Gyrokinetics from variational averaging: existence and error bounds

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(Dated: 6 June 2018)

The gyrokinetic paradigm in the long wavelength regime is reviewed from the perspective of variational averaging (VA). The VA-method represents a third pillar for averaging kinetic equations with highly-oscillatory characteristics, besides classical averaging and Hilbert expansions. VA operates on the level of the Lagrangian function and preserves the Hamiltonian structure of the characteristics at all orders. We discuss the methodology of VA in detail by means of charged-particle motion in a strong magnetic field. The application of VA to a broader class of highly-oscillatory problems can be envisioned. For the charged particle, we prove the existence of a coordinate map in phase space that leads to a gyrokinetic Lagrangian at any order of the expansion, for general external fields. We compute this map up to third order, independent of the electromagnetic gauge. Moreover, an error bound for the solution of the derived gyrokinetic equation with respect to the solution of the Vlasov equation is provided, allowing to estimate the quality of the VA-approximation in this particular case.

PACS numbers: 02.30.Hq, 02.30.Mv, 02.40.Hw, 52.20.Dq, 52.25.Dg, 52.25.Xz, 52.30.Gz

I. INTRODUCTION

Charged particles in a strong magnetic field are spiraling around their “center of motion”, the gyro-center (GY). The stronger the magnetic field, the smaller the gyro-radius $\rho_g$ and the larger the gyro-frequency $\omega_g$ of the spiraling motion; the charged-particle dynamics is usually a multiscale problem. Low-frequency (1) phenomena become of interest for instance in space physics or in magnetically confined fusion devices\textsuperscript{30}. The modeling of these phenomena relies on averaging the gyro-motion, leading to reduced dynamics for the GY. This is a prototypical example of perturbation theory in (nearly-) periodic dynamical systems\textsuperscript{40}, with important consequences for practical applications. The GY-dynamics are appealing for numerical simulations of large ensembles of charged particles, giving rise to gyrokinetic equations. Such models are implemented in many state-of-the art computer codes for plasma turbulence simulations\textsuperscript{8,22,33–36}.

Given a large ensemble of charged particles, the kinetic (Vlasov) equation for the phase space distribution $f$ of these particles reads

$$\frac{d}{dt} f(x^\varepsilon(t),v^\varepsilon(t),t) = 0. \quad (1)$$

The solution $f$ is constant along the characteristics $x^\varepsilon(t) \in \mathbb{R}^3$, $v^\varepsilon(t) \in \mathbb{R}^3$, which satisfy Newton’s equations of motion under the Lorentz force. The high-dimensionality of the problem makes its numerical solution rather arduous. Moreover, in strong magnetic fields the characteristics $x^\varepsilon(t)$, $v^\varepsilon(t)$ are highly-oscillatory with a period $\varepsilon \ll 1$, i.e. $x^\varepsilon(t) = x(t/\varepsilon)$ leading to severe time step restrictions in numerical solvers. Instead of following the exact trajectory, reduced dynamics for the GY have proven to be useful in numerical experiments. Gyrokinetic equations have been derived on three different levels:

1. directly from the kinetic equation (1) via a Hilbert expansion of the solution, $f^\varepsilon = f^0 + \varepsilon f^1 + \ldots$,
2. from the characteristics by averaging the dynamical system $x^\varepsilon(t)$, $v^\varepsilon(t)$,
3. on the level of the Lagrangian via “variational averaging” (VA).

In this work we shall focus on the third approach, variational averaging. VA places the emphasis on the Hamiltonian structure of the dynamical system, which is preserved in the process. The formal theory of VA has been developed in the early eighties in the plasma physics community\textsuperscript{8,22,33–36}. A pioneering work for averaging nearly-periodic Hamiltonian systems has been given by Kruskal\textsuperscript{31}. Up to now mathematically rigorous results for VA are lacking, which is surprising considering its importance for numerical plasma physics. In this work we shall close this mathematical gap and establish several cornerstones of variational GY-theory:

- existence of a GY-transformation leading to reduced dynamics;
- gauge-invariance;
- definition of a gyrokinetic equation;
- strong error estimate for the gyrokinetic solution with respect to $f$, solution of (1).

We are able to prove existence with a new ansatz for the GY-transformation as a finite power series in

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\( \varepsilon \), algebraic in the generating functions, in contrast to the usual Lie-transform approach, which relies on operator exponentials of Poisson brackets. Prerequisites for understanding existing formal VA-theories\(^9,22,23,39,44\) include a firm knowledge about exterior calculus, differential forms and Lie transforms, with rare exceptions\(^34,41\). Our theory does not rely on these concepts and is thus more accessible for non-specialists. The long wavelength regime is considered, hence the inclusion of finite-Larmor radius effects postponed to a future work. We stress the non-uniqueness of transformations leading to GY-Lagrangians, which is overlooked in the existing VA-theories. A new GY-transformation is presented which leads to simpler equations of motion; this is possible due to the freedom of “unloading” complicated terms into the transformation (the generating functions), rather than keeping them in the Lagrangian.

The methodology of VA is carefully developed in this work. The concept of the “tangent map” between two coordinate representations of a manifold’s tangent bundle is introduced in detail. We then shift the focus to a particular class of Lagrangian functions of the form (14), linear in the tangent vectors. The VA-theory developed here could in principle be applied to a large class of highly-oscillatory problems, formulated in terms of this generic Lagrangian. The charged particle is a prototypical example and treated in detail.

Historically, the first approach towards reduced GY-models stems from averaging Newton’s equation of motion for the charged particle\(^28,37\). Assuming a uniform static magnetic field, these can be solved exactly to yield the spiraling motion around the straight field lines. In this case the GY is well-defined and its trajectory follows a magnetic field line. Adding a static perpendicular electric field gives rise to a drift across field lines, but the GY is still well-defined. The problem complicates when the fields have curvature (non-homogeneous case). In this case several new drifts appear, for instance the curvature drift and the grad-\(B\) drift\(^26\). On top of that, the GY is no longer well-defined: the center of the spiral cannot be computed in closed form, its location only approximated by an infinite series. In the non-homogeneous case the GY-dynamics are thus truncated dynamics (perturbation theory). For the validity of the theory it is thus essential to control the error that arises from truncation.

VA is based on a variational principle from which the equations of motion can be derived. Since the variational principle is coordinate independent it is particularly suited for averaging, which is nothing else than a change of coordinates, with a minimum amount of algebra. VA has the advantage that the Hamiltonian structure of the particle dynamics is not destroyed in the process. This leads in particular to conservation of a truncated energy and to conservation of a truncated phase space volume, which are easily identified. These and other conservation properties related to the Hamiltonian structure are beneficial for stability and accuracy of long-time numerical simulations.

Let us briefly mention some mathematical results on averaged particle dynamics in strong magnetic fields, not related to VA: Frénod and Sonnendrücker\(^17,18\) use two-scale convergence to establish limit models of the Vlasov-Poisson system in strong magnetic fields. The asymptotic behavior of the Vlasov-Maxwell system in strong magnetic fields has been considered by Bostan\(^1\)–\(^6\), relying on energy methods or averaging techniques. The transition from Vlasov to gyrokinetic equations has also been studied for example in\(^15,19,25\). Stroboscopic averaging is applied to the GY-problem in\(^13,14\). A WKB-based approach with emphasis on gyro-gauge has been presented in\(^42\).

The article is organized as follows: in a preliminary section we clarify some notation in II A and introduce the equations of motion and their normalization in II B. In II C we discuss the corresponding variational formulation; the notion of a Lagrangian function defined on the tangent bundle of the underlying manifold is presented in detail, along with a change of coordinates via the tangent map. We formulate the guiding-center problem as well as the full problem with electromagnetic external fields in the extended phase space. In section II D we introduce the method of VA, which is based on the concept of the tangent map. Our new ansatz for the algebraic GY-transformation is stated here. The main results are collected in section III, which is split into three subsections: in the preliminary part III A we define an equivalence relation for Lagrangians, a normal form of guiding-center Lagrangians, the gyro-average operation and we state the existence of solutions for the charge-particle dynamics. Section III B contains the existence results for the GY-transformation in Theorem 1. The gyrokinetic equation is defined in section III C; its strong solution is compared to the solution of the Vlasov equation in Theorem 2. Explicit expressions for the GY-transformation, the corresponding GY-Hamiltonian and the generalized magnetic moment are given in section IV. Proofs that require a lot of algebra have been put into section V. We summarize the article and discuss future perspectives in section VI.

### II. PRELIMINARIES

#### A. Notation

The vector product in \( \mathbb{R}^3 \) is denoted by ‘\( \times \)’. The symbol ‘\( \nabla \)’ denotes the usual gradient operator in \( \mathbb{R}^3 \), hence \( \nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})^\top \). For a vector field \( A : \mathbb{R}^3 \to \mathbb{R}^3 \), \( A = (A_1, A_2, A_3)^\top \), we write \( \nabla \times A \) to denote the curl operator. Given a map \( \tau : \mathbb{R}^n \to \mathbb{R}^n \), the Jacobian is denoted by \( D\tau \), i.e. \( (D\tau)_{i,j} = \partial_{\tau_i}/\partial x_j \). For \( n = 3 \) we denote the transpose Jacobian by

\[
\nabla A := \left( \frac{\partial A_j}{\partial x_i} \right)_{1 \leq i,j \leq 3} = \left( \frac{\partial A}{\partial x} \right)^\top = (DA)^\top.
\]
The dot ‘·’ denotes the scalar product in Euclidean space; it is also used to denote matrix-vector multiplication in \( \mathbb{R}^n \). For \( b \in \mathbb{R}^3 \) for example
\[
(b \cdot \nabla)A = b \cdot \nabla A = (\nabla A)^\top : b.
\]

B. Equations of motion and scaling

Newton’s equation of motion for a non-relativistic charged particle in an electromagnetic field can be written as
\[
\frac{dx}{dt} = v, \quad \frac{dv}{dt} = \frac{e}{m} \left[ v \times B(x,t) + E(x,t) \right].
\] (2)

Here, \( x \) stands for the particle position, \( v \) its velocity, \( e \) the particle’s charge, \( m \) its mass and \( B \) and \( E \) denote external magnetic and electric fields. The right-hand-side in the equation for \( v \) is the Lorentz force, hence gravitational and other effects are neglected. Our first task is to formulate Newton’s equation of motion in dimensionless form. For example, we write the solution \( x \) as \( x(t) = \hat{x}x'(t') \), where \( \hat{x} \) denotes the characteristic size (scale or unit) of the particle position and \( x' \) is a dimensionless function of \( t' = t/\hat{t} \), the time in units of \( \hat{t} \). The characteristic size \( \hat{x} \) could be for instance the diameter of our domain of study and \( \hat{t} = \hat{\omega}^{-1} \), where \( \hat{\omega} \) characterizes the frequency domain of interest. Similarly, \( B(x,t) = \hat{B}B'(x',t') \) for the fields. Hence,
\[
\frac{dx'}{dt'} = \frac{\hat{v}}{\hat{x}\hat{\omega}} v',
\]
\[
\frac{dv'}{dt'} = \frac{\hat{e}}{m\hat{\omega}} \left[ v' \times B'(x',t') + \frac{\hat{E}}{\hat{v}} B'(x',t') \right].
\]

The characteristic cyclotron frequency of the problem is \( \hat{\omega}_c = e \hat{B}/m \). We simplify via
\[
\text{a) } \hat{v} = \frac{\hat{x}}{\hat{x}\hat{\omega}} \hat{\omega}, \quad \text{b) } \varepsilon := \frac{\hat{\omega}}{\hat{\omega}_c}, \quad \text{c) } \varepsilon_\delta := \frac{\hat{E}}{\hat{v}\hat{B}}.
\] (4)

In assumption a) we relate the velocity scale \( \hat{v} \) to the chosen time- and space scales \( \hat{x}\hat{\omega}^{-1} \) and \( \hat{x} \). In b) we introduce a first parameter \( \varepsilon \) if \( \varepsilon \ll 1 \) one enters the low-frequency regime, which means that the frequency of interest \( \hat{\omega} \) is much smaller than the cyclotron frequency \( \hat{\omega}_c \). A second parameter \( \varepsilon_\delta \) is introduced in c); it represents the ratio of the \( E\times B \)-velocity to the characteristic velocity \( \hat{v} \). This parameter will also appear in the magnetic field, which we assume to be composed of two parts:
\[
B(x,t) = B_0(x) + \varepsilon_\delta B_1(x,t),
\] (5)
a so-called “guide field” \( B_0 \), which is static and nonhomogeneous and a dynamical part \( B_1 \) with amplitude \( \varepsilon_\delta \). Thus \( \varepsilon_\delta \) signifies the amplitude of the dynamical fields \( E/v \) and \( B_1 \) with respect to the static guide field \( B_0 \). We introduce a third parameter \( \varepsilon_B \) which measures the degree of inhomogeneity of the guide field (\( || \cdot || \) is some matrix norm):
\[
\varepsilon_B := \frac{\hat{x}}{|\nabla B_0|}/|B_0|.
\] (6)

Two cases of \( \varepsilon_B \) shall be addressed in this paper: \( \varepsilon_B = 1 \), which signifies that the guide field variations are on the same scale \( \hat{x} \), and \( \varepsilon_B = \varepsilon \) which corresponds to less important variations of the guide field.

Let us now insert the above definitions of \( \varepsilon \)-parameters into Newton’s equations (3) and omit the primes to obtain
\[
\frac{dx}{dt} = v,
\]
\[
\frac{dv}{dt} = v \times \left[ \frac{B_0(\varepsilon_B x)}{\varepsilon} + \frac{\varepsilon_\delta}{\varepsilon} B_1(x,t) \right] + \frac{\varepsilon_\delta}{\varepsilon} E(x,t).
\] (7)

Two orderings shall be addressed in this work:
\[
\begin{align*}
1) & \quad \varepsilon_\delta = \varepsilon, \quad \varepsilon_B = 1, \quad \varepsilon_B = \varepsilon. \quad (8)
\end{align*}
\]

Ordering 1) is called the “maximal ordering”\(^9\) but is rarely applied in practical GY-theory, because curvature terms tend to be rather involved at higher orders of perturbation theory. We also point out that the ordering 2) is implemented in most the aforementioned gyrokinetic computer models, because of its relative simplicity with respect to the case \( \varepsilon_B = 1 \) at the second order of expansion, as seen from the results in section IV.

C. Variational formulation

1. Problem statement

Under the scaling assumptions from the previous section, the initial-value problem (IVP) we consider reads
\[
\frac{dx}{dt} = v,
\]
\[
\frac{dv}{dt} = v \times \left[ \frac{B_0(\varepsilon_B x)}{\varepsilon} + B_1(x,t) \right] + E(x,t),
\] (9a)
wher at the initial time \( t_0 \),
\[
x(t_0) = x_0, \quad v(t_0) = v_0.
\] (9b)

Here, we assume \( x, x_0 \in \Omega_x \subset \mathbb{R}^3 \), \( v, v_0 \in \Omega_v \subset \mathbb{R}^3 \), \( \Omega = \Omega_x \times \Omega_v \) open and bounded and \( 0 < \varepsilon \leq \varepsilon_{\text{max}} \). For \( \varepsilon \ll 1 \) system (9) represents a multi-scale problem with a fast, nearly-periodic motion around \( B_0 \). Classical averaging\(^4\) can be applied to extract reduced dynamics free of the fast scale. However, system (9) is also rich in structure, a so-called Hamiltonian system. In order to see the structure we need to study its variational formulation.
2. Lagrangian functions and tangent maps

The variational formulation of (9) is based on a Lagrangian function, simply called the “Lagrangian”. Lagrangians are defined on the tangent bundle of the underlying manifold, which in our study is the phase space, and map into the real numbers. In this work we will consider different coordinate representations of the same Lagrangian on the tangent bundle. We shall now clarify these notions in more detail.

Let \( M \) denote an \( n \)-dimensional differentiable manifold, described for simplicity by one chart \((M, \varphi)\). Here, \( \varphi : M \to U \subset \mathbb{R}^n \) is a homeomorphism and \( q = \varphi(m) \) are the coordinates of the point \( m \in M \) under the chart \( (M, \varphi) \). Suppose that \((M, \chi), \chi : M \to V \subset \mathbb{R}^n\) is a different chart defining the coordinates \( l = \chi(m) \). The transition map, or change of coordinates, \( \tau : U \to V, l = \tau(q) \), is defined by \( \tau := \chi \circ \varphi^{-1} \).

