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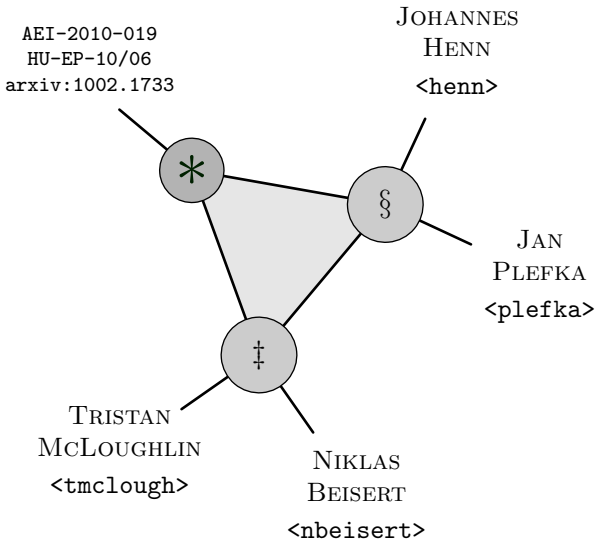
One-Loop Superconformal and Yangian Symmetries* of Scattering Amplitudes in $\mathcal{N} = 4$ Super Yang-Mills



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Abstract

Recently it has been argued that tree-level scattering amplitudes in $\mathcal{N} = 4$ Yang-Mills theory are uniquely determined by a careful study of their superconformal and Yangian symmetries. However, at one-loop order these symmetries are known to become anomalous due to infrared divergences. We compute these one-loop anomalies for amplitudes defined through dimensional regularisation by studying the tree-level symmetry transformations of the unitarity branch cuts, keeping track of the crucial collinear terms arising from the holomorphic anomaly. We extract the superconformal anomalies and show that they may be cancelled through a universal one-loop deformation of the tree-level symmetry generators which involves only tree-level data. Specialising to the planar theory we also obtain the analogous deformation for the level-one Yangian generator of momentum. Explicit checks of our one-loop deformation are performed for MHV and the 6-point NMHV amplitudes.

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1 Introduction

Maximally supersymmetric $\mathcal{N} = 4$ Yang–Mills theory (SYM) [1] is an important testing ground for the foundations of four-dimensional quantum field theories. On the one hand, the model is based on highly non-trivial interactions which are reasonably similar to those appearing in the standard model of particle physics. On the other hand, a host of surprising features make the theory much more tractable than many others. By maximally exploiting these features, we hope to gain access to previously unexplored regions of the model, e.g. the finite coupling regime. Such insights would not only be beneficial to the study of the particular model, but they could also teach us about the (qualitative) behaviour of four-dimensional quantum field theory in general.

Maximal supersymmetry turns out to constrain the model uniquely up to the choice of a gauge group and two coupling constants. It improves the quantum behaviour and leads to various cancellations and simplifications. For instance, the model is "finite" [2, 3] in that its beta function vanishes exactly, leading to unbroken (super)conformal symmetry at the quantum level. Finiteness can be traced back to the large amount of supersymmetry, but there are also a number of curious features not following from maximal supersymmetry, at least not immediately. The AdS/CFT correspondence, [4], [5], claiming exact duality to IIB strings on $AdS_5 \times S^5$ is doubtlessly the most influential property of $\mathcal{N} = 4$ SYM which is far from obvious in the standard quantum field theoretical formulation. Another important insight is that the planar limit is apparently exactly integrable [6, 7] (for reviews see [8]). This enables one to compute efficiently the spectrum of scaling dimensions of local operators using the Bethe ansatz and related techniques instead of elaborate higher-loop QFT machinery. Integrability in the guise of dual superconformal symmetry was also identified as the underlying reason for non-trivial simplifications observed in the computation of planar scattering amplitudes: Hints of this hidden, dual conformal symmetry were first seen in [9], were further studied in [10–12] and extended to

dual superconformal symmetry in [13]. It was shown in [14,15] that this is indeed a symmetry of the tree-level amplitudes. On the string theory side the dual symmetries were identified as the symmetries of a T-dual model [16,17] and shown to be a part of the integrable hierarchy [18,17, 19]. For scattering amplitudes, integrability serves thus as an enhancement of superconformal symmetry to an infinite-dimensional Hopf algebra called the Yangian [20]. As is commonly the case in integrable systems, one may hope that the large amount of symmetry highly constrains physical observables and that it predicts a unique S-matrix for planar $\mathcal{N} = 4$ SYM.

Now it is well-known that scattering amplitudes are not properly defined in a conformal field theory, so how to make sense of the above statements? Typically one introduces a regulator, e.g. by going away from $D = 4$ to $D = 4 - 2\epsilon$ spacetime dimensions in a dimensional regularisation scheme. Alternatively one can regulate the theory by introducing small masses for the particles by going off-shell [10] or higgsing the theory e.g. [16,21]. In every case the regulator breaks conformal symmetry and scattering amplitudes become well-defined. After renormalisation one tries to remove the regulator in order to return to the original model. For example, for local operators the dimensional regularisation procedure leads to perfectly finite answers showing the desired conformal behaviour, albeit with a non-trivial spectrum of anomalous dimensions. Scattering amplitudes, however, remain divergent in the limit of vanishing regulator, hence they are problematic as anticipated. The main distinction between local operators and scattering amplitudes is the following: The former introduce short-distance (UV) singularities due to multiple fields at coincident spacetime points, while the latter introduce long-distance (IR) singularities due to collinear massless particles. We know well how to renormalise UV singularities by redefining local operators but this is not the case for the IR singularities. The structure of these divergences is of course well known from the study of QCD — they factorise and exponentiate [22–25]. Furthermore in properly defined physical observables, [26], such as inclusive cross-sections or hadronic event shapes we expect that all such divergences cancel (see [27] and [28] for recent discussions of such observables in the current context). However, they cannot be removed from the S-matrix itself. For the special case of the *dual* conformal symmetry we do have control over the apparent breaking of the symmetry via the relation between light-like Wilson loops and amplitudes [16, 29, 10, 30, 11]. As the IR divergences of the amplitudes are mapped to the UV divergences, due to cusps, in the Wilson loops an all-order anomalous Ward identity can be derived [11, 12]. This in turn strongly constrains the form of the amplitudes.

