

Minimizers of the dynamical Boulatov model

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We study the the Euler-Lagrange equation of the dynamical Boulatov model, which is a simplicial model for 3D gravity augmented by a Laplace-Beltrami operator. We provide all its solutions on the space of left and right invariant functions that render the interaction of the model an equilateral tetrahedron. Surprisingly, for a nonlinear equation, the solution space forms a vector space. This space distinguishes three classes of solutions: saddle points, global and local minima of the action. Our analysis shows that there exists one parameter region of coupling constants, for which the action admits degenerate global minima.

I. Introduction

In three dimensions, gravity can be formulated as a BF -theory [1]. Its functional integral quantization discretized over simplicial complexes leads to the Ponzano-Regge model [2, 3], which can be regarded as a quantum gravity model of discrete geometry. A corner stone of this approach is then to recover continuous geometry with all desired requirements and properties of our space-time. Such a description, however, as well as a mechanism which could successfully lead to it, remains an open problem in any background independent approach to quantum gravity.

The Boulatov model of group field theory (GFT) [4, 5] provides one way to address this issue. The model is formally defined by the gen-

erating functional,

$$Z[J] = \int \mathcal{D}\varphi e^{-S(\varphi) + \int J\varphi}, \quad (1)$$

where $S(\varphi)$ denotes the Boulatov action [6]. The striking fact about this generating functional is that its Feynman graphs correspond to simplicial complexes and its Feynman amplitudes coincide with Ponzano-Regge spin foam amplitudes [2, 3]. This leads to the conclusion that a perturbative expansion of (1) provides a discrete model of quantum gravity. It is then expected, that a description of continuous quantum geometry will necessarily include effects beyond the perturbative regime and will require a nonperturbative understanding of (1).

The construction of a full nonperturbative quantum field theory is rarely possible, but often it is already enough to construct a perturbation theory around a nonperturbative vacuum [7]. Moreover, if quantum fluctuations are not too strong, a nonperturbative vacuum can be reasonably well approximated by the minimum of the classical action S , called the minimizer. In

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that case, the mean-field approximation around the minimizer will lead to an effective field theory that will capture the nonperturbative regime of the model. For that reason, a study of minimizers of the Boulatov action is an important step towards a better understanding of continuous quantum geometry.

Despite their importance, however, the extrema of the Boulatov action are poorly understood in the literature. This is mostly due to the fact that the Euler-Lagrange equations of the Boulatov action are nonlinear differential equations that also involve integrals. Such equations are called integro-differential equations; generally, they are notoriously difficult to solve. In the Boulatov model these integro-differential equations can be formulated in terms of integral equations with an integral kernel given by the Wigner 6J-symbol. A solution of the extremal equations then requires full control of zeros of the 6J-symbol, which remains an open problem despite many decades of research [8–11]. This makes the complete analysis of the full problem out of reach.

In addition to this, there seems to be no uniform agreement on the signs of the coupling constants in GFT models. For instance, the convention used in renormalization analysis [12] is opposite to the one used in the context of the condensate cosmology investigations [13–21]. Despite the differences in the sign convention both analysis rely on the existence of global or at least local minimizers and for that reason require a good understanding of the extrema in GFT.

In this work, we address the minimizers of the Boulatov action augmented by a Laplace-Beltrami operator, hereafter called dynamical Boulatov action [22]. To make the problem tractable, we look for minimizers in the space of left and right invariant fields corresponding to equilateral triangles. Section II gives the definition of the model and the space of functions considered in this paper. On this space the Euler-Lagrange equations of the action become solvable, allowing us to provide a full characterization of solutions in section III A. We then identify the parameter regimes in which the action admits minima and characterize the minimizers in section III B. Our main result regarding the

extrema is presented in theorem 1 and the subsequent discussion. The characterization of minimizers is provided in theorem 2. Implications of our results on the quantum theory is discussed in section IV. A closing appendix gathers useful identities and the proofs of some statements in the text.

II. The dynamical Boulatov action

Let $\mathcal{C}^\infty(M)$ be the space of smooth, real-valued functions defined on the compact Lie group $M = \text{SU}(2)^{\times 3}$. The components of elements of M are denoted by a subindex such that

$$x = (x_1, x_2, x_3) \in M = \text{SU}(2)^{\times 3}. \quad (2)$$

Define the space of functions \mathcal{S} , such that any function f in \mathcal{S} satisfies:

Right invariance: for any $R \in \text{SU}(2)$ and any $x \in M$, $f(x_1 R, x_2 R, x_3 R) = f(x_1, x_2, x_3)$.

Cyclicity: for any $x \in M$,

$$f(x_1, x_2, x_3) = f(x_2, x_3, x_1) = f(x_3, x_1, x_2).$$

We call the space \mathcal{S} , the space of right and cyclic invariant functions.

The dynamical Boulatov action is a function $S_{m,\lambda}$ on \mathcal{S} , given by the integral

$$\begin{aligned} S_{m,\lambda}(\varphi) = & \frac{1}{2} \int_M dx \varphi(x) (-\Delta + m^2) \varphi(x) \quad (3) \\ & + \frac{\lambda}{4!} \int_{M^{\times 4}} dx dy dz dw \text{Tet}(x, y, z, w) \\ & \times \varphi(x) \varphi(y) \varphi(z) \varphi(w), \end{aligned}$$

where m^2 and λ are real, possibly negative, coupling constants, dx is the Haar measure on M , Δ is the Laplace-Beltrami operator on M with the canonical metric,¹ and the integral kernel Tet is

¹ We include the Laplace-Beltrami operator in the action, for a consistent implementation of a renormalization scheme [22].

given by

$$\begin{aligned} \text{Tet}(x, y, z, w) &= \delta(x_1 y_1^{-1}) \delta(x_2 z_1^{-1}) \delta(x_3 w_1^{-1}) \\ &\quad \times \delta(y_2 w_3^{-1}) \delta(y_3 z_2^{-1}) \\ &\quad \times \delta(z_3 w_2^{-1}). \end{aligned} \quad (4)$$

This kernel encodes the combinatorics of a tetrahedron (fig. 1) and is symmetric under cyclic permutations of its arguments.

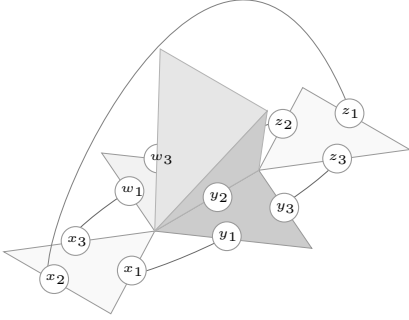


Figure 1: Combinatorics of a tetrahedron

To address the variational problem, we introduce a topology on the space \mathcal{S} , that is the one given by the family of semi-norms

$$\|f\|_n \doteq \sup_{x \in M} |\Delta^n f(x)|. \quad (5)$$

With this topology, \mathcal{S} is a locally convex topological space, with the neighborhood base given by semi-balls [23],

$$N_{\epsilon, n}(0) = \{\|f\|_n < \epsilon \mid f \in \mathcal{S}\}, \quad (6)$$

for $n \in \mathbb{N}$ and $\epsilon > 0$.

By the Peter-Weyl theorem, every $f \in \mathcal{S}$ can be written as

$$f(x) = \sum_{J \in \mathfrak{J}} f^J \mathcal{X}^J(x). \quad (7)$$

where $J = (j_1, j_2, j_3)$ belongs to $\mathfrak{J} \doteq \left(\frac{\mathbb{N}}{2}\right)^{\times 3}$, the space of triplets of positive half-integers and $\{\mathcal{X}^J\}_{J \in \mathfrak{J}}$ is a set of left and right invariant matrix coefficients on M (see appendix A 3 for the definition of \mathcal{X}^J). By [23, theorem 3] the sequence

of coefficients $(f^J)_{J \in \mathfrak{J}}$ is a rapidly decreasing sequence of real numbers and the equality is understood such that the right hand side of the equation converges to $f(x)$ in the aforementioned topology.

