

Moduli Anomalies and Local Terms in the Operator Product Expansion

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ABSTRACT

Local terms in the Operator Product Expansion in Superconformal Theories with extended supersymmetry are identified. Assuming a factorized structure for these terms their contributions are discussed.

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1 Introduction

Conformal Field Theories (CFT) are specified by the list of primary operators with their dimensions and the structure constants, i.e. the normalizations of the correlators of three primary operators. Generically from this data any correlator in the theory can be reconstructed.

In the present paper we will discuss additional input data which is needed for certain classes of primaries. This additional data can in principle be recovered from the symmetries of the CFT. The possible need for additional data is suggested by the presence of “semilocal terms” in certain correlators. These terms were discussed recently from various points of view [1, 2, 3]. We will call “semilocal” in a generic sense those terms which do not have the full analytic structure in momentum space expected in a correlator with singularities in all kinematical invariants. The semilocal terms have a singularity (logarithmic or powerlike) in one kinematical variable while the dependence on the other invariants is polynomial. We will distinguish these terms from the ordinary local terms which are polynomial in all invariants and which appear in specific correlators. In position space locality means that all operator insertion points coincide while in semi-local correlators only some of them do. The generic situation which shows the full analytic structure in momentum space is when they are all different.

We will use alternatively the operatorial or the covariant (i.e. based on the analytic structure of the correlators) language and will verify their compatibility in each case. We start by defining the local terms in the Operator Product Expansion (OPE) and necessary conditions for their appearance.

In a CFT the information about the structure constants can be reconstructed from the operator product expansion which we will use in the following: given two primaries $\mathcal{O}_1, \mathcal{O}_2$ with scaling dimensions d_1 and d_2 , their OPE is

$$\mathcal{O}_1(x) \mathcal{O}_2(y) = c_{123} ((x-y)^2)^{\frac{1}{2}(d_3-d_1-d_2)} \mathcal{O}_3(y) + \dots \quad (1.1)$$

Generically this is singular for $(x-y)^2 \sim 0$. In momentum space the singularity produces an analytic function of the kinematical invariant p^2 with a branch point at $p^2 = 0$, which in general has both an imaginary and a real part. A special situation arises when the singular function has the form $\partial^n \delta^d(x-y)$. Then, in momentum space, we have a purely real (polynomial) dependence on p^2 , and in configuration space the singular function is “local”. For such a situation to occur, the following conditions have to be satisfied:

- a) the dimensions should fulfill $d_1 + d_2 - d_3 = \text{integer} \geq d$ (the space-time dimension), following from the above form of the local singular function;
- b) the ordinary structure constant/OPE coefficient accompanying the singular function $((x-y)^2)^{-\frac{1}{2}(d+2n)}$ which, in even dimensions, corresponds in momentum space to $(p^2)^n \log \frac{p^2}{\Lambda^2}$, must vanish. If this were not the case its contribution to the three point function would be the most singular and would “mask” the semilocal structure which contains just the polynomial.

While the momentum dependence involving $\mathcal{O}_1, \mathcal{O}_2$ will be polynomial, the singular behaviour in the correlators is obtained by singularities of \mathcal{O}_3 with other operators.

We remark a basic difference between the ordinary terms in the OPE and the local ones. For the ordinary OPE terms coming from a unique three point function there is a relation between the three possible orderings. In contrast to this, the local terms in the OPE for the three possible orderings are independent: they produce the three operator correlators through convolutions with different two point functions.

A basic issue which will be important in our discussion is whether the local term in the OPE can be removed and its effect in the various correlators absorbed in a redefinition in the ordinary framework of the CFT. Alternatively this reduces to the question whether the local term can be formulated in a universal regularization independent way.

To illustrate this setup and the issue of universality we review the well known case of the Zamolodchikov metric on the conformal manifold in $d = 2$. This was studied a long time ago in [4]. We will denote moduli, which are exactly marginal operators, by M_i . They have dimension d , which we assume to be even, and their structure constants vanish by their requirement of being moduli.

In the present paper global properties of the moduli space do not play an essential role and we will not discuss them. We will limit the range of moduli couplings to the vicinity of a fiducial CFT where the moduli space does not have singularities. In this range, where the conformal deformation of the CFT by adding the moduli multiplied by

couplings is well defined, we will treat the couplings as particular x -independent values of the sources. They are local coordinates on the moduli space which is also referred to as the conformal manifold. If the space-time dependent sources are denoted by $J^i(x)$, the conformal deformation is obtained by adding

$$\delta S = \frac{1}{\pi^{d/2}} \int d^d x \sqrt{g} J^i(x) M_i(x) \quad (1.2)$$

to the action of the fiducial CFT.

The conditions a) and b) are satisfied for the moduli and we can search for the local contributions in their OPE. In our approach these are a consequence of the type B trace anomaly in the CFT. To be specific, consider the situation in $d = 2$ [5]: if the x -dependent sources of the moduli are denoted by J^i , the anomaly is

$$\mathcal{A} = -\frac{1}{4\pi} \int d^2 x \sqrt{g} \sigma G_{ij}(J) \partial_\mu J^i \partial^\mu J^j \quad (1.3)$$

$G_{ij}(J)$ is the Zamolodchikov metric. The universal information contained in (1.3), like in every type B anomaly [6, 7], is the logarithmically divergent counterterm

$$\frac{1}{8\pi} \log(\Lambda^2) \int d^2 x \sqrt{g} G_{ij}(J) \partial_\mu J^i \partial^\mu J^j \quad (1.4)$$

which contains semilocal correlators of the moduli, the singularity in momentum space being inherited from the two point function.

Taking three functional derivatives with respect to J and Fourier transforming, we obtain

$$\langle M_i(p_1) M_j(p_2) M_k(p_3) \rangle = \frac{\pi^2}{4} \log(\Lambda^2) (p_3^2 \Gamma_{ij,k} + \text{cyclic permutations}) \quad (1.5)$$

Here Γ is the Christoffel connection for the Zamolodchikov metric, evaluated at the point in moduli space we are studying. Since the two point function at that same point in moduli space is

$$\langle M_i(p) M_j(-p) \rangle = \frac{\pi}{4} G_{ij} p^2 \log p^2 / \Lambda^2 \quad (1.6)$$

the logarithmically divergent contribution in the three point function (1.5) will be reproduced by a local term in the OPE:

$$M_i(p_1) M_j(p_2) = \frac{\pi}{4} \Gamma_{ij}^k M_k(p_1 + p_2) \quad (1.7)$$

as proposed in [4]. In configuration space this corresponds to

$$M_i(x) M_j(y) = \frac{\pi}{4} \delta^2(x - y) \Gamma_{ij}^k M_k(x) \quad (1.8)$$

In the previous argument we used universal features of the logarithmically divergent counterterm. This will be part of our approach, i.e. we will always start with the semilocal

term in the correlator which has all the symmetries and analyticity properties of the theory and derive from it the local contribution to the OPE needed to reproduce it. In particular the transformation properties under source reparametrizations reflect the covariance of the Zamolodchikov metric defined by the counterterm.

We now discuss the issue of the universality of the local term in the OPE above. The key is the behaviour of this term under source reparametrization, which is a symmetry of the theory. The term found is normalized by the Christoffel connection of the Zamolodchikov metric. It transforms inhomogeneously under reparametrizations of the sources which suggests that it is not universal. Indeed, using Gaussian normal coordinates at a given point in moduli space, the connection can be put to zero. This shows that the semilocal term which it represents can be obtained from the ordinary set up of the theory without the need of local terms in the OPE. Moreover in [8] an explicit procedure in a special regularization is given which shows how one could recover the reparametrization invariant information contained in the Zamolodchikov metric from correlators calculated without using additional local terms in the OPE. An example where some local contributions to the OPE can be removed while others not was discussed recently in [9].