The tangent space at point \( m \in M \), denoted by \( T_m M \), can be thought of as the space of all vectors originating from \( m \) that are tangent to the manifold at \( m \). More precisely, the elements of \( T_m M \) can be defined either as equivalence classes of curves through \( m \), two curves being equivalent when they are tangent to each other at \( m \), or as directional derivatives at \( m \), hence \( T_m M \cong \mathbb{R}^n \). For an open interval \( I \subset \mathbb{R} \), let \( c : I \to U \) denote a curve in \( U \) with \( c(0) = q \); then \( \varphi^{-1}(c) \) is a curve on the manifold \( M \) passing through \( m \) at \( t = 0 \). The tangent at \( q \) is \( \dot{q} := \frac{dc}{dt}(0) \). Using the transition map \( \tau \), we can derive the corresponding tangent at \( l \in V \):

\[
i = \frac{d\tau(c(t))}{dt}
|_{t=0} = \sum_j \frac{\partial \tau}{\partial q_j} \frac{dc_j(0)}{dt} = D\tau(q) \cdot \dot{q}.
\]

Since the Jacobian \( D\tau(q) \) is invertible, there is a one-to-one correspondence between \( l \in \mathbb{R}^n \) and \( \dot{q} \in \mathbb{R}^n \). The columns of \( D\tau(q) \) form thus a basis of \( \mathbb{R}^n \), which we denote by \( \partial_j := \frac{\partial \varphi}{\partial q_j} \) (covariant basis). Therefore,

\[
i = D\tau(q) \cdot \dot{q} = \sum_j \dot{q}_j \partial_j.
\]

The union of all tangent spaces “attached” to \( M \) is called the tangent bundle \( TM \); its elements are tangent vectors. The chart \( \varphi \) induces coordinates in the tangent bundle which we denote by \((q, \dot{q})\). The first coordinate \( m \to q \) identifies the tangent space and the second coordinate \( \dot{q} \) identifies an element in that particular tangent space.

Definition 1. (Tangent map.) Given a change of coordinates \( \tau : U \to V, q \to l \) on the manifold \( M \), the associated “tangent map” \( T\tau : (q, \dot{q}) \to (l, \dot{l}) \) relating two coordinate systems of the tangent bundle \( TM \) is defined by

\[
(l, \dot{l}) = T\tau(q, \dot{q}) := (\tau(q)), D\tau(q) \cdot \dot{q}.
\]

The dual to the tangent space \( T^*_m M \) is the cotangent space \( T^*_m M \cong \mathbb{R}^n \). Its elements are covectors or linear forms \( \alpha : T_m M \to \mathbb{R} \), mapping tangents at \( m \) into the real numbers. Coordinates of a linear form are usually stated in the contravariant basis of \( \mathbb{R}^n \), induced by the transition map \( \tau \) as follows: given the basis vectors \( \partial_j \), the dual basis \( d_j \) is defined by the property \( d_j(\partial_j) = \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker delta. Since for Jacobians we have \( D\tau^{-1}D\tau = I_n \) where \( I_n \) is the identity matrix, we deduce that the lines of \( D\tau^{-1} \) are the sought dual basis, thus \( d_j := \nabla \varphi_j^{-1} \). Identifying the covariant basis vectors in \( \mathbb{R}^n \) with the corresponding tangent vectors in \( T_m M \), and identifying the contravariant basis vectors in \( \mathbb{R}^n \) with the corresponding covectors in \( T_m^* M \), for general \( \alpha = \sum \gamma_i d_i \in T_m^* M \) and \( \xi = \sum \dot{q}_j \partial_j \in T_m M \) we have

\[
\alpha(\xi) = \sum_i \gamma_i \dot{q}_j (\partial_j) = \sum_i \gamma_i \dot{q}_i = \gamma \cdot \dot{q}.
\]

Hence the natural pairing between elements of the tangent space \( T_m M \) and elements of its dual \( T^*_m M \) can be written as a scalar product in \( \mathbb{R}^n \). We shall use this convenient notation throughout this work. The union of all cotangent spaces of \( M \) is called the cotangent bundle and is denoted by \( T^* M \).

We are now equipped to define a Lagrangian function on the tangent bundle of the manifold \( M \). We shall consider dynamical systems defined by a particular class of Lagrangians \( L : TM \to \mathbb{R} \) which, in coordinates \((q, \dot{q})\) defined by some chart \((M, \varphi)\), can be written as

\[
L(q, \dot{q}) = \gamma(q) \cdot \dot{q} - H(q).
\]

Here, \( H : U \to \mathbb{R} \) is called the Hamiltonian and \( \gamma \in \mathbb{R}^n \) is the coordinate representation of the symplectic form \( \alpha \in T^* M \). We will now discuss how the charged-particle problem (9) can be deduced from a Lagrangian of the form (14) by a variational principle.

3. The action principle

Given the Lagrangian (14) the dynamics follow from a variational principle on curves in \( U \). Let us denote such curves by \( q(s) \), or more precisely by \( q : I \to U \) for some open interval \( I \subset \mathbb{R} \). Let us further define the following functional on the space of curves,

\[
\mathcal{A}[q] := \int_I L(q(s), \frac{d}{ds}q(s)) \, ds.
\]

The variational (action) principle \( \delta \mathcal{A} = 0 \) yields the Euler-Lagrange equations

\[
\frac{\partial L}{\partial q} - \frac{d}{ds} \frac{\partial L}{\partial \dot{q}} = 0,
\]

which, for \( L \) given by (14), become

\[
\omega \cdot \frac{dq}{ds} = \frac{\partial H}{\partial \dot{q}},
\]

where \( \omega \) is the symplectic form.
where \( \omega := (D\gamma)^T - D\gamma \) is called the Lagrange matrix. We assume that \( \omega \) is invertible on \( U \) and write \( J := \omega^{-1} \). Then system (17) can be written as

\[
\frac{dq}{ds} = (q, H),
\]
where \( \{G, H\} := \partial G/\partial q \cdot J \cdot \partial H/\partial q \) denotes the Poisson bracket, defined for differentiable functions \( G, H : U \to \mathbb{R} \). The bracket is bilinear, anti-symmetric and satisfies the Jacobi identity

\[
\{F, \{G, H\}\} + \{H, \{F, G\}\} + \{G, \{H, F\}\} = 0.
\]

Systems of the form (17) where \( \omega \) is invertible are called non-canonical symplectic systems, which belong to the larger class of Hamiltonian systems. It is an immediate consequence of (18) that \( H(q) \) is a constant of the motion, \( \frac{d}{ds} H = 0 \). Moreover, it can be shown that the flow of (18) conserves the phase space volume \( \sqrt{\det \omega} \), computed from the determinant of the Lagrange matrix \( \omega \). Other constants of the motion are the so-called Casimirs and the momentum maps\(^{1,2}\). Exact conservation of these invariants on the discrete level leads to improved long-term stability and accuracy of numerical schemes. An example of such a symplectic integrator is the well-known Störmer-Verlet scheme\(^{24}\).

4. The guiding-center problem

If the dynamical fields \( E \) and/or \( B_1 \) in (9) are zero, the problem of averaging reduces to the so-called guiding-center (GC) problem. In this case the system (9) is autonomous and a Lagrangian of the generic form (14) can be formulated in the coordinates \( q = (x, v) \), hence with \( U = \Omega \). It reads

\[
L_a = \left[ v + \frac{A_0(x)}{\varepsilon} \right] \cdot \dot{x} - \frac{|v|^2}{2}.
\]

Here, \( A_0 \) is the vector potential related to the guide field via \( B_0 = \nabla \times A_0 \). In terms of the generic form (14) we have

\[
\gamma = \gamma_a = \left[ v + \frac{A_0(x)}{\varepsilon}, 0, 0, 0 \right] \in \mathbb{R}^4,
\]
\[
H = H_a = \frac{|v|^2}{2}.
\]

The velocity components of the symplectic form are zero. It can be easily checked that the Euler-Lagrange equations corresponding to \( L_a \) yield the equations (9) without dynamical fields. Moreover, we know that this system is non-canonical symplectic because its Lagrange matrix is invertible. The kinetic energy \( H_a \) is conserved during the motion.

Variational averaging of the Lagrangian (19) has been studied extensively on the formal level; the first rigorous results are presented in this work. A review can be found in\(^{12}\). Higher-order computations of the asymptotic GC-expansion have recently been reported\(^{10,43}\). The computations in this paper will reproduce the standard GC-results up to second order in the GC-Hamiltonian and GC-symplectic form. The first-order GC-Lagrangian is defined in (39).

5. Full problem with dynamical fields

In case that the dynamical fields \( E \) and/or \( B_1 \) are not zero the system (9) is non-autonomous. Nevertheless, it can be written as an autonomous system in the extended phase space \( U = \Omega \times \mathbb{R}^2 \) with coordinates \( q = (x, v, t, w) \). Here, the time \( t \) and the energy \( w \) are dependent variables and the independent variable is denoted by \( s \in \mathbb{R} \). Symplectic form and Hamiltonian are introduced as

\[
\gamma_{\text{ext}} := \left[ \gamma_a, 0, 0 \right] + \left[ A_1(x, t), 0, 0, 0, -w, 0 \right] \in \mathbb{R}^8,
\]
\[
H_{\text{ext}} := H_a + \phi(x, t) - w.
\]

Here, the dynamical electromagnetic potentials \( A_1 \) and \( \phi \) are such that

\[
B_1 = \nabla \times A_1, \quad E = -\nabla \phi - \frac{\partial A_1}{\partial t}.
\]

The Lagrangian is of the generic form (14) with \( \gamma = \gamma_{\text{ext}} \) and \( H = H_{\text{ext}} \), hence

\[
L_{\text{ext}} = \left[ v + \frac{A_0(x)}{\varepsilon} + A_1(x, t) \right] \cdot \dot{x} - w \dot{t}
\]
\[
- \frac{|v|^2}{2} - \phi(x, t) + w.
\]

The corresponding Lagrange matrix is invertible and the system is non-canonical symplectic with conserved energy \( H_{\text{ext}} \). The Euler-Lagrange equation for \( w \) automatically yields \( \frac{d}{ds} t = 1 \) and thus \( t = s \). The charged-particle dynamics are found to occur on the hyper-surface \( H_{\text{ext}} = 0 \) of the extended coordinate space. For simplicity during variational averaging we directly impose \( H_{\text{ext}} = 0 \) which means \( H_1 := |v|^2/2 + \phi = w \); this leads to the “extended Lagrangian”\(^2\)

\[
L_1 := L_{\text{ext}} \big|_{w=H_1} = \left[ v + \frac{A_0(x)}{\varepsilon} + A_1(x, t) \right] \cdot \dot{x} - H_1 \dot{t},
\]

where the coordinate space is \( \Omega_1 := \Omega \times \mathbb{R} \) with elements \( q = (x, v, t) \). The Lagrangian (24) is written as

\[
\gamma_1 := \left[ v + \frac{A_0(x)}{\varepsilon} + A_1(x, t), 0, 0, 0, -H_1 \right] \in \mathbb{R}^7
\]

is the well-known Poincaré-Cartan form; it is the starting point for any gyro-averaging theory in the variational framework.
D. Change of coordinates

1. What is variational averaging?

The aim of variational GY-theory is to preserve the symplectic structure of the charged-particle dynamics (9) when averaging the fast scale due to the \( v \times B_0 \) motion. This structure originates from the generic form (14) of the particle Lagrangian (23). Hence, averaging directly on the level of the Lagrangian while keeping the generic form is the favorable strategy, as outlined in (14). “Averaging” in this context is not an integration over some coordinate, but rather a change of coordinates such that one of the new coordinates is an adiabatic invariant. At lowest order, this adiabatic invariant is the particle’s magnetic moment. The procedure of VA consists of the following steps:

1. Identify the fast variable, here the gyro-angle, that changes on the time scale \( \varepsilon \) due to the \( v \times B_0 \) motion. This is done by a “preliminary map” in the extended Lagrangian (24).

2. Construct a change of coordinates as a (finite) power series in \( \varepsilon \) that eliminates the fast variable from the Lagrangian, order by order in \( \varepsilon \), up to the desired order \( \varepsilon^N \).

3. The "slow dynamics" can now be decoupled from the fast scale by truncating the new Lagrangian at order \( N \), which means neglecting terms of order \( \varepsilon^{N+1} \):

\[
L^\varepsilon = L_{\varepsilon}^{(N)} + L_{\varepsilon}^{N+1} + \ldots.
\]

4. The GY-equations of motion are the Euler-Lagrange equations stemming from the truncated Lagrangian \( L_{\varepsilon}^{(N)} \) defined in (26); they are independent of one of the new coordinates, namely the GY-angle \( \alpha \), by construction. The term “averaged dynamics” refers to the dynamics of the variables other than \( \alpha \). Moreover, the Euler-Lagrange equation (16) for \( \alpha \) yields

\[
\frac{d}{ds} \frac{\partial L_{\varepsilon}^{(N)}}{\partial \dot{\alpha}} = 0,
\]

which states the exact conservation of the “generalized magnetic moment” \( \vec{\mu} = \frac{\partial L_{\varepsilon}^{(N)}}{\partial \dot{\alpha}} \), which is merely an adiabatic invariant in the exact dynamics.

We identify three fundamental questions related to the above approach:

- How does the truncation error of order \( O(\varepsilon^{N+1}) \) in the Lagrangian \( L_{\varepsilon}^{(N)} \) translate to errors in the equations of motion?
- In what way can the averaged equations be used to derive a gyrokinetic equation?

These questions, among others, shall be addressed in the course of this work. The main tool for variational averaging will be the tangent map defined in (12), which allows us to transform Lagrangians defined on tangent bundles. It will be used to transform the extended Lagrangian (24) via the gyro-transformation \( \tau_{\varepsilon} : \Omega_{\varepsilon} \rightarrow \Omega_{\varepsilon} \), \( q_{\varepsilon} \mapsto \bar{q} = (x, v, t) \), as follows:

\[
L_1(q, \dot{q}) = \gamma_1(q) \gamma_{\varepsilon}(q_{\varepsilon}) : D\tau_{\varepsilon}(q_{\varepsilon}) \cdot \dot{q}_{\varepsilon}
\]

Here, \( q_{\varepsilon} \) are called the GY-coordinates and we almost accidentally uncovered the transformation law of covectors (elements of the cotangent space), \( \gamma_1 \circ \tau_{\varepsilon} = (D\tau_{\varepsilon})^{-T} \gamma_{\varepsilon} \). Variational averaging is built on the fact that in (28) the generic form of the extended Lagrangian \( L_1 \) is preserved under the tangent map. Moreover, from the transformation law of covectors we can deduce that the new Lagrange matrix \( \Omega_{\varepsilon} = (D\gamma_{\varepsilon})^T - D\gamma_{\varepsilon} \) is invertible, and hence the symplectic structure preserved.

2. Preliminary transformation

We apply a preliminary coordinate map to the extended Lagrangian \( L_1 : T\Omega \rightarrow \mathbb{R} \) from (24) for the purpose of identifying the fast variable (gyro-angle), which is then subject to averaging. We start from a local, orthonormal basis \((e_1(x), e_2(x), b_0(x))\) that satisfies \( e_1 \times e_2 = b_0 \) such that \( b_0 \cdot e_1 = e_2 = 1 \) and the basis is right-handed at each \( x \in \Omega_x \). The particular choice of \( e_1 \) and \( e_2 \) is arbitrary and is called the “gyro-gauge.” The here derived GY-dynamics are independent of the gyro-gauge. New velocity coordinates are introduced as

\[
\begin{align*}
\tilde{v}_\parallel &:= v \cdot b_0(x), \\
\tilde{v}_\perp &:= |b_0(x) \times v \times b_0(x)| = |v - v \cdot b_0(x) b_0(x)|, \\
\theta &:= -\arctan \left( \frac{v \cdot e_2(x)}{v \cdot e_1(x)} \right),
\end{align*}
\]

such that \( v = \tilde{v}_\parallel b_0 + \tilde{v}_\perp c_0 \), where \( c_0 := e_1 \cos \theta - e_2 \sin \theta \). Together with the unit vector \( b_0 := e_1 \sin \theta + e_2 \cos \theta \), the triple \((a_0, b_0, c_0)\) is an orthonormal basis of \( \Omega_x \) at each \( x \in \Omega_x \). Moreover, one has the identities

\[
\begin{align*}
b_0 \times v &= \tilde{v}_\perp a_0, \\
b_0 \times v \times b_0 &= \tilde{v}_\perp c_0.
\end{align*}
\]
Now let Ω′₀ denote the extended phase space with velocity coordinates (29), i.e. for q′ ∈ Ω′₀ we have \( q′ = (x,v∥,v⊥,θ,t) \). The preliminary map is thus
\[
\tau′ : Ω′₀ \rightarrow Ω₀, \quad q′ \mapsto q,
\] (31)
defined by
\[
x = x, \quad v = v∥b₀(x) + v⊥c₀(x,θ), \quad t = t,
\] (32)
with Jacobian determinat \(-v⊥\). The transformed Lagrangian \( L′ \) is obtained from (24) by inserting (32),
\[
L′ = \left[ v∥b₀(x) + v⊥c₀(x,θ) + \frac{A₀(x)}{ε} + A₁(x,t) \right] \cdot \dot{x}
- \left[ \frac{v²}{2} + \frac{v⊥²}{2} + \phi(x,t) \right] i,
\] (33)
It is straightforward to show from the Euler-Lagrange equations
\[
\frac{∂L′}{∂q′} - \frac{d}{ds} \frac{∂L′}{∂\dot{q}′} = 0
\] (34)
that \( θ \) is the fast gyro-angle, changing on the time scale \( ε \).

