\psi}^{(n)}(\vec{k})$, see equation (2). In the remainder of the paper, we will make use of this field and of the geometrical interpretation of the polarization to provide a recipe to visualize the Chern number of the band.

A typical polarization pattern is shown in figure 3(a). There, it can be seen that the polarization gets perfectly linear at the interfaces between the regions with left- and right-handed polarization. These interfaces are therefore called L lines. They are a generic feature of continuous polarized fields. L lines have been studied especially in polarized random fields [5, 6]. They are structurally stable upon small perturbations $\vec{\psi}(\vec{k}) \rightarrow \vec{\psi}(\vec{k}) + \delta\vec{\psi}(\vec{k})$. Their robustness can be motivated by a topological argument: because the mapping on the Poincaré sphere is continuous, any path between two points on different hemispheres, i. e. opposite handedness, has to inevitably cross the equator where the polarization is linear.

Below, we explain how to simply read off the Chern numbers by inspecting the L lines of the polarization field. This is possible under a basic assumption: there should be no amplitude vortex in the polarization field. Amplitude vortices are the points where S_0 vanishes. For a two-band model, this condition is equivalent to a degeneracy point, i. e. the band gap has to be closed there, which implies that in this case the Chern number is not well-defined anyway. Otherwise, it can always be argued that amplitude vortices are not topologically stable in two dimensions because they require that four parameters vanish simultaneously (the real and imaginary part of ψ_1 and ψ_2). Thus, they do not appear in generic random fields. While they could emerge as a consequence of some lattice point symmetry (for three or more bands), they will disappear in the presence of a small perturbation that breaks that symmetry (and does not change the Chern number). Alternatively, one could eliminate such structures even without modifying the underlying Hamiltonian, by choosing different projections to define the auxiliary polarization field $\vec{\psi}^{(n)}(\vec{k})$. For the above reasons, the scenario analyzed here where no amplitude vortex is present is not a special case but rather a generic one.



In view of establishing a connection between the L lines and the Chern numbers, we define the winding number z of the polarization azimuth along one L line. This is defined as follows: we observe how the linear polarization direction changes as we traverse the closed L line in a specified sense. If the polarization turns around the origin counterclockwise (clockwise), z acquires a positive (negative) sign, and its absolute value is determined by the number of full turns. This value can be half-integer, since the linear polarization direction is defined only modulo 180° . For our purposes, we specify that the traversal direction of the L line should always be chosen to match the handedness of the polarization field in the enclosed region (clockwise in the example of figure 3(a)).

We now state (and later prove) one of the main messages of this paper: the winding number z of the polarization azimuth along this L line is directly connected to the Chern number. In the simplest case with one L line per Brillouin zone, it is directly given by $C_n = -2z$. If there are multiple L lines per Brillouin zone, their contributions simply add up

$$C_n = -2 \sum_{l \in \mathcal{L}_n} z_l. \quad (6)$$

In particular, the Chern number is automatically zero if there are no L lines at all.

4. C point classifications and Chern numbers

Next, we show that the Chern numbers can also be related to the properties of the so-called C points [6, 7]. The C points are the points where the polarization gets perfectly circular. In other words, they correspond to the poles of the Poincaré sphere, and they are the nodes of the scalar field [71]

$$\sigma = S_1 + iS_2 = S_0 \cos(2\chi) e^{2i\theta} \quad (7)$$

whose complex phase is directly related to the polarization azimuth θ .

It turns out that the C points can be classified according to four different criteria [7, 72, 73]. Below, we show that two of these criteria are relevant in determining the Chern number. The first relevant criterium is based on their handedness, i. e. whether the polarization is purely left-handed ($S_3 > 0$) or right-handed ($S_3 < 0$). The other relevant criterium is based on the so-called I classification: the polarization azimuth θ is not well-defined at a C point, and the index I counts its winding around the corresponding singularity (in counter-clockwise direction)

$$I = \frac{1}{2\pi} \oint d\theta \in \frac{1}{2}\mathbb{Z}. \quad (8)$$

Its half-integer nature emerges from the fact that a half rotation of θ corresponds to a full rotation around the S_3 axis, i. e. it is already sufficient to restore the original elliptical motion pattern. We note that a C point, by definition, emerges at any crossing of the contours $S_1 = 0$ and $S_2 = 0$. Whenever any random perturbation is introduced, all the crossings with $I \neq \pm\frac{1}{2}$ are splitted into several C points (the sum over all C points of the I indexes is conserved in this process). In contrast, those C points with $I = \pm\frac{1}{2}$ are structurally stable, i. e. they will not split, and they cannot be created or destroyed spontaneously; pairwise creation and annihilation is however possible.