Unfortunately we do not know how to make use of the (super)conformal symmetries or integrability to constrain the S-matrix at loop level. Yet it has become clear that the divergent and finite contributions to the scattering amplitudes can be computed unambiguously and have a physical interpretation. Furthermore conformal symmetry in $\mathcal{N} = 4$ SYM is non-anomalous. Consequently it is our firm belief that all symmetries apply to every physical observable such as the S-matrix, even if some symmetries are obscured by quantum effects.

At first sight superconformal and Yangian symmetry appear to be good symmetries of the tree level S-matrix whereas at loop level they are broken beyond repair. Fortunately, both statements are not true. Even at tree level the conformal symmetry is superficially broken at singular configurations where massless particles become collinear. Interestingly, the existence of collinear particles is closely related to the subtleties in defining asymptotic states and scattering amplitudes

in a conformal field theory: Multiple quanta with collinear momenta are physically indistinguishable from a single particle with the same overall momentum. Hence the Fock space description for asymptotic states is not quite adequate, one ought to factor out collinear configurations. Unfortunately, such a projective space of asymptotic states is technically hard to realise and instead it is easier to work in the larger Fock space with some projective structure implied. That the S-matrix respects the projective structure can be inferred from the well-studied collinear behaviour determined by the splitting functions [31,32]. Therefore the S-matrix is indeed a proper physical object even in the presence of conformal symmetry or massless states. The problem rather lies with conformal symmetry, because naïvely it does not respect the projective structure on Fock space. Luckily the free representation of superconformal symmetry on scattering amplitudes can be deformed in such a way as to make it compatible with the projective structure. In particular this makes the tree level S-matrix exactly conformal.

One can see that naïve conformal symmetry is broken by a holomorphic anomaly due to collinear particles [33–36]. At tree level this happens exclusively at singular particle configurations which is why the anomaly can be ignored to large extent. At loop level the situation is more complicated because particles running in loops can become collinear with others. Due to the integration over loop momenta the anomaly is smeared over all particle configurations and thus it always requires proper treatment.

In this paper we consider the superconformal and Yangian symmetries of one-loop scattering amplitudes in $\mathcal{N} = 4$ SYM. We will show how to deform the representation of superconformal symmetry, in a manner generalising the tree-level construction [35], such that it annihilates one-loop scattering amplitudes including the IR singularities as well as the finite contributions. Importantly, we will work in a strictly on-shell framework for the S-matrix. As we will see, such a framework is provided by generalised unitarity which was introduced in [37, 38] and further developed in [39, 40]. These methods, based on studying the behaviour of amplitudes across branch cut singularities [41] relate loop-level amplitudes to on-shell tree level amplitudes. Symmetries of the latter are under full control and will dictate the structure of the symmetries at one loop [42,36,43,44]. Although any other self-consistent regulator could be used in principle, we shall choose a dimensional regularisation scheme for convenience. The majority of perturbative results are formulated in this scheme where they take a reasonably compact form.

Our proposal shares several features with a recent proposal [43] which however uses a very different framework of off-shell amplitudes and a particular massive particle regularisation scheme. Although the previous proposal is very elegant and economical, the application to the on-shell S-matrix including its divergences appears to be subtle. In our framework we thus have to choose different deformations whose action we can however define straight-forwardly on the IR-singular on-shell S-matrix.

For the reader's convenience we outline the contents of subsequent sections. We start in Sec. 2 with a review of the on-shell superspace description of tree-level scattering amplitudes. We discuss the symmetries of the amplitudes and how they can be deformed to account for the holomorphic anomaly which arises in collinear configurations. In Sec. 3 we turn to our main topic: Symmetries of one-loop amplitudes. After an outline of our general method we analyse the portion of the superconformal anomaly for a generic amplitude arising from unitarity cuts. We argue that the

anomaly of the full amplitude can be trivially lifted from the cut contribution and propose a set of deformations of the representation that annihilates all one-loop amplitudes. Restricting this deformation to the planar limit we check by explicit calculation that all MHV, Sec. 4, and the six-point NMHV, Sec. 5, amplitudes are indeed invariant with respect to the deformed generators. In Sec. 6 we perform the analogous analysis for the level one momentum generator of the Yangian and outline the necessary procedure for the level one supersymmetry generator. In Sec. 7 we discuss the propagator $i\epsilon$ prescription used in our definition of amplitudes. This allows us to compare our proposal to that of [43]. We close with a discussion of our conclusions and several appendices with further calculational details and conventions.

2 Tree Amplitudes and Their Symmetries

We start by reviewing tree-level scattering amplitudes and their symmetries in the on-shell superspace formulation. This provides a context for our later discussion of one-loop amplitudes and allows us to fix our notation.

2.1 On-Shell Superspace and Generating Functional

We will be concerned with the n-particle scattering amplitudes of $\mathrm{U}(N_\mathrm{c})$ $\mathcal{N}=4$ super-Yang-Mills (see [45, 46] for recent, relevant reviews). These amplitudes can be conveniently expanded in a basis of colour structures. At tree level only single-trace structures appear[1]

$$A_n^{a_1\dots a_n}(1,\dots,n) = \sum_{\sigma\in S_n/\mathbb{Z}_n} \mathrm{Tr}\left(T^{a_{\sigma(1)}}\dots T^{a_{\sigma(n)}}\right) A_n(\sigma(1),\dots,\sigma(n)), \tag{2.1}$$

where T^a is an $\mathrm{U}(N_\mathrm{c})$ generator in the fundamental representation and $a, b, \dots$ denote indices for the adjoint representation.