The situation in $d = 4$ is similar: we start from the anomaly [5]

$$\mathcal{A} = \frac{1}{192\pi^2} \int d^4x \sqrt{g} \sigma \left(G_{ij}(J) \hat{\square} J^i \hat{\square} J^j - 2G_{ij}(J) \partial_\mu J^i \left(R^{\mu\nu} - \frac{1}{3} g^{\mu\nu} R \right) \partial_\nu J^j \right) \quad (1.9)$$

where $\hat{\square} J^i = \square J^i + \Gamma_{jk}^i \nabla^\mu J^j \nabla_\mu J^k$. This reflects the counterterm (in flat space-time) proportional to

$$\log(\Lambda^2) \int d^4x G_{ij}(J) \hat{\square} J^i \hat{\square} J^j \quad (1.10)$$

and one gets again (1.5) through (1.8), with the only difference (apart from the numerical factor) that p^2 is replaced by $(p^2)^2$ and one has a four-dimensional δ -function in (1.8). The generalization of these equations to arbitrary even dimensions is obvious¹. Again, the apparent addition to the OPE specified above is removable and therefore not universal.

In this paper we will apply the same logic to identify new local contributions to the OPE involving currents and moduli. However, the terms which we single out cannot be removed and are therefore universal. This will be a consequence of supersymmetry, i.e. we will study $\mathcal{N} = (2, 2)$ superconformal theories in $d = 2$ and $\mathcal{N} = 2$ SCFTs in $d = 4$. The common feature we will find is that the local terms in the OPE will be normalized by the Zamolodchikov metric itself and therefore cannot be removed by source reparametrization.

Once the additional terms in the OPE are identified we study whether factorization can be used for the local terms in a manner analogous to the decomposition in terms of the ordinary structure constants. The local terms generate classes of contributions to certain correlators of the theory in addition to the usual one. We check various requirements for these contributions, in particular their consistency with supersymmetry. While for $(2, 2)$ theories in $d = 2$ we have a complete and consistent construction, in $d = 4$ we have

¹but not so the generalization of (1.9) which can, however, be worked out case by case.

to face an ambiguity in the separation of certain correlators into ordinary contributions (the result of combining the usual three point structures) and the ones produced through factorization from the local terms.

The paper is organized as follows: in Section 2 we will review the structure of moduli anomalies for $\mathcal{N} = 2$ superconformal theories in four dimensions and identify the logarithmically divergent counterterm involving currents. We identify a contribution to the trace anomaly originating in the correlator of a $U(1)_R$ current and moduli and applying the logic outlined above we identify the local term in the OPE of two moduli giving the current.

In Section 3 we discuss $\mathcal{N} = (2, 2)$ superconformal theories in $d = 2$. In the general expression for the superconformal anomalies we identify a contribution to the $U(1)$ anomaly and we determine the local contribution in the OPE needed to reproduce it. Using special features of $d = 2$ we construct the full anomalous part of the effective action which incorporates terms obtained through factorization from the local additions to the OPE.

In Section 4, where we present our conclusions, we discuss the possibility of using the local terms in the OPE in a factorized manner, the consequences of such an assumption and the consistency checks needed.

In Appendix A we give a proof for the absence of ordinary structure constants of moduli and conserved currents. In Appendix B we work out explicitly the local contributions for the simplest example of an $\mathcal{N} = 2$ theory in $d = 4$: the free Maxwell gauge supermultiplet. We identify in a Feynman diagram calculation the contribution of redundant operators which leads to the semilocal structures we find. This suggests that in a diagrammatic calculation the local terms in the OPE can be replaced by redundant (i.e. vanishing on shell) operators. We check the consistency of these factorized contributions by calculating the anomalous four moduli correlator consistent with supersymmetry in this model as a sum of factorized and ordinary contributions. In Appendix C we work out a similar field theoretic model realizing the structures we found for the $\mathcal{N} = (2, 2)$ theories in two dimensions.

2 $\mathcal{N} = 2$ in $d = 4$

In this section we discuss four-dimensional $\mathcal{N} = 2$ superconformal theories with moduli. Superconformal theories in $d = 2$ with $(2, 2)$ supersymmetry, which have several special, simplifying features, will be discussed in the next section.

Conformal field theories with $\mathcal{N} = 2$ supersymmetry have an $SU(2) \times U(1)$ \mathcal{R} -symmetry, of which the $U(1)$ factor is anomalous. The basic result of this section is that this anomalous $U(1)$ \mathcal{R} -current appears as a local term in the $M \overline{M}$ operator product. This follows from the structure of a counterterm related to a type B Weyl-anomaly. This counterterm is required by supersymmetry. In Appendix B we will verify some of our general results and claims by looking at pure $\mathcal{N} = 2$ $U(1)$ gauge theory where all features appear at one

loop order and can be explicitly computed.

We gauge the global symmetries and couple the CFT to an external metric and gauge fields, the sources for the energy momentum tensor and for the \mathcal{R} -symmetry currents, respectively. And, of course, we also have the sources for the moduli. The anomalous Ward identities are then most succinctly incorporated in the effective action, which is the non-local functional of the sources obtained by integrating out the CFT. It necessarily violates some of the symmetries and the anomaly \mathcal{A} is the variation of the generating functional under these transformations. Here the anomalous symmetries are super-Weyl transformations, which are parametrized by a chiral superfield Σ , whose lowest component $\Sigma| = \sigma + i\alpha$ parametrizes Weyl (σ) and $U(1)_{\mathcal{R}}$ transformations (α).

The super-Weyl anomaly is therefore the variation of the effective action with the chiral superfield parameter Σ [5]

$$\begin{aligned} \mathcal{A} = \frac{1}{16\pi^2} & \left(\int d^4x d^4\theta \mathcal{E} \Sigma (a \Xi + (c-a) W^{\alpha\beta} W_{\alpha\beta}) + \text{c.c.} \right) \\ & + \frac{1}{192\pi^2} \int d^4x d^4\theta d^4\bar{\theta} E (\Sigma + \bar{\Sigma}) K(J, \bar{J}) \end{aligned} \quad (2.1)$$

c and a are the Weyl anomaly coefficients which are characteristic of a particular SCFT. The normalization of the last term is fixed by the two-point function of the moduli, which is given by the Zamolodchikov metric for which K is the Kähler potential. The component expansion of the above expression is [10, 5]²

$$\begin{aligned} \mathcal{A} = \frac{1}{16\pi^2} \int d^4x \sqrt{g} & \left\{ -a \sigma \left(E_4 - \frac{2}{3} \square R \right) + c \sigma C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} - 2c \sigma F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} c \sigma \text{tr}(H^{\mu\nu} H_{\mu\nu}) \right. \\ & + (a-c) \alpha R^{\mu\nu\rho\sigma} \tilde{R}_{\mu\nu\rho\sigma} + 2(c-a) \alpha F_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{1}{2} (2a-c) \alpha \text{tr}(H_{\mu\nu} \tilde{H}^{\mu\nu}) \\ & \left. + 4a \nabla^\mu A_\mu \square \alpha - 8a A^\mu \left(R_{\mu\nu} - \frac{1}{3} R g_{\mu\nu} \right) \nabla^\nu \alpha - 8a F_{\mu\nu} A^\mu \nabla^\nu \sigma \right\} \\ & + \frac{1}{96\pi^2} \int d^4x \sqrt{g} \left\{ \sigma \mathcal{R}_{i\bar{k}j\bar{l}} \nabla^\mu J^i \nabla_\mu J^j \nabla^\nu \bar{J}^k \nabla_\nu \bar{J}^l + \sigma G_{i\bar{j}} \left(\hat{\square} J^i \hat{\square} \bar{J}^j - 2 \left(R^{\mu\nu} - \frac{1}{3} R g^{\mu\nu} \right) \partial_\mu J^i \partial_\nu \bar{J}^j \right) \right. \\ & + \frac{1}{2} K \square^2 \sigma + \frac{1}{6} K \partial^\mu R \partial_\mu \sigma + K \left(R^{\mu\nu} - \frac{1}{3} R g^{\mu\nu} \right) \nabla_\mu \nabla_\nu \sigma - 2 G_{i\bar{j}} \nabla^\mu J^i \nabla^\nu \bar{J}^j \nabla_\mu \nabla_\nu \sigma \\ & + i G_{i\bar{j}} \left(\hat{\nabla}^\mu \hat{\nabla}^\nu J^i \nabla_\nu \bar{J}^j - \hat{\nabla}^\mu \hat{\nabla}^\nu \bar{J}^j \nabla_\nu J^i \right) \partial_\mu \alpha - \nabla^\mu A_\mu \square \alpha + 2 \mathcal{A}^\mu \left(R_{\mu\nu} - \frac{1}{3} R g_{\mu\nu} \right) \partial^\nu \alpha \\ & \left. - \sigma F_{\mu\nu} \mathcal{F}^{\mu\nu} + 2 F_{\mu\nu} \mathcal{A}^\mu \nabla^\nu \sigma + F_{\mu\nu} \nabla^\mu K \nabla^\nu \alpha \right\} \end{aligned} \quad (2.2)$$

