LMU-ASC 76/17

Ricci-flat metrics on the cone over $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$

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Abstract. We describe a framework for constructing the Ricci-flat metrics on the total space of the canonical bundle over $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ (the del Pezzo surface of rank one). We construct explicitly the first-order deformation of the so-called 'orthotoric metric' on this manifold. We also show that the deformation of the corresponding conformal Killing-Yano form does not exist.

Whereas Ricci-flat metrics on *compact* Calabi-Yau manifolds are difficult to construct, there exist many explicitly known Ricci-flat metrics on *noncompact* Calabi-Yau manifolds (the first examples being [1], [2], [3]). The reason is that these latter metrics possess sufficiently many isometries. The role of these metrics is that they describe the geometry of the compact Calabi-Yau manifold in the vicinity of a singularity, after it has been resolved.

One particular type of singularity that can occur for a complex Calabi-Yau threefold is that of a complex cone over a complex surface. In this article we will be dealing with a particular case, when the surface is the del Pezzo surface of rank one (also known as a Hirzebruch surface F_1) — the blow-up of $\mathbb{CP}^2$ at one point. Topologically, the blow-up of $\mathbb{CP}^2$ at one point is the same as the connected sum $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ [4], where $\overline{\mathbb{CP}^2}$ means $\mathbb{CP}^2$ with inverted orientation. In fact, one explicit Ricci-flat metric on

$$Y := \text{ Total space of the canonical bundle over } \mathbb{CP}^2\#\overline{\mathbb{CP}^2} \tag{1}$$

is known [5] – it is a metric that can be obtained by the so-called 'orthotoric ansatz' [6] and later will be referred to as the orthotoric metric. This ansatz follows from the requirement that the corresponding metric possesses a conformal Killing-Yano form of type $(1,1)$ with respect to the Hodge decomposition [6, 7]. The main results of the present paper concern the study of the first-order deformation of the orthotoric metric:

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Proposition 1. **There exists a first-order Ricci-flat deformation δg of the orthotoric metric on Y. This deformation corresponds to a change $\delta\omega$ of the Kähler class of the metric that lies in the compactly supported cohomology group $\delta\omega \in H^2_c(Y, \mathbb{R})$. The metric, before and after the deformation, is asymptotic to the metric cone over the Sasaki-Einstein manifold $Y^{2,1}$.**

Proposition 2. **There does not exist a deformation of the conformal Killing-Yano tensor, corresponding to the deformation of the metric.**

The structure of the paper is as follows.

In § 1 for completeness of the exposition we recall the salient aspects of toric differential geometry, which are well-known but necessary for the foregoing discussion. Most importantly, we introduce the 'master' function that determines the metric on a toric Kähler manifold – the so-called symplectic potential G.

In § 2 we introduce the manifold Y as a toric manifold.

In § 3 we write out a Ricci-flatness equation for the metric on Y. In § 3.1 we introduce the moment polytope for a $U(1)^3$ action on Y. We explain that most of the information is in fact encoded in a two-dimensional slice of this polytope, which is an unbounded polygon. We describe its topological properties and, in particular, determine the normal bundles of the two $\mathbb{CP}^1$'s embedded in the corners of the polygon.

In § 4.1 we review a particular solution of the Ricci-flatness equation – it has the form of a metric cone, i.e. it defines a metric of the type $ds^2 = dr^2 + r^2\widetilde{ds^2}$. The expression for $\widetilde{ds^2}$ can be found explicitly and leads to the Sasakian manifolds $Y^{p,q}$. In § 4.2 we show how the topology of the underlying del Pezzo cone fixes the Sasakian manifold to be $Y^{2,1}$.

In § 5 we prove that the solution of the Ricci-flatness equation is unique, once the moment polytope is specified. This is similar in spirit to the proof of [8], the main difference being in the analysis of the behavior at infinity – the issue arises due to the non-compactness of the cone. The key technical result is the lower bound for the first non-zero eigenvalue of the Laplacian on $Y^{2,1}$, which is the subject of Lemma 3. The result of this section implies that the only potential moduli of the metric are the moduli of the moment polytope.

In oder to introduce the known metric on Y – the orthotoric metric – we come in § 6 to the discussion of conformal Killing-Yano forms (CKYF), with particular emphasis on such forms on Calabi-Yau manifolds. In § 6.1.1 we show that the $(2, 0)$ part of such a form is highly constrained – we show that a vector 'dual' to the $(2, 0)$ part of a conformal Killing-Yano form has to be a zero-vector of the Riemann tensor (Proposition 5). If one insists that the $(2, 0)$ part is zero, i.e. the form is of type $(1, 1)$, one arrives at an object termed twistor form or Hamiltonian 2-form[1], and the existence of such an object severely constrains the metric [6]. We review the calculations of [6] in § 6.2, the main results summarized in Lemmas 5-8. The expression for the orthotoric metric (which is the metric that admits a Hamiltonian 2-form) is given in § 6.3.

In [9], [10] a claim was put forward that the Calabi-Yau theorem holds for asymptotically-

[1] These are not exactly the same but related to each other in a simple way [7].

conical non-compact Ricci-flat manifolds, of which Y is an example. This is a generalization of the asymptotically-locally-Euclidean (ALE) case previously considered in [11]. Since we have the explicit orthotoric metric at hand, we may test the proposal directly, by deforming the metric. In § 7 we construct a first-order deformation of the orthotoric metric, compatible with the topological properties of Y. We show that the corresponding variation of the Kähler form belongs to the compactly-supported cohomology group $H^2_c(Y, \mathbb{R})$.

The next question to be answered is whether the deformed metric as well admits a conformal Killing-Yano form. For that to be the case, the $(2, 0)$-part of the deformed form would have to be non-zero, as the CKYF of type $(1, 1)$ completely fixes the metric to be of orthotoric form. As we proved earlier in § 6.1.1, however, that would imply that the Riemann tensor of the orthotoric metric has a zero-vector. In § 7.3 we show that this is not the case. Therefore the deformed metric does not admit a conformal Killing-Yano form.

The question of whether the first-order deformation of the metric may be extended to a finite one could, at least in principle, also be answered with the help of our methods. An affirmative answer would then constitute (at least locally in Kähler moduli space) an alternative proof to the Calabi-Yau theorem for the manifold Y. To this end, one should recall that the first-order deformation can be extended to a finite one by means of an inverse function argument. It turns out, however, that in the language we use – the one of a symplectic potential G defined on a domain, which is the moment polytope of Y, – the linearized equation is a degenerate elliptic equation (the corresponding quadratic form degenerates at the boundary of the moment polytope), and there does not seem to be a readily available answer to the question of whether this operator may be inverted in the relevant weighted Banach spaces (despite a long history of the subject of boundary-degenerate problems, which started with the seminal work [12]).

There are several appendices:
A. We present an explicit derivation of the metric (59).
B. We find a canonical form for the vector fields generating $U(2) \times U(1)$ action on a three-dimensional (complex) manifold.
C. Contains some technical results pertaining to § 4.2.
D. We find a rational parametrization for the space of polynomials of the form $x^3 - \frac{3}{2}x^3 + d, d \in \mathbb{R}$, encountered in the Ricci-flat metrics built using the orthotoric ansatz.
E. Contains the derivation of a one-parametric generalization of the 'unresolved' solution (with a conical singularity), discussed in § 4.1.
F. We show how the Ricci-flatness equation for a Kähler metric with the relevant symmetries may be obtained from a variational problem, akin to the one of optimal transport theory.
G. We review the formal definition of a conformal Killing-Yano tensor (form).
H. At the example of Taub-NUT we discuss the possibility of having non-holomorphic Killing vector fields on Calabi-Yau twofolds.

1. Aspects of toric differential geometry

Most of the statements in this section may be easily generalized to an arbitrary number of dimensions, but for concreteness we will limit ourselves to complex threefolds. On a toric threefold we may choose the complex coordinates (u_1, u_2, u_3) in such a way that the torus $U(1)^3$ acts simply by shifts of these coordinates: $u_k \to u_k + i\,\beta_k$ ($\beta_k \in \mathbb{R}$), i.e. the holomorphic Killing vector fields are $K_j = \mathrm{Re}\left(i\,\frac{\partial}{\partial u_j}\right)$. The Kähler potential that is preserved under these shifts has the form

$$K = K(\underbrace{u_1 + \bar{u}_1}_{:=x_1}, \underbrace{u_2 + \bar{u}_2}_{:=x_2}, \underbrace{u_3 + \bar{u}_3}_{:=x_3})\,. \tag{2}$$

The moment maps are $\mu_j = \frac{\partial K}{\partial x_j}$. It is convenient to introduce the dual symplectic potential G – the Legendre dual of K:

$$G(\mu_1, \mu_2, \mu_3) = \sum_{j=1}^{3} \mu_j\, x_j - K(x_1, x_2, x_3)\,. \tag{3}$$

In terms of G, the metric corresponding to the Kähler potential (2) has the form (here $\phi_i = \mathrm{Im}\, u_i$)

$$ds^2 = \frac{1}{4}\, G_{ij} d\mu^i d\mu^j + (G^{-1})^{ij} d\phi_i d\phi_j\,. \tag{4}$$

The Kähler form is $g_K = \sum\limits_{k=1}^{3} d\mu_k \wedge d\phi_k$. The potential G for flat space $\mathbb{C}^3$ is

$$G_{\text{flat}} = \sum_{k=1}^{3} \mu_k\,(\log \mu_k - 1)\,. \tag{5}$$

On a Kähler manifold the only non-zero Christoffel symbols are Γ^i_{jk} and $\Gamma^{\bar{i}}_{\bar{j}\bar{k}}$. The only non-zero components of the Riemann tensor are, accordingly,

$$R^i{}_{jk\bar{n}} = -\partial_{\bar{n}} \Gamma^i_{jk} \tag{6}$$

and their complex conjugates. The Kähler metric, Christoffel symbols and the curvature tensor (6) of a toric manifold have a particularly simple expression in the moment map variables:

$$g_{i\bar{j}} = \frac{\partial^2 K}{\partial x_i \partial x_j}, \qquad \Gamma^i_{jk} = \frac{\partial G^{-1}_{jk}}{\partial \mu_i}, \qquad R^i{}_{jk\bar{n}} = -\sum_s G^{-1}_{ns}\, \frac{\partial^2 G^{-1}_{jk}}{\partial \mu_s \partial \mu_i}\,. \tag{7}$$

Here G^{-1}_{jk} means the jk-component of the matrix inverse to the Hessian of G. It is also useful to write out the expression for the Riemann tensor with all lower indices:

$$R_{\bar{m}jk\bar{n}} = -\sum_{s,t} G^{-1}_{ns}\, \frac{\partial^2 G^{-1}_{jk}}{\partial \mu_s \partial \mu_t}\, G^{-1}_{tm}. \tag{8}$$

One can check directly that it has all the correct symmetry properties of the Riemann tensor[2]. A useful immediate check is the verification that the curvature vanishes for the symplectic potential (5) of flat space $\mathbb{C}^3$.

[2]Note also the following additional symmetry property. Since the Riemann tensor is real in real coordinates, one has, in general, $R^*_{\bar{m}jk\bar{n}} = R_{\bar{j}mn\bar{k}}$. In the particular toric coordinates that we are using, however, the Hermitian components (8) of the Riemann tensor are real as well, therefore we have the symmetry property $R_{\bar{m}jk\bar{n}} = R_{\bar{j}mn\bar{k}}$, i.e. a symmetry under the simultaneous exchange $m \leftrightarrow j, k \leftrightarrow n$. It is not immediately obvious from the expression (8) but can be checked directly.

The Ricci tensor of the metric (4) is obtained from (7) by contracting indices:

$$R_{i\bar{j}} = \sum_{p,s} G^{-1}_{js} \frac{\partial}{\partial \mu_s} \left(G^{-1}_{ip} \frac{\partial}{\partial \mu_p} \log \text{Det Hess}\, G \right) . \tag{9}$$

The Ricci-flatness equation $R_{i\bar{j}} = 0$ may be integrated to give

$$\text{Det Hess}\, G = a\, e^{\sum\limits_k b^k \frac{\partial G}{\partial \mu^k}} \tag{10}$$

One of the benefits of using the symplectic potential G in place of the Kähler potential K is that the domain in μ-space, on which G is defined, is the moment polytope of the toric manifold. From the perspective of the equation (10), it is the singularities of the function G that determine the polytope. It is known [13] that in the simplest case of a (generally non-Ricci-flat) metric induced by a Kähler quotient of flat space with respect to an action of a complex torus, the potential G takes the form of a superposition of 'hyperplanes':

$$G_{\text{toric}} = \sum_{i=1}^{M} \ell_i \,(\log \ell_i - 1) \quad \text{with} \quad \ell_i = \sum_k c_{ik}\, \mu_k + d_i\, . \tag{11}$$

In general, a potential G satisfying (58) will not have this form. However, we will assume that it has the corresponding *asymptotic* behavior at the faces of the moment polytope. More exactly, when we approach an arbitrary face ℓ_i, i.e. when $\ell_i \to 0$, we impose the asymptotic condition

$$G = \ell_i \,(\log \ell_i - 1) + \dots \quad \text{as} \quad \ell_i \to 0, \tag{12}$$

where the ellipsis indicates terms regular at $\ell_i \to 0$. Despite being subleading, they are important for the equation (58) to be consistent even in the limit $\ell_i \to 0$.

Moment polytopes of toric symplectic manifolds are rather constrained – they must possess Delzant properties [14]:

- They are simple: at every vertex exactly N faces meet, $2N$ being the dimension of the manifold. In our case $N = 3$.
- The normals to the faces c_{ik} are integer-valued: $c_{ik} \in \mathbb{Z}$.
  Moreover, the normals to the three faces meeting at each vertex form a basis of $\mathbb{Z}^3$.

Let us consider a vertex of the polytope and label the three hyperplanes meeting at this vertex as $\ell_1 = 0$, $\ell_2 = 0$, $\ell_3 = 0$. Then, in the notations of (11) the second Delzant property means that $\sum\limits_{i=1}^{3} f_{ji}\, c_{ik} = \delta_{jk}$, where $f_{ji} \in \mathbb{Z}$. Therefore $c, f \in GL(3, \mathbb{Z})$. In other words, the inverse of the matrix c is integer-valued as well. The meaning of the condition of integrality of the normal vectors may be understood by analyzing the metric (4) in the vicinity of an angle of the moment polytope, defined by $\ell_1 = \ell_2 = \ell_3 = 0$. According to (12), near such an angle the potential is asymptotically approximated by $G = \sum\limits_{k=1}^{3} \ell_k \,(\log \ell_k - 1) + \dots$ We may now make a linear change of variables from (μ_1, μ_2, μ_3) to (ℓ_1, ℓ_2, ℓ_3). The metric (4) then reads

$$ds^2 = \sum_{i=1}^{3} \left(\frac{d\ell_i^2}{4\,\ell_i} + \ell_i\, d\widetilde{\phi}_i^2 \right) + \dots \qquad \text{where} \qquad \widetilde{\phi}_i = \sum_j f_{ji}\, \phi_j\, . \tag{13}$$

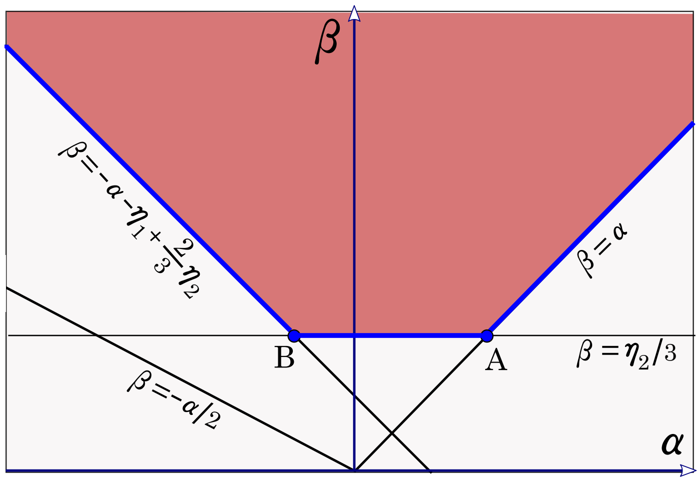

Figure 1: The (α, β) section of the moment polytope of Y. The marked points have coordinates $A = (\frac{\eta_2}{3}, \frac{\eta_2}{3}), B = (\frac{\eta_2}{3} - \eta_1, \frac{\eta_2}{3})$. The moment polytope of Y has five faces whenever the projection has three edges. This happens when the two conditions in (19) are satisfied.

The metric in brackets is the metric of flat space, if $\widetilde{\phi}_i$ have periodicity 2π. Otherwise the metric has a conical singularity. The map $\phi \to \widetilde{\phi}$ is an automorphism of the torus $\mathbb{T}^3$ if and only if $c \in GL(3, \mathbb{Z})$, which is precisely the second Delzant condition.

2. The resolved cone over the del Pezzo surface

In this paper we will be constructing Ricci-flat metrics on the manifold Y introduced in (1). The manifold $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ is diffeomorphic to the del Pezzo surface of rank one (or, equivalently, of degree 8) [4] – the blow-up of $\mathbb{CP}^2$ at one point. This surface is further denoted by $\mathbf{dP}_1$, and we will mostly use this abbreviation in what follows. It is a compact simply-connected Kähler manifold of complex dimension 2, such that $H^2(\mathbf{dP}_1, \mathbb{Z}) = \mathbb{Z}^2$, and the intersection pairing on $H^2(\mathbf{dP}_1, \mathbb{Z})$ has the form $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Denoting the corresponding de-Rham generators of $H^2(Y, \mathbb{R}) \simeq H^2(\mathbf{dP}_1, \mathbb{R})$ by ω_1, ω_{-1}, we may write the Kähler class $\Omega = [g_K]$ of the metric g on Y as follows:

$$\Omega = a\,\omega_1 + b\,\omega_{-1}\,, \quad a, b \in \mathbb{R}\,. \tag{14}$$

The space Y has a representation in terms of a GIT quotient (see [15], Table 1):

$$Y = \mathbb{C}^5/(\mathbb{C}^*)^2, \tag{15}$$

with the charge vectors given by

$$\vec{v}_1 = (0, 0, 1, 1, -2), \qquad \vec{v}_2 = (1, 1, 1, 0, -3) \tag{16}$$

Y is a toric Kähler manifold, and the representation (15)-(16) allows to build the associated moment polytope $\triangle$:

$$\begin{aligned} \triangle = \{&\mu_3 + \mu_4 - 2\mu_5 = \eta_1, \\ &\mu_1 + \mu_2 + \mu_3 - 3\,\mu_5 = \eta_2\} \subset \mathbb{R}^5_+ = \{\mu_i \geqslant 0\} \end{aligned} \tag{17}$$

Clearly, the equations in figure brackets define a three-dimensional space, which we will parametrize by means of the coordinates α, β, γ. These are related to μ_k as follows:

$$\begin{aligned} &\mu_1 = \alpha + \beta + \gamma, \quad \mu_2 = \beta - \gamma, \quad \mu_3 = \beta - \alpha, \\ &\mu_4 = \eta_1 - \tfrac{2\eta_2}{3} + \beta + \alpha, \quad \mu_5 = -\tfrac{\eta_2}{3} + \beta \end{aligned} \tag{18}$$

The inequalities defining the polytope are now $\mu_k \geqslant 0, k = 1 \ldots 5$. In the (α, β) projection we have the following inequalities:

$$\mu_1 + \mu_2 = 2\beta + \alpha \geqslant 0, \quad \mu_3 = \beta - \alpha \geqslant 0,$$
$$\mu_4 = \eta_1 - \tfrac{2\,\eta_2}{3} + \beta + \alpha \geqslant 0, \quad \mu_5 = -\tfrac{\eta_2}{3} + \beta \geqslant 0$$

The relevant chamber in the (η_1, η_2)-space is where the polytope has five faces – this chamber is defined by (see Fig. 1)

$$0 < \eta_1 < \eta_2 \,. \tag{19}$$

The parameters η_1, η_2 are related to the cohomological parameters a, b of (14). Speaking more invariantly, the Kähler moduli are moduli of the moment polytope. To find the relation, we can build a Kähler quotient metric on Y. The corresponding Kähler form is

$$\Omega = d\alpha \wedge d\varphi_1 + d\beta \wedge d\varphi_2 + d\gamma \wedge d\varphi_3, \tag{20}$$

where $(\varphi_1, \varphi_2, \varphi_3)$ are global angular variables associated to the moment map variables (α, β, γ).

In section 3.1 we will find that, as generators of $H_2(Y, \mathbb{R})$ one can take the two spheres, which are the edges of the moment polytope lying at points A and B orthogonal to the section shown in Fig. 1. Under the isomorphism $H_2(Y, \mathbb{R}) \simeq H^2(Y, \mathbb{R})$, the corresponding generators are the forms ω_1, ω_{-1} from (14), obeying the following relations:

$$\int\limits_{\mathbb{CP}^1_A} \omega_1 = 1, \quad \int\limits_{\mathbb{CP}^1_B} \omega_{-1} = 1, \quad \int\limits_{\mathbb{CP}^1_B} \omega_1 = 0, \quad \int\limits_{\mathbb{CP}^1_A} \omega_{-1} = 0. \tag{21}$$

Therefore we see from (14) that a and b are integrals of the Kähler form over the corresponding cycles:

$$a = \int\limits_{\mathbb{CP}^1_A} \Omega, \qquad b = \int\limits_{\mathbb{CP}^1_B} \Omega \tag{22}$$

Let us calculate, for instance, the integral over $\mathbb{CP}^1_A$. The corresponding edge of the polytope is defined by $\mu_3 = \mu_5 = 0$. Since on $\mathbb{CP}^1_A$ we have $d\alpha = d\beta = 0$, the integral is

$$a = 2\pi \int\limits_{\mu_1 = 0}^{\mu_2 = 0} d\gamma = 2\pi \int\limits_{\frac{-2\eta_2}{3}}^{\frac{\eta_2}{3}} d\gamma = 2\pi\, \eta_2 \tag{23}$$

Analogously

$$b = 2\pi \left(\eta_2 - \eta_1 \right), \tag{24}$$

hence we have $a > 0$, $b > 0$ and the ratio $\frac{b}{a}$ is bounded as follows:

$$0 < \frac{b}{a} < 1 \,. \tag{25}$$

The bound (25), together with (14), define the Kähler cone of Y.

§ 2.1. Compactly supported cohomology

An interesting refined description of the cohomology of Y may be found in [16]. To explain it, we will have to slightly jump ahead in our exposition and accept the fact (explained in subsequent sections) that, at infinity, the metric on Y has the form of a Riemannian cone over a Sasaki-Einstein manifold S (for the particular case that we are considering we will have $S = Y^{2,1}$, see § 4.1 and § 4.2 for definitions, as well as [17] for a comprehensive review of Sasaki-Einstein manifolds), i.e. the metric at infinity is of the form

$$(ds^2)_\infty = dr^2 + r^2\,(\widetilde{ds^2})_S\,. \tag{26}$$

Here r is a certain function on Y. In particular, this function has the property that, for r_0 sufficiently large, the set $Y_{r_0} := \{r \leqslant r_0 \subset Y\}$ is a compact manifold with boundary S, i.e. $\partial Y_{r_0} \simeq S$. One can then consider the relative cohomology $H^2(Y_{r_0}, S, \mathbb{R})$, which, by definition, is the compactly supported cohomology $H^2_c(Y) := H^2(Y_{r_0}, S, \mathbb{R})$. Using the long exact sequence for relative cohomology and certain facts about Y and S, one derives [16] that the following sequence is exact:

$$0 \to H^2_c(Y,\mathbb{R}) \to H^2(Y,\mathbb{R}) \to H^2(S,\mathbb{R}) \to 0\,. \tag{27}$$

As we will see below, in the case of interest we have $S = Y^{2,1}$, and topologically $Y^{2,1} \simeq S^2 \times S^3$, hence $H^2(S,\mathbb{R}) \simeq \mathbb{R}$. Since, as we discussed above, $H^2(Y,\mathbb{R}) = \mathbb{R}^2$, we deduce that $H^2_c(Y,\mathbb{R}) = \mathbb{R}$.