Leading the analysis further, we will restrict the space \mathcal{S} by requiring two additional conditions:

Left invariance: for any $L \in \text{SU}(2)$, $x \in M$ and $f \in \mathcal{S}$, $f(Lx_1, Lx_2, Lx_3) = f(x_1, x_2, x_3)$.

Equilateral condition: let $f \in \mathcal{S}$, then f is an equilateral function if its non-vanishing Peter-Weyl coefficients are of the form $(f^{(j,j,j)})_{j \in \mathbb{N}}$.

We denote the restriction of \mathcal{S} to left invariant equilateral functions by \mathcal{S}_{EL} and the space of equilateral triples by $\mathfrak{J}_{\text{EL}} = \{(j, j, j) \mid j \in \mathbb{N}\}$. Note, that \mathfrak{J}_{EL} contains only integer multi-indices, since for any half-integer j the matrix coefficients vanish (see appendix A 3),

$$\mathcal{X}^{(j,j,j)} = 0 \quad \text{with } j = \frac{2n+1}{2} \quad n \in \mathbb{N}.$$

In the following we will sometimes use the notation $f \in \mathcal{S}_{(\text{EL})}$ and f^J with $J \in \mathfrak{J}_{(\text{EL})}$ to signal that the statement holds equally for \mathcal{S} and \mathcal{S}_{EL} and, correspondingly, with a set of indices belonging to \mathfrak{J} or to \mathfrak{J}_{EL} . For clearer notation we also define the square of the triple J as

$$J^2 \doteq j_1(j_1 + 1) + j_2(j_2 + 1) + j_3(j_3 + 1),$$

and its modulus as

$$|J| \doteq j_1 + j_2 + j_3. \quad (8)$$

Definition 1. A local minimizer of the action $S_{m, \lambda}$ on \mathcal{S}_{EL} is a field $\varphi \in \mathcal{S}_{\text{EL}}$, that for some $n \in \mathbb{N}$ and $\epsilon > 0$ satisfies

$$S_{m, \lambda}(\phi) \geq S_{m, \lambda}(\varphi), \quad (9)$$

for any $\phi \in N_{\epsilon, n}(\varphi) \cap \mathcal{S}_{\text{EL}}$. If condition (9) is satisfied on the whole space \mathcal{S}_{EL} we call the minimizer global.

In the following we will characterize all minimizers of the action $S_{m,\lambda}$ on \mathcal{S}_{EL} for the four different parameter regions

$$(a) \begin{matrix} m^2 < 0 \\ \lambda < 0 \end{matrix} \quad (b) \begin{matrix} m^2 > 0 \\ \lambda < 0 \end{matrix} \quad (c) \begin{matrix} m^2 > 0 \\ \lambda > 0 \end{matrix} \quad (d) \begin{matrix} m^2 < 0 \\ \lambda > 0 \end{matrix}$$

For each of the parameter regions we will characterize all extrema of the action $S_{m,\lambda}$ on \mathcal{S}_{EL} and identify, which of the extrema are minimizers.

A. Physical interpretation of the setting

We now briefly motivate the restrictions made in our analysis and point out the geometrical considerations behind the use of the space \mathcal{S}_{EL} .

The space \mathcal{S} : The space \mathcal{S} is the space of field configurations of the Boulatov model. Its graphical interpretation in terms of closed triangles is well-known in the GFT literature [4]. By the Peter-Weyl theorem we can decompose any smooth field f on $\text{SU}(2)$ in modes such that

$$f(x) = \sum_{J \in \mathfrak{J}} \sum_{\alpha, \beta = -J}^J f_{\alpha, \beta}^J \mathfrak{D}_{\alpha, \beta}^J(x), \quad (10)$$

where $\mathfrak{D}_{\alpha, \beta}^J$ are the Wigner-matrix coefficients for the product representation of M (see appendix A 2). To gain an intuition on the construction,

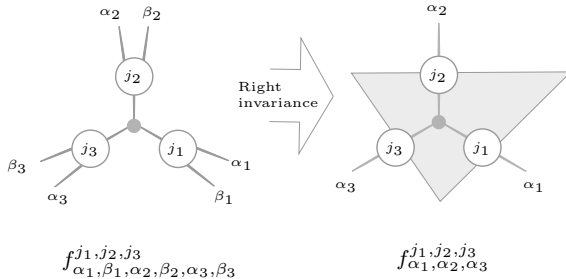


Figure 2: Graphical representation of right invariance.

we depict the Peter-Weyl coefficients by stranded lines, emanating from a single point. Then the right invariance of f ensures a closure of the dual edges to form a triangle (fig. 2). Hence, the right

invariance is necessary to give a geometric interpretation to the fields and it is thus crucial for the connection between the Boulatov group field theory and the Ponzano-Regge spin-foam model [1, 2, 6]. For this reason, the original Boulatov model, as well as any other geometrical model in GFT, includes this symmetry in their definition. In addition, the cyclic relabeling of the field arguments ensures the invariance of the triangles under cyclic relabeling of its edges (fig. 3).

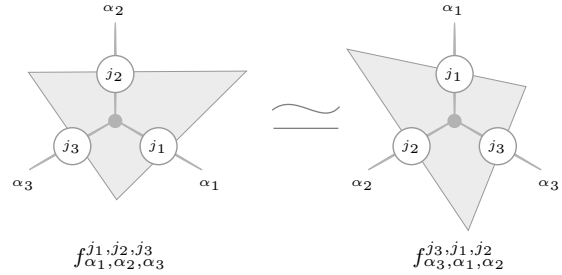


Figure 3: Graphical representation of invariance under cyclic permutations of its arguments.

The space \mathcal{S}_{EL} : The right invariance of smooth functions saturates one magnetic index per spin in the Peter-Weyl coefficients. The remaining magnetic indices connect to the right-invariant Wigner-matrix products. Hence, the associated triangles still carry one magnetic index for each j . This index “anchors” the triangle to the reference frame given by the Wigner-matrix coefficients (fig. 4), and therefore destroys the invariance of the triangles under rotations. To enforce rotational symmetry of the triangles, one requires the additional left invariance of the field [15, 16], such that for any $h \in \text{SU}(2)$ the field f satisfies

$$f(hx_1, hx_2, hx_3) = f(x). \quad (11)$$

This implements the invariance of f under rotation of all its arguments, which is interpreted as rotational invariance of the triangles described by the Peter-Weyl coefficients.

Some applications of GFT to cosmology demonstrated that this symmetry is needed to identify the domain space of the fields with the superspace of homogeneous spatial geometries [15]. Hence, this restriction plays a pivotal role

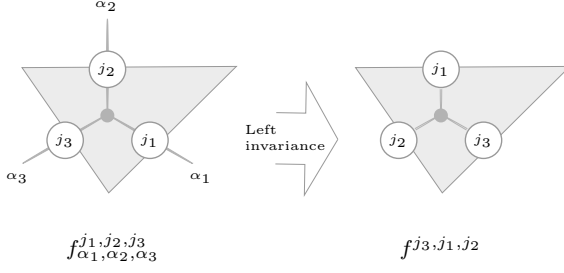


Figure 4: Graphical representation of left invariance.

to model the homogeneity of resulting cosmologies [13–17].

For such configurations, the equilateral condition of the triangles is then claimed to relate to their isotropy. The argument for this restriction states that we need to set all edges of the triangle to equal length in order to ensure equality in all directions (fig. 5). This condition is again required in the context of GFT condensate cosmology and is crucial for the recovery of a Friedmann-like dynamics [16, 17, 20, 21].

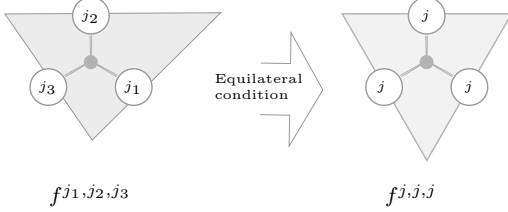


Figure 5: Graphical representation of the equilateral condition.