The tree-level amplitudes take a particularly simple form when written as functions of the on-shell spinor-helicity superspace coordinates $(\lambda_i^\alpha, \tilde\lambda_i^{\dot\alpha}, \eta_i^A)$ [47] (see also e.g. [48, 13, 49]). Here $\alpha,\beta = 1,2$ and $\dot\alpha,\dot\beta = 1,2$ are fundamental indices for two distinct $\mathfrak{su}(2)$'s and $A, B = 1,2,3,4$ are indices for $\mathfrak{su}(4)$. The commuting spinors λ^α and $\tilde\lambda^{\dot\alpha}$ parametrise the on-shell momenta in spinor notation $p^{\beta\dot\alpha} = \sigma_\mu^{\beta\dot\alpha} p^\mu$

$$p_i^{\beta\dot\alpha} = \lambda_i^\beta \tilde\lambda_i^{\dot\alpha}, \qquad P^{\beta\dot\alpha} = \sum_{i=1}^n p_i^{\beta\dot\alpha} = \sum_{i=1}^n \lambda_i^\beta \tilde\lambda_i^{\dot\alpha}, \tag{2.2}$$

and can be used to form the usual invariants, i.e. $\langle i,j\rangle = \varepsilon_{\alpha\gamma}\lambda_i^\alpha\lambda_j^\gamma$ and $[i,j] = \varepsilon_{\dot\alpha\dot\gamma}\tilde\lambda_i^{\dot\alpha}\tilde\lambda_j^{\dot\gamma}$. In Minkowski signature the spinors are related by complex conjugation and for positive (negative) energy particles we have $\tilde\lambda = +(-)\bar\lambda$. The Graßmann variable η^A allows one to combine the

[1] The trace in this expression can be expanded into $\mathrm{U}(N_\mathrm{c})$ structure constants when making use of the explicit form of the colour-ordered amplitude A_n. As such it becomes valid for generic gauge groups.

on-shell states of $\mathcal{N}=4$ SYM — the two gluon helicity states $G^{\pm}$, the fermions $\Gamma_A/\bar{\Gamma}^A$, and scalars S_{AB} — into a single superwavefunction [3, 50]

$$\begin{aligned}\Phi(\lambda,\tilde{\lambda},\eta) = G^+(\lambda,\tilde{\lambda}) + \eta^A\Gamma_A(\lambda,\tilde{\lambda}) + \tfrac{1}{2}\eta^A\eta^B S_{AB}(\lambda,\tilde{\lambda}) \\ + \tfrac{1}{6}\varepsilon_{ABCD}\eta^A\eta^B\eta^C\bar{\Gamma}^D(\lambda,\tilde{\lambda}) + \tfrac{1}{24}\varepsilon_{ABCD}\eta^A\eta^B\eta^C\eta^D G^-(\lambda,\tilde{\lambda})\,. \end{aligned} \tag{2.3}$$

Using the η's one can form the supermomentum q_i of a particle which serves as the fermionic partner to the momentum p_i (Q is the overall supermomentum)

$$q_i^{\beta A} = \lambda_i^\beta \eta_i^A, \qquad Q^{\beta A} = \sum_{i=1}^n q_i^{\beta A} = \sum_{i=1}^n \lambda_i^\beta \eta_i^A. \tag{2.4}$$

It will often be convenient to refer to all spinor-helicity superspace coordinates in a collective fashion through $\Lambda := (\lambda^\alpha, \tilde{\lambda}^{\dot\alpha}, \eta^A)$. These can represent particles with both positive and negative energies depending on the choice $\tilde{\lambda} = \pm\bar{\lambda}$. Related to this we introduce the compact notation $\bar{\Lambda}$ for flipping the energy as well as all other components of the momentum p and supermomentum q

$$\bar{\Lambda} := (+\lambda, -\tilde{\lambda}, -\eta), \qquad e^{i\varphi}\Lambda := (e^{+i\varphi}\lambda, e^{-i\varphi}\tilde{\lambda}, e^{-i\varphi}\eta). \tag{2.5}$$

The second notation $e^{i\varphi}\Lambda$ corresponds to a helicity rotation about the particle axis. Finally, we introduce the canonical measure on superspace $d^{4|4}\Lambda := d^4\lambda\, d^4\eta$. The bosonic integral $d^4\lambda := d^2\lambda\, d^2\bar{\lambda}$ is equivalent to the Lorentz-invariant on-shell integral and an integral over the particle phase

$$d^4\lambda = d\varphi\, d^4p\, \delta(p^2) = d\varphi\, \frac{d^3p}{2E}\,. \tag{2.6}$$

The fermionic integral $d^4\eta$ implements the sum over all particle types. We shall assume that integration is over positive *and* negative energies. To restrict the integral to the forward or backward light-cone we shall use the notation $d^{4|4}_{\pm}\Lambda$.