Here \mathcal{A}_μ is the Kähler connection, defined as

$$\mathcal{A}_\mu = \frac{i}{2} \left(\partial_j K \partial_\mu J^j - \partial_{\bar{j}} K \partial_\mu \bar{J}^{\bar{j}} \right) \quad (2.3)$$

²In [5] the $U(1)_{\mathcal{R}}$ and $SU(2)_{\mathcal{R}}$ gauge fields were set to zero.

and $\mathcal{F}_{\mu\nu}$ its field strength which depends only on the Kähler metric and is therefore invariant under Kähler shifts and covariant under holomorphic coordinate changes on the conformal manifold. F is the field strength of the $U(1)$ gauge field A and H that of the $SU(2)$ gauge field. Were it not for supersymmetry, many terms in the component expression would be cohomologically trivial and could be dropped, but as it is obvious from the (three irreducible) superspace expressions, supersymmetry demands that they accompany the cohomologically non-trivial terms. Supersymmetry is also responsible for the appearance of the target space Riemann tensor in the fourth line. In a bosonic theory it could be replaced by an arbitrary tensor with the correct symmetries and would still be a non-trivial solution to the Wess-Zumino consistency condition. But $\mathcal{N} = 2$ supersymmetry requires that this tensor is the Riemann tensor. In the general form this anomaly first appeared in [11] and we therefore refer to it as the Osborn anomaly.

One can write down a non-local action, both in superspace and in components, whose super-Weyl variation reproduces (4.3), but one is faced with the same problem as for the ordinary Weyl anomaly in four dimensions that this ‘‘Riegert’’ action does not have the correct analyticity properties [12]. It therefore differs from the unknown ‘true’ effective action by unknown non-local Weyl invariant terms.

The anomaly polynomial is invariant under a combined (field dependent) super-Weyl transformation and a Kähler transformation if their parameters are related as [5]

$$\tilde{\Sigma} = \frac{1}{24a} F \tag{2.4}$$

This is easy to verify for the superspace action (4.3) and can also be verified for the component expression (2.2). It is readily observed that every term in the first three lines with a bare gauge field A_μ , i.e. not appearing in the gauge invariant combination $F_{\mu\nu}$, has a counterpart in the last two lines if we replace $A_\mu \rightarrow -\frac{1}{24a} \mathcal{A}$. This reflects the invariance of $A_\mu - \frac{1}{24a} \mathcal{A}$ under a joint gauge and Kähler transformation with (2.4). In the ‘‘Conclusions’’ section we will reformulate this symmetry in terms of the Kähler shift variation of the effective action.

Let us now analyse the anomaly polynomial (2.2) further. Consider the first two terms in the last line. The second one vanishes for $\sigma = \text{const.}$ and therefore will not contribute to the following argument. The first term is a type B Weyl-anomaly and corresponds to a counterterm, in the same way as was described in the Introduction. As such it contains information about non-local terms in correlation functions. Taking functional derivatives with respect to J^i , $\bar{J}^{\bar{j}}$ and A_μ , one finds the correlator

$$\langle M_i(k_1) \bar{M}_{\bar{j}}(k_2) j_\mu(-k_1 - k_2) \rangle = -\frac{\pi^2}{192} G_{i\bar{j}}(q^2 r_\mu - q \cdot r q_\mu) \log \Lambda^2 \tag{2.5}$$

where $G_{i\bar{j}}$ is evaluated for constant sources. We have defined $q = k_1 + k_2$ and $r = k_1 - k_2$. Of course the same counterterm also generates correlation functions of one current and an arbitrary number of moduli, always via the current-current two-point function. The term cannot come from an ordinary three point function since, as we show in Appendix A, the

moduli being neutral under $U(1)_R$ the structure constant vanishes. This indicates that the $U(1)$ \mathcal{R} -current j_μ appears in a contact term in the $M\overline{M}$ operator product. Since it is proportional to the Zamolodchikov metric it cannot be removed by a reparametrization of the conformal manifold. The fact that the Zamolodchikov metric appears is a consequence of supersymmetry. If it were not for supersymmetry, the counterterm which is responsible for this correlator could be omitted and one could adopt a scheme where there are no local terms in the operator product of two moduli.

If we normalize the $U(1)_R$ current such that the coupling in the microscopic theory is normalized to $\int A_\mu j^\mu$, we find the local terms in the $M\overline{M}$ OPE

$$M_i(x)\overline{M}_j(y) \sim \frac{\pi^4}{48c} G_{ij} \left(\partial_\mu^{(x)} \delta^4(x-y) j^\mu(y) - \partial_\mu^{(y)} \delta^4(x-y) j^\mu(y) \right) + \dots \quad (2.6)$$

There could be other local terms in this operator product, but they do not contribute to the three point function with the \mathcal{R} -symmetry current. We will give their specific form for the particular case of the free Maxwell theory in Appendix B.

Once we formulated the local term in operatorial language we can translate it into a covariant one: the local term in the OPE will give a contribution to any correlator involving moduli by coupling the moduli to the $U(1)_R$ current. The correlators of \mathcal{R} -currents are represented by terms in the effective action containing its source A_μ . Therefore the contribution of the local term in the OPE to correlators with moduli is obtained by replacing A_μ in any term in the generating functional by $\frac{1}{24c} \mathcal{A}_\mu$. This is the general formulation of factorization we are using. The normalization follows from comparing the last term in the last line with the third term in the first line of (2.2).

One might wonder to what extent factorization determines the form of the anomaly polynomial. An explicit calculation in $\mathcal{N} = 2$ super-Maxwell theory shows that the counterterm in (2.2) which involves the Riemann tensor on the conformal manifold (the ‘‘Osborn anomaly’’), is not completely accounted for by factorization. The same calculation however shows that without the local term in the above operator product the Riemann tensor would not appear, but it is required by $\mathcal{N} = 2$ supersymmetry.

More generally if the anomaly polynomial were given completely by factorization all the terms would contain the combination $A_\mu + \frac{1}{24c} \mathcal{A}_\mu$. This is clearly not the case for (2.2) which contains invariant field strengths of A_μ without the corresponding terms constructed from \mathcal{A}_μ . This seems to be dictated by supersymmetry, because there is no way to supersymmetrize e.g. $\alpha \mathcal{F}_{\mu\nu} \tilde{\mathcal{F}}^{\mu\nu}$.³ We will elaborate on this point in the concluding section, but already draw the partial conclusion that while the factorized contributions of the moduli are needed, the typical situation is that they come together with ordinary contributions obtained ignoring the local terms. An interesting connection appears in the explicit example discussed in Appendix B: the local terms in the OPE can be replaced

³At least as long as we use only chiral multiplets to represent the sources J^i . This might be reminiscent of the situation in $d = 2$ where the coupling of chiral multiplets to a target space B -field cannot be accomplished off-shell. To do this one has to use semi-chiral multiplets. We did not pursue the generalization of this possibility to $d = 4$.

by including in the moduli “redundant operators” in the covariant calculation of the correlators. Here “redundant” means operators in a lagrangian CFT which vanish if one uses the equations of motion.

3 $\mathcal{N} = (2, 2)$ in $d = 2$

We begin with a review of the basic features of moduli anomalies in $\mathcal{N} = (2, 2)$ superconformal theories [5]. Extended supersymmetry implies additional global symmetries which in this case are the $U(1)_A \times U(1)_V$ \mathcal{R} -symmetries. We can choose to preserve either one of the two $U(1)$ factors. The second factor then belongs to the multiplet of anomalous currents. Due to this choice we have two possible types of theories.