3. Algebraic GY-transformations

The second step of variational averaging requires a coordinate map \( \tau^ε : Ω_{GY} → Ω′₀, \) \( \tau^ε \rightarrow q′ \) which eliminates the fast variable \( α \rightarrow θ \) from the Lagrangian (33), order by order in \( ε \). The second transformation is thus assumed to be a finite power series in \( ε \), defined by
\[
q′ = \tau^ε(q_{GY}) := q_{GY} + \sum_{n=1}^{N+1} ε^n G_n(q_{GY}),
\] (35)
where \( N \geq 0 \) denotes the order of the transformation and the \( G_n : \Omega_{GY} → Ω′₀ \) are smooth maps, the so-called “generating functions” or generators of the transformation. They should be bounded uniformly in \( ε \), such that \( \lim_{ε→0} τ^ε = τ^0 \) is the identity. Note that one needs \( N+1 \) generators in the \( N \)-th order transformation and that these generators occur merely as coefficients in the \( ε \)-series (algebraic dependence on the generators). Component-wise, we denote
\[
q′ = \begin{pmatrix} x \\ v∥ \\ v⊥ \\ θ \\ t \end{pmatrix}, \quad q_{GY} = \begin{pmatrix} r \\ q∥ \\ q⊥ \\ α \\ t \end{pmatrix}, \quad G_n = \begin{pmatrix} q_n \\ G_n∥ \\ G_n⊥ \\ G_nα \\ 0 \end{pmatrix},
\] (36)
where \( x, r, q_n \in \mathbb{R}^3 \) denote the particle position, GY-position and \( n \)-th order position generator, respectively, \( q∥ \) and \( q⊥ \) are the respective parallel and perpendicular GY-velocities and \( α \) stands for the gyro-angle. The time coordinate \( t \) rests untransformed since we assume its generators to be zero at all orders. Moreover, from the definition of the tangent map one obtains
\[
\dot{x} = \dot{r} + \sum_{n=1}^{N+1} ε^n \dot{q}_n(q_{GY}, q_{GY}'), \quad \dot{q}_n := \frac{∂q_n}{∂q_{GY}'} \cdot \dot{q}_{GY}.
\] (37)
Starting from (33) the tangent map leads to the extended Lagrangian \( L' \) in the variables \( q_{GY} \),
\[
L'(q_{GY}, q'_{GY}) := L'(T\tau^ε(q_{GY}, q_{GY}')).
\] (38)

If we assume sufficiently regular potentials \( A \) and \( φ \), the definition of \( τ^ε \) as a power series in (35) translates to a Taylor expansion of \( L' \) around \( (q_{GY}, q_{GY}') \), leading to a representation of \( L' \) in the form (26). The generators \( G_n \) are still undetermined in this formulation. As outlined by Kruskal and Littlejohn\textsuperscript{31,34}, they can be chosen order by order such that the truncated Lagrangian \( L_{GY}^{(N)} \) is independent of the GY-angle \( α \).

Remark 1. The gyro-transformation (GT) that leads to the Lagrangian (26) will be composed of two transformations, \( τ^ε_{GY} = τ^ε ∘ τ^ε \), where \( τ^ε \) is the “preliminary” transformation (31), independent of \( ε \), and \( τ^ε \) denotes the algebraic GY-transformation (35). Even though \( τ^ε_{GY} \) is a composition, it must not be confused with the “two-step” GT\textsuperscript{9}, where only the static \( B₀ \) is considered at first (guiding-center problem) and only after the dynamical fields \( B₁ \) and \( E \) are taken into account. Indeed, the two-step GT is really a three-step GT since the preliminary transformation \( τ' \) is applied also in this case. Our procedure corresponds to what is known as the “one-step” GT.

III. MAIN RESULTS

A. Preliminaries

The main results have been arranged into three categories which are presented in the following three subsections: the present section contains a preliminary result on the existence of solutions to the initial value problem (9), under the here used regularity assumptions on the electromagnetic potentials:

Assumption 1. For \( N \geq 0 \) we suppose \( A₀ ∈ C^{N+3}(Ωₓ) \), \( A₁ ∈ C^{N+2}(Ωₓ × \mathbb{R}) \) and \( φ ∈ C^{N+1}(\mathbb{R}; C^{N+2}(Ωₓ)) \).

Theorem 1 of section III B states that this regularity is sufficient for the existence of a truncated Lagrangian \( L_{GY}^{(N)} \) in (26), independent of the gyro-angle, for arbitrary order \( N \). Finally, section III C concerns the error in the averaged dynamics due to truncation of the true Lagrangian. We give an exact definition of a gyrokinetic equation and compute a strong error bound for its solution with respect to the solution of the Vlasov equation (1) in our second main result, Theorem 2. Let us start with some useful definitions:
Definition 2. (Equivalence of Lagrangians.) Two Lagrangians \( L^*, L \) defined on \( TM \) are equivalent, \( L^* \sim L \), if there exists a function \( S : M \to \mathbb{R} \) such that \( L^* = L + \partial S / \partial q \cdot \dot{q} \) in some coordinates \( q \). Equivalent Lagrangians lead to the same Euler-Lagrange equations.

Definition 3. (GC-Lagrangian in normal form.) We define the normal form of guiding-center (GC) Lagrangians as

\[
L_{gc}(q_{\text{gy}}, \dot{q}_{\text{gy}}) := \left[ q_1 b_0(r) + A_0(r) \right] \cdot \dot{r} + \frac{q_2^2}{2B_0(r)} \dot{\alpha} - \frac{q_2^2}{2} i.
\]

The equivalence relation from Definition 2 allows us to pick representatives for each class of Lagrangians to work with. In this paper we pick (39) as the representative of GC-Lagrangians. Indeed, this is a particularly simple representative for first order theory. Geometric effects due to curvature of the magnetic background are added to this Lagrangian at higher orders of the expansion. The above normal form occurs in any gyrokinetic theory.

Definition 4. (Gyro-average.) The gyro-average and fluctuations of a function \( G \) that is \( 2\pi \)-periodic in \( \alpha \) are defined by

\[
\langle G \rangle(\cdot) := \frac{1}{2\pi} \int_0^{2\pi} G(\cdot, \alpha) \, d\alpha, \quad \tilde{G} := G - \langle G \rangle.
\]

Assumption 2. In the initial value problem (9) we denote by \( \rho \) the radius of the largest ball in \( \Omega_x \) containing \( x_0 \), that is \( \rho := \sup_{R \in \mathbb{R}} \{ x \in \Omega_x : |x - x_0| < R \} \). Moreover, the velocity space is bounded by a maximal kinetic energy, \( \Omega_v = \{ v \in \mathbb{R}^3 : |v|^2 < v_{\text{max}}^2 \} \), and \( \rho_{\text{kin}} := (v_{\text{max}}^2 - |v_0|^2)/2 \).

Lemma 1. Let \( \varepsilon > 0 \). Under the assumptions 1 and 2, the initial value problem (9) has a unique solution which exists for \( t \in [t_0, t_0 + T] \) with \( T = \min(\rho/v_{\text{max}}, \rho_{\text{kin}}/(E_{\text{max}}v_{\text{max}})) \), independent of \( \varepsilon \).

Proof. Due to assumption 1 the fields in (9) are continuous on a bounded domain \( \Omega_x \); hence the electric field has an upper bound, \( |E| < E_{\text{max}} \). We only need to check at which time the solution will leave \( \Omega \). Integrating the first equation and taking the norm yields

\[
|x(t) - x_0| \leq (t - t_0)v_{\text{max}} < \rho \implies (t - t_0) < \frac{\rho}{v_{\text{max}}},
\]

Multiplying the second equation by \( v \) and integrating in time leads to

\[
\frac{1}{2} (|v(t)|^2 - |v_0|^2) \leq (t - t_0)E_{\text{max}}v_{\text{max}} < \rho_{\text{kin}}
\]

\[
\implies (t - t_0) < \frac{\rho_{\text{kin}}}{E_{\text{max}}v_{\text{max}}},
\]

which determines the time interval \( T \).

B. Existence of an algebraic GY-map \( \tau^\varepsilon \)

In what follows we set \( \varepsilon_B = 1 \) and consider the maximal ordering defined in (8).

Proposition 1. (Series-expansion of \( L^\varepsilon \).) Under the assumption 1, the Lagrangian \( L^\varepsilon \) defined in (38) via the tangent map \( T\tau^\varepsilon \) is equivalent to the series expansion

\[
L^\varepsilon \sim \frac{1}{\varepsilon}L_{-1} + L_0 + \sum_{n=1}^N \varepsilon^n L_n + O(\varepsilon^{N+1}),
\]

with the terms

\[
L_{-1} = A_0 \cdot \dot{r},
\]

\[
L_0 = \left( q_1 b_0 + q_\perp c_0 - q_\parallel B_0 + A_1 \right) \cdot \dot{r} - \left( q_2^2/2 + q_\perp^2/2 + \varepsilon \right) i,
\]

\[
L_{1 \leq n \leq N} = \left( G_{1\parallel} b_0 + G_{1\perp} c_0 - \varepsilon_{n+1} \times B_0 - q_\parallel x B_1 \right) \cdot \dot{r} - \left( q_2 G_{n\parallel} + q_\perp G_{n\perp} - \varepsilon_{n} \cdot E \right) \dot{i} + Q_n \cdot \dot{\alpha},
\]

Here, the linear form \( Q_n \) and the Lagrangian \( L_n \) are given in (106) and (107), respectively. For \( n = 1 \) and \( n = 2 \) they can be written in terms of the fields \( B_0, B_1 \) and \( E \) (gauge-invariance).

Proof. The proof is written in section V A.

Theorem 1. (Existence of \( \tau^\varepsilon \).) Under the assumption 1, for all orders \( N \geq 1 \), there exist generators \( G_{n \leq N+1} \in C^2(\Omega_{\text{gy}}) \) of the algebraic GY-transformation \( \tau^\varepsilon \) such that \( L^\varepsilon \sim L_{\text{gy}}^{(N)} + O(\varepsilon^{N+1}) \) in (41), where the GY-Lagrangian reads

\[
L_{\text{gy}}^{(N)} = L_{\text{gc}} + A_1 \cdot \dot{r} - \phi \dot{i} + \varepsilon^2 \delta \mu^{(N)} \dot{\alpha}.
\]

Here, \( L_{\text{gc}} \) denotes the basic GC-Lagrangian defined in (39) and \( \delta \mu^{(N)} : \Omega_{\text{gy}} \to \mathbb{R} \) is the \( N \)-th order correction to the magnetic moment \( \mu := q_\perp^2/(2|B_0|) \), independent of the gyro-angle \( \alpha \).

Proof. The proof is written in section V B.

Remark 2. The existence result from Theorem 1 does not imply that the transformation \( \tau^\varepsilon \) exists as \( N \to \infty \), because we cannot say that the series (35) converges in this limit. Convergence of the series would mean that a gyro-center of the motion exists globally. This is true for the constant field case \( B = \text{const.} \) and \( E = \text{const.} \) but it is not clear in the non-homogeneous case. In practice, however, only low orders \( N \leq 2 \) are important for numerical purposes.
C. An error estimate for gyrokinetics

Due to the Euler-Lagrange equation (27), the result in Theorem 1 leads to the conservation of the generalized magnetic moment \( \tilde{\mu} \) during the GY-motion, where

\[
\tilde{\mu} := \frac{q_{\perp}^2}{2|B_0|} + \varepsilon \delta \mu^{(N)}. \tag{43}
\]

In order to reduce the dimension of the problem, \( \tilde{\mu} \) must be adopted as one of the coordinates. In particular, we shall assume that there is a one-to-one correspondence \( \tilde{\tau} : \tilde{\mu} \mapsto q_{\perp} \), which is the case in all of the results presented in section IV. The full GY-transformation from \( x \)-v-t-coordinates (extended phase space \( \Omega_t \)) to the GY-coordinates \( \tilde{q}_{gy} \in \tilde{\Omega}_{gy} \) with generalized magnetic moment, hence \( \tilde{q}_{gy} = (r, q_{\parallel}, \tilde{\mu}, \alpha, t) \), is given by the composition

\[
\tilde{\tau} : \tilde{\Omega}_t \to \tilde{\Omega}_{gy}, \quad \tilde{\tau} \circ \tau = \tau' \circ \tau' \circ \tilde{\tau}. \tag{44}
\]

It follows from Theorem 1 that the exact dynamics can be obtained from the Lagrangian

\[
L^\varepsilon \sim L_{gy}^{(N)} + O(\varepsilon^{N+1}), \tag{45}
\]

which is now written in terms of the coordinates \( \tilde{q}_{gy} \) as

\[
L_{gy}^{(N)} = \frac{1}{\varepsilon} A^* \cdot \dot{r} - H_{gy} \dot{t} + \varepsilon \tilde{\mu} \alpha, \tag{46}
\]

with the auxiliary potential \( A^* \) and the GY-Hamiltonian \( H_{gy} \) defined by

\[
A^* := A_0 + \varepsilon A_1 + \varepsilon q_{\parallel} b_0, \quad H_{gy} := \frac{q_{\perp}^2}{2} + \tilde{\mu}|B_0| + \phi + \varepsilon \delta H^{(N)}. \]

Here, the Hamiltonian correction \( \delta H^{(N)} \) stems from the transformation of the term \( q_{\perp}^2/2 \) under the map \( \tilde{\tau} \). It is a remarkable feature that \( \delta H^{(N)} \) is the only term in the Lagrangian that changes with the order \( N \) of the transformation. This means in particular that, in the coordinates \( \tilde{q}_{gy} \), the non-time components \( (A^*/\varepsilon, 0, 0, \varepsilon \tilde{\mu}) \) of the GY-symplectic form are the same for all \( N \).

Setting \( \dot{t} = 1 \) in (46), a straightforward computation yields the Euler-Lagrange equations of (45), here stated in terms of \( B^* = \nabla \times A^* \), \( B_{\parallel}^* = B^* \cdot b_0 \) and \( E^* := E - \tilde{\mu} \nabla |B_0| - \varepsilon \nabla \delta H^{(N)} \),

\[
\begin{align*}
\frac{d\tilde{\tau}}{dt} &= \left( q_{\parallel} + \varepsilon \frac{\delta H^{(N)}}{\partial q_{\parallel}} \right) B_{\parallel}^* + \varepsilon \frac{E^* \times b_0}{B_{\parallel}^*} + O(\varepsilon^{N+2}), \\
\frac{dq_{\parallel}}{dt} &= B_{\parallel} \cdot E^* + O(\varepsilon^{N+1}), \\
\frac{d\tilde{\mu}}{dt} &= O(\varepsilon^N), \\
\frac{d\alpha}{dt} &= \frac{|B_0|}{\varepsilon} + \frac{\partial \delta H^{(N)}}{\partial \tilde{\mu}} + O(\varepsilon^N),
\end{align*}
\tag{47}
\]

with the initial conditions at time \( t_0 \) computed from (9) via the map \( \tau_{gy} \). Let \( f_{gy} : \Omega_{gy} \to \mathbb{R}_+ \) denote the unique function which is constant along the solutions of (47), with initial condition \( f_{gy}(t = t_0) = f_{0,gy} \) strictly positive. Since (47) is merely a reformulation of the initial-value problem (9) in the coordinates \( \tilde{q}_{gy} \), we have \( f_{gy} = f \circ \tau_{gy} \), where \( f \) is the unique solution of the Vlasov equation (1) with initial condition \( f_0 = f_{0,gy} \circ (\tau_{gy})^{-1} \). Regarding existence of \( f_{gy} \) we remark:

**Lemma 2.** A solution of the problem (47) exists, is unique and continuous on the interval \([t_0,t_0+T]\), where \( T \) is given in Lemma 1.

**Proof.** This follows from Lemma 1 and the fact that (47) is equivalent to the initial value problem (9) if \( \tilde{q}_{gy,0} = (\tau_{gy})^{-1}(x_0, v_0, t_0) \). The inverse of the algebraic GY-map \( \tau_{gy} = \tau' \circ \tau' \circ \tilde{\tau} \) exists at least locally due to the fact \( \tau' \) is continuously differentiable because all generators are (Theorem 1).

If one truncates the residual terms of order \( O(\varepsilon^{k\geq N}) \) in (47), one obtains the “decoupled dynamics”, which are the Euler-Lagrange equations of the truncated Lagrangian (46):

\[
\begin{align*}
\frac{d\tilde{\tau}}{dt} &= \left( q_{\parallel} + \varepsilon \frac{\delta H^{(N)}}{\partial q_{\parallel}} \right) B_{\parallel}^* + \varepsilon \frac{E^* \times b_0}{B_{\parallel}^*}, \\
\frac{dq_{\parallel}}{dt} &= B_{\parallel} \cdot E^*, \\
\frac{d\tilde{\mu}}{dt} &= 0, \\
\frac{d\alpha}{dt} &= \frac{|B_0|}{\varepsilon} + \frac{\partial \delta H^{(N)}}{\partial \tilde{\mu}}.
\end{align*}
\tag{48}
\]

Note that the system (48) is obtained from the exact dynamics (47) by truncation, not by performing some sort of averaging operation. We shall distinguish the solutions of the truncated equations (48) from the solutions of the exact equations (47) by an overbar. Moreover, some well-known particle drifts are contained in the equation for the gyro-center \( \tau \): the electric cross-field drift and the grad-B drift can be obtained by expanding the term \( E^* \times b_0/B_{\parallel}^* \) in powers of \( \varepsilon \) and the curvature drift is obtained by expanding \( B^*/B_{\parallel}^* \) in powers of \( \varepsilon \).

