

Local Hamiltonians for one-dimensional critical models

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Conformal field theory has turned out to be a powerful tool to derive interesting lattice models with analytical ground states. Here, we investigate a class of critical, one-dimensional lattice models of fermions and hardcore bosons related to the Laughlin states. The Hamiltonians of the exact models involve interactions over long distances that are difficult to realize experimentally. This motivates us to study the properties of models with the same type of interactions, but now only between nearest and possibly next-nearest neighbor sites. Based on computations of wavefunction overlaps, entanglement entropies, and two-site correlation functions for systems of up to 32 sites, we find that the ground state is close to the ground state of the exact model. There is also a high overlap per site between the lowest excited states for the local and the exact models, although the energies of the low-lying excited states are modified to some extent for the system sizes considered. We also briefly discuss possibilities for realizing the local models in ultracold atoms in optical lattices.

I. INTRODUCTION

The physics that can be realized in strongly-correlated quantum many-body systems is rich and far from being fully understood. The systems are challenging to investigate, because the resources needed to do numerical studies of the systems are often at the borders of or beyond what can be done. Analytical models therefore play an important role in getting insight into the physics of the systems.

It has turned out that conformal field theory provides an interesting technique to construct different quantum many-body models on lattices, where the ground state, and sometimes also excited states, are known analytically [1, 2]. The idea is to interpret certain conformal field theory correlation functions as ground state wavefunctions and then use the mathematical properties of these objects to derive parent Hamiltonians of the states. The resulting Hamiltonians typically consist of terms that are few-body and nonlocal. The method has, for instance, been used to obtain 2D lattice models with ground states that are lattice versions of the Laughlin state at filling factor $1/q$, where $q = 2, 3, \dots$ [3]. These states are topologically ordered and can host Abelian anyons.

One can also obtain 1D versions of the lattice Laughlin models by doing the same derivation in one dimension [3]. The resulting models are critical and have interactions that decay as the inverse of the square of the distance ($1/r^2$). The model for $q = 2$ turns out to coincide with the famous Haldane-Shastry model [4–6]. The Haldane-Shastry model is interesting, since all the excited states of the model can be found analytically, and since the model allows for semionic excitations. It is likely that the excited states of the one-dimensional models for other values of q are also interesting and related to the anyons that can be obtained in the 2D versions of the models. One-dimensional quantum many-body models with nonlocal

interactions of the kind $1/r^2$, as well as truncated versions of them, have gained considerable interest over the years [1–12], exploring interesting aspects of integrable systems. A common feature of quantum $1/r^2$ systems is that the ground state can be exactly represented by a Jastrow-type wavefunction [1–12], namely a product of two-body functions.

One purpose of constructing analytical models of quantum many-body systems with interesting properties is to use them as a guide to find ways to implement the physics experimentally. In many cases, it may not be possible to directly implement the analytical model itself, but with some suitable modifications one may be able to simulate a model displaying the same physics. Ultracold atoms in optical lattices provide an interesting and flexible setup for simulating quantum models, and there is a rapid development in the field [13, 14]. As far as the models derived from conformal field theory are concerned, a challenging aspect for experimental implementation is that the models involve nonlocal interactions. This poses the question, whether the same physics can be realized in a model with only local interactions, and whether one can use the form of the exact model to predict a suitable choice of local interactions. In this article, we investigate this question for the family of 1D models related to the Laughlin states. We do the investigation with exact diagonalization, and we find that it is possible to obtain local Hamiltonians, whose low-energy eigenstates have high overlap per site with the eigenstates of the exact Hamiltonians.

Investigations of whether a nonlocal 1D Hamiltonian derived from conformal field theory can be replaced by a local Hamiltonian have been done for other systems previously with affirmative conclusions [1, 2, 15, 16]. These investigations have focused mainly on models with $SU(2)$ or $SU(n)$ symmetry or spin models with a symmetry when flipping all the spins and have looked only at ground state properties and mainly ground state overlaps. In the present article, we do a more thorough investigation for a family of models without these symmetries (the model with $q = 2$ has $SU(2)$ symmetry, but the others do not).

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The presence of symmetries simplifies the study, since the symmetry reduces the number of possible local interactions, and this makes it interesting to ask the question also for models with less symmetry. In addition to ground state overlap, we compute entanglement entropy and correlation functions for the ground states of the local models, which are important quantities for determining the physics of the models. The low-energy excited states are, however, also crucial when considering dynamics or systems at low, but nonzero, temperature. We therefore also compute the low-lying part of the energy spectra and overlaps for the low-lying excited states. Finally, we briefly discuss possibilities for implementing the local models in ultracold atoms in optical lattices. Implementing the models would allow the power law decay of the correlation functions and the logarithmic growth of entanglement entropy with subsystem size to be measured [17–19]. In addition, one could investigate the low-lying excited states, such as the semionic excitations for $q = 2$.

The results presented in this article suggest that the interesting physics of the exact models can be realized in models with local interactions. The studies are limited to systems of up to 32 lattice sites, as we are using exact diagonalization to compute the eigenstates of the local models. Implementing the models experimentally would allow even larger systems to be considered and could hence provide further insight into how well the local models reproduce the physics of the exact models.

The paper is organized as follows. In Sec. II, we introduce the family of exact Hamiltonians and their analytical ground states. We show that the interaction strengths decay with distance, and we propose different local models with nearest-neighbor (NN) and possibly next nearest-neighbor (NNN) couplings. We also briefly discuss possibilities for realizing the local models in ultracold atoms in optical lattices. In Sec. III, we quantify how well the local models capture the physics of the nonlocal models by computing properties of the ground states, including overlap, entanglement entropy, and correlation functions. In Sec. IV, we compute the lowest energies of the models and the overlap between excited states of the local and the exact models. Section V concludes the paper.

II. MODEL HAMILTONIANS

A. Exact model

Our starting point is a family of exact 1D models with nonlocal interactions that have been constructed using tools from a conformal field theory in [3]. The models are defined on a 1D lattice with periodic boundary conditions, and it is convenient to think of the N lattice sites as sitting on a unit circle in the complex plane with the j th site at $z_j = e^{i2\pi j/N}$. Each site can be either empty or occupied by one particle, and n_j denotes the number of particles on site j . The members of the family are

labeled by a positive integer q , and for q odd (even), the particles are fermions (hardcore bosons).