For concreteness, we will only discuss the theory which preserves the $U(1)_A$ \mathcal{R} -symmetry. In this case the anomaly is⁴

$$\mathcal{A} = -\frac{1}{2\pi} \int d^2x \left(\sigma G_{i\bar{j}} \partial_\mu J^i \partial^\mu \bar{J}^{\bar{j}} - \frac{1}{2} \square \sigma K + \alpha \partial^\mu \mathcal{A}_\mu + \frac{c}{12} (\sigma R + \alpha \epsilon^{\mu\nu} F_{\mu\nu}) \right) \quad (3.1)$$

In this expression, whose superspace version will be given later, K is the Kähler potential on the conformal manifold, a real function of the sources J and \bar{J} and

$$\mathcal{A}_\mu = \frac{i}{2} \left(\partial_j K \partial_\mu J^j - \partial_{\bar{j}} K \partial_\mu \bar{J}^{\bar{j}} \right) \quad (3.2)$$

is again the Kähler connection. $F_{\mu\nu}$ is the field strength of the $U(1)_A$ gauge field A_μ . Under local $U(1)_V$ transformations it transforms as $\delta A_\mu = \epsilon_\mu{}^\nu \partial_\nu \alpha$. If we define $V_\mu = \epsilon_\mu{}^\nu A_\nu$, $\delta V_\mu = \partial_\mu \alpha$ and $\epsilon^{\mu\nu} F_{\mu\nu} = 2 \nabla^\mu V_\mu$. σ parametrizes local Weyl transformations and c is the Virasoro central charge⁵. The relative coefficients in (3.1) are dictated by supersymmetry. The invariance of the effective action (nonlocal and local terms) under the axial ($U(1)_A$) gauge transformation $\delta A_\mu = \partial_\mu \beta$ is part of the definition of the theory. As a consequence the terms $\alpha \partial^\mu \mathcal{A}_\mu$ and $\alpha \epsilon^{\mu\nu} F_{\mu\nu}$, which can be obtained by the gauge variation of $A_\mu \mathcal{A}^\mu$ and $A_\mu A^\mu$ respectively, remain cohomologically nontrivial since the addition of these local terms to the effective action would violate the $U(1)_A$ symmetry, i.e. the definition of the theory. Note also a very special feature of $d = 2$: the chiral anomaly can be seen not only in odd parity correlators like in all even dimensions but also as a “clash” between conservation in the even parity vector-vector and axial-axial correlators when the vector and axial currents are related by a duality transformation.

We now analyse the $\alpha \partial^\mu \mathcal{A}_\mu$ term of the anomaly polynomial (3.1). It represents an anomaly in the correlator of the $U(1)_V$ current and at least one modulus and one antimodulus or, equivalently, a non-invariance of the corresponding terms in the effective action under a vector gauge transformation of the gauge field A_μ . The momentum space

⁴For simplicity we only consider chiral primary moduli. For the general case, which includes also twisted chiral primary moduli, we refer to [5], where further details of the notation can also be found.

⁵Diffeomorphism invariance requires $c = c_L = c_R$, i.e. absence of a gravitational anomaly.

structure of the term which leads to this anomaly is $\frac{q_\mu q_\nu}{q^2}$, where q is the momentum carried by the axial current to which A_μ is coupled. By an argument similar to the one in $d = 4$ which is discussed in Appendix A, such a contribution cannot come from a modulus-antimodulus-current coupling since the moduli are neutral under the R-symmetries. In this case the semilocal structure involves not a logarithm like in $d = 4$ but the characteristic $\frac{1}{q^2}$ pole multiplied by a polynomial in the momenta of the moduli. In order to reproduce it we need to assume the existence of a local term in the OPE

$$M_1(x) \overline{M}_2(y) \sim \frac{6\pi^2}{c} G_{i\bar{j}} (\partial_\mu^{(x)} \delta^2(x-y) \tilde{j}^\mu(y) - \partial_\mu^{(y)} \delta^2(x-y) \tilde{j}^\mu(x)) \quad (3.3)$$

where \tilde{j}_μ is the anomalous vector current. We could use instead equivalently the non-anomalous axial current j_μ related to \tilde{j}_μ by a duality transformation. Combining then the local term in the OPE with the correlator of two vector currents:

$$\langle \tilde{j}_\mu(q) \tilde{j}_\nu(-q) \rangle = \epsilon_\mu^\rho \epsilon_\nu^\sigma \langle j_\rho(q) j_\sigma(-q) \rangle = -\frac{c}{6} \frac{q_\mu q_\nu}{q^2} \quad (3.4)$$

we obtain

$$\langle M_1(k_1) \overline{M}_2(k_2) \tilde{j}_\mu(-k_1 - k_2) \rangle = -\frac{i}{16\pi^2} G_{i\bar{j}} \frac{q \cdot r}{q^2} q_\mu \quad (3.5)$$

As in $d = 4$ this term in the OPE cannot be removed by reparametrizations of the sources and therefore it is universal. It leads through factorization to classes of calculable contributions to the effective action. These factorized contributions can be calculated following the rule analogous to the one used in $d = 4$ i.e. wherever the gauge field A_μ is coupled to the axial current we should replace it by the combination $A_\mu + \frac{6}{c} \epsilon_{\mu\nu} \mathcal{A}^\nu$. This combination is manifest in the anomaly polynomial (3.1) and comparing the terms proportional to α it is clear that the anomaly involving the moduli is reproduced.

The combination which appears is invariant under a joint transformation of the Kähler potential which generates \mathcal{A}_μ by $f(J) + \bar{f}(\bar{J})$ and a vector gauge transformation of A_μ with parameter $\alpha = -\frac{3i}{c}(f - \bar{f})$. The consistency between the combinations selected by factorization and the invariance of the anomaly polynomial is a special feature of the two dimensional theory.

Since the local terms in the OPE are factorized we treat this effective coupling on equal terms with A_μ , i.e. for every term in the effective action involving A_μ we can get a term involving correlators of moduli by the above replacement. As discussed in Section 2 in $d = 4$ the typical situation is that both factorized local OPE contributions and ordinary ones are needed to reproduce the total supersymmetric expressions.

In $d = 2$ due the special kinematical features we are able to give a complete description of the anomalous part of the effective action and to check including only the factorized contributions we get an answer consistent with supersymmetry and the Kähler structure.

We proceed now to construct the anomalous part of the effective action. We start with the first building block involving the Zamolodchikov anomaly, i.e. eq.(1.3). Generically a type-B trace anomaly is induced by a logarithmically divergent term, but the anomaly

itself appears in the Weyl variation of a finite correlator which involves the sources in the divergent term and the metric which is coupled to the energy momentum tensor [6, 7]. For a general type B anomaly there are no closed expressions for the finite part to all orders in the external sources. Even for the standard c -trace anomaly in $d = 4$ the finite, non local correlator, whose Weyl variation is the anomaly, is known only in the leading order i.e. a correlator of three energy momentum tensors [12]. For the case considered here, i.e. the finite part reproducing the Zamolodchikov anomaly in $d = 2$ containing any number of moduli and external metric perturbations, the problem can be exactly solved. Let us start with the first correlator which contributes to the finite part: a correlator of two moduli and one energy momentum. In a convenient basis the correlator has the kinematical decomposition

$$\begin{aligned} \langle M_i(k_1)M_j(k_2)T_{\mu\nu}(-q) \rangle &= A(q^2, k_1^2, k_2^2)(\eta_{\mu\nu} q^2 - q_\mu q_\nu) \\ &\quad + B(q^2, k_1^2, k_2^2)(-\eta_{\mu\nu} q^2 + 2q_\mu q_\nu) + C(q^2, k_1^2, k_2^2)(-\eta_{\mu\nu} r^2 + 2r_\mu r_\nu) \end{aligned} \quad (3.6)$$