One way to distinguish a compactly supported two-form ϖ is by its decay rate at infinity ($r \to \infty$). Indeed, let g_0 be the conical metric, i.e. $(ds^2)_{g_0} = dr^2 + r^2\,(\widetilde{ds^2})_S$. Then we have the following result:

Lemma 1. [16] **Suppose $\|\varpi\|_{g_0} = O(r^{-\alpha})$ for $\alpha > 2$. Then $\varpi \in H^2_c(Y)$.**

Proof.
According to (27), a two-form ϖ lies in $H^2_c(Y)$ whenever it is in the kernel of the map $H^2(Y,\mathbb{R}) \to H^2(S,\mathbb{R})$. This map, in turn, is the 'restriction to the boundary' map. Therefore to check whether $\varpi \in H^2_c(Y)$, we need to check whether its restriction $\varpi\big|_S$ is trivial in $H^2(S,\mathbb{R})$. On the other hand, $\varpi\big|_S$ is trivial if for any three-form $\Lambda \in H^3(S,\mathbb{R})$ one has $\int\limits_S \varpi\big|_S \wedge \Lambda = 0$. Now, here by $\varpi\big|_S$ we actually mean the restriction $\varpi\big|_{r=r_1}$ for some sufficiently large r_1. We may now extend the form $\varpi\big|_{r=r_1} \wedge \Lambda$, defined on S, to a form $\varpi \wedge \Lambda$ defined on $S \times I_r$, where I_r is a segment with coordinate r: $I_r = [r_1, r_2]$. The form $\Lambda \in H^3(S,\mathbb{R})$ is extended trivially, and the form ϖ is closed on $S \times I_r$, since it was closed from the start. Therefore, by Stokes theorem, $\int \varpi\big|_{r=r_1} \wedge \Lambda = \int \varpi\big|_{r=r_2} \wedge \Lambda$. We may now use the decay rate of ϖ to calculate the integral in the limit $r_2 \to \infty$. Since $\varpi\big|_{r=r_2} \wedge \Lambda = (\varpi, *\Lambda)$ is a (point-wise) scalar product between two-forms on S, we may use the Cauchy inequality

$$\Big|\int \varpi\big|_{r=r_2} \wedge \Lambda\Big| \leqslant \int\limits_{S=\partial Y_{r_2}} \|\varpi\|_{\tilde g} \cdot \| * \Lambda\|_{\tilde g} \cdot \mathrm{vol}_S\,, \tag{28}$$

where $\tilde g$ is the metric on S, entering the formula (26) above, and the Hodge star $*$ again refers to $\tilde g$. Note that the metric $\tilde g$ does not depend on r_2. On the other hand, we have the bound for $\|\varpi\|_{g_0}$, rather than $\|\varpi\|_{\tilde{g}}$, but we can easily relate the two. Clearly, $\|\varpi\|^2_{g_0} = g_0^{\alpha\beta} g_0^{\mu\nu}\, \varpi_{\alpha\mu}\, \varpi_{\beta\nu} \geqslant \frac{1}{r^4}\, \|\varpi\|^2_{\tilde{g}}$, hence $\|\varpi\|_{\tilde{g}} \leqslant r^2\, \|\varpi\|_{g_0} \leqslant \frac{\text{const.}}{r^{-(\alpha-2)}}$. Substituting this in (28) above, we obtain

$$\Big|\int \varpi\big|_{r=r_2} \wedge \Lambda\Big| \leqslant \frac{\text{const.}}{r_2^{-(\alpha-2)}} \int\limits_S \|\ast\Lambda\|_{\tilde{g}} \cdot \mathrm{vol}_S\,, \tag{29}$$

Supposing $\alpha > 2$ and sending $r_2 \to \infty$, we find that $\int \varpi\big|_{S=\partial Y_{r_0}} \wedge \Lambda = 0$ for all $\Lambda \in H^3(S,\mathbb{R})$. As a result, we find that $[\varpi\big|_{S=\partial Y_{r_0}}] = 0 \in H^2(S,\mathbb{R})$, which, as explained earlier, implies $\varpi \in H^2_c(Y,\mathbb{R})$. ■

Another view at the compactly supported cohomology group is via Poincaré duality. In fact, this can be described more clearly if we slightly generalize the setup. Let Y be the total space of a vector bundle V of rank m over a surface X. The surface is embedded in Y as the zero section, $i: X \hookrightarrow Y$. Using Poincaré duality, we can construct the dual compactly-supported form $[i(X)]^\vee \in H^m_c(Y,\mathbb{R})$. It is a classic fact that the restriction of this form to the zero section is the Euler class of the bundle: $[i(X)]^\vee\big|_{i(X)} = eu(V)$ (see [18], Propositions 6.24 (b) and 6.41). In the case that V is a complex vector bundle, $eu(V) = c_m(V)$. Returning back to our case, we have $m = 1$, and moreover $V = K_X$ – the canonical bundle of X. Therefore we have the result $[i(X)]^\vee\big|_{i(X)} = c_1(K_X) = -c_1(X)$. Apart from that, one has $H^2_c(Y,\mathbb{R}) \simeq H_4(Y,\mathbb{R}) \simeq \mathbb{R}$, since the homology of Y is the same as that of the base of the bundle, the surface X, for which we have of course $H_4(X,\mathbb{R}) \simeq \mathbb{R}$. Therefore, as $H^2_c(Y,\mathbb{R})$ is one-dimensional, $[i(X)]^\vee\big|_{i(X)} \in H^2_c(Y,\mathbb{R})$ is its generator over the real numbers. Summarizing, to find out whether a given two-form ϖ belongs to $H^2_c(Y,\mathbb{R})$, we may restrict it to X and check whether it is proportional to $c_1(X)$. To facilitate future use, let us express $c_1(X)$ in terms of the generators $\omega_{\pm1}$ featuring in (14):

$$c_1(X) = -3\,\omega_1 - \omega_{-1}\,. \tag{30}$$

Here by $\omega_{\pm1}$ we mean, again, the restrictions of these forms to X. This is essentially the formula $K = -3H + E$, where K is the canonical divisor, H is the hyperplane divisor and E the exceptional divisor of the blow-up (the (-1)-curve). The relative sign in front of ω_{-1} is due to our normalizations (21). The result of this discussion may be reformulated as follows: for a form ϖ to belong to $H^2_c(Y,\mathbb{R})$, one should have

$$\frac{\int\limits_{\mathbb{CP}^1_A} \varpi|_X}{\int\limits_{\mathbb{CP}^1_B} \varpi|_X} = 3\,. \tag{31}$$

The theory that we have reviewed was used in [9] to formulate a version of the Calabi-Yau theorem relevant for the case of asymptotically-conical manifolds:

Proposition 3. [9], [10] **Let Y_0 be the manifold with a conical singularity, equipped with the metric (26), that we will denote g_0. Let $\pi: Y \to Y_0$ be the Ricci-flat resolution of the conical singularity. Then in every Kähler class in $H^2_c(Y,\mathbb{R}) \subset H^2(Y,\mathbb{R})$ there is a unique Ricci-flat Kähler metric g asymptotic to g_0 as follows**

$$|\pi_* g - g_0|_{g_0} = O\left(\frac{1}{r^6}\right) \qquad \textbf{for} \qquad r \to \infty\,. \tag{32}$$

Furthermore, in every Kähler class in $H^2(Y,\mathbb{R})\backslash H^2_c(Y,\mathbb{R})$ there is a Ricci-flat metric g asymptotic to g_0 with the following decay estimate:

$$|\pi_* g - g_0|_{g_0} = O\left(\frac{1}{r^2}\right) \qquad \textbf{for} \qquad r \to \infty\,. \tag{33}$$

In both cases the derivatives of the metric decay appropriately.

The decay estimates of the type above were introduced in [11] in a proof of an analogous Calabi-Yau type theorem for asymptotically locally-Euclidean spaces. The Proposition above is a generalization thereof for asymptotically-conical manifolds. A significant part of the present paper will be dedicated to certain explicit checks and illustrations for the statements contained in the Proposition.

§ 2.2. Example. The total space of the canonical bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$.

A simple example where most of the above assertions may be checked directly is that of a cone over the surface $X_0 := \mathbb{CP}^1 \times \mathbb{CP}^1$. The explicit Ricci-flat metric on such manifold was constructed in [19] by means of the same ansatz that was used earlier in [3]. The ansatz for the Kähler potential has the form:

$$K = a\,\log(1+|z|^2) + K_0\left(\underbrace{|u|^2(1+|z|^2)(1+|w|^2)}_{:=x}\right)\,, \tag{34}$$

where a is a certain parameter (Kähler modulus) whose meaning will be clarified later. The vector fields $k_1 = \mathrm{Re}\left(i\,z\,\frac{\partial}{\partial z}\right)$, $k_2 = \mathrm{Re}\left(i\,w\,\frac{\partial}{\partial w}\right)$ and $k_3 = \mathrm{Re}\left(i\,u\,\frac{\partial}{\partial u}\right)$, generating phase rotations for the local complex variables z, w, u, are clearly Killing. If we denote $|z|^2 = e^t, |w|^2 = e^s, |u|^2 = e^v$, the moment maps are simply derivatives of the Kähler potential w.r.t. the corresponding real variables:

$$\mu_1 = \frac{\partial K}{\partial t} = \frac{e^t}{1+e^t}\,(a + x\,K_0')\,, \quad \mu_2 = \frac{\partial K}{\partial s} = \frac{e^s}{1+e^s}\,x\,K_0', \quad \mu = \frac{\partial K}{\partial v} = x\,K_0'\,. \tag{35}$$

The Ricci-flatness equation is most conveniently expressed in terms of the function $\mu(x)$:

$$\mu\,(a+\mu)\,\mu' = \beta\,x, \qquad \beta = \text{const.}, \tag{36}$$

which may be integrated to give

$$\frac{\mu^3}{3} + a\,\frac{\mu^2}{2} = \beta\,\frac{x^2}{2} - \frac{\kappa}{3}\,. \tag{37}$$

The K_0-part of the Kähler potential may be then obtained from the definition (35):

$$K_0 = \int \frac{\mu}{x}\,dx = \int \frac{\mu^2\,(a+\mu)}{\beta\,x^2}\,d\mu = \text{using (37)} = \frac{3}{2}\int \frac{\mu^2\,(a+\mu)\,d\mu}{\mu^3 + \frac{3a}{2}\,\mu^2 + \kappa} \tag{38}$$

Upon taking the integral, we obtain the following expression for the symplectic potential G ($\simeq$ means 'up to a linear function', which is irrelevant):

$$\begin{aligned} &G := \mu\, v + \mu_1\, t + \mu_2\, s - K \simeq \\ &\simeq \frac{1}{2}\sum_{i=1}^{3}(\mu-\lambda_i)\,\log(\mu-\lambda_i) - \mu\,\log\,\mu - (\mu+a)\,\log(\mu+a) + \\ &+\mu_1\,\log\,\mu_1 + \mu_2\,\log\,\mu_2 + (\mu-\mu_2)\,\log(\mu-\mu_2) + (\mu-\mu_1+a)\,\log(\mu-\mu_1+a)\,, \end{aligned} \tag{39}$$

where λ_i, $i = 1, 2, 3$ are the roots of the polynomial

$$f(\mu) = \mu^3 + \frac{3a}{2}\,\mu^2 + \kappa\,, \tag{40}$$

which enters the denominator of the integrand in (38). We choose the ordering $\lambda_3 \geqslant \lambda_2 \geqslant \lambda_1$ if all roots are real, otherwise λ_3 denotes the real root.

The region in the parameter space, which corresponds to the manifold being the total space of the canonical bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$, is the following:

$$\lambda_3 > 0, \qquad a + \lambda_3 > 0\,. \tag{41}$$

Indeed, in this case the metric on the underlying surface $\mathbb{CP}^1 \times \mathbb{CP}^1$ may be recovered from (34) by taking the limit $x \to 0$. This corresponds to sending $\mu \to \lambda_3$, see (37). Since $\mu = xK_0'$, in the limit $x \to 0$ we have $K_0 \simeq \lambda_3\,\log\,x + \ldots$, therefore the full Kähler potential reduces to[3]

$$K \simeq (a+\lambda_3)\,\log(1+|z|^2) + \lambda_3\,\log(1+|w|^2) + \ldots \tag{42}$$

We see that λ_3 and $a + \lambda_3$ are the squared radii of the two spheres and therefore have to be positive, leading to (41). Note in passing, that once this bound is established, the equations defining the moment polytope may be read off from (39):

$$\mu \geqslant \lambda_3, \qquad \mu + a \geqslant \mu_1 \geqslant 0, \qquad \mu \geqslant \mu_2 \geqslant 0\,. \tag{43}$$

In particular, going to infinity corresponds to sending $\mu \to \infty$ with the ratios $\frac{\mu_1}{\mu}$, $\frac{\mu_2}{\mu}$ bounded. In the formula (39) a and λ_3 are the resolution parameters, and in the limit the are effectively set to zero[4]. The limiting function will be denoted by G_0. One can show that this is the symplectic potential defining a metric cone over $T^{1,1} := \frac{SU(2)\times SU(2)}{U(1)}$ – the manifold introduced in [20].

For the bounds (32)-(33) to make sense, the difference $\pi_* G - G_0$ should be non-singular at infinity in the first place. The potential singularity is at $\mu = \mu_1$. The hyperplane $\mu = \mu_1$ lies outside the moment polytope for $a < 0$. Let us first analyze this case. Since λ_3 is a root of the polynomial $f(\mu)$ from (40), in the limit $\lambda_3 \to 0$, $a \to 0$ the other two roots of

[3] Omitting the contribution $\lambda_3\,\log\,|u|^2$, which does not affect the metric.

[4] Note that the parameter κ may be related to λ_3 but it is more convenient to treat λ_3 as the independent parameter.

the polynomial vanish as well, $\lambda_1, \lambda_2 \to 0$. Therefore we may expand (39) to first order in $\lambda_1, \lambda_2, \lambda_3, a$ as follows

$$G - G_0 \simeq -\frac{1}{2}\left(\sum_{i=1}^{3}\mu_i\right)\log\mu - a\,\log\mu + a\log(\mu-\mu_1) + \ldots =$$
$$= -\frac{a}{4}\,\log\mu + a\,\log(\mu-\mu_1) + \ldots \tag{44}$$

When passing to the second line, we used the expression $\sum_{i=1}^{3}\mu_i = -\frac{3a}{2}$ for the sum of roots, which follows from (40). Let us denote by g_0 the metric given by the symplectic potential G_0. It is now easy to check that

$$\|g-g_0\|_{g_0} = O\left(\frac{a}{\mu}\right) = O\left(\frac{a}{r^2}\right) \qquad \text{for} \qquad r\to\infty \quad \text{and} \quad a\neq 0\,. \tag{45}$$

The case $a > 0$ may be analyzed similarly, if one makes the change of variables $\mu \to \mu - a$ in the function G, before comparing it to G_0 (this corresponds to a choice of map π in the Proposition). In this case the moment polytope is defined by the inequalities $\mu \geqslant \lambda_3 + a,\ \mu \geqslant \mu_1 \geqslant 0,\ \mu - a \geqslant \mu_2 \geqslant 0$. The potential singularity is now at $\mu = \mu_2$ and lies outside of the moment polytope, and the analysis above can be carried through. Note also that, from the point of view of the polynomial $f(\mu)$, the replacement $\mu \to \mu - a$ amounts to a redefinition of κ and the required flip of the sign $a \to -a$.

The remaining interesting case to be considered is $a = 0$. In this case $\lambda_3 = (-\kappa)^{1/3}$, and we have from (39):

$$G - G_0 = \frac{\lambda_3^3}{4\,\mu^2} + \ldots\,, \tag{46}$$

and one easily obtains

$$\|g-g_0\|_{g_0} = O\left(\frac{\lambda_3^3}{\mu^3}\right) = O\left(\frac{\lambda_3^3}{r^6}\right) \qquad \text{for} \qquad r\to\infty \quad \text{and} \quad a = 0\,. \tag{47}$$

According to the Proposition, the decay estimates (45) for $a \neq 0$ and (47) for $a = 0$ correspond to the Kähler form being in $H^2(Y,\mathbb{R})\backslash H^2_c(Y,\mathbb{R})$ and in $H^2_c(Y,\mathbb{R})$ respectively. In the case $a = 0$ the two spheres at the base of the cone (i.e. the zero section) have equal radii, and therefore the induced metric on the zero section is Kähler-Einstein, meaning that its Kähler class is indeed proportional to $c_1(\mathbb{CP}^1 \times \mathbb{CP}^1)$.

3. The equation of Ricci-flatness

In the previous section we introduced the variety Y as a Kähler quotient of flat space. This definition allows constructing a Kähler metric on Y. However, this metric is by no means Ricci-flat. In the remainder of the paper we will be looking for a Ricci-flat metric on Y. To this end, we will start with the most general Kähler potential compatible with the symmetries of the problem, and then solve the Ricci-flatness equation that this Kähler potential has to satisfy.

We mentioned above that the del Pezzo surface of rank one $\mathbf{dP}_1$ may be thought of as the blow-up of one point on $\mathbb{CP}^2$. Without loss of generality let us choose this point to be $(0:0:1) \in \mathbb{CP}^2$. The choice of a distinguished point reduces the automorphism group $\mathbb{P}GL(3,\mathbb{C})$ of $\mathbb{CP}^2$ to the automorphism group of $\mathbf{dP}_1$:

$$Aut(\mathbf{dP}_1) = \mathbb{P}\left\{ G \in GL(3,\mathbb{C}) : G = \begin{pmatrix} \bullet & \bullet & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{pmatrix} \right\}, \tag{48}$$

The linear part of the group of automorphisms of the affine cone over $\mathbf{dP}_1$ (w.r.t. the anti-canonical embedding) is the maximal parabolic subgroup $\mathrm{H} \subset GL(3,\mathbb{C})$ defined by matrices of the form (48) (forgetting the projectivization) (see [21], in particular Proposition 2.15 and Theorem 1.5). The resolved cone inherits these automorphisms as well, as the equation of the blow-up is linear in the embedding coordinates.

We will be looking for a Kähler metric on Y with the isometry group being the maximal compact subgroup of H:

$$\mathrm{Isom}(Y) = U(2) \times U(1) \tag{49}$$

We will choose local coordinates z_1, z_2, u, in which the $\mathfrak{u}(2) \oplus \mathfrak{u}(1)$ action uniformizes, i.e. the holomorphic vector fields generating this action have the form

$$\begin{gathered} v_0^{[u(2)]} = iz_1\tfrac{\partial}{\partial z_1} + iz_2\tfrac{\partial}{\partial z_2}, \qquad v_1^{[u(2)]} = z_1\tfrac{\partial}{\partial z_2} + z_2\tfrac{\partial}{\partial z_1}, \\ v_2^{[u(2)]} = iz_1\tfrac{\partial}{\partial z_2} - iz_2\tfrac{\partial}{\partial z_1}, \qquad v_3^{[u(2)]} = iz_1\tfrac{\partial}{\partial z_1} - iz_2\tfrac{\partial}{\partial z_2}, \qquad v^{[u(1)]} = u\tfrac{\partial}{\partial u}\ . \end{gathered} \tag{50}$$

This is always possible: see Appendix B. The $U(2) \times U(1)$-invariant Kähler potential depends on the two combinations of these variables:

$$K = K(|z_1|^2 + |z_2|^2, |u|^2) \tag{51}$$

The corresponding Kähler form is $\Omega = i\partial\bar{\partial}K$ and the metric is $g_{i\bar{j}} = \partial_i\bar{\partial}_j K$. Since the Ricci tensor is related to the metric of a Kähler manifold as $R_{i\bar{j}} = -\partial_i\bar{\partial}_j \log\det g$, the Ricci-flatness (Calabi-Yau) condition $R_{i\bar{j}} = 0$ implies that the determinant of the Hermitian metric g has to factorize in a holomorphic and conjugate antiholomorphic pieces: $\det g = |f(z_1,z_2,u)|^2$. As $\det g$ is $U(2) \times U(1)$-invariant, it means that $\det g = a\,|u|^{2l}$ for some constants a, l. On the other hand, a direct calculation of $\det g$ for a metric arising from the Kähler potential (51) gives

$$\det g = 8\,e^{-t-s}\;K_t\;\left(K_{tt}K_{ss} - K_{ts}^2\right), \tag{52}$$

where

$$e^{\frac{t}{2}} = |z_1|^2 + |z_2|^2 \quad \text{and} \quad e^s = |u|^2. \tag{53}$$

The Ricci-flatness condition is reduced to the following equation:

$$K_t\;\left(K_{tt}K_{ss} - K_{ts}^2\right) = \frac{a}{8}\,e^{t+(l+1)\,s} \tag{54}$$

It turns out useful to perform a Legendre transform, passing from the variables $\{t, s\}$ to the new independent variables

$$\mu = \frac{\partial K}{\partial t}, \qquad \nu = \frac{\partial K}{\partial s} \tag{55}$$

and from the Kähler potential $K(t,s)$ to the dual potential $G(\mu,\nu)$:

$$G = \mu\, t + \nu\, s - K \tag{56}$$

The usefulness of the new variables (55) to a large extent relies on the fact that they have a transparent geometric meaning – these are the moment maps for the following two $\mathfrak{u}(1)$ actions on Y:

$$\mathfrak{u}(1)_\mu : \quad \delta z_1 = i\,\epsilon_1\, z_1, \quad \delta z_2 = i\,\epsilon_1\, z_2, \qquad \mathfrak{u}(1)_\nu : \quad \delta u = i\,\epsilon_2\, u\,. \tag{57}$$

In this paper we will leave aside the case $l+1=0$ (l is the parameter entering the exponent in (54)) and assume that $l+1\neq 0$. In this case we can get rid of the l dependence by a rescaling $\nu \to (l+1)\,\nu$, so in what follows we effectively set $l=0$. Then we obtain from (54) a Monge-Ampere equation for the dual potential G – a function of two variables μ,ν – of the following form:

$$\boxed{e^{\frac{\partial G}{\partial \mu}+\frac{\partial G}{\partial \nu}}\left(\frac{\partial^2 G}{\partial \mu^2}\frac{\partial^2 G}{\partial \nu^2}-\left(\frac{\partial^2 G}{\partial \mu \partial \nu}\right)^2\right)=\tilde{a}\,\mu} \tag{58}$$

Denoting (μ,ν) by (μ_1,μ_2), we can recover the metric from the dual potential G [22] using the formula (see also Appendix A)

$$ds^2 = \mu\, g_{\mathbb{CP}^1} + \frac{1}{4}\sum_{i,j=1}^{2}\frac{\partial^2 G}{\partial \mu_i \partial \mu_j}\, d\mu_i\, d\mu_j + \sum_{i,j=1}^{2}\left(\frac{\partial^2 G}{\partial \mu^2}\right)^{-1}_{ij}(d\phi_i - 2A_i)\,(d\phi_j - 2A_j)\,, \tag{59}$$

where $g_{\mathbb{CP}^1}$ is the standard round metric on $\mathbb{CP}^1$ of volume 2π (i.e. $g_{\mathbb{CP}^1} = \frac{2\,dw\,d\bar{w}}{(1+\|w\|^2)^2}$), $A_2=0$ and A_1 is the 'Kähler current' of $\mathbb{CP}^1$, i.e. a connection, whose curvature is the Fubini-Study form of $\mathbb{CP}^1$: $dA_1 = \frac{i\,dw\wedge d\bar{w}}{(1+\|w\|^2)^2}$.

Comment 1. Note that the parameter $\tilde{a}$ in (58) is irrelevant, since one can effectively set $\tilde{a}=1$ by a *linear* redefinition of the potential G, i.e. $G \to G + \nu \log(\tilde{a})$. Such a linear redefinition does not affect the metric (59), which depends only on the second derivatives of G. The only requirement is that $\tilde{a}>0$, since this is necessary for the positive-definitiveness of the metric (59).

Comment 2. There is a group of motions in the (μ,ν)-plane, under which the equation (58) is invariant. It is generated by the transformations

$$\nu \to \nu + \delta, \tag{60}$$

$$\mu \to \sigma\mu, \quad \nu \to \sigma\nu, \quad G \to \sigma\, G + \nu\, \log(\sigma^3), \tag{61}$$

$$\delta = \text{const.}, \quad 0 \neq \sigma = \text{const.}$$

The metrics, which differ by the transformation (60), are isometric, whereas the ones, which differ by (61), are related by an overall rescaling.

