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Quantum Electrodynamical Bloch Theory with Homogeneous Magnetic Fields

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Probing electronic properties of periodic systems by arbitrary homogeneous magnetic fields has unravelled fundamental new phenomena in condensed matter physics. Much theoretical work has been devoted to describe those systems in different regimes, still a general first principles modeling of such fundamental effects is lacking. Here we propose a solution to the problem of Bloch electrons in a homogeneous magnetic field by including the quantum fluctuations of the photon field. A generalized quantum electrodynamical (QED) Bloch theory from first principles is presented. As an application we show how the well known Landau physics shows up in this framework and we derive quantum corrections to the Landau levels.

Cavity QED materials is a growing research field bridging quantum optics [1, 2], polaritonic chemistry [3–7], and materials science, such as light-induced new states of matter achieved with classical laser fields [8, 9]. Photon-matter interactions have recently been suggested to modify electronic properties of solids, such as superconductivity and electron-phonon coupling [10–14]. On the other hand, materials in classical magnetic fields are known to give rise to a variety of novel phenomena such as the Landau levels [15], the integer [16, 17] and the fractional quantum Hall effect [18], and the quantum fractal of the Hofstadter butterfly [19] which can be now accessed experimentally with high resolution [20–22]. One of the open questions in this field is whether Bloch theory is applicable for solids in the presence of a homogeneous magnetic field. The homogeneous magnetic field breaks explicitly translational symmetry [17, 23, 24]. This issue was solved to some extent by introducing the magnetic translation group [23, 24]. However, the magnetic translation group introduces fundamental limitations on the possible directions and values of the strength of the magnetic field [23, 24].

In this Letter, by combining QED with solid-state physics, we provide a consistent and comprehensive theory for solids interacting with homogeneous electromagnetic fields, both classical and quantum. Our main findings are as follows: (i) The quantum fluctuations of the electromagnetic field allow us to restore translational symmetry that is broken due to an external homogeneous magnetic field (see Fig. 1). (ii) We generalize Bloch theory and provide a Bloch central equation for electrons in a solid in the presence of a homogeneous magnetic field and its quantum fluctuations. (iii) In the case of *no* periodic potential we identify quantum corrections to the Landau levels [15]. The spectrum of an electron in the presence of a homogeneous magnetic field, in the z-direction, and the fluctuations of the field, in atomic units, is

$$E_{j,k_z,k_w} = k_z^2/2 + k_w^2/2M + \Omega\,(j + 1/2)\,. \qquad (1)$$

The frequency Ω of the modified Landau levels, defined by $\Omega^2 = \omega_c^2 + \omega_p^2$, depends on both the cyclotron frequency ω_c and the plasma frequency ω_p. This modification is due to the quantum fluctuations of the field. The kinetic energy $k_w^2/2M$ is a polaritonic kinetic term which will be explained in detail in what follows.

Non-Relativistic QED for Interacting Electrons in Solids.—We start by assuming an arbitrary large but fi-

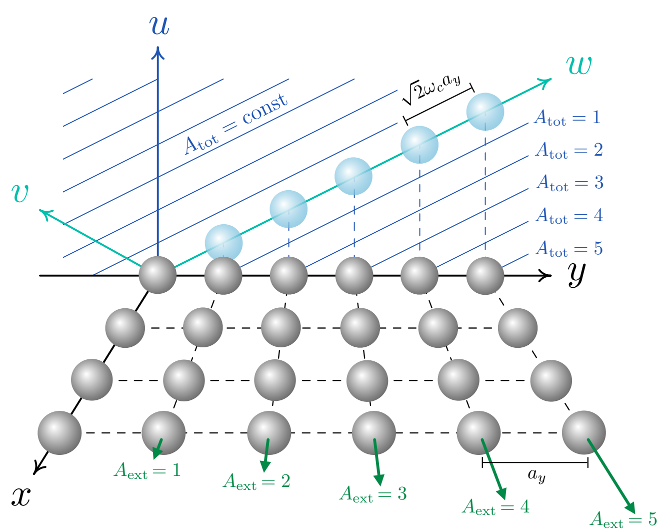

FIG. 1. An external vector potential $\mathbf{A}_{\rm ext}$ in Landau gauge (in arbitrary units) breaks periodicity along y of an otherwise periodic material in the (x, y) plane with lattice constant a_y. Periodicity is restored by including the photon fluctuations $\hat{\mathbf{A}}$ proportional to the photonic coordinate u, which leads to the total vector potential $\hat{\mathbf{A}}_{\rm tot} = \mathbf{A}_{\rm ext} + \hat{\mathbf{A}}$. The total vector potential is constant in the polaritonic direction w, which makes the combined system periodic along this coordinate with lattice constant $\sqrt{2}\omega_c a_y$, where ω_c is the cyclotron frequency. In the case that electrons and photons decouple, the polaritonic coordinate v merges with y and w vanishes.

