Cyclicity of All Anti-NMHV and N^{2}MHV Tree Amplitudes in $\mathcal{N}=4$ SYM

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ABSTRACT: This essay proves the cyclic invariance of anti-NMHV and N^{2}MHV tree amplitudes in $\mathcal{N}=4$ SYM up to any number of external particles as an interesting exercise. In the proof the two-fold simplex-like structures introduced in 1609.08627 (and reviewed in 1712.10000) play a key role, as the cyclicity of amplitudes also induces similar simplex-like structures for the boundary generators of homological identities. For this purpose, we only need a part of all distinct boundary generators, and the relevant identities only involve BCFW-like cells.

KEYWORDS: Amplitudes, Positive Grassmannian.

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Contents

1. Introduction

Tree amplitudes in planar $\mathcal{N}=4$ super Yang-Mills theory have an impressive simplicity in the language of positive Grassmannian in momentum twistor space, namely the so-called two-fold simplex-like structures [1], a concise review can be referred in [2]. In terms of Grassmannian geometry representatives specifying linear dependencies of different ranks and empty slots for null columns, information of amplitudes can be compactly captured by finite numbers of fully-spanning cells and their growing parameters. Given a fixed k, as $(k+2)$ is the number of negative helicities, there is no new full cell beyond $n=4k+1$, then after we identify all full cells with their growing parameters at this critical n, N^kMHV amplitudes are known once for all up to any number of external particles.

With the aid of this purely geometric description, homological identities can be understood in a much more intuitive way, and most of them turn out to be the secret incarnation of the simple NMHV identity. A part of these identities are crucial for interconnecting different BCFW cells, and hence different BCFW recursion schemes [3]. Explicitly in this essay, we would like to manifest the cyclicity of amplitudes of two specific classes: the anti-NMHV and the N^2MHV families, by applying the simplex-like structures of both the amplitudes and boundary generators of identities. From [1] we have fully understood the structures of anti-NMHV, NMHV, N^2MHV and N^3MHV families while only the cyclicity of NMHV family and $n=7,8$ anti-NMHV amplitudes has been shown. Now it is a helpful warmup exercise to reconsider the cyclicity of NMHV family in a more formal way as presented below, which precedes the main body of this work.

Recall the NMHV $n=6$ amplitude in the default recursion scheme is given by

$$Y_6^1 = [6] + [4] + [2], \tag{1.1}$$

so the difference between itself and its cyclicly shifted (by $+1$) counterpart is

$$Y_6^1 - Y_{6,+1}^1 = -[1] + [2] - [3] + [4] - [5] + [6] \equiv I_{123456}, \tag{1.2}$$

here the 6-term NMHV identity of labels $1,2,3,4,5,6$ is defined as I_{123456}. According to the simplex-like structures of amplitudes, or more explicitly, the 2-mode of Y_6^1 with growing parameters $(6,4,2)$, we have the following relation for $n=7$:

$$Y_7^1 - Y_{7,+1}^1 = \left(\begin{array}{rrr} & & [23] \\ & [27] & [25] \\ [67] & [47] & [45] \end{array}\right) - \left(\begin{array}{rrr} & & [34] \\ & [31] & [36] \\ [71] & [51] & [56] \end{array}\right) = \left(\begin{array}{rr} & [3]\,I_{124567} \\ [7]\,I_{123456} & [5]\,I_{123467} \end{array}\right), \tag{1.3}$$

which completes the proof of cyclicity for all NMHV amplitudes since the growing parameters $(7,5,3)$ of I_{123456} have been identified. We see that $(7,5,3)$ are closely related to $(6,4,2)$ of Y_6^1, and this shows how the cyclicity of amplitudes induces similar simplex-like structures for the relevant homological identities. As we will see, this intriguing feature appears in a much more nontrivial form for N^2MHV amplitudes.