The super-amplitudes $A_n(\Lambda_1,\ldots,\Lambda_n)$ are polynomials in the η_i's whose coefficients are the amplitudes of the various component fields. For reasons of $\mathfrak{su}(4)$-invariance the η's must come in sets of four leading to the helicity classification

$$A_n = \sum_{k=2}^{n-2} A_{n,k}, \qquad A_{n,k} \sim \eta^{4k}. \tag{2.7}$$

The terms $A_{n,m+2}$ are called N^mMHV subamplitudes. Furthermore, conservation of momentum and supermomentum as well as conformal transformations enforce that all amplitudes have a common prefactor

$$A_{n,k} = \frac{\delta^4(P)\,\delta^8(Q)}{\langle 12\rangle\ldots\langle n1\rangle}\, R_{n,k}. \tag{2.8}$$

The remainder functions $R_{n,m+2}$ are homogeneous of degree $4m$ in the η's. The first term $R_{n,2}$ is simply 1, so we recover the well-known formula for MHV amplitudes [47, 48],

$$A_{n,\mathrm{MHV}} := A_{n,2} = \frac{\delta^4(P)\,\delta^8(Q)}{\langle 12\rangle\ldots\langle n1\rangle}\,, \tag{2.9}$$

and explicit, though somewhat more complicated, expressions for all other terms at tree level can be found in [15].

In considering the symmetries it is useful to combine all amplitudes into a single generating functional (see also [51]). We introduce a source field $J(\Lambda)$ conjugate to the superspace field $\Phi(\Lambda)$ and we shall use the compressed notation $J_i := J(\Lambda_i)$ for the source field corresponding to particle i. The generating functional of tree amplitudes reads

$$\mathcal{A}[J] = \sum_{n=4}^{\infty} \int \prod_{i=1}^{n} (d^{4|4}\Lambda_i)\, \frac{1}{n}\, \mathrm{Tr}\big(J_1 \dots J_n\big) A_n(\Lambda_1, \dots, \Lambda_n)\ . \tag{2.10}$$

The n-particle super-amplitude can be extracted by taking functional derivatives

$$\check{J}(\Lambda) := \frac{\delta}{\delta J(\Lambda)} \tag{2.11}$$

of the generating functional

$$A_n^{a_1 \dots a_n}(\Lambda_1, \dots, \Lambda_n) = \check{J}^{a_1}(\Lambda_1) \dots \check{J}^{a_n}(\Lambda_n) \mathcal{A}[J]\big|_{J=0}, \tag{2.12}$$

and we note that the commutative variations naturally account for the sum over all permutations in (2.1).

2.2 Free Symmetries

We will be interested in how the superconformal algebra, $\mathfrak{psu}(2,2|4)$, is realised on the scattering amplitudes. This algebra comprises the Lorentz rotations L, $\bar{\mathrm{L}}$, the internal symmetry rotations R, momentum generators P, special conformal generators K, the dilatation generator D, the Poincaré supercharges Q, $\bar{\mathrm{Q}}$ and special conformal supercharges S, $\bar{\mathrm{S}}$. Using the on-shell superspace notation the free representation carried by a single on-shell superparticle can be written very compactly [48]

$$\begin{aligned}
\mathrm{L}^{\alpha}{}_{\beta} &= \lambda^{\alpha}\partial_{\beta} - \tfrac{1}{2}\delta^{\alpha}_{\beta}\lambda^{\gamma}\partial_{\gamma}, & \bar{\mathrm{L}}^{\dot\alpha}{}_{\dot\beta} &= \tilde\lambda^{\dot\alpha}\tilde\partial_{\dot\beta} - \tfrac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\tilde\lambda^{\dot\gamma}\tilde\partial_{\dot\gamma},\\
\mathrm{R}^{A}{}_{B} &= \eta^{A}\partial_{B} - \tfrac{1}{4}\delta^{A}_{B}\eta^{C}\partial_{C}, & \mathrm{D} &= 1 + \tfrac{1}{2}\lambda^{\gamma}\partial_{\gamma} + \tfrac{1}{2}\tilde\lambda^{\dot\gamma}\tilde\partial_{\dot\gamma},\\
\mathrm{Q}^{aB} &= \lambda^{\alpha}\eta^{B}, & \mathrm{S}_{aB} &= \partial_{\alpha}\partial_{B},\\
\bar{\mathrm{Q}}^{\dot a}_{B} &= \tilde\lambda^{\dot\alpha}\partial_{B}, & \bar{\mathrm{S}}^{B}_{\dot a} &= \eta^{B}\tilde\partial_{\dot\alpha},\\
\mathrm{P}^{\beta\dot\alpha} &= \lambda^{\beta}\tilde\lambda^{\dot\alpha}, & \mathrm{K}_{\beta\dot\alpha} &= \partial_{\beta}\tilde\partial_{\dot\alpha},
\end{aligned} \tag{2.13}$$

where we abbreviate $\partial_a = \partial/\partial\lambda^a$, $\tilde\partial_{\dot a} = \partial/\partial\tilde\lambda^{\dot a}$ and $\partial_A = \partial/\partial\eta^A$. Furthermore, there is a central charge C

$$\mathrm{C} = 1 + \tfrac{1}{2}\lambda^{\gamma}\partial_{\gamma} - \tfrac{1}{2}\tilde\lambda^{\dot\gamma}\tilde\partial_{\dot\gamma} - \tfrac{1}{2}\eta^{C}\partial_{C}. \tag{2.14}$$

It acts as the constraint that every physical particle must be uncharged under it, which follows from (2.3) and the helicities of the various fields.

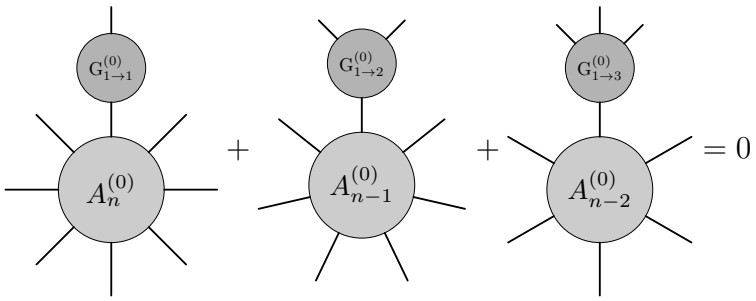

Figure 1: Statement of exact invariance of tree amplitudes under the deformed superconformal representation.