The Hamiltonian takes the form

$$H_{1D} = \sum_{i \neq j} \left[(q-2)w_{ij} - w_{ij}^2 \right] d_i^\dagger d_j - \frac{1}{2}(q^2 - q) \sum_{i \neq j} w_{ij}^2 n_i n_j. \quad (1)$$

Here, $d_i^\dagger d_j$ is the particle hopping operator from site j to site i , the operator $n_i n_j$ denotes the density-density interaction terms between sites i and j , and $w_{ij} = \frac{z_i + z_j}{z_i - z_j}$. Explicitly,

$$d_j = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_2 \otimes \dots \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{j-1} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_j \otimes \begin{pmatrix} (-1)^q & 0 \\ 0 & 1 \end{pmatrix}_{j+1} \otimes \dots \otimes \begin{pmatrix} (-1)^q & 0 \\ 0 & 1 \end{pmatrix}_N. \quad (2)$$

The Hamiltonian conserves the number of particles, and we shall assume throughout that the number of particles in the system is fixed to N/q .

It has been shown analytically in [3] that the state

$$|\Psi_{\text{Exact}}\rangle = \sum_{n_1, n_2, \dots, n_N} \Psi_{\text{Exact}}(n_1, n_2, \dots, n_N) |n_1, n_2, \dots, n_N\rangle \quad (3)$$

with

$$\Psi_{\text{Exact}}(n_1, n_2, \dots, n_N) \propto \delta_n \chi_n \prod_{i < j} (z_i - z_j)^{q n_i n_j - n_i - n_j}, \quad (4)$$

is an eigenstate of the Hamiltonian (1) with energy

$$E_0 = -\frac{(q-1)}{6q} N[3N + (q-8)], \quad (5)$$

and numerical investigations for small systems show that this state is the unique ground state in the subspace with N/q particles. The factor δ_n is defined by

$$\delta_n = \begin{cases} 1 & \text{for } \sum_i n_i = N/q \\ 0 & \text{otherwise,} \end{cases} \quad (6)$$

and it fixes the number of particles in the state to N/q . The factor $\chi_n = (-1)^{\sum_j (j-1)n_j}$ is a sign factor. A subset of the excited states of the model can also be found analytically as discussed in [3].

Monte Carlo simulations for systems with a few hundred sites have already shown that the exact model is critical with a ground state entanglement entropy that grows logarithmically with the size of the subsystem and ground state two-site correlation functions that decay as a power law [3]. In the following, we concentrate on the

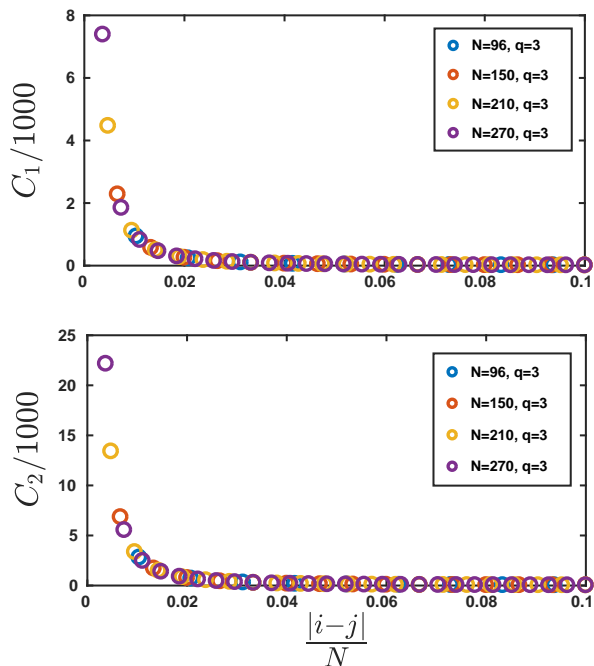


FIG. 1. (Color online) Behavior of the coefficients $C_1 = (q - 2)w_{ij} - w_{ij}^2$ (upper) associated with the operator $d_i^\dagger d_j$ and $C_2 = -\frac{1}{2}(q^2 - q)w_{ij}^2$ (lower) associated with the operator $n_i n_j$ as a function of distance for the Hamiltonian (1) with $q = 3$. The system sizes are as specified in the legends.

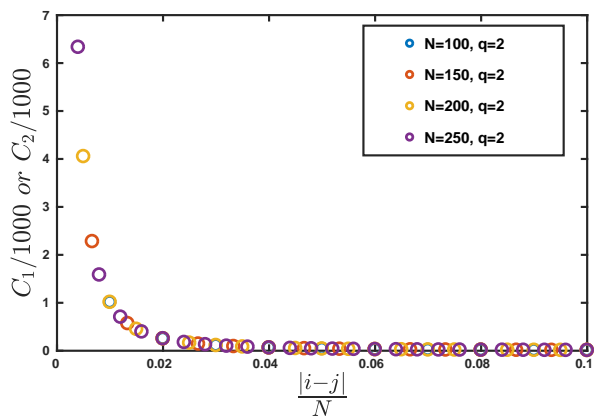


FIG. 2. (Color online) Behavior of the coefficients $C_1 = C_2 = -w_{ij}^2$ associated with the operators $d_i^\dagger d_j$ or $n_i n_j$ as a function of distance for the Hamiltonian (1) with $q = 2$. The legend shows the system sizes considered.

cases $q = 3, 2$, and 4 . We do so, because we want to study both the fermionic and the hardcore bosonic case, and because the $q = 2$ model has an $SU(2)$ invariant ground state, while the $q = 4$ model does not.