2. Cyclicity of Anti-NMHV Amplitudes

Before moving to the N^2MHV family, let's first consider the cyclicity of all anti-NMHV amplitudes, since this is in fact the nontrivial starting point for all N^kMHV cases. More explicitly, recall that for a given k non-vanishing amplitudes start with the anti-MHV sector $n=k+4$, which contains just one top cell, then the first interesting case is the anti-NMHV sector $n=k+5$. It can be rearranged in the similar form of a triangle-shape sum as its parity conjugate (the NMHV sector).

The anti-NMHV triangle-like pattern can be clearly observed in the series of examples below

$$Y_6^1 = \left([2] \; \left\{\begin{array}{r} [6] \\ [4] \; \end{array}\right. \right), \tag{2.1}$$

$$Y_7^2 = \left([2] \;\; (23) \left\{\begin{array}{r} (67) \\ (45) \;\; \end{array}\right. \;\; \left\{\begin{array}{r} [7] \\ (45)(71) \\ [5] \;\; \end{array}\right. \right), \tag{2.2}$$

$$Y_8^3 = \left([2] \;\; (23) \left\{\begin{array}{r} (678) \\ (456) \;\; \end{array}\right. \;\; (234) \left\{\begin{array}{r} (78) \\ (456)(781) \\ (56) \;\; \end{array}\right. \;\; \left\{\begin{array}{r} [8] \\ (456)(81) \\ (56)(812) \\ [6] \;\; \end{array}\right. \right), \tag{2.3}$$

$$Y_9^4 = \left([2] \;\; (23) \left\{\begin{array}{r} (6789) \\ (4567) \;\; \end{array}\right. \;\; (234) \left\{\begin{array}{r} (789) \\ (4567)(7891) \\ (567) \;\; \end{array}\right. \;\; (2345) \left\{\begin{array}{r} (89) \\ (4567)(891) \\ (567)(8912) \\ (67) \;\; \end{array}\right. \;\; \left\{\begin{array}{r} [9] \\ (4567)(91) \\ (567)(912) \\ (67)(9123) \\ [7] \;\; \end{array}\right. \right), \tag{2.4}$$

and its general form can be proved by induction. From [1] it is already known that

$$Y_7^2 - Y_{7,+1}^2 = -\,\partial(23) - \partial(56) - \partial(71), \tag{2.5}$$

$$Y_8^3 - Y_{8,+1}^3 = \partial(23) + \partial(67) + \partial(81) + \partial(234)(567) + \partial(567)(812) + \partial(781)(234), \tag{2.6}$$

which manifest the cyclicity of Y_7^2 and Y_8^3. These results can be rearranged in a more suggestive form as

$$Y_7^2 - Y_{7,+1}^2 = -\,\partial \left((23) \; \left\{ \begin{array}{l} \quad (71) \\ (56) \end{array} \right. \right), \tag{2.7}$$

$$Y_8^3 - Y_{8,+1}^3 = \partial \left((23) \;\; (234) \left\{ \begin{array}{l} \quad (781) \\ (567) \end{array} \right. \; \left\{ \begin{array}{c} (81) \\ (567)(812) \\ (67) \end{array} \right. \right), \tag{2.8}$$

as well as a further extension of this pattern, for which $k = 4$:

$$Y_9^4 - Y_{9,+1}^4 = -\,\partial \left((23) \;\; (234) \left\{ \begin{array}{l} \quad (7891) \\ (5678) \end{array} \right. \;\; (2345) \left\{ \begin{array}{c} (891) \\ (5678)(8912) \\ (678) \end{array} \right. \; \left\{ \begin{array}{c} (91) \\ (5678)(912) \\ (678)(9123) \\ (78) \end{array} \right. \right), \tag{2.9}$$

where the sign factor $(-)^{k+1}$ for each of these relations follows the convention of [3, 4]. And the types of homological identities used in (2.7) and (2.8), as already proved in [1], include

$$\partial(23) = -\,[2] + [3] - (23)(45) + (23)(56) - (23)(67) + (23)(71), \tag{2.10}$$

for $k = 2$ (for comparison we have used boundary generator (23) instead of (12) in [1], and similar below), as well as