The corresponding representation of a generic generator $\mathrm{G}^{\mathrm{free}}$ on n particles is simply given by the sum over actions on individual particles $\mathrm{G}_i^{\mathrm{free}}$,

$$\mathrm{G}^{\mathrm{free}} = \sum_{i=1}^{n} \mathrm{G}_i^{\mathrm{free}}. \tag{2.15}$$

2.3 Exact Tree-Level Symmetries

Invariance of the amplitude A_n under the generator G is the statement

$$\mathrm{G}A_n = 0. \tag{2.16}$$

As was discussed at length in [35] the free representation does not exactly annihilate tree-level amplitudes, but rather must be deformed by generators which change the number of external legs. These non-linear contributions to the generators in the interacting *classical* theory are generically of the form, see Fig. 1,

$$\mathrm{G} = \mathrm{G}_{1\to1} + \mathrm{G}_{1\to2} + \mathrm{G}_{1\to3}\ . \tag{2.17}$$

The first term $\mathrm{G}_{1\to1} = \mathrm{G}^{\mathrm{free}}$ is the free generator discussed above, which simply takes a single leg and returns a single modified leg. The correction terms compensate for the contributions occurring at values of the external momenta where particles become collinear. The deformation $\mathrm{G}_{1\to2}$ can be found by explicitly calculating the action of generators involving derivatives in λ or $\tilde{\lambda} = \mathrm{sign}(E(\lambda))\bar{\lambda}$ on the n-point amplitude, A_n and by carefully accounting for the holomorphic anomaly terms, which arise in $(3,1)$ spacetime signature, i.e.

$$\frac{\partial}{\partial\tilde{\lambda}^{\dot{\alpha}}}\frac{1}{\langle\lambda,\mu\rangle} = 2\pi\,\mathrm{sign}\big(E(\lambda)E(\mu)\big)\,\varepsilon_{\dot{\alpha}\dot{\gamma}}\tilde{\mu}^{\dot{\gamma}}\delta^2\big(\langle\lambda,\mu\rangle\big)\,. \tag{2.18}$$

It was shown in [35] that this anomaly is equivalent to attaching an anomaly three-vertex G_3 to an amplitude with one leg less. The anomaly is thus cancelled by deforming the naïve free generator (left in Fig. 1) by a term $\mathrm{G}_{1\to2}$ which attaches the same vertex to the amplitude but

with the opposite sign (middle in Fig. 1). For a three-vertex of massless particles the support must be on configurations with all three momenta collinear $p_1^\mu \sim p_2^\mu \sim p_3^\mu$, i.e.

$$G_3 \sim \delta^4(\lambda_{\bar{1}} - e^{-i\varphi}\lambda_3 \sin\alpha)\, \delta^4(\lambda_{\bar{2}} - e^{-i\vartheta}\lambda_3 \cos\alpha). \tag{2.19}$$

Furthermore, their colour structure equals the structure constants f^{abc}

$$G_3^{abc}(\Lambda_1, \Lambda_2, \Lambda_3) = i f^{abc} G_3(\Lambda_1, \Lambda_2, \Lambda_3). \tag{2.20}$$

The third term $\mathrm{G}_{1\to 3}$ corresponds to a four-vertex G_4. Luckily, $\mathrm{G}_{1\to 3}$ arises only in the closure of the algebra. For this reason it is not necessary to specify the vertex G_4 explicitly; it consists of a combination of two G_3's. To complete the picture it is instructive to note that also the free representation corresponds to a vertex, but now with two legs

$$G_2^{ab}(\Lambda_1, \Lambda_2) = \delta^{ab} G_2(\Lambda_1, \Lambda_2), \qquad G_2(\Lambda_1, \Lambda_2) = \mathrm{G}_1^{\mathrm{free}} \delta^{4|4}(\Lambda_1 - \bar{\Lambda}_2). \tag{2.21}$$

Here $\mathrm{G}_1^{\mathrm{free}}$ is the free generator acting as a differential operator on the spinor-helicity superspace with label 1.

In the language of sources and generating functionals the deformations are very natural

$$\begin{aligned} \mathrm{G}_{1\to 1} &= \int (d^{4|4}\Lambda)^2\, G_2^{ab}(\Lambda_1, \Lambda_2) J^a(\Lambda_1) \check{J}^b(\bar{\Lambda}_2), \\ \mathrm{G}_{1\to 2} &= \tfrac{1}{2} \int (d^{4|4}\Lambda)^3\, \mathrm{sign}(E_1 E_2)\, G_3^{abc}(\Lambda_1, \Lambda_2, \Lambda_3) J^a(\Lambda_1) J^b(\Lambda_2) \check{J}^c(\bar{\Lambda}_3). \end{aligned} \tag{2.22}$$

The sign in $\mathrm{G}_{1\to 2}$ for opposite energy states 1 and 2 is required to match the sign in (2.18). Invariance of the amplitude as a whole now becomes the statement $\mathrm{G}\mathcal{A}[J] = 0$.