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nite box of length L and volume $V = L^3$ in which we quantize the electromagnetic field. This leads to the so-called Pauli-Fierz Hamiltonian that describes electrons minimally coupled to photons [1, 2, 25]

$$\hat{H} = \frac{1}{2m_\mathrm{e}} \sum_{j=1}^{N} \left(\mathrm{i}\hbar\nabla_j + \frac{e}{c}\hat{\mathbf{A}}(\mathbf{r}_j)\right)^2 + \frac{1}{4\pi\epsilon_0} \sum_{j<k}^{N} \frac{e^2}{|\mathbf{r}_j - \mathbf{r}_k|} + \sum_{j=1}^{N} v_\mathrm{ext}(\mathbf{r}_j) + \sum_{\mathbf{n},\lambda} \hbar\omega_n \left(\hat{a}^\dagger_{\mathbf{n},\lambda}\hat{a}_{\mathbf{n},\lambda} + \frac{1}{2}\right), \quad (2)$$

where we neglected the Pauli (Stern-Gerlach) term and use SI units, unless stated otherwise. Here $\hat{\mathbf{A}}(\mathbf{r})$ is the quantized vector potential of the electromagnetic field in Coulomb gauge, $\nabla \cdot \hat{\mathbf{A}}(\mathbf{r}) = 0$, given by

$$\hat{\mathbf{A}}(\mathbf{r}) = \left(\frac{\hbar c^2}{\epsilon_0 V}\right)^{\frac{1}{2}} \sum_{\mathbf{n},\lambda} \frac{\boldsymbol{\epsilon}_{\mathbf{n},\lambda}}{\sqrt{2\omega_n}} \left[\hat{a}_{\mathbf{n},\lambda} e^{\mathrm{i}\boldsymbol{\kappa}_\mathbf{n}\cdot\mathbf{r}} + \hat{a}^\dagger_{\mathbf{n},\lambda} e^{-\mathrm{i}\boldsymbol{\kappa}_\mathbf{n}\cdot\mathbf{r}}\right]. \quad (3)$$

Further, $\boldsymbol{\kappa}_\mathbf{n} = 2\pi\mathbf{n}/L$ are wave vectors with $\mathbf{n} = (n_x, n_y, n_z) \in \mathbb{Z}^3$, $\omega_n = c|\boldsymbol{\kappa}_\mathbf{n}|$ are the allowed frequencies, ϵ_0 the vacuum permittivity, and $\boldsymbol{\epsilon}_{\mathbf{n},\lambda}$ are the transversal polarization vectors of each photon mode [1, 2]. The operators $\hat{a}_{\mathbf{n},\lambda}$ and $\hat{a}^\dagger_{\mathbf{n},\lambda}$ are annihilation and creation operators, respectively, and obey canonical commutation relations $[\hat{a}_{\mathbf{n},\lambda}, \hat{a}^\dagger_{\mathbf{m},\kappa}] = \delta_{\mathbf{nm}}\delta_{\lambda\kappa}$. By introducing the displacement coordinates $q_{\mathbf{n},\lambda}$ and their conjugate momenta $\partial/\partial q_{\mathbf{n},\lambda}$ we can define $\hat{a}_{\mathbf{n},\lambda} = [q_{\mathbf{n},\lambda} + \partial/\partial q_{\mathbf{n},\lambda}]/\sqrt{2}$ and $\hat{a}^\dagger_{\mathbf{n},\lambda} = [q_{\mathbf{n},\lambda} - \partial/\partial q_{\mathbf{n},\lambda}]/\sqrt{2}$.

In Bloch theory [26] the external potential is assumed periodic, $v_\mathrm{ext}(\mathbf{r}) = v_\mathrm{ext}(\mathbf{r} + \mathbf{R}_\mathbf{n})$, where $\mathbf{R}_\mathbf{n}$ is a Bravais lattice vector. It is obvious that in general the vector potential (3) is not invariant under the translation $\mathbf{r} \to \mathbf{r}+\mathbf{R}_\mathbf{n}$. As a consequence the Pauli-Fierz Hamiltonian (2) is not periodic. This implies that Bloch's theorem and the usual band theory are not applicable to the Pauli-Fierz Hamiltonian.