where $q^\mu \equiv k_1^\mu + k_2^\mu$ and $r^\mu \equiv k_1^\mu - k_2^\mu$. The conservation and trace Ward identities completely determine A, B, C . The diffeomorphism Ward Identity (conservation of the energy-momentum tensor) determines the B and C amplitudes in terms of the two-point function of the moduli

$$Q(k^2) \equiv \langle M_i(k)M_j(-k) \rangle \quad (3.7)$$

evaluated at k_1^2 and k_2^2 respectively, while the Weyl transformation (trace of energy momentum) Ward identity determines the A amplitude in terms of the trace of the energy momentum tensor, i.e. the anomaly which we denote by \mathcal{B} :

$$A(q^2, k_1^2, k_2^2) = -\frac{\mathcal{B}}{q^2} \quad (3.8)$$

Therefore the B, C part of the decomposition obeys the diffeomorphism Ward identity and it is traceless as seen from the explicit decomposition. That part contains through the two point correlator of moduli the logarithmic divergence. It follows that due to the very special kinematical features of $d = 2$, the three point function of two moduli and one energy momentum tensor splits into a non-anomalous part and a completely explicit anomalous part represented by the A amplitude. We remark the $\frac{1}{q^2}$ structure in the anomalous part which is surprising, since a priori one would expect singularities combining the three kinematical invariants. Once this lowest correlator is understood it is easy to write the result for any number of energy momentum tensors and moduli by simply making the result covariant in space-time and using covariance under source reparametrizations for the moduli. The result for the anomalous part of the effective action which has the correct Zamolodchikov trace anomaly is:

$$W_a = \frac{1}{4\pi} \int d^2x \sqrt{g} \mathcal{G} \frac{1}{\square} R \quad (3.9)$$

where we have defined

$$\mathcal{G} = G_{i\bar{j}} \partial^\mu J^i \partial_\mu \bar{J}^{\bar{j}} \quad (3.10)$$

This can be combined in a single expression with the Polyakov trace anomaly since they have the same structure, i.e. $1/\square$.

The supersymmetrization is now straightforward by replacing the scalar curvature in the Polyakov anomaly and the Zamolodchikov anomaly with their superspace generalizations in a single linear combination. The relative normalization is fixed by the linear combination selected through factorization for the gauge components since the superspace curvature contains the gauge field A while the superspace Zamolodchikov anomaly contains the Kähler $U(1)$ field \mathcal{A} . Then the anomalous part of the effective action in superspace is

$$W = -\frac{c}{48\pi} \int d^2x \int d^4\theta E \left(\bar{R} - \frac{6}{c} \nabla^2 K \right) \frac{1}{\square} \left(R - \frac{6}{c} \bar{\nabla}^2 K \right) \quad (3.11)$$

whose super-Weyl variation is

$$\mathcal{A} = \int d^2x d^2\theta \mathcal{E} \Sigma \left(-\frac{c}{24\pi} R + \frac{1}{4\pi} \bar{\nabla}^2 K \right) + \text{c.c.} \quad (3.12)$$

Its component expansion is (3.2). That (3.12) follows from (3.11) can be checked using

$$\begin{aligned} \delta R &= -\Sigma R + 2 \bar{\nabla}^2 \bar{\Sigma} \\ \delta \nabla^2 &= -\bar{\Sigma} \nabla^2, \quad \square = \bar{\nabla}^2 \nabla^2 \end{aligned} \quad (3.13)$$

Invariance under a joint Kähler shift $K \rightarrow K + f + \bar{f}$ and super-Weyl transformations with $\Sigma = \frac{6}{c} f$ is also manifest. Here Σ is a chiral superfield which parametrizes super-Weyl transformations; its lowest component is $\Sigma| = \sigma + i\alpha$ and R is the curvature chiral superfield, whose top component contains the Ricci scalar (also denoted by R) and the $U(1)_V$ field strength F . The sources J^i are chiral superfields and K is a real function of the sources, the Kähler potential on the conformal manifold. For further details on the geometry of $\mathcal{N} = (2, 2)$ supergravity we refer to [13]. The anomalous effective action in super-conformal gauge was given in [5]. The symmetry under a joint Kähler shift and a correlated Weyl transformation, which acts on the anomaly polynomial, is promoted here to a symmetry of the anomalous part of the effective action. We remark that the anomalous part as we defined it through factorization, contains a local Weyl invariant piece $\sim \int d^2x \int d^4\theta E K^2$. To this we should add the fully Weyl invariant nonlocal contribution. There is an additional freedom we have since the Weyl invariance is anomalous: one is allowed to add local Weyl noninvariant functionals of the external sources respecting all the other symmetries:

$$\int d^2x d^2\theta \mathcal{E} H(J) R + \text{c.c.} + \int d^2x d^4\theta E I(J, \bar{J}) \quad (3.14)$$

It is instructive to have the anomalous effective action also in components:

$$\begin{aligned} W = \int d^2x \sqrt{g} \left(\frac{1}{4\pi} \mathcal{G} \frac{1}{\square} R - \frac{1}{4\pi} F \frac{1}{\square} \nabla^\mu \mathcal{A}_\mu - \frac{c}{96\pi} \left(R \frac{1}{\square} R + F \frac{1}{\square} F \right) - \frac{1}{8\pi} K R \right. \\ \left. + \frac{3}{2\pi c} \left(\mathcal{G} K - \mathcal{G} \frac{1}{\square} \mathcal{G} - \frac{1}{4} K \square K - \nabla^\mu \mathcal{A}_\mu \frac{1}{\square} \nabla^\nu \mathcal{A}_\nu \right) \right) \end{aligned} \quad (3.15)$$

where we have defined $F = \epsilon^{\mu\nu} F_{\mu\nu}$. Note that it contains the gauge field and the Kähler connection only in the combination $\mathbb{A}_\mu = \mathcal{A}_\mu + \frac{c}{6}\epsilon_\mu{}^\nu A_\nu$.

The last term in (3.15) represents correlators of moduli which are only induced through the factorization assumption. The local term in the OPE which through factorization produced the \mathcal{A} dependent terms above was defined in terms of $G_{i\bar{j}}$, but after translating it to the covariant formalism we ended up with an explicit dependence on \mathcal{A}_μ . Since the field strength \mathcal{F} corresponding to \mathcal{A}_μ , which is the pull-back of the Kähler form, contains only $G_{i\bar{j}}$, it is clear that in order to recover the original information we should impose a gauge invariance of \mathcal{A} . Such a gauge invariance is induced by a Kähler shift $K \rightarrow K + f(J) + \bar{f}(\bar{J})$ i.e.

$$\delta\mathcal{A}_\mu = \frac{i}{2}\partial_\mu(f - \bar{f}) \quad (3.16)$$

We are therefore led to study the behaviour of the anomalous part of the effective action under a Kähler shift. Since the effective action is by construction invariant under a joint transformation by a Kähler shift and a Weyl transformation with $\Sigma = \frac{6}{c}f$, the transformation under a Kähler shift can easily be calculated, simply replacing in the anomaly calculation Σ by f :

$$\sim \int d^2x d^2\theta \mathcal{E} f(J) R \quad (3.17)$$

The result of the Kähler shift can be absorbed in a change in the local Weyl noninvariant term by $H(J) \rightarrow H(J) + f(J)$

In summary in this class of theories through factorization the local terms in the OPE produce contributions to the effective action consistent with $(2, 2)$ supersymmetry, but the Weyl anomalous part of the effective action is not invariant under a Kähler shift, its variation being local.

4 Conclusions

In [5] the behaviour of the anomaly polynomial under a Kähler shift was studied. In this section, for the discussion of the implications of factorization, we find it convenient to discuss the behaviour under a Kähler shift of the effective action itself. In this way we are able to isolate universal features of the terms generated by factorization which are not invariant under Kähler shift.