Let us investigate the behavior of the coefficients $C_1 = (q - 2)w_{ij} - w_{ij}^2$ of the hopping terms and $C_2 = -\frac{1}{2}(q^2 - q)w_{ij}^2$ of the density-density interaction terms of the exact

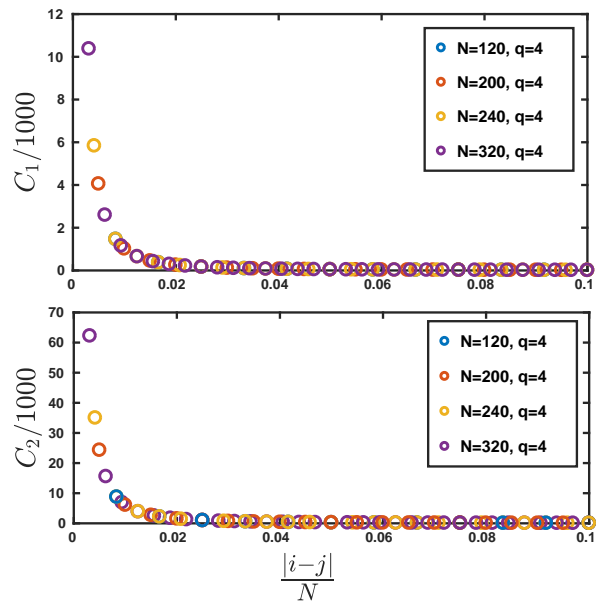


FIG. 3. (Color online) The coefficients $C_1 = (q - 2)w_{ij} - w_{ij}^2$ (upper) associated with the operator $d_i^\dagger d_j$ and $C_2 = -\frac{1}{2}(q^2 - q)w_{ij}^2$ (lower) associated with the operator $n_i n_j$ as a function of distance for the Hamiltonian (1) with $q = 4$. The different system sizes we consider are shown in the legends.

Hamiltonian (1) with respect to distance. First we note that

$$w_{jk} = \frac{z_j + z_k}{z_j - z_k} = \frac{-i}{\tan[\frac{\pi}{N}(j - k)]}. \quad (7)$$

From this and the expressions for C_1 and C_2 , we observe that the strength of the interactions decay as $|i - j|^{-2}$ for short distances ($\pi|i - j|/N \ll 1$), and for long distances ($|i - j| \approx N/2$) both C_1 and C_2 approach zero. Plots of C_1 and C_2 for different system sizes are given in Figs. 1, 2, and 3 for $q = 3, q = 2$, and $q = 4$, respectively. The fact that the interaction strengths decay fast with distance suggest that it may be possible to truncate the nonlocal Hamiltonian to a local Hamiltonian without significantly altering the low-energy physics of the model.

B. Local models

Replacing the exact Hamiltonian with a local one is an advantage experimentally, both because it removes the need to engineer couplings between distant sites and because it reduces the number of terms in the Hamiltonian. Ultracold atoms in optical lattices provide an interesting framework for simulating quantum physics. By trapping ultracold atoms in a 1D optical lattice, one naturally gets a model with hopping between NN sites and on-site interactions [20]. The hardcore constraint can be implemented by ensuring that the on-site interaction is large

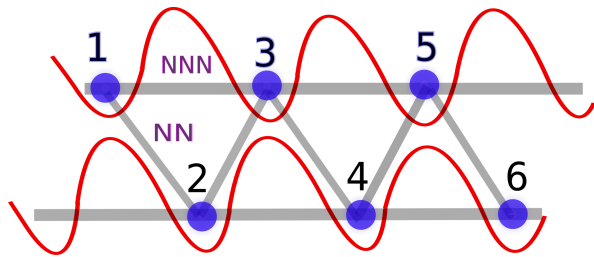


FIG. 4. (Color online) Schematic diagram showing a zigzag lattice. The geometry allows for adjusting the strengths of the NNN terms relative to the strengths of the NN terms.

enough. Density-density interactions between neighboring sites can be achieved through dipole-dipole interactions, see e.g. [21–23].

If one changes the geometry of the 1D lattice into a zigzag lattice, one can have couplings between both NN and NNN sites, as schematically depicted in Fig. 4. There are already more methods available to realize zigzag lattices with cold atoms, see e.g. [24–27]. Anisimovas *et al.* have proposed a scheme for realizing a zigzag lattice [27] using ultracold bosons. In their study, the two legs of the ladder correspond to a synthetic dimension given by two spin states of the atoms, and the tunneling between them can be realized by a laser-assisted process. Subsequently, by employing a spin-dependent optical lattice with the site position depending on the internal atomic state, a zigzag ladder can be achieved. In another experimental proposal, Zhang *et al.* have described a feasible method to achieve the zigzag lattice in one dimension [26]. They use a superlattice generated by commensurate wavelengths of light beams to realize tunable geometries including zigzag and sawtooth configurations. Also, in a theoretical study, Greschner *et al.* have suggested that the zigzag chain may be formed by an incoherent superposition between a triangular lattice and a 1D superlattice [24].

In the following, we will therefore restrict ourselves to models with at most NNN couplings in the Hamiltonian. Specifically, we study four different models. The first one is the NN model defined by the Hamiltonian

$$H_{\text{NN}} = \sum_{\langle i,j \rangle} \left[(q-2)w_{ij} - w_{ij}^2 \right] d_i^\dagger d_j - \frac{1}{2}(q^2 - q) \sum_{\langle i,j \rangle} w_{ij}^2 n_i n_j. \quad (8)$$

Here, the sum over i and j is restricted to NN terms as denoted by the symbol $\langle \dots \rangle$. (Note that we include both, e.g., $(i, j) = (1, 2)$ and $(i, j) = (2, 1)$ in the sum, which ensures that the Hamiltonian is hermitian.)

The second model is the model obtained by truncating the full Hamiltonian to only include terms up to NNN

TABLE I. Optimal values of U_{NN} and U_{NNN} .

q	U_{NN}	U_{NNN}
2	1.00	1.00
3	1.70	0.70
4	5.36	0.60

distance, i.e.,

$$H_{\text{NNN}} = \sum_{\langle\langle i,j \rangle\rangle} \left[(q-2)w_{ij} - w_{ij}^2 \right] d_i^\dagger d_j - \frac{1}{2}(q^2 - q) \sum_{\langle\langle i,j \rangle\rangle} w_{ij}^2 n_i n_j, \quad (9)$$

where $\langle\langle \dots \rangle\rangle$ is the sum over NN and NNN terms.