Among the superconformal generators, only $\mathrm{G} = \mathrm{S}$, $\bar{\mathrm{S}}$ and K receive deformations at tree level. The latter follows from the algebra and hence we do not need to consider it further. The two former generators receive only corrections of the type $\mathrm{G}_{1\to 2}$. In App. A we present a formal derivation of the corresponding anomaly vertices. The resulting anomaly vertices read

$$\begin{aligned} (S_3)_{\alpha B}(\Lambda_1, \Lambda_2, \Lambda_3) = -2 \int d^{4|4}\Lambda'\, \delta^{4|4}(\Lambda')\, \varepsilon_{\alpha\gamma} \lambda_3^\gamma \partial'_B \int d\alpha\, d\varphi\, d\vartheta\, e^{-i\varphi - i\vartheta} \\ \cdot \delta^{4|4}(e^{-i\varphi}\Lambda_3 \sin\alpha + e^{i\vartheta}\Lambda' \cos\alpha - \bar{\Lambda}_1) \\ \cdot \delta^{4|4}(e^{-i\vartheta}\Lambda_3 \cos\alpha - e^{i\varphi}\Lambda' \sin\alpha - \bar{\Lambda}_2) + \begin{matrix}\text{two cyclic} \\ \text{images}\end{matrix}, \\ (\bar{S}_3)^B_{\dot{\alpha}}(\Lambda_1, \Lambda_2, \Lambda_3) = -2 \int d^{4|4}\Lambda'\, \delta^4(\lambda')\, \varepsilon_{\dot{\alpha}\dot{\gamma}} \tilde{\lambda}_3^{\dot{\gamma}} \eta'^B \int d\alpha\, d\varphi\, d\vartheta\, e^{i\varphi + i\vartheta} \\ \cdot \delta^{4|4}(e^{-i\varphi}\Lambda_3 \sin\alpha + e^{i\vartheta}\Lambda' \cos\alpha - \bar{\Lambda}_1) \\ \cdot \delta^{4|4}(e^{-i\vartheta}\Lambda_3 \cos\alpha - e^{i\varphi}\Lambda' \sin\alpha - \bar{\Lambda}_2) + \begin{matrix}\text{two cyclic} \\ \text{images}\end{matrix}. \end{aligned} \tag{2.23}$$

The ranges for integration read $0 \le \alpha \le \pi/2$ and $0 \le \varphi, \vartheta < 2\pi$. The cyclic images account for the different combinations of energy signatures as described in App. A.

For comparison, the two-vertices for the free generators $\mathrm{S}, \bar{\mathrm{S}}$ read, cf. (2.13)

$$\begin{aligned}(S_2)_{\alpha B}(\Lambda_1, \Lambda_2) &= \partial_{1,\alpha}\partial_{1,B}\delta^{4|4}(\Lambda_1 - \bar{\Lambda}_2)\,,\\ (\bar{S}_2)^B_{\dot{\alpha}}(\Lambda_1, \Lambda_2) &= \tilde{\partial}_{1,\dot{\alpha}}\eta_1^B\delta^{4|4}(\Lambda_1 - \bar{\Lambda}_2)\,.\end{aligned} \tag{2.24}$$

An important point is that the deformed tree-level symmetries relate all tree amplitudes. As described above we can label the amplitudes by the number of η's, $A_{n,k} \sim \eta^{4k}$. The correction to $\bar{\mathrm{S}}$ keeps k fixed while increasing n by one whereas the correction to S increases both k and n by one. Thus by use of the generators we find relations between the variations of all amplitudes.

It has recently been shown, [36], that the free symmetries (including dual superconformal symmetries) alone are insufficient to uniquely fix the tree amplitudes. They merely determine that the amplitudes be linear combinations of dual superconformal invariants. However, the demand of correct collinear behaviour (*or* the absence of so-called spurious poles) is, under certain mild assumptions, sufficient to fix all relative coefficients and so uniquely determine the full tree-level amplitude [52, 36]. Equivalently, demanding that the corrected generators are exact symmetries, fixes all tree-level amplitudes. Of course all tree-level amplitudes have already been explicitly determined in [15], by use of the BCFW recursion relations [53], and their generalisations [54, 14, 55, 49]. However, the extent to which the symmetries fix the amplitudes is an important question, particularly beyond tree level where we no longer have such efficient methods as BCFW.

A related analysis of the symmetries of tree level amplitudes was performed in [43]. This work made use of the CSW [56] approach to constructing scattering amplitudes and, with some assumptions regarding the regularisation of divergences, could be generalised to loop level. We will comment on the relation of our proposals to that of [43] in Sec. 7.

3 Superconformal Symmetry at One Loop

We now wish to consider scattering amplitudes beyond tree level in order to account for the new features to which loops give rise. One important aspect of massless theories is the existence of infra-red divergences which necessitate the introduction of a regulator. In part, these divergences originate from virtual particles in loops becoming collinear with external legs. These divergences cannot be removed, but rather cancel only when calculating physical observables, and so the amplitudes will explicitly depend on the regulator.

We will expand the amplitudes using the loop counting parameter[2] [3]

$$g^2 = \frac{g^2_{\mathrm{YM}}C_{\mathrm{A}}}{16\pi^2}\,. \tag{3.1}$$

[2]We shall use a minimal subtraction scheme without absorbing predictable numerical combinations like γ or $\log 4\pi$ into the coupling constant. Instead we will carry them along in a constant $c_\epsilon = 1 + \mathcal{O}(\epsilon)$, see (F.2).

[3]For general gauge groups we write the loop counting parameter in terms of the quadratic Casimir. For $\mathrm{U}(N_c)$ gauge group and with normalisation $\mathrm{Tr}(t^a t^b) = \delta^{ab}$, $C_{\mathrm{A}} = N_{\mathrm{c}}$.