**Definition 5.** (Gyrokinetic equation.) The solution \( F : \Omega_{gy} \to \mathbb{R}_+ \) of a gyrokinetic equation is a strictly positive function, constant along the solutions of the decoupled dynamics (48),

\[
\frac{\partial F}{\partial t} + \frac{\partial \tau}{\partial t} \cdot \frac{\partial F}{\partial \tau} + \frac{\partial q_{\parallel}}{\partial t} \frac{\partial F}{\partial q_{\parallel}} + \frac{\partial \tilde{\mu}}{\partial t} \frac{\partial F}{\partial \tilde{\mu}} + \frac{\partial \alpha}{\partial t} \frac{\partial F}{\partial \alpha} = 0, \tag{49}
\]

\( F(t_0) = F_0 \).
Lemma 3. The gyro-average \( \langle F \rangle \) and fluctuations \( \bar{F} \) of a solution \( F = F(r, q_\parallel, \mu, \alpha, t) \) to (49) evolve independently in time (decoupling); they satisfy

\[
\frac{\partial \langle F \rangle}{\partial t} + \frac{d\tau}{dt} \cdot \frac{\partial \langle F \rangle}{\partial r} + \frac{d\eta_\parallel}{dt} \cdot \frac{\partial \langle F \rangle}{\partial \eta_\parallel} = 0, \tag{50}
\]

and

\[
\frac{\partial \bar{F}}{\partial t} + \frac{d\tau}{dt} \cdot \frac{\partial \bar{F}}{\partial r} + \frac{d\eta_\parallel}{dt} \cdot \frac{\partial \bar{F}}{\partial \eta_\parallel} + \frac{d\tau}{dt} \cdot \frac{\partial \bar{F}}{\partial \alpha} = 0, \tag{51}
\]

\( \bar{F}(t_0) = \bar{F}_0 \).

Proof. This is an immediate consequence of the definition 4 of the gyro-average and the fact that the flow field (ie. the right-hand-side) of (48) is independent of the gyro-angle \( \tau \).

In what follows we denote by \( z := (r, q_\parallel, \mu) \) the first five components of the phase space variables \( \tilde{a}_{\text{gy}} \). Correspondingly, in the decoupled dynamics (48), the “slow trajectories” \( z(t) := (\tau(t), \eta_\parallel(t), \mu(t)) \) evolve independently of the gyro-angle \( \tau \), which varies with the frequency \( 1/\varepsilon \). From (47) and (48) we can extract two subproblems, namely

\[
\frac{dz}{dt} = \Lambda(z, t, \varepsilon) + \varepsilon^N S(z, \alpha, t, \varepsilon), \tag{52}
\]

\( z(t_0) = z_0 \),

and, respectively,

\[
\frac{d\tau}{dt} = \Lambda(z, t, \varepsilon), \tag{53}
\]

\( z(t_0) = z_0 \).

Here, \( \Lambda \) stands for the flow field of the slow variables in (48) and \( S \) comprises the residual terms \( O(\varepsilon^{k+N}) \) of (47).

Remark 3. The flow field \( \Lambda \) is independent of the gyro-angle \( \tau \) by construction. From assumption 1 we deduce that it is a \( C^1 \)-function of \( (z, t, \varepsilon) \). In particular, \( \Lambda \) is Lipschitz in \( z \), uniformly in \( (t, \varepsilon) \),

\[
\| \Lambda(y, t, \varepsilon) - \Lambda(z, t, \varepsilon) \| \leq \ell_\Lambda \| y - z \|,
\]

for some vector norm \( \| \cdot \| \), where the Lipschitz constant \( \ell_\Lambda \) is independent of \( (t, \varepsilon) \).

Remark 4. The residual term \( S \) depends on \( \alpha \). It is obtained from the \( O(\varepsilon^{N+1}) \)-terms in the Lagrangian (45) via the Euler-Lagrange equations. Since these \( O(\varepsilon^{N+1}) \)-terms are the residuals in the Taylor expansion (60), they are \( C^1 \)-functions of \( (z, \alpha, t) \). Therefore \( S \), being a power series in \( \varepsilon \), is continuous in \( (z, \alpha, t, \varepsilon) \).

Lemma 4. Consider the IVPs (52) and (53) for the “slow” GY-variables on the interval \( t \in I = [t_0, t_1] \) with \( t_1 \leq T \) (Lemma 2), then

\[
\| \langle F \rangle(t) - \langle F \rangle(t_0) \| \leq \varepsilon^N \| S \|_A (e^{\kappa(t-t_0)} - 1) + \| z_0 - z_0 \| e^{\kappa(t-t_0)},
\]

where \( \| S \| = \max_{t_0} |S| \| S \| = O(1) \) as \( \varepsilon \to 0 \).

Proof. The proof is written in section V.C.

Theorem 2. Let \( \langle F \rangle \) stand for the unique solution of the averaged gyrokinetic equation (50) on the interval \( I = [t_0, t_1] \). Moreover, let \( f \) denote the unique solution of the Vlasov equation (1) with initial data \( f_0 \), and introduce \( f_{\text{gy}} := f \circ \tau_{\text{gy}}^\varepsilon \), the exact kinetic solution in GY-phase space, where \( \tau_{\text{gy}}^\varepsilon \) denotes the transformation (44) of order \( N \), with initial condition \( f_{\text{gy}}(t_0) = f_0 \circ \tau_{\text{gy}}^\varepsilon \). Suppose

1. \( \langle F \rangle(t_0) = \langle f_{\text{gy}} \rangle \), is Lipschitz with constant \( \ell_0 \),

2. \( f_{\text{gy}}(t_0) + \varepsilon^N \tilde{f}_{\text{gy}} \), with continuous perturbation \( \tilde{f}_{\text{gy}} \) of order one as \( \varepsilon \to 0 \).

Then, denoting \( z = (r, q_\parallel, \mu) \), for \( t \in I \) one has

\[
\max_{z, \alpha} \| \langle F \rangle(z, \alpha, t) - f_{\text{gy}}(z, \alpha, t) \| \leq \varepsilon^N C(t),
\]

with

\[
C(t) = \ell_0 \| S \|_A (e^{\kappa(t-t_0)} - 1) + \max_{z, \alpha} | \tilde{f}_{\text{gy}}(z, \alpha, t) |,
\]

where \( | S | \) is defined in Lemma 4.

Proof. Let \( \Phi_{s,t} : \tilde{q}_{\text{gy}} \to \tilde{q}_{\text{gy}} \) stand for the local flow map of problem (47), i.e. \( \Phi_{s,t}(z, \alpha) \) is the solution of (47) at time \( s \) which is at \( (z, \alpha) \) at time \( t \), and \( \Phi_{t,t} = \text{id}_{\tilde{q}_{\text{gy}}} \). We shall denote the “slow” components of the flow by \( Z_{s,t} \), hence \( Z_{t,t}(z, \alpha) = z \) and \( Z_{t,t}(z_0, \alpha_0) = z(t) \), solution of the subproblem (52). Using that \( f_{\text{gy}} \) is constant along solutions of (47) we may write

\[
f_{\text{gy}}(z, \alpha, t) = f_{\text{gy}}(\Phi_{t,t}(z, \alpha), t)
\]

\[
= f_{0,\text{gy}}(\Phi_{t,0}(z, \alpha))
\]

\[
= \langle f_{0,\text{gy}} \rangle(\tilde{Z}_{t,0}(z, \alpha)) + \varepsilon^N \tilde{f}_{\text{gy}}(\Phi_{t,0}(z, \alpha)).
\]

Our aim is to compare this expression to \( \langle F \rangle \), solution of (50). For this let us denote by \( \tilde{Z}_{s,t} \) the flow map of the subproblem (53), hence \( \tilde{Z}_{t,t}(z) = z \) and \( \tilde{Z}_{t,t}(z_0, \alpha_0) = z(t) \). Since \( \langle F \rangle \) is constant along solutions of (53) we have

\[
\langle F \rangle(z, t) = \langle F \rangle(\tilde{Z}_{t,t}(z), t)
\]

\[
= \langle F \rangle(\tilde{Z}_{0,t}(z))
\]

\[
= \langle f_{0,\text{gy}} \rangle(\tilde{Z}_{t,0}(z)).
\]
Therefore, using the Lipschitz continuity of \( f_{0,\text{gy}} \) as well as the continuity of \( \tilde{f}_{0,\text{gy}} \), we obtain
\[
\left| \langle f'(z,t) - f_{\text{gy}}(z,\alpha,t) \rangle \right| \\
\leq \left| \langle f_{0,\text{gy}}(Z_{t_0},t)(z) - f_{0,\text{gy}}(Z_{t_0},t)(\alpha) \rangle \right| + O(\varepsilon^N) \\
\leq \ell_0 \left| Z_{t_0},t(z) - Z_{t_0},t(\alpha) \right| + O(\varepsilon^N).
\]
The difference in the flow functions can be estimated from Lemma 4, thus \( \forall (z,\alpha,t) \in \Omega_{\text{gy}} \cap I, \)
\[
\left| \tilde{Z}_{t_0},t(z) - \tilde{Z}_{t_0},t(\alpha) \right| \leq \varepsilon^N \left( \frac{S}{\ell_\Lambda} (e^{\lambda(t-t_0)} - 1) \right).
\]
\[\square\]

IV. EXPRESSIONS FOR \( \delta\mu^{(N)}, \delta H^{(N)} \) AND THE GENERATORS

Theorem 1 states the existence of algebraic GY-maps \( \pi^1 \) that lead to the reduced dynamics implied by the Lagrangian (42). Here we give some concrete examples of such transformations for the two different scalings (8) of \( \varepsilon_B \) and for the orders \( N = 1 \) and \( N = 2 \), respectively. We stress that the choice for the generators in these transformations is not unique for two reasons:

1. the equivalence of Lagrangians that differ by a “total time derivative” allows us to add or subtract such a term,

2. gyro-averages of the generators could be kept in the Lagrangian rather than in the transformation; one then still obtains a set of decoupled equations of motion, albeit a rather different one, cf. the discussion in remark 5.

Moreover, we state the derived expressions for the correction \( \delta\mu^{(N)} \) to the magnetic moment, the map \( \tilde{\tau} \) between \( \tilde{\mu} \) and \( q_\perp \), as well as the correction \( \delta H^{(N)} \) to the Hamiltonian. The proof of the following Lemmas is left as an exercise to the reader; it consists of performing the steps described in the proof of Theorem 1.

Lemma 5. (Small background variations \( \varepsilon_B = \varepsilon \), \( N = 1 \).) In this case possible generators read
\[
\begin{align*}
\varrho_1 &= \frac{q_\perp}{|B_0|} a_0, \\
G_1 &= -\frac{q_\perp}{|B_0|} B_1 \cdot c_0, \\
G_1^\parallel &= \frac{q_\perp}{|B_0|^2} B_1 \cdot c_0 + \frac{1}{|B_0|} E \cdot a_0, \\
\varrho_2 &= \left( \frac{q_\perp}{|B_0|^2} B_1 \cdot c_0 - \frac{q_\perp}{|B_0|^2} B_1 \cdot b_0 + \frac{1}{|B_0|^2} E \cdot a_0 \right) a_0 \\
&\quad + (\varrho_2 \cdot b_0) b_0 + \frac{q_\perp}{|B_0|^2} G_1^\parallel c_0,
\end{align*}
\]
where \( \varrho_2 \cdot b_0 \) and \( G_1^\parallel \) are arbitrary. This choice of the generators leads to \( \delta\mu^{(1)} = 0 \), therefore \( \tilde{\mu} = \mu \), the map \( \tilde{\tau} : \tilde{\mu} \mapsto q_\perp \) is given by \( q_\perp = \sqrt{2\tilde{\mu}/|B_0|} \) and the Hamiltonian correction is zero,
\[
\delta H^{(1)} = 0.
\]

Lemma 6. (Small background variations \( \varepsilon_B = \varepsilon \), \( N = 2 \).) In this case possible generators are the functions in (54) along with
\[
\begin{align*}
\varrho_2 \cdot b_0 &= \frac{\partial S_2^{*\ast}}{\partial q_\parallel}, \\
G_1^\parallel &= \frac{|B_0|}{q_\perp} \frac{\partial S_2^{*\ast}}{\partial q_\perp}, \\
G_2 &= (\varrho_2 \times B_1)_0 - Q_2^{*\ast} \cdot b_0, \\
G_2^\parallel &= -\frac{q_\perp}{q_\perp} G_2^\parallel + \frac{1}{q_\perp} \varrho_2 \cdot E + \frac{q_\perp}{2|B_0|^2} a_0 \cdot \nabla E \cdot a_0 \\
&\quad + \frac{1}{q_\perp} \frac{\partial S_2^{*\ast}}{\partial t}, \\
\varrho_3 &= \frac{1}{|B_0|} \left[ G_2^\parallel + (\varrho_2 \times B_1)_0 \cdot c_0 + Q_2^{*\ast} \cdot c_0 \right] a_0 \\
&\quad + (\varrho_3 \cdot b_0) b_0 + \frac{1}{|B_0|} \left( (\varrho_2 \times B_1)_0 \cdot a_0 - Q_2^{*\ast} \cdot a_0 \right) c_0,
\end{align*}
\]
where \( Q_2^{*\ast} \) is given by
\[
Q_2^{*\ast} = \frac{q_\perp^2}{2|B_0|^2} (a_0 \cdot \nabla B_0) \times a_0 + \frac{q_\perp^2}{2|B_0|^2} (a_0 \cdot \nabla B_1) \times a_0 \\
- \frac{q_\perp}{|B_0|} a_0 \times (\nabla \times b_0) - \frac{q_\perp^2}{|B_0|} a_0 \times (\nabla \times c_0) \\
- G_1^\parallel G_1^\parallel a_0 - q_\perp G_2^\parallel a_0 \\
- \frac{q_\perp}{2} (G_1^\parallel)^2 c_0 - \frac{q_\perp^2}{2|B_0|^2} R,
\]
with \( R = \nabla a_0 \times c_0 = \nabla e_2 \cdot e_1 \) denoting the gyro-gauge term, \( \varrho_3 \cdot b_0 \) and \( G_2^\parallel \) are arbitrary and \( S_2^{*\ast} \) reads
\[
S_2^{*\ast} = -\frac{q_\perp}{|B_0|^2} \varrho_1 \cdot B_1 \cdot a_0 + \frac{q_\perp}{|B_0|^2} E \cdot c_0.
\]
This leads to \( \delta\mu^{(2)} = \mu \sigma^{**} \) with \( \sigma^{**} := -B_1 \cdot b_0/|B_0| \), independent of \( q_\perp \). Therefore, \( \tilde{\mu} = \mu(1 + \varepsilon \sigma^{**}) \), the map \( \tilde{\tau} : \tilde{\mu} \mapsto q_\perp \) is given by \( q_\perp = \sqrt{2\tilde{\mu}/|B_0|}/(1 + \varepsilon \sigma^{**}) \) and the Hamiltonian correction reads
\[
\delta H^{(2)} = -\frac{\tilde{\mu}}{|B_0|} \frac{\sigma^{**}}{1 + \varepsilon \sigma^{**}}.
\]

Remark 5. Standard second-order gyrokinetic Lagrangians in the long-wavelength approximation\(^{44}\) used for example in the codes GENE and ORB5\(^7,20\), can be recovered from Lemma 6. However, our choice of the generators differs from the conventional ones and leads to simpler equations of motion. For example, the polarization
generators read 

Lemma 7. \((\varepsilon_B = 1, N = 1.)\) In this case possible generators read

\[
\phi_1 = \frac{q_\perp}{|B_0|} a_0, \\
G_1^\parallel = -\frac{q_\perp}{|B_0|} B_1 \cdot c_0 + \frac{q_\perp^2}{2|B_0|^2} a_0 \cdot \nabla b_0 \cdot c_0 \\
- \frac{q_\perp^2 q_\perp}{|B_0|^3} (\nabla \times b_0) \cdot c_0 - \frac{q_\perp^2}{2|B_0|} R \cdot b_0, \\
G_1^\perp = \frac{q_\perp}{|B_0|} G_1^\perp + \frac{1}{|B_0|} a_0 \cdot E, \\
\phi_2 = \left[ \frac{G_1^\perp}{|B_0|} - \frac{q_\perp}{|B_0|^2} B_1 \cdot b_0 - \frac{q_\perp^2}{2|B_0|^3} a_0 \cdot \nabla |B_0| \\
- \frac{q_\perp^2 q_\perp}{|B_0|^3} (\nabla \times b_0) \cdot b_0 + \frac{q_\perp^2}{2|B_0|^2} R \cdot c_0 \right] a_0 \\
+ (\phi_2 \cdot b_0) b_0 + \left[ \frac{q_\perp q_\perp}{|B_0|^2} + \frac{q_\perp^2}{2|B_0|^2} R \cdot a_0 \right] c_0, \\
\]

where \(R = \nabla a_0 \cdot c_0 = \nabla a_0 \cdot e_1\) is the gyro-gauge term and \(\phi_2 \cdot b_0\) and \(G_1^\perp\) are arbitrary. This leads to \(\delta H^{(1)} = 0\), therefore \(\tilde{\mu} = \mu\), the map \(\tilde{\mu} : q_\perp \mapsto q_\perp\) is given by \(q_\perp = \sqrt{2} \tilde{\mu} |B_0|\) and the Hamiltonian correction is zero,

\[
\delta H^{(1)} = 0. 
\]