Instead of simply cutting the Hamiltonian as done above, one could try to partially compensate for the removed terms by adjusting the relative strengths of the couplings. We therefore also consider the optimized NN model

$$H_{\text{NN}}^{\text{opt}} = \sum_{\langle i,j \rangle} \left[(q-2)w_{ij} - w_{ij}^2 \right] d_i^\dagger d_j - \frac{U_{\text{NN}}}{2}(q^2 - q) \sum_{\langle i,j \rangle} w_{ij}^2 n_i n_j, \quad (10)$$

and the optimized NNN model

$$H_{\text{NNN}}^{\text{opt}} = \sum_{\langle\langle i,j \rangle\rangle} \left[(q-2)w_{ij} - w_{ij}^2 \right] d_i^\dagger d_j - \frac{U_{\text{NNN}}}{2}(q^2 - q) \sum_{\langle\langle i,j \rangle\rangle} w_{ij}^2 n_i n_j. \quad (11)$$

In these models, U_{NN} and U_{NNN} are parameters that are chosen to maximize the overlap between the ground state of $H_{\text{NN}}^{\text{opt}}$ or $H_{\text{NNN}}^{\text{opt}}$ with the analytical state $|\psi_{\text{Exact}}\rangle$ in (4). The optimized values are given in Tab. I. We find that the optimal value is independent of N . For the case of $q = 2$, the optimal value is unity, and the optimized models hence coincide with the models that are not optimized. This probably happens because when $U_{\text{NN}} = U_{\text{NNN}} = 1$, $q = 2$, and the number of particles is $N/2$, the Hamiltonian is SU(2) invariant like the ground state.

III. PROPERTIES OF THE GROUND STATES OF THE LOCAL MODELS

We now compute properties of the ground states of the local models to quantify how well the local models reproduce the physics of the exact models. The results are

TABLE II. Overlap Δ and overlap per site $\Delta^{1/N}$ between the state $|\psi_{\text{Exact}}\rangle$ in (4) and the ground state of H_{NN} , H_{NNN} , $H_{\text{NN}}^{\text{opt}}$, or $H_{\text{NNN}}^{\text{opt}}$ for $q = 3$. The different rows are for different numbers of lattice sites N .

N	Δ				$\Delta^{1/N}$			
	NN	NNN	NN _{opt}	NNN _{opt}	NN	NNN	NN _{opt}	NNN _{opt}
15	0.953	0.981	0.965	0.996	0.9968	0.9987	0.9976	0.9998
18	0.939	0.973	0.954	0.995	0.9965	0.9985	0.9974	0.9997
21	0.926	0.966	0.944	0.994	0.9963	0.9984	0.9973	0.9997
24	0.913	0.959	0.934	0.992	0.9962	0.9983	0.9971	0.9997
27	0.899	0.953	0.923	0.991	0.9961	0.9982	0.9971	0.9997
30	0.880	0.944	0.912	0.987	0.9957	0.9981	0.9970	0.9996

obtained using exact diagonalization, and we hence consider systems with $q = 2, 3$, and 4 with at most 32 lattice sites and N/q particles. We first show that overlaps per site higher than 0.999 can be obtained for all these cases with at most optimized NNN interactions. We then show that the behavior of important properties like entanglement entropy and two-site correlation functions is also reproduced well (possibly except for a bit of discrepancy for $q = 3$ and N even).

A. Overlap between ground state wavefunctions

We have calculated the overlap Δ between the ground state $|\Psi_{\text{Local}}\rangle$ of the local Hamiltonian (Eq. (8), (9), (10) or (11)) and the analytical wavefunction $|\Psi_{\text{Exact}}\rangle$ in (4). For a non-degenerate ground state, we define the overlap as

$$\Delta = |\langle \Psi_{\text{Local}} | \Psi_{\text{Exact}} \rangle|^2. \quad (12)$$

The overlap is expected to decay exponentially with system size even if the states deviate only a bit from each other, since the dimension of the Hilbert space increases exponentially in N . We therefore also consider the overlap per site $\Delta^{1/N}$, which is expected to remain roughly constant with system size. We find that the ground state is non-degenerate for all the system sizes we consider.

The overlap for the fermionic case ($q = 3$) is given in Table II. For a complete analysis, these Δ values are explicitly calculated with and without optimization for different system sizes. The overlaps are higher for the NNN Hamiltonian than for the NN Hamiltonian, and optimization also improves the overlaps. The overlap decreases with system size, but the overlap per site is almost constant. The overlap per site is already around 0.996 for

TABLE III. Overlap Δ and overlap per site $\Delta^{1/N}$ between the state $|\psi_{\text{Exact}}\rangle$ in (4) and the ground state of H_{NN} or H_{NNN} for $q = 2$. The different rows are for different numbers of lattice sites N .

N	Δ		$\Delta^{1/N}$	
	NN	NNN	NN	NNN
16	0.9930	0.9960	0.9996	0.9997
18	0.9917	0.9950	0.9995	0.9997
20	0.9904	0.9940	0.9995	0.9997
22	0.9891	0.9931	0.9995	0.9997
24	0.9878	0.9922	0.9995	0.9997
26	0.9864	0.9912	0.9995	0.9997
28	0.9852	0.9903	0.9995	0.9997
30	0.9839	0.9894	0.9995	0.9996

TABLE IV. Overlap Δ and overlap per site $\Delta^{1/N}$ between the state $|\psi_{\text{Exact}}\rangle$ in (4) and the ground state of H_{NN} , H_{NNN} , $H_{\text{NN}}^{\text{opt}}$, or $H_{\text{NNN}}^{\text{opt}}$ for $q = 4$. The different rows are for different numbers of lattice sites N .