§ 3.1. The moment 'biangle'

Since (μ, ν) are moment maps for the $\mathfrak{u}(1)^2$ action, they define a map to $\mathbb{R}^2$. The domain in $\mathbb{R}^2$ on which the potential $G(\mu, \nu)$ is defined is the moment polygon for this $\mathfrak{u}(1)^2$ action. In addition, there is yet another $\mathfrak{u}(1)$ action given by

$$\delta z_1 = i\,\epsilon_3\, z_1\,, \qquad \delta z_2 = -i\,\epsilon_3\, z_2\;. \tag{62}$$

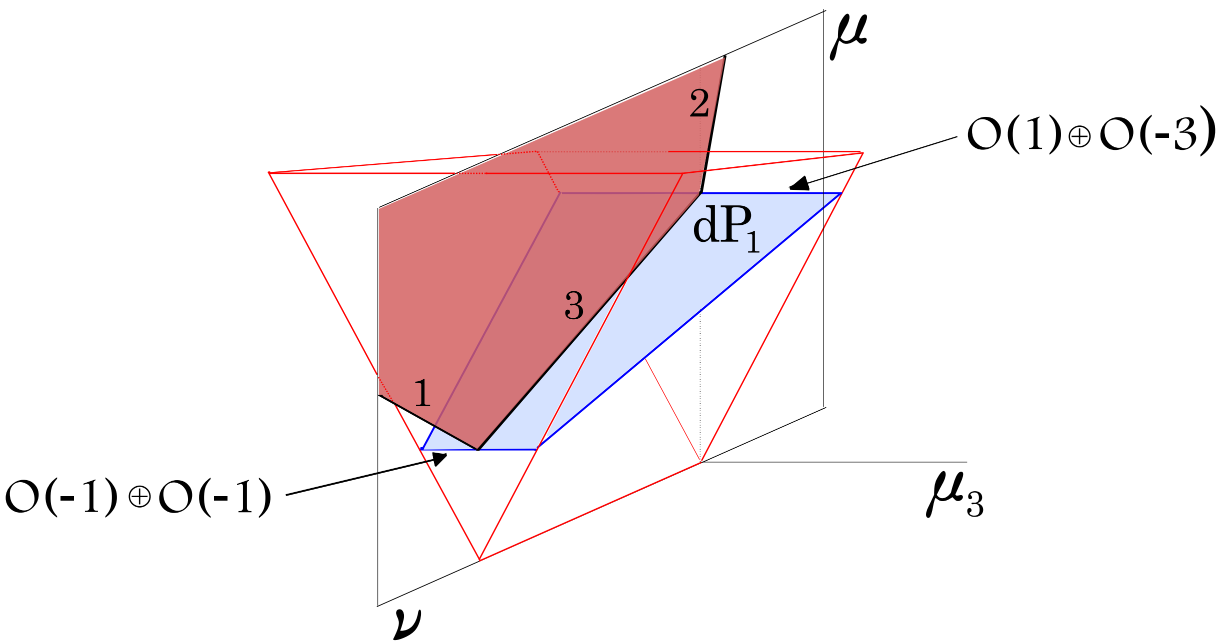

Figure 2: The moment polytope of Y. The blue polygon is the moment polygon of the del Pezzo surface – it may be obtained from the moment polygon of $\mathbb{CP}^2$ (the triangle) by 'cutting a corner'.

Denoting the dual moment map by μ_3, we obtain:

$$\mu_3 = \frac{\partial}{\partial\tau} K(e^{\tau}|z_1|^2 + e^{-\tau}|z_2|^2, |u|^2)\big|_{\tau=0} = \frac{|z_1|^2 - |z_2|^2}{|z_1|^2 + |z_2|^2}\,\mu\;, \tag{63}$$

therefore $-\mu \leqslant \mu_3 \leqslant \mu\,.$

The full three-dimensional moment polytope is shown in Fig. 2. The two-dimensional (μ, ν)-section, shown in red, is an unbounded domain with two vertices that we will call a 'biangle'. The points in Y mapping to a generic point of the biangle – a point in the interior – constitute a $\mathbb{CP}^1 \times \mathbb{T}^2$. The third $U(1)$ action (62) corresponds to the rotation of the sphere $\mathbb{CP}^1$ around its axis. Now, the points in Y mapping to a point at the edge of the biangle constitute a $\mathbb{CP}^1 \times S^1$, and, finally, the points mapping to one of the two corners constitute a $\mathbb{CP}^1$. These two $\mathbb{CP}^1$'s, which map to the corners of the biangle, will be crucial in the foregoing discussion. We will see shortly that their normal bundles fully determine the topology of the moment polytope (i.e. its Delzant properties). For this reason we will not attempt to preserve explicitly the integrality of the normals to the facets of the polytope. The (μ, ν) variables, which we are working with, are related to the moment map coordinates, in which the normals are integral, by a linear transformation.

This is the reason that the moment biangle shown in Fig. 2 in red color is only congruent to the (integral) one shown in Fig. 1. As we just stated, the integrality properties will be automatically accounted for, once we make sure the normal bundles to the spheres at the corners of the polytope are the right ones.

Let us analyze what constraint the behavior (12) of G at a facet of the polytope and the Ricci-flatness equation (58) impose on the facet itself. Suppose the facet is given by $\ell_i = h_i\mu + k_i\nu + p_i = 0$. We have the following lemma:

LEMMA 2. **The 'slope' of the facet is constrained to satisfy $h_i + k_i = 1$.**

Proof.
Indeed, let $G = \ell(\log \ell - 1) + f(\tilde{\ell}, \ell)$ in the vicinity of a facet $\ell = 0$, where $\tilde{\ell}$ is another linear combination of μ, ν such that $d\ell \wedge d\tilde{\ell} \neq 0$, and f is a smooth function at $\ell \to 0$. Substituting G in (58), one obtains in the limit $\ell \to 0$: $\ell^{h_i+k_1-1} \frac{\partial^2 f}{\partial \tilde{\ell}^2}\big|_{\ell=0} \sim \mu(\tilde{\ell})\big|_{\ell=0}$. The non-degeneracy of the induced metric on the facet implies $\frac{\partial^2 f}{\partial \tilde{\ell}^2}\big|_{\ell=0} \neq 0$. We assume $\mu|_{\ell=0} \not\equiv 0$, which leads to $h_i + k_i = 1$. ■

We will now demonstrate how the angles of the moment polytope are detemined by the normal bundles to the two $\mathbb{CP}^1$'s 'located' at the corners.

A corner of the moment polytope may be given by the equations

$$\ell_1 = 0, \quad \ell_2 = 0, \tag{64}$$

where

$$\ell_i = h_i\mu + k_i\nu + p_i, \quad i = 1, 2 \tag{65}$$

are two linear forms. Moreover, according to the discussion above we assume that the behavior of the potential G near the corner is as follows:

$$G = \ell_1(\log \ell_1 - 1) + \ell_2(\log \ell_2 - 1) + \ldots, \tag{66}$$

where ... denotes less singular terms. Compatibility with the Ricci-flatness condition (58) implies

$$h_i + k_i = 1, \quad i = 1, 2 \tag{67}$$

We wish to determine what the behavior (66) implies for the *metric* near a given embedded $\mathbb{CP}^1$. To this end we will insert the asymptotic form (66) of the symplectic potential (omitting the subleading terms denoted by the ellipsis) into the expression for the metric (59). To simplify the calculation, it will be useful to pass to the new 'moment map' coordinates ℓ_1, ℓ_2 instead of μ, ν. The Hessian $\frac{\partial^2 G}{\partial \mu^2}$ then undergoes the standard transformation $\frac{\partial^2 G}{\partial \mu^2} = S^{\mathrm{T}} \frac{\partial^2 G}{\partial \ell^2} S$, where $S = \frac{\partial \ell}{\partial \mu} = \begin{pmatrix} h_1 & k_1 \\ h_2 & k_2 \end{pmatrix}$. The virtue of this change of variables, clearly, is that $\frac{\partial^2 G}{\partial \ell^2}$ is a diagonal matrix: $\frac{\partial^2 G}{\partial \ell^2} = \begin{pmatrix} \frac{1}{\ell_1} & 0 \\ 0 & \frac{1}{\ell_2} \end{pmatrix}$. The metric (59) acquires the form

$$ds^2 = \mu(\ell_1,\ell_2)\, g_{\mathbb{CP}^1} + \left(\frac{d\ell_1^2}{4\ell_1} + \frac{d\ell_2^2}{4\ell_2}\right) + \ell_1\mathcal{A}_1^2 + \ell_2\mathcal{A}_2^2 \tag{68}$$

$$\begin{pmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \end{pmatrix} = (S^{\mathrm{T}})^{-1} \circ \begin{pmatrix} d\phi_1 - A_1 \\ d\phi_2 \end{pmatrix} \tag{69}$$

$$\mu(\ell_1,\ell_2) = \frac{\ell_1 k_2 - \ell_2 k_1}{k_2 - k_1} + \frac{p_2 k_1 - p_1 k_2}{k_2 - k_1} \tag{70}$$

Introducing the angular variables

$$\varphi_1 = \frac{k_2\phi_1 - h_2\phi_2}{k_2 - k_1}, \qquad \varphi_2 = \frac{h_1\phi_2 - k_1\phi_1}{k_2 - k_1}, \tag{71}$$

we can write the metric as

$$ds^2 = \mu(\ell_1,\ell_2)\, g_{\mathbb{CP}^1} + \left(\frac{d\ell_1^2}{4\ell_1} + \frac{d\ell_2^2}{4\ell_2}\right) + \ell_1(d\varphi_1 - nA_1)^2 + \ell_2(d\varphi_2 - mA_1)^2, \tag{72}$$

where

$$\boxed{n = \frac{2k_2}{k_2 - k_1}, \quad m = -\frac{2k_1}{k_2 - k_1},} \tag{73}$$

In appropriate coordinates the Kähler potential of the above metric is

$$K = \kappa\ \log\left(1 + |w|^2\right) + \left(1 + |w|^2\right)^n\ |x|^2 + \left(1 + |w|^2\right)^m\ |y|^2, \qquad \kappa = \frac{p_2 k_1 - p_1 k_2}{k_2 - k_1}. \tag{74}$$

For $\kappa > 0$ the formulas (72) or (74) imply that the normal bundle $N_{\mathbb{CP}^1}$ to the $\mathbb{CP}^1$ parametrized by the inhomogeneous coordinate w and located in a given corner of the moment polytope is[5]

$$N_{\mathbb{CP}^1} = \mathcal{O}(-n)\ \oplus\ \mathcal{O}(-m), \qquad n + m = 2 \tag{75}$$

Note that $n + m = 2$ is essentially a consequence of the Calabi-Yau condition

$$\det N_{\mathbb{CP}^1} = \text{ the canonical class of } \mathbb{CP}^1\ = \mathcal{O}(-2) \tag{76}$$

In the del Pezzo cone case the two corners of the moment biangle in the (μ,ν)-plane correspond to the two bases of the trapezium representing the moment polygon of the del Pezzo surface itself, which serves as the base of the cone. This is emphasized in Fig. 2, where the moment polygon of the del Pezzo surface is shown in blue. The two bases of the trapezium correspond to the two $\mathbb{CP}^1$'s embedded in the del Pezzo surface:

- One $\mathbb{CP}^1$ is inherited from $\mathbb{CP}^2$, i.e. it is the standard embedding $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2$, hence the normal bundle inside $\mathbf{dP}_1$ is $N = \mathcal{O}(1)$. This implies that the normal bundle inside the *cone over* $\mathbf{dP}_1$ is $N = \mathcal{O}(1) \oplus \mathcal{O}(-3)$

[5] See [23] for a detailed discussion of how the Kähler potential encodes the normal bundle to a $\mathbb{CP}^1$ in the analogous situation, when the $\mathbb{CP}^1$ is embedded in a complex surface.

- The second $\mathbb{CP}^1$ is the exceptional divisor of the blow-up and is embedded with normal bundle $N = \mathcal{O}(-1)$. The normal bundle inside the *cone over* $\mathbf{dP}_1$ is therefore $N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

These two spheres generate the second homology group of the del Pezzo surface, and their intersection matrix is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The diagonal ± 1 entries encode the normal bundles to the spheres.

4. The 'three-line' solution

§ 4.1. The metric cone

We start by solving the equation (58) at infinity. We are looking for solutions, which asymptotically have the form of a metric cone:

$$(ds^2)_\infty = dr^2 + r^2\, \widetilde{ds^2} \qquad \text{as} \qquad r \to \infty, \tag{77}$$

where $\widetilde{ds^2}$ is a Sasakian metric on a 5-manifold. From the point of view of the function G, this behavior translates to the following one:

$$G_\infty = 3\,\nu\,(\log\nu - 1) + \nu\, P_0(\xi), \qquad \text{where} \qquad \xi = \frac{\mu}{\nu} \tag{78}$$

This leads to a metric with the following 'radial' part ($r = 2\sqrt{3\nu}$):

$$\left[ds^2\right]_\mu := \frac{\partial^2 G_\infty}{\partial\mu_i \partial\mu_j}\, d\mu_i d\mu_j = 3\,\frac{d\nu^2}{\nu} + \nu\, P_0''(\xi)\, d\xi^2 = dr^2 + r^2\, \frac{P_0''}{12}\, d\xi^2 \tag{79}$$

In particular, we see that positivity of the metric requires $P_0'' > 0$.

Substituting (78) in (58), we obtain the ordinary differential equation

$$P_0'' = \frac{a}{3}\,\xi\, e^{(\xi-1)P_0' - P_0}, \tag{80}$$

which has the solution

$$P_0(\xi) = \log\left(\frac{a}{9}\right) - \sum_{i=0}^{2} \frac{\xi - \xi_i}{\xi_i - 1}\, \log|\xi - \xi_i|, \tag{81}$$

where ξ_i are the roots of the polynomial

$$Q(\xi) = \xi^3 - \frac{3}{2}\xi^2 + d, \tag{82}$$

and d is a constant of integration, which plays a crucial geometric role that we will reveal in the next section.

The singular case $\xi_1 = 1$ (and hence $\xi_2 = 1$) corresponds to the situation, when the physical region shown in Fig. 3 shrinks to zero (see next section). We will therefore omit it in our discussion.

Comment. In Appendix E we construct a one-parametric generalization of the solution (78), (81). The virtue of this generalization is that its isothermal coordinates may be related in a simple way to the 'orthotoric variables' that follow from the existence of a conformal Killing-Yano tensor (see § 6 and in particular § 6.3).

§ 4.2. Topological considerations

The potential (78) may as well be written in the original (μ, ν) variables (up to a linear function, which does not affect the metric):

$$G_\infty = \sum_{i=0}^{2} \frac{\mu - \xi_i\,\nu}{1-\xi_i}\;\left(\log|\mu - \xi_i\,\nu| - 1\right) \tag{83}$$

One sees that the slopes of the three lines involved are defined by the roots ξ_i:

$$\text{Slope}_i = \left(\frac{\mu}{\nu}\right)_i = \xi_i \tag{84}$$

It is important to mention that the three lines appearing in (83) are *not* the three edges of the (μ, ν) moment polytope depicted in Fig. 2 in red. (Otherwise we would have already constructed the desired metric.) In fact, two of the lines, associated with the roots ξ_1, ξ_2, do correspond to the two semi-infinite edges of the red polygon, however the line associated with the root ξ_0 is auxiliary and does not have a direct geometric interpretation.

In the notations (65) of the moment polytope, which we used before, one has

$$\xi_1 = -\frac{k_1}{1-k_1} \quad \text{and} \quad \xi_2 = -\frac{k_2}{1-k_2} \tag{85}$$

On the other hand, k_1 and k_2 are both related to k_3 (the indices $1, 2, 3$ correspond to the numbering of lines in Fig. 2) through the normal bundle formulas (73), which therefore implies that there is a relation between ξ_1 and ξ_2. This geometric relation fixes the parameter d of the polynomial $Q(\xi)$.

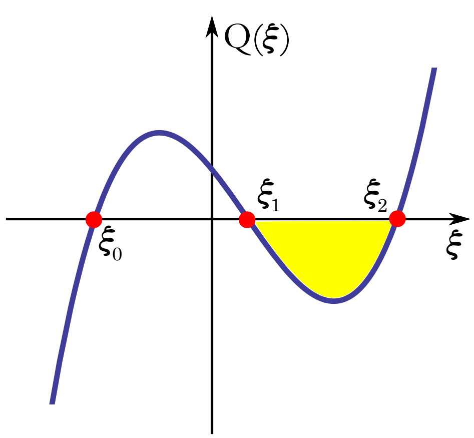

Figure 3: Yellow shading indicates the physical interval $\xi \in (\xi_1, \xi_2)$.

Indeed, from the normal bundle formulas (73) and Fig. 2 it follows that

$$1 - \frac{k_2}{k_3} = -2, \quad 1 - \frac{k_1}{k_3} = 2 \tag{86}$$

Hence $\frac{k_2}{k_1} = -3$. This implies the following relation for ξ_1, ξ_2:

$$-\frac{\xi_2}{1-\xi_2} = \frac{3\xi_1}{1-\xi_1} \tag{87}$$

One can show (see Appendix C) that it has two solutions: $(\xi_1^{(1)}, \xi_2^{(1)})$, $(\xi_1^{(2)}, \xi_2^{(2)})$. However, for $\xi \in (\xi_1^{(2)}, \xi_2^{(2)})$ one has $P_0'' < 0$ and for $\xi \in (\xi_1^{(1)}, \xi_2^{(1)})$ one has $P_0'' > 0$, so the positivity of the metric requires that we choose the first solution. It corresponds to

$$d = \frac{16 + \sqrt{13}}{64}\,. \tag{88}$$

The third root of $Q(\xi)=0$, which we will denote ξ_0, is smaller than the two other roots (see Fig. 3).

One can check directly that the Sasakian metric $\widetilde{ds^2}$ in (77), which may be reconstructed from (78), (81) with the value of the parameter d given in (88), defines the Sasaki-Einstein manifold $Y^{2,1}$, which is one of a family of manifolds found in [24] and termed $Y^{p,q}$. The fact that the $Y^{p,q}$ manifolds are the only compact simply-connected Sasaki-Einstein five-manifolds of cohomogeneity one with respect to the action of the isometry group was proven in [25].

In what follows we will denote the roots of $Q(\xi)$ by ξ_0, ξ_1, ξ_2 so that $Q(\xi) = \prod_{i=0}^{2}(\xi-\xi_i)$ and we will take into account that the 'physical' region corresponds to $\xi \in (\xi_1, \xi_2)$.

5. Uniqueness for a fixed polytope

The goal of this section is to prove the following proposition:

Proposition 4. **The solution of equation (58) with the behavior (12) at the edges of the moment polytope and asymptotic to a real cone over $Y^{2,1}$ at infinity cannot be smoothly deformed.**

Proof.
First we consider a more general equation $\det \mathrm{Hess}\, G = f(\mu,\nu)\, e^{\nabla_a G}$, where $a \in \mathbb{R}^2$ is a constant vector. Suppose G_0 is a solution with the correct asymptotic properties. Construct a first order deformation $G = G_0 + \epsilon H$. The linearized equation has the form

$$G_0^{ij}\, \partial_i \partial_j H = \nabla_a H \tag{89}$$

We can rewrite it as

$$\partial_i\left(G_0^{ij}\, \partial_j H\right) = \left(\partial_i (G_0^{ij}) + a^j\right) \partial_j H \tag{90}$$

We will now use the identity $\partial_i G_0^{ij} = -G_0^{jm} \partial_m \log\det G_0$, which is valid if $(G_0)_{ij} = \partial_i\partial_j G_0$. Since $\log\det G_0 = \log(f(\mu,\nu)) + a^i \partial_i G_0$ and a is a constant vector, we have $\partial_i(G_0^{ij}) = -G_0^{jm}\partial_m \log(f(\mu,\nu)) - G_0^{jm}(G_0)_{mi}a^i = -G_0^{jm}\partial_m \log(f(\mu,\nu)) - a^j$. Therefore the linearized equation acquires the form $\partial_i\left(G_0^{ij}\, \partial_j H\right) = -G_0^{jm}\partial_m \log(f(\mu,\nu))\partial_j H$, which can also be rewritten in divergence form:

$$\frac{\partial}{\partial \mu_i}\left(f(\mu,\nu)\, G_0^{ij}\, \frac{\partial H}{\partial \mu_j}\right) = 0 \tag{91}$$

It is rather nontrivial that the linearized equation has the self-adjoint form, and this relies on the fact that the r.h.s. of the original equation is an exponential of a linear combination of the derivatives of G with constant coefficients.

Multiplying (91) by H and integrating over the moment polygon, we obtain upon integration by parts:

$$0 = -\int d\mu\, d\nu\, f(\mu,\nu) \|\nabla H\|^2_{G_0} + \text{boundary terms} \tag{92}$$

We will now show that, once the asymptotic conditions for G_0 are satisfied and H is smooth, the boundary terms vanish. Clearly, the boundary consists of four segments: three edges of the polygon and 'a segment at infinity'. First, consider the boundary term for each of the edges. Near such an edge one can make a linear change of coordinates $(\mu,\nu) \to (\ell,\tilde{\ell})$, such that the edge is at $\ell = 0$ (same as in the proof of Lemma 3.1). Therefore the relevant boundary term is

$$B\Big|_{\ell=0} = f(\mu,\nu)\left(G_0^{\ell\ell}\,\frac{\partial H}{\partial \ell} + G_0^{\ell\tilde{\ell}}\,\frac{\partial H}{\partial \tilde{\ell}}\right)\Big|_{\ell=0} \tag{93}$$

Since the asymptotic form of G_0 is $G_0 = \ell(\log \ell - 1) + G_0^{reg}(\ell,\tilde{\ell})$, one finds

$$G_0^{\ell\ell} = \ell\,(1+\ldots) \qquad \text{and} \qquad G_0^{\ell\tilde{\ell}} = \ell\left(-\frac{(G_0^{reg})_{\ell\tilde{\ell}}}{(G_0^{reg})_{\tilde{\ell}\tilde{\ell}}} + \ldots\right). \tag{94}$$

In particular, both vanish at the boundary. Since we have assumed that H is regular at the boundary, it follows that

$$B\Big|_{\ell=0} = 0 \tag{95}$$

Let us now have a look at the boundary term, corresponding to the boundary segment γ at infinity, B_∞:

$$B_\infty = \int\limits_{\gamma} dl\,\mu\,H\,n_i G_0^{ij}\,\frac{\partial H}{\partial \mu_j}, \qquad (f(\mu,\nu) = \mu)\,, \tag{96}$$

where $\vec{n}$ is the unit vector, normal to γ, and dl is the infinitesimal length element along γ. In order to estimate the value of B_∞, one needs to know the behavior of H at infinity. Therefore we will start by solving the equation (91) at infinity. To this end we need to recall the asymptotic behavior of the solution G_0 at infinity, discussed in 4.2:

$$G_\infty = \sum_{i=0}^{2} \frac{\mu - \xi_i \nu}{1-\xi_i}\,\left(\log|\mu - \xi_i\nu| - 1\right) \tag{97}$$

One checks that the equation (91), with G_0 replaced by its asymptotic value G_∞, takes the form

$$\frac{\partial}{\partial \mu_i}\left(f(\mu,\nu)\,(G_\infty)^{ij}\,\frac{\partial H}{\partial \mu_j}\right) = \frac{1}{3}\left[-\frac{\partial}{\partial \xi}\left(Q(\xi)\,\frac{\partial H}{\partial \xi}\right) + \frac{\xi}{\nu}\,\frac{\partial}{\partial \nu}\,\left(\nu^3\,\frac{\partial H}{\partial \nu}\right)\right] = 0 \tag{98}$$

Clearly, the variables separate, so one can use the asymptotic ansatz $H = \nu^{-m}\,h(\xi)$ to obtain

$$-\frac{d}{d\xi}\left(Q(\xi)\,\frac{dh}{d\xi}\right) + m(m-2)\,\xi\,h(\xi) = 0 \tag{99}$$

Multiplying by $h(\xi)$ and integrating over the interval $\xi \in (\xi_1,\xi_2)$, one obtains, upon integration by parts

$$\int\limits_{\xi_1}^{\xi_2} d\xi\,\left(Q(\xi)\,\left(\frac{dh}{d\xi}\right)^2 + m(m-2)\,\xi\,h(\xi)^2\right) = 0 \tag{100}$$

In the chosen segment $Q(\xi) < 0$, so (100) leads to the condition

$$m(m-2) > 0\,. \tag{101}$$

Assuming that the metric defined by G_0 is subleading to the conical metric defined by G_∞ at infinity (i.e. $|\mathrm{Hess}\, G_0 - \mathrm{Hess}\, G_\infty|_{G_\infty} \to 0$), we wish that $G_0 + \epsilon\, H$ is subleading to G_∞ as well. This implies $|\mathrm{Hess}\, H|_{G_\infty} \to 0$, and we have two possibilities:

- I. $m > 2$ or
- II. $-1 < m < 0$

Case I.
It is easy to see that the boundary contribution (96) vanishes, as

$$|B_\infty| < \frac{A}{\nu^{2m-2}} \quad \text{for} \quad \nu \to \infty \tag{102}$$

Here by ν we mean the average value of ν on the boundary γ. To arrive at (102) one should take into account that $(G_0^\infty)^{ij} \sim \nu$ and the length of γ behaves as $\int\limits_\gamma dl \sim \nu$.