It turns out that also the sum of the I indexes weighted by the corresponding handedness (the sign of $S_3(\vec{k}_c)$) is conserved for all these continuous transformations. In other words, such a weighted sum over all C points is a topological invariant. In the present setting where the polarization field is associated to the n th energy band of a translationally invariant system, the topological invariant discussed above can be identified with the Chern number

$$C_n = -\sum_{c \in \mathcal{C}} I_c \cdot \text{sgn}(S_3(\vec{k}_c)) = \sum_{c \in \mathcal{C}_-} 2I_c = -\sum_{c \in \mathcal{C}_+} 2I_c, \quad (9)$$

where \mathcal{C}_+ (\mathcal{C}_-) contains all the C points with $\text{sgn}(S_3(\vec{k}_c)) > 0$ (< 0) and $\mathcal{C} = \mathcal{C}_+ \cup \mathcal{C}_-$.

Equation (9) can be obtained from equation (6) by using that the winding of the polarization azimuth equals the sum of the enclosed C point indices. The two last conversions in equation (9) are possible because the winding number along the boundaries of the Brillouin zone has to vanish (contributions on opposite edges will always cancel), so the unweighted index sum always vanishes: $\sum_{c \in \mathcal{C}} I_c = 0$.

Like equation (6), equation (9) applies only in the absence of amplitude vortices. Again, as we already discussed above in the context of L lines, such amplitude vortices are not topologically stable and do not appear generically. They can typically be eliminated by choosing a different projection. However, if needed, it is easy to extend equation (9) by their contributions

$$C_n = -\sum_{c \in \mathcal{C}} I_c \cdot \text{sgn}(S_3(\vec{k}_c)) - \sum_{a \in \mathcal{A}} J_a, \quad (10)$$

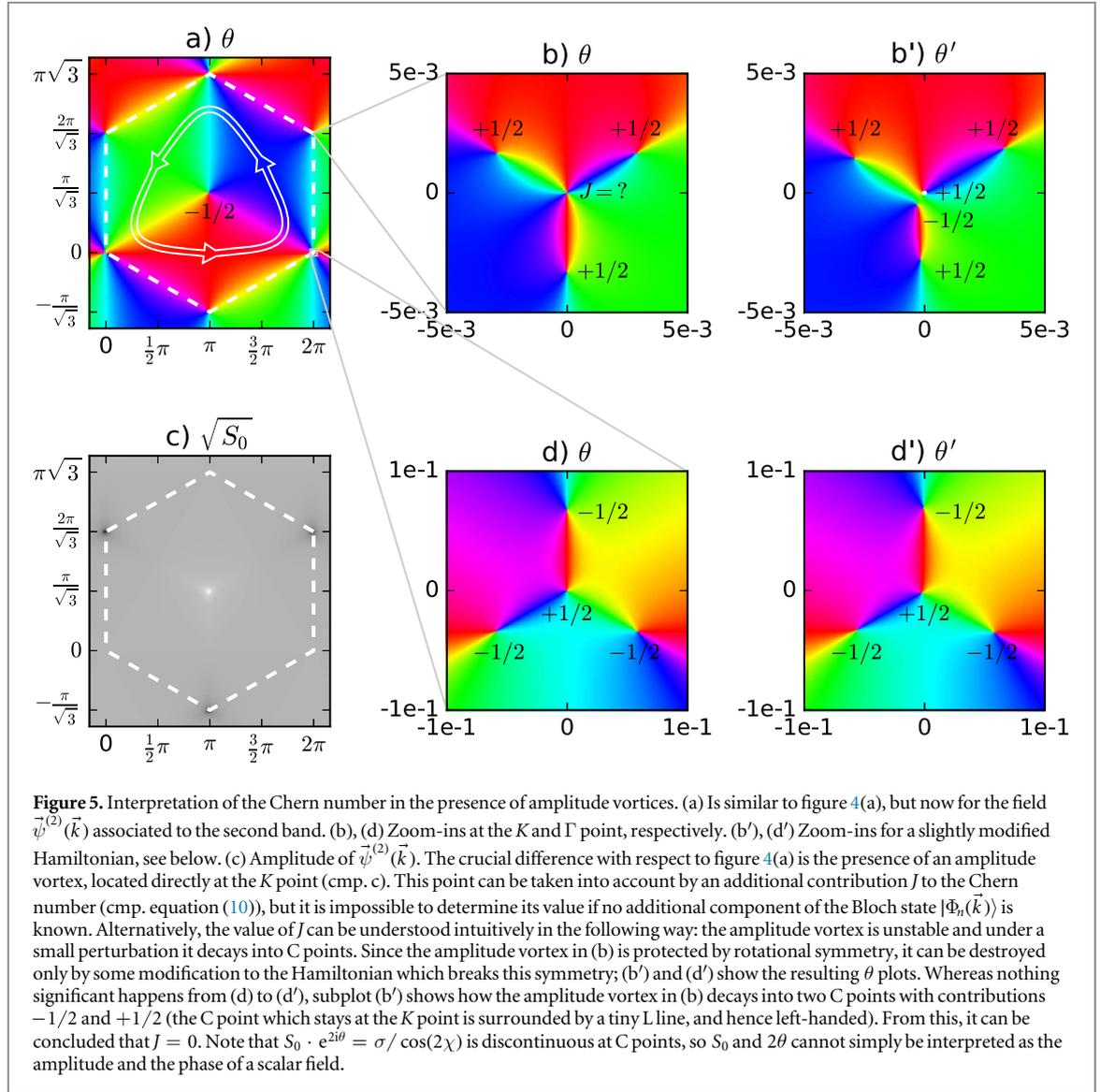
where \mathcal{A} consists of the labels for the amplitude vortices. The concrete value of J_a cannot be deduced from the values of $\Psi_{s_1}^{(n)}(\vec{k})$ and $\Psi_{s_2}^{(n)}(\vec{k})$ alone, but it can be determined if an additional projection $\Psi_{s_3}^{(n)}(\vec{k})$ is known (which must not vanish at the same point \vec{k}). Alternatively, J_a can be interpreted as the contributions of the C points into which the amplitude vortex decays under a small perturbation of the Hamiltonian. The treatment of amplitude vortices is illustrated in figure 5.