$$\partial(23) = +\,[2] - [3] + (23)(456) - (23)(567) + (23)(678) - (23)(781), \tag{2.11}$$

$$\begin{aligned} \partial(234)(567) = &+ (23)(567) - (34)(567) + (234)(56) - (234)(67) \\ &+ (234)(567)(781) - (234)(567)(812), \end{aligned} \tag{2.12}$$

for $k = 3$, while those for $k = 4$ used in (2.9) are new, as given by

$$\partial(23) = -\,[2] + [3] - (23)(4567) + (23)(5678) - (23)(6789) + (23)(7891), \tag{2.13}$$

$$\begin{aligned} \partial(234)(5678) = &- (23)(5678) + (34)(5678) - (234)(567) + (234)(678) \\ &- (234)(5678)(7891) + (234)(5678)(8912), \end{aligned} \tag{2.14}$$

$$\begin{aligned} \partial(2345)(678) = &- (234)(678) + (345)(678) - (2345)(67) + (2345)(78) \\ &- (2345)(678)(8912) + (2345)(678)(9123), \end{aligned} \tag{2.15}$$

$$\begin{aligned} \partial(2345)(5678)(8912) = &- (234)(5678)(8912) + (345)(5678)(8912) - (2345)(567)(8912) \\ &+ (2345)(678)(8912) - (2345)(5678)(891) + (2345)(5678)(912), \end{aligned} \tag{2.16}$$

and they can be proved by using the similar matrix approach as done in [1]. The examples of anti-NMHV family again show how the cyclicity of amplitudes induces similar structures for the relevant identities.

3. Cyclicity of N^2MHV Amplitudes

Now we will start with the cyclicity of N^2MHV $n=7$ amplitude, namely (2.7), to explore its generalization towards $n\geq 8$. Recall the N^2MHV full cells along with their growing parameters are given by

$$G_{7,0} = \begin{cases} (45)(71) \\ \quad [5] \end{cases} \qquad (5) \tag{3.1}$$

$$G_{7,1} = (23)\begin{cases} \quad (67) \\ (45) \end{cases} \qquad (6,4) \tag{3.2}$$

$$G_{8,1} = \begin{cases} (234)_2(678)_2 & (7,4) \\ (456)_2(781)_2 & (7,5) \\ (23)(456)_2(81) & (6,4) \end{cases} \tag{3.3}$$

$$G_{9,2} = \begin{cases} (2345)_2(6789)_2 \\ (23)(4567)_2(891)_2 \end{cases} \qquad (8,6,4) \tag{3.4}$$

and for notational convenience, below we will suppress subscript '2' for consecutive vanishing 2×2 minors, such as $(234)_2 \equiv (234) = (23)(34)$, which is unambiguous as we restrict the discussion to the N^2MHV sector from now on. Given the information above (or refer to [1]), the N^2MHV $n=8$ amplitude is then

$$Y_8^2 = S_{8,2} + S_{8,1} + S_{8,0}, \tag{3.5}$$

where we have separated the terms containing $2,1,0$ empty slots respectively as

$$S_{8,2} = \begin{pmatrix} & & [23] \\ & [28] & [26] \\ [78] & [58] & [56] \end{pmatrix}, \quad S_{8,1} = \begin{pmatrix} & [2](56)(81) \\ & [2](34)\begin{cases}(78)\\(56)\end{cases} \\ [8](45)(71) & [5](46)(81) \\ [8](23)\begin{cases}(67)\\(45)\end{cases} & [6](23)\begin{cases}(78)\\(45)\end{cases} \\ & [4](23)\begin{cases}(78)\\(56)\end{cases} \end{pmatrix}, \quad S_{8,0} = \begin{cases} (234)(678) \\ (456)(781) \\ (23)(456)(81) \end{cases}, \tag{3.6}$$

as indicated by the second subscript of $S_{n,i}$. Now the cyclicity of Y_8^2 is separated into three different parts:

$$Y_8^2 - Y_{8,+1}^2 = (S_{8,2} - S_{8,2,+1}) + (S_{8,1} - S_{8,1,+1}) + (S_{8,0} - S_{8,0,+1}), \tag{3.7}$$

where the third subscript '+1' similarly denotes the cyclic shift. Straightforwardly we find

$$S_{8,2} - S_{8,2,+1} = \left.\begin{cases} [8]\,(-\partial(23) - \partial(56) - \partial(71)) \\ [6]\,(-\partial(23) - \partial(57) - \partial(81)) \\ [3]\,(-\partial(24) - \partial(67) - \partial(81)) \end{cases}\right|_1, \tag{3.8}$$

and '$|_1$' denotes the truncation that only keeps terms containing one empty slot. For example, (2.10) can be separated as

$$\partial(23)\,|_1 = -[2]+[3], \quad \partial(23)\,|_0 = -(23)(45)+(23)(56)-(23)(67)+(23)(71). \tag{3.9}$$

Knowing growing parameters $(8,6,3)$ of $n\!=\!7$ boundary generators, or identities $(-\partial(23)-\partial(56)-\partial(71))$, we can denote this result as (B_n stands for boundary generators first induced by the cyclicity of Y_n^2)

$$B_7 = -(23)-(56)-(71) \qquad (8,6,3) \tag{3.10}$$

so that

$$S_{8,2}-S_{8,2,+1} = (\partial B_7\,|_1)_1, \quad S_{8,1}-S_{8,1,+1} = (\partial B_7\,|_0)_1 + \partial B_8\,|_1, \quad S_{8,0}-S_{8,0,+1} = \partial B_8\,|_0, \tag{3.11}$$

where $(\partial B_7\,|_1)_1$ denotes the counterpart of $\partial B_7\,|_1$ when n increases from 7 to 8, according to its simplex-like growing patterns. In this way, we can figure out $\partial B_8\,|_1$ (and hence B_8) via the second relation above, namely (in the 4th and 7th lines below we have added $+[4](56)(81)$ and $-[4](56)(81)$ respectively)

$$\begin{aligned}
\partial B_8\,|_1 = {} & (S_{8,1}-S_{8,1,+1}) - (\partial B_7\,|_0)_1 \\
= {} & -[7](81)(34)+[8](71)(34)-[1](78)(34) \\
& +[2](34)(56)-[3](24)(56)+[4](23)(56) \\
& +[2](34)(78)-[3](24)(78)+[4](23)(78) \\
& -[3](56)(81)+[4](56)(81)-[5](34)(81)+[6](34)(81)+[8](34)(56)-[1](34)(56) \\
& +[8](12)(56)-[1](82)(56)+[2](81)(56) \\
& -[5](67)(34)+[6](57)(34)-[7](56)(34) \\
& -[4](56)(81)+[5](46)(81)-[6](45)(81) \\
= {} & \partial\,(+(781)(34)-(234)(56)-(234)(78)+(34)(56)(81)+(567)(34)-(812)(56)+(456)(81))\,|_1 .
\end{aligned} \tag{3.12}$$

After identifying B_8 (by trial and error), we can check the third relation above, as

$$\begin{aligned}
\partial B_8\,|_0 = {} & +(781)(234)-(781)(345)+(781)(34)(56) \\
& -(234)(567)+(234)(56)(78)-(234)(56)(81) \\
& -(234)(56)(78)+(234)(678)-(234)(781) \\
& -(34)(567)(81)-(34)(56)(781)+(34)(56)(812)+(234)(56)(81) \\
& +(567)(81)(34)-(567)(12)(34)+(567)(234) \\
& -(812)(34)(56)+(812)(456)-(812)(567) \\
& +(456)(781)-(456)(812)+(456)(81)(23) \\
= {} & S_{8,0}-S_{8,0,+1}
\end{aligned} \tag{3.13}$$

nicely obeys the required consistency. The identities used above can be referred in appendix A and they can be classified into five distinct types at $n=8$.