Case II.
We will show that in this case the equation (99) cannot have a solution, real analytic at the two singular points in question, ξ_1 and ξ_2. This is a result of the following lemma:

Lemma 3. **The smallest non-zero eigenvalue λ of the Laplacian $\triangle_\xi = -\frac{d}{d\xi}\left(Q(\xi)\,\frac{dh}{d\xi}\right)$, entering the equation $\triangle_\xi h + \lambda\,\xi\, h = 0$, is $\lambda = 3$.**

Proof.
The fact that $\lambda = 0$ and $\lambda = 3$ are eigenvalues of the Laplacian is almost obvious, since one can write out the corresponding eigenfunctions directly: for $\lambda = 0$ one has $h = 1$, and for $\lambda = 3$ one has $h = \xi - 1$.

The equation at hand is a Heun equation – a Fuchsian ODE with four regular singular points: the three roots of the polynomial $Q(\xi)$ and the point at infinity. In order to make a more canonical 'centering' of the Heun equation we make a change of variables

$$\xi \to \frac{\xi_1+\xi_2}{2} - \frac{\xi_2-\xi_1}{2}\,\xi, \tag{103}$$

bringing the equation to the canonical form

$$\frac{d}{d\xi}\left((1-\xi^2)(\xi-t)\frac{dh}{d\xi}\right) - \lambda\,(s-\xi)\,h = 0 \tag{104}$$

with

$$t = \frac{\xi_1+\xi_2-2\,\xi_0}{\xi_2-\xi_1} \quad \text{and} \quad s = \frac{\xi_2+\xi_1}{\xi_2-\xi_1} \tag{105}$$

We will use the method of solving the eigenvalue problem for the Heun equation using an expansion in hypergeometric (Jacobi) polynomials, which goes back to Svartholm [26] (see also [27] as a general reference on Heun's equations). In our case, since the exponents of the corresponding singular points are zero, the Jacobi polynomials reduce to Legendre polynomials. We expand h in the Legendre polynomials

$$h = \sum_{k=0}^{\infty} a_k\, L_k(\xi). \tag{106}$$

For a function $h(\xi)$, *analytic on the closed segment* $\xi \in [-1,1]$, the expansion (106) is convergent in an ellipse having ± 1 as its foci ([28], p. 245; [29], p. 322). Note that the shape of the ellipse depends on the nearest singularities of $h(\xi)$.

Substituting the expansion (106) in the equation (104), obtain the recurrence relation

$$g_k \, a_{k+1} - f_k \, a_k + j_k \, a_{k-1} = 0 \tag{107}$$

with

$$g_k = \frac{(k+1)\left((k+1)^2 - \lambda - 1\right)}{2k+3} \tag{108}$$

$$f_k = t\, k(k+1) - s\, \lambda \tag{109}$$

$$j_k = \frac{k\left(k^2 - \lambda - 1\right)}{2k-1} \tag{110}$$

Introducing the new variable $\tau_k = \frac{a_{k-1}}{a_k}$, we can rewrite the recurrence relation (107) as follows:

$$\frac{g_k}{\tau_{k+1}} + j_k \tau_k - f_k = 0 \tag{111}$$

and take

$$\tau_0 = 0 \tag{112}$$

as the initial condition for our recursion.

It is easy to solve the recurrence relation in the limit $k \to \infty$. Indeed, in this case we obtain a quadratic equation for τ_∞:

$$\tau_\infty^2 - 2t\tau_\infty + 1 = 0, \tag{113}$$

which has the solutions

$$(\tau_\infty)_\pm = t \pm \sqrt{t^2 - 1} \tag{114}$$

The solution of the recurrence relation (107) therefore behaves at large k as

$$a_k \sim s_+ \left(\frac{1}{(\tau_\infty)_+}\right)^k + s_- \left(\frac{1}{(\tau_\infty)_-}\right)^k \tag{115}$$

It is easy to check, using (105), that $t > 1$, therefore $(\tau_\infty)_- < 1$ and $(\tau_\infty)_+ > 1$. Looking back at the expansion (106), and taking into account that $L_k(1) = 1, L_k(-1) = (-1)^k$, we see that the requirement of regularity of the function h at the points $\xi = 0, 1$ is equivalent to the condition $s_- = 0$. We will prove below that this is not so, i.e. that the solution in fact grows as $a_k \sim \left(\frac{1}{(\tau_\infty)_-}\right)^k$, where $\frac{1}{(\tau_\infty)_-} > 1$. The proof is by induction: assuming that $0 < \tau_k < a$ for a suitable constant a, we will show that $0 < \tau_{k+1} < a$. If one can take $a < 1$, this is sufficient to prove that the sequence $\{a_k\}$ is exponentially growing.

The key technical inequality that we will need to prove is the following:

$$f_k - a j_k - \frac{1}{a} g_k > 0 \quad \text{for all} \quad k \geqslant 2 \quad \text{and some} \ a: \quad 0 < a < 1\,, \tag{116}$$

where f_k, j_k, g_k have been defined in (108)-(110). The relevant values of s and t are

$$s = \frac{4+\sqrt{13}}{3} \qquad \text{and} \qquad t = \sqrt{13}\,. \tag{117}$$

Once we have proven (116), suppose $0 < \tau_k < a$. Then

$$\tau_{k+1} = \frac{g_k}{f_k - j_k \tau_k} > 0, \tag{118}$$

since $j_k > 0$ and $g_k > 0$ for $k > 2 \geqslant \sqrt{\lambda+1}$, and it follows from (116) that $\frac{f_k}{j_k} > a > \tau_k$. Besides, since, according to (116), $f_k - \tau_k j_k > f_k - a j_k > \frac{1}{a} g_k$, (118) implies

$$\tau_{k+1} < a. \tag{119}$$

In order to prove (116), first of all we make some elementary estimates:

$$g_k < \frac{1}{2}\left((k+1)^2 - \lambda - 1\right) \tag{120}$$

$$j_k < \left(\frac{1}{2} + \epsilon\right)(k^2 - \lambda - 1) < \left((k+1)^2 - \lambda - 1\right), \qquad 0 < \epsilon \ll 1, \tag{121}$$

hence

$$f_k - a j_k - \frac{1}{a} g_k > t\,k(k+1) - s\,\lambda - b\left((k+1)^2 - \lambda - 1\right) := \phi_k \tag{122}$$

with

$$b = a + \frac{1}{2a}, \quad (b \geqslant \sqrt{2})\,. \tag{123}$$

ϕ_k, defined in (122), is a quadratic function of k, so in order to prove that $\phi_k > 0$ for $k \geqslant 2$ we will show that $\phi_2 > 0$ and $\phi'_k > 0$ for $k > 0$. First of all,

$$\phi_2 = 6\,t - s\,\lambda - b\,(8 - \lambda) \tag{124}$$

Since we are interested in the segment $\lambda \in (0, 3)$, and ϕ_2 is a linear function of λ, it suffices to require that the values of ϕ_2 at the ends of the segment are positive: $\phi_2\big|_{\lambda=0} = 6\,t - 8\,b > 0$ and $\phi_2\big|_{\lambda=3} = 6\,t - 3\,s - 5\,b > 0$. Therefore we need to take $b < \min\left(\frac{3\,t}{4}, \frac{6\,t - 3\,s}{5}\right) = \frac{3\,t}{4}$ for the values of t and s given in (117). To ensure that ϕ_k is a growing parabola we require $b < t$ and, since $\phi'_k = (2k+1)\,t - 2b\,(k+1)$, for $b < \frac{1}{2}t$ the bottom of the parabola lies at $k < 0$. Therefore for $b < \frac{1}{2}t$ we have $\phi_k > 0$ for $k \geqslant 2$, implying

$$f_k - a j_k - \frac{1}{a} h_k > 0. \tag{125}$$

Now, the requirement $b < \frac{1}{2}t$ means that

$$a^2 - \frac{t}{2} a + \frac{1}{2} < 0 \tag{126}$$

This is easily satisfied for $a = \frac{1}{2}$, since $t = \sqrt{13} > 3$. What remains to be checked is that $\tau_2 < a = \frac{1}{2}$. This is true, since $\tau_1 = \frac{1}{3s}$ and $\tau_2 = \frac{2}{5}\frac{1}{s - \frac{1}{3s}} < \frac{1}{2}$. Therefore we have proven that $\tau_k < \frac{1}{2}$, so that $\frac{a_k}{a_{k-1}} > 2$, which implies in particular $\lim\limits_{k\to\infty} \frac{a_k}{a_{k-1}} > 1$. The expansion (106) is thus divergent at the two singular points of interest: $\xi = 0$, $\xi = 1$.

This completes the proof of the **Lemma** ■

In equation (99) one has $\lambda = m(m-2)$. Case II corresponds to the range $-1 < m < 0$, which is equivalent to $0 < \lambda < 3$. It follows from the lemma that there are no eigenvalues of $\triangle_\xi$ lying in this range. This completes the proof of **Proposition** 4 ■

Comment. Proposition 3 implies that the equation (58) does not require the specification of any boundary values, apart from (12) and the condition that the metric is asymptotic to a real cone over $Y^{2,1}$. This is related to the singular nature of the boundary condition (12). A Monge-Ampere equation with a similar boundary behavior of the solution was considered in [30].

6. The Killing-Yano forms

Now that we have proven that the solution is unique, once the moment polytope has been specified, we may ask if a solution exists. It turns out that there is a closed expression for G, and hence for the metric, in the case of a particularly chosen moment polytope — this is the metric obtained in [5], as well as in [31], and it has the so-called 'orthotoric' form [6]. This form of metric arises naturally from the requirement of existence of a conformal Killing-Yano form of type $(1,1)$ on the manifold. We therefore start by reviewing the concept of conformal Killing-Yano forms on Kähler manifolds. For a general review of Killing and Killing-Yano tensors the reader is referred to [32], [33].

§ 6.1. Conformal Killing-Yano forms on a Calabi-Yau threefold

First we consider a manifold $\mathcal{M}$ of arbitrary dimension D. By definition, a conformal Killing-Yano form (CKYF) is a 2-form ω_{jk} on $\mathcal{M}$ satisfying an equation of the form (see the derivation in Appendix G)

$$\widetilde{\mathscr{D}}\omega = 0, \qquad \text{where} \tag{127}$$

$$(\widetilde{\mathscr{D}}\omega)_{ijk} := \nabla_i\omega_{jk} - \frac{1}{3}T_{ijk} + \frac{1}{D-1}\left(g_{ik}\, g^{mn}\nabla_m\omega_{nj} - g_{ij}g^{mn}\nabla_m\omega_{nk}\right) \tag{128}$$

$$\text{and} \qquad T_{ijk} = \nabla_i\omega_{jk} + \nabla_k\omega_{ij} - \nabla_j\omega_{ik}\,. \tag{129}$$

The tensor T here, which is anti-symmetric in all pairs of indices, is proportional to the exterior derivative of ω, i.e. $T \propto d\omega$.

Let us now specialize to the case of a Calabi-Yau manifold $\mathcal{M}$ of complex dimension 3, i.e. $D = 6$. Since $\mathcal{M}$ is Calabi-Yau, its volume form may be decomposed as

$$\mathrm{vol}_{\mathcal{M}} = i\,\Omega \wedge \bar{\Omega}\,, \tag{130}$$

where Ω is a holomorphic non-vanishing 3-form

$$\Omega := \Omega_{abc}(z)\, dz^a \wedge dz^b \wedge dz^c\,. \tag{131}$$

It also follows from the above two equalities that Ω is covariantly constant:

$$\nabla_m \Omega = 0, \qquad \nabla_{\bar m} \Omega = 0\,. \tag{132}$$

Let us introduce a poly-vector $\widetilde{\Omega}^{abc}$ by raising the indices of the form Ω. This poly-vector is 'inverse' to Ω in the following sense:

$$\widetilde{\Omega}^{ijk}\Omega_{i'j'k} = \delta^i_{i'}\,\delta^j_{j'} - \delta^i_{j'}\,\delta^j_{i'}, \qquad \widetilde{\Omega}^{\bar i\bar j\bar k}\Omega_{\bar i'\bar j'\bar k} = \delta^{\bar i}_{\bar i'}\,\delta^{\bar j}_{\bar j'} - \delta^{\bar i}_{\bar j'}\,\delta^{\bar j}_{\bar i'}, \tag{133}$$

$$\widetilde{\Omega}^{ijk}\Omega_{i'jk} = 2\,\delta^i_{i'}, \qquad \widetilde{\Omega}^{\bar i\bar j\bar k}\Omega_{\bar i'\bar j\bar k} = 2\,\delta^{\bar i}_{\bar i'}\,. \tag{134}$$

To see that this is the case, we write Ω in components as $\Omega_{ijk} = \epsilon_{ijk}\,q(z)$, where $|q|^2 = \mathrm{Det}(g)$. Analogously, $\Omega_{\bar i\bar j\bar k} = \epsilon_{\bar i\bar j\bar k}\,\bar q(\bar z)$. Therefore the dual poly-vector $\widetilde{\Omega}^{ijk} = \epsilon^{ijk}\,(q(z))^{-1}$ is holomorphic, and $\widetilde{\Omega}^{\bar i\bar j\bar k} = \epsilon^{\bar i\bar j\bar k}\,(\bar q(\bar z))^{-1}$ is anti-holomorphic.

Using the form Ω and its inverse $\widetilde{\Omega}$, we can dualize vectors to forms and vice versa, for example we can view $\widetilde{\Omega}$ as a map

$$\widetilde{\Omega}: \quad (T^*)^{(1,0)} \wedge (T^*)^{(1,0)} \to T^{(1,0)} \tag{135}$$

On the other hand, on a Kähler manifold, the Killing-Yano form may be disassembled into its Hodge components:

$$\omega = \omega^{(2,0)} \bigoplus \omega^{(1,1)} \bigoplus \omega^{(0,2)}\,. \tag{136}$$

We will be denoting the vector of type $(1,0)$, dual to $\omega^{(2,0)}$, by the same letter ω:

$$\omega^s := \widetilde{\Omega}^{sjk}\omega_{jk}\,. \tag{137}$$

The goal of the following paragraphs §§ 6.1.1 and 6.1.2 will be in proving the following proposition:

Proposition 5. **Let $\mathcal{M}$ be a Ricci-flat complex manifold, $\dim_{\mathbb{C}}\mathcal{M} = 3$, without parallel vector fields. Then the vector $\omega^m \frac{\partial}{\partial z_m}$ of type $(1,0)$, dual to the $(2,0)$-part $\omega^{(2,0)}$ of a conformal Killing-Yano two-form on $\mathcal{M}$, satisfies the following equation:**

$$R^n_{\;mj\bar k}\,\omega^m = 0\,. \tag{138}$$

§§ 6.1.1. The $(2,0)$-part of the Killing-Yano form

We first concentrate on the $(2,0)$ component of the Killing-Yano form. The equation (128) with all indices holomorphic gives

$$\nabla_i \omega_{jk} = \frac{1}{3} T_{ijk} \tag{139}$$

Here T is a totally skew-symmetric tensor of type $(3,0)$.

I. T has to be proportional to the Calabi-Yau 3-form:

$$T = f \cdot \Omega, \tag{140}$$

where f is a scalar function on $\mathcal{M}$. Dualizing the eq. (139) in the (j,k) indices and using the notation (137), we obtain

$$\nabla_i \omega^s = \frac{2}{3}\, f \cdot \delta_i^s \,. \tag{141}$$

Since on a Kähler manifold the holomorphic covariant derivatives commute, $[\nabla_i, \nabla_j] = 0$, we obtain a consistency condition

$$\partial_j f\, \delta_i^s - \partial_i f\, \delta_j^s = 0 \qquad \Rightarrow \qquad \partial_j f = 0\,, \tag{142}$$

i.e. $f = f(\bar z)$ is anti-holomorphic.

II. To proceed further let us introduce the one-form

$$\tilde\lambda := g^{s\bar t}\, \nabla_{\bar t}\omega_{js}\, dz^j + g^{t\bar s}\, \nabla_t \omega_{\bar j \bar s}\, d\bar z^j \tag{143}$$

and the dual vector field

$$\lambda = \nabla_{\bar t}\tilde\omega^{\bar j \bar t}\, \frac{\partial}{\partial \bar z^j} + \nabla_t \tilde\omega^{jt}\, \frac{\partial}{\partial z^j}\,. \tag{144}$$

Let us now act on (139) by $\nabla_{\bar j}$, contract the equation with $g^{j\bar j}$ and commute the covariant derivatives to obtain an equation for the divergence of ω, i.e. for λ:

$$g^{j\bar j}\nabla_{\bar j}\nabla_i \omega_{jk} = g^{j\bar j}\, R^p_{\ j\bar j i}\, \omega_{pk} + g^{j\bar j}\, R^p_{\ k\bar j i}\, \omega_{jp} - \nabla_i \tilde\lambda_k = \frac{1}{3} g^{j\bar j}\partial_{\bar j} f(\bar z)\cdot \Omega_{ijk} \tag{145}$$

Both terms involving the Riemann tensor are zero. The first one is zero since it is proportional to the Ricci tensor, and the second one is equal $R^{p\ j}_{\ k\ i}\, \omega_{jp} = 0$ as a contraction of the symmetric (in the (p,j) indices) Riemann tensor with the skew-symmetric tensor ω. Therefore we get

$$\nabla_i \tilde\lambda_k = -\frac{1}{3} g^{j\bar j}\partial_{\bar j} f(\bar z)\cdot \Omega_{ijk} \tag{146}$$

Since Ω is skew-symmetric, we see that

$$\nabla_i \tilde\lambda_k + \nabla_k \tilde\lambda_i = 0\,. \tag{147}$$

Moreover, the corresponding dual vector field is holomorphic ($\partial_{\bar i}\lambda^j = 0 \Leftrightarrow \nabla_i \tilde\lambda_k = 0$) if and only if $f(\bar z) = \text{const.} = f_0$.

§§ 6.1.2. The $(1,1)$-part of the Killing-Yano form

Equation (141) (and its complex-conjugate) is therefore the only constraint on the $(2,0)$ and $(0,2)$ parts of the Killing-Yano form. We now turn to the analysis of the remaining equations (128), which constrain the $(1,1)$ part of the form.

$$i, j \text{ hol.}, \quad k \text{ anti-hol.}: \qquad \frac{2}{3}\nabla_i\, \omega_{j\bar k} + \frac{1}{3}(\nabla_j \omega_{i\bar k} - \nabla_{\bar k}\omega_{ij}) + \frac{1}{5} g_{i\bar k}\, g^{\mu\nu}\nabla_\mu \omega_{\nu j} = 0$$

$$i \text{ anti-hol.}, \qquad j, k \text{ hol.}, \quad i \leftrightarrow k:$$

$$-\frac{2}{3}\nabla_{\bar k}\omega_{ij} + \frac{1}{3}(\nabla_i \omega_{j\bar k} - \nabla_j \omega_{i\bar k}) + \frac{1}{5}(g_{i\bar k}\, g^{\mu\nu}\nabla_\mu \omega_{\nu j} - g_{j\bar k}\, g^{\mu\nu}\nabla_\mu \omega_{\nu i}) = 0\,.$$

Expressing $\nabla_j \omega_{i\bar k}$ from the second equation and substituting in the first one, we obtain an equation for $\omega_{j\bar k}$:

$$\nabla_i \omega_{j\bar k} = \nabla_{\bar k}\omega_{ij} - \frac{2}{5} g_{i\bar k}\, g^{\mu\nu}\nabla_\mu \omega_{\nu j} + \frac{1}{5} g_{j\bar k}\, g^{\mu\nu}\nabla_\mu \omega_{\nu i}\,. \tag{148}$$

Contracting it with $g^{j\bar{k}}$, we obtain the following:

$$\frac{1}{5}\, g^{\mu\nu}\nabla_\mu \omega_{\nu i} = \partial_i h - \tilde{\lambda}_i\,, \qquad h := g^{j\bar{k}}\omega_{j\bar{k}} = \mathrm{Tr}(\omega)\,.$$

Substituting this in (148), we get

$$\nabla_i \omega_{j\bar{k}} = g_{j\bar{k}}\,\partial_i h - 2g_{i\bar{k}}\,\partial_j h + \left(\nabla_{\bar{k}}\omega_{ij} + 2g_{i\bar{k}}\,\tilde{\lambda}_j - g_{j\bar{k}}\,\tilde{\lambda}_i\right)\,. \tag{149}$$

Note that the terms in brackets depend only on the $(2,0)$ part of the KY-form. Another important equation is the complex-conjugate one. To write it, note that, since $\omega = \omega_{ij}\, dz_i \wedge dz_j + \omega_{\bar{i}\bar{j}}\, d\bar{z}_i \wedge d\bar{z}_j + 2\,\omega_{i\bar{j}}\, dz_i \wedge d\bar{z}_j$ is a real form, $\omega^*_{i\bar{j}} = -\omega_{j\bar{i}}$ and $\omega^*_{ij} = \omega_{\bar{i}\bar{j}}$. By analogous arguments, $g^*_{i\bar{j}} = g_{j\bar{i}}$ and therefore $h^* = -h$. Hence the complex conjugation of (149) gives, upon the interchange $j \leftrightarrow k$,

$$\nabla_{\bar{i}} \omega_{j\bar{k}} = g_{j\bar{k}}\,\partial_{\bar{i}} h - 2g_{j\bar{i}}\,\partial_{\bar{k}} h - \left(\nabla_{j}\omega_{\bar{i}\bar{k}} + 2g_{j\bar{i}}\,\tilde{\lambda}_{\bar{k}} - g_{j\bar{k}}\,\tilde{\lambda}_{\bar{i}}\right)\,. \tag{150}$$

A potential obstruction to the solvability of equations (149)-(150) lies in the commutators $[\nabla_i, \nabla_{\bar{i}}]$ and $[\nabla_i, \nabla_j] = 0$. We will first analyze the commutator $[\nabla_i, \nabla_{\bar{i}}]$. In particular, its trace is the Ricci tensor $\mathrm{Ric} = g^{i\bar{i}}[\nabla_i, \nabla_{\bar{i}}]$, acting on the two-form ω. Since the manifold $\mathcal{M}$ is Calabi-Yau, we have $\mathrm{Ric} = 0$, therefore we get the necessary condition for the solvability of (149)-(150):

$$\begin{aligned} &g^{i\bar{i}}\,\nabla_{\bar{i}}\left(g_{j\bar{k}}\,\partial_i h - 2g_{i\bar{k}}\,\partial_j h + \left(\nabla_{\bar{k}}\omega_{ij} + 2g_{i\bar{k}}\,\tilde{\lambda}_j - g_{j\bar{k}}\,\tilde{\lambda}_i\right)\right) = \\ &= g^{i\bar{i}}\,\nabla_{i}\left(g_{j\bar{k}}\,\partial_{\bar{i}} h - 2g_{j\bar{i}}\,\partial_{\bar{k}} h - \left(\nabla_{j}\omega_{\bar{i}\bar{k}} + 2g_{j\bar{i}}\,\tilde{\lambda}_{\bar{k}} - g_{j\bar{k}}\,\tilde{\lambda}_{\bar{i}}\right)\right)\,. \end{aligned} \tag{151}$$

The terms involving h cancel out. We may rewrite the equation (151) term by term as

$$-\,\nabla_{\bar{k}}\tilde{\lambda}_j + 2\,\nabla_{\bar{k}}\tilde{\lambda}_j - g_{j\bar{k}}\,g^{i\bar{i}}\,\nabla_{\bar{i}}\tilde{\lambda}_i = \nabla_j\tilde{\lambda}_{\bar{k}} - 2\,\nabla_j\tilde{\lambda}_{\bar{k}} + g_{j\bar{k}}\,g^{i\bar{i}}\,\nabla_i\tilde{\lambda}_{\bar{i}}\,.$$

It is easily seen from (143) that $g^{i\bar{i}}\,\nabla_{\bar{i}}\tilde{\lambda}_i = g^{i\bar{i}}\,\nabla_i\tilde{\lambda}_{\bar{i}} = 0$ (it is the 'double divergence' of the two-form $\omega^{(2,0)}$ or $\omega^{(0,2)}$). Therefore what we get is

$$\nabla_{\bar{k}}\tilde{\lambda}_j + \nabla_j\tilde{\lambda}_{\bar{k}} = 0\,. \tag{152}$$

The equations (147), (152) imply the following lemma:

LEMMA 4. **The vector field λ defined in (144), i.e. the divergence of the two-form $\omega^{(0,2)} + \omega^{(2,0)}$, is Killing.**

As discussed earlier, λ is holomorphic if and only if $f = \mathrm{const.} = f_0$. We will now prove the following statement:

PROPOSITION 6. **On a Calabi-Yau threefold without parallel vectors any Killing vector field is holomorphic[6].**

[6]On a Calabi-Yau twofold the situation is different, see Appendix H for more details.