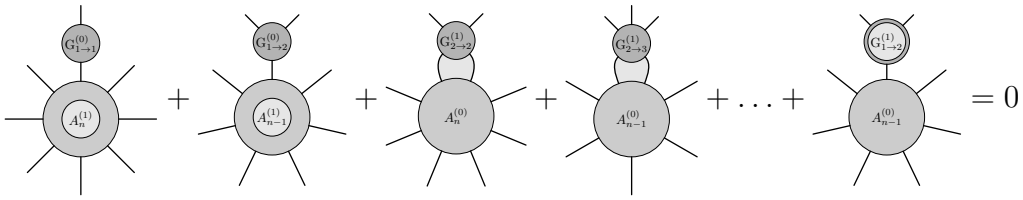

Figure 2: Deformations of generators necessary for invariance of scattering amplitudes at loop level.

so that

$$A_n^{a_1\dots a_n}(1,\dots,n) = \sum_{\ell=0}^{\infty} g^{2\ell} \left(A^{(\ell)}\right)_n^{a_1\dots a_n}(1,\dots,n)\ . \tag{3.2}$$

In principle one can further expand the amplitudes in an appropriate basis of colour structures at each order. For example, at one loop there are double traces in addition to the single traces seen already at tree level. However we will for the most part treat the general case and only simplify to specific colour structures in considering the planar limit.

In explicit calculations of amplitudes it is common to make use of dimensional regularisation[4] where $D = 4 - 2\epsilon$ and for concrete calculations this is the regularisation we will consider. The amplitudes will have singularities as $\epsilon \to 0$, typically $1/\epsilon^2$ per loop level for a conformal theory. The structure of these divergences is well understood, see e.g. [23, 25], being determined by an evolution equation which follows from gauge invariance and the factorisation of processes separated by energy scales.

The introduction of the regulator breaks conformal symmetry and thus the divergent parts of the amplitude will manifestly fail to be invariant.[5] Generically even the finite parts of the loop-level amplitudes will not be annihilated by the tree-level generators. However, by introducing further deformations of the generators we can account for these effects and show that the amplitudes are indeed invariant. Said deformations will in general involve more external legs and are schematically of the form

$$\mathrm{G}_{m\to n} \sim \int \mathrm{Tr}\big(\underbrace{J\dots J}_{n}\underbrace{\check{J}\dots\check{J}}_{m}\big). \tag{3.3}$$

Such an operator grabs m legs of an amplitude and replaces them by n. Acting on an ℓ-loop amplitude with p external legs this could cancel a term arising from the free generator acting on an $\ell + m - 1$ loop amplitude with $p + n$ external legs. In addition to creating loops by acting on multiple legs of an amplitude, deformations can contain loops within themselves. The general structure of the deformations necessary to annihilate the one-loop amplitudes is shown in Fig. 2. It is the goal of the subsequent sections to find the explicit form of these deformations.

[4] In order to maintain consistency with the supersymmetric Ward identities a supersymmetric variant, such as dimensional reduction [57], should be chosen.

[5] In fact, the most divergent parts actually are invariant, but for the subleading (including finite) parts invariance fails

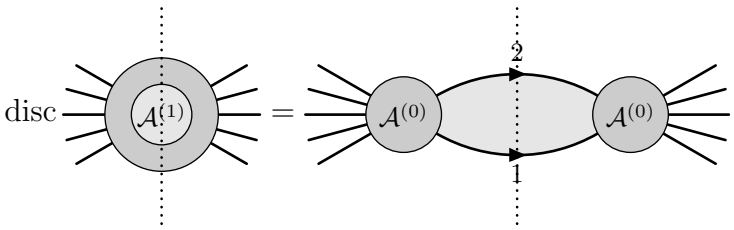

Figure 3: Discontinuity of $\mathcal{A}^{(1)}$

3.1 General One-Loop Anomaly of Cuts

Considered as functions of the kinematic invariants, loop-level amplitudes have branch cuts, in addition to collinear singularities and multi-particle poles which appear already at tree level. The form of the discontinuity across a given cut is determined by unitarity and at one loop can be expressed as a phase space integral over products of tree-level amplitudes. In fact, in supersymmetric theories one can reconstruct the entire amplitude from its cuts. Such unitarity methods, introduced in [37, 38] and further developed in [39, 40], have proved tremendously powerful calculational tools and are a convenient method for uncovering the structure of the symmetries at one loop [42, 36, 43, 44].

As was the case at tree level it is convenient to make use of the generating functional language as it naturally allows for the length-changing deformations and includes the sum over all cuts. We start with with the definition of the unitarity cut, see Fig. 3,[6]

$$\operatorname{disc}\mathcal{A}^{(1)} = -\tfrac{1}{4}\left(\frac{i}{4\pi^2 C_{\mathrm{A}}}\right)\int (d^{4|4}\Lambda)^2\,\Delta_{12}^{\epsilon}(\check{J}_{\bar{1}}^{a}\check{J}_{\bar{2}}^{b}\mathcal{A}^{(0)})(\check{J}_{1}^{a}\check{J}_{2}^{b}\mathcal{A}^{(0)}). \tag{3.4}$$

The above unitarity relation is to be understood as follows: The generating functionals $\mathcal{A}^{(0)}$ both represent a tree level subamplitude with an indeterminate number of legs (saturated with source fields J). From each subamplitude we grab two legs by action with the source variation $\check{J}$. We then perform an on-shell integration over the momenta for each pair of legs. The measure Δ_{12}^{ϵ} is non-zero only where the energies of the two particles have equal signs. Technically it is achieved by the following step function θ of the two-particle invariant $s_{j,k}$

$$\Delta_{12}^{\epsilon} = \theta(-s_{12}) + \mathcal{O}(\epsilon), \qquad s_{j,k} = (p_j + p_k)^2 = \langle j,k\rangle[k,j]. \tag{3.5}$$

This step function also specifies which particular two-particle channel we are talking about. In some cases the above integral is divergent so that the $\mathcal{O}(\epsilon)$ contributions to the measure Δ_{12}^{ϵ} become important and will serve as a regulator. We will specify its precise form later where we need it.