5. Connection to other methods and proof

We will now present a proof for our recipe in section 3 to obtain the Chern number from the L lines. In addition, we will relate the schemes presented in sections 3 and 4 with another, well-known technique to visualize the Chern number, the Skyrmion configuration method, and discuss common features and differences.

5.1. Kohmoto method

We will prove our main result formula, equation (6), for the Chern number by making a connection to the well-known analytical method for calculating the Chern numbers which is due to Kohmoto [18]. In Kohmoto's method, one fixes a gauge by requiring that the overlap between the Bloch eigenstates of a particular band with a fixed state, for example $|j, s\rangle$, is a real positive number. When this prescription is well defined in the whole Brillouin zone, the Chern number can be interpreted as the flux of a curl piercing a closed surface (the Brillouin zone which is a torus), see equation (5). In this case, Stokes theorem ensures that the Chern number will be zero. Thus, in all topologically non-trivial cases the prescription will be ill-defined for one or more values of the quasi-momentum \vec{k} . These points are commonly referred to as obstructions. The reason why the gauge prescription is ill-defined at an obstruction is simply that the corresponding overlap matrix element vanishes there. In this situation, it is necessary to divide the Brillouin zone into several regions where different gauge conditions are



applied. Then, the Chern number is encoded in the phase mismatches φ_{AB} at the interface between the different regions, $\exp(i\varphi_{AB}(\vec{k})) := \langle \Phi_A(\vec{k}) | \Phi_B(\vec{k}) \rangle$ where $|\Phi_{A,B}(\vec{k})\rangle$ is the wavefunction in the gauge chosen in the regions A and B , respectively. If the subdivision into regions is chosen such that there are no trijunctions, i. e. all the interfaces are closed loops, the Chern number is

$$C_n = \frac{1}{2\pi} \sum_{A \subset B} \oint d\varphi_{AB}, \quad (11)$$

where the sum is taken over interfaces between neighboring regions A, B (where A is enclosed by B), and the integral is along the respective interface line. In practice, an effective route to analytically calculate the Chern number is to choose one gauge $|\Phi_B(\vec{k})\rangle$, determine the corresponding obstructions, and, for each obstruction j , fix a suitable gauge $|\Phi_{A_j}(\vec{k})\rangle$ in a small region around it. This procedure reduces the task of calculating the Chern number to computing the phase mismatch in a few infinitesimally small regions. This often paves the way to an analytical treatment.