N	Δ				$\Delta^{1/N}$			
	NN	NNN	NN _{opt}	NNN _{opt}	NN	NNN	NN _{opt}	NNN _{opt}
16	0.837	0.988	0.865	0.997	0.9881	0.9993	0.9910	0.9998
20	0.785	0.984	0.820	0.996	0.9880	0.9992	0.9901	0.9998
24	0.736	0.980	0.777	0.994	0.9873	0.9992	0.9895	0.9998
28	0.690	0.976	0.736	0.992	0.9868	0.9992	0.9891	0.9997
32	0.672	0.968	0.702	0.973	0.9877	0.9990	0.9890	0.9991

only NN interactions without optimization, and for the NNN Hamiltonian with optimization it is around 0.9997.

Next, we report the overlap values for half-filling ($q = 2$) and quarter-filling ($q = 4$) of the lattice with hardcore bosons. These results are shown in Tables III and IV, respectively. For $q = 2$, the NN model is enough to obtain overlaps per site around 0.9995, and these overlaps are slightly improved by considering the NNN model.

The scenario is different for $q = 4$. In this case, the overlap per site is around 0.987 for the NN model. Adding NNN interactions increases the overlap per site to

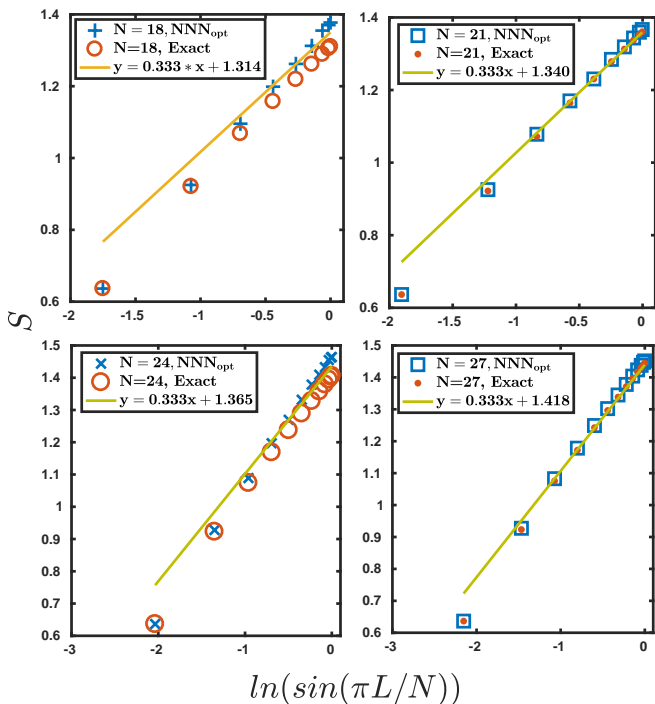


FIG. 5. (Color online) Entanglement entropy S as a function of subsystem size L for $q = 3$. The numerical results for S are computed using the ground state of $H_{\text{NNN}}^{\text{opt}}$ and (4), respectively. The different plots are for different system sizes as specified in the legends.

around 0.999. The optimized NN model produces lower overlaps than the NNN model. Optimizing the NNN model gives overlaps slightly better than for the NN model. For $q = 4$, the average distance between particles is four sites, which could be part of the explanation why NNN interactions are more important in this case than for $q = 2$.

From the above analysis of the overlap, we conclude that for the $q = 2$ case, it is sufficient to consider the local Hamiltonian with only NN interactions to study the ground state properties, whereas for the $q = 3$ and $q = 4$, the local NNN Hamiltonian with optimization is an appropriate choice.

B. Entanglement Entropy

We next consider the entanglement entropy. The entanglement entropy quantifies bipartite entanglement between two subsystems and is a typical measure of entanglement [28]. Besides its own fundamental theoretical interest, a principal reason for the success of the entanglement entropy as an entanglement measure in extended quantum systems is its universal scaling at 1D conformal critical points. It has already been investigated that the entanglement entropy and its spectra carry signatures of the underlying conformal field theory: In 1D, the entan-

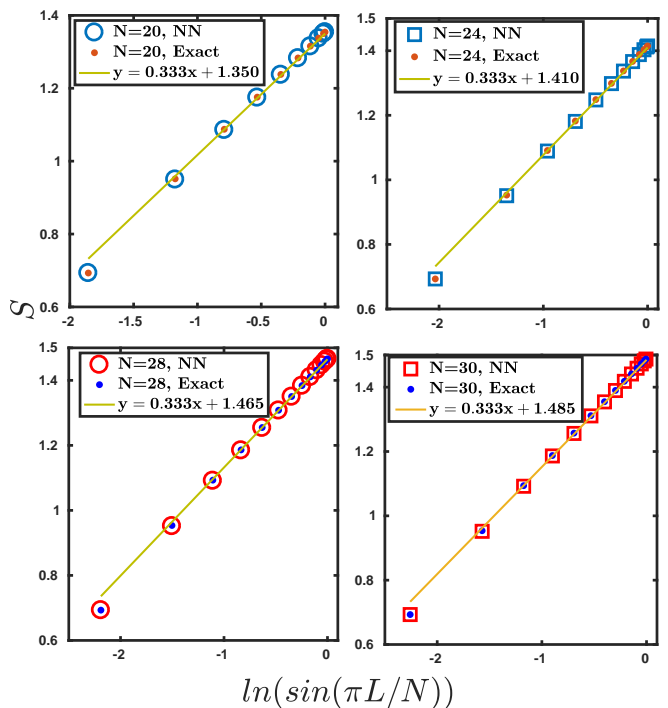


FIG. 6. (Color online) Entanglement entropy S for a subsystem of L consecutive sites and $q = 2$. The discrete data points are for the ground state of H_{NN} and for the analytical state (4). The different plots are for different system sizes as specified in the legends.

glement entropy of a conformal field theory exhibits a logarithmic violation of the area law and this characteristic has also been observed in lattice systems [29, 30].