Lemma 8. \((\varepsilon_B = 1, N = 2.)\) In this case possible generators are the functions in (56) along with

\[
\phi_2 \cdot b_0 = \frac{\partial S_2}{\partial q_\parallel} - \frac{q_\perp^2}{2|B_0|^2} a_0 \cdot \nabla b_0 \cdot a_0, \\
G_1^\parallel = \frac{|B_0|}{q_\perp} \frac{\partial S_2}{\partial q_\perp} + \frac{q_\perp}{2|B_0|} a_0 \cdot R, \\
G_2^\parallel = (\phi_2 \times B_1) \cdot b_0 - Q_2^* \cdot b_0, \\
G_2^\perp = -\frac{q_\perp}{q_\perp} G_2^\perp + \frac{1}{q_\perp} \phi_2 \cdot E + \frac{q_\perp}{2|B_0|^2} a_0 \cdot \nabla E \cdot a_0 \\
+ \frac{1}{q_\perp} \partial S_2^*, \\
\phi_3 = \frac{1}{|B_0|} \left[ G_2^\perp - (\phi_2 \times B_1) \cdot c_0 + Q_2^* \cdot c_0 \right] a_0 \\
+ (\phi_3 \cdot b_0) b_0 + \frac{1}{|B_0|} \left[ (\phi_2 \times B_1) \cdot a_0 - Q_2^* \cdot a_0 \right] c_0, \\
\]

where \(\phi_2 \cdot b_0\) and \(G_2^\parallel\) are arbitrary and \(S_2^*\) reads

\[
S_2^* = -\frac{q_\perp^2}{|B_0|^2} B_1 \cdot a_0 + \frac{q_\perp}{|B_0|^2} E \cdot c_0 \\
+ \frac{q_\perp^2}{4|B_0|^2} (a_0 \cdot \nabla b_0 \cdot a_0) - \frac{q_\perp^2}{2|B_0|^2} (\nabla \times b_0) \cdot a_0 \\
- \frac{q_\perp^3}{3|B_0|^3} c_0 \cdot \nabla |B_0| + \frac{q_\perp^3}{2|B_0|^2} (\nabla \times a_0) \cdot b_0. \\
\]

This leads to \(\delta \mu^{(2)} = \mu \sigma^*\) with

\[
\sigma^* := -\frac{1}{|B_0|} \left[ B_1 \cdot b_0 + \frac{1}{2} q_\parallel (\nabla \times b_0) \cdot b_0 \right] + \frac{2q_\parallel}{|B_0|} R \cdot b_0, \\
\]

independent of \(q_\perp\). Therefore, \(\tilde{\mu} = \mu(1 + \varepsilon \sigma^*)\), the map \(\tilde{\mu} : q_\perp \mapsto q_\perp\) is given by \(q_\perp = \sqrt{2} \tilde{\mu} |B_0|/\left(1 + \varepsilon \sigma^*\right)\) and the Hamiltonian correction reads

\[
\delta H^{(2)} = -\tilde{\mu} |B_0| \frac{\sigma^*}{1 + \varepsilon \sigma^*}. \\
\]

The term \(Q_2^*\) is given by
Taylor’s theorem to write a series expansion in \( L \), the Lagrangian

strategy in mind: first, for \( (\alpha, \phi) \) estimated the remainder. In part two we compute the

V. PROOFS

A. Proof of Proposition 1

The proof is split into three sections, with the following strategy in mind: first, for \((q_{gy}, \dot{q}_{gy})\) fixed, we consider the Lagrangian \( L^\varepsilon \) as a function of \( \varepsilon \) and apply

Taylor’s theorem to write a series expansion in \( \varepsilon \) and estimate the remainder. In part two we compute the

coefficients of this series expansion. This step involves a

lot of algebra. Finally, we rewrite the series expansion

of the Lagrangian so as to arrive at a gauge-invariant

representation. Let us introduce the following notation

for the gyro-transformation \( \tau^\varepsilon \) defined in (35)-(36):

\[
Q_2 = \frac{q}{2|B_0|} (q_2 \cdot \nabla B_0) \times a_0 + \frac{q}{2|B_0|} (a_0 \cdot \nabla B_0) \times q_2 + \frac{q^2}{2|B_0|^2} (a_0 \cdot \nabla B_1) \times a_0
- G^\varepsilon_2 \frac{q}{|B_0|} a_0 \times (\nabla \times b_0) - G^\varepsilon_2 \frac{q}{|B_0|} a_0 \times (\nabla \times c_0) - G^\varepsilon_2 \frac{q}{|B_0|} a_0
- q_2 \times (\nabla \times b_0) - q_1 \times (\nabla \times c_0) - q_2 \times \dot{q}_2 \cdot b_0 a_0
- G^\varepsilon_2 \frac{q^3}{|B_0|^3} (\nabla a_0 \times a_0) \cdot (a_0 \cdot \nabla B_0) - \frac{q^3}{2|B_0|^2} (\nabla a_0 \times a_0) \cdot B_1
- \frac{q^2}{2|B_0|^2} (\nabla a_0 \times a_0) \cdot (\nabla \times b_0) - \frac{q^2}{2|B_0|^2} (\nabla a_0 \times a_0) \cdot (\nabla \times c_0) + \frac{\partial S_2^\varepsilon}{\partial \tau},
\]

where \( \nabla \) only acts on \( \nabla b_0 \) and on \( \nabla c_0 \), respectively.

The time coordinate \( t \) rests untransformed. For \((q_{gy}, \dot{q}_{gy})\) fixed, we consider the Lagrangian \( (38) \) as a function of \( \varepsilon \), split into three parts,

\[
L^\varepsilon = \varphi_0(\varepsilon)/\varepsilon + \varphi_1(\varepsilon) - \varphi_2(\varepsilon) \varepsilon, \tag{58}
\]

with

\[
\varphi_0(\varepsilon) := \dot{x} \cdot A_0(x), \tag{59a}
\]

\[
\varphi_1(\varepsilon) := v_{||,\varepsilon} \cdot b_0(x) + v_{\perp,\varepsilon} \cdot c_0(x, \theta_x) + \dot{x} \cdot A_1(t, x), \tag{59b}
\]

\[
\varphi_2(\varepsilon) := \frac{v^2_{\perp,\varepsilon}}{2} + \frac{v^2_{||,\varepsilon}}{2} + \phi(t, x). \tag{59c}
\]

From assumption 1 we have \( A_0 \in C^{N+3}(\Omega_x) \) and thus \( \varphi_0 \in C^{N+3}([0, \varepsilon_{\max}]) \), since \( \dot{x} \varepsilon \) is a polynomial in \( \varepsilon \). Also, \( b_0, c_0, A_1, \phi \in C^{N+2}(\Omega_x) \) which implies \( \varphi_1, \varphi_2 \in C^{N+2}([0, \varepsilon_{\max}]) \). Hence, we may apply Taylor’s theorem and write

\[
\varphi_0(\varepsilon) = \sum_{j=0}^{N+1} \frac{\varepsilon^j}{j!} \varphi_0^{(j)}(0) + O(\varepsilon^{N+2}),
\]

\[
\varphi_1(\varepsilon) = \sum_{j=0}^{N} \frac{\varepsilon^j}{j!} \varphi_1^{(j)}(0) + O(\varepsilon^{N+1}), \tag{60}
\]

\[
\varphi_2(\varepsilon) = \sum_{j=0}^{N} \frac{\varepsilon^j}{j!} \varphi_2^{(j)}(0) + O(\varepsilon^{N+1}),
\]

where \( \varphi^{(j)} \) denotes the \( j \)-th derivative of \( \varphi \) with respect to \( \varepsilon \). The remainders are still \( C^1 \). This kind of regularity
is necessary in the proof of Lemma 4 where, in order to apply the Grönwall’s lemma, one needs the residual terms in the Euler-Lagrange equations to be continuous, which is guaranteed by the remainders being in $C^1$.

1. **Taylor coefficients**

Let us now compute the coefficients of the $\varepsilon$-polynomials (60). For $j = 0$ one has

\[
\varphi_0(0) = A_0(r) \cdot \dot{r}, \\
\varphi_1(0) = \left[ q_0 \cdot b_0(r) + q_1 c_0(r) + A_1(t, r) \right] \cdot \dot{r}, \\
\varphi_2(0) = q_1^2 + \frac{q_2^2}{2} + \phi(t, r).
\]

In order to fix ideas, let us carry out the computation leading to the zeroth order Lagrangian $L_0$ explicitly (the Lagrangian $L_{-1}$ is readily given by $\varphi_0(0)$). Due to (58) and (60), we need the derivative of $\varphi_0(\varepsilon)$, evaluated at $\varepsilon = 0$. Hence,

\[
\varphi^{(1)}_0(0) = \left[ \frac{\partial \varphi}{\partial \varepsilon} \right]_{\varepsilon=0} + \left[ \dot{x} \cdot \left( \frac{\partial \varphi}{\partial \varepsilon} \cdot \nabla A_0 \right) \right]_{\varepsilon=0} \\
= \dot{\varphi}_0 \cdot A_0 + \dot{\varphi}_1 \cdot [q_0 \cdot (q_1 \cdot \nabla A_0)] \\
= \frac{d}{ds} [q_0 \cdot A_0(r)] - \nabla A_0 \cdot \dot{\varphi}_1 + \dot{\varphi}_1 \cdot \nabla A_0 \cdot \dot{r} \\
\sim -[q_1 \times B_0(r)] \cdot \dot{r}.
\]

In the third line we used the product rule for elements of a tangent space and, to get to the fourth line, we applied the equivalence relation from Definition 2 to remove a “total time derivative”, and used that $\nabla A_0 \cdot \dot{\varphi}_0 + \dot{\varphi}_1 \cdot \nabla A_0 = \dot{q}_1 \cdot (\nabla \times A_0)$. The zeroth order Lagrangian from Proposition 1 is now readily obtained as $L_0 = \varphi^{(1)}_0(0) + \varphi^{(1)}_1(0) - \varphi^{(2)}_2(0) \dot{t}$. The first order Lagrangian is computed in a similar fashion. The second derivative of $\varphi_0$ evaluated at $\varepsilon = 0$ reads

\[
\frac{\varphi^{(2)}(0)}{2} = \dot{\varphi}_2 \cdot A_0 + \dot{\varphi}_1 \cdot [q_2 \cdot (q_1 \cdot \nabla A_0)] \\
+ \dot{\varphi}_1 \cdot \nabla A_0 \cdot \dot{\varphi}_1 + \frac{1}{2} (q_1 \cdot \nabla A_0^2) \cdot \dot{r}.
\]

Here, in the first line we apply the same equivalence as in (62). In the second line we have the term

\[
q_1 \cdot \nabla A_1 \cdot \dot{\varphi}_1 = \frac{1}{2} q_1 \cdot \nabla A_1 \cdot \dot{\varphi}_1 + \frac{1}{2} \frac{d}{ds} (q_1 \cdot \nabla A_0 \cdot \dot{\varphi}_1) \\
- \frac{1}{2} (\dot{q}_1 \cdot \nabla A_0 \cdot \dot{\varphi}_1 + q_1 \cdot (\dot{\varphi}_1 \cdot \nabla A_0) \cdot \dot{\varphi}_1) \\
\sim -\frac{1}{2} \dot{\varphi}_1 \cdot (q_1 \times B_0) - \frac{1}{2} \dot{\varphi}_1 \cdot \nabla (\dot{\varphi}_1 \cdot \nabla A_0) \cdot \dot{\varphi}_1.
\]

Therefore, using the equivalence (64) in (63) leads to

\[
\frac{\varphi^{(2)}(0)}{2} \sim -[q_2 \times B_0(r)] \cdot \dot{r} - \frac{1}{2} \dot{\varphi}_1 \cdot [q_1 \times B_0(r)] \\
+ \frac{1}{2} [(q_1 \cdot \nabla B_0(r)) \times \dot{\varphi}_1] \cdot \dot{r}.
\]

The first derivatives of $\varphi_1$ and $\varphi_2$, respectively, evaluated at $\varepsilon = 0$, read

\[
\varphi^{(1)}_1(0) = G_1 \cdot \dot{r} \cdot b_0 + q_1 \dot{\varphi}_1 + q_1 \cdot \nabla b_0 \\
+ G_1 \cdot \dot{r} \cdot c_0 + q_1 \dot{\varphi}_1 + \dot{\varphi}_1 \cdot (\nabla c_0 - G_1^a a_0) \\
+ \dot{\varphi}_1 \cdot A_1 + \dot{r} \cdot (q_1 \cdot \nabla A_1), \\
\varphi^{(1)}_2(0) = q_1 [G_1^a + q_1 G_1^a] + \dot{\varphi}_1 \cdot \nabla \phi.
\]

In the last line of (66) we can use

\[
\dot{\varphi}_1 A_1 + \dot{r} \cdot (q_1 \cdot \nabla A_1) \sim -[q_1 \times B_1] \cdot \dot{r} - \varphi_1 \cdot \frac{\partial A_1}{\partial t} \cdot \dot{t}.
\]

Moreover, in the first line of (66) we have

\[
q_1 \dot{\varphi}_1 + q_1 \cdot \nabla b_0 \sim -q_1 \dot{\varphi}_1 \cdot b_0 \\
- q_1 \left[ \varphi_1 \times (\nabla \times b_0) \right] \cdot \dot{r}.
\]

and in the second line of (66) we have

\[
q_1 \dot{\varphi}_1 + q_1 \cdot \nabla c_0 \sim -q_1 \dot{\varphi}_1 \cdot c_0 \\
+ q_1 \cdot \dot{\varphi}_1 \cdot (q_1 \cdot \nabla c_0) - q_1 \left[ \varphi_1 \times (\nabla \times c_0) \right] \cdot \dot{r}.
\]

Using these last three formulas in (66), the Lagrangian $L_1 = \varphi^{(2)}_0(0) + \varphi^{(1)}_1(0) - \varphi^{(2)}_2(0) \dot{t}$ corresponds exactly to the result stated in Proposition 1.

Let us now consider the case of arbitrary order $n$. To compute the $\varepsilon$-derivatives of order $j$ of products, we use the Leibniz rule,

\[
(\varphi \chi)^{(j)} = \sum_{l=0}^{j} \binom{j}{l} \varphi^{(j-l)} \chi^{(l)}.
\]

For polynomials in $\varepsilon$ we use the formula

\[
\left[ \sum_{n=0}^{N+1} \varepsilon^n \varphi_n \right]^{(j)} = \sum_{n=j}^{N+1} \frac{n!}{(n-j)!} \varepsilon^{n-j} \varphi_n,
\]

which leads to

\[
\left[ \sum_{n=0}^{N+1} \varepsilon^n \varphi_n \right]^{(j)}(0) = j! \varphi_j.
\]

The Taylor expansion of a function $b_0(r + s)$ around $r$ reads

\[
b_0(r + s) = b_0(r) + s \cdot \nabla b_0(r) \\
+ \frac{1}{2} (s \cdot \nabla)^2 b_0(r) + \frac{1}{6} (s \cdot \nabla)^3 b_0(r) + \ldots
\]
With the Leibniz rule (71), for \( j \geq 1 \) one computes
\[
b_0^{(j)}(x_\varepsilon) = (x_\varepsilon^{(1)} \cdot \nabla) b_0^{(j-1)}
\]
\[
= \sum_{k_1=0}^{j-1} \binom{j-1}{k_1} (x_\varepsilon^{(j-k_1)} \cdot \nabla) b_0^{(k_1)}
\] (74)
\[
= x_\varepsilon^{(j)} \cdot \nabla b_0 + \sum_{k_1=1}^{j-1} \binom{j-1}{k_1} (x_\varepsilon^{(j-k_1)} \cdot \nabla) b_0^{(k_1)}.
\]

Here and in the following, for sums we will use the convention
\[
\sum_{k=k_{\text{start}}}^{j} \ldots = 0 \quad \text{if} \quad j < k_{\text{start}}.
\] (75)

By iteration, applying the rule (74) to the last term \( b_0^{(k_1)} \) in (74) yields
\[
b_0^{(j)}(x_\varepsilon) = x_\varepsilon^{(j)} \cdot \nabla b_0 + \sum_{k_1=1}^{j-1} \binom{j-1}{k_1} (x_\varepsilon^{(j-k_1)} \cdot \nabla) (x_\varepsilon^{(k_1)} \cdot \nabla) b_0
\]
\[
+ \sum_{k_2=1}^{k_1} \binom{k_1-1}{k_2} (x_\varepsilon^{(k_1-k_2)} \cdot \nabla) b_0^{(k_2)}.
\] (76)

We remark that \( x_\varepsilon^{(k)} \) is regarded here solely as a function of \( \varepsilon \), such that \( \nabla \) only acts on \( b_0 \). Applying the rule (74) again, namely to the last term \( b_0^{(k_2)} \) in (76), leads to
\[
b_0^{(j)}(x_\varepsilon) = x_\varepsilon^{(j)} \cdot \nabla b_0 + \sum_{k_1=1}^{j-1} \binom{j-1}{k_1} (x_\varepsilon^{(j-k_1)} \cdot \nabla) (x_\varepsilon^{(k_1)} \cdot \nabla) b_0
\]
\[
+ \sum_{k_2=1}^{k_1} \binom{k_1-1}{k_2} (x_\varepsilon^{(k_1-k_2)} \cdot \nabla) b_0^{(k_2)}
\]
\[
+ \sum_{k_3=1}^{k_2} \binom{k_2-1}{k_3} (x_\varepsilon^{(k_2-k_3)} \cdot \nabla) b_0^{(k_3)}.
\] (77)