Next, for the cyclicity of Y_9^2 we similarly have

$$S_{9,2} - S_{9,2,+1} = (\partial B_7\,|_0)_2 + (\partial B_8\,|_1)_1, \quad S_{9,1} - S_{9,1,+1} = (\partial B_8\,|_0)_1 + \partial B_9\,|_1, \quad S_{9,0} - S_{9,0,+1} = \partial B_9\,|_0, \tag{3.14}$$

where the simplex-like growing patterns give

$$S_{9,1} = \begin{pmatrix} [2]\begin{cases}(345)(789)\\(567)(891)\\(34)(567)(91)\end{cases} & \begin{cases}[7](234)(689)\\ [4](235)(789)\end{cases} \\ & \begin{cases}[7](456)(891)\\ [5](467)(891)\end{cases} \\ [9]\begin{cases}(234)(678)\\(456)(781)\\(23)(456)(81)\end{cases} & \begin{cases}[6](23)(457)(91)\\ [4](23)(567)(91)\end{cases} \end{pmatrix}, \quad S_{9,0} = \begin{cases}(2345)(6789)\\(23)(4567)(891)\end{cases}, \tag{3.15}$$

and via the first relation above we can figure out $(\partial B_8\,|_1)_1$, or the growing parameters of B_8, as

$$\begin{aligned} B_8 =& + (781)(34) - (234)(56) - (234)(78) + (34)(56)(81) && (9,7,5,3) \\ &+ (567)(34) && (9,8,6,3) \\ &- (812)(56) && (9,6,3) \\ &+ (456)(81) && (9,6,4) \end{aligned} \tag{3.16}$$

so that via the second relation we can similarly figure out $\partial B_9\,|_1$ (and hence B_9), and the third one again serves as a consistency check. Explicitly, we find

$$\begin{aligned} B_9 =& - (2345)(789) - (7891)(345) \\ &+ (8912)(567) - (5678)(234) - (5678)(91)(34) + (5678)(12)(34) \\ &- (912)(34)(567) + (4567)(891). \end{aligned} \tag{3.17}$$

Following exactly the same logic, for the cyclicity of Y_{10}^2 we have

$$S_{10,2} - S_{10,2,+1} = (\partial B_8\,|_0)_2 + (\partial B_9\,|_1)_1, \quad S_{10,1} - S_{10,1,+1} = (\partial B_9\,|_0)_1 + \partial B_{10}\,|_1, \quad S_{10,0} - S_{10,0,+1} = \partial B_{10}\,|_0, \tag{3.18}$$

where the simplex-like growing patterns give

$$S_{10,1} = \begin{pmatrix} [2]\begin{cases}(3456)(789\,10)\\(34)(5678)(9\,10\,1)\end{cases} & [8]\begin{cases}(2345)(679\,10)\\(23)(4567)(9\,10\,1)\end{cases} \\ & [6]\begin{cases}(2345)(789\,10)\\(23)(4578)(9\,10\,1)\end{cases} \\ [10]\begin{cases}(2345)(6789)\\(23)(4567)(891)\end{cases} & [4]\begin{cases}(2356)(789\,10)\\(23)(5678)(9\,10\,1)\end{cases} \end{pmatrix}, \tag{3.19}$$

note that $S_{10,0}=0$ since there is no new full cell at $n\geq 10$, and hence $\partial B_{10}\,|_0=0$. Explicitly, we find

$$\begin{aligned} B_9 =&-(2345)(789)-(7891)(345) && (10,8,5,3)\\ &+(8912)(567)-(5678)(234)-(5678)(91)(34)+(5678)(12)(34) && (10,8,6,3)\\ &-(912)(34)(567) && (10,7,5,3)\\ &+(4567)(891) && (10,7,5) \end{aligned} \tag{3.20}$$

as well as

$$B_{10}=+(789\,10\,1)(3456)-(23456)(789\,10)+(5678)(9\,10\,1\,2)(34)+(45678)(9\,10\,1\,2)-(45678)(9\,10\,1)(23). \tag{3.21}$$