We will consider terms in the effective action which depend on the moduli through a Kähler potential K . When the $\mathcal{N} = 2$ theory is the result of a compactification from a six dimensional theory on a Riemann surface and K has an ab initio geometric meaning [15, 14], this is the case for the full effective action. For a generic $\mathcal{N} = 2$ theory K is defined by the moduli trace anomalies and therefore we are really discussing only the

Weyl anomalous part of the effective action. One expects a ‘‘Kähler shift invariance’’ for the transformation

$$K(J, \bar{J}) \rightarrow K(J, \bar{J}) + f(J) + \bar{f}(\bar{J}) \quad (4.1)$$

This transformation induces on the pulled back universal $U(1)$ Kähler form a gauge transformation

$$\mathcal{A}_\mu \rightarrow \mathcal{A}_\mu + \frac{i}{2} \partial_\mu (f - \bar{f}) \quad (4.2)$$

This Kähler shift can give a nonvanishing result. We will treat it as an anomaly with the understanding that it originates just in those terms in the effective action which depend on K . Then like for any other anomaly one should look for nontrivial solutions of the appropriate cohomological problem. A partial solution is given by

$$\mathcal{A}_f = \frac{1}{16\pi^2} \left(\int d^4x d^4\theta \mathcal{E} f(a' \Xi + b' W^{\alpha\beta} W_{\alpha\beta}) + \text{c.c.} \right) \quad (4.3)$$

or, in components:

$$\begin{aligned} \mathcal{A}_f = \frac{1}{32\pi^2} \int d^4x \sqrt{g} \left\{ -a' (f + \bar{f}) \left(E_4 - \frac{2}{3} \square R \right) + (a' + b') (f + \bar{f}) C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} \right. \\ - 2(a' + b') (f + \bar{f}) F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} (a' + b') (f + \bar{f}) \text{tr}(H^{\mu\nu} H_{\mu\nu}) - 2i b' (f - \bar{f}) F_{\mu\nu} \tilde{F}^{\mu\nu} \\ + i b' (f - \bar{f}) R^{\mu\nu\rho\sigma} \tilde{R}_{\mu\nu\rho\sigma} - \frac{i}{2} (a' - b') (f - \bar{f}) \text{tr}(H_{\mu\nu} \tilde{H}^{\mu\nu}) - 4i a' \nabla^\mu A_\mu \square (f - \bar{f}) \\ \left. + 8i a' A^\mu \left(R_{\mu\nu} - \frac{1}{3} R g_{\mu\nu} \right) \nabla^\nu (f - \bar{f}) - 8a' F_{\mu\nu} A^\mu \nabla^\nu (f + \bar{f}) \right\} \quad (4.4) \end{aligned}$$

A third candidate

$$\int d^4x d^8\theta E(f + \bar{f}) K = \frac{1}{2} \delta_f \int d^4x d^8\theta E K^2 \quad (4.5)$$

is omitted, being cohomologically trivial in superspace. It is an open question if there are possible additional terms in the anomaly equation in which some of the dependence on the $\mathcal{N} = 2$ supergravity multiplet fields is replaced by a dependence on some fields derived from K itself. We will discuss this aspect in more detail below.

The Kähler shift anomaly has the special feature that there are counterterms present with the same structure as (4.3) with arbitrary chiral coefficient functions and the anomaly shifts these coefficients. These terms, though local, are chiral and therefore cannot cancel the Kähler shift produced starting from K and therefore this feature does not change the way we treat the anomaly.

The Kähler shift anomaly and Weyl anomalies have Wess-Zumino type consistency conditions which follow from the commutativity of the two transformations:

$$\delta_f \delta_\Sigma W = \delta_\Sigma \delta_f W \quad (4.6)$$

This equation fixes a' in terms of the moduli contribution to the Weyl anomaly,

$$a' = -\frac{1}{24} \tag{4.7}$$

This condition is equivalent to the invariance of the anomaly polynomial under a joint transformation with correlated f and Σ [5].

On the other hand b' is left unfixed since the expression it multiplies is Weyl invariant. It follows immediately from the above discussion that if there are additional terms in the Kähler shift anomaly polynomial they should be Weyl invariant since there is no term which could match its Weyl variation.

We now come to the role of the local terms in the OPE. Through factorization for every term in the effective action involving A_μ , we should get a corresponding term with $\frac{1}{24c}\mathcal{A}_\mu$ replacing A_μ . We will discuss the implications for just the Kähler shift anomaly polynomial:

1) Replacing one A_μ by \mathcal{A}_μ in the Weyl anomalous generating functional generates under Kähler shifts terms with the same structure as those in (4.4) which contain $f - \bar{f}$. For a general $\mathcal{N} = 2$ theory in $d = 4$ their normalization is however incompatible with the relative normalization obtained by the Wess-Zumino condition. This implies that the local terms in the OPE, while contributing to the anomalous correlators, do not account for the complete answer. For this we need to add the ordinary contributions.

In Appendix B we describe the explicit check of a similar situation for the Maxwell supermultiplet: the Osborn anomaly is completely fixed by $\mathcal{N} = 2$ supersymmetry in terms of the Riemann tensor computed from the Zamolodchikov metric. This was obtained as the sum of two terms, one representing the factorized contributions of the local terms in the OPE and the other one the ordinary contribution. Interestingly the two terms had even different index structures and only their sum gave the Riemann tensor of the Zamolodchikov metric.

2) If we want to replace more than one A_μ , we should limit ourself to the anomalous term involving three A_μ which generate the $U(1)_R$ chiral anomaly. Using factorization, terms depending on A_μ in the anomaly polynomial could generate terms in the Kähler shift anomaly polynomial. We will assume in the following discussion that the Weyl anomaly polynomial is “complete” i.e. the new anomalies suggested by factorization will appear only in the Kähler shift. This can always be achieved by adding variations of local counterterms. Then replacing two A_μ we get a new term in the Kähler anomaly polynomial $(f - \bar{f})\mathcal{F}_{\mu\nu}\tilde{F}^{\mu\nu}$. This term should be made compatible with $\mathcal{N} = 2$ supersymmetry, i.e. obtained from an appropriate superspace expansion. If the supersymmetrization turns out to be impossible, the factorized contribution should be cancelled by an ordinary term.

Finally, by replacing all three A_μ we have the new term $(f - \bar{f})\mathcal{F}_{\mu\nu}\tilde{\mathcal{F}}^{\mu\nu}$. This again should be supersymmetrized and the previous discussion applies. We remark that in order that the structures discussed under this point would appear, one needs at least four (real) moduli: otherwise the contributions vanish or are cohomologically trivial. In summary

in $d = 4$, while the factorized local OPE contributions are needed, they seem to act always together with the ordinary terms and their normalization therefore does not have unambiguous predictive power.

In $d = 2$ the situation is different. Due to the specific two dimensional kinematical simplifications and $(2, 2)$ supersymmetry, the Weyl anomalous part determines completely the Kähler shift anomalies in this component of the effective action. The Weyl anomalous part of the effective action can be separated unambiguously from the Weyl invariant part and it depends on an explicit combination of the curvature superfield and the Kähler potential. The normalization of this combination is determined by factorization and therefore the Kähler shift anomalies can be understood to follow entirely from Weyl anomalies combined with factorization. One cannot exclude of course that the Weyl invariant part of the effective action produces under the Kähler shift an additional contribution with the same anomaly structure but it is a consistent assumption that the Weyl invariant part is also Kähler shift invariant.

Finally we would like to comment on the possible role of local terms in the OPE in the conformal bootstrap. For theories with extended supersymmetries the local terms should be included as additional couplings to the usual conformal blocks. The constraints following from crossing symmetry should give interesting relations between the contributions of local terms and the ordinary ones.