In each line of (77) the starting value of the indices \( k_1, k_2, \ldots \) gets raised by one, due to the sum convention (75). One can thus iterate the rule (74) \( j-1 \) times, until the last sum becomes
\[
\sum_{k_1=j-1}^{j-1} \sum_{k_2=j-2}^{j-2} \ldots \sum_{k_{j-1}=1}^{1} \binom{j-1}{k_1} \binom{k_1-1}{k_2} \ldots \binom{1}{k_{j-1}}
\]
\[
\left( x_\varepsilon^{(j-k_1)} \cdot \nabla \right) \left( x_\varepsilon^{(k_1-k_2)} \cdot \nabla \right) \ldots \left( x_\varepsilon^{(k_{j-2}-k_{j-1})} \cdot \nabla \right) \left( x_\varepsilon^{(k_{j-1})} \cdot \nabla \right) b_0
\]
\[
= \left( x_\varepsilon^{(1)} \cdot \nabla \right)^{j} b_0.
\]

Therefore, for we can write the result in the compact form
\[
b_0^{(j)}(x_\varepsilon) = x_\varepsilon^{(j)} \cdot \nabla b_0 + \mathbb{D}_{j-1} b_0, \quad j \geq 1,
\] (78)

with the differential operator \( \mathbb{D}_{j-1} \) defined by
\[
\mathbb{D}_{j-1} := \sum_{i=1}^{j-1} \prod_{l=1}^{i} \sum_{k_l=i-l+1}^{k_l-1} \binom{k_l-1}{k_l} (x_\varepsilon^{(k_l-k_l)} \cdot \nabla) (x_\varepsilon^{(k_l)} \cdot \nabla)^{\delta_{i,l}}.
\] (79)

Here, the operator \( \Pi^* \) denotes a non-commutative product such that sums over \( k_l \) are inside sums over \( k_{l-1}, k_0 := j \) and \( \delta_{i,l} \) stands for the Kronecker delta. Here, \( x_\varepsilon^{(k)} \) is regarded solely as a function of \( \varepsilon \), thus not affected by \( \nabla \). Since
we need the derivatives (78) evaluated at \( \varepsilon = 0 \), using (73) leads to
\[
\left. b_{0}^{(j)}(x_{\varepsilon}) \right|_{\varepsilon = 0} = j! \left[ \partial_{j} \cdot \nabla b_{0}(r) + \mathcal{D}_{j-1} b_{0} \right], \quad j \geq 1, \tag{80}
\]
with the differential operator \( \mathcal{D}_{j-1} \) defined by
\[
\mathcal{D}_{j-1} := \frac{1}{j!} \mathcal{D}_{j-1}^{j} \bigg|_{\varepsilon = 0} = \sum_{i=1}^{j-1} \prod_{l=1}^{i} \sum_{k_{l}=i-l+1}^{j-k_{l}-1} \left( \frac{j!}{k_{l}!} \right) (k_{l}!)(k_{l}!\delta_{li}) \left( \theta k_{l-1-k_{l}} \cdot \nabla \right) \left( \theta k_{l} \cdot \nabla \right) \delta_{li}. \tag{81}
\]

The computation for \( c_{0}^{(j)}(x_{\varepsilon}, \theta_{\varepsilon}) \) is done in analogy to (74), hence we start from
\[
c_{0}^{(j)}(x_{\varepsilon}, \theta_{\varepsilon}) = \left( x_{\varepsilon} \cdot \nabla c_{0} + \theta_{(j-1)} \frac{\partial}{\partial \theta} c_{0} \right)^{(j-1)}\]
\[
= \sum_{k_{1}=0}^{j-1} \left( x_{\varepsilon}^{(j-1)} \cdot \nabla + \theta_{(j-1)} \frac{\partial}{\partial \theta} \right) c_{0}^{(k_{1})}
\]
\[
= x_{\varepsilon}^{(j)} \cdot \nabla c_{0} + \theta_{(j)} \frac{\partial}{\partial \theta} c_{0} + \sum_{k_{1}=1}^{j-1} \left( x_{\varepsilon}^{(j-k_{1})} \cdot \nabla + \theta_{(j-k_{1})} \frac{\partial}{\partial \theta} \right) c_{0}^{(k_{1})}, \tag{82}
\]
Applying the rule (82) iteratively \( j - 1 \) times, and using that \( \partial c_{0}/\partial \alpha = -a_{0} \), we eventually arrive at
\[
c_{0}^{(j)}(x_{\varepsilon}, \theta_{\varepsilon}) \bigg|_{\varepsilon = 0} = j! \left[ \partial_{j} \cdot \nabla c_{0}(r, \theta) - G_{j}^{\theta} a_{0}(r, \theta) + \mathcal{D}_{j-1}^{\alpha} c_{0} \right], \quad j \geq 1, \tag{83}
\]
with the differential operator \( \mathcal{D}_{j-1}^{\alpha} \) defined by
\[
\mathcal{D}_{j-1}^{\alpha} := \sum_{i=1}^{j-1} \prod_{l=1}^{i} \sum_{k_{l}=i-l+1}^{j-k_{l}-1} \left( \frac{j!}{k_{l}!} \right) (k_{l}!)(k_{l}!\delta_{li}) \left( \theta k_{l-1-k_{l}} \cdot \nabla + G_{k_{l}}^{\theta} \frac{\partial}{\partial \alpha} \right) \delta_{li}, \tag{84}
\]
where the generators \( g_{k} \) and \( G_{k}^{\theta} \) are viewed as functions of \( \varepsilon \) solely and are thus not affected by differentiation with respect to \( r \) and \( \alpha \).

We have now all the material to compute the \( j \)-h derivative at \( \varepsilon = 0 \) of \( \varphi_{0}, \varphi_{1} \) and \( \varphi_{1} \), defined in (59), which are the coefficients of the Taylor expansions (60). Applying the Leibniz rule (71) twice, for the first term in (59b) we obtain
\[
\left[ v_{\parallel, \varepsilon} \cdot \dot{x}_{\varepsilon} \cdot b_{0}(x_{\varepsilon}) \right]^{(j)}(0) = \left\{ \sum_{l=0}^{j} \left[ \left. v_{\parallel, \varepsilon} \cdot \dot{x}_{\varepsilon} \right|_{\varepsilon = 0} \right]^{(j-l)} \cdot b_{0}^{(l)}(x_{\varepsilon}) \right\}^{(0)} = \left\{ \sum_{l=0}^{j} \left[ \left. v_{\parallel, \varepsilon} \cdot \dot{x}_{\varepsilon} \right|_{\varepsilon = 0} \right]^{(j-l-m)} \cdot b_{0}^{(l)}(x_{\varepsilon}) \right\}^{(0)}.
\]
Since \( b_{0}^{(l)}(x_{\varepsilon}) \) at \( \varepsilon = 0 \) is given by the formula (80) for \( l \geq 1 \), we single out the summand with \( l = 0 \), and insert (73) to obtain, for \( j \geq 1 \),
\[
\left[ v_{\parallel, \varepsilon} \cdot \dot{x}_{\varepsilon} \cdot b_{0}(x_{\varepsilon}) \right]^{(j)}(0) = \sum_{m=0}^{j} \frac{j!}{m!(j-m)!} (j-m)! G_{j-m}^{\parallel, m} \cdot \dot{\varphi}_{m} \cdot b_{0}
\]
\[
+ \sum_{l=1}^{j} \frac{j!}{(j-l)!} \left[ \sum_{m=0}^{j-l-m} \frac{(j-l)!}{m!(j-l-m)!} (j-l-m)! G_{j-l-m}^{\parallel, m} \cdot \dot{\varphi}_{m} \right] \cdot \left[ \partial_{l} \cdot \nabla b_{0} + \mathcal{D}_{l-1} b_{0} \right]
\]
\[
= j! \sum_{m=0}^{j} G_{j-m}^{\parallel} \cdot \dot{\varphi}_{m} \cdot b_{0} + j! \sum_{l=1}^{j} \left[ \sum_{m=0}^{j-l-m} G_{j-l-m}^{\parallel} \cdot \dot{\varphi}_{m} \right] \cdot \left[ \partial_{l} \cdot \nabla b_{0} + \mathcal{D}_{l-1} b_{0} \right],
\]
With the same reasoning, using the result (83), for the second term in (59b) one obtains
\[
\left[ v_{\perp, \varepsilon} \cdot \dot{x}_{\varepsilon} \cdot c_{0}(x_{\varepsilon}, \theta_{\varepsilon}) \right]^{(j)}(0) = j! \sum_{m=0}^{j} G_{j-m}^{\parallel} \cdot \dot{\varphi}_{m} \cdot c_{0} + j! \sum_{l=1}^{j} \left[ \sum_{m=0}^{j-l-m} G_{j-l-m}^{\parallel} \cdot \dot{\varphi}_{m} \right] \cdot \left[ \partial_{l} \cdot \nabla c_{0} - G_{l}^{\theta} a_{0} + \mathcal{D}_{l-1}^{\alpha} c_{0} \right], \tag{86}
\]
For the third term in (59b) we apply the Leibniz rule (71) once to compute

\[
\left[ \dot{x}_t \cdot A_1(t, x_t) \right]^{(j)}(0) = \left\{ \sum_{l=0}^{j} \binom{j}{l} \dot{x}_t^{(j-l)} \cdot A_1^{(l)}(t, x_t) \right\}_{\epsilon=0}
\]

\[
= j! \dot{\theta}_j \cdot A_1 + j! \sum_{l=1}^{j} \dot{\theta}_{j-l} \cdot \left[ \theta_l \cdot \nabla A_1 + \mathbb{D}_{l-1} A_1 \right].
\]

From this result we can also compute the derivatives of \(\varphi_0\) given in (59a),

\[
\frac{\varphi_0^{(j)}(0)}{j!} = \dot{\theta}_j \cdot A_0 + \sum_{l=1}^{j} \dot{\theta}_{j-l} \cdot \left[ \theta_l \cdot \nabla A_0 + \mathbb{D}_{l-1} A_0 \right].
\]

For \(\varphi_1\) given in (59b), using the previous results (85)-(87), we obtain

\[
\frac{\varphi_1^{(j)}(0)}{j!} = \sum_{m=0}^{j} G_j^{\bot-m} \dot{\theta}_m \cdot b_0 + \sum_{l=1}^{j} \left[ \sum_{m=0}^{j-l} G_j^{\bot-l-m} \dot{\theta}_m \right] \cdot \left[ \theta_l \cdot \nabla b_0 + \mathbb{D}_{l-1} b_0 \right]
\]

\[
+ \sum_{m=0}^{j} G_j^{\bot-m} \dot{\theta}_m \cdot c_0 + \sum_{l=1}^{j} \left[ \sum_{m=0}^{j-l} G_j^{\bot-l-m} \dot{\theta}_m \right] \cdot \left[ \theta_l \cdot \nabla c_0 - G_l^0 a_0 + \mathbb{D}_{l-1}^0 c_0 \right] \quad (89)
\]

\[
+ \dot{\theta}_j \cdot A_1 + \sum_{l=1}^{j} \dot{\theta}_{j-l} \cdot \left[ \theta_l \cdot \nabla A_1 + \mathbb{D}_{l-1} A_1 \right].
\]

For \(\varphi_2\) given in (59c), using (73) and (80) with \(\phi\) instead of \(b_0\), we obtain

\[
\frac{\varphi_2^{(j)}(0)}{j!} = q_0 G_j^{\bot} + q_1 G_j^{\bot} + \dot{\theta}_j \cdot \nabla \phi(r) + \mathbb{I}_{j-1} \phi.
\]

We shall transform the expressions (88)-(89) a bit further. In particular, in the sum over \(l\) in (88) and in the last line of (89) we single out the term with \(l = j\). Additionally, in the sums over \(m\) in (89) we single out the terms with \(m = 0\). For \(j \geq 1\), this leads to

\[
\frac{\varphi_0^{(j)}(0)}{j!} = \dot{\theta}_j \cdot A_0 + \dot{r} \cdot \left[ \theta_j \cdot \nabla A_0 + \mathbb{D}_{j-1} A_0 \right] + \sum_{l=1}^{j-1} \dot{\theta}_{j-l} \cdot \left[ \theta_l \cdot \nabla A_0 + \mathbb{D}_{l-1} A_0 \right]
\]

(91)

and to

\[
\frac{\varphi_1^{(j)}(0)}{j!} = G_j^{\bot} \dot{r} \cdot b_0 + \sum_{m=1}^{j} G_j^{\bot} \dot{\theta}_m \cdot b_0 + \sum_{l=1}^{j} G_j^{\bot} \left[ \theta_l \cdot \nabla b_0 + \mathbb{D}_{l-1} b_0 \right] \cdot \dot{r}
\]

\[
+ \sum_{l=1}^{j-1} \left[ \sum_{m=1}^{j-l} G_j^{\bot} \dot{\theta}_m \right] \cdot \left[ \theta_l \cdot \nabla b_0 + \mathbb{D}_{l-1} b_0 \right]
\]

\[
+ G_j^{\bot} \dot{r} \cdot c_0 + \sum_{m=1}^{j} G_j^{\bot} \dot{\theta}_m \cdot c_0 + \sum_{l=1}^{j} G_j^{\bot} \left[ \theta_l \cdot \nabla c_0 - G_l^0 a_0 + \mathbb{D}_{l-1}^0 c_0 \right] \cdot \dot{r}
\]

\[
+ \sum_{l=1}^{j-1} \left[ \sum_{m=1}^{j-l} G_j^{\bot} \dot{\theta}_m \right] \cdot \left[ \theta_l \cdot \nabla c_0 - G_l^0 a_0 + \mathbb{D}_{l-1}^0 c_0 \right]
\]

\[
+ \dot{\theta}_j \cdot A_1 + \dot{r} \cdot \left[ \theta_j \cdot \nabla A_1 + \mathbb{D}_{j-1} A_1 \right] + \sum_{l=1}^{j-1} \dot{\theta}_{j-l} \cdot \left[ \theta_l \cdot \nabla A_1 + \mathbb{D}_{l-1} A_1 \right].
\]

(92)

It will be convenient to eliminate the terms \(\dot{\theta}_m\) from the first and the third line of (92), using the equivalence of
Lagrangians from Definition 2:

\[
\sum_{m=1}^{j} \left( G_{j-m}^\parallel \rho_m \cdot b_0 + \sum_{l=1}^{j} \left( \frac{d}{ds} G_{j-l}^\parallel (\rho_m \cdot b_0) - \dot{G}_{j-m}^\parallel (\rho_m \cdot b_0) - G_{j-m}^\parallel \cdot (\rho_m \times (\nabla \times b_0)) \right) \right) \\
= \sum_{m=1}^{j} \left[ \frac{d}{ds} (G_{j-m}^\parallel (\rho_m \cdot b_0)) - \dot{G}_{j-m}^\parallel (\rho_m \cdot b_0) - G_{j-m}^\parallel \cdot (\rho_m \times (\nabla \times b_0)) \right] \\
\sim \sum_{m=1}^{j} \left[ - \dot{G}_{j-m}^\parallel (\rho_m \cdot b_0) - G_{j-m}^\parallel \cdot (\rho_m \times (\nabla \times b_0)) \right],
\]

(93)

and, similarly,

\[
\sum_{m=1}^{j} G_{j-m}^\perp \rho_m \cdot c_0 + \sum_{l=1}^{j} \left( \rho_l \cdot \nabla c_0 \cdot \dot{r} \right) \\
\sim \sum_{m=1}^{j} \left[ - \dot{G}_{j-m}^\perp (\rho_m \cdot c_0) - G_{j-m}^\perp \cdot (\rho_m \times (\nabla \times c_0)) + G_{j-m}^\perp \cdot (\rho_m \cdot a_0) \right],
\]

(94)

This allows us to write (92) as

\[
\frac{\phi^{(j)}_1(0)}{j!} \sim \left( G_{j-m}^\parallel \dot{\rho}_m \cdot b_0 + G_{j-m}^\parallel \dot{\rho}_m \cdot c_0 - \sum_{m=1}^{j} \left[ G_{j-m}^\parallel (\rho_m \cdot b_0) + G_{j-m}^\parallel (\rho_m \cdot c_0) \right] \right)
\]

\[
+ \sum_{l=1}^{j-1} \left[ \sum_{m=1}^{j-l} G_{j-l-m}^\parallel \dot{\rho}_m \cdot \left( \rho_l \cdot \nabla b_0 + \mathbb{D}_{l-1} b_0 \right) + \sum_{l=2}^{j} G_{j-l}^\parallel \mathbb{D}_{l-1} b_0 \cdot \dot{r} \right]
\]

\[
+ \sum_{m=1}^{j} \left[ \dot{G}_{j-m}^\parallel (\rho_m \cdot c_0) + G_{j-m}^\parallel \cdot (\rho_m \times (\nabla \times c_0)) \right]
\]

\[
- \sum_{l=2}^{j} \left[ \dot{G}_{j-l}^\parallel \mathbb{D}_{l-1} c_0 \cdot \dot{r} \right]
\]

\[
+ \dot{\rho}_j \cdot A_1 + \dot{r} \cdot \left[ \rho_j \cdot \nabla A_1 + \mathbb{D}_{j-1} A_1 \right] + \sum_{l=1}^{j-1} \dot{\rho}_{j-l} \cdot \left[ \rho_l \cdot \nabla A_1 + \mathbb{D}_{l-1} A_1 \right].
\]

(95)