For a general 1D system in a state $|\Psi\rangle$, one can define the entanglement entropy by partitioning the whole system into two subsystems A and B . The reduced density operator of subsystem A (B) is

$$\rho_A = \text{Tr}_B(\rho), \quad \rho_B = \text{Tr}_A(\rho), \quad (13)$$

where $\rho = |\Psi\rangle\langle\Psi|$ is the density operator of the full system. The von Neumann entanglement entropy of subsystem A is then defined as

$$S(A) = -\text{Tr}_A[\rho_A \ln(\rho_A)]. \quad (14)$$

It can be shown that $S \equiv S(A) = S(B)$, so that only the partitioning of the system is important.

One can also express the von Neumann entanglement entropy in terms of the eigenvalues $\lambda_{A,i}$ of ρ_A as

$$S = -\sum_i \lambda_{A,i} \ln(\lambda_{A,i}). \quad (15)$$

We use this formula to calculate the von Neumann entanglement entropy for our 1D system. We consider subsystems that consist of L consecutive sites. Due to the translational invariance of the models we are considering, all such entanglement entropies can be expressed in terms

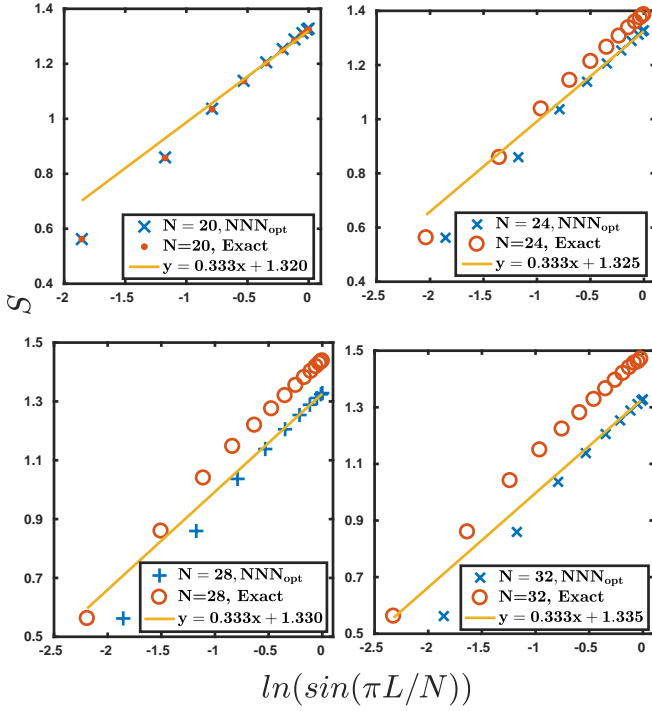


FIG. 7. (Color online) Entanglement entropy S as a function of subsystem size L for $q = 4$. The S values are calculated using the ground state of $H_{\text{NNN}}^{\text{opt}}$ and (4), respectively. The different plots are for different system sizes as specified in the legends.

of the entanglement entropies obtained when region A is site number 1 to site number L with $L \in \{1, 2, \dots, N/2\}$ for N even and $L \in \{1, 2, \dots, (N+1)/2\}$ for N odd.

The von Neumann entanglement entropy of a critical system in 1D is given by [31]

$$S = \frac{c}{3} \ln[\sin(\pi L/N)] + \text{constant}, \quad (16)$$

when L is large compared to 1. The central charge c was found to be $c = 1$ for the exact model in [3]. Plotting S as a function of $\ln[\sin(\pi L/N)]$, we should hence compare to a line of slope $1/3$.

Plots of the entanglement entropy for $q = 3, 2$, and 4 and different system sizes are shown in Figs. 5, 6, and 7, respectively. For $q = 2$, the data for the NN model and the exact model are seen to fit very well and also fit well with a line of slope $1/3$. For $q = 3$, there is a good match for N odd, while there is a bit of deviation for N even. For $q = 4$, the data for the local model are displaced by a constant, but the slope for L not too small fits well with $1/3$. The central charge is hence the same as in the exact model.

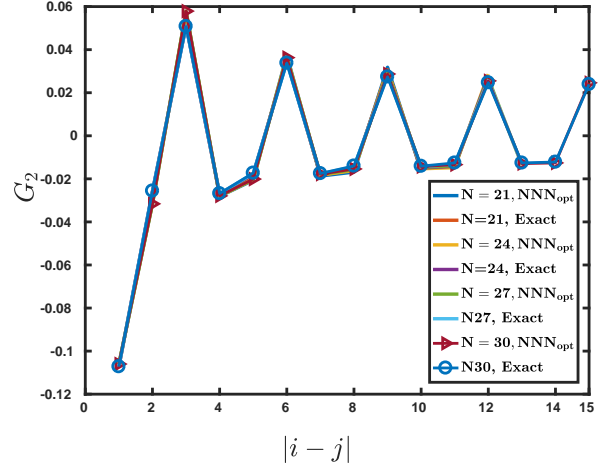


FIG. 8. (Color online) G_2 for the ground state of $H_{\text{NNN}}^{\text{opt}}$ and for $|\Psi_{\text{Exact}}\rangle$ for $q = 3$.

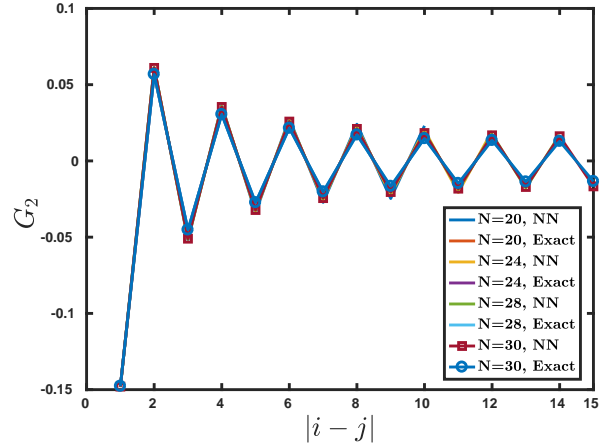


FIG. 9. (Color online) G_2 for the ground state of H_{NN} and for $|\Psi_{\text{Exact}}\rangle$ for $q = 2$.