Finally, for the cyclicity of Y^2_{11} we have

$$S_{11,2}-S_{11,2,+1}=(\partial B_9\,|_0)_2+(\partial B_{10}\,|_1)_1,\;\; S_{11,1}-S_{11,1,+1}=(\partial B_{10}\,|_0)_1+\partial B_{11}\,|_1,\;\; S_{11,0}-S_{11,0,+1}=\partial B_{11}\,|_0, \tag{3.22}$$

and explicitly we find

$$\begin{aligned} B_{10} =&+(789\,10\,1)(3456)-(23456)(789\,10)+(5678)(9\,10\,1\,2)(34) && (11,9,7,5,3)\\ &+(45678)(9\,10\,1\,2)-(45678)(9\,10\,1)(23) && (11,9,7,5) \end{aligned} \tag{3.23}$$

which leads to $(\partial B_{10}\,|_0)_1=0$, and hence $\partial B_{11}\,|_1=0$. From $S_{11,0}=0$ we also have $\partial B_{11}\,|_0=0$, therefore it is safe to conclude that $B_{11}=0$. We can summarize these intriguing results as

$$B_7 = -(23)-(56)-(71) \qquad (8,6,3) \tag{3.24}$$

$$\begin{aligned} B_8 =&+(781)(34)-(234)(56)-(234)(78)+(34)(56)(81) && (9,7,5,3)\\ &+(567)(34) && (9,8,6,3)\\ &-(812)(56) && (9,6,3)\\ &+(456)(81) && (9,6,4) \end{aligned} \tag{3.25}$$

$$\begin{aligned} B_9 =&-(2345)(789)-(7891)(345) && (10,8,5,3)\\ &+(8912)(567)-(5678)(234)-(5678)(91)(34)+(5678)(12)(34) && (10,8,6,3)\\ &-(912)(34)(567) && (10,7,5,3)\\ &+(4567)(891) && (10,7,5) \end{aligned} \tag{3.26}$$

$$\begin{aligned} B_{10} =&+(789\,10\,1)(3456)-(23456)(789\,10)+(5678)(9\,10\,1\,2)(34) && (11,9,7,5,3)\\ &+(45678)(9\,10\,1\,2)-(45678)(9\,10\,1)(23) && (11,9,7,5) \end{aligned} \tag{3.27}$$

which terminate at $n=10$ like the full cells. With B_7, B_8, B_9, B_{10} and the growing parameters of relevant boundary generators identified, the cyclicity of Y^2_n for any n is proved. These identities are classified into $1,5,6,4$ distinct types with respect to B_7, B_8, B_9, B_{10} in appendix A.

A final remark is, not all N^{2}MHV homological identities are required for this proof. Especially, those involving the quadratic cell at $n=8$, namely (12)(34)(56)(78), or the composite-linear cell at $n=9$, namely (123)(456)(789), are irrelevant. These two non-BCFW-like cells will result in extra non-unity factors along with the 5-brackets [1], which cannot be generated by recursion. Therefore it is desirable to find that they do not appear at all in the proof of cyclicity for N^{2}MHV amplitudes, not even appear as canceling pairs in the intermediate steps.

A. Relevant N^{2}MHV Homological Identities

Below we list all distinct N^{2}MHV homological identities that are relevant in this work. Note that we have discarded boundary cells that fail to have kinematical supports in terms of momentum twistors, but still we abuse the term 'homological' here while the actual kinematics also matters [1, 2].