The Lagrangian \( L^\varepsilon \) from (58) now reads

\[
L^\varepsilon = \frac{L_{j+1}}{\varepsilon} + L_0 + \sum_{j=1}^{N} \varepsilon^j \left[ \frac{\phi^{(j+1)}_0(0)}{(j+1)!} + \frac{\phi^{(j)}_1(0)}{(j)!} - \frac{\phi^{(j)}_2(0)}{(j)!} t \right] + O(\varepsilon^{N+1}),
\]

(96)

where we used (60) to estimate the remainder and the terms \( L_1, L_0 \) (and \( L_1 \)) have already been computed in (61)-(70). For \( j \geq 1 \), from (90),(91) and (95) we write in compact form

\[
\frac{\phi^{(j+1)}_0(0)}{(j+1)!} + \frac{\phi^{(j)}_1(0)}{(j)!} - \frac{\phi^{(j)}_2(0)}{(j)!} t = \dot{\rho}_{j+1} \cdot A_0 + \dot{\rho}_{j+1} \cdot \nabla A_0 \cdot \dot{r} + \dot{\rho}_j \cdot A_1 + \dot{\rho}_j \cdot \nabla A_1 \cdot \dot{r} + \left[ G_j^\parallel b_0 + G_j^\perp c_0 + \mathcal{Q}^*_j(q_{kv}) \right] \cdot \dot{r} - \left[ q_{kv} G_j^\parallel + q_{kv} G_j^\perp + q_{kv} \cdot \nabla \phi \right] t + L^*_{j}(q_{kv}, \dot{q}_{kv}),
\]

(97)

with

\[
\mathcal{Q}^*_j(q_{kv}) := \mathbb{D}_{j} A_0 - \sum_{m=1}^{j} \left[ G_{j-m}^\parallel (\rho_m \times (\nabla \times b_0)) + G_{j-m}^\parallel (\rho_m \times (\nabla \times c_0)) \right]
\]

\[
+ \mathbb{D}_{j-1} A_1 + \sum_{l=2}^{j} \left[ G_{j-l}^\parallel \mathbb{D}_{l-1} b_0 + G_{j-l}^\parallel \mathbb{D}_{l-1} c_0 \right] - \sum_{m=1}^{j} G_{j-m}^\perp \rho_m a_0,
\]

(98)
and

\[
\mathcal{L}_j(q_{gy}, \dot{q}_{gy}) := \sum_{l=1}^{j} \dot{q}_{j+1-l} \cdot \left[ \dot{q}_l \cdot \nabla A_0 + \mathbb{D}_{l-1} A_0 \right] - \sum_{m=1}^{j} \left[ \dot{G}^\parallel_{j-m}(\varrho_m \cdot b_0) + \dot{G}^\perp_{j-m}(\varrho_m \cdot c_0) - G^\perp_{j-m} \dot{\varrho}_m (\varrho_m \cdot a_0) \right] + \sum_{l=1}^{j-1} \left[ \sum_{m=1}^{j-l} \dot{G}^\parallel_{j-l-m} \dot{\varrho}_m \right] \cdot \left[ \varrho_l \cdot \nabla b_0 + \mathbb{D}_{l-1} b_0 \right] + \sum_{l=1}^{j-1} \left[ \sum_{m=1}^{j-l} \dot{G}^\perp_{j-l-m} \dot{\varrho}_m \right] \cdot \left[ \varrho_l \cdot \nabla c_0 - G^\perp_l \dot{a}_0 + \mathbb{D}^\perp_{l-1} c_0 \right] + \sum_{l=1}^{j-1} \dot{q}_{j-l} \cdot \left[ \varrho_l \cdot \nabla A_1 + \mathbb{D}_{l-1} A_1 \right] - \mathbb{D}_{j-1} \phi \dot{t}.
\]

(99)

Here, \(\mathbb{D}\) and \(\mathbb{D}^\alpha\) have been defined in (81) and (84), respectively, and we remind the reader of the sum convention (75).

2. Gauge-invariant formulation

It will be convenient to write (97) in terms of the electromagnetic fields \(E, B\) rather than the potentials \(\phi, A\). For this, consider the product rule on the tangent space,

\[
\frac{d}{ds} (q_j \cdot A) = \dot{q}_j \cdot A + q_j \cdot \dot{A} = \dot{q}_j \cdot A + (\dot{\varrho} \cdot \nabla A + \dot{\varrho} \partial_i A) \cdot q_j.
\]

where we used the tangent map to express \(A\). One can thus use the identity

\[
\nabla A \cdot q_j - q_j \cdot \nabla A = q_j \times (\nabla \times A) = q_j \times B
\]

(100)

to obtain

\[
\dot{q}_j \cdot A + q_j \cdot \nabla A \cdot \dot{t} = \frac{d}{ds} (q_j \cdot A) - \dot{\varrho} \cdot (q_j \times B) - \dot{\varrho} \partial_i A \cdot q_j.
\]

Therefore, by Definition 2 one has the equivalence

\[
\dot{q}_j \cdot A_0 + q_j \cdot \nabla A_0 \cdot \dot{t} \sim -\dot{\varrho} \cdot (q_j \times B_0),
\]

\[
\dot{q}_j \cdot A_1 + q_j \cdot \nabla A_1 \cdot \dot{t} - q_j \cdot \nabla \phi \dot{t} \sim -\dot{\varrho} \cdot (q_j \times B_1) + q_j \cdot E \dot{t}.
\]

The only remaining terms featuring the electromagnetic potentials are the first terms in each line of (98), as well as the first and the last line of (99). Since \(\mathbb{D}\) is linear, these terms are of the generic form

\[
\mathbb{D}_{j-1} (A \cdot \dot{t} - \varrho \dot{t}) + \sum_{l=1}^{j-1} \dot{q}_{j-l} \cdot \left[ \varrho_l \cdot \nabla A + \mathbb{D}_{l-1} A \right].
\]

(101)

In the differential operator \(\mathbb{D}_{j-1}\), defined in (81), we single out the term with \(i = 1\) and group the remaining sums together in an operator \(\mathbb{D}^I_{j-1}\), hence

\[
\mathbb{D}_{j-1} = \sum_{k_1=1}^{j-1} \frac{j-k_1}{j} (q_{j-k_1} \cdot \nabla) (q_{k_1} \cdot \nabla) + \mathbb{D}^I_{j-1}
\]

\[
= \frac{1}{2} \sum_{k_1=1}^{j-1} \left[ \frac{j-k_1}{j} (q_{j-k_1} \cdot \nabla) (q_{k_1} \cdot \nabla) + \frac{k_1}{j} (q_{k_1} \cdot \nabla) (q_{j-k_1} \cdot \nabla) \right] + \mathbb{D}^I_{j-1}
\]

(102)

\[
= \frac{1}{2} \sum_{k_1=1}^{j-1} (q_{j-k_1} \cdot \nabla) (q_{k_1} \cdot \nabla) + \mathbb{D}^I_{j-1}.
\]
Moreover, in the sum of (101) we have
\[
\sum_{l=1}^{j-1} g_l \cdot \nabla A \cdot \dot{g}_{j-l} = \frac{1}{2} \sum_{l=1}^{j-1} (g_l \cdot \nabla A \cdot \dot{g}_{j-l} + g_{j-l} \cdot \nabla A \cdot \dot{g}_l) \\
= \frac{1}{2} \sum_{l=1}^{j-1} \left[ g_l \cdot \nabla A \cdot \dot{g}_{j-l} + \frac{d}{ds} (g_{j-l} \cdot \nabla A \cdot g_l) - g_{j-l} \cdot \nabla A \cdot g_l - g_{j-l} \cdot \nabla (\dot{r} \cdot \nabla A + \dot{t} \partial_t A) \cdot g_l \right] \\
\approx \frac{1}{2} \sum_{l=1}^{j-1} \left[ - \dot{g}_{j-l} \cdot (g_l \times B) - g_{j-l} \cdot \nabla (\dot{r} \cdot \nabla A + \dot{t} \partial_t A) \cdot g_l \right].
\]

Combining this result with the singled-out operator in (102) applied to \( A \cdot \dot{r} - \phi \dot{t} \) yields
\[
\frac{1}{2} \sum_{l=1}^{j-1} \left[ (g_{j-l} \cdot \nabla) (g_l \cdot \nabla)(A \cdot \dot{r} - \phi \dot{t}) - g_{j-l} \cdot \nabla (\dot{r} \cdot \nabla A + \dot{t} \partial_t A) \cdot g_l \right] \\
= \frac{1}{2} \sum_{l=1}^{j-1} \left[ (g_{j-l} \cdot \nabla) (g_l \cdot \nabla A - \nabla A \cdot g_l) \cdot \dot{r} \right] - \frac{1}{2} \sum_{l=1}^{j-1} g_{j-l} \cdot \nabla (\nabla \phi \dot{t} + \dot{t} \partial_t A) \cdot g_l \\
= \frac{1}{2} \sum_{l=1}^{j-1} \left[ (g_{j-l} \cdot \nabla B_0) \times g_l \right] \cdot \dot{r} + \frac{1}{2} \sum_{l=1}^{j-1} g_{j-l} \cdot \nabla E \cdot g_l \dot{t}.
\]

Hence we arrived at
\[
\mathbb{D}_{j-1}(A \cdot \dot{r} - \phi \dot{t}) + \sum_{l=1}^{j-1} \dot{g}_{j-l} \cdot \left[ g_l \cdot \nabla A + \mathbb{D}_{l-1} A \right] \\
\approx \frac{1}{2} \sum_{l=1}^{j-1} \left[ (g_{j-l} \cdot \nabla B) \times g_l \right] \cdot \dot{r} + \frac{1}{2} \sum_{l=1}^{j-1} g_{j-l} \cdot \nabla E \cdot g_l \dot{t} - \frac{1}{2} \sum_{l=1}^{j-1} \dot{g}_{j-l} \cdot (g_l \times B) \\
+ \mathbb{D}_{j-1}^3(A \cdot \dot{r} - \phi \dot{t}) + \sum_{l=1}^{j-1} \dot{g}_{j-l} \cdot \mathbb{D}_{l-1} A.
\]

The terms in the last line still contain the electromagnetic potentials instead of the fields. We were not able to prove that a field representation of these terms exists at all orders. However, we can easily prove it for \( j = 3 \):
\[
\mathbb{D}_2(A \cdot \dot{r} - \phi \dot{t}) + \dot{g}_1 \cdot \mathbb{D}_1 A \\
= \frac{1}{3!} (g_1 \cdot \nabla)^3 (A \cdot \dot{r} - \phi \dot{t}) + \frac{1}{2!} (g_1 \cdot \nabla)^2 A \cdot \dot{g}_1 \\
= \frac{1}{6} (g_1 \cdot \nabla) (g_1 \cdot \nabla (A \cdot \dot{r} - \phi \dot{t}) + A \cdot \dot{g}_1) + \frac{1}{3} (g_1 \cdot \nabla)^2 A \cdot \dot{g}_1 \\
= \frac{1}{6} (g_1 \cdot \nabla)^2 \left[ g_1 \cdot \nabla (A \cdot \dot{r} - \phi \dot{t}) + \frac{d}{ds} (A \cdot g_1) - (\dot{r} \cdot \nabla A + \dot{t} \partial_t A) \cdot g_1 \right] + \frac{1}{3} (g_1 \cdot \nabla)^2 A \cdot \dot{g}_1 \\
= -\frac{1}{6} \dot{r} \cdot [g_1 \times (g_1 \cdot \nabla) B] + \frac{1}{6} (g_1 \cdot \nabla) (g_1 \cdot \nabla E \cdot g_1) \dot{t} + \frac{1}{6} \frac{d}{ds} [(g_1 \cdot \nabla)^2 A \cdot g_1] \\
- \frac{1}{3} (g_1 \cdot \nabla) (\dot{g}_1 \cdot \nabla A \cdot g_1) + \frac{1}{3} (g_1 \cdot \nabla)^2 A \cdot \dot{g}_1 \\
\approx -\frac{1}{6} \dot{r} \cdot [g_1 \times (g_1 \cdot \nabla) B] + \frac{1}{6} (g_1 \cdot \nabla) (g_1 \cdot \nabla E \cdot g_1) \dot{t} - \frac{1}{3} \dot{g}_1 \cdot [g_1 \times (g_1 \cdot \nabla B)].
\]

We conjecture that such field representations can be derived at every order and leave the proof for later. In summary,
the above algebra leads to the following representation of (97):

\[
\frac{\varphi^{(n+1)}_0(0)}{(n+1)!} + \frac{\varphi^{(n)}_1(0)}{(n)!} - \frac{\varphi^{(n)}_2(0)}{(n)!} \, i \sim \left[ G^\parallel_n b_0 + G^\perp_n c_0 - q_{n+1} \times B_0 - q_n \times B_1 + Q_n(q_{\text{GY}}) \right] \cdot \dot{r} - \left( q \| G^\parallel_n + q \perp G^\perp_n - q_n \cdot E \right) \dot{t} + L_n(q_{\text{GY}}, q_{\text{GY}}),
\]

with

\[
Q_n(q_{\text{GY}}) := \frac{1}{2} \sum_{l=1}^n (q_{n+1-l} \cdot \nabla B_0) \times \dot{q}_l + \frac{1}{2} \sum_{l=1}^{n-1} (q_{n-l} \cdot \nabla B_1) \times \dot{q}_l - \sum_{m=1}^n \left[ G^\parallel_{n-m} (q_m \times (\nabla \times b_0)) + G^\perp_{n-m} (q_m \times (\nabla \times c_0)) \right]
+ \sum_{l=2}^n \left[ G^\parallel_{n-l} D_{l-1} b_0 + G^\perp_{n-l} D_{l-1}^\perp b_0 \right] - \sum_{m=1}^n G^\perp_{n-m} G^\parallel_m a_0,
\]

and

\[
L_n(q_{\text{GY}}, q_{\text{GY}}) := -\frac{1}{2} \sum_{l=1}^n \dot{q}_{n+1-l} \cdot (\dot{q}_l \times B_0) - \frac{1}{2} \sum_{l=1}^{n-1} \dot{q}_{n-l} \cdot (\dot{q}_l \times B_1) + \frac{1}{2} \sum_{l=1}^{n-1} q_{n-l} \cdot \nabla E \cdot \dot{q}_l - \sum_{m=1}^n \left[ G^\parallel_{n-m} (q_m \cdot \nabla \cdot B_0) + G^\perp_{n-m} (q_m \cdot c_0) - G^\perp_{n-m} \dot{\alpha} (q_m \cdot a_0) \right]
+ \sum_{l=1}^{n-1} \left[ \sum_{m=1}^{n-l} G^\parallel_{n-l-m} \dot{q}_m \cdot [q_l \cdot \nabla b_0 + D_{l-1} b_0] \right]
+ \sum_{l=1}^{n-1} \left[ \sum_{m=1}^{n-l} G^\perp_{n-l-m} \dot{q}_m \cdot [q_l \cdot \nabla c_0 - G^\parallel_l a_0 + D_{l-1}^\perp c_0] \right]
+ \sum_{l=2}^{n-1} \dot{q}_{n-l} \cdot \nabla D_{l-1} A_0 \cdot \dot{r} + \sum_{l=2}^n \dot{q}_{n+1-l} \cdot D_{l-1} A_0 + D_{n-1} (A_1 \cdot \dot{r} - \phi \, \dot{\phi}) + \sum_{l=2}^{n-1} \dot{q}_{n-l} \cdot D_{l-1} A_1.
\]

Here, the differential operators $D$, $D^\perp$ and $D^\parallel$ have been defined in (81), (84) and (102), respectively.

\[\boxend\]

**B. Proof of Theorem 1**

The $N$-th order GY-Lagrangian in (42) can be written as

\[
L^{(N)}_{\text{GY}} = \frac{1}{\varepsilon} L^*_{-1} + L^*_0 + \sum_{n=1}^N \varepsilon^n L^*_n,
\]

with

\[
L^*_0 = A_0 \cdot \dot{r},
L^*_1 = q_0 b_0 + A_1 \cdot \dot{r} - \left( \frac{q^2_0}{2} + \frac{q^2_1}{2} + \phi \right) \dot{t},
L^*_n = \mu_{n-1} \dot{\alpha},
\]

and $\mu_0 \equiv \mu = q^2_1/(2|B_0|)$. We compare this to the series expansion of the exact Lagrangian $L^*$ in Proposition 1, $L^* = L_{-1}/\varepsilon + L_0 + \varepsilon L_1 + \ldots$, and will show that generators can be chosen such that $L_n \sim L^*_n$ for $-1 \leq n \leq N$.

At lowest order one has $L_{-1} = L^*_{-1}$ and nothing needs to be done. At zeroth order we choose

\[
q_1 = \frac{q_1}{|B_0|} b_0 \times c_0 = \frac{q_1}{|B_0|} a_0,
\]

which yields $L_0 = L^*_0$. For the higher orders we proof the following:

**Lemma 9.** In the Lagrangian $L_{1 \leq n \leq N}$ from (41) one can choose generators $G_n$ and $g_n$ such that $L_n \sim L^*_n$, where $b_0 \cdot g_{n+1}$ and $G^\parallel_n$ are still arbitrary. Moreover, $\bar{\mu}_n = 0$ for $n \geq 1$.