There are other reasons to consider the restricted space \mathcal{S}_{EL} . In GFT the action $S_{m,\lambda}$ defines statistical weights of a generating functional by means of a functional integral,

$$Z[J] = \int \mathcal{D}\varphi e^{-S_{m,\lambda}(\varphi) + \int J\varphi}. \quad (12)$$

It has been shown, however, that on \mathcal{S} the action $S_{m,\lambda}$ is generally not bounded from below, regardless of the parameter region [24]. For this reason, the above integral is dominated by those

field configurations that make the action $S_{m,\lambda}$ arbitrarily negative making (12) ill-defined. As we will show below, this problem gets resolved on \mathcal{S}_{EL} , where global minimizers of the action exist (at least for some parameter regions). This allows us to define (12), at least perturbatively, around these minimal field configurations. From this perspective, a restriction to the space \mathcal{S}_{EL} could lead to a well-defined statistical theory.

B. Extremal conditions and minimizers

Let $I \subset \mathbb{R}$ denote an interval containing zero; for $t \in I$ and $\varphi, f \in \mathcal{S}_{\text{EL}}$ a necessary condition for φ to be a local minimizer on \mathcal{S}_{EL} is given by

$$S'_{m,\lambda}(\varphi, f) \doteq \partial_t S_{m,\lambda}(\varphi + tf)|_0 = 0, \quad (13)$$

$$S''_{m,\lambda}(\varphi, f) \doteq \partial_t^2 S_{m,\lambda}(\varphi + tf)|_0 \geq 0, \quad (14)$$

for any $f \in \mathcal{S}_{\text{EL}}$.

In the following we will investigate the extremal condition (13) for the model (3). We will then check if some solutions are minimal and thus fulfill (14) and the condition in definition 1.

Proposition 1. $\varphi \in \mathcal{S}_{(\text{EL})}$ is an extremum of S if and only if the Peter-Weyl coefficients of φ — denoted φ^J — satisfy for any $J \in \mathfrak{J}_{(\text{EL})}$,

$$0 = (J^2 + m^2)\varphi^J + \frac{\lambda}{3!} \sum_{K \in \mathfrak{J}_{(\text{EL})}} \varphi^{j_1 k_2 k_3} \varphi^{j_2 k_3 k_1} \varphi^{j_3 k_1 k_2} \begin{Bmatrix} j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{Bmatrix}^2, \quad (15)$$

where $K = (k_1, k_2, k_3) \in \mathfrak{J}_{(\text{EL})}$.

Proof. see appendix B 1. \square

III. Extrema and minimizers

The extremal condition (15) is a nonlinear tensor equation with an integral kernel given by the 6J-symbol squared. To this issue comes the fact that the nontrivial zeros of the 6J-symbol are still under investigation, making (15) inherently difficult to solve in full generality. Some specific solutions for the case without the Laplace-Beltrami

operator and $\lambda < 0$ have been introduced in [25], but a systematic analysis of extrema was not performed therein.

Although the extremal condition (15) is difficult to solve on \mathcal{S} , it turns out to be solvable on \mathcal{S}_{EL} , because in this case the 6J-symbol significantly simplifies.

A. Extrema

In the following we will denote the Wigner 6J-symbol for $J \in \mathfrak{J}_{(\text{EL})}$ by

$$\{6J\} \doteq \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_1 & j_2 & j_3 \end{Bmatrix}, \quad (16)$$

and define the space $\mathfrak{J}_{(\text{EL})}^{\text{S}}$ of J 's such that

$$\mathfrak{J}_{(\text{EL})}^{\text{S}} = \{J \in \mathfrak{J}_{(\text{EL})} \mid \text{with } \{6J\} \neq 0\}. \quad (17)$$

In order to characterize the extrema of the action, we define the space of extremal sequences. Let $C = (C^J)_{J \in \mathfrak{J}_{(\text{EL})}}$ denote the sequence of (possibly complex) numbers such that for $J \in \mathfrak{J}_{(\text{EL})}^{\text{S}}$

$$C^J \in \left\{ 0, \pm \frac{1}{|\{6J\}|} \sqrt{-\frac{3!}{\lambda} (J^2 + m^2)} \right\} \quad (18)$$

and for $J \in \mathfrak{J}_{(\text{EL})} / \mathfrak{J}_{(\text{EL})}^{\text{S}}$.

$$C^J = \begin{cases} r \in \mathbb{R} & \text{if } J^2 = -m^2 \\ 0 & \text{otherwise} \end{cases} \quad (19)$$

Since $J^2 > 0$, the first case in (19) can happen only when m^2 is negative and for $J \in \mathfrak{J}_{(\text{EL})}$, m^2 has to be an even integer. For simplicity, we will exclude this case in the following analysis, because it requires a strong fine-tuning on the parameter m^2 . It is convenient to define the length ℓ of the sequence C such that

$$\ell(C) = \sum_{J \in \mathfrak{J}_{\text{EL}}} |\text{sgn}(C^J)|, \quad (20)$$

with the convention $\text{sgn}(0) = 0$.

Definition 2. We define the space of extremal sequences as

$$\mathcal{E}_{m,\lambda} = \left\{ C = (C^J)_{J \in \mathfrak{J}_{\text{EL}}} \mid C^J \in \mathbb{R}, \ell(C) < \infty \right\},$$

where the coefficients of each sequence are of the form (18).

This space of course depends on the values of m^2 and λ , since different choices of these parameters may violate the reality condition $C^J \in \mathbb{R}$.

$\mathcal{E}_{m,\lambda}$ fully characterizes the space of extrema of the action as states the following theorem.

Theorem 1. For any $C \in \mathcal{E}_{m,\lambda}$ the field $\varphi \in \mathcal{S}_{\text{EL}}$

$$\varphi(x) = \sum_{J \in \mathfrak{J}_{\text{EL}}} C^J \mathcal{X}^J(x) \quad (21)$$

is an extremum of the action $S_{m,\lambda}$. Moreover, every equilateral extremum of $S_{m,\lambda}$ is of the above form.

Proof. see appendix B2. \square

We denote the space of extremal functions by $\tilde{\mathcal{E}}_{m,\lambda}$. It is worth mentioning that, despite the nonlinearity of the Euler-Lagrange equations, its solutions form a vector space over $(\mathbb{Z}_3, +, \cdot)$.

Corollary. The space $\tilde{\mathcal{E}}_{m,\lambda}$ is a vector space over the discrete algebraic field $(\mathbb{Z}_3, +, \cdot)$.

Proof. Denote the space of sequences with finitely many non-zero elements over \mathbb{Z}_3 by $c_{00}(\mathbb{Z}_3)$. Clearly, it is a vector space over \mathbb{Z}_3 . Consider the map

$$\begin{aligned} \mathcal{I} : \tilde{\mathcal{E}}_{m,\lambda} &\rightarrow c_{00}(\mathbb{Z}_3) \\ \varphi &\mapsto (\text{sgn}(C^1), \text{sgn}(C^2), \dots), \end{aligned}$$

with the convention $\text{sgn}(0) = 0$. \mathcal{I} is one-to-one on its image, however, it may not be onto $c_{00}(\mathbb{Z}_3)$ simply because the nontrivial zeros of the 6J-symbol are not fully characterized. Nevertheless, the image of \mathcal{I} is algebraically closed and forms a subspace of $c_{00}(\mathbb{Z}_3)$. For any $s = (s_0, s_1, \dots) \in \mathcal{I}(\tilde{\mathcal{E}}_{m,\lambda})$, the inverse mapping is given by

$$\mathcal{I}^{-1} : s \mapsto [\mathcal{I}^{-1}s](x) = \sum_{j \in \mathbb{N}} \text{sgn}(s_j) |C^{J_j}| \mathcal{X}^{J_j}(x),$$

where $J_j = (j, j, j)$, $j \in \mathbb{N}$, with

$$|C^J| = \frac{1}{|\{6J\}|} \left| \sqrt{-\frac{3!}{\lambda} (J^2 + m^2)} \right|. \quad (22)$$

Since there are only finitely many non-zero coefficients, $s_j \neq 0$, the sum trivially converges in \mathcal{S}_{EL} . Since \mathcal{I} is linear it is an isomorphism between $\tilde{\mathcal{E}}_{m,\lambda}$ and $\mathcal{I}(c_{00}(\mathbb{Z}_3))$.