C. Correlation function

Correlation functions are also an important characteristic property for determining the physics of a system. Here, we compute the two-site correlation function defined as

$$G_2 = \langle \Psi | n_i n_j | \Psi \rangle - \langle \Psi | n_i | \Psi \rangle \langle \Psi | n_j | \Psi \rangle, \quad (17)$$

where $|\Psi\rangle$ is the state of the system. In our case, $\langle \Psi | n_j | \Psi \rangle = 1/q$ for all j . In the exact model, G_2 decays with the distance between the points to the power $-2/q$ and also shows oscillations with period q . For the local models, it is difficult to determine the precise behavior of the decay, since we cannot go to sufficiently large system sizes. Instead we simply compare the values of G_2 for the local and the exact models.

Plots of G_2 are shown for $q = 3, 2$, and 4 and different system sizes in Figs. 8, 9, and 10, respectively. In all

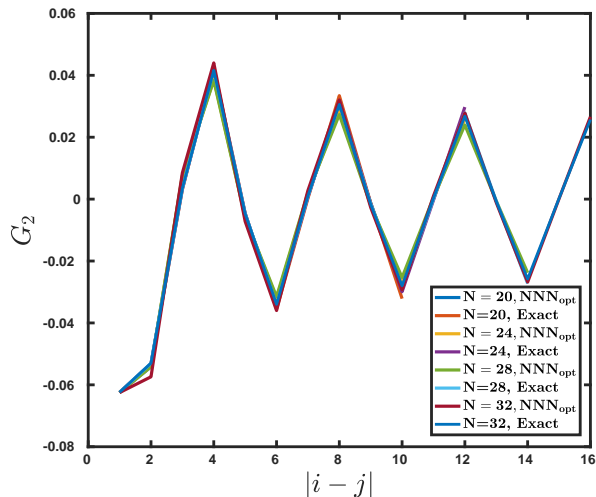


FIG. 10. (Color online) G_2 for the ground state of $H_{\text{NNN}}^{\text{opt}}$ and for $|\Psi_{\text{Exact}}\rangle$ for $q = 4$.

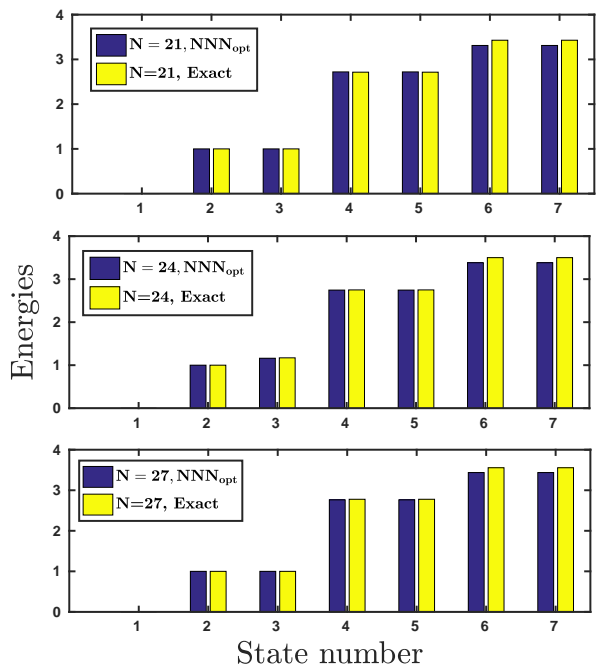


FIG. 11. (Color online) Energies of the 5 lowest states of the NNN_{opt} model (11) and of the exact model (1) for $q = 3$ (the ground state energy has been set to zero, and the energies have been scaled by a constant factor such that the first excited state is at energy 1). The different plots are for 21, 24, and 27 sites, respectively, as indicated.

cases, there is a good agreement between the local and the exact models.

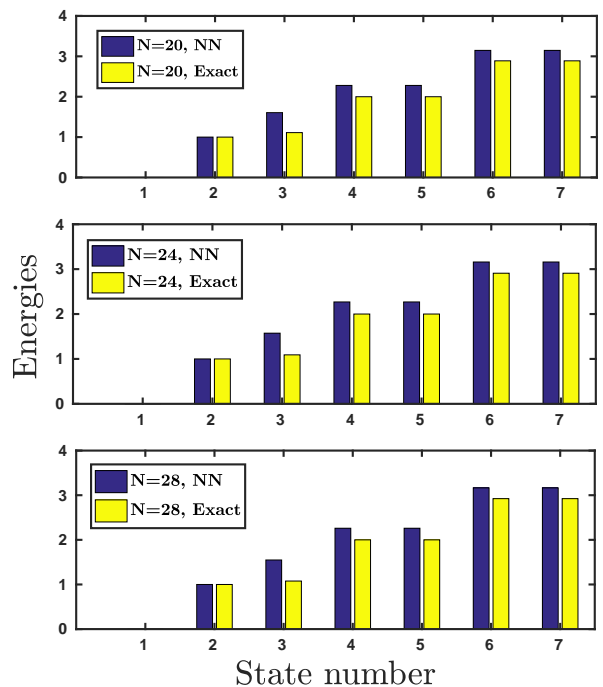


FIG. 12. (Color online) Energies of the 7 lowest states of the NN model (8) and of the exact model (1) for $q = 2$ (the ground state energy has been set to zero, and the energies have been scaled by a constant factor such that the first excited state is at energy 1). The different plots are for 20, 24, and 28 sites, respectively, as indicated.

IV. LOW-LYING EXCITED STATES

Achieving a good overlap for the ground state wavefunctions is a good starting point, but if the system is at nonzero temperature or we are interested in dynamics, it is also important that the low-lying excited states are not affected significantly in going from the exact to the local Hamiltonian, and we now investigate this question.

The energies of the seven lowest states are plotted in Figs. 11, 12, and 13 for $q = 3, 2$ and 4 , respectively, for both a local model and the exact model. We have added a constant to the energies and divided by a constant factor to fix the ground state energy to zero and the energy of the first excited state to one. For $q = 3$, the differences in energy between the local and the exact models are small, in particular for the first four excited states. For $q = 2$ and $q = 4$, the differences are a bit larger. It is interesting to note that the plots for a given value of q are almost independent of N for $q = 2$ and $q = 3$, but not for $q = 4$. Finite size effects hence play a role.