Our scheme discussed in section 3 can be viewed as a variation of Kohmoto's method where rather than trying to fix the same gauge in the whole Brillouin zone and inserting patches where this does not work, we focus from the very beginning on two different gauge choices. For this purpose, we define the left- and right-handed components of an eigenstate $|\Phi\rangle$ (here, we omit the band index n) using the corresponding auxiliary polarization field from equation (2)

$$\Psi_L[\Phi] := \frac{1}{\sqrt{2}}(\Psi_{s_1}[\Phi] - i\Psi_{s_2}[\Phi]) \quad \Psi_R[\Phi] := \frac{1}{\sqrt{2}}(\Psi_{s_1}[\Phi] + i\Psi_{s_2}[\Phi]). \quad (12)$$

Demanding either Ψ_L or Ψ_R to be real, we obtain two distinct gauge conditions for the state $|\Phi\rangle$

$$\Psi_L[\Phi_L(\vec{k})] \stackrel{!}{\in} \mathbb{R}^+ \quad \Psi_R[\Phi_R(\vec{k})] \stackrel{!}{\in} \mathbb{R}^+. \quad (13)$$

This implicitly defines the gauge Φ_L (Φ_R) by demanding its component Ψ_L (Ψ_R) to be real-positive; starting from an arbitrary gauge $|\Phi_0(\vec{k})\rangle$, this can always be achieved by the choices

$$|\Phi_L(\vec{k})\rangle = \frac{\Psi_L^*[\Phi_0(\vec{k})]}{|\Psi_L[\Phi_0(\vec{k})]|} |\Phi_0(\vec{k})\rangle \quad |\Phi_R(\vec{k})\rangle = \frac{\Psi_R^*[\Phi_0(\vec{k})]}{|\Psi_R[\Phi_0(\vec{k})]|} |\Phi_0(\vec{k})\rangle. \quad (14)$$

We will fix the gauge using the condition for Ψ_L when the state $|\Phi\rangle$ is in the northern hemisphere of the Poincaré sphere (corresponding to $|\Psi_L| > |\Psi_R|$, i. e. left-handed polarization), and otherwise the one for Ψ_R . This construction is possible because we have assumed that there are no amplitude vortices, so the larger component can never become zero.

By applying equation (11) to this situation, we see the remarkable fact that knowing two components of the Bloch band $|\Phi\rangle$, here Ψ_L and Ψ_R , is sufficient to determine its Chern number as they fix the gauges Φ_L and Φ_R , respectively. This already confirms our claim from section 2 that we need the projections along only two linearly independent polarization directions.

What remains is to show that the phase mismatch between these two gauges, defined by $\exp(i\varphi_{RL}(\vec{k})) := \langle \Phi_R(\vec{k}) | \Phi_L(\vec{k}) \rangle$, is directly related to the polarization azimuth: $\varphi_{RL} = 2\theta$. For this purpose, we first apply $\langle \Phi_R | \Phi_L \rangle = \Psi_L^* \Psi_R / |\Psi_L^* \Psi_R|$. By inserting equation (12) into the numerator and comparing the result with the definition of σ in equation (7) rewritten in terms of Ψ_{s_1} and Ψ_{s_2} using equation (4), we find that $\sigma = 2\Psi_L^* \Psi_R$. So, $\exp(i\varphi_{RL}(\vec{k})) = \sigma / |\sigma| = \exp(2i\theta)$ according to the last equality in equation (7). This directly leads to the result of this calculation which has been already stated in equation (6).

5.2. Skyrmion number

It has been discussed before [26, 27] that in a two-band model, the Chern number can be related to the skyrmion number: every eigenstate is represented by a Bloch vector \vec{d} which is analogous to the vector (S_1, S_2, S_3) here; the skyrmion number

$$\frac{1}{4\pi} \iint_{\text{BZ}} \vec{d} \cdot \left(\frac{\partial \vec{d}}{\partial x} \times \frac{\partial \vec{d}}{\partial y} \right) d^2k \quad (15)$$

counts how many times the mapping $\vec{d}(\vec{k})$ wraps the Bloch sphere, and is identical to the Chern number of the corresponding energy band. The skyrmion number has been developed in the context of magnetic skyrmions, quasiparticles which appear in special magnetic materials [74].

In that sense, our method generalizes this ‘skyrmion method’ to models with an arbitrary number of bands. In addition, it connects the Chern number to the theory of L lines and C points, and it provides an interpretation of the Chern number in terms of directly visible features.