We define the sum on $\tilde{\mathcal{E}}_{m,\lambda}$ by

$$\varphi_1 +_{\mathbb{Z}_3} \varphi_2 \doteq \mathcal{I}^{-1}(\mathcal{I}(\varphi_1) + \mathcal{I}(\varphi_2)) \quad (23)$$

□

We now discuss the space of extremal sequences according to different parameter regions, whose major difference is captured by the sign of the radicand in (18). We obtain the four cases:

(a) $m^2 < 0$, $\lambda < 0$: the radicand is positive only if

$$J^2 - |m^2| = 3j(j+1) - |m^2| \geq 0, \quad (24)$$

which is the case when j satisfies

$$j_{\min} = \left\lceil \frac{1}{6} \left(\sqrt{9 + 12|m^2|} - 3 \right) \right\rceil \leq j, \quad (25)$$

where $\lceil \cdot \rceil$ denotes the ceiling function. The space of extremal sequences contains infinitely many sequences of the form

$$(0, \dots, 0, C^{J_{\min}}, C^{J_{\min}+1}, \dots),$$

where we used the notation $J_{\min} + n \doteq (j_{\min} + n, j_{\min} + n, j_{\min} + n)$ for $n \in \mathbb{N}$, with finitely many non-zero elements C^J .

(b) $m^2 > 0$, $\lambda < 0$: all coefficients C^J are real. The space of extremal sequences can be written as

$$\mathcal{E}_{m,\lambda} = \left\{ \left(C^{(0,0,0)}, C^{(1,1,1)}, \dots \right) \mid \ell(C) < \infty \right\}.$$

(c) $m^2 > 0$, $\lambda > 0$: the reality condition $C^J \in \mathbb{R}$ then requires $C^J = 0$ for all $J \in \mathfrak{J}_{\text{EL}}$. The space of extremal sequences contains a single zero-sequence

$$\mathcal{E}_{m,\lambda} = \{(0, 0, 0, \dots)\}.$$

(d) $m^2 < 0$, $\lambda > 0$: the radicand is positive only if

$$3j(j+1) - |m^2| \leq 0, \quad (26)$$

or equivalently for j satisfying,

$$0 \leq j \leq \left\lfloor \frac{1}{6} \left(\sqrt{9 + 12|m^2|} - 3 \right) \right\rfloor = j_{\max} \quad (27)$$

where $\lfloor \cdot \rfloor$ denotes the floor function. In this case $\mathcal{E}_{m,\lambda}$ contains finitely many sequences of the form

$$\left(C^{(0,0,0)}, \dots, C^{J_{\max}}, 0, 0, \dots \right),$$

where $J_{\max} = (j_{\max}, j_{\max}, j_{\max}) \in \mathfrak{J}_{\text{EL}}$.

At this point, a few comments are in order: according to the geometrical interpretation in the previous section, each Fourier mode can be interpreted as a triangle with the edge length given by j . The area of the triangle is then measured in terms of J^2 . In the parameter regime (d) relation (26) provides an upper bound on the possible j 's for the extrema of the action. Hence, in this case $|m^2|$ can be interpreted as the bound on the area of the triangles determined by the extremal solutions. This is an interesting geometrical fact that deserves further investigation.

A second remark is that the method of resolution restricting to equilateral configurations used to tackle (15) certainly exports to GFT models on higher dimensional manifolds $M = G^{\times D}$ with $G = \text{SU}(2), \text{SO}(4)$ and $D \in \mathbb{N}$. We expect that a similar results as in (18) will hold if we replace the 6J-symbol by the appropriate Wigner symbol and replace the square root by the $D-2$ root. However, the search of minimizers for these theories as performed in the subsequent analysis might be different.

B. Minimizers

We now seek the minimizers of the action and show that only two parameter regions admit global minimizers.

First, notice that in the case, $m^2 < 0$, $\lambda > 0$, the value of $|m^2|$ can determine, whether or not the action $S_{m,\lambda}$ is bounded from below. To agree with this, assume the first nontrivial zero of the 6J-symbol to be at $J_0 \in \mathfrak{J}_{EL}$ and choose a function $f(x) \doteq f^{J_0} \mathcal{X}^{J_0}(x)$ with $f^{J_0} \in \mathbb{R}$. Then, for $|m^2| > J_0^2$ the action evaluated at f yields

$$S_{m,\lambda}(f) = (f^{J_0})^2 (J_0^2 - |m^2|) < 0. \quad (28)$$

Hence, the action can become arbitrarily negative and thus is unbounded from below. On the other hand, for $|m^2| < J_0^2$ the action has a global minimum as we will show in the following.

In order to give a general classification of solutions we need to exclude cases when the 6J-symbol vanishes. A quick numerical analysis shows that for $|m^2| \leq 10^9$, the space of nontrivial zeros of the 6J-symbol with $J^2 \leq |m^2|$ is empty. Therefore, theorem 2 captures all possible solutions up to this order. In fact, we conjecture that for equilateral configurations, $\mathfrak{J}_{EL}/\mathfrak{J}_{EL}^S = \emptyset$, and our theorem holds for any value of $|m^2|$.

Theorem 2. *Let $|m^2|$ be such that for $j \leq j_{\max}$ every $J \in \mathfrak{J}_{EL}^S$ and such that there is no $J \in \mathfrak{J}_{EL}/\mathfrak{J}_{EL}^S$ such that $J^2 - |m^2| = 0$. Then the equilateral extrema of the dynamical Boulatov action are of the following type:*

- (a) For $m^2 < 0$, $\lambda < 0$, all extrema are saddle points.
- (b) For $m^2 > 0$, $\lambda < 0$, all nontrivial extrema are saddle points and the trivial extremum, $\varphi = 0$, is a local minimizer on \mathcal{S}_{EL} .
- (c) For $m^2 > 0$, $\lambda > 0$ the unique trivial extremum is a global minimizer on \mathcal{S}_{EL} .
- (d) For $m^2 < 0$, $\lambda > 0$ there are $2^{j_{\max}}$ global minimizers on \mathcal{S}_{EL} given by extremal sequences $C \in \mathcal{E}_{m,\lambda}$ with maximal length, $\ell(C) = j_{\max}$. Any other extremum of length $\ell(C) < j_{\max}$ is a saddle point.

Proof of theorem 2. In the following, let $\varphi(x)$ denote an extremum and let $f \in \mathcal{S}_{EL}$ be a generic function with the Peter-Weyl decomposition given by $f(x) = \sum_{J \in \mathfrak{J}_{EL}} f^J \mathcal{X}^J(x)$. We

remind here that a necessary condition for an extremum $\varphi(x)$ to be a minimizer (maximizer, respectively) is given by

$$S''_{m,\lambda}(\varphi, f) \geq 0 \quad \left(S''_{m,\lambda}(\varphi, f) \leq 0, \text{ resp.} \right),$$

for any $f \in \mathcal{S}_{EL}$. In the Peter-Weyl decomposition the second variation recasts as

$$S''_{m,\lambda}(\varphi, f) = \sum_{J \in \mathfrak{J}_{EL}} (f^J)^2 \left((J^2 + m^2) - \frac{\lambda}{2} \sum_{K \in \mathfrak{J}_{EL}^S} \delta_{J,K} \varphi^K \{6K\}^2 \right),$$

where φ^K is the Peter-Weyl coefficient of the extremum φ . The above condition is necessary but not sufficient, nevertheless, it turns out to be useful to exclude some extrema.