When we look at the overlaps (see Tabs. V, VI, and VII), there is an excellent match between states in the local model and corresponding states in the exact model. The overlap per site is higher than 0.998 in all the considered cases.

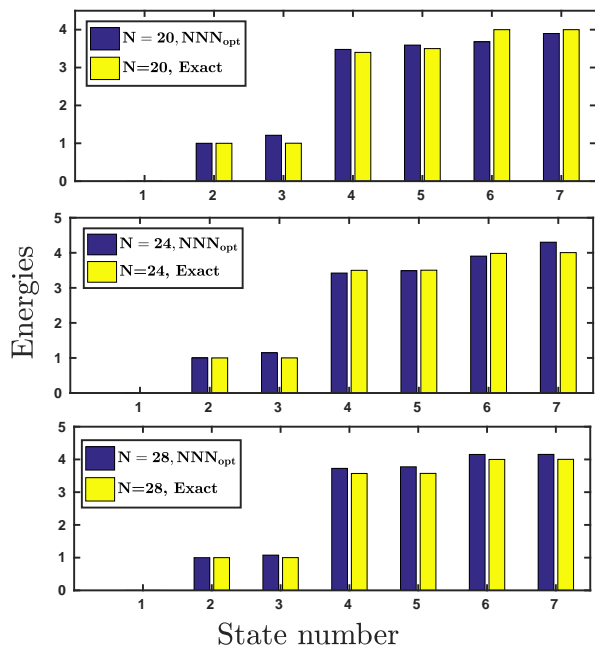


FIG. 13. (Color online) Energies of the 5 lowest states of the NNN_{opt} model (11) and of the exact model (1) for $q = 4$ (the ground state energy has been set to zero, and the energies have been scaled by a constant factor such that the first excited state is at energy 1). The different plots are for 20, 24, and 28 sites, respectively, as indicated.

V. CONCLUSION

We have started from a class of one-dimensional models derived from conformal field theory and used exact diagonalization to investigate to what extent the same physics is realized by different models containing only NN and possibly NNN couplings. For the $q = 2$ model we found that already the NN model gives a good description, while for $q = 3$ and $q = 4$ an optimized model with both NN and NNN couplings is preferred. With these models we find that the overlap per site between the ground states of the local models and the ground states of the exact models is larger than 0.999 for the considered system sizes. The overlaps per site for the low-lying excited state are larger than 0.998. We also find that the ground state entanglement entropy and the correlation functions are reproduced well by the local models, except that there is a bit of discrepancy for $q = 3$ for N even for the entanglement entropy. The low energy spectra are modified to some extent in the local models compared to the exact models. For $q = 4$, the modification varies with system size, so finite size effects play a role for the considered system sizes.

The high agreement between the ground state properties and the excited states suggest that most of the physics will remain the same in the local models. This paves the way to experimentally realize the physics in optical lattices using ultracold atoms, since the local mod-

TABLE V. Overlap Δ and overlap per site $\Delta^{1/N}$ between the first six excited states of the NNN_{opt} Hamiltonian and the first six excited states of the nonlocal Hamiltonian (1) for $q = 3$. The number of sites is $N = 21, 24$, and 27 , respectively.

State	Δ			$\Delta^{1/N}$		
	$N = 21$	$N = 24$	$N = 27$	$N = 21$	$N = 24$	$N = 27$
2	0.9933	0.9921	0.9908	0.9997	0.9997	0.9997
3	0.9933	0.9921	0.9908	0.9997	0.9997	0.9997
4	0.9894	0.9887	0.9878	0.9995	0.9995	0.9995
5	0.9894	0.9887	0.9878	0.9995	0.9995	0.9995
6	0.9893	0.9887	0.9795	0.9995	0.9995	0.9992
7	0.9893	0.9887	0.9795	0.9995	0.9995	0.9992

TABLE VI. Overlap Δ and overlap per site $\Delta^{1/N}$ between the first six excited states of the NN Hamiltonian and the first six excited states of the nonlocal Hamiltonian (1) for $q = 2$. The number of sites is $N = 20, 24$, and 28 , respectively.

State	Δ			$\Delta^{1/N}$		
	$N = 20$	$N = 24$	$N = 28$	$N = 20$	$N = 24$	$N = 28$
2	0.9900	0.9879	0.9856	0.9995	0.9995	0.9995
3	0.9707	0.9668	0.9634	0.9985	0.9986	0.9987
4	0.9914	0.9800	0.9772	0.9996	0.9992	0.9992
5	0.9914	0.9800	0.9772	0.9996	0.9992	0.9992
6	0.9894	0.9763	0.9739	0.9995	0.9990	0.9991
7	0.9894	0.9763	0.9739	0.9995	0.9990	0.9991

els are simpler to realize experimentally than the corresponding exact models derived from conformal field theory. The ingredients needed to realize the different terms in the Hamiltonian are already available experimentally. The observation that the local models resemble the exact models also shows that conformal field theory can not just be used to obtain exact, nonlocal models, but is also a powerful tool to obtain interesting, local models.

TABLE VII. Overlap Δ and overlap per site $\Delta^{1/N}$ between the first six excited states of the NNN_{opt} Hamiltonian and the first six excited states of the nonlocal Hamiltonian (1) for $q = 4$. The number of sites is $N = 20, 24$, and 28 , respectively.

State	Δ			$\Delta^{1/N}$		
	$N = 20$	$N = 24$	$N = 28$	$N = 20$	$N = 24$	$N = 28$
2	0.9915	0.9906	0.9893	0.9996	0.9996	0.9996
3	0.9933	0.9917	0.9900	0.9997	0.9997	0.9996
4	0.9860	0.9845	0.9828	0.9993	0.9993	0.9994
5	0.9883	0.9858	0.9835	0.9994	0.9994	0.9994
6	0.9632	0.9678	0.9743	0.9981	0.9986	0.9991
7	0.9697	0.9715	0.9624	0.9985	0.9988	0.9986

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