6. Application to mechanics and optics

The illustrative examples in figures 3–5 have actually been calculated for a specific model which has already been implemented for gyroscopic pendula [2] and proposed for photon-exciton polaritons [66–68, 75]. This tight-binding model features a \mathbb{C}^2 polarization degree of freedom on a honeycomb lattice. The on-site polarization degree of freedom represents two in-plane directions of motion of a two-dimensional mechanical (e. g. gyroscopic) oscillator or two directions of the field distributions in the case of photon-exciton polaritons. Because of the two sublattices in a honeycomb lattice, there are in total four energy/frequency bands in such a tight-binding model. For a pair of sites, the longitudinal (transversal) mode is defined such that it is parallel (perpendicular) to the connecting line. The longitudinal (transversal) modes of nearest neighbor sites are coupled with coupling strength t_L (t_T). The toy model for this interaction in mechanics are two-dimensional mass-spring networks. In this setting, additional on-site Zeeman splitting—breaking the time-reversal symmetry—gives rise to AQHE edge channels [38, 75].

Because of its simple conception and the possibility to directly observe the polarization patterns in a mechanical system, such a system is a promising platform to demonstrate the determination and visualization of the Chern number with the technique described in this article. Whereas a macroscopic implementation of this model seems straightforward, a mesoscopic or microscopic one is more challenging due to the difficulties to fabricate springs at these lengthscales. However, there might be some more easily fabricated systems which intrinsically have this kind of interaction, e. g. arrays of nanopillars [3, 4]. Unpublished experimental data [76]

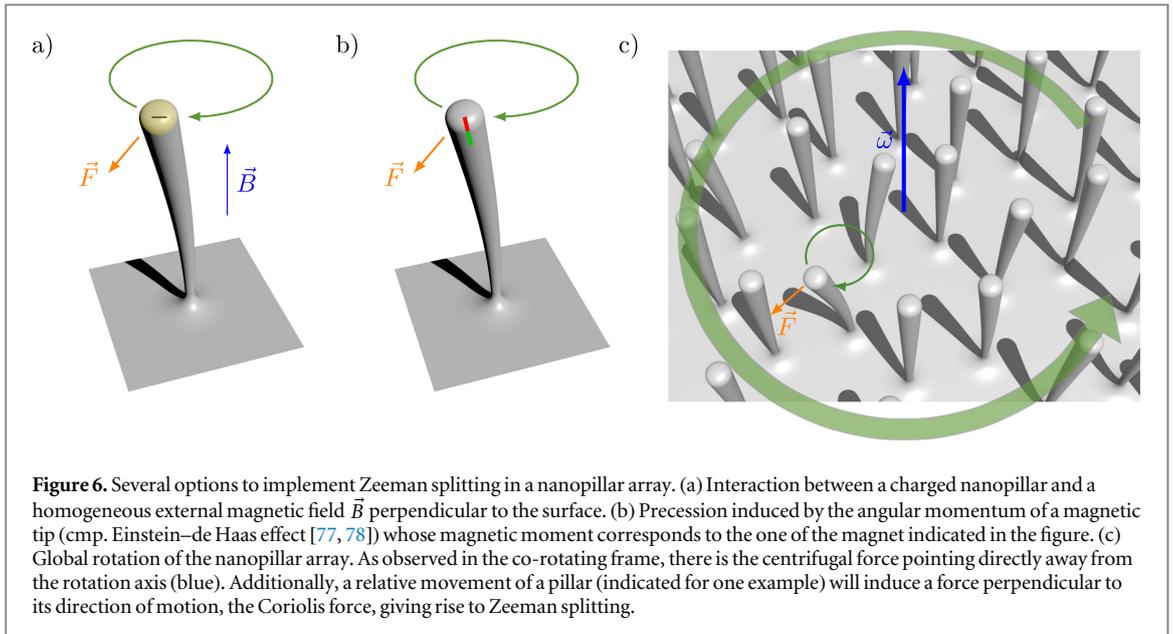


Figure 6. Several options to implement Zeeman splitting in a nanopillar array. (a) Interaction between a charged nanopillar and a homogeneous external magnetic field \vec{B} perpendicular to the surface. (b) Precession induced by the angular momentum of a magnetic tip (cmp. Einstein–de Haas effect [77, 78]) whose magnetic moment corresponds to the one of the magnet indicated in the figure. (c) Global rotation of the nanopillar array. As observed in the co-rotating frame, there is the centrifugal force pointing directly away from the rotation axis (blue). Additionally, a relative movement of a pillar (indicated for one example) will induce a force perpendicular to its direction of motion, the Coriolis force, giving rise to Zeeman splitting.

indicate that the motion of neighboring nanopillars is coupled, and the described coupling type is the only linear interaction model which respects all the present symmetries.

Conceptually, there are different options to implement Zeeman splitting including charged nanopillars in a homogeneous magnetic field, the conserved angular momentum [2] of magnetic tips in analogy to the Einstein–de Haas effect [77, 78], and the Coriolis force [38] (cmp. figure 6). However, order-of-magnitude estimates indicate that in a nanopillar setup only the latter one can be strong enough to overcome the effect of dissipation and disorder.