Case (a) ($m^2 \leq 0$, $\lambda \leq 0$): By theorem 1, extremal solutions contain only finitely many non-zero Fourier coefficients. Therefore it is possible to find $J_{>} \in \mathfrak{J}_{EL}$ such that $J_{>}^2 - |m^2| > 0$ and $\varphi^{J_{>}} = 0$. Choosing $f_{>}(x) \doteq f^{J_{>}} \mathcal{X}^{J_{>}}(x)$ the second variation gives

$$S''_{m,\lambda}(\varphi, f_{>}) = (f^{J_{>}})^2 (J_{>}^2 - |m^2|) > 0, \quad (29)$$

which violates the maximizer condition.

To see that the minimizer condition is also violated, choose $f_{<}(x) \doteq f^{J_{<}} \mathcal{X}^{J_{<}}(x)$ such that $J_{<}^2 - |m^2| < 0$. Then the second variation is written as

$$S''_{m,\lambda}(\varphi, f_{<}) = (f^{J_{<}})^2 (J_{<}^2 - |m^2|) \leq 0. \quad (30)$$

Hence, each extremum in this parameter region violates the minimizer and the maximizer condition and therefore is a saddle point.

Case (b) ($m^2 \geq 0$, $\lambda \leq 0$): For the nontrivial minimizer the above argument can also be applied in this case. Choosing the functions $f_{>}(x)$ and $f_{<}(x)$ as above we find

$$S''_{m,\lambda}(\varphi, f_{>}) = (f^{J_{>}})^2 (J_{>}^2 + |m^2|) > 0,$$

and

$$S''_{m,\lambda}(\varphi, f_{<}) = -2 (f^{J_{<}})^2 (J_{<}^2 + |m^2|) < 0.$$

Hence, nontrivial extrema are saddle points. For the trivial extremum the second variation of $S_{m,\lambda}$ reads for any $f \in \mathcal{S}_{\text{EL}}$

$$S''_{m,\lambda}(0, f) = \sum_{J \in \mathfrak{J}_{\text{EL}}} (f^J)^2 (J^2 + |m^2|) \geq 0,$$

and the necessary condition is satisfied. Indeed, the trivial extremum is a local minimum. To prove this we first notice that the Peter-Weyl transform is a topological isomorphism from \mathcal{S}_{EL} to the space of rapidly decreasing sequences $\mathcal{S}(\mathbb{N})$ with topology given by the family of semi-norms [23, theorem 4],

$$\| (f^J)_{J \in \mathfrak{J}_{\text{EL}}} \|_n = \sup_{J \in \mathfrak{J}_{\text{EL}}} |J^n f^J|. \quad (31)$$

The action evaluated at f becomes

$$S_{m,\lambda}(f) = \sum_{J \in \mathfrak{J}_{\text{EL}}} (f^J)^2 (J^2 + m^2) - \frac{\lambda}{4!} \sum_{J \in \mathfrak{J}_{\text{EL}}} (f^J)^4 \{6J\}^2 \quad (32)$$

Since the Wigner-6J-symbol is upper-bounded by 1, we can estimate

$$\begin{aligned} S_{m,\lambda}(f) &\geq \sum_{J \in \mathfrak{J}_{\text{EL}}} (f^J)^2 \left((J^2 + |m^2|) - \frac{\lambda}{4!} (f^J)^2 \right) \\ &\geq \sum_{J \in \mathfrak{J}_{\text{EL}}} (f^J)^2 \left(m^2 - \frac{\lambda}{4!} (f^J)^2 \right). \end{aligned} \quad (33)$$

Since Peter-Weyl transform is a topological isomorphism, we get for any $f \in \mathcal{S}_{\text{EL}}$ with $\|f\|_0 \leq \sqrt{\frac{4!m^2}{|\lambda|}}$, an estimate on the Fourier coefficients

$$|f^J| \leq \| (f^J)_{J \in \mathfrak{J}_{\text{EL}}} \|_0 \leq \sqrt{\frac{4!m^2}{|\lambda|}}. \quad (34)$$

Inserting this bound in (33) we obtain

$$S_{m,\lambda}(f) \geq 0 = S_{m,\lambda}(0). \quad (35)$$

Hence, in the neighborhood $N_{\epsilon,0} \cap \mathcal{S}_{\text{EL}}$ with $\epsilon = \sqrt{\frac{4!m^2}{|\lambda|}}$ the trivial extremum is a minimizer.

Case (c) ($m^2 > 0, \lambda > 0$): In this case the space of extremal sequences contains only the zero-sequence, leading to the trivial extremum $\varphi(x) = 0$. Denoting the quadratic part of the action in (3) by $Q_m(f)$ and the interaction part by $\lambda I(f)$ such that

$$S_{m,\lambda}(f) = Q_m(f) + \lambda I(f), \quad (36)$$

we have for any $f \in \mathcal{S}_{\text{EL}}$

$$\begin{aligned} Q_m(f) &= \sum_{J \in \mathfrak{J}_{\text{EL}}} (f^J)^2 (J^2 + m^2) \geq 0 \\ \lambda I(f) &= \frac{\lambda}{4!} \sum_{J \in \mathfrak{J}_{\text{EL}}} (f^J)^4 \{6J\}^2 \geq 0. \end{aligned}$$

Hence,

$$S_{m,\lambda}(0) = 0 \leq S_{m,\lambda}(f) \quad \forall f \in \mathcal{S}_{\text{EL}}. \quad (37)$$

We obtain a global minimizer, since the minimal condition is satisfied on the whole \mathcal{S}_{EL} .

Case (d) ($m^2 < 0, \lambda > 0$): For any $f \in \mathcal{S}_{\text{EL}}$ the action evaluated at f gives

$$\begin{aligned} S_{m,\lambda}(f) &= \frac{1}{2} \sum_{J \in \mathfrak{J}_{\text{EL}}} (f^J)^2 (J^2 - |m^2|) \\ &\quad + \frac{\lambda}{4!} \sum_{J \in \mathfrak{J}_{\text{EL}}} (f^J)^4 \{6J\}^2 \end{aligned}$$

Splitting f such that $f(x) = f^-(x) + f^+(x)$ with

$$\begin{aligned} f^-(x) &= \sum_{\substack{|J| \leq 3j_{\text{max}} \\ J \in \mathfrak{J}_{\text{EL}}}} f^J \mathcal{X}^J(x) \\ f^+(x) &= \sum_{\substack{|J| > 3j_{\text{max}} \\ J \in \mathfrak{J}_{\text{EL}}}} f^J \mathcal{X}^J(x), \end{aligned}$$

we have

$$S_{m,\lambda}(f) = S_{m,\lambda}(f^- + f^+) \geq S_{m,\lambda}(f^-).$$

Hence, verifying the minimizer condition, it is enough to show that

$$S_{m,\lambda}(\varphi) \leq S_{m,\lambda}(f^-). \quad (38)$$

The space of functions of the form f^- is finite-dimensional and we can use the usual minimization procedure for functions. More specifically, let $s_J : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that

$$s_J(f^J) = (f^J)^2 \left[\frac{1}{2} (J^2 - |m^2|) + \frac{\lambda}{4!} (f^J)^2 \{6J\}^2 \right].$$

The action $S_{m,\lambda}(f^-)$ is smallest when each s_J is minimal on \mathbb{R} for each $J \leq J_{\max}$. Taking the first and second derivative of s_J we see that the minimum is achieved by the coefficients C^J from (18). Hence, an extremum given by an extremal sequence of maximal length is a global minimizer on the whole \mathcal{S}_{EL} .