$n=7$

$$\partial(12) = -[1] + [2] - (12)(34) + (12)(45) - (12)(56) + (12)(67). \tag{A.1}$$

$n=8$

$$\partial(123)(45) = -[1](23)(45) + [2](13)(45) - [3](12)(45) + (123)(456) - (123)(45)(67) + (123)(45)(78). \tag{A.2}$$

$$\partial(123)(56) = -[1](23)(56) + [2](13)(56) - [3](12)(56) + (123)(456) - (123)(567) + (123)(56)(78). \tag{A.3}$$

$$\partial(123)(67) = -[1](23)(67) + [2](13)(67) - [3](12)(67) + (123)(45)(67) - (123)(567) + (123)(678). \tag{A.4}$$

$$\partial(123)(78) = -[1](23)(78) + [2](13)(78) - [3](12)(78) + (123)(45)(78) - (123)(56)(78) + (123)(678). \tag{A.5}$$

$$\begin{aligned}\partial(12)(34)(67) = &-[1](34)(67) + [2](34)(67) - [3](12)(67) + [4](12)(67) + [6](12)(34) - [7](12)(34)\\ &- (12)(345)(67) - (12)(34)(567) + (12)(34)(678) + (812)(34)(67).\end{aligned} \tag{A.6}$$

$n=9$

$$\begin{aligned}\partial(1234)(567) = &-[1](234)(567) + [2](134)(567) - [3](124)(567) + [4](123)(567)\\ &- (1234)(5678) + (1234)(567)(89).\end{aligned} \tag{A.7}$$

$$\begin{aligned}\partial(1234)(678) = &-[1](234)(678) + [2](134)(678) - [3](124)(678) + [4](123)(678)\\ &- (1234)(5678) + (1234)(6789).\end{aligned} \tag{A.8}$$

$$\begin{aligned}\partial(1234)(789) = &-[1](234)(789) + [2](134)(789) - [3](124)(789) + [4](123)(789)\\ &- (1234)(56)(789) + (1234)(6789).\end{aligned} \tag{A.9}$$

$$\begin{aligned}\partial(1234)(56)(89) = &-[1](234)(56)(89) + [2](134)(56)(89) - [3](124)(56)(89) + [4](123)(56)(89)\\ &- (1234)(567)(89) + (1234)(56)(789).\end{aligned} \tag{A.10}$$

$$\begin{aligned}\partial(1234)(67)(89) = &-[1](234)(67)(89)+[2](134)(67)(89)-[3](124)(67)(89)+[4](123)(67)(89)\\ &-(1234)(567)(89)+(1234)(6789).\end{aligned} \tag{A.11}$$

$$\begin{aligned}\partial(123)(45)(678) = &-[1](23)(45)(678)+[2](13)(45)(678)-[3](12)(45)(678)\\ &+[4](123)(678)-[5](123)(678)+[6](123)(45)(78)\\ &-[7](123)(45)(68)+[8](123)(45)(67)\\ &+(123)(45)(6789)-(9123)(45)(678).\end{aligned} \tag{A.12}$$

$n=10$

$$\begin{aligned}\partial(12345)(6789) = &-[1](2345)(6789)+[2](1345)(6789)-[3](1245)(6789)\\ &+[4](1235)(6789)-[5](1234)(6789)+(12345)(6789\,10).\end{aligned} \tag{A.13}$$

$$\begin{aligned}\partial(12345)(789\,10) = &-[1](2345)(789\,10)+[2](1345)(789\,10)-[3](1245)(678\,10)\\ &+[4](1235)(789\,10)-[5](1234)(789\,10)+(12345)(6789\,10).\end{aligned} \tag{A.14}$$

$$\begin{aligned}\partial(12345)(678)(9\,10) = &-[1](2345)(678)(9\,10)+[2](1345)(678)(9\,10)-[3](1245)(678)(9\,10)\\ &+[4](1235)(678)(9\,10)-[5](1234)(678)(9\,10)+(12345)(6789\,10).\end{aligned} \tag{A.15}$$

$$\begin{aligned}\partial(1234)(5678)(9\,10) = &-[1](234)(5678)(9\,10)+[2](134)(5678)(9\,10)-[3](124)(5678)(9\,10)\\ &+[4](123)(5678)(9\,10)-[5](1234)(678)(9\,10)+[6](1234)(578)(9\,10)\\ &-[7](1234)(568)(9\,10)+[8](1234)(567)(9\,10)\\ &-[9](1234)(5678)+[10](1234)(5678).\end{aligned} \tag{A.16}$$

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