A possible solution to this problem is to enforce the periodicity of the external potential on the vector potential $\hat{\mathbf{A}}(\mathbf{r}) = \hat{\mathbf{A}}(\mathbf{r} + \mathbf{R}_\mathbf{n})$, which results in the condition $\boldsymbol{\kappa}_\mathbf{n} \cdot \mathbf{R}_\mathbf{n} = 2\pi l$ for the allowed frequencies. This means that the smallest allowed frequencies are extremely large since for typical crystals the lattice constant is on the order of a few ångströms. As a consequence the corresponding allowed wavelengths are very small and the optical part of the spectrum in which many experiments are performed would be out of reach. In order to overcome this problem we apply the dipole approximation, also known as the optical limit. In the optical limit the vector potential is assumed uniform and has no spatial dependence. As a consequence the Pauli-Fierz Hamiltonian, after expanding the covariant kinetic energy, takes the form

$$\hat{H} = -\frac{\hbar^2}{2m_\mathrm{e}} \sum_{j=1}^{N} \nabla_j^2 + \frac{\mathrm{i}e\hbar}{m_\mathrm{e}c}\hat{\mathbf{A}} \cdot \sum_{j=1}^{N} \nabla_j + \frac{1}{4\pi\epsilon_0} \sum_{j<k}^{N} \frac{e^2}{|\mathbf{r}_j - \mathbf{r}_k|} + \sum_{j=1}^{N} v_\mathrm{ext}(\mathbf{r}_j) + \frac{Ne^2}{2m_\mathrm{e}c^2}\hat{\mathbf{A}}^2 + \sum_\alpha \hbar\omega_\alpha \left(\hat{a}^\dagger_\alpha \hat{a}_\alpha + \frac{1}{2}\right) \quad (4)$$

where ω_α are the allowed frequencies in the optical limit ($\alpha \equiv (\mathbf{n}, \lambda)$). The vector potential is then given by

$$\hat{\mathbf{A}} = \left(\frac{\hbar c^2}{\epsilon_0 V}\right)^{\frac{1}{2}} \sum_\alpha \frac{\boldsymbol{\epsilon}_\alpha}{\sqrt{\omega_\alpha}} q_\alpha. \quad (5)$$

Thus, in the optical limit translational invariance with respect to the electronic coordinates is preserved. But what exactly does the optical limit mean for a solid? The optical limit is valid in cases where the wavelength of the electromagnetic field is much larger than the size of the electronic system. But solids compared to the size of an atom are infinitely large systems. This holds especially within the context of Bloch theory where full periodicity is assumed. This implies that in the optical limit the wavelength of the field should be infinite and the frequency should tend to zero, $\omega_\alpha \to 0$. Naively, taking $\omega_\alpha \to 0$ in (5) seems to lead to divergencies in Eq. (4). However, if the limit is performed consistently, which means that we take into account the back-reaction of matter due to the square of vector potential, no divergencies arise.

In the following we discuss how this procedure has to be performed. For simplicity we consider the case of a monochromatic electromagnetic field of frequency ω. We isolate the purely photonic part of $\hat{H}$ which includes only one bare photon mode of frequency ω plus the square of the vector potential $\hat{H}_p = \hbar\omega\left(\hat{a}^\dagger\hat{a} + 1/2\right) + \hat{\mathbf{A}}^2 Ne^2/2m_\mathrm{e}c^2$. In terms of the photonic coordinate q and the conjugate momentum $\partial_q = \partial/\partial q$ it is $\hat{H}_p = \hbar\omega/2\left(-\partial_q^2 + q^2\right) + q^2 Ne^2\hbar/2m_\mathrm{e}\omega\epsilon_0 V$. By introducing the dressed frequency parameter given by $\tilde{\omega}^2 = \omega^2 + \omega_p^2$ and the coordinate $u = q\sqrt{\tilde{\omega}/\omega}$, the Hamiltonian $\hat{H}_p$ takes the form of a harmonic oscillator