Another possible application, in the context of optics, are dielectric materials with a discrete translational symmetry in two dimensions; chiral edge channels analogous to the QHE have already gained experimental verification in photonic crystals [48]. In these systems, the (complex) electric field $\vec{E}(x, y)$ takes over the role of the electron wave function. If arbitrary quasimomentum states can be excited and the resultant polarization of the electric field $\vec{E}(x_0, y_0)$ can be measured at one single point (x_0, y_0) which is fixed previously, mapping these polarization patterns onto the Brillouin zone allows to directly read off the Chern number. Note that although the degree of polarization of the field \vec{E} is only two, the dimension of the Bloch space is infinite due to the dependency on the two (continuous) coordinates x and y . This is one example where the generalization to an arbitrary number of bands, compared to the Skyrmion method (see section 5), becomes important.

7. Measuring the Chern number

Due to its outstanding role in the description of topologically protected edge transport, the measurement of the Chern number may be a crucial point in an experimental analysis of a Chern insulator.

One possible approach to obtain the Chern numbers is to measure the quantized Hall conductance in the band gaps. For example, this measurement works with very high precision in the original setup, the quantum Hall effect [13, 14]. However, there might be some situations in which wrong results are obtained due to technical limitations, e. g. if edge states in one band gap exist in principle, but they have a too large penetration length compared to the system size to be detectable. Or, if there is only a local, but not a global band gap, the Chern number is well-defined, but cannot be measured this way. More importantly, this method is by a fundamental reason restricted to fermionic systems as it relies on the quantized Hall conductance. Since there is no equivalent quantity in a bosonic system, different approaches to measure the Chern number for those had to be developed [28–32].

Our method could serve as an alternative to these techniques as it offers a very direct approach: experimental techniques like Fourier transform spectroscopy or the excitation with quasimomentum modes give full access to the Bloch states; this is sufficient input to identify the L lines and C points, and to determine their relevant properties. Therefore, straightforward application of the schemes in sections 3 and 4 can also be used to measure the Chern numbers for any type of Chern insulator.

Its universality and directness could be a great advantage in comparison to other strategies. We also emphasize that, as opposed to, for example, a numerical calculation using a band structure simulation (there are

both analytical [18, section III] and numerical [79] methods), our method requires neither precise knowledge of the actual system parameters nor even the understanding of the underlying microscopic mechanisms.

8. Conclusion

The central aspect of this article is the connection between two topological concepts in modern physics: we have related the Chern number, one of the central quantities in the context of the QHE and the AQHE, to L lines and C points, the structurally stable objects known from random fields in polarized optics. What we obtain is a graphical interpretation of the Chern number which makes this abstract quantity a bit more tangible.

The term polarization is fundamental in the geometric description of oscillating electromagnetic fields. Even more, we can directly see what it means to a mechanical system by looking at the trajectories. This intuition makes these two classes of systems play an exclusive role for the scheme presented here.

Besides the visualization, we have also discussed the possibility to use this technique as a tool to measure the Chern number in an experiment.

Acknowledgments

This work was supported by European Research Council Starting Grant OPTOMECH (to V.P. and F.M.). This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant: 732894 (Hybrid Optomechanical Technologies) (to V.P. and F.M.). This work was also supported by the Julian Schwinger Foundation with the grant: JSF-16-03-0000 (to V.P.).