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A Non-zero structure constant implies non-zero charge

Consider in $d = 4$ the correlator of a conserved current $J_\mu(z)$ with two dimension four operators $M_1(x)$ and $M_2(y)$. We assume that M_1, M_2 are not orthogonal to each other but we do not assume anything about their charge under J . From conformal invariance the coordinate dependence is completely fixed [16] for non-coinciding coordinates

$$\langle M_1(x)M_2(y)J_\mu(z) \rangle = c \frac{1}{(x-y)^6} \left(\frac{1}{(z-x)^2} \frac{(z-x)_\mu}{(z-x)^4} - x \leftrightarrow y \right) \quad (\text{A.1})$$

except for the structure constant c . The OPE between $J_\mu(z)$ and $M_1(x)$ can be extracted from the above correlator assuming that the representation holds also when one coordinate

approaches another. We put $x = 0$ and z infinitesimally close to 0 while y is kept fixed with the component in the direction μ chosen to be 0. Then the OPE has the form

$$J_\mu(z)M_1(0) \sim c \frac{z_\mu}{z^4} M_1(0) \quad (\text{A.2})$$

Continuing to Minkowski space we obtain

$$T(Q(t)M_1(0)) \sim c \text{sign}(t) M_1(0) \quad (\text{A.3})$$

where Q is the charge operator $\int d^3z J_0(t, \vec{z})$ and T is time ordering. Considering the relation above for $t = \pm\epsilon$ we find

$$[Q, M_1(0)] \sim c M_1(0) \quad (\text{A.4})$$

i.e. M is necessarily charged if the structure constant c is not zero.

B The Maxwell case

A simple toy model is four-dimensional $\mathcal{N} = 2$ supersymmetric $U(1)$ gauge theory. A useful reference is Appendix B of [17] whose notation we follow in this appendix. The field content are the gauge field, a $SU(2)_{\mathcal{R}}$ doublet of Weyl spinors and a complex scalar. There is also a $SU(2)_{\mathcal{R}}$ triplet of auxiliary fields. They play no role in our analysis. The action is

$$S = -\frac{1}{g^2} \int d^4x \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{g^2}{32\pi^2} \theta F_{\mu\nu} \tilde{F}^{\mu\nu} + i \bar{\lambda}_i \bar{\sigma}^\mu \partial_\mu \lambda^i + \partial_\mu \phi \partial^\mu \bar{\phi} \right) \quad (\text{B.1})$$

The fermions carry $U(1)_{\mathcal{R}}$ charge +1 while the scalar has charge +2. All other fields are neutral. The $U(1)_{\mathcal{R}}$ current is therefore

$$j_\mu = -\bar{\lambda}_i \bar{\sigma}_\mu \lambda^i + 2i(\phi \partial_\mu \bar{\phi} - \bar{\phi} \partial_\mu \phi) \quad (\text{B.2})$$

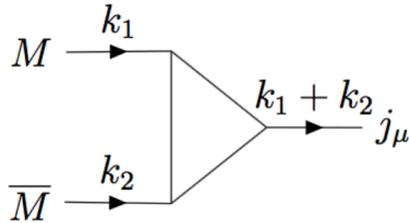
This theory has a complex modulus, i.e. an exactly marginal operator,

$$M = \frac{i\pi}{2} \left(\frac{1}{8} F_{\mu\nu}^+ F^{+\mu\nu} + i \bar{\lambda}_i \bar{\sigma}^\mu \partial_\mu \lambda^i - \bar{\phi} \square \phi \right) \quad (\text{B.3})$$

where $F^\pm = F \pm i\tilde{F}$. This operator is neutral under $U(1)_{\mathcal{R}}$ and one might expect that the $\langle M \bar{M} j_\mu \rangle$ correlation function vanishes. But this is not quite true and it has, in fact, an imaginary part. Note that the last two terms in (B.3) vanish on-shell. The reason why we are not allowed to set these redundant operators to zero is supersymmetry. As we will show they contribute in an essential way to the three-point function. When inserted into a Feynman diagram they cancel a propagator, but the diagram still retains a nontrivial analytic structure. In the four-moduli correlator which we will compute below, the redundant operators contribute in a similar way and their contribution is required

in order to get the result which is consistent with supersymmetry. On the other hand it is also clear that their contribution to the two-point function is completely real and therefore for the Zamolodchikov metric only the gauge field part of the moduli is relevant.

In this free field theory the $\langle M \bar{M} j_\mu \rangle$ correlation function is given by triangle diagrams. Only the fermions and the scalar contribute and among the different possible contractions those where the propagator between the M and \bar{M} insertions is cancelled, have an imaginary part. This implies a local term in the $M(x) \bar{M}(y)$ operator product expansion which is proportional to the current and the $\langle j_\mu j_\nu \rangle$ two-point function is responsible for the logarithm.



Explicit calculation of the one-loop triangle diagram gives

$$\langle M(k_1) \bar{M}(k_2) j_\mu(-k_1 - k_2) \rangle = -\frac{1}{64} (q^2 r_\mu - q \cdot r q_\mu) \log \frac{\Lambda^2}{q^2} + \text{local} \quad (\text{B.4})$$

where $q = k_1 + k_2$ and $r = k_1 - k_2$.

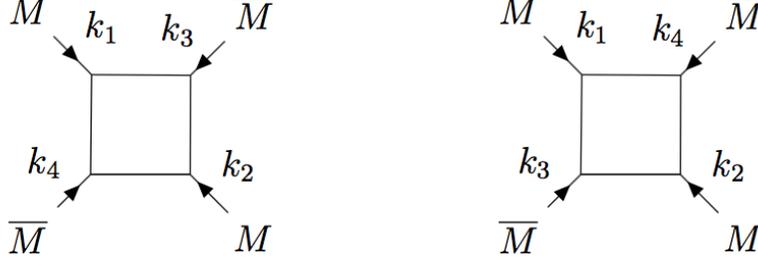
Some comments/observations are in order: (i) The correlation function is not gauge anomalous. This is consistent with (2.2) where all moduli dependent terms with α are cohomologically trivial. (ii) It follows from the calculation that the logarithmic divergence is due to the cancelled propagators. Those contractions where this does not happen, do not contribute. (iii) The kinematical structure of the diagrams corresponds to $\int d^4x F_{\mu\nu} \mathcal{F}^{\mu\nu}$. This reflects the general structure of contact terms in this simplest example of a free theory.

We can also compute the logarithmically divergent part of the four-point function. For the non-supersymmetric theory, where the modulus consists only of the spin one part in (B.3), this was done by Osborn [18]. His result cannot be cast into the form dictated by $\mathcal{N} = 2$ supersymmetry, which contains the target space Riemann tensor (cf. [5], or the fourth line of eq. (2.2)), which for the Zamolodchikov metric $g_{\tau\bar{\tau}} = \frac{1}{2\tau_2^2}$ is $\mathcal{R}_{\tau\bar{\tau}\tau\bar{\tau}} = \frac{1}{4\tau_2^4}$. Here τ is the single source in this case and τ_2 its imaginary part. The difference between these two expressions is proportional to

$$2 (\nabla^\mu \tau \nabla_\mu \bar{\tau})^2 - 5 |\nabla^\mu \tau \nabla_\mu \tau|^2 \quad (\text{B.5})$$

where now τ is the fluctuation around a constant value of the source and we have only kept terms up to $\mathcal{O}(\tau^4)$. This difference must be accounted for by spin 0 and spin 1/2

contributions of M and \bar{M} via the cancelled propagator argument. In each case, there are two Feynman diagrams which contribute⁶



For the spin zero part these two diagrams evaluate to

$$\frac{\pi^2}{384} \left(m_1^2 m_2^2 + m_1^2 m_3^2 + m_1^2 m_4^2 + m_2^2 m_3^2 + m_2^2 m_4^2 + m_3^2 m_4^2 - (s+u)(m_1^2 + m_2^2 + m_3^2 + m_4^2) + s^2 + su + u^2 \right) \log \Lambda^2 + \text{finite} \quad (\text{B.6})$$

while for the fermions one computes

$$-\frac{\pi^2}{384} \left(2m_1^2 m_2^2 + 2m_3^2 m_4^2 - m_1^2 m_3^2 - m_1^2 m_4^2 - m_2^2 m_3^2 - m_2^2 m_4^2 + (s+u)(m_1^2 + m_2^2 + m_3^2 + m_4^2) - s^2 - 4su - u^2 \right) \log \Lambda^2 + \text{finite} \quad (\text{B.7})$$

We have expressed the amplitudes in terms of an independent set of kinematical invariants

$$m_i^2 = k_i^2, \quad s = (k_1 + k_3)^2, \quad u = (k_1 + k_4)^2 \quad (\text{B.8})$$

Their sum is proportional to

$$m_1^2 m_2^2 + m_3^2 m_4^2 - 2(m_1^2 m_3^2 + m_1^2 m_4^2 + m_2^2 m_3^2 + m_2^2 m_4^2) + 2(s+u)(m_1^2 + m_2^2 + m_3^2 + m_4^2) - 2(s^2 + u^2) - 5su \quad (\text{B.9})$$

which is precisely the kinematical structure derived from (B.5). The overall normalization can be fixed by an appropriate rescaling of the source τ .