$$\hat{H}_p = \hbar\tilde{\omega}/2\left(-\partial_u^2 + u^2\right), \quad (6)$$

where the frequency ω_p is the well-known plasma frequency which depends on the electron density $n_\mathrm{e} = N/V$ and is given by $\omega_p = \sqrt{n_\mathrm{e}e^2/m_\mathrm{e}\epsilon_0}$. The vector potential as a function of the new coordinate u is given by $\hat{\mathbf{A}} = u\boldsymbol{\epsilon}\sqrt{\hbar c^2/\tilde{\omega}\epsilon_0 V}$. Substituting $\hat{H}_p$ and $\hat{\mathbf{A}}$ back into Eq. (4) leads to

$$\hat{H} = -\frac{\hbar^2}{2m_\mathrm{e}} \sum_{j=1}^{N} \nabla_j^2 + \frac{\mathrm{i}\hbar^{\frac{3}{2}}\omega_p}{\sqrt{m_\mathrm{e}N\tilde{\omega}}} u\boldsymbol{\epsilon} \cdot \sum_{j=1}^{N} \nabla_j + \sum_{j=1}^{N} v_\mathrm{ext}(\mathbf{r}_j) + \frac{1}{4\pi\epsilon_0} \sum_{j<k}^{N} \frac{e^2}{|\mathbf{r}_j - \mathbf{r}_k|} + \frac{\hbar\tilde{\omega}}{2}\left(-\partial_u^2 + u^2\right). \quad (7)$$

We have thus rewritten Eq. (4) in a form such that the bare frequency ω does no longer appear explicitly, which makes the optical limit specifically simple. In the optical limit the dressed frequency $\tilde{\omega}$ goes to the plasma frequency ω_p, $\tilde{\omega} \to \omega_p$, and as a consequence the Hamiltonian in the optical limit is

$$\hat{H}_{\rm opt} = -\frac{\hbar^2}{2m_{\rm e}}\sum_{j=1}^{N}\nabla_j^2 + {\rm i}\sqrt{\frac{\hbar^3\omega_p}{m_{\rm e}N}}u\boldsymbol{\epsilon}\cdot\sum_{j=1}^{N}\nabla_j + \sum_{j=1}^{N}v_{\rm ext}(\mathbf{r}_j) + \frac{1}{4\pi\epsilon_0}\sum_{j<k}^{N}\frac{e^2}{|\mathbf{r}_j-\mathbf{r}_k|} + \frac{\hbar\omega_p}{2}\left(-\partial_u^2+u^2\right). \quad (8)$$

Translational Symmetry with Homogeneous Magnetic Fields.—We now discuss the case where in addition to the quantized electromagnetic field the electrons inside the solid also interact with an external homogeneous magnetic field. Since the translational symmetry is independent of the number of electrons or modes, in the following we can restrict our considerations without loss of generality to the case of only one electron and one mode. Let us start now from the Hamiltonian of Eq. (4) before taking the optical limit.

In order to describe a macroscopic homogeneous magnetic field we need to add an external part $\mathbf{A}_{\rm ext}(\mathbf{r})$ to the quantized vector potential $\hat{\mathbf{A}}$. We choose the external vector potential in Landau gauge [15] $\mathbf{A}_{\rm ext}(\mathbf{r}) = -\mathbf{e}_x By$. This external vector potential induces a homogeneous magnetic field along the z-direction $\mathbf{B}_{\rm ext} = \nabla\times\mathbf{A}_{\rm ext}(\mathbf{r}) = \mathbf{e}_z B$. For the quantum vector potential we choose the same polarization. Thus, after these considerations the total vector potential is

$$\hat{\mathbf{A}}_{\rm tot}(\mathbf{r},q) = \hat{\mathbf{A}} + \mathbf{A}_{\rm ext}(\mathbf{r}) = \mathbf{e}_x\left[q\sqrt{\hbar c^2/\epsilon_0 V\omega} - By\right] \quad (9)$$

The Hamiltonian $\hat{H}$ of Eq. (4) with the new total vector potential, which still satisfies the Coulomb gauge condition, after expanding the covariant kinetic energy is

$$\hat{H} = -\frac{\hbar^2}{2m_{\rm e}}\nabla^2 + \frac{{\rm i}e\hbar}{m_{\rm e}c}\left(\hat{\mathbf{A}} + \mathbf{A}_{\rm ext}(\mathbf{r})\right)\cdot\nabla + v_{\rm ext}(\mathbf{r}) + \frac{e^2}{2m_{\rm e}c^2}\left(\hat{\mathbf{A}} + \mathbf{A}_{\rm ext}(\mathbf{r})\right)^2 + \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \quad (10)$$