ORCID iDs

Vittorio Peano  <https://orcid.org/0000-0003-1931-5080>

Florian Marquardt  <https://orcid.org/0000-0003-4566-1753>

References

- [1] Süsstrunk R and Huber S D 2015 *Science* **349** 47–50
- [2] Nash L M, Kleckner D, Read A, Vitelli V, Turner A M and Irvine W T 2015 *Proc. Natl Acad. Sci. USA* **112** 14495–500
- [3] Paulitschke P 2011 Mechanische Charakterisierung einzelner & periodisch angeordneter nanoskaliger GaAs-Säulen *PhD Thesis* Ludwig-Maximilians-Universität München
- [4] Paulitschke P, Seltner N, Lebedev A, Lorenz H and Weig E M 2013 *Appl. Phys. Lett.* **103** 261901
- [5] Nye J F 1983 Polarization effects in the diffraction of electromagnetic waves: the role of disclinations *Proc. R Soc.* **387** 105–32
- [6] Dennis M R, O'Holleran K and Padgett M J 2009 *Prog. Opt.* **53** 293–363
- [7] Nye J F 1983 Lines of circular polarization in electromagnetic wave fields *Proc. R Soc.* **389** 279–90
- [8] Berry M V, Dennis M R and Lee R L Jr 2004 *New J. Phys.* **6** 162
- [9] Flossmann F, Kevin O, Dennis M R and Padgett M J 2008 *Phys. Rev. Lett.* **100** 203902
- [10] Banzer P, Neugebauer M, Aiello A, Marquardt C, Lindlein N, Bauer T and Leuchs G 2013 The photonic wheel—demonstration of a state of light with purely transverse angular momentum *J. Eur. Opt. Soc.* **8** 13032
- [11] Aiello A, Banzer P, Neugebauer M and Leuchs G 2015 *Nat. Photon.* **9** 789–95
- [12] Aiello A and Banzer P 2016 *J. Opt.* **18** 085605
- [13] von Klitzing K, Dorda G and Pepper M 1980 *Phys. Rev. Lett.* **45** 494
- [14] von Klitzing K 1986 *Rev. Mod. Phys.* **58** 519
- [15] Haldane F D M 1988 *Phys. Rev. Lett.* **61** 2015
- [16] Thouless D J, Kohmoto M, Nightingale M P and den Nijs M 1982 *Phys. Rev. Lett.* **49** 405
- [17] Avron J E, Seiler R and Simon B 1983 *Phys. Rev. Lett.* **51** 51
- [18] Kohmoto M 1985 *Ann. Phys., NY* **160** 343–54
- [19] Kane C L and Mele E J 2005 *Phys. Rev. Lett.* **95** 226801
- [20] Bernevig B A, Hughes T L and Zhang S C 2006 *Science* **314** 1757–61
- [21] König M, Wiedmann S, Brüne C, Roth A, Buhmann H, Molenkamp L W, Qi X L and Zhang S C 2007 *Science* **318** 766–70
- [22] Qi X L and Zhang S C 2011 *Rev. Mod. Phys.* **83** 1057
- [23] Bansil A, Lin H and Das T 2016 *Rev. Mod. Phys.* **88** 021004
- [24] Kane C L and Mele E J 2005 *Phys. Rev. Lett.* **95** 146802
- [25] Hasan M Z and Kane C L 2010 *Rev. Mod. Phys.* **82** 3045
- [26] Fradkin E 2013 *Field Theories of Condensed Matter Physics* (Cambridge: Cambridge University Press)
- [27] Bernevig B A and Hughes T L 2013 *Topological Insulators and Topological Superconductors* (Princeton, NJ: Princeton University Press)
- [28] Aidelsburger M, Lohse M, Schweizer C, Atala M, Barreiro J T, Nascimbène S, Cooper N, Bloch I and Goldman N 2015 *Nat. Phys.* **11** 162–6
- [29] Price H M, Zilberberg O, Ozawa T, Carusotto I and Goldman N 2016 *Phys. Rev. B* **93** 245113
- [30] Mittal S, Ganeshan S, Fan J, Vaezi A and Hafezi M 2016 *Nat. Photon.* **10** 180–3
- [31] Bardyn C E, Huber S D and Zilberberg O 2014 *New J. Phys.* **16** 123013