If φ is given by an extremal sequence C of length $\ell(C) < j_{\max}$, then there exists a \mathcal{X}^{J_0} with $J_0 \leq J_{\max}$ and $\varphi^{J_0} = 0$. For $\delta \in \mathbb{R}$ define the function

$$g(x) = \varphi(x) + \delta \cdot \mathcal{X}^{J_0}(x). \quad (39)$$

Inserting g into the action we get

$$S_{m,\lambda}(g) = S_{m,\lambda}(\varphi) + \delta^2 \left[\frac{1}{2} (J_0^2 - |m^2|) + \frac{\lambda}{4!} \delta^2 \{6J_0\}^2 \right].$$

If δ^2 is in the range $0 < \delta < 2C^{J_0}$ the square bracket is negative and it follows

$$S_{m,\lambda}(g) < S_{m,\lambda}(\varphi). \quad (40)$$

Moreover, for any $\epsilon > 0$ and $\delta < \frac{\epsilon}{J_0^{2n}}$ we have

$$\begin{aligned} \|g - \varphi\|_n &= \delta \sup_{x \in M} |\Delta^n \mathcal{X}^{J_0}(x)| \\ &= \delta \sup_{x \in M} |J_0^{2n} \mathcal{X}^{J_0}(x)| < \epsilon, \end{aligned}$$

since the characters are bounded by one, $|\mathcal{X}^{J_0}(x)| \leq 1$. Hence, $g \in N_{\epsilon,n}(\varphi)$. For any $\epsilon > 0$ choosing $\delta < \min\left(\frac{\epsilon}{J_0^{2n}}, C^{J_0}\right)$ we get

$$S_{m,\lambda}(f) < S_{m,\lambda}(\varphi). \quad (41)$$

This shows that we can find a function g in any neighborhood of φ that decreases the value of the action, and hence, φ is not a minimizer. \square

IV. Conclusion

We investigated the minimizers of the dynamical Boulatov action in four different parameter regions of the coupling constants. Our analysis is restricted to the space of smooth, equilateral, left and right invariant functions, also invariant under cyclic permutations of its variables, \mathcal{S}_{EL} . This restriction ensures that the action is bounded from below for some parameter regions. Moreover, it is motivated by cosmology studies on GFT.

In this article, we have shown that the very same restrictions allow us to solve the Euler-Lagrange equations for the dynamical Boulatov action and lead to a complete characterization of minimizers on the restricted space. Our result characterizes the space of solutions by extremal sequences of finite length and shows that it forms a vector space over \mathbb{Z}_3 , which is surprising for the set of solutions to a nonlinear integro-differential equation. Furthermore, in the most interesting parameter region (d), the non-vanishing Fourier modes of extremal solutions are bounded by the coupling constant m^2 , which suggests a connection between m^2 and the area of the triangle of the largest Peter-Weyl mode of the GFT field.

Our analysis shows that the region (a) does not have any minimizers on \mathcal{S}_{EL} , which makes this parameter region perhaps the least suitable for the definition of the statistical measure in (1). For the parameter regions (b) and (c) there is a single (local respectively global) minimizer given by the trivial extremum, $\varphi = 0$. Finally, in the region (d) the action has $2^{j_{\max}}$ degenerate global minimizers, where j_{\max} is a function of the coupling constant m^2 . The rich structure of global minima makes this region most interesting for further investigations, especially for the statistical theory.

On the space of equilateral functions only two possible parameter regions (c) and (d) allow for the presence of global minimizers, and hence could lead to a meaningful definition of a non-perturbative statistical measure.

Case (c) admits a single global minimizer $\varphi = 0$. Perturbation theory around this minimizer leads to the perturbation theory in the coupling constant λ and is used in the GFT literature to

draw a connection to spin-foam models. Hence, our analysis would suggest that this regime is suitable for such relation.

Case (d), on the other hand, may suggest more structure for the quantum theory: a degenerate global minimum could lead to instantons or symmetry breaking in the corresponding statistical field theory in the following sense:

Instantons: The full nonperturbative formulation of a model is given by the minimizer of its quantum effective action. The latter is commonly assumed to be convex [26] and therefore admits a single, unique minimizer. Hence, the difference between the minimizers of the classical and the quantum effective action becomes apparent, especially in the case when the classical action admits degenerate minimizers. In this case, a perturbative description around any of the minimizers of the classical action does not capture the nonperturbative effects of the theory. In quantum field theory these nonperturbative effects can be understood as “tunneling” between the perturbative vacua, where the tunneling probability is described by the instanton action. Thus, the degenerate structure of global minimizers in our case, suggests the necessity of instantons in the statistical formulation of GFT at least for the parameter region (d) (for a similar result see [19]).

Symmetry breaking: this mechanism happens when the classical action admits degenerate global minimizers — related by a symmetry of the classical action — but the tunneling probability between them vanishes. As we already mentioned, the tunneling probability is described by the instanton action, which in ordinary field theory is often proportional to the volume of the base manifold. On a manifold with finite volume the tunneling probability is therefore finite. This often leads to the statement that spontaneous symmetry breaking can not occur in quantum field theories in a box. This realization, however, contains further assumptions that are satisfied in ordinary field theories but do not hold for GFT. In fact, it has been recently shown that even on the compact base manifold, $M = \text{SU}(2)^d$ the tunneling between different perturbative minima can vanish [27], leading to a similar phenomenon of symmetry breaking. In order to talk about sym-

metry breaking, we need to identify the symmetry, which in our case, is given by a flip of the sign of at least one of the modes in the Peter-Weyl decomposition of the minimizer (this can be modeled as a \mathbb{Z}_2 -symmetry). Since the action is of even power in the fields, such a flip will not affect the value of the action and will correspond to a discrete symmetry. For this reason it is possible that the global minimizers of the action lead to the breaking of sign-flip symmetry. This needs to be investigated more rigorously in future work.

For ordinary local quantum field theories, a symmetry breaking mechanism can sometimes be related to a phase transition and the formation of a condensate. In particular, this could be the signal of a Bose-Einstein condensation just as expected for cosmology studies in GFT. A closer look at the solutions found for sector (d) shows that these might lead to intriguing perspectives. Indeed, the ‘particle’ number, used in cosmology, is computable in terms of the L^2 -norm of the minimizer. In the present situation, that very number proves to be bounded by the parameter m^2 :

$$\begin{aligned} N \doteq \|\varphi\|_{L^2} &= \frac{3!}{\lambda} \sum_{J \in \mathfrak{J}_{\text{EL}}^S}^{|J| \leq 3j_{\text{max}}} \frac{1}{\{6J\}^2} |J^2 - |m^2|| \\ &\leq \frac{3! |m^2|}{\lambda} \sum_{J \in \mathfrak{J}_{\text{EL}}^S}^{|J| \leq 3j_{\text{max}}} \frac{1}{\{6J\}^2} \\ &\leq \frac{3! |m^2|}{\lambda} j_{\text{max}} C_{\text{max}}, \end{aligned}$$

with $C_{\text{max}} = \max_{J \in \mathfrak{J}_{\text{EL}}^S} (\{6J\}^{-2})$. For $|m^2| \gg 1$ we can approximate j_{max} further as $j_{\text{max}} \leq 2|m^2|$ and obtain a simpler bound on the L^2 -norm of the minimizers

$$N \leq \frac{12|m^4|}{\lambda} C_{\text{max}}. \quad (42)$$

The coupling constant m^2 (or $|m^4|/\lambda \gg 1$) could be large but that itself is not enough to ensure $N = \infty$. Nevertheless, starting from our solutions, a divergent parameter m^2 is a necessary condition for the divergent L^2 -norm.

We should mention here that minimizers with a divergent L^2 -norm are not captured by our analysis (dealing only with integrable functions), and

some modifications will be in order to also take into consideration these cases. One necessary modification would be to relax the smoothness condition of the minimizers and use the space of tempered distributions instead. This could be particularly interesting for GFT models without Laplace-Beltrami operator, which correspond to a topological BF-theory. Due to the distributional nature of minimizers their L^2 -norm will sometimes diverge making them potentially interesting for cosmological studies [19, 27] and spin-foam models [25]. The solutions of these GFT models must be addressed in a different way but certainly deserve further attention.