In the case of solids the external potential is periodic $v_{\rm ext}(\mathbf{r}) = v_{\rm ext}(\mathbf{r}+\mathbf{R_n})$ where $\mathbf{R_n} = \mathbf{x}_n + \mathbf{y}_m + \mathbf{z}_l$ is a Bravais lattice vector with $\mathbf{n} = (n,m,l) \in \mathbb{Z}^3$. In order to analyze conveniently the vector potential, which depends on the electronic coordinates only in y-direction and is polarized along the x-direction, we choose the lattice vectors as follows: $\mathbf{x}_n = x_n\mathbf{e}_x = na_x\mathbf{e}_x$, $\mathbf{y}_m = y_m\mathbf{e}_y = ma_y\mathbf{e}_y$ and $\mathbf{z}_l = z_l\mathbf{e}_z = la_z\mathbf{e}_z$. Having a periodic external potential and a uniform magnetic field one would expect a periodic solution using Bloch theory. However, the Hamiltonian $\hat{H}$ of Eq. (10) is not periodic because the vector potential $\hat{\mathbf{A}}_{\rm tot}(\mathbf{r},q)$ depends linearly on the coordinate y. We note that the vector potential also depends linearly on the photonic coordinate q. This implies that the vector potential is invariant under the combined translation $(y,q) \longrightarrow (y+y_m, q+By_m\sqrt{\epsilon_0 V\omega}/\sqrt{\hbar c^2})$. The fact that such a translation exists makes all terms of the Hamiltonian (10) *except* for the last one invariant under this generalized translation.

We propose that the problem of the broken translational symmetry can be resolved in the optical limit, where the bare frequency ω of the electromagnetic field goes to zero. We already showed how this procedure has to be performed in the many electron case. The purely photonic part of the Hamiltonian (10) in the one electron case $\hat{H}_p = \hbar\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right) + \hat{\mathbf{A}}^2e^2/2m_{\rm e}c^2$ can be written in the form of a harmonic oscillator already given by (6). The frequency $\omega_p = \sqrt{e^2/m_{\rm e}\epsilon_0 V}$ is the plasma frequency in the one electron case. The vector potential of Eq. (9) in terms of the new coordinate u is $\hat{\mathbf{A}}_{\rm tot}(\mathbf{r},u) = \mathbf{e}_x\left[u\sqrt{\hbar c^2/\epsilon_0 V\tilde{\omega}} - By\right]$. If we now perform the optical limit $\tilde{\omega} \to \omega_p$ and introduce the cyclotron frequency $\omega_c = eB/m_{\rm e}c$ the Hamiltonian (10) is

$$\hat{H}_{\rm opt} = \frac{-\hbar^2}{2m_{\rm e}}\nabla^2 + {\rm i}\hbar\mathbf{e}_x\left(u\sqrt{\hbar\omega_p/m_{\rm e}} - y\omega_c\right)\cdot\nabla + v_{\rm ext}(\mathbf{r}) + \frac{m_{\rm e}}{2}\left(u\sqrt{\hbar\omega_p/m_{\rm e}} - y\omega_c\right)^2 - \frac{\hbar\omega_p}{2}\partial_u^2. \quad (11)$$

For a periodic potential $\hat{H}_{\rm opt}$ is periodic under the generalized translation

$$(\mathbf{r},u) \longrightarrow (\mathbf{r}+\mathbf{R_n}, u + y_m\omega_c\sqrt{m_{\rm e}}/\sqrt{\hbar\omega_p}). \quad (12)$$

This proves our claim that in the optical limit the broken translational symmetry, induced by the homogeneous magnetic field, gets restored. In the new set of coordinates (as is depicted in Fig. 1) we can make use of Bloch's theorem in order to treat electrons in solids which interact with a homogeneous magnetic field as well as fluctuations of the electromagnetic field. We further note, that if we include a time-dependent homogeneous external vector potential as well [27–29], we can treat a solid subject to a homogeneous electric as well as magnetic field.