- [32] Schroer M, Kolodrubetz M, Kindel W, Sandberg M, Gao J, Vissers M, Pappas D, Polkovnikov A and Lehnert K 2014 *Phys. Rev. Lett.* **113** 050402
- [33] Prodan E and Prodan C 2009 *Phys. Rev. Lett.* **103** 248101
- [34] Zhang L, Ren J, Wang J S and Li B 2010 *Phys. Rev. Lett.* **105** 225901
- [35] Peano V, Brendel C, Schmidt M and Marquardt F 2015 *Phys. Rev. X* **5** 031011
- [36] Wang P, Lu L and Bertoldi K 2015 *Phys. Rev. Lett.* **115** 104302
- [37] Yang Z, Gao F, Shi X, Lin X, Gao Z, Chong Y and Zhang B 2015 *Phys. Rev. Lett.* **114** 114301
- [38] Kariyado T and Hatsugai Y 2015 *Sci. Rep.* **5** 18107
- [39] Chen Z G and Wu Y 2016 *Phys. Rev. Appl.* **5** 054021
- [40] Fleury R, Khanikaev A B and Alù A 2016 *Nat. Commun.* **7** 11744
- [41] Matlack K H, Serra-Garcia M, Palermo A, Huber S D and Daraio C 2016 arXiv:1612.02362
- [42] Mitchell N P, Nash L M, Hexner D, Turner A and Irvine W 2016 arXiv:1612.09267
- [43] Souslov A, van Zuijden B C, Bartolo D and Vitelli V 2017 Topological sound in active-liquid metamaterials *Nat. Phys.* **13** 1091–4
- [44] Süssstrunk R and Huber S D 2016 *Proc. Natl Acad. Sci.* **113** E4767–75
- [45] Lu L, Joannopoulos J D and Soljačić M 2014 *Nat. Photon.* **8** 821–9
- [46] Haldane F and Raghu S 2008 *Phys. Rev. Lett.* **100** 013904
- [47] Raghu S and Haldane F 2008 *Phys. Rev. A* **78** 033834
- [48] Wang Z, Chong Y, Joannopoulos J D and Soljačić M 2009 *Nature* **461** 772–5
- [49] Koch J, Houck A A, Le Hur K and Girvin S 2010 *Phys. Rev. A* **82** 043811
- [50] Umucalilar R and Carusotto I 2011 *Phys. Rev. A* **84** 043804
- [51] Hafezi M, Demler E A, Lukin M D and Taylor J M 2011 *Nat. Phys.* **7** 907–12
- [52] Fang K, Yu Z and Fan S 2012 *Nat. Photon.* **6** 782–7
- [53] Petrescu A, Houck A A and Le Hur K 2012 *Phys. Rev. A* **86** 053804
- [54] Hafezi M, Mittal S, Fan J, Migdall A and Taylor J 2013 *Nat. Photon.* **7** 1001–5
- [55] Schmidt M, Kessler S, Peano V, Painter O and Marquardt F 2015 *Optica* **2** 635–41
- [56] Peano V, Houde M, Marquardt F and Clerk A A 2016 *Phys. Rev. X* **6** 041026
- [57] Anderson B M, Ma R, Owens C, Schuster D I and Simon J 2016 *Phys. Rev. X* **6** 041043
- [58] Jotzu G, Messer M, Desbuquois R, Lebrat M, Uehlinger T, Greif D and Esslinger T 2014 *Nature* **515** 237–40
- [59] Goldman N et al 2014 *Rep. Prog. Phys.* **77** 126401
- [60] Onose Y, Ideue T, Katsura H, Shiomi Y, Nagaosa N and Tokura Y 2010 *Science* **329** 297–9
- [61] Shindou R, Matsumoto R, Murakami S and Ohe J I 2013 *Phys. Rev. B* **87** 174427
- [62] Zhang L, Ren J, Wang J S and Li B 2013 *Phys. Rev. B* **87** 144101
- [63] Mook A, Henk J and Mertig I 2014 *Phys. Rev. B* **90** 024412
- [64] Chisnell R, Helton J, Freedman D, Singh D, Bewley R, Nocera D and Lee Y 2015 *Phys. Rev. Lett.* **115** 147201
- [65] Hafezi M and Rabl P 2012 *Opt. Express* **20** 7672–84
- [66] Jacqmin T, Carusotto I, Sagnes I, Abbarchi M, Solnyshkov D, Malpuech G, Galopin E, Lemaître A, Bloch J and Amo A 2014 *Phys. Rev. Lett.* **112** 116402
- [67] Sala V et al 2015 *Phys. Rev. X* **5** 011034
- [68] Nalitov A, Malpuech G, Terças H and Solnyshkov D 2015 *Phys. Rev. Lett.* **114** 026803
- [69] Born M and Wolf E 1959 *Principles of Optics. Electromagnetic Theory of Propagation, Interference and Diffraction of Light* (Oxford: Pergamon)
- [70] Brosseau C 1998 *Fundamentals of Polarized Light: A Statistical Optics Approach* (New York: Wiley)
- [71] Konukhov A I and Melnikov L A 2001 *J. Opt. B: Quantum Semiclass. Opt.* **3** S139
- [72] Berry M V and Hannay J 1977 *J. Phys. A: Math. Gen.* **10** 1809
- [73] Dennis M R 2008 *Opt. Lett.* **33** 2572–4
- [74] Nagaosa N and Tokura Y 2013 *Nat. Nanotechnol.* **8** 899–911
- [75] Nalitov A, Solnyshkov D and Malpuech G 2015 *Phys. Rev. Lett.* **114** 116401
- [76] Weig E M 2017 private communications
- [77] Einstein A 1915 *Naturwissenschaften* **3** 237–8
- [78] Einstein A and de Haas W J 1915 Experimental proof of the existence of Amperes molecular currents *Proc. KNAW* **181** 696
- [79] Fukui T, Hatsugai Y and Suzuki H 2005 *J. Phys. Soc. Japan* **74** 1674–7