**Proof.** We proceed by induction. For $n = 1$ we have

\[
Q_1 = \frac{1}{2} (q_1 \cdot \nabla B_0) \times \dot{q}_1 - q_0 (q_1 \times (\nabla \times b_0) - q_1 \times (\nabla \times c_0) - q_\perp G^\parallel_1 a_0,
\]
and
\[ \mathcal{L}_1 = -\frac{1}{2} \dot{\phi}_1 \cdot (\dot{q}_1 \times B_0) - \dot{q}_\parallel (q_1 \cdot b_0) \]
\[ - \frac{q_\perp}{|B_0|} \dot{\alpha} (q_1 \cdot a_0). \]

Now, from the result (109) for \( q_1 \) we compute
\[ \dot{q}_1 = \frac{q_\perp}{|B_0|} a_0 - \frac{q_\perp}{|B_0|^2} (\dot{\gamma} \cdot \nabla |B_0|) a_0 \]
\[ + \frac{q_\perp}{|B_0|} \dot{\alpha} a_0 + \frac{q_\perp}{|B_0|} \dot{a}_0. \]

Moreover, from \( a_0 \times b_0 = c_0 \),
\[ \dot{q}_1 \cdot (\dot{q}_1 \times B_0) = q_\perp \dot{q}_\perp c_0 = q_\perp^2 \dot{\gamma} \cdot \nabla a_0 c_0 + \frac{q_\perp^2}{|B_0|} \dot{\alpha}, \]
where we recognize the gyro-gauge \( R = \nabla a_0 c_0 = \nabla e_2 \cdot e_1 \).
Inserting this into \( \mathcal{L}_1 \) yields
\[ L_1 = (G_1^\parallel b_0 + G_1^\perp c_0 - q_\perp \gamma n_1 \cdot B_0 - \frac{q_\perp}{|B_0|} a_0 \times B_1) \cdot \dot{r} \]
\[ + \left( Q_1 - \frac{q_\perp^2}{2|B_0|} R \right) \cdot \dot{r} \]
\[ - \left( q_\perp^2 G_1^\parallel + q_\perp^2 G_1^\perp - q_\perp a_0 \cdot E \right) \dot{t} + \frac{q_\perp^2}{|B_0|} \dot{\alpha}. \]
We can eliminate the Hamiltonian multiplying \( \dot{t} \) by setting
\[ G_1^\perp = -\frac{q_\perp}{q_\parallel} G_1^\parallel + \frac{1}{|B_0|} a_0 \cdot E, \]
and write the remainder of \( L_1 \) as
\[ L_1 = (G_1^\parallel b_0 - q_\perp \gamma n_1 \cdot B_0 - \frac{q_\perp^2}{2|B_0|} \dot{\alpha}, \]
thereby implicitly defining the term \( \gamma_1 \).
Any vector \( v \in \mathbb{R}^3 \) can be written as \( v = v_\parallel b_0 + v_\perp \), where \( v_\parallel = v \cdot b_0 \) and \( v_\perp = b_0 \times v \times b_0 \); hence by setting
\[ q_\parallel = \frac{b_0 \times \gamma_1}{|B_0|}, \quad G_1^\parallel = -\gamma_1, \]
we obtain
\[ L_1 = \frac{q_\perp^2}{2|B_0|} \dot{\alpha} = \mu_0 \dot{\alpha} = L_1^*. \]
Moreover, \( b_0 \cdot q_\parallel \) and \( G_1^\parallel \) are still arbitrary; thus we proved that the statement of the lemma holds for \( n = 1 \).

Suppose now that the statement holds for some \( n \geq 1 \).
From Proposition 1 we write the Lagrangian at order \( n + 1 \) as
\[ L_{n+1} = (G_{n+1}^\parallel b_0 + G_{n+1}^\perp c_0 - q_{n+2} \gamma n_1 \cdot B_0) \cdot \dot{r} \]
\[ + \left( - q_{n+1} \gamma n_1 \cdot B_1 + Q_{n+1} \right) \cdot \dot{r} + L_{n+1} \]
\[ - (q_\perp G_{n+1} + q_\perp G_{n+1} - q_{n+1} a_0 \cdot E) \dot{t}, \]
Let us examine the term \( L_{n+1} \) from (107) a bit more careful; in particular, let us single out two terms:
\[ \text{• the term} \ -q_\perp (q_{n+1} \cdot b_0) \text{ from the second line} \]
\[ (m = n + 1), \]
\[ \text{• the term} \ -q_\perp G_{n}^\theta (q_1 \cdot a_0) \text{ from the fourth line} \]
\[ (l = n). \]
From (110) we obtain
\[ \dot{q}_1 \cdot a_0 = \frac{q_\perp}{|B_0|} \left( \frac{1}{|B_0|^2} (\dot{\gamma} \cdot \nabla |B_0|) \right). \]
Therefore,
\[ L_{n+1} = -q_\perp (q_{n+1} \cdot b_0) - q_\perp G_{n}^\theta \dot{q}_\perp + \text{terms}. \]
Moreover, let us add to the Lagrangian \( L_{n+1} \) the “total time derivative” of some arbitrary function \( S_{n+1} : \Omega_{xy} \to \mathbb{R} \), and let us write it in compact notation similarly to (113),
\[ L_{n+1} \sim \left( G_{n+1}^\parallel b_0 - q_{n+2} \gamma n_1 \cdot B_0 + \gamma n_1 \cdot r \right) \cdot \dot{r} \]
\[ - (q_{n+1} \cdot b_0 + \gamma n_1, s) \dot{q}_\parallel \]
\[ - \left( \left( q_\perp \gamma n_1 + q_{n+1} \right) \dot{q}_\perp \right) \]
\[ + \left( \frac{\partial S_{n+1}}{\partial \alpha} + \gamma n_1, s \right) \dot{\alpha} \]
\[ - \left( q_\perp G_{n+1} + \gamma n_1, t \right) \dot{t}, \]
where we implicitly defined the terms \( \gamma_{n+1} \) by comparison to (116). Let us treat each component of the Poincaré-Cartan form in (117) separately:
\[ \text{• The component of} \ \dot{r} \ \text{is zero for} \]
\[ q_{n+2} = \frac{b_0 \times \gamma_{n+1}}{|B_0|}, \quad G_{n+1}^\parallel = -\gamma_{n+1, s}. \]
\[ \text{• Since} \ b_0 \cdot q_{n+1} \text{is still undetermined by the inductive hypothesis, the component of} \ \dot{q}_\parallel \text{is zero for} \]
\[ b_0 \cdot q_{n+1} = -\gamma_{n+1, s}. \]
\[ \text{• Noting that} \ G_{n}^\theta \text{is still undetermined by the inductive hypothesis, the component of} \ \dot{q}_\perp \text{is zero for} \]
\[ G_{n}^\theta = -\frac{|B_0|}{q_\perp} \gamma_{n+1}. \]
\[ \text{• The component of} \ \dot{t} \ \text{the (Hamiltonian) is zero for} \]
\[ G_{n+1}^\perp = -\frac{\gamma_{n+1, t}}{q_\perp}. \]
\[ \text{• The term with} \ \dot{\alpha} \ \text{is rewritten as} \]
\[ \left( \frac{\partial S_{n+1}}{\partial \alpha} + \gamma_{n+1, s} + \frac{\dot{\gamma}_{n+1, s}}{\dot{\gamma}_{n+1, s}} \right) \dot{\alpha}, \]
where \( \gamma_{n+1,\alpha} \) has been decomposed into gyro-
average and fluctuations. The equation
\[
\frac{\partial S_{n+1}}{\partial \alpha} + \gamma_{n+1,\alpha} = 0
\]
has 2\( \pi \)-periodic solutions \( S_{n+1} \). We pick one of
those solutions to obtain
\[
\left( \frac{\partial S_{n+1}}{\partial \alpha} + \gamma_{n+1,\alpha} \right) \dot{\alpha} = \langle \gamma_{n+1,\alpha} \rangle \dot{\alpha}.
\]

Hence, with the above choices for the generators, all that
remains from (117) is
\[
L_{n+1} \sim \langle \gamma_{n+1,\alpha} \rangle \dot{\alpha} =: \mu_n \dot{\alpha} = L^*_n+1.
\]
Noting that \( b_0, \varrho_n+2 \) and \( G^\parallel_{n+1} \) are still arbitrary and that
\( \mu_n = 0 \), we showed that the statement of the lemma holds
for \( n+1 \) and thus completed the proof by induction. \( \square \)

Considering the regularity of the generators \( G_n \) the
following is true:

**Lemma 10.** \( G_n, \varrho_{n+1} \in C^{N+2-n}(\Omega_{gy}) \) for \( 1 \leq n \leq N \).

**Proof.** The proof is again achieved by induction. For
\( n = 1 \) the generators \( G_1^0, G_2^\parallel, G_2^\perp \) and \( \varrho_{2,\perp} \) are
given in Lemma 7 and \( G^\parallel_1 \) is given in Lemma 8,
respectively. From assumption 1 we deduce \( G_1, \varrho_{2,\perp} \in
C^{N+1}(\Omega_{gy}) \). Assuming the statement holds for some
\( n \leq N - 1 \), it follows from the proof of Lemma 9 that
the generators \( \varrho_{n+2,\perp}, G^\parallel_{n+1}, G^\perp_{n+1} \) and \( b_0 \cdot \varrho_n + 1 \)
have the same regularity as the Lagrangian \( L_{n+1} \) written
in (116). The fact that \( L_{n+1} \in C^{N+2-(n+1)} \) follows
from \( \varrho_{n+1,\perp} \in C^{N+2-n-1} \) due to the inductive
hypothesis (needed in the first term of \( L_{n+1} \), equation
(107)) as well as from \( \mathbb{D}_n b_0, \mathbb{D}_n^2 c_0, \mathbb{D}_n+1 A_0 \) all being in
\( C^{N+2-(n+1)} \).

It remains to determine the regularity of the generator
\( G^\parallel_{n+1} \), which is the same as the one of the terms multi-
plying \( \dot{q}_{\perp} \) in \( L_{n+2} \), according to the proof of Lemma 9.

A close inspection of (107) reveals that the least regular
of those terms can only stem from \( \dot{q}_{n+2,\perp}, \hat{q}_n+1, G^\parallel_{n+1}
\) and \( \dot{G}^\perp_{n+1} \). But derivation with respect to \( q_{\perp} \) does not
change the regularity since everything is \( C^\infty \) in the
velocities; therefore, \( G^\parallel_{n+1} \in C^{N+2-(n+1)} \) and the proof is
complete. \( \square \)

Taking the statement from Lemma 10 for \( n = N \) we have
\( G_N, \varrho_{N+1} \in C^2(\Omega_{gy}) \). According to Lemma 9 all
other generators at the level \( N+1 \) can be set to zero and
thus Theorem 1 is proved. \( \square \)

**C. Proof of Lemma 4**

**Lemma 11.** (Gronwall\textsuperscript{10}) Suppose for \( t \in [t_0, t_0 + T] \),
\[
\varphi(t) \leq b(t - t_0) + a \int_{t_0}^t \varphi(s)ds + c,
\]
with \( \varphi(t) \) continuous, \( \varphi(t) \geq 0 \) for \( t \in [t_0, t_0 + T] \) and
constants \( a > b, c \geq 0 \), then
\[
\varphi(t) \leq \left( \frac{b}{a} + c \right) e^{a(t-t_0)} - \frac{b}{a}
\]
for \( t \in [t_0, t_0 + T] \).

In order to set the framework necessary to apply Gron-
wall’s lemma, let us write (52) as integral equations,
\[
\begin{align*}
\zeta(t) &= \zeta_0 + \int_{t_0}^t [\Lambda(z, s, \varepsilon) + \varepsilon N S(z, s, \varepsilon)]ds, \\
\zeta(t) &= \zeta_0 + \int_{t_0}^t \Lambda(z, s, \varepsilon)ds.
\end{align*}
\]

Subtracting the equations and taking the norm yields
\[
|\zeta(t) - \zeta(t)| = ||\zeta_0 - \zeta_0 + \int_{t_0}^t [\Lambda(\zeta, s, \varepsilon) - \Lambda(\zeta, s, \varepsilon) - \varepsilon N S(\zeta, \alpha, s, \varepsilon)]ds||
\leq ||\zeta_0 - \zeta_0|| + \int_{t_0}^t ||\Lambda(\zeta, s, \varepsilon) - \Lambda(\zeta, s, \varepsilon)||ds + \varepsilon N \int_{t_0}^t ||S(\zeta, \alpha, s, \varepsilon)||ds.
\]
The residual \( S \) is continuous; a solution \( \zeta(s) \) of (47) is too (Remark 2) and lives in a bounded domain with size \( O(1) \)
as \( \varepsilon \rightarrow 0 \). Therefore,
\[
||S(\zeta(s), \alpha, s, \varepsilon)|| \leq |S| := \max_{\Omega_{gy} \times (0, \varepsilon_{\max})} ||S(\zeta, \alpha, s, \varepsilon)|| = O(1).
\]

Moreover, \( \Lambda \) is Lipschitz with constant \( \ell_\Lambda \) and we may estimate
\[
|\zeta(t) - \zeta(t)| \leq ||\zeta_0 - \zeta_0|| + \ell_\Lambda \int_{t_0}^t ||\zeta(s) - \zeta(s)||ds + \varepsilon N (t-t_0)|S|.
\]
We now apply Gronwall’s lemma with ϕ(t) = ||ζ(t) − z(t)||, a = ℓ, b = εN|S| and c = ||ζ0 − z0|| to obtain
\[ ||ζ(t) − z(t)|| ≤ \left( \varepsilon^N |S| \frac{\ell}{\varepsilon} + ||ζ0 − z0|| \right) e^{\ell (t−t_0)} − \varepsilon^N \frac{|S|}{\ell}. \]

VI. CONCLUSION

Gyrokinetics is a prevalent theory in plasma physics; it enables the numerical simulation of sophisticated multiscale physics on long timescales. The contribution of this work is to build a mathematically sound foundation for gyrokinetics by means of averaging systems of differential equations on the level of the Lagrangian function, hence the name variational averaging (VA). The formal theory of VA is well-known for almost three decades; the most important results with emphasis on applications in plasma physics have been gathered in recent reviews. The theory has here been made rigorous in the following sense:

- The theory starts from the normalized set of equations (9) and stays consistent with this scaling thorough all orders of the perturbation expansion.
- The gyro-transformations (35) employed in this work exist, c.f. Theorem 1. This is in contrast to the formal theories, where transformations are infinite series of which the convergence cannot be established.
- We state the unambiguous definition of a gyrokinetic equation in (49) by means of the “decoupled dynamics” (48), which stem from a truncated Lagrangian function.
- For the first time we give an error estimate for gyrokinetics in Theorem 2.

The method of VA is thus well-established for the charged particle motion. It seems plausible that this technique could be applied also to other problems of averaging, where the Hamiltonian structure of the equations is important and which are non-canonical symplectic, i.e. with a Lagrangian of the form (14). For instance, an application of VA to the semi-classical limit of the Schrödinger equation could be envisioned. But also other fields like liquid crystal dynamics might be an interesting possibility for application of the VA-method. Moreover, the relation between VA and other averaging methods, in particular normal forms, should be clarified.

Regarding the charged particle, let us comment on some of the practical implications of the here derived results. In view of the GY-Lagrangian from Theorem 1, repeated in equation (46) with the generalized magnetic moment ̃μ as one of the coordinates, we remark that only the Hamiltonian \( H_{gy} \) depends on the order \( N \) of the perturbation expansion, whereas the symplectic form remains unchanged through all orders. This is remarkable because we did not make any particular effort to achieve this; in conventional GY-theories this is usually enforced by an ansatz for the GY-transformation in the form of a Lie-series. Here, the formalism is considerably simpler.

Expressions for the Hamiltonian \( H_{gy} \) have been computed for \( N = 2 \) in the Lemmas 6 and 8. They differ from the conventional GY-Hamiltonians as was pointed out in Remark 5. This is not a surprise considering the amount of freedom within the VA methodology: at each order \( n \), there is a choice to be made which terms of the Lagrangian \( L_n \) in the series (41) should be attached to the generators, and thus appear in the transformation, and which should be kept in the Lagrangian, and thus appear in the dynamics. In conventional GY-theories the gyro-average of the Lagrangian constitutes the dynamics, while the fluctuating part disappears into the generators. However, this is not mandatory. Our approach was to attach as many terms as possible to the generators, even gyro-averaged terms, thereby keeping the dynamics simpler. This could be beneficial for a certain class of numerical codes, in particular particle-based codes, in which an efficient particle pusher is important. We plan the implementation of such a scheme in a forthcoming work.

Finally, the error estimate in Theorem 2 relies on the assumption that the gyrokinetic initial condition has gyro-fluctuations of the order \( O(\varepsilon^N) \); this is called a “well-prepared” initial condition. In the estimate we compare the solution of the averaged part (50) of the gyrokinetic equation to the solution of the Vlasov equation, transformed to the new coordinates, which depends on the gyro-angle \( \alpha \). It is thus clear that the error is small only when the \( \alpha \)-dependence of the Vlasov solution \( f \) is.

ACKNOWLEDGMENTS

I am thankful to Eric Sonnendrücker for encouraging this work and for his valuable scientific input. I also thank Francis Filbet, Michael Kraus and Edoardo Zoni for the inspiring discussion I had with them as well as Roman Hatzky for reading the manuscript with care.


We have $\Phi_{s,t} = \Phi_{s}^{-1}$, since $\int_{t}^{s} = \Phi_{t,s} \circ \Phi_{s,t}$ by the semi-group property of the local flow.