QED-Bloch Theory with Homogeneous Magnetic Fields.— Having restored the broken translational symmetry we will move a step further and derive a Bloch central equation for periodic solids in homogeneous magnetic fields. For convenience we work in atomic units where $\hbar = m_{\rm e} = e = 1$. The Hamiltonian $\hat{H}_{\rm opt}$ is invariant under the translation given by Eq. (12) that acts on both, the electronic and photonic configuration space. In order to describe properly this symmetry we switch to a new set of *polaritonic* coordinates given by

$$v = \frac{\sqrt{\omega_p}u - \omega_c y}{\sqrt{2}}, w = \frac{m_p\sqrt{\omega_p}u + m_c\omega_c y}{\sqrt{2}M}, \quad (13)$$

where the mass parameters are $m_p = 1/\omega_p^2$, $m_c = 1/\omega_c^2$, and $M = (m_p + m_c)/2$. In this coordinate system the Hamiltonian $\hat{H}_{\rm opt}$ becomes

$$\hat{H}_{\rm opt} = -\left(\partial_x^2 + \partial_z^2 + \partial_w^2/M\right)/2 + {\rm i}\sqrt{2}v\partial_x + v_{\rm ext}(\mathbf{r}) - \Omega^2\partial_v^2/4 + v^2 \quad (14)$$

with $\Omega^2 = 1/m_c + 1/m_p = \omega_c^2 + \omega_p^2$ and $\mathbf{r} = \left(x, w/\sqrt{2}\omega_c - m_p v/\sqrt{2}M\omega_c, z\right)$. The coordinates v and w are independent since the respective momenta and positions commute. The Hamiltonian $\hat{H}_{\text{opt}}$ includes a harmonic oscillator $\hat{H}_v = -\Omega^2\partial_v^2/4 + v^2$ which has the Hermite functions as eigen-states

$$\phi_j(v) = \frac{(2/\Omega)^{\frac{1}{4}}}{\pi^{\frac{1}{4}}\sqrt{2^j j!}} e^{-v^2/\Omega} H_j(\sqrt{2/\Omega}v), \tag{15}$$

and its spectrum is $\mathcal{E}_j = \Omega(j + 1/2)$ with $j \in \mathbb{N}_0$. $\hat{H}_v$ can be written equivalently in terms of annihilation and creation operators $\hat{H}_v = \Omega(\hat{b}^\dagger\hat{b} + \frac{1}{2})$, $\hat{b} = v/\sqrt{\Omega} + \sqrt{\Omega}\partial_v/2$ and $\hat{b}^\dagger = v/\sqrt{\Omega} - \sqrt{\Omega}\partial_v/2$ with $[\hat{b}, \hat{b}^\dagger] = 1$. The Hamiltonian $\hat{H}_{\text{opt}}$ of Eq. (14) is invariant under the translation $(x, w, z) \longrightarrow (x + x_n, w + \sqrt{2}\omega_c y_m, z + z_l)$, implying that we can make use of Bloch's theorem in those coordinates. Thus, the eigen-functions of the Hamiltonian $\hat{H}_{\text{opt}}$ can be written with the ansatz

$$\Psi_{\mathbf{k}}(\mathbf{r}_w, v) = e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}_w} U^{\mathbf{k}}(\mathbf{r}_w, v) \tag{16}$$

where $\mathbf{r}_w = (x, w, z)$. Here $U^{\mathbf{k}}(\mathbf{r}_w, v)$ is periodic along $\mathbf{r}_w = (x, w, z)$ with periodicities a_x, $\sqrt{2}\omega_c a_y$, and a_z, respectively. One important aspect of our version of the Bloch ansatz above is that it is a *polaritonic* Bloch ansatz, in the sense that both coordinates v and w are combined coordinates. The crystal momentum $\mathbf{k} = (k_x, k_w, k_z)$, corresponds to $\mathbf{r}_w$, and k_w is a *polaritonic* quantum number. Moreover, the *polaritonic* unit cell, as it is shown in Fig. 1, in w-direction scales linearly with the strength of the magnetic field. The same feature appears also in the case of the so-called magnetic unit cell, but the magnetic unit cell allows only field strengths which generate a rational magnetic flux through a unit cell [17]. This is a consequence of invariance under the magnetic translation group introduced by Zak [23, 24]. On the contrary, the polaritonic unit cell puts no restrictions on the allowed magnetic strengths.