Proof.
A vector v is Killing if $\nabla_\mu v_\nu + \nabla_\nu v_\mu = 0$, where v_μ is the dual one-form. On a Kähler manifold this equation may be split into two:

$$\nabla_i v_j + \nabla_j v_i = 0 \tag{153}$$
$$\nabla_i v_{\bar{j}} + \nabla_{\bar{j}} v_i = 0\,. \tag{154}$$

A holomorphic Killing field is the one that satisfies $\partial_i v^{\bar{j}} = 0$ or, in terms of the dual one-form, as $\nabla_i v_j = 0$. Therefore for a holomorphic vector field the two terms in (153) are separately zero.

Imagine, however, that the Killing vector field v is not necessarily holomorphic. In this case the quantity characterizing its non-holomorphicity is $F_{ij} = \nabla_i v_j - \nabla_j v_i$. In fact, it arises naturally in the Lie derivative of the Kähler form ϖ w.r.t. the Killing field v. Indeed, one calculates $i_v \varpi = \varpi_{a\bar{a}} (v^{\bar{a}}\, dz^a - v^a\, dz^{\bar{a}}) = i\,(v_a\, dz^a - v_{\bar{a}}\, dz^{\bar{a}})$, where we used the fact that the Hermitian components of the Kähler form and of the metric are related simply as $\varpi_{a\bar{a}} = i\, g_{a\bar{a}}$. therefore

$$\mathfrak{L}_v \varpi = i\, \partial_b v_a\, dz^b \wedge dz^a - i\, \partial_{\bar{b}} v_{\bar{a}}\, dz^{\bar{b}} \wedge dz^{\bar{a}} - i\,(\partial_a v_{\bar{b}} + \partial_{\bar{b}} v_a)\, dz^a \wedge dz^{\bar{b}}\,. \tag{155}$$

The term in brackets vanishes due to one of the Killing conditions[7] (154). As a result,

$$\mathfrak{L}_v \varpi = \frac{i}{2} (F - F^*)\,. \tag{156}$$

We see that the Lie derivative $\mathfrak{L}_v \varpi \in \Omega^{(2,0)}(\mathcal{M}) \oplus \Omega^{(0,2)}(\mathcal{M})$ is uniquely characterized by the two-form F. Let us derive constraints on this form, starting from the defining equations (153), (154). Since on a Kähler manifold $[\nabla_i, \nabla_k] = 0$ we have from (153)

$$0 = \nabla_k \nabla_j v_i - \nabla_i \nabla_j v_k = \nabla_j F_{ki}. \tag{157}$$

By the same token from (154) we get

$$0 = \nabla_k \nabla_{\bar{j}} v_i - \nabla_i \nabla_{\bar{j}} v_k = \nabla_{\bar{j}} F_{ki} + R^n_{\ ik\bar{j}} v_n - R^n_{\ ki\bar{j}} v_n \tag{158}$$

On a Kähler manifold, since the $(2,0)$-components of the Riemann tensor are zero, $R^n_{\ \bar{j}ik} = 0$, the cyclic Bianchi identity implies the symmetry property $R^n_{\ ik\bar{j}} = R^n_{\ ki\bar{j}}$, therefore the above equation is simplified to

$$\nabla_{\bar{j}} F_{ki} = 0\,. \tag{159}$$

The two equations (157), (289) together imply that the $(2,0)$ two-form F is parallel:

$$\nabla_\mu F_{ij} = 0\,. \tag{160}$$

Clearly, its complex conjugate, which is a form of type $(0,2)$, is parallel as well: $\nabla_\mu F_{\bar{i}\bar{j}} = 0\,.$

On a Calabi-Yau 3-fold there is a nowhere-vanishing holomorphic 3-form Ω_{ijk}. If it is normalized so that the volume form is $\mathrm{vol} = i\,\Omega \wedge \bar{\Omega}$, then Ω is also parallel: $\nabla_\mu \Omega = 0$.

[7] The covariant derivative in that expression may in fact be replaced by an ordinary derivative as the mixed Christoffel symbols are zero.

Raising the indices, we also obtain the dual poly-vector $\widetilde{\Omega}^{abc}$, which is parallel as well. Using this poly-vector, we can dualize the $(2,0)$ two-form F to a $(1,0)$ vector field $f^a := \frac{1}{2}\,\widetilde{\Omega}^{abc}\,F_{bc}$. It follows from (160) that this vector field is parallel:

$$\nabla_\mu\, f^a = 0\,. \tag{161}$$

On a Kähler manifold the parallel vector fields come in pairs, since $I \circ f$ is parallel as well due to the fact that $\nabla\, I = 0$. For $f \neq 0$ this implies the reduction of the holonomy group $SU(3) \to SU(2)$, and the manifold is $\mathcal{M}_3 \simeq \mathbb{R}^2 \times \mathcal{M}_2$. ■

We have thus proven that the manifold $\mathcal{M}$ has no non-holomorphic isometries, so that we may set $f = f_0$ and assume that λ is a *holomorphic* vector field.

To derive a further constraint on λ, let us calculate $\Omega(\bullet,\bullet,\lambda)$:

$$\Omega_{abc}\,\lambda^c = \text{by } (144) = \Omega_{abc}\,\nabla_{\bar t}\omega^{ct} = \text{by } (\Omega - \text{dualization}) = \Omega_{abc}\widetilde{\Omega}^{cts}\,\nabla_{\bar t}\omega_s =$$
$$= \text{by } (133) = \nabla_a\omega_b - \nabla_b\omega_a = (\partial\omega)_{ab}\,.$$

Since $\Omega(\bullet,\bullet,\lambda) := \Omega_{abc}\,\lambda^c\, dz_a \wedge dz_b$ is a holomorphic 2-form (both Ω and λ are holomorphic), $\bar\partial\Omega(\bullet,\bullet,\lambda) = 0$. According to the above, one also has $\partial\Omega(\bullet,\bullet,\lambda) = 0$. To summarize,

$$\mathcal{L}_\lambda\Omega = 0\,, \tag{162}$$

i.e. λ is a holomorphic Killing vector field that preserves the Calabi-Yau 3-form Ω.

The requirement (162) is an additional condition on λ, i.e. it is not satisfied for an arbitrary holomorphic Killing field: consider the case of $\mathcal{M} = \mathbb{C}$ with Kähler form (=volume form) $\mathrm{vol} = i\, dz \wedge d\bar z$, holomorphic one-form $\Omega := dz$ and holomorphic Killing field $\lambda := \mathrm{Re}\,(i\,z\,\frac{\partial}{\partial z})$. One has $\mathcal{L}_\lambda\Omega = i\,\Omega \neq 0$.

Since λ is a holomorphic Killing vector field, it preserves the Kähler form and one can introduce the corresponding moment map τ by means of the following equation

$$d\tau = \mathcal{J} \circ \tilde\lambda,$$

where $\mathcal{J}$ is the complex structure. In components, $\tilde\lambda_k = i\,\partial_k\tau$ and $\tilde\lambda_{\bar k} = -i\,\partial_{\bar k}\tau$. We can now rewrite the equation (149) as

$$\nabla_i\omega_{j\bar k} = g_{j\bar k}\,\partial_i\widehat{h} - 2g_{i\bar k}\,\partial_j\widehat{h} + \nabla_{\bar k}\omega_{ij}\,, \tag{163}$$
$$\text{where} \qquad \widehat{h} = h - i\,\tau\,. \tag{164}$$

Let us now discuss the integrability conditions for (163). If i and m are both holomorphic indices, one has $[\nabla_i, \nabla_m] = 0$, therefore one has the following condition:

$$\nabla_m(g_{j\bar k}\,\partial_i\widehat{h} - 2g_{i\bar k}\,\partial_j\widehat{h} + \nabla_{\bar k}\omega_{ij}) - \nabla_i(g_{j\bar k}\,\partial_m\widehat{h} - 2g_{m\bar k}\,\partial_j\widehat{h} + \nabla_{\bar k}\omega_{mj}) = 0 \tag{165}$$

Contracting this with $g^{i\bar k}$, we get

$$4\nabla_m\partial_j\widehat{h} + \nabla_m\tilde\lambda_j + g^{i\bar k}\,\nabla_i\nabla_{\bar k}\omega_{mj} = 0 \tag{166}$$

Since λ is a holomorphic vector field, $\nabla_m\tilde{\lambda}_j = 0$. Noting the following equality:

$$\nabla_{\bar{k}}\nabla_i\omega_{mj} = 0, \tag{167}$$

which follows from (139), (140), the assumption $f = f_0 = \text{const.}$ and the fact that Ω is a holomorphic form, we deduce that $g^{i\bar{k}}\,\nabla_i\nabla_{\bar{k}}\omega_{mj} = g^{i\bar{k}}\,\nabla_{\bar{k}}\nabla_i\omega_{mj} + (\mathrm{Ric}\circ\omega)_{mj} = \frac{f_0}{3}\,g^{i\bar{k}}\,\nabla_{\bar{k}}\Omega_{imj} = 0$. Hence (166) leads to

$$\nabla_m\partial_j\widehat{h} = 0\,. \tag{168}$$

Substituting this in (165), we obtain

$$\nabla_m\nabla_{\bar{k}}\omega_{ij} - \nabla_i\nabla_{\bar{k}}\omega_{mj} = 0$$

Commuting the derivatives, using (167) and the symmetry property of the Riemann tensor on a Kähler manifold: $R^n_{\;\;jm\bar{k}} = R^n_{\;\;mj\bar{k}}$ (which follows from the cyclic Bianchi identity), we get

$$R^n_{\;\;mj\bar{k}}\,\omega_{in} = R^n_{\;\;ij\bar{k}}\,\omega_{mn}\,. \tag{169}$$

Dualizing the two-form ω to a vector using the three-form Ω, i.e. $\omega_{ij} = \frac{1}{2}\Omega_{ijk}\,\omega^k$, subsequently dualizing the equation (169) (which is skew-symmetric in the (i,m) indices) by means of multiplication by $\widetilde{\Omega}^{ims}$ and using (133), we get

$$\boxed{R^n_{\;\;mj\bar{k}}\,\omega^m = 0} \tag{170}$$

i.e. the Riemann tensor has a 'null-vector'. This completes the proof of the **Proposition**. ■

§ 6.2. CKYF of type $(1,1)$

For the time being we will make an additional assumption that the CKYF 2-form ω is of type $(1,1)$, i.e. we set $\omega^{(2,0)} = \omega^{(0,2)} = 0$ (which trivially satisfies (170)). We therefore find ourselves in the situation studied in [6, 7] – in this section we mainly review the results of these papers. The equation (149) simplifies to

$$\nabla_a\omega_{b\bar{c}} = \left(g_{b\bar{c}}\partial_a h - 2\,g_{a\bar{c}}\partial_b h\right)\,, \quad \text{where} \quad h = g^{a\bar{b}}\omega_{a\bar{b}} \tag{171}$$

It is convenient to introduce the 'shifted' 2-form of type $(1,1)$ $\Omega_{b\bar{c}} = \omega_{b\bar{c}} - h\,g_{b\bar{c}}$, which brings the equation to the form

$$\nabla_a\Omega_{b\bar{c}} = -2\,g_{a\bar{c}}\,\partial_b h\,. \tag{172}$$

The complex conjugate equation is[8]

$$\nabla_{\bar{a}}\Omega_{b\bar{c}} = -2\,g_{b\bar{a}}\,\partial_{\bar{c}}h\,. \tag{175}$$

[8] Using the notation $X = X^a\,\partial_a + X^{\bar{a}}\,\partial_{\bar{a}}$ for a vector field X as well as for the corresponding dual 1-form $X = X_a dz^a + X_{\bar{a}}dz^{\bar{a}}$, one has

$$\nabla_X\Omega_{b\bar{c}} = 2\,\left(X_{\bar{c}}\,\partial_b h + X_b\,\partial_{\bar{c}}h\right)\,. \tag{173}$$

This equation is the defining equation of a so-called Hamiltonian 2-form [6], and may be rewritten invariantly as follows:

$$\nabla_X\Omega = \mathcal{J}\circ dh\wedge X - dh\wedge\mathcal{J}\circ X\,. \tag{174}$$

The equations defining a Killing-Yano tensor of type $(1,1)$ have the form (172)-(175) for a manifold $\mathcal{M}$ of arbitrary complex dimension $M > 2$, up to a rescaling of the function h. Therefore for the moment we will relax the assumption that $\dim_{\mathbb{C}} \mathcal{M} = 3$ and consider this more general situation. Let us assume that the Hamiltonian 2-form Ω has a maximum number of *distinct* (real) eigenvalues $\lambda_1 \ldots \lambda_M$ with corresponding eigenvectors $v^{(1)} \ldots v^{(M)}$, i.e.

$$\Omega_{m\bar{n}}\,(v^{(i)})^{\bar{n}} = \lambda_i\, g_{m\bar{n}}\,(v^{(i)})^{\bar{n}}, \qquad (v^{(i)})^m\,\Omega_{m\bar{n}} = \lambda_i\,(v^{(i)})^m\, g_{m\bar{n}}\,. \tag{176}$$

We will also assume that the eigenvalues $\lambda_1 \ldots \lambda_M$, which are functions on Y, are functionally independent, i.e. $d\lambda_1 \wedge \ldots \wedge d\lambda_M \not\equiv 0$.

LEMMA 5. [6] **The gradients of the eigenvalues λ_i are mutually orthogonal. The eigenvalues λ_i are in involution w.r.t. the Poisson bracket.**

Proof.
Multiplying the first equality in (176) by $(v^{(j)})^m$ and using the second equality, we get

$$(\lambda_i - \lambda_j)\,(v^{(j)})^m\, g_{m\bar{n}}\,(v^{(i)})^{\bar{n}} = 0\,, \tag{177}$$

hence the eigenvectors corresponding to different eigenvalues are orthogonal (in the Hermitian sense):

$$(v^{(j)})^m\, g_{m\bar{n}}\,(v^{(i)})^{\bar{n}} = 0 \qquad \text{for} \quad i \neq j\,. \tag{178}$$

Let us now multiply the equation (172) defining the Hamiltonian 2-form Ω by $v^b\, v^{\bar{c}}$, where v is a unit-normalized eigenvector of Ω ($\|v\|^2 := v^b\, v^{\bar{c}}\, g_{b\bar{c}} = 1$) corresponding to eigenvalue λ. Using

$$\begin{aligned} v^b\, v^{\bar{c}}\, \nabla_a \Omega_{b\bar{c}} &= \partial_a(\lambda\,\|v\|^2) - \Omega_{b\bar{c}}\,(\nabla_a v^b\, v^{\bar{c}} + v^b\, \nabla_a v^{\bar{c}}) = \\ &= \partial_a\,(\lambda\,\|v\|^2) - \lambda\,\partial_a \|v\|^2 = (\text{since } \|v\|^2 = 1) = \partial_a \lambda\,, \end{aligned}$$

we then get

$$\partial_a \lambda = -2\, g_{a\bar{c}}\, v^{\bar{c}} \cdot (v^b \partial_b h)\,. \tag{179}$$

This formula, together with (178), implies that the gradients of the eigenvalues λ_i are mutually orthogonal. Recall also that the Poisson bracket defined by the Kähler form is $\{f_1, f_2\} = g^{a\bar{b}}\,\partial_a f_1\,\partial_{\bar{b}} f_2 - g^{a\bar{b}}\,\partial_a f_2\,\partial_{\bar{b}} f_1$. It follows easily from (178) and (179) that the λ_i are in involution with respect to this Poisson bracket. ■

We will now show that it is possible to associate to the λ_i a set of commuting holomorphic Killing vector fields (i.e. vector fields preserving both the metric and the complex structure). To this end, we construct the elementary symmetric polynomials of λ_i (up to $\pm$ signs), which we call μ_k:

$$\prod_{k=1}^{n} (\vartheta - \lambda_k) = \sum_{k=0}^{n} \vartheta^k\, \mu_{k+1}, \qquad \mu_{n+1} = 1. \tag{180}$$

LEMMA 6. [6] **The vector fields $\xi_i := \mathcal{J} \circ \nabla \mu_i$ are commuting holomorphic Killing vector fields.**

Proof.
First, let us write out these vector fields and the dual one-forms in components:

$$(\xi_i)^a = i\, g^{a\bar b}\partial_{\bar b}\mu_i, \quad (\xi_i)^{\bar b} = -i\, g^{a\bar b}\partial_a\mu_i \quad \Rightarrow \quad (\xi_i)_a = -i\,\partial_a\mu_i, \quad (\xi_i)_{\bar b} = i\,\partial_{\bar b}\mu_i\,. \tag{181}$$

Let us start with $\mu_n = -\sum_{k=1}^{n}\lambda_k = -g^{m\bar n}\Omega_{m\bar n} := -\mathrm{Tr}(\Omega)$ (Here we have used the partition of unity

$$g^{m\bar n} = \sum_{j=1}^{M} (v^{(j)})^m\, (v^{(j)})^{\bar n} \tag{182}$$

formed out of the eigenvectors $v^{(i)}$ of Ω). From the definition of the Hamiltonian 2-form we obtain:

$$\nabla_a\Omega_{b\bar c} = -g_{a\bar c}\partial_b\mu_n \tag{183}$$

We have already seen in (168) that $\nabla_a\partial_b\mu_n = 0$, which may be also written as the holomorphicity of the vector field ξ_n (defined in (181)):

$$\partial_a(\xi_n)^{\bar c} = 0. \tag{184}$$

Note that the vector field $\xi_i := \mathcal{J}\circ\nabla\mu_i$ is Hamiltonian by definition, i.e. it preserves the Kähler form. The Killing condition $\mathfrak{L}_\xi g_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = 0$ is then automatically satisfied:

$$\nabla_a(\xi_n)_{\bar b} + \nabla_{\bar b}(\xi_n)_a = 0 \Rightarrow \nabla_a\partial_{\bar b}\mu_n - \nabla_{\bar b}\partial_a\mu_n \equiv 0 \quad \text{since } \Gamma^{\bar c}_{a\bar b} = 0 \tag{185}$$

$$\nabla_a(\xi_n)_b + \nabla_b(\xi_n)_a = 0 \quad (\text{follows from } \partial_a(\xi_n)^{\bar c} = 0\,). \tag{186}$$

So far we have shown that $\xi_n := \mathcal{J}\circ\nabla\mu_n$ is a holomorphic Killing vector field. Now we will prove inductively that $\nabla\mu_i$ are holomorphic Killing for all i. Suppose that $\partial_b(\xi_j)^{\bar a} = 0$ for $j = n, \ldots, k$.

Lemma 7. **One has the following recurrence relation:**

$$(\xi_{k-1})^{\bar a} = \mu_k(\xi_n)^{\bar a} + \Omega^{\bar a}_{\ \bar c}(\xi_k)^{\bar c}, \tag{187}$$

where $\Omega^{\bar a}_{\ \bar c} := g^{a\bar a}\,\Omega_{a\bar c}$. (This gives the matrix elements of the operator Ω in the basis of vectors $\{\xi_k\}$.)

Proof.
According to (179), one has for the gradients of λ_i the following formula:

$$\nabla^{\bar a}\lambda_i := g^{a\bar a}\,\partial_a\lambda_i = -2\, v_i^{\bar a}\cdot(v_i^b\,\partial_b h)\,. \tag{188}$$

Using it, we calculate the gradient of the logarithm of (180):

$$\sum \frac{-\nabla^{\bar a}\lambda_k}{\vartheta - \lambda_k} = \frac{\sum \vartheta^k\,\nabla^{\bar a}\mu_{k+1}}{\sum \vartheta^k\,\mu_{k+1}}\,. \tag{189}$$

Acting on it by $-i\,(\vartheta\,\delta^{\bar c}_{\bar a} - \Omega^{\bar c}_{\ \bar a})$, using (188), the definition (176) in the form $\Omega^{\bar c}_{\ \bar a}\,v_i^{\bar a} = \lambda_i\,v_i^{\bar c}$ and the definition (181) ($-i\,\nabla^{\bar a}\mu_{k+1} \equiv (\xi_{k+1})^{\bar a}$), we have

$$-\,2i\,\nabla^{\bar c}h = \frac{\vartheta\sum\vartheta^k\,(\xi_{k+1})^{\bar c} - \sum\vartheta^k\,\Omega^{\bar c}_{\ \bar a}\,(\xi_{k+1})^{\bar a}}{\sum\vartheta^k\,\mu_{k+1}} \tag{190}$$

To simplify the l.h.s., we have used the partition of unity (182). Recalling the definition $(\xi_n)^{\bar{a}} = -i\,\nabla^{\bar{a}}\mu_n = -2i\,\nabla^{\bar{a}}h$, we get from (190) the recurrence relation (187). ■

Continuation of proof of Lemma 6.
Let us now calculate the holomorphic derivative of (187):

$$\partial_b(\xi_{k-1})^{\bar{a}} = \partial_b\mu_k(\xi_n)^{\bar{a}} + (\partial_b\Omega^{\bar{a}}_{\bar{c}})(\xi_k)^{\bar{c}} = ig_{b\bar{c}}\,(\xi_k)^{\bar{c}}\,(\xi_n)^{\bar{a}} + (\partial_b\Omega^{\bar{a}}_{\bar{c}})(\xi_k)^{\bar{c}} \tag{191}$$

In order to evaluate the last term we multiply the definition (183) of the Hamiltonian 2-form by $(\xi_k)^{\bar{c}}$ to obtain $\partial_b(\Omega^{\bar{a}}_{\bar{c}})\xi^{\bar{c}}_k = -g_{b\bar{c}}(\xi_k)^{\bar{c}}\nabla^{\bar{a}}\mu_n = -ig_{b\bar{c}}(\xi_k)^{\bar{c}}\,(\xi_n)^{\bar{a}}$. Plugging this in the above expression, we get

$$\partial_b(\xi_{k-1})^{\bar{a}} = 0\,. \tag{192}$$

We have proven that the vector fields ξ_i are Killing. Moreover, they commute, as these are Hamiltonian vector fields, whose corresponding Hamiltonians are $\mu_i = \mu_i(\{\lambda_k\})$, and we have shown above that λ_k's are in involution. (We use the property of the Hamiltonian vector fields $[X_f, X_g] = X_{\{f,g\}}$.) This completes the proof of **Lemma 6.2**. ■

Let us analyze the consequences of lemmas 5 and 6, first at the example of the toric metric (4) and assuming the maximal number (three) of linearly independent vector fields ξ_k.