There are several models using tensor fields (with interesting properties such as perturbative renormalizability) which do not impose strong symmetry conditions on the fields. These models' interactions could also be radically different from that of Boulatov [28]. Their corresponding Euler-Lagrange equation (without 6J-symbols) still involves a nonlinear tensor like equation and it remains a difficult task to solve them. In this case, an approach to circumvent the nonlinearity and to obtain solution fields which are more general than equilateral configurations is to consider symmetric tensor fields and to decompose the field into its traceless part and the rest, namely vector-like components [29]. Such a decomposition could help to solve the extremal conditions on \mathcal{S} which might find applications in GFT studies of inhomogeneous and anisotropic quantum cosmologies.

On the other hand, the existence of global minima on \mathcal{S}_{EL} suggests that we can define a self-consistent statistical theory using only this space. This theory could potentially be well-defined due to the bound of the action on \mathcal{S}_{EL} and may have implications for cosmological studies of GFT.

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Appendix

A. Harmonic analysis on $SU(2)$

This appendix gathers the main identities on the harmonic analysis on $SU(2)$ repeatedly used throughout the text.

1. Properties of the Wigner matrices

In the following we present a list of properties for the Wigner matrix coefficients:

1. the Wigner matrices, denoted by $D^j(x)$, define an irreducible representation of $x \in SU(2)$ of dimension $d_j = 2j + 1$, with $j \in \{0, \frac{1}{2}, 1, \dots\}$. We denote the coefficients of this matrix by $D_{mn}^j(x)$ with $m, n \in \{-j, \dots, j\}$;
2. the set of Wigner matrix coefficients $\{D_{mn}^j(x)\}_{j,m,n}$ forms an orthogonal basis in L^2 such that

$$\int dx D_{m_1 n_1}^{j_1}(x) \bar{D}_{m_2 n_2}^{j_2}(x) = \frac{1}{d_{j_1}} \delta^{j_1 j_2} \delta_{m_1 m_2} \delta_{n_1 n_2}, \quad (\text{A1})$$

where $\bar{}$ denotes the complex conjugation;

3. the Wigner matrix coefficients form a basis of eigenfunctions for the Laplace-Beltrami operator Δ (defined with the canonical metric), such that

$$-\Delta D_{mn}^j(x) = j(j+1) D_{mn}^j(x); \quad (\text{A2})$$

4. the characters of the Wigner representation are defined by $\chi^j(x) \doteq \text{tr}(D^j(x)) = \sum_m D_{mm}^j(x)$;
5. they are smooth real-valued functions that satisfy $\chi^j(x) = \chi^j(x^{-1})$;

6. for $x_1, x_2 \in \text{SU}(2)$ they also obey the convolution relation

$$\int dh \chi^j(hx_1)\chi^l(x_2h) = \frac{\delta_{jl}}{d_j} \chi^j(x_2x_1^{-1}) \quad (\text{A3})$$

from which the orthogonality relation $\int dh \chi^j(h)\chi^l(h) = \delta_{jl}$ follows;

7. the Wigner 6J-symbol can be defined in terms of characters as [30],

$$\left\{ \begin{matrix} l_{01} & l_{02} & l_{03} \\ l_{23} & l_{13} & l_{12} \end{matrix} \right\}^2 = \int (dh)^4 \prod_{i < j}^3 \chi^{l_{ij}}(h_j h_i^{-1}). \quad (\text{A4})$$

2. Peter-Weyl transform

We briefly recall the most important properties of the Peter-Weyl transform and Wigner matrices, needed for the harmonic analysis on $\text{SU}(2)$. Let $\mathcal{C}^\infty(\text{SU}(2))$ be the space of smooth functions f on $\text{SU}(2)$ which is equipped with the topology given by semi-norms

$$\|f\|_n = \sup_{g \in \text{SU}(2)} |\Delta^n f(g)|, \quad (\text{A5})$$

with Δ the Laplace-Beltrami operator and $n \in \mathbb{N}$.

For any $f \in \mathcal{C}^\infty(\text{SU}(2))$ there exists a sequence of complex numbers (f_{mn}^j) with $j \in \frac{\mathbb{N}}{2}$ and $m, n \in \{-j, \dots, j\}$ and $D_{mn}^j(x)$ denote the Wigner matrix coefficients with $d_j = 2j + 1$ such that

$$\lim_{N \rightarrow \infty} \sum_{j=0}^N \sum_{m, n=-j}^j \sqrt{d_j} f_{mn}^j D_{mn}^j = f, \quad (\text{A6})$$

in the above topology. The sequence of Fourier coefficients (f_{mn}^j) is rapidly decreasing, i.e. for any $K \in \mathbb{N}$

$$\sup_j \left| j^K \sum_{\alpha, \beta=-j}^j \bar{f}_{\alpha\beta}^j f_{\alpha\beta}^j \right| < \infty. \quad (\text{A7})$$

If we call the space of rapidly decreasing sequences $\mathcal{S}(\mathbb{N})$, then (A7) defines a family of

semi-norms on $\mathcal{S}(\mathbb{N})$ and in the corresponding topology it becomes a Fréchet space. Then the Peter-Weyl transform $\mathcal{F} : \mathcal{C}^\infty(\text{SU}(2)) \rightarrow \mathcal{S}(\mathbb{N})$ is a topological isomorphism between the space of smooth functions and the space of rapidly decreasing sequences [23].

In our work, we deal with functions on three copies of $\text{SU}(2)$. For this reason, we introduce $M = \text{SU}(2)^{\times 3}$ as a Lie group with points (x_1, x_2, x_3) . The representations of M are given by product representations such that

$$\mathfrak{D}^{(j_1, j_2, j_3)} : \text{SU}(2)^{\times 3} \rightarrow L(V^{j_1} \otimes V^{j_2} \otimes V^{j_3}) \quad (\text{A8})$$

with

$$\mathfrak{D}^{(j_1, j_2, j_3)} = D^{j_1} \otimes D^{j_2} \otimes D^{j_3}, \quad (\text{A9})$$

where $L(V)$ denotes the space of linear maps on V a vector space.

It follows by the Peter-Weyl theorem that the matrix coefficients $\mathfrak{D}_{\alpha, \beta}^J(x)$ are dense in the space of smooth functions on M , where now J, α and β are multi-indices such that $J = (j_1, j_2, j_3)$ with $j_1, j_2, j_3 \in \frac{\mathbb{N}}{2}$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3), \beta = (\beta_1, \beta_2, \beta_3)$ such that $\alpha_i, \beta_i \in \{-j_i, \dots, j_i\}$ for $i \in \{1, 2, 3\}$.

3. Basis for left and right invariant functions

In the above notations, the left and right invariant functions on $M = \text{SU}(2)^{\times 3}$ are given by group averaging, such that for any $f \in \mathcal{C}^\infty(M)$, and any $x = (x_1, x_2, x_3) \in M$

$$\int dLdR f(Lx_1R, Lx_2R, Lx_3R), \quad (\text{A10})$$

with $L, R \in \text{SU}(2)$. In the Peter-Weyl decomposition a left and right invariant function f assumes the form

$$f(x) = \sum_J f^J \sqrt{d_{j_1} d_{j_2} d_{j_3}} \times \int dh \chi^{j_1}(x_1h) \chi^{j_2}(x_2h) \chi^{j_3}(x_3h), \quad (\text{A11})$$

where $J = (j_1, j_2, j_3) \in (\frac{\mathbb{N}}{2})^{\times 3}$, and χ^{j_i} denotes the character of representation of $SU(2)$ with dimension d_{j_i} . We denote the integral of the product of three characters by

$$\mathcal{X}^J(x) \doteq \sqrt{d_{j_1} d_{j_2} d_{j_3}} \int_M dh \chi^{j_1}(x_1 h) \chi^{j_2}(x_2 h) \chi^{j_3}(x_3 h).$$

It can be easily checked that \mathcal{X}^J has the following properties:

1. Using (A3) and reality of characters, the \mathcal{X}^J 's are real valued and form an orthonormal family with respect to the $L^2(M, dx)$ scalar product:

$$\int_M dx \mathcal{X}^J(x) \mathcal{X}^K(x) = \delta_{J,K}; \quad (\text{A12})$$