Since the function $U^{\mathbf{k}}(\mathbf{r}_w, v)$ is periodic in $\mathbf{r}_w$ we can expand it in a Fourier series along $\mathbf{r}_w$. For the v coordinate of the polaritonic Bloch ansatz we use the eigenfunctions of the harmonic oscillator $\hat{H}_v$ given by (15). Thus, our polaritonic Bloch ansatz becomes

$$\Psi_{\mathbf{k}}(\mathbf{r}_w, v) = e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}_w} \sum_{\mathbf{n},j} U^{\mathbf{k}}_{\mathbf{n},j} e^{\mathrm{i}\mathbf{G}_{\mathbf{n}}\cdot\mathbf{r}_w}\phi_j(v), \tag{17}$$

where $\mathbf{G}_{\mathbf{n}} = (G^x_n, G^w_m, G^z_l) = 2\pi(n/a_x, m/\sqrt{2}\omega_c a_y, l/a_z)$ is the reciprocal lattice vector. The external potential is expanded in a Fourier series as well.

$$v_{\text{ext}}(\mathbf{r}) = \sum_{\mathbf{n}} V_{\mathbf{n}} e^{\mathrm{i}\mathbf{G}_{\mathbf{n}}\cdot\mathbf{r}_w} e^{-\mathrm{i}G^w_m m_p v/M} \tag{18}$$

Substituting Eqs. (17) and (18) into the Hamiltonian $\hat{H}_{\text{opt}}$ of Eq. (14), and then acting from the left with $\langle\phi_i|$ and eliminating the plane waves we obtain

$$\begin{aligned}&\left[\frac{(k_x + G^x_n)^2}{2} + \frac{(k_w + G^w_m)^2}{2M} + \frac{(k_z + G^z_l)^2}{2} + \mathcal{E}_i - E_{\mathbf{k}}\right] U^{\mathbf{k}}_{\mathbf{n},i} \\ &-\sqrt{2}\,(k_x + G^x_n)\sum_j \langle\phi_i|v|\phi_j\rangle U^{\mathbf{k}}_{\mathbf{n},j} \\ &+\sum_j\sum_{\mathbf{n}'} V_{\mathbf{n}-\mathbf{n}'} U^{\mathbf{k}}_{\mathbf{n}',j}\langle\phi_i|e^{-\mathrm{i}G^w_{m-m'}m_p v/M}|\phi_j\rangle = 0.\end{aligned} \tag{19}$$

Using the Hermite recursion relations we find for the matrix $\langle\phi_i|v|\phi_j\rangle = \sqrt{\Omega}[\sqrt{j}\delta_{i,j-1} + \sqrt{j+1}\delta_{i,j+1}]/2$. The plane wave in the last term of Eq. (19) can be written as a displacement operator using $\hat{b}$ and $\hat{b}^\dagger$,

$$e^{-\mathrm{i}G^w_{m-m'}m_p v/M} = e^{\alpha_{mm'}\hat{b} - \alpha^*_{mm'}\hat{b}^\dagger} = \hat{D}(\alpha_{mm'}) \tag{20}$$

where $\alpha_{mm'} = -\mathrm{i}G^w_{m-m'}m_p\sqrt{\Omega}/2M$. The matrix representation of this displacement operator in the basis $\{\phi_i(v)\}$ is given by [30]

$$\langle\phi_i|\hat{D}(\alpha_{mm'})|\phi_j\rangle = \sqrt{\frac{j!}{i!}}\alpha^{i-j}_{mm'} e^{-\frac{|\alpha_{mm'}|^2}{2}} L^{(i-j)}_j(|\alpha_{mm'}|^2), \tag{21}$$

where $i \geq j$ and $L^{(i-j)}_j(|\alpha_{mm'}|^2)$ are the associated Laguerre polynomials. Using Eq. (21) and the expression for the matrix $\langle\phi_i|v|\phi_j\rangle$ we obtain the generalized Bloch central equation

$$\begin{aligned}&\left[\frac{(k_x + G^x_n)^2}{2} + \frac{(k_w + G^w_m)^2}{2M} + \frac{(k_z + G^z_l)^2}{2} + \mathcal{E}_i - E_{\mathbf{k}}\right] U^{\mathbf{k}}_{\mathbf{n},i} \\ &-\frac{(k_x + G^x_n)\sqrt{\Omega}}{\sqrt{2}}\left[\sqrt{i+1}U^{\mathbf{k}}_{\mathbf{n},i+1} + \sqrt{i}U^{\mathbf{k}}_{\mathbf{n},i-1}\right] \\ &+\sum_{\mathbf{n}',j} V_{\mathbf{n}-\mathbf{n}'} U^{\mathbf{k}}_{\mathbf{n}',j}\sqrt{\frac{j!}{i!}}\alpha^{i-j}_{mm'} e^{-\frac{|\alpha_{mm'}|^2}{2}} L^{(i-j)}_j(|\alpha_{mm'}|^2) = 0.\end{aligned} \tag{22}$$