Lemma 8. [6] **Suppose the metric (4) admits a Killing-Yano form of type $(1,1)$, and the Killing vector fields $\xi_i, i = 1, 2, 3$, generated by this form, as defined in Lemma 6, coincide with $\frac{\partial}{\partial\phi_i}$. Then in the metric (4) one has**

$$G_{ij} = \sum_{k=1}^{n} Q_k(\lambda)\,\frac{\partial\lambda_k}{\partial\mu_i}\,\frac{\partial\lambda_k}{\partial\mu_j}\,, \qquad \textbf{where} \qquad Q_p = f_p(\lambda_p)\,\prod_{t\neq p}(\lambda_p - \lambda_t)\,. \tag{193}$$

The potential G can be expressed as (up to a function linear in $\{\mu_k\}$)

$$G = -\sum_m \int^{\lambda_m} dy_m\, f_m(y_m)\,\prod_k (y_m - \lambda_k)\,. \tag{194}$$

Proof.
We proved in Lemma 5.1 that the gradients of λ_k are orthogonal, therefore in the new variables $\{\lambda_k\}$ the metric should be of orthogonal form:

$$G_{ij}d\mu^i d\mu^j = \sum_{k=1}^{n} Q_k(\lambda)\,d\lambda_k^2\,.$$

This is clearly the same as (193). Let us now use the condition $\partial_k G_{ij} = \partial_i G_{kj}$. Multiplying the resulting equation by $\mathfrak{J}^j_p \cdot \mathfrak{J}^i_s \cdot \mathfrak{J}^k_t$, where $\mathfrak{J}$ is the Jacobian $\mathfrak{J}^i_s := \frac{\partial\mu_i}{\partial\lambda_s}$, we obtain:

$$\delta_{ps}\frac{\partial Q_p}{\partial\lambda_t} - \delta_{pt}\frac{\partial Q_p}{\partial\lambda_s} + Q_s\,T^s_{pt} - Q_t\,T^t_{ps} = 0, \qquad T^s_{pt} := \sum_{i,j}\frac{\partial\mu_j}{\partial\lambda_p}\frac{\partial\mu_i}{\partial\lambda_t}\,\frac{\partial^2\lambda_s}{\partial\mu_i\partial\mu_j}$$

Setting in the above equation $p = s \neq t$, we get

$$t \neq p \quad \Rightarrow \quad \frac{\partial Q_p}{\partial\lambda_t} + Q_p\,T^p_{pt} - Q_t\,T^t_{pp} = 0 \tag{195}$$

In order to calculate T^p_{pt} and T^t_{pp} we use the defining equation $\prod\limits_{k=1}^{n}(\vartheta-\lambda_k)=\sum\limits_{k=0}^{n}\vartheta^k\,\mu_{k+1}$. Differentiating it w.r.t. μ_j and sending $\vartheta\to\lambda_i$, we get

$$\frac{\partial\lambda_i}{\partial\mu_j}=-\frac{\lambda_i^{j-1}}{\prod\limits_{k\neq i}(\lambda_i-\lambda_k)}$$

It is also easy to calculate the second derivative $\frac{\partial}{\partial\lambda_k}\left(\frac{\partial\lambda_i}{\partial\mu_j}\right)$ for $k\neq i$:

$$k\neq i\quad\Rightarrow\quad\frac{\partial}{\partial\lambda_k}\left(\frac{\partial\lambda_i}{\partial\mu_j}\right)=\frac{1}{\lambda_i-\lambda_k}\,\frac{\partial\lambda_i}{\partial\mu_j}\,.$$

Using this, we get

$$t\neq p\quad\Rightarrow\quad T^p_{pt}=\frac{\partial\mu_i}{\partial\lambda_p}\,\frac{\partial}{\partial\lambda_t}\left(\frac{\partial\lambda_p}{\partial\mu_i}\right)=\frac{1}{\lambda_p-\lambda_t},\qquad T^t_{pp}=0$$

The equations (195) therefore are $\frac{\partial Q_p}{\partial\lambda_t}+\frac{1}{\lambda_p-\lambda_t}\,Q_p=0\,(p\neq t)$ and have the solution

$$Q_p=f_p(\lambda_p)\,\prod_{t\neq p}(\lambda_p-\lambda_t)\tag{196}$$

Using the above results, one can integrate (193) to obtain an expression for the symplectic potential G. Indeed, since $G_{ij}=\frac{\partial\lambda_m}{\partial\mu_j}\,\frac{\partial}{\partial\lambda_m}\left(\frac{\partial G}{\partial\mu_i}\right)$, we have from (193):

$$\frac{\partial}{\partial\lambda_m}\left(\frac{\partial G}{\partial\mu_i}\right)=\frac{\partial\lambda_m}{\partial\mu_i}\,Q_p(\lambda)=-\lambda_m^{i-1}\,f_m(\lambda_m)\,.$$

Integrating, we get $\frac{\partial G}{\partial\mu_i}=-\sum\limits_m\int\limits^{\lambda_m}dy_m\,y_m^{i-1}\,f_m(y_m)$. Once again passing to the λ-variables in the l.h.s., we get

$$\frac{\partial G}{\partial\lambda_n}=-\sum_i\frac{\partial\mu_i}{\partial\lambda_n}\sum_m\int^{\lambda_m}dy_m\,y_m^{i-1}\,f_m(y_m)=-\sum_m\int^{\lambda_m}dy_m\,f_m(y_m)\left(\sum_i\frac{\partial\mu_i}{\partial\lambda_n}y_m^{i-1}\right)=$$

$$=\text{using the definition (180)}=\sum_m\int^{\lambda_m}dy_m\,f_m(y_m)\,\prod_{k\neq n}(y_m-\lambda_k)\,.$$

This is easily integrated to give (194). ■

Let us now see what form the metric (4) takes. To this end, recall that we have already seen that the matrix $\mathfrak{J}$ diagonalizes the metric G_{ij}, i.e. $\mathfrak{J}^T\circ G\circ\mathfrak{J}=\mathrm{Diag}\{Q_1,\ldots,Q_n\}$. Therefore

$$ds^2=\frac{1}{4}\sum_{k=1}^{3}Q_k\,d\lambda_k^2\,+\sum_{k=1}^{3}\frac{1}{Q_k}\,(\mathfrak{J}^i_k\,d\phi_i)\,(\mathfrak{J}^j_k\,d\phi_j),\tag{197}$$

where the functions Q_k are of the form (196). The Jacobian $\mathfrak{J}=\frac{\partial\mu}{\partial\lambda}$ is easily found by differentiating the definition (180) w.r.t. λ_m:

$$-\prod_{k=1,k\neq m}^{n}(\vartheta-\lambda_k)=\sum_{k=0}^{n-1}\vartheta^k\,\frac{\partial\mu_{k+1}}{\partial\lambda_m}$$

It follows that $\mathfrak{J}^{k+1}_m = \frac{\partial \mu_{k+1}}{\partial \lambda_m}$ is minus the elementary symmetric polynomial of degree $n-1-k$ in the variables $\lambda_1, \ldots, \widehat{\lambda_m}, \ldots, \lambda_n$ with λ_m omitted.

The expression for the metric (193) and for the potential (194) essentially reproduce the formulae of Proposition 11 in [6]. We also note that an expression identical to (196) was obtained in the investigation of metrics possessing Killing tensors in [32].

Now, it is not difficult to see that the fully orthotoric metric (197) cannot describe the geometry (59) of Y. Indeed, consider the sphere $\mathbb{CP}^1 \subset Y$ lying at the intersection of two hyperplanes, say $\ell_1 = 0$ and $\ell_2 = 0$. According to (12), at each of these hyperplanes one should have $\mathrm{Det}(\mathrm{Hess}\, G)^{-1} = 0$. From the point of view of the orthotoric metric, this means that one of the functions $\{\frac{1}{f_p(\lambda_p)}\}$ has to vanish at this hyperplane. Therefore at the intersection of two hyperplanes two functions vanish, and we may assume $\frac{1}{f_1(\lambda_1^{(0)})} = \frac{1}{f_2(\lambda_2^{(0)})} = 0$. The induced metric on the sphere is therefore determined by the remaining function $f_3(\lambda)$. For this metric to be the standard round metric one has to require $Q_3(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3) \sim \frac{1}{1-\lambda_3^2}$. This choice is however incompatible with the Ricci-flatness of the metric (197) (the conditions that Ricci-flatness imposes on the functions $f_p(\lambda_p)$ are given in [6], see the Theorem in the introduction). The resolution of this problem lies in relaxing the requirement of having three linearly independent vector fields $\{\xi_k\}$ and considering instead the situation when one of the eigenvalues of the Killing-Yano tensor is constant, and the other two give rise to two linearly independent Killing vector fields. In this case, as it was shown in [6], the metric takes the form

$$\begin{aligned} ds^2 = \quad & -x\, y \cdot \left(g_V\, d\zeta\, d\bar{\zeta}\right) + \frac{y-x}{4}\left(f_2(y)\, dy^2 - f_1(x)\, dx^2\right) + \\ & + \frac{1}{y-x}\left(\frac{1}{f_2(y)}\,(d\phi_1 + x\,\omega)^2 - \frac{1}{f_1(x)}\,(d\phi_1 + y\,\omega)^2\right) \\ & \omega = d\phi_2 - A, \quad dA = i\, g_V\, d\zeta \wedge d\bar{\zeta}\,. \end{aligned} \tag{198}$$

Here $g_V(\zeta, \bar{\zeta})\, d\zeta\, d\bar{\zeta}$ is a Kähler metric on a Riemann surface, which may be seen as the metric on the Kähler quotient w.r.t. the vector fields $\frac{\partial}{\partial \phi_i}$. If one assumes that g_V is the standard round metric on $\mathbb{CP}^1$, $g_V\, d\zeta\, d\bar{\zeta} = \frac{2\, d\zeta\, d\bar{\zeta}}{(1+|\zeta|)^2}$,the metric (198) becomes compatible with (59), and in fact may be obtained from it by carrying out the above derivations word-by-word for the part of (59), 'transverse' to the $\mathbb{CP}^1$. In particular, the formula (194) now gives the 'transverse', or 'reduced', symplectic potential G referred to in (59).

§ 6.3. The orthotoric metric

The dual potential for the Ricci-flat metric may be obtained from (194) by setting

$$\begin{aligned} & f_1(x) := -\frac{3\,x}{P(x)}, \qquad f_2(y) := -\frac{3\,y}{Q(y)}, \qquad \text{where} \\ & P(x) = x^3 - \frac{3}{2}x^2 + c = \prod_{i=1}^{3}(x - x_i), \qquad Q(y) = y^3 - \frac{3}{2}y^2 + d = \prod_{i=1}^{3}(y - y_i)\,. \end{aligned} \tag{199}$$

This form of the functions follows from the Ricci-flatness equation. Upon the identification $\lambda_1 = x$ and $\lambda_2 = y$ we get

$$\boxed{G_{\rm o} = -\sum_{i=1}^{3} \frac{(x-x_i)(y-x_i)}{1-x_i} \log|x-x_i| - \sum_{i=1}^{3} \frac{(x-y_i)(y-y_i)}{1-y_i} \log|y-y_i| + 3\,(x+y)\,.} \tag{200}$$

The roots are labeled in increasing order, i.e. $x_- := x_1 < x_2 < x_3$ and $y_1 < y_2 < y_3$. Note that $y_i = \xi_i$ are the roots of $Q(y)$ that we encountered before, in §§ 4.1 and 4.2. The domain in the (x, y) space is as follows (for details see [31]):

$$y \in [y_1, y_2], \qquad x \leqslant x_- < 0\,. \tag{201}$$

Note that in this domain $f_1(x) \leqslant 0$ and $f_2(y) \geqslant 0$, therefore the metric (198) is positive-definite. The moment maps μ, ν are related to the auxiliary 'orthotoric' variables x, y by means of the following formulas, which follow essentially from (180):

$$\mu = -x\,y, \quad \nu = -(x+y)\,. \tag{202}$$

The meaning of these variables was explained in detail in the previous paragraphs. The minus signs are needed, since in the (μ, ν) variables 'infinity' corresponds to $\mu, \nu \to +\infty$, whereas in the (x, y) variables it corresponds to $x \to -\infty$, y bounded and positive. The potential (200), expressed in terms of μ, ν, satisfies the Ricci-flatness equation (58) with $a = 9$.

Let us first of all expand the potential $G_{\rm ortho}$ at 'infinity', i.e. at $\nu \to \infty$ with $\xi = \frac{\mu}{\nu}$ fixed. It is easy to see from (202) that this corresponds, in terms of the x, y variables, to the limit $x \to -\infty$, y fixed. We obtain:

$$G_{\rm o} \to -3\,x\,\log|x| - x\sum_{i=1}^{3} \frac{(y-y_i)}{1-y_i} \log|y-y_i| + \ldots \tag{203}$$

This should be compared with formula (78). In particular, this means that the parameter d of the orthotoric metric coincides with the corresponding parameter d from (82). Its value is therefore given by (88):

$$d = \frac{16+\sqrt{13}}{64}\,. \tag{204}$$

It might seem that the orthotoric potential $G_{\rm o}$ still possesses one nontrivial parameter c, the free term of the polynomial $P(x)$. However, it turns out that this parameter has to be fixed to a particular value by the requirement that the 3-rd line of the biangle in Fig. 2 passes at a correct angle with respect to the other two lines (meaning that the topology of the manifold is indeed the one of a cone over $\mathbf{dP}_1$). In fact, the value of c may be deduced from the above formulas (86). Indeed, we calculate $k_1 = -\frac{\xi_1^{(1)}}{1-\xi_1^{(1)}}$, $k_2 = -\frac{\xi_2^{(1)}}{1-\xi_2^{(1)}}$, where the values of $\xi_{1,2}^{(1)}$ are given in (274). Assuming that the lower line of the moment polygon is given by $x = x_-$ and remembering that $k_3 = -\frac{x_-}{1-x_-}$, we can then calculate x_- from either of the relations (86): $x_- = \frac{1}{2}(4+\sqrt{13})$. Since x_- has to be a root of the polynomial $P(x)$, we obtain $c = \frac{3}{2}\,x_-^2 - x_-^3$, which numerically turns out to be

$$c = -\frac{1}{8}(133+37\sqrt{13})\,. \tag{205}$$

7. Deformation of the moment polytope

In this section we will show directly that there is a first-order deformation of the orthotoric metric that reflects an infinitesimal deformation of the moment polytope. The deformation of the polytope in question is the ϵ-shift of its lower side, as shown in Fig. 4. The canonical way to deal with this problem is to keep the domain unchanged and to introduce the explicit dependence on ϵ in the equation itself. This can be done by explicitly mapping the new domain to the old one, which can be achieved as follows:

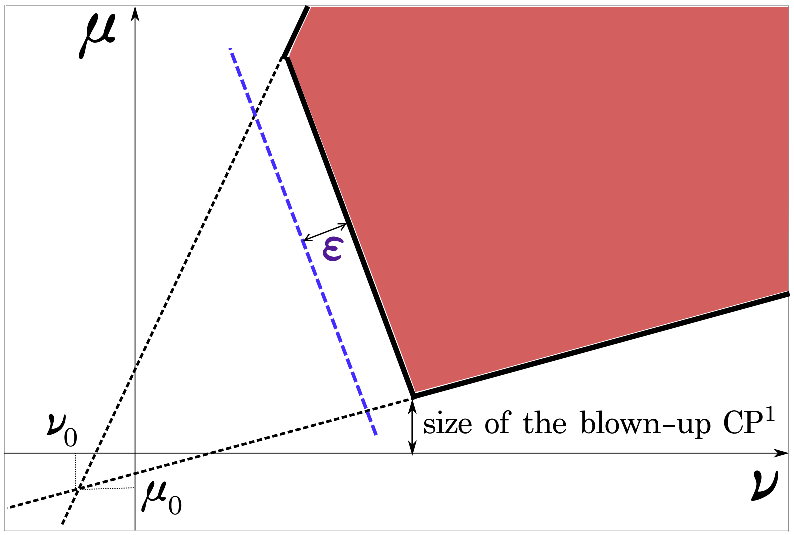

Figure 4: Deformation of the moment polytope.

- Shift the variables (μ, ν) so that the new origin is located at the intersection point of the dashed lines shown in Fig. 4
- Rescale the variables infinitesimally i.e. $\mu \to (1+\epsilon)\,\mu, \;\; \nu \to (1+\epsilon)\,\nu$. Clearly, this maps the dashed lines to themselves (since they pass through the origin) and moves the lower line of the polytope parallel to itself by a distance of order ϵ.
- Shift the variables (μ, ν) back .

The net effect is in the following change of variables:

$$\bar{\mu} = \frac{\mu + \epsilon\,\mu_0}{1+\epsilon}, \qquad \bar{\nu} = \frac{\nu + \epsilon\,\nu_0}{1+\epsilon}\,, \tag{206}$$

where (μ_0, ν_0) are the coordinates of the intersection point of the dashed lines in the original coordinates:

$$\mu_0 = -\xi_1\xi_2, \qquad \nu_0 = -(\xi_1+\xi_2)\,. \tag{207}$$

It is also convenient to pass to a new unknown function $\bar{G}$:

$$G = (1+\epsilon)\,(3\,\bar{\nu}\,\log(1+\epsilon) + \bar{G})\,. \tag{208}$$

One then has the following equation for $\bar{G}$:

$$e^{\frac{\partial \bar{G}}{\partial \bar{\mu}} + \frac{\partial \bar{G}}{\partial \bar{\nu}}} \times \mathrm{Det}\,\overline{\mathrm{Hess}}\,\bar{G} = a\left(\bar{\mu} - \frac{\epsilon\mu_0}{1+\epsilon}\right)\,. \tag{209}$$

What we have achieved is that the domain of definition here is the same as the original moment polytope. In particular, for $\epsilon = 0$ we know the (unique) solution to (209) – it is given by the orthotoric potential (200) above (for the case $a = 9$). Since the latter is most conveniently expressed in the orthotoric (x, y) coordinates, let us pass to these coordinates in the eq. (209), using the formulas (202). We then obtain:

$$\mathcal{R}[\bar{G}, \epsilon] = 0, \qquad \text{where} \tag{210}$$

$$\mathcal{R}[\bar{G}, \epsilon] := e^{\left(\frac{1-y}{y-x}\bar{G}_y - \frac{1-x}{y-x}\bar{G}_x\right)} \underbrace{\frac{1}{(x-y)^2}\left(\bar{G}_{xx}\bar{G}_{yy} - \left(\bar{G}_{xy} + \frac{\bar{G}_x - \bar{G}_y}{x-y}\right)^2\right)}_{=\det \overline{\mathrm{Hess}}_{(\bar{\mu},\bar{\nu})}\bar{G}} + 9\left(xy + \frac{\epsilon\,\mu_0}{1+\epsilon}\right).$$

The first order of perturbation theory in ϵ for the solution $\bar{G}$ may be constructed as follows:

$$\begin{aligned} &\bar{G} = G_\mathrm{o} + \epsilon\, H(x, y), \\ &\frac{\delta \mathcal{R}[\bar{G}, 0]}{\delta \bar{G}}\Big|_{\bar{G} = G_\mathrm{o}} \circ (\epsilon\, H) + \mathcal{R}[G_\mathrm{o}, \epsilon] = 0\,. \end{aligned} \tag{211}$$

Taking into account that

$$\frac{\delta \mathcal{R}[\bar{G}, 0]}{\delta \bar{G}}\Big|_{\bar{G} = G_\mathrm{o}} \circ H = \frac{3}{x-y}\left(y\frac{\partial}{\partial x}\left(P(x)\frac{\partial H}{\partial x}\right) - x\frac{\partial}{\partial y}\left(Q(y)\frac{\partial H}{\partial y}\right)\right), \tag{212}$$

we can write the linearized equation as

$$\frac{1}{x-y}\left(y\frac{\partial}{\partial x}\left(P(x)\frac{\partial H}{\partial x}\right) - x\frac{\partial}{\partial y}\left(Q(y)\frac{\partial H}{\partial y}\right)\right) + 3\,\mu_0 = 0\,. \tag{213}$$

The variables separate, and one can look for the solution in the form

$$H(x, y) = h_1(x) + h_2(y)\,. \tag{214}$$

The functions $h_1(x), h_2(y)$ then satisfy the following equations:

$$h_1'(x) = \frac{\mathrm{B} + 3\,\mu_0\, x + \mathrm{A}\, x^2}{P(x)}, \qquad h_2'(y) = \frac{\widetilde{\mathrm{B}} + 3\,\mu_0\, y + \mathrm{A}\, y^2}{Q(y)}, \tag{215}$$

where $\mathrm{A}, \mathrm{B}, \widetilde{\mathrm{B}}$ are constants. Since we wish the function H to be regular at the sides of the moment polytope, i.e. at $y = \xi_1, \xi_2$ and $x = x_-$, we have to require that the numerators of the fractions in the right hand sides of the equations vanish at the prescribed points, which can be formulated as

$$\widetilde{\mathrm{B}} + 3\,\mu_0\, y + \mathrm{A}\, y^2 = \mathrm{A}\,(y - \xi_1)\,(y - \xi_2), \qquad \mathrm{B} + 3\,\mu_0\, x_- + \mathrm{A}\, x_-^2 = 0\,. \tag{216}$$

Using the relations between the roots ξ_0, ξ_1, ξ_2 summarized in Appendix C, we find

$$\mathrm{A} = -3\,\xi_0, \quad \mathrm{B} = 3\,x_-\,\xi_0\,(x_- - \xi_1 - \xi_2), \quad \widetilde{\mathrm{B}} = 3\,d, \tag{217}$$

so that

$$h_1'(x) = -\frac{3\,\xi_0\,(x - \tilde{x})}{(x - x_1)(x - x_2)}, \quad h_2(y) = -3\,\xi_0\,\log|y - \xi_0|\,, \quad \tilde{x} = \xi_1 + \xi_2 - x_-\,. \tag{218}$$

The formulas (208), (211), (214) and (218) together give the first-order Ricci-flat deformation of the metric, corresponding to the change of the polytope depicted in Fig. 4.

§ 7.1. Asymptotic behavior at infinity

We wish to quantify the deviation of the ϵ-corrected metric from the conical metric (77) at infinity. To do so, first we write the symplectic potential as (see (208))

$$G = (1+\epsilon)\,(G_o(\bar{x},\bar{y}) - 3\,\log(1+\epsilon)\,(\bar{x}+\bar{y}) + \epsilon\,H(\bar{x},\bar{y})), \tag{219}$$

where $\bar{x}, \bar{y}$ are the orthotoric variables which themselves depend on ϵ via the formulas (206):

$$-\,\bar{x}\,\bar{y} = \frac{\mu + \epsilon\,\mu_0}{1+\epsilon}, \qquad -(\bar{x}+\bar{y}) = \frac{\nu + \epsilon\,\nu_0}{1+\epsilon} \tag{220}$$

Since we are interested in the first order in ϵ, we may construct a perturbation theory for the variables $\bar{x}, \bar{y}$. If we set $\bar{x} = x + \epsilon\,\delta x, \quad \bar{y} = y + \epsilon\,\delta y$, from (220) we easily find

$$\delta x = \frac{(x-\xi_1)(x-\xi_2)}{y-x}, \qquad \delta y = -\frac{(y-\xi_1)(y-\xi_2)}{y-x}\,. \tag{221}$$

From (219) we find that the first order (in ϵ) correction to the orthotoric potential $G_o(x,y)$ can be expressed as

$$G = G_o(x,y) + \epsilon\left(\underbrace{G_o(x,y) + \frac{\partial G_o(x,y)}{\partial x}\,\delta x + \frac{\partial G_o(x,y)}{\partial y}\,\delta y + H(x,y) - 3\,(x+y)}_{:=\delta G(x,y)}\right) + \dots \tag{222}$$

Now, x and y are related to μ, ν by means of the standard formulas (220) with $\epsilon = 0$, i.e. $\mu = -x\,y, \nu = -(x+y)$. We are interested in the behavior of $\delta G(x,y)$ at infinity, i.e. $x \to -\infty$, y bounded. A direct calculation shows that

$$\delta G(x,y) = \frac{\alpha}{x^2} + o\left(\frac{1}{x^2}\right), \qquad \alpha = \text{const}. \tag{223}$$

Since for large negative x we have $|x| = O(\mu) = O(\nu) = O(r^2)$, we find

$$|g - g_{\text{ortho}}|_{g_{\text{ortho}}} = O\left(\frac{1}{r^6}\right)\,. \tag{224}$$

According to the general theory introduced in § 2.1 (see Lemma 1), this implies that the variation of the Kähler form

$$[\delta\omega] \in H^2_c(Y,\mathbb{R}) \tag{225}$$

lies in the compactly supported cohomology group.