We remark that analysing the above diagrams in terms of the OPE we identify two additional local terms specific to this model which contribute: denoting by the $S(x) \equiv \phi(x)\bar{\phi}(x)$ the dimension two scalar operator and by κ_μ the (conserved) vector operator which differs from j_μ only by the relative sign between the bosonic and fermionic contributions to (B.2) and which can be shown to have vanishing two-point function with j_μ ,

⁶The spin 0 and 1/2 parts of M do not contribute non-local parts to $M^3 \bar{M}$ or M^4 correlators. As for the other orderings around the box digram, there are always at least three cancelled propagators and therefore the Cutkosky rules give zero imaginary part and therefore no logarithm.

we find

$$\begin{aligned}
M(x)\bar{M}(y) \sim & \frac{\pi^2}{32} \left(\partial_\mu^{(x)} \delta^4(x-y) (3j^\mu(y) - \kappa^\mu(y)) - \partial_\mu^{(y)} \delta^4(x-y) (3j^\mu(x) - \kappa^\mu(x)) \right) \\
& + \frac{\pi^2}{8} \left(\square_{(x)} \delta^4(x-y) S(y) - \partial_{(x)}^\mu \delta(x-y) \partial_\mu S(y) \right)
\end{aligned} \tag{B.10}$$

C Free theory with moduli in $d = 2$

We want to study the free (2,2) SCFT of a single twisted chiral superfield Φ with unperturbed superspace action $\int \bar{\Phi}\Phi$. $\Phi = (\phi, \psi_+, \bar{\psi}_-, F)$ being twisted chiral means that it satisfies $\bar{D}_+\Phi = D_-\Phi = 0$ and the complex conjugate relations $D_+\bar{\Phi} = \bar{D}_-\bar{\Phi} = 0$. Here D_\pm and \bar{D}_\pm are flat superspace covariant derivatives. We deform the theory by a chiral primary operator

$$M = \bar{D}_+\bar{\Phi} \bar{D}_-\Phi \tag{C.1}$$

We then couple the deformed CFT to $U(1)_A$ supergravity. If J is the chiral source superfield, the deformed action is

$$\int d^2x d^4\theta E \bar{\Phi}\Phi + \left(\int d^2x d^2\theta \mathcal{E} J M + \text{c.c.} \right) = \int d^2x d^4\theta E (1 + J + \bar{J}) \bar{\Phi}\Phi \tag{C.2}$$

A useful reference for flat (2,2) superspace is Chapter 12 of [19]. For curved superspace we follow [13]. With the help of results obtained there, we find the following component action

$$\begin{aligned}
\int d^2x \sqrt{g} \left(-\phi \square \bar{\phi} + \frac{i}{2} (1 + J + \bar{J}) \bar{\psi} \gamma^\mu \overleftrightarrow{\nabla}_\mu \psi + (1 + J + \bar{J}) \bar{\psi} \gamma^\nu \psi (\mathcal{A}_\nu + \frac{1}{2} \epsilon_\nu{}^\mu A_\mu) \right. \\
\left. + J (\partial^\mu \phi \partial_\mu \bar{\phi} - \epsilon^{\mu\nu} \partial_\mu \phi \partial_\nu \bar{\phi}) + \bar{J} (\partial^\mu \bar{\phi} \partial_\mu \phi + \epsilon^{\mu\nu} \partial_\mu \bar{\phi} \partial_\nu \phi) \right)
\end{aligned} \tag{C.3}$$

Here we have set the gravitini to zero. The auxiliary scalar in the gravity multiplet drops out. We have defined the Dirac spinor $\psi = (\psi_-, \psi_+)^T$ and J is now the lowest component of the source superfield. The other components are set to zero as is the auxiliary field F contained in Φ ; it vanishes on-shell. A_μ is the $U(1)_A$ gauge field of the SUGRA multiplet and \mathcal{A}_μ is the Kähler connection computed from the potential⁷

$$K = -\ln(1 + J + \bar{J}) \tag{C.4}$$

Note in (C.3) the relative factor $\frac{1}{2}$ in the coupling to the $U(1)$ -current $j^\mu \bar{\psi} \gamma^\mu \gamma \psi = \bar{\psi} \gamma^\nu \psi \epsilon_\nu{}^\mu$. It is $\frac{c}{6}$ for $c = 3$, the central charge of the twisted chiral multiplet and we see that besides the coupling to gravity, the fermions couple precisely to the combination of

⁷The relation with the more familiar Kähler potential for the metric on the upper half-plane, $K = -\ln(\tau - \bar{\tau})$ is established with the coordinate transformation $J = i\tau - \frac{1}{2}$.

the $U(1)_A$ and the Kähler connection which was discussed in Section 3. From the action we can also read off the moduli operators as the coefficients of the sources. In a flat background ($g_{\mu\nu} = \eta_{\mu\nu}$, $A_\mu = 0$) they are

$$\begin{aligned} M &= \partial^\mu \phi \partial_\mu \bar{\phi} - \epsilon^{\mu\nu} \partial_\mu \phi \partial_\nu \bar{\phi} + i \bar{\psi} \gamma^\mu \partial_\mu \psi \\ \bar{M} &= \partial^\mu \phi \partial_\mu \bar{\phi} + \epsilon^{\mu\nu} \partial_\mu \phi \partial_\nu \bar{\phi} - i \partial_\mu \bar{\psi} \gamma^\mu \psi \end{aligned} \tag{C.5}$$

The fermionic contribution vanishes on-shell i.e. it is redundant and will only contribute via the cancelled propagator argument, already familiar from the discussion of the free Maxwell theory. The bosonic part accounts for the ordinary contributions to correlators. As in the free Maxwell theory, only this non-redundant part contributes to the logarithmic divergence of the $\langle M \bar{M} \rangle$ two-point function and therefore to the Zamolodchikov metric. If we expand the action around constant moduli, $J = \lambda + \delta J$ and compute $\langle M \bar{M} \rangle$, we find that it is proportional to $(1 + \lambda + \bar{\lambda})^{-2}$, from the normalization of the kinetic term of ϕ and the fact that the one-loop diagram which computes it has two propagators. This is the Kähler metric derived from (C.4). Again as in the $\mathcal{N} = 2$ Maxwell theory in $d = 4$, the redundant piece of M is responsible for the non-vanishing of $\langle M \bar{M} j_\mu \rangle$ and the cancelled propagator localizes the $M(x) \bar{M}(y)$ operator product on the $U(1)$ current.

If we integrate out ϕ and ψ , we recover the non-local effective action. It is easy to integrate out the fermions. They can be rescaled to eliminate the $(1 + J + \bar{J})$ factor. What is left are free fermions coupled to an external gauge field $\mathbb{A}_\mu \equiv \mathcal{A}_\mu + \frac{\epsilon}{6} \epsilon_\mu{}^\nu A_\nu$. This leads to a term

$$\int d^2x \partial^\mu \mathbb{A}_\mu \frac{1}{\square} \partial^\nu \mathbb{A}_\nu \tag{C.6}$$

in the effective action, in agreement with (3.15).

It is special to this simple model that the microscopic action $\int d^4\theta e^{-K(J, \bar{J})} \Phi \bar{\Phi}$ formally depends on the sources through the Kähler potential on the moduli space. However the explicit expansion in components shows that due to the fact that the scalar fields without derivatives acting on them are not legal operators, the actual dependence on the sources does not necessarily reflect the coupling of the potential. For this model therefore one can see explicitly that while the Weyl anomalous part of the action is defined by the Kähler potential with its potentially anomalous shift invariance, the couplings of the Weyl invariant part effectively do not contain anymore the Kähler potential.

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