Equation (22) gives the spectrum and the eigen-functions for electrons inside a 3D solid under the influence of a constant magnetic field, in the case where quantum fluctuations of the field due to the electron density are also taken into account. In the limit where the fluctuations go to zero, $\omega_p \to 0$, the parameters take the values, $M \to \infty$, $\Omega \to \omega_c$, and $\alpha_{mm'} \to -\mathrm{i}\pi\sqrt{2}(m - m')/\sqrt{\omega_c}a_y$. In this limit Eq. (22) can be derived directly from the Hamiltonian of Eq. (10) without any photonic degrees of freedom by a similar Bloch ansatz as Eq. (16) without the coordinate w and with $v = -y\omega_c/\sqrt{2}$. This shows that the physics of periodic structures in homogeneous magnetic fields, previously considered by [17, 19, 23, 24], is recovered as a limit of our theory.

Modification of Landau Levels.—In the following we investigate the case of *no* external potential, $v_{\text{ext}}(\mathbf{r}) = 0$. Then $\hat{H}_{\text{opt}}$ of Eq. (14) can be diagonalized analytically. For the part of the Hamiltonian depending on the coordinates $\mathbf{r}_w = (x, w, z)$ the eigen-functions are plane waves given by $e^{\mathbf{ik}\cdot\mathbf{r}_w}$ and by applying the Hamiltonian $\hat{H}_{\text{opt}}$ on these plane waves we obtain

$$\hat{H}_{\text{opt}}[\mathbf{k}] = k_z^2/2 + k_w^2/2M - \Omega^2\partial_v^2/4 + \left(v - k_x/\sqrt{2}\right)^2.$$

The eigen-functions of the shifted harmonic oscillator are the Hermite functions $\phi_j(v - k_x/\sqrt{2})$ given by Eq. (15) with spectrum $\mathcal{E}_j = \Omega\,(j + 1/2)$. Thus, the energy spectrum of $\hat{H}_{\text{opt}}$ is given by Eq. (1) and the eigen-functions are

$$\Psi_{\mathbf{k},j}(\mathbf{r}_w, v) = e^{\mathbf{ik}\cdot\mathbf{r}_w}\phi_j\left(v - k_x/\sqrt{2}\right). \qquad (23)$$

The spectrum given by Eq. (1) is similar to the one derived by Landau [15], but there are significant differences due to the fluctuations of the electromagnetic field. The most important difference is that the frequency of the quantum corrected Landau levels depend on both, the cyclotron and the plasma frequency via $\Omega^2 = \omega_c^2 + \omega_p^2$. Secondly, we see that there exists a second kinetic term in the spectrum which contributes along w that combines y and u. This means that the quantum fluctuations make the spectrum dependent on the momentum along the y coordinate. Yet, in the limit where the plasma frequency goes to zero, $\omega_p \to 0$, we obtain the original Landau levels since $\Omega \to \omega_c$ and $M \to \infty$.

Conclusions.—In this Letter we presented how the translational symmetry can be restored for Bloch electrons in the presence of a homogeneous magnetic field, when the fluctuations of the field are taken into account. We derived a Bloch central equation which gives the spectrum of electrons in solids with a homogeneous magnetic field in the presence but also in the absence of the field fluctuations. The solutions of this equation in the limit where the fluctuations go to zero will reproduce the already known results of Bloch electrons in magnetic fields, such as the quantum Hall effect [16, 17] and the spectrum of the Hofstadter butterfly [19]. The central equation derived puts no limitations on the strength of the magnetic field and thus allows to scan through the whole continuum of field strengths for the first time. The problem of the restricted magnetic field strengths due to the magnetic unit cell has been highlighted by Hofstadter in [19] and our theory provides a solution. The presence of fluctuations of the field coming from the electron density in our theory implies that there will be modifications to phenomena such as the quantum Hall effects and the Hofstadter butterfly. Many of these phenomena are studied in layered or 2D materials. We propose that cavity QED confinement of 2D materials will allow for the observation of quantum fluctuation corrections. We are presently exploring this line of research.

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