§ 7.2. An alternative derivation of the deformation

There is also a simpler way to evaluate the decay rate of the first-order deformation. Indeed, instead of first mapping the new moment polytope to the old one, one can try to construct directly a deformation of the potential G as follows:

$$G = G_{\text{o}} + \epsilon\,H \tag{226}$$

Expanding the Ricci-flatness equation (58) around the orthotoric solution to the first order in the deformation H, or simply using (213) that follows from the linearization of (210), we obtain the following remarkably simple linear equation:

$$\frac{1}{x}\frac{\partial}{\partial x}\left(P(x)\,\frac{\partial H}{\partial x}\right)-\frac{1}{y}\frac{\partial}{\partial y}\left(Q(y)\,\frac{\partial H}{\partial y}\right)=0 \tag{227}$$

The price that we will have to pay for not working in a fixed domain (as we did in the previous paragraph, by mapping the new domain to the old one) is that the deformation H will be affected by the domain shift in a singular way. Indeed, this has to be the case, since we proved in § 5 that there cannot be a deformation that is smooth at all sides of the polytope (this would imply that there is a deformation of the Ricci-flat metric with the same moment polytope, i.e. within the same Kähler class).

We recall that near any one of its edges, let us say the one defined by $\ell = 0$, the potential G behaves as follows:

$$G = \ell\,(\log \ell - 1) + \ldots \tag{228}$$

For our application we think of $\ell = 0$ as being the lower line of the moment polytope depicted in Fig. 4. Transporting the line defined by $\ell = 0$ parallel to itself means changing ℓ by $\ell + \epsilon$, where ϵ is a constant. Therefore after the shift

$$G = (\ell+\epsilon)\,(\log(\ell+\epsilon)-1)+\ldots = \ell\,(\log\ell-1)+\epsilon\,\log\ell+\ldots \tag{229}$$

One sees that the deformation is formally proportional to $\log \ell$, which is singular, however the important point is that the coefficient of proportionality is a constant (ϵ), whereas in general it could be a function of μ, ν. We come to the conclusion that the admissible deformation of the potential G, i.e. the one that can be resummed into a smooth potential G defined on a deformed moment polygon, is the one which has the form

$$H \sim \beta\,\log\ell + \ldots, \qquad \beta = \text{const.} \tag{230}$$

In the x, y variables this means that we are looking for a deformation of the form

$$H \sim \beta\,\log|x - x_-| + \ldots \quad \text{as} \quad x \to x_- \tag{231}$$

Since the left and right sides of the moment polytope in Fig. 4 are not shifted, we are looking for solutions of (227), non-singular at $y = y_1, y = y_2$. The general solution of (227) has the form:

$$H(x,y) = \sum_{\lambda} h_\lambda(x)\,g_\lambda(y), \tag{232}$$

where h_λ and g_λ are eigenfunctions of the Heun operators,

$$\frac{\partial}{\partial x}\left(P(x)\,\frac{\partial h_\lambda(x)}{\partial x}\right) - \lambda\,x\,h_\lambda(x) = 0, \tag{233}$$

$$\frac{\partial}{\partial y}\left(Q(y)\,\frac{\partial g_\lambda(y)}{\partial y}\right) - \lambda\,y\,g_\lambda(y) = 0 \tag{234}$$

Moreover, $g_\lambda(y)$ is an eigenfunction of the Sturm-Liouville problem, namely it has to be real-analytic at $y = y_1, y = y_2$. It then follows from Lemma 3 that $\lambda = 0$ or $\lambda \geqslant 3$. Then the standard Frobenius analysis of the equation (233) shows that $h_\lambda(x)$ decays at least as $\frac{1}{x^2}$ at infinity[9]. Now, if we assume $h_\lambda(x)$ regular at $x = x_-$, multiplying (233) by $h_\lambda(x)$ and integrating by parts, we find that $h_\lambda(x) = \text{const.}$ for $\lambda = 0$ and $h_\lambda(x) = 0$ for $\lambda > 0$. Therefore a regular deformation (different from a constant) does not exist. To have a nontrivial deformation, we have to assume that $h_\lambda(x)$ is singular at $x = x_-$, moreover the Frobenius analysis shows that it behaves as

$$h_\lambda(x) = a_\lambda \log|x - x_-| + \ldots, \qquad a_\lambda \neq 0 \tag{235}$$

Thus, we see that the solution $H(x, y)$ behaves at $x = x_-$ as

$$H(x, y) \sim q(y) \log|x - x_-| + \ldots \quad \text{with} \quad q(y) = \sum_\lambda a_\lambda g_\lambda(y) \tag{236}$$

According to the condition (231), $q(y) = \beta = \text{const.}$, so that

$$\beta = \sum_\lambda a_\lambda g_\lambda(y) \tag{237}$$

It is clear from (234) that one can take $g_0(y) = 1$. Moreover, by standard Sturm-Liouville theory arguments, the eigenfunctions $g_{\lambda_1}(y), g_{\lambda_2}(y)$ for $\lambda_1 \neq \lambda_2$ are orthogonal with respect to the weight function $y > 0$:

$$\int_{y_1}^{y_2} dy\, y\, g_{\lambda_1}(y)\, g_{\lambda_2}(y) = 0 \quad \text{for} \quad \lambda_1 \neq \lambda_2 \tag{238}$$

It follows that in (237) $a_0 = \beta$, $a_{\lambda \neq 0} = 0$, hence

$$H(x, y) = -\beta \int_x^\infty \frac{d\hat{x}}{P(\hat{x})} = -\frac{\beta}{2\, x^2} + \ldots \qquad \text{for} \quad x \to -\infty\,. \tag{239}$$

This confirms the decay estimate (224).

We may also use an alternative criterion, given by formula (31), to confirm that the variation of the Kähler form lies in $H^2_c(Y, \mathbb{R})$. It involves the calculation of the integrals of $\delta\omega$ over the homologically non-trivial cycles, which we called $\mathbb{CP}^1_A$ and $\mathbb{CP}^1_B$. These spheres are embedded in the del Pezzo surface with normal bundles $\mathcal{O}(1)$ for $\mathbb{CP}^1_A$ and $\mathcal{O}(-1)$ for $\mathbb{CP}^1_B$. From the point of view of the plot shown in Fig. 4, the sphere $\mathbb{CP}^1_A$ is located transversely to the plot at the upper angle of the polygon, and the sphere $\mathbb{CP}^1_B$ – at the lower angle. It follows from the expression for the metric (59) that the integrals of the Kähler form over these cycles are proportional to their μ-coordinates in the plot. The integrals of $\delta\omega$ – the deformation of the Kähler form – are then the differences of the μ-coordinates of the corners in the original and shifted polytopes. Therefore we arrive at the following equations:

$$\mathbb{CP}^1_A: \qquad \delta\mu_A - \xi_2\, \delta\nu_A = 0, \qquad \delta\mu_A - x_-\, \delta\nu_A - \epsilon = 0 \quad \Rightarrow \quad \delta\mu_A = \frac{\xi_2\, \epsilon}{\xi_2 - x_-}$$

$$\mathbb{CP}^1_B: \qquad \delta\mu_B - \xi_1\, \delta\nu_B = 0, \qquad \delta\mu_B - x_-\, \delta\nu_B - \epsilon = 0 \quad \Rightarrow \quad \delta\mu_B = \frac{\xi_1\, \epsilon}{\xi_1 - x_-}$$

[9]If we exclude the growing case, which would then not be subleading to the conical metric at infinity.

Using the actual values for the roots $\xi_{1,2}$ and x_-[10], which may be obtained from (199), (204), (205), we find

$$\frac{\delta\mu_A}{\delta\mu_B} = 3\,, \tag{240}$$

which, according to the criterion (31), implies $\delta\omega \in H^2_c(Y,\mathbb{R})$.

Comment. Interestingly, the same calculation shows that the moment polygon, corresponding to the compactly supported Kähler class, is the one where the two semi-infinite sides in Fig. 4 intersect precisely at $\mu = 0$. The polygon may be freely translated in the ν-direction, so we may assume that in this case the two semi-infinite sides intersect at the origin. The three lines are then given by the equations $\mu = \xi_1\,\nu, \mu = \xi_2\,\nu, \mu = x_-\,\nu + a$, and the calculation again gives the answer $\frac{\delta\mu_A}{\delta\mu_B} = 3$ for the ratio of the volumes of the two $\mathbb{CP}^1$'s located at the angles of the polygon.

§ 7.3. Deformation of the Killing-Yano form

PROPOSITION 7. **The curvature tensor of the orthotoric metric does not possess a null vector, i.e. equation (170) is only satisfied for $\omega = 0$.**

Proof.
The statement that the Riemann tensor has a null vector can be formulated in two equivalent ways:

$$R^i_{\;jk\bar{n}}\,\omega^k = 0, \qquad R^i_{\;jk\bar{n}}\,\omega^{\bar{n}} = 0\,. \tag{241}$$

The two are effectively related by complex conjugation and invoking the symmetry properties of the Riemann tensor. We will use the second form and the expression (8) for the Riemann tensor, which can be seen to imply

$$\sum_t \frac{\partial^2 G^{-1}_{jk}}{\partial\mu_s\partial\mu_t}\,\widehat{\omega}^t = 0 \quad \text{for all} \quad j,k,s, \qquad \widehat{\omega}^t = G^{-1}_{tn}\,\omega^{\bar{n}}\,. \tag{242}$$

In particular, the necessary condition is

$$\mathrm{Det}\left\{\frac{\partial^2 G^{-1}_{jk}}{\partial\mu_s\partial\mu_t}\right\}_{s,t} = \mathrm{Det\,Hess}(G^{-1}_{jk}) = 0 \qquad \text{for all} \quad j,k\,. \tag{243}$$

We will now prove that this does not hold for the orthotoric metric. First, we specialize to the case that the metric has $U(2)\times U(1)$ symmetry, rather than $U(1)^3$, i.e. we assume the form (51) of the Kähler potential. One can check that, for the dual potential, this implies the following form:

$$G = \left(\frac{\mu}{2}+\tau\right)\log\left(\frac{\mu}{2}+\tau\right) + \left(\frac{\mu}{2}-\tau\right)\log\left(\frac{\mu}{2}-\tau\right) - \mu\log\mu + \widetilde{G}(\mu,\nu)$$
$$\mu = \mu_1+\mu_2, \qquad \tau = \frac{\mu_1-\mu_2}{2}, \qquad \nu = \mu_3\,.$$

Here $\widetilde{G}(\mu,\nu)$ is the 'reduced' potential used everywhere above – it does not depend on τ. In our application we have in mind, of course, that

$$\widetilde{G} = G_\circ. \tag{244}$$

[10]These values are: $x_- = \frac{1}{2}(4+\sqrt{13})$, $\xi_1 = \frac{1}{8}(1+\sqrt{13})$, $\xi_2 = \frac{1}{8}(7+\sqrt{13})$.

The Hessian G_{ij} for the potential G of the above form is (the ordering of rows/columns is τ, μ, ν):

$$\mathrm{Hess}\, G = \{G_{ij}\} = \begin{pmatrix} \frac{\mu}{\frac{\mu^2}{4}-\tau^2} & \frac{-\tau}{\frac{\mu^2}{4}-\tau^2} & 0 \\ \frac{-\tau}{\frac{\mu^2}{4}-\tau^2} & \frac{\tau^2}{\mu(\frac{\mu^2}{4}-\tau^2)} + \widetilde{G}_{\mu\mu} & \widetilde{G}_{\mu\nu} \\ 0 & \widetilde{G}_{\mu\nu} & \widetilde{G}_{\nu\nu} \end{pmatrix} \tag{245}$$

$$\mathrm{Det\,Hess}\, G = \frac{\mu}{\frac{\mu^2}{4}-\tau^2}\, \mathrm{Det\,Hess}\, \widetilde{G}\,. \tag{246}$$

One easily calculates

$$G_{33}^{-1} \equiv ((\mathrm{Hess}\, G)^{-1})_{33} = \frac{\widetilde{G}_{\mu\mu}}{\mathrm{Det\,Hess}\, \widetilde{G}}\,. \tag{247}$$

Since G_{33}^{-1} is independent of τ, $\mathrm{Hess}\, G_{33}^{-1}$ is degenerate and has a null-vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Let us check that this is the only null-vector of $\mathrm{Hess}\, G_{33}^{-1}$. To this end, we need to show that

$$\mathrm{Det}\,\underset{\mu,\nu}{\mathrm{Hess}}\,(G_{33}^{-1}) \not\equiv 0\,. \tag{248}$$

Since in the case of the orthotoric metric everything is expressed in terms of the (x, y) variables, we will be using the following formulas describing the change of variables, valid for an arbitrary function $F(\mu, \nu)$:

$$\begin{aligned} F_{\mu\mu} &= \frac{F_{xx}+F_{yy}}{(x-y)^2} - \frac{2}{(x-y)^2}\left(F_{xy} + \frac{F_x - F_y}{x-y}\right), \\ F_{\nu\nu} &= \frac{x^2\, F_{xx}+y^2\, F_{yy}}{(x-y)^2} - \frac{2\,x\,y}{(x-y)^2}\left(F_{xy} + \frac{F_x - F_y}{x-y}\right) \\ F_{\mu\nu} &= -\frac{x\, F_{xx}+y\, F_{yy}}{(x-y)^2} + \frac{x+y}{(x-y)^2}\left(F_{xy} + \frac{F_x - F_y}{x-y}\right), \\ \mathrm{Det\,Hess}\, F &= \frac{1}{(x-y)^2}\left(F_{xx}F_{yy} - \left(F_{xy} + \frac{F_x - F_y}{x-y}\right)^2\right)\,. \end{aligned} \tag{249}$$

We write out the derivatives of the orthotoric symplectic potential G_{o}:

$$(G_{\mathrm{o}})_{xx} = \frac{3x(x-y)}{P(x)}, \quad (G_{\mathrm{o}})_{yy} = \frac{3y(y-x)}{Q(y)}, \quad (G_{\mathrm{o}})_{xy} + \frac{(G_{\mathrm{o}})_x - (G_{\mathrm{o}})_y}{x-y} = 0\,, \tag{250}$$

where $P(x) = \prod_{i=1}^{3}(x - x_i) = x^3 - \frac{3}{2}x^2 + c$, $Q(y) = \prod_{i=1}^{3}(y - y_i) = y^3 - \frac{3}{2}y^2 + d$.

Setting in (247) $\widetilde{G} = G_{\mathrm{o}}$ and using the above formulas, we obtain

$$G_{33}^{-1} = \frac{1}{3\,(x-y)}\left(\frac{P(x)}{x} - \frac{Q(y)}{y}\right)\,. \tag{251}$$

Substituting in (249) $F = G_{33}^{-1}$, one finds explicitly

$$\begin{aligned} &\mathrm{Det}\,\underset{\mu,\nu}{\mathrm{Hess}}\, G_{33}^{-1} = \\ &= \frac{4(c-d)\left(y^3\left(c\,(x+y)\left(10x^2-5xy+y^2\right)+3x^3(x-y)^5\right) - d\,x^3(x+y)\left(x^2-5xy+10y^2\right)\right)}{3x^3y^3(x-y)^{10}} \not\equiv 0 \end{aligned} \tag{252}$$

To summarize, we have proven that the only null-vector of Hess G^{-1}_{33} is $\begin{pmatrix}1\\0\\0\end{pmatrix}$. For equation (242) to have a non-zero solution $\widehat{\omega}$, this vector would have to be a null-vector of *all* matrices Hess G^{-1}_{ij}, which would imply $\partial_\tau^2 G^{-1}_{ij} = 0$ for all i, j. We compute, however,

$$G^{-1}_{11} = \frac{1}{\mu}\left(\frac{\mu^2}{4} - \tau^2\right) + \frac{\tau^2}{\mu^2}\,\frac{\widetilde{G}_{\nu\nu}}{\mathrm{Det\,Hess}\,\widetilde{G}}\,. \tag{253}$$

Therefore

$$\partial_\tau^2 G^{-1}_{11} = \frac{2}{\mu^2}\left(\frac{\widetilde{G}_{\nu\nu}}{\mathrm{Det\,Hess}\,\widetilde{G}} - \mu\right) \not\equiv 0 \tag{254}$$

meaning that the only solution of (242) is $\widehat{\omega} = 0 = \omega$. ■

8. Summary

In the present paper we analyzed the space of Ricci-flat metrics on the non-compact manifold Y – the total space of the canonical bundle over the del Pezzo surface of rank one. This surface is the blow-up of the projective plane $\mathbb{CP}^2$ at one point. As we explained, a version of the Calabi-Yau for the space Y requires that the metric contain two real parametes, which are the Kähler moduli representing the sizes of the original $\mathbb{CP}^2$ and the blown-up sphere $\mathbb{CP}^1$. The only explicitly known (so-called 'orthotoric') metric, however, has just one parameter (the overall scale). In the paper we have related this to the fact that the orthotoric metric admits a conformal Killing-Yano form. We have shown that, although the metric allows a first-order deformation, which preserves the Ricci-flatness (as it should, by the Calabi-Yau theorem), the deformed metric will no longer admit a conformal Killing-Yano tensor.

Acknowledgements. I would like to thank Dmitri Ageev for a collaboration at an initial stage of this project and for many useful conversations. I am grateful to Sergey Frolov, Ulrich Menne, Osvaldo Santillan, Armen Sergeev, Stefan Theisen, Konstantin Zarembo for discussions. I would like to thank the Institut des Hautes Études Scientifiques and in particular Vasily Pestun for hospitality during my stay, during which a part of this work was done. I am indebted to Prof. A.A.Slavnov and to my parents for support and encouragement. My work was supported by the ERC Advanced Grant No. 320045 "Strings and Gravity" (Principal Investigator Prof. D. Lüst).

Appendices

A. Derivation of the metric (59)

Here we will derive the formula (59) for the metric, starting from the $U(2)\times U(1)$-invariant Kähler potential (51): $K = K(|z_1|^2 + |z_2|^2, |u|^2)$. Denoting by K_0 the Kähler potential of the sphere $\mathbb{CP}^1$, $K_0 = \log(|z_1|^2 + |z_2|^2)$, and introducing the real variables t, s via $e^{\frac{t}{2}} = |z_1|^2 + |z_2|^2$ and $e^s = |u|^2$, we obtain the following formulas:

$$\frac{\partial^2 K}{\partial z_i \partial \bar{z}_j} = 4\frac{\partial^2 K}{\partial t^2}\partial_i K_0 \bar{\partial}_j K_0 + 2\frac{\partial K}{\partial t}\partial_i \bar{\partial}_j K_0, \tag{255}$$

$$\frac{\partial^2 K}{\partial u \partial \bar{u}} = \frac{1}{|u|^2}\frac{\partial^2 K}{\partial s^2}, \qquad \frac{\partial^2 K}{\partial z_i \partial \bar{u}} = 2\frac{\partial^2 K}{\partial t \partial s}\frac{1}{\bar{u}}\partial_i K_0\,. \tag{256}$$

Taking into account that $\frac{\partial K}{\partial t} = \mu$ and $2\,\partial_i \bar{\partial}_j K_0\, dz_i d\bar{z}_j = g_{\mathbb{CP}^1}$, we obtain the following expression for the line element:

$$ds^2 = \mu\, g_{\mathbb{CP}^1} + 4\tfrac{\partial^2 K}{\partial t^2}\partial_i K_0 \bar{\partial}_j K_0\, dz_i d\bar{z}_j + \tfrac{1}{|u|^2}\tfrac{\partial^2 K}{\partial s^2} du d\bar{u} + 2\tfrac{\partial^2 K}{\partial t \partial s}\tfrac{1}{\bar{u}}\partial_i K_0\, dz_i d\bar{u} + 2\tfrac{\partial^2 K}{\partial t \partial s}\tfrac{1}{u}\bar{\partial}_i K_0\, du d\bar{z}_j \tag{257}$$

We now introduce the following combinations:

$$\mathcal{A} = \frac{i}{2}\left(\bar{\partial}_i K_0\, d\bar{z}_i - \partial_i K_0\, dz_i\right) \tag{258}$$

$$dt = 2\left(\partial_i K_0\, dz_i + \bar{\partial}_i K_0\, d\bar{z}_i\right) \tag{259}$$

Therefore

$$\bar{\partial}_i K_0\, d\bar{z}_i = \frac{dt}{4} - i\,\mathcal{A}, \qquad \partial_i K_0\, dz_i = \frac{dt}{4} + i\,\mathcal{A} \tag{260}$$

We also parametrize the variable u as follows:

$$u = e^{\frac{s}{2} - i\phi} \tag{261}$$

Substituting these expressions into (257), we obtain

$$\begin{aligned} ds^2 &= \mu\, g_{\mathbb{CP}^1} + \frac{\partial^2 K}{\partial t^2}\left(\frac{dt^2}{4} + 4\mathcal{A}^2\right) + \frac{\partial^2 K}{\partial s^2}\left(\frac{ds^2}{4} + d\phi^2\right) + \frac{\partial^2 K}{\partial t \partial s}\left(\frac{ds\, dt}{2} - 4\, d\phi\, \mathcal{A}\right) = \\ &= \mu\, g_{\mathbb{CP}^1} + \frac{1}{4}\frac{\partial^2 K}{\partial t_i \partial t_j} dt_i dt_j + \frac{\partial^2 K}{\partial t_i \partial t_j}\mathcal{A}_i \mathcal{A}_j, \end{aligned} \tag{262}$$

where $(t_1, t_2) = (t, s)$ and $\mathcal{A}_1 = -2\mathcal{A}, \mathcal{A}_2 = d\phi$. Let us choose the following parametrization for (z_1, z_2): $(z_1, z_2) = \rho\, e^{\frac{-i\tilde{\phi}}{2}}\,(1, w)$. Then the current $\mathcal{A}$ can be rewritten as

$$\mathcal{A} = \frac{i}{2}\frac{z_i d\bar{z}_i - \bar{z}_i dz_i}{|z_1|^2 + |z_2|^2} = -\frac{d\tilde{\phi}}{2} + \frac{i}{2}\frac{w d\bar{w} - \bar{w} dw}{1 + |w|^2} \tag{263}$$

Taking into account that $\frac{\partial^2 K}{\partial t_i \partial t_j} dt_i dt_j = \frac{\partial^2 G}{\partial \mu_i \partial \mu_j} d\mu_i d\mu_j$, we may rewrite (262) as

$$ds^2 = \mu\, g_{\mathbb{CP}^1} + \frac{1}{4}\frac{\partial^2 G}{\partial \mu_i \partial \mu_j} d\mu_i d\mu_j + \left(\frac{\partial^2 G}{\partial \mu^2}\right)^{-1}_{ij} (d\phi_i - 2A_i)(d\phi_j - 2A_j), \tag{264}$$

where $(\phi_1, \phi_2) = (\tilde{\phi}, \phi)$, $A_1 = \frac{i}{2}\frac{w d\bar{w} - \bar{w} dw}{1 + |w|^2}$, $A_2 = 0$. We have thus arrived at the desired result, formula (59). Note that, in these notations, $g_{\mathbb{CP}^1} = \frac{2\, dw d\bar{w}}{(1 + |w|)^2}$.