2. \mathcal{X}^J is proportional to the 3J-Wigner symbol with three equal j 's and sum over the magnetic indices and hence vanishes if j is not an integer;
3. Using (A11), the family of \mathcal{X}^J 's is dense in the space of left and right invariant functions, such that any left and right invariant function f can be written as

$$f(x) = \sum_{J \in \mathfrak{J}} f^J \mathcal{X}^J(x); \quad (\text{A13})$$

4. Using (A4), the 6J-symbol is given by a \mathcal{X}^J integral as

$$\begin{aligned} & \delta_{j_1, k_1} \delta_{j_2, l_1} \delta_{j_3, q_1} \delta_{q_2, l_3} \delta_{k_2, q_3} \delta_{k_2, l_3} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ q_2 & l_2 & k_2 \end{matrix} \right\}^2 \\ &= \int dx dy dz dw \text{Tet}(x, y, z, w) \\ & \quad \times \mathcal{X}^J(x) \mathcal{X}^K(y) \mathcal{X}^L(z) \mathcal{X}^Q(w), \end{aligned} \quad (\text{A14})$$

with $J = (j_1, j_2, j_3)$, $K = (k_1, k_2, k_3)$, $L = (l_1, l_2, l_3)$, $Q = (q_1, q_2, q_3)$.

Since we are interested in functions that are invariant under cyclic permutation we need to symmetrize the characters $\mathcal{X}^J(x)$. To achieve this,

we introduce the symmetrization operator

$$P\mathcal{X}^J(x) = \frac{1}{3} \sum_{\sigma \in \text{Cyc}} \mathcal{X}^{(j_{\sigma(1)}, j_{\sigma(2)}, j_{\sigma(3)})}(x), \quad (\text{A15})$$

where Cyc denotes the set of cyclic permutations of $\{1, 2, 3\}$. All aforementioned properties of \mathcal{X}^J can be adapted to $P\mathcal{X}^J(x)$ by including a normalized sum over cyclic permutations of indices. Since for the equilateral case we have $P\mathcal{X}^J(x) = \mathcal{X}^J(x)$, we simply use the notation $\mathcal{X}^J(x)$ for symmetric characters on \mathcal{S}_{EL} and on \mathcal{S} .

B. Proofs

1. Proof of proposition 1

Consider the action $S_{m,\lambda}$ (3) and $S'_{m,\lambda}$ (13); $\mathcal{S}_{(\text{EL})}$ means either \mathcal{S} (space of right invariant functions) or \mathcal{S}_{EL} (space of left and right invariant and equilateral functions). The following statement holds:

Lemma 1. *The field $\varphi \in \mathcal{S}_{(\text{EL})}$ is an extremum of $S_{m,\lambda}$ iff*

$$S'_{m,\lambda}(\varphi, \mathcal{X}^J) = 0 \quad (\text{B1})$$

for all $J \in \mathfrak{J}_{(\text{EL})}$.

Proof. Let φ be an extremum of $S_{m,\lambda}$, then the ‘‘only if’’ direction is obvious since for any $J \in \mathfrak{J}_{(\text{EL})}$ the functions \mathcal{X}^J are in $\mathcal{S}_{(\text{EL})}$.

For the ‘‘if’’ direction we observe the following: since the set $\{\mathcal{X}^J\}_{J \in \mathfrak{J}_{(\text{EL})}}$ is dense in $\mathcal{S}_{(\text{EL})}$, for any $f \in \mathcal{S}_{(\text{EL})}$ there exists a family of real numbers $\{f^J\}_{J \in \mathfrak{J}_{(\text{EL})}}$ such that the sequence of functions given for all $N \in \mathbb{N}$ as

$$f_N(x) = \sum_{J \in \mathfrak{J}_{(\text{EL})}}^{|J| < N} f^J \mathcal{X}^J(x), \quad (\text{B2})$$

converges to f . Then

$$c = \sup_{x \in M} \sup_{N \in \mathbb{N}} |f_N(x)|, \quad (\text{B3})$$

exists and dominates each f_N such that, $|f_N| \leq c$. Moreover, c , seen as a constant function on M , is integrable since M is compact.

For any $f \in \mathcal{S}_{(EL)}$ the extremal condition for the action $S_{m,\lambda}$ reads as

$$S'_{m,\lambda}(\varphi, f) = \int_M dx f(x) (-\Delta + m^2) \varphi(x) + \frac{\lambda}{3!} \int_{M \times 4} dx dy dz dw \text{Tet}(x, y, z, w) \times f(x) \varphi(y) \varphi(z) \varphi(w).$$

Using the Peter-Weyl decomposition for f , we can interchange the limit and the integral by the dominant convergence theorem (using the bound c) and obtain

$$S'_{m,\lambda}(\varphi, f) = \lim_{N \rightarrow \infty} \sum_J f^J S'_{m,\lambda}(\varphi, \mathcal{X}^J) = 0,$$

for any $f \in \mathcal{S}_{(EL)}$, from which the statement follows. \square

Corollary. $\varphi \in \mathcal{S}_{(EL)}$ is an extremum of S if and only if the Peter-Weyl coefficients of φ — denoted by φ^J — satisfy for any $J \in \mathfrak{J}_{(EL)}$,

$$(J^2 + m^2) \varphi^J + \frac{\lambda}{3!} \sum_{k_i} \varphi^{j_1 k_2 k_3} \varphi^{j_2 k_3 k_1} \varphi^{j_3 k_1 k_2} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{matrix} \right\}^2 = 0.$$

Proof. From lemma 1 the extremal condition is given by the variation in the basis direction \mathcal{X}^J for any $J \in \mathfrak{J}_{(EL)}$. Inserting the Peter-Weyl decomposition of φ in the action $S_{m,\lambda}(\varphi)$, interchanging the limit with the integral by the dominant convergence theorem and using the relation in (A14) we obtain the desired statement. \square

2. Proof of theorem 1

Theorem. For any $C \in \mathcal{E}_{m,\lambda}$ the field $\varphi \in \mathcal{S}_{EL}$

$$\varphi(x) = \sum_{J \in \mathfrak{J}_{EL}} C^J \mathcal{X}^J(x) \quad (\text{B4})$$

is an extremum of the action $S_{m,\lambda}$. Moreover, every equilateral extremum of $S_{m,\lambda}$ in \mathcal{S}_{EL} is of the above form.

Proof. To show that φ solves the extremal condition we need to show, by proposition 1, that each C^J satisfies (15), which follows by direct calculation.

Conversely, every equilateral function can be written as

$$f(x) = \sum_{J \in \mathfrak{J}_{EL}} A_J \mathcal{X}^J(x), \quad (\text{B5})$$

with $(A^J)_{J \in \mathfrak{J}_{EL}}$ being a rapidly decreasing sequence [23]. Using proposition 1, we find that the extremal solutions have coefficients A^J which satisfy

$$A^J \in \left\{ \pm \frac{1}{|\{6J\}|} \sqrt{-\frac{3!}{\lambda} (J^2 + m^2)}, 0 \right\}, \quad (\text{B6})$$

or $A^J \in \mathbb{R}$ for $J \in \mathfrak{J}_{EL}/\mathfrak{J}_{EL}^S$ with $J^2 + m^2 = 0$. If A^J is not trivial we can estimate its growth using the asymptotic behavior of 6J-symbols [31] as

$$A^J \sim \frac{j}{|\{6J\}|} \sim j^{\frac{5}{2}}. \quad (\text{B7})$$

However, for $(A^J)_{J \in \mathfrak{J}_{EL}}$ to be a rapidly decreasing sequence, the coefficients have to satisfy for any $n \in \mathbb{N}$,

$$\lim_{j \rightarrow \infty} |j|^n |A^J| \rightarrow 0. \quad (\text{B8})$$

This is only possible if $A^J = 0$ for all but finitely many $J \in \mathfrak{J}_{EL}$. \square

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