B. Vector fields generating the $\mathfrak{u}(2) \oplus \mathfrak{u}(1)$ action

To see what restrictions the enhanced $U(2) \times U(1)$ symmetry imposes on the toric metric (4), let us start from the holomorphic Killing vector fields generating the toric subgroup[11], i.e. $v_1 := w_1 \frac{\partial}{\partial w_1}, v_2 := w_2 \frac{\partial}{\partial w_2}, v_3 := w_3 \frac{\partial}{\partial w_3}$. Moreover, let us assume that v_1 is the generator of the Cartan subgroup $\mathfrak{u}(1) \subset \mathfrak{su}(2)$ (one can always make a change of variables to make sure this is fulfilled). We wish to construct the remaining generators $L_\pm$ of $\mathfrak{su}(2)$, defined by the following properties:

$$[v_1, L_\pm] = \pm L_\pm, \quad [L_+, L_-] = 2\, v_1, \quad [L_\pm, v_2] = [L_\pm, v_3] = 0\,. \tag{265}$$

The first of these commutation relations leads to the following form of $L_\pm$:

$$L_+ = w_1^2 \frac{\partial}{\partial w_1} + w_1 \left(a\, w_2 \frac{\partial}{\partial w_2} + b\, w_3 \frac{\partial}{\partial w_3} \right), \tag{266}$$

$$L_- = -\frac{\partial}{\partial w_1} - \frac{1}{w_1} \left(c\, w_2 \frac{\partial}{\partial w_2} + d\, w_3 \frac{\partial}{\partial w_3} \right). \tag{267}$$

A priori a, b, c, d are functions of w_2, w_3, however the last two commutation relations in (265) imply that they are constants. It follows from the remaining commutation relation that $c = -a, d = -b$. There are two distinct possibilities:

A) $a = b = 0$, in which case $L_+ = w_1^2 \frac{\partial}{\partial w_1}$, $L_- = -\frac{\partial}{\partial w_1}$. The orbit of the $SU(2)$ action is given by familiar fractional-linear transformations, $w_1 \to \frac{a w_1 + b}{c w_1 + d}$.

B) $a \neq 0$ or $b \neq 0$. A linear change of $(\log w_2, \log w_3)$-variables brings the vector fields to the form $L_+ = w_1^2 \frac{\partial}{\partial w_1} + w_1 w_2 \frac{\partial}{\partial w_2}$, $L_- = -\frac{\partial}{\partial w_1} + \frac{w_2}{w_1} \frac{\partial}{\partial w_2}$. Changing variables again according to $w_2' = \frac{1}{(w_1 w_2)^{1/2}}, w_1' = w_1$, we arrive at the canonical form of the generators:

$$L_- = -\frac{\partial}{\partial w_1'}, \quad L_+ = w_1'^2 \frac{\partial}{\partial w_1'} - w_1' w_2' \frac{\partial}{\partial w_2'}, \quad v_1 = w_1' \frac{\partial}{\partial w_1'} - \frac{1}{2} w_2' \frac{\partial}{\partial w_2'} \tag{268}$$

The orbit is $\left\{ w_1' \to \frac{a w_1' + b}{c w_1' + d},\ w_2' \to (c w_1' + d)\, w_2 \right\}$.

In order to build a Kähler metric with the corresponding isometries, one can construct a Kähler potential $K(w)$, which, under the transformations, is shifted as $K \to K + f(w) + \overline{f(w)}$. The only such possibility in case A is to have a potential of the form $K = \log(1 + |w_1|^2) + \tilde{K}(|w_2|^2, |w_3|^2)$, but this implies that the manifold is a product of a sphere and some complex surface. In case B, however, the most general choice is $K = K(|w_2|^2(1 + |w_1|^2), |w_3|^2)$, which coincides with (51) after an obvious change of variables.

C. Determining the physical roots ξ_1, ξ_2 of $Q(\xi) = 0$

We showed in § 4.2 that the normal bundles of the spheres embedded in the cone require that

$$-\frac{\xi_2}{1 - \xi_2} = \frac{3\xi_1}{1 - \xi_1}, \tag{269}$$

[11] Representations of various Lie algebras through vector fields in 3 variables were thoroughly studied in [34]. The discussion presented here is sufficient for us, due to the fact that we have two additional $\mathfrak{u}(1)$ actions on top of the $\mathfrak{su}(2)$.

where ξ_1 and ξ_2 are *both* roots of the polynomial $Q(\xi)$. This means that

$$\xi_1 + \xi_2 + \xi_0 = \tfrac{3}{2}, \tag{270}$$

$$\xi_1\xi_2 + \xi_1\xi_0 + \xi_2\xi_0 = 0, \tag{271}$$

$$\xi_2 = \tfrac{3\xi_1}{4\xi_1 - 1} \tag{272}$$

Eliminating the variables ξ_0 and ξ_2 we arrive at a cubic equation for ξ_1, which, however, factorizes:

$$(\xi_1 - 1)(16\xi_1^2 - 4\xi_1 - 3) = 0 \tag{273}$$

As we mentioned in § 4.2, the case $\xi_1 = 1$ corresponds to the case when the physical region shrinks to zero (i.e. the planes $1, 2$ in Fig. 2 merge), so we assume that $\xi_1 \neq 1$. Then we have the two solutions:

$$\xi_1^{(1)} = \frac{1}{8}(1 + \sqrt{13}), \quad \xi_2^{(1)} = \frac{1}{8}(7 + \sqrt{13}) \tag{274}$$

$$\xi_1^{(2)} = \frac{1}{8}(1 - \sqrt{13}), \quad \xi_2^{(2)} = \frac{1}{8}(7 - \sqrt{13}) \tag{275}$$

Since $P_0'' = -\frac{3\xi}{Q(\xi)}$, in order for the metric at infinity (79) to be positive-definite, we ought to determine in which of these segments $(\xi_1^{(i)}, \xi_2^{(i)})$ the function $\frac{\xi}{Q(\xi)}$ is negative (in the whole segment). An elementary check shows that this is so only for the first segment, $(\xi_1^{(1)}, \xi_2^{(1)})$. This leads to the following value of d:

$$d = \frac{16 + \sqrt{13}}{64}\,. \tag{276}$$

D. The space of polynomials $y^3 - \frac{3}{2}y^2 + d$

In most calculations one encounters the roots ξ_i of the polynomials of the form

$$Q(y) = y^3 - \frac{3}{2}y^2 + d \tag{277}$$

These can be written out explicitly in terms of Cardano's formula, however this expression is rather complicated. A better approach is to use a *rational* parametrization for the space of polynomials of the form (277). Indeed, denoting the roots of such a polynomial by ξ_0, ξ_1, ξ_2 (as we did in the body of the paper), polynomials of the type (277) are defined by the following relations:

$$\xi_0 + \xi_1 + \xi_2 = \frac{3}{2}, \qquad \xi_0\xi_1 + \xi_0\xi_2 + \xi_1\xi_2 = 0 \tag{278}$$

Reparametrizing the roots as $\xi_1 = \lambda_1\xi_0$, $\xi_2 = \lambda_2\xi_0$, we arrive at a simple equation $(\lambda_1 + 1)(\lambda_2 + 1) = 1$, which can be 'solved' as follows: $\lambda_1 + 1 = u$, $\lambda_2 + 1 = \frac{1}{u}$, where u is a new variable. In terms of this variable the roots are parametrized as

$$\xi_0 = \frac{3}{2}\,\frac{1}{u + \frac{1}{u} - 1}, \quad \xi_1 = \frac{3}{2}\,\frac{u - 1}{u + \frac{1}{u} - 1}, \quad \xi_2 = \frac{3}{2}\,\frac{\frac{1}{u} - 1}{u + \frac{1}{u} - 1}, \tag{279}$$

whereas the parameter d of the polynomial $Q(y)$ is expressed as

$$d = \frac{27}{8}\,\frac{(u-1)^2}{u}\,\frac{1}{\left(u + \frac{1}{u} - 1\right)^3} \tag{280}$$

E. The general three-line solution

In sections 4.1, 4.2 we studied the simplest solution of the Monge-Ampere equation, which is the metric cone over a Sasakian manifold. The structure of the solution (83) hints at the possibility of using the following more general ansatz:

$$G = \sum_{i=0}^{2} \ell_i \left(\log|\ell_i| - 1\right) - (\log\kappa)\,\nu, \qquad\qquad \kappa > 0\,, \tag{281}$$

where $\ell_i = 0, i = 0, 1, 2$ are three a priori arbitrary lines in the (μ, ν)-plane:

$$\ell_i = a_i\,\mu + b_i\,\nu + c_i \tag{282}$$

and κ is a constant. Asymptotically, when $\mu, \nu \to \infty$ with $\xi = \frac{\mu}{\nu}$ fixed,

$$G = \nu\log(\nu)\left(\sum_{i=0}^{2} b_i + \xi\sum_{i=0}^{2} a_i\right) + \nu\left(\sum_{i=0}^{2}(b_i + a_i\xi)\,\left(\log|b_i + a_i\xi| - 1\right) - \log\kappa\right) + \ldots \tag{283}$$

Compatibility with the conical asymptotics (77), (78) requires

$$\sum_{i=0}^{2} a_i = 0, \qquad \sum_{i=0}^{2} b_i = 3\,. \tag{284}$$

Substituting the ansatz (281) in the Ricci-flatness condition (58) (with $\tilde{a} = 1$), one arrives at the following system of equations for the parameters of the ansatz:

$$a_i + b_i = 1 \quad \text{(already encountered in Lemma 2)} \tag{285}$$

$$(a_1 - a_2)^2\,a_3 + (a_1 - a_3)^2\,a_2 + (a_2 - a_3)^2\,a_1 = \pm\kappa \tag{286}$$

$$(a_1 - a_2)^2\,b_3 + (a_1 - a_3)^2\,b_2 + (a_2 - a_3)^2\,b_1 = 0 \tag{287}$$

$$(a_1 - a_2)^2\,c_3 + (a_1 - a_3)^2\,c_2 + (a_2 - a_3)^2\,c_1 = 0\,. \tag{288}$$

The sign $\pm$ in the r.h.s. of (286) is defined by $\pm = \mathrm{sgn}(\ell_0\,\ell_1\,\ell_2)$. Using the equation $\sum_{i=0}^{2} a_i = 0$ from (284) (the second equation in (284) now being a consequence of the first one and (285)), eq. (286) can be rewritten as follows:

$$(a_1 + a_2 + a_3)(a_1a_2 + a_1a_3 + a_2a_3) - 9\,a_1a_2a_3 = \pm\kappa \quad\Rightarrow\quad a_1a_2a_3 = \mp\frac{\kappa}{9} \tag{289}$$

Eq. (287) can be brought to the following form, using $b_i = 1 - a_i$:

$$\pm\kappa = (a_1 - a_2)^2 + (a_1 - a_3)^2 + (a_2 - a_3)^2 = 2(a_1 + a_2 + a_3)^2 - 6(a_1a_2 + a_1a_3 + a_2a_3)$$
$$\Rightarrow\quad a_1a_2 + a_1a_3 + a_2a_3 = \mp\frac{\kappa}{6} \tag{290}$$

Since $\kappa > 0$, we see that the first equation has real solutions, only if one chooses the sign $+$ in the l.h.s. This implies

$$\ell_0\,\ell_1\,\ell_2 > 0\,. \tag{291}$$

It follows from (284), (289), (290) that a_1, a_2, a_3 are roots of the equation

$$a^3 - \frac{\kappa}{6} a + \frac{\kappa}{9} = 0 \tag{292}$$

Upon a change of variables $a = \frac{1}{1-\xi}$, we obtain

$$Q(\xi) := \xi^3 - \frac{3}{2}\xi^2 + d = 0, \quad \text{where} \quad d = \frac{1}{2} - \frac{9}{\kappa}\,. \tag{293}$$

We therefore obtain the following parametrization for the constants a_i, b_i, c_i:

$$a_i = \frac{1}{1-\xi_i}, \qquad b_i = -\frac{\xi_i}{1-\xi_i}, \qquad c_i = \frac{\sigma_1 \xi_i^2 + \sigma_2 \xi_i}{1-\xi_i}, \tag{294}$$

where $\sigma_{1,2}$ are arbitrary constants, and ξ_i are the solutions of the equation $Q(\xi) = 0$.

If $\sigma_1 = 0$, the solution differs from (83) by a trivial shift of $\nu \to \nu + \sigma_2$. The interesting case is $\sigma_1 \neq 0$ – in this situation we can as well shift ν to set $\sigma_2 = 0$, arriving at the solution

$$G_{3L} = \sum_{i=0}^{2} \frac{\mu - \xi_i\, \nu + \sigma_1 \xi_i^2}{1-\xi_i} \left(\log|\mu - \xi_i\, \nu + \sigma_1 \xi_i^2| - 1\right) \tag{295}$$

This is a one-parametric generalization of (83). For the case of the manifold $Y^{2,1}$, taking into account that $a_0 > 0, a_1 > 0, a_2 < 0$, the analogue of our former requirement $\xi \in (\xi_1, \xi_2)$ is $\ell_0 > 0, \ell_1 > 0, \ell_2 > 0$, which is compatible with (291).

Since the 'radial' part of the metric $\left[ds^2\right]_\mu := \frac{\partial^2 G_{3L}}{\partial\mu_i \partial\mu_j} d\mu_i d\mu_j$ is two-dimensional, one may introduce isothermal coordinates to simplify it. One can check that the orthotoric coordinates (x, y) (see § 6.3) serve this purpose. Indeed, if one makes the change of variables

$$\mu = \sigma_1 x y, \nu = \sigma_1 (x + y)\,, \tag{296}$$

the radial part of the metric acquires the form

$$\frac{\partial^2 G_{3L}}{\partial\mu_i \partial\mu_j} d\mu_i d\mu_j = \frac{3x(x-y)\sigma_1}{Q(x)}\, dx^2 + \frac{3y(x-y)\sigma_1}{Q(y)}\, dy^2\,. \tag{297}$$

This is a special case of the orthotoric metric, which arises if one makes the polynomials $P(x), Q(y)$ in (199) identical, i.e. if one equates the parameters $c = d$.

F. The variational problem

Interestingly, the Monge-Ampere equation (58) may be obtained from a variational principle. In fact, although the Ricci-flatness equation $R_{ij} = 0$ can be obtained through the extremization of the Einstein-Hilbert functional $\mathcal{S}_{EH} = \int d^n x \sqrt{g}\, R = \int d^n x\, \mathcal{L}$, this is no longer true if one restricts to the class of Kähler manifolds. It turns out that in this case the Lagrangian $\mathcal{L}$ is a total derivative:

$$\mathcal{L}\,(g \;\; \text{Kähler}) = \sqrt{g}\, R = -\det g_H \cdot g_H^{i\bar{j}}\, \partial_i \bar{\partial}_j \log\det g_H = -\bar{\partial}_j (g_H^{i\bar{j}} \partial_i \det g_H) \tag{298}$$

Here g_H is the Hermitian metric associated with the real metric g. One concludes that the action only depends on the values of the (derivatives of the) Kähler potential at the boundary and does not give rise to any equation in the bulk.

In order to obtain an equation of the type (58) one should consider the following action:

$$\mathcal{S} = \int d\mu\, d\nu\, f(\mu,\nu)\, G(\mu,\nu) + \int ds\, dt\, g(s,t)\, K(s,t), \tag{299}$$

where the variables (μ,ν) and (s,t), as well as the functions $G(\mu,\nu), K(s,t)$, are Legendre dual to each other, just as in Section 3. Indeed, passing to a single set of variables, say (μ,ν), we obtain:

$$\mathcal{S} = \int d\mu\, d\nu\, \left(f(\mu,\nu)\, G(\mu,\nu) + \left(G_{\mu\mu}G_{\nu\nu} - G_{\mu\nu}^2\right) g(G_\mu, G_\nu)(\mu\, G_\mu + \nu\, G_\nu - G)\right) \tag{300}$$

Variation of this action with respect to G produces the following equation:

$$G_{\mu\mu}G_{\nu\nu} - G_{\mu\nu}^2 = \frac{f(\mu,\nu)}{g(G_\mu, G_\nu)} \tag{301}$$

The equation (58) is a particular case, when $f = \tilde{a}\,\mu$ and $g = e^{G_\mu + G_\nu}$.

Remark. The variational problem above may be related to one of optimal transport theory [35]. In the latter setup the relevant problem is to maximize the functional

$$\mathcal{S} = \int d\mu\, d\nu\, f(\mu,\nu)\, \tilde{G}(\mu,\nu) + \int ds\, dt\, g(s,t)\, \tilde{K}(s,t), \tag{302}$$

with $f > 0, g > 0$, subject to the condition

$$\tilde{G}(\mu,\nu) + \tilde{K}(s,t) \leqslant (\mu - s)^2 + (\nu - t)^2 \tag{303}$$

Changing variables to

$$\tilde{G}(\mu,\nu) = \mu^2 + \nu^2 - G(\mu,\nu), \quad \tilde{K}(s,t) = s^2 + t^2 - K(s,t), \tag{304}$$

one is to minimize

$$\tilde{\mathcal{S}}[G,K] = \int d\mu\, d\nu\, f(\mu,\nu)\, G(\mu,\nu) + \int ds\, dt\, g(s,t)\, K(s,t) \tag{305}$$

subject to

$$G(\mu,\nu) + K(s,t) \geqslant \mu s + \nu t \tag{306}$$

It clearly follows that $G(\mu,\nu) \geqslant \max\limits_{(s,t)}(\mu s + \nu t - K(s,t)) := K^\vee(\mu,\nu)$ and $K(s,t) \geqslant \max\limits_{(\mu,\nu)}(\mu s + \nu t - G(\mu,\nu)) := G^\vee(s,t)$. Therefore $\tilde{\mathcal{S}}[G,K] \geqslant \tilde{\mathcal{S}}[G, K = G^\vee]$, hence in the minimizing configuration $K = G^\vee$, meaning that K and G are Legendre dual.

G. Killing-Yano forms: the definition

First of all, a Killing-Yano form is a 2-form ω_{jk} on $\mathcal{M}$ satisfying the equation $\nabla_i\omega_{jk}+\nabla_j\omega_{ik}=0$. By definition, a conformal Killing-Yano form (CKYF) is a 2-form ω_{jk} on $\mathcal{M}$ satisfying an equation of the form

$$\mathscr{D}\omega=0, \tag{307}$$

where $\mathscr{D}\omega$ is a 3-tensor, which is a linear combination of covariant derivatives $\nabla_i\omega_{jk}$, symmetric w.r.t. the first pair of indices and fully traceless. Being symmetric w.r.t. $i\leftrightarrow j$, we can write it as follows:

$$(\mathscr{D}\omega)_{ijk}=\nabla_i\omega_{jk}+\nabla_j\omega_{ik}+a\,g_{ij}\,g^{mn}\nabla_m\omega_{nk}+b\,(g_{ik}\,g^{mn}\nabla_m\omega_{nj}+g_{jk}g^{mn}\nabla_m\omega_{ni}) \tag{308}$$

Requiring this 3-tensor to be completely traceless, i.e. $(\mathscr{D}\omega)^i{}_{ik}=(\mathscr{D}\omega)^i{}_{ji}=0$, we get

$$b+1+\frac{aD}{2}=0,\qquad -1+a+b\,(D+1)=0\quad\Rightarrow\quad b=\frac{1}{D-1},\quad a=-\frac{2}{D-1}\,. \tag{309}$$

Therefore the CKYF condition takes the form

$$(\mathscr{D}\omega)_{ijk}=\nabla_i\omega_{jk}+\nabla_j\omega_{ik}+\frac{1}{D-1}\,(g_{ik}\,g^{mn}\nabla_m\omega_{nj}+g_{jk}g^{mn}\nabla_m\omega_{ni}-2\,g_{ij}\,g^{mn}\nabla_m\omega_{nk})=0 \tag{310}$$

We can give an equivalent definition by requiring that $(D\omega)_{ijk}$ is skew-symmetric w.r.t. the interchange $j\leftrightarrow k$, i.e.

$$(\widetilde{\mathscr{D}}\omega)_{ijk}:=\frac{1}{3}((\mathscr{D}\omega)_{ijk}-(\mathscr{D}\omega)_{ikj})= \tag{311}$$

$$=\nabla_i\omega_{jk}-\frac{1}{3}T_{ijk}+\frac{1}{D-1}\,(g_{ik}\,g^{mn}\nabla_m\omega_{nj}-g_{ij}g^{mn}\nabla_m\omega_{nk})=0 \tag{312}$$

$$\text{where}\qquad T_{ijk}=\nabla_i\omega_{jk}+\nabla_k\omega_{ij}-\nabla_j\omega_{ik} \tag{313}$$

The two conditions (310), (312) are equivalent. The tensor T here, which is anti-symmetric in all pairs of indices, is proportional to the exterior derivative of ω, i.e. $T\propto d\omega$.

H. Non-holomorphic Killing vector fields on Calabi-Yau twofolds

In Proposition 6 we showed that on a Calabi-Yau threefold without parallel vector fields every Killing vector is holomorphic. In complex dimension two, i.e. for a Calabi-Yau 2-fold, the situation is different. In that case we have a parallel holomorphic 2-form Ω_{ij}, and the dualization of (160) gives

$$\nabla_\mu\,h=0,\qquad\text{where}\qquad h=\widetilde{\Omega}^{ij}\,F_{ij}\,. \tag{314}$$

In particular, in this case h is a function, and the above equation implies

$$h=h_0=\text{const.} \tag{315}$$

One can see how this scenario is realized in practice. The relevant example is the Taub-NUT space (see [36] for a detailed discussion of the Kähler structure of this space), which has the metric

$$ds^2 = V\,(dx^2 + dy^2 + dz^2) + V^{-1}\,(dt + A)^2\,, \qquad V = a + \frac{1}{r}, \qquad dA = *\,dV\,, \qquad a > 0\,. \tag{316}$$

One can define an integrable complex structure as a map

$$\mathcal{J}: \qquad dx \to V^{-1}\,(dt + A), \qquad dy \to dz\,. \tag{317}$$

Two Killing vectors $\frac{\partial}{\partial t}$ and $y\,\frac{\partial}{\partial z} - z\,\frac{\partial}{\partial y}$, generating translations along t and rotations in the (y, z)-plane, are holomorphic. On the other hand, the metric (316) has isometry $SO(3) \times U(1)$, where $U(1)$ is the group of translations along the periodic direction t, and $SO(3)$ is generated by rotations in the (x, y, z)-space[12].

The Killing vectors that lie in $\mathfrak{so}(3)\backslash\mathfrak{u}(1)$ – the complement to the subgroup $\mathfrak{u}(1)$ of rotations in the (y, z) plane, are *not* holomorphic. Their action may be characterized by using the fact that the Taub-NUT is a hyper-Kähler manifold, with three symplectic forms $\varpi = \varpi_1, \varpi_2, \varpi_3$, each of which is Hermitian w.r.t. its own complex structure, I, J or K. The vector fields generating the $\mathfrak{so}(3)$ rotate the three Kähler forms, i.e.

$$\mathfrak{L}_v\,(\varpi_1, \varpi_2, \varpi_3) = \; a_v \circ (\varpi_1, \varpi_2, \varpi_3) \qquad a_v \in \mathfrak{so}(3)\,. \tag{318}$$

For the form $\varpi = \varpi_1$ this implies

$$\mathfrak{L}_v\,\varpi = \alpha\,\varpi_2 + \beta\,\varpi_3\,, \qquad \alpha, \beta = \text{const.} \tag{319}$$

Here we assume that v is non-holomorphic, in which case α and β are not simultaneously zero. On the other hand, it is known that on a hyper-Kähler manifold, the complex two-form $\Omega := \varpi_2 + i\,\varpi_3$ is of type $(2, 0)$ w.r.t. the complex structure I – the one, in which ϖ is Hermitian. The formula above may now be recast in the form

$$\mathfrak{L}_v\,\varpi = \frac{\alpha - i\,\beta}{2}\,\Omega + \text{c.c.} \tag{320}$$

Comparing with (156), we find

$$F \sim \Omega\,, \tag{321}$$

with a constant proportionality factor. Finally, it is easily seen that Ω is the Calabi-Yau two-form, as $\Omega \wedge \bar{\Omega} \sim$ vol. To check this, one might recall that $\omega_1, \omega_2, \omega_3$ are the three Kähler forms for the same metric, therefore $\omega_1 \wedge \omega_1 = \omega_2 \wedge \omega_2 = \omega_3 \wedge \omega_3 = \frac{1}{2}\,\Omega \wedge \bar{\Omega}$. As a result,

$$\widetilde{\Omega}^{ij}\,F_{ij} = \text{const.}\,, \tag{322}$$

as required by (314).

[12]The gauge field A is in general not invariant under such transformations, but rather shifts by $A \to A + d\Phi$, where Φ is a function of (x, y, z). Therefore we also need to appropriately compensate by shifts of the t-variable, $t \to t - \Phi$.)

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