The tree amplitude functionals $\mathcal{A}^{(0)}[J]$ in the cut are both invariant, so the tree generator $\mathrm{G}^{(0)}$ will see only the source variations $\check{J}$

$$\mathrm{G}^{(0)}\operatorname{disc}\mathcal{A}^{(1)} = -\tfrac{1}{2}\left(\frac{i}{4\pi^2 C_{\mathrm{A}}}\right)\int (d^{4|4}\Lambda)^2\,\Delta_{12}^{\epsilon}(\check{J}_{\bar{1}}^{a}\check{J}_{\bar{2}}^{b}\mathcal{A}^{(0)})\big([\mathrm{G}^{(0)},\check{J}_{1}^{a}\check{J}_{2}^{b}]\mathcal{A}^{(0)}\big). \tag{3.6}$$

[6] In anticipation of taking the planar limit for the $\mathrm{U}(N_{\mathrm{c}})$ case, we have included a factor of the quadratic Casimir, C_{A}, in the loop counting parameter g^2, see (3.1), and thus we need to cancel it here.

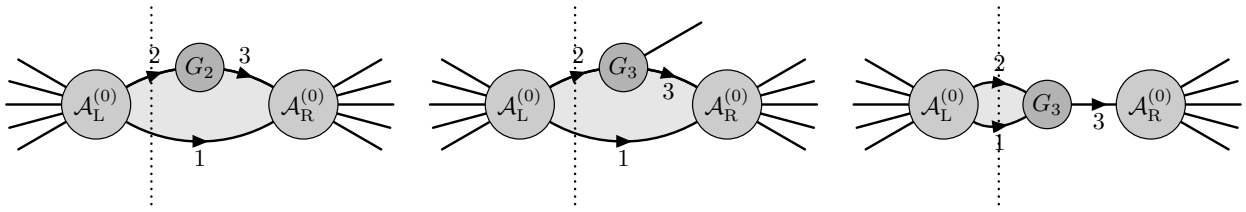

Figure 4: Three types of anomalies on the cut.

We can now substitute the definition of the deformed generic generator at tree level $\mathrm{G}^{(0)} = \mathrm{G}^{(0)}_{1\to 1} + \mathrm{G}^{(0)}_{1\to 2}$ with

$$\mathrm{G}^{(0)}_{1\to 1} = \int (d^{4|4}\Lambda)^2 G^{ab}_{1\bar{2}} J^a_1 \check{J}^b_2, \qquad \mathrm{G}^{(0)}_{1\to 2} = \tfrac{1}{2}\int (d^{4|4}\Lambda)^3 \,\mathrm{sign}(E_1 E_2) G^{abc}_{12\bar{3}} J^a_1 J^b_2 \check{J}^c_3. \tag{3.7}$$

Here, and subsequently, we use an analogous notation for the kernels as was introduced for the sources e.g. $G^{ab}_2(\Lambda_1, \bar{\Lambda}_2) = G^{ab}_{1\bar{2}}$. The action of the generator on the $\check{J}^a_{\bar{1}}\check{J}^b_{\bar{2}}\mathcal{A}^{(0)}$ subamplitudes yields

$$\begin{aligned}
&\left(\frac{i}{4\pi^2 C_{\rm A}}\right)\int (d^{4|4}\Lambda)^3 \Delta^{\epsilon}_{12} G^{bc}_{2\bar{3}} (\check{J}^a_{\bar{1}}\check{J}^b_{\bar{2}}\mathcal{A}^{(0)})(\check{J}^a_1 \check{J}^c_3 \mathcal{A}^{(0)}) \\
&+ \left(\frac{i}{4\pi^2 C_{\rm A}}\right)\int (d^{4|4}\Lambda)^4 \Delta^{\epsilon}_{12} \,\mathrm{sign}(E_4 E_2) G^{dbc}_{42\bar{3}} (\check{J}^a_{\bar{1}}\check{J}^b_{\bar{2}}\mathcal{A}^{(0)})(J^d_4 \check{J}^a_1 \check{J}^c_3 \mathcal{A}^{(0)}) \\
&+ \tfrac{1}{2}\left(\frac{i}{4\pi^2 C_{\rm A}}\right)\int (d^{4|4}\Lambda)^3 \Delta^{\epsilon}_{12} \,\mathrm{sign}(E_1 E_2) G^{abc}_{12\bar{3}} (\check{J}^a_{\bar{1}}\check{J}^b_{\bar{2}}\mathcal{A}^{(0)})(\check{J}^c_3 \mathcal{A}^{(0)}).
\end{aligned} \tag{3.8}$$

The three types of contributions are depicted in Fig. 4, they correspond to

- For the generators we are interested in, for example the dilatation generator D, or the superconformal boost $\bar{\mathrm{S}}$, the first term can be integrated by parts to write it as a commutator of the generator with the measure. For the case of the generators without a holomorphic anomaly, e.g. the dilatation generator, this is the only anomalous contribution.

- The second term corresponds to the case where the anomaly sits partially inside the loop integral. It occurs where one external and one internal leg become collinear.

- The third term occurs when two internal legs become collinear; that is when the anomaly vertex sits entirely inside the loop integral. As we will see below this term corresponds to a one-loop correction to the collinear limit and is only non-trivial for the two particle cut.

Let us consider these various contributions in more detail.

3.2 Anomaly of the Measure

We start with the first type of terms where we substitute the definition of the two-vertex G_2 (2.21) and write it in a symmetric fashion as

$$\begin{aligned}&\tfrac{1}{4}\left(\frac{i}{4\pi^2 C_{\rm A}}\right)\int (d^{4|4}\Lambda)^2 \Delta_{12}^{\epsilon}\big((\mathrm{G}_{\check{1}}^{\rm free}+\mathrm{G}_{\check{2}}^{\rm free})\check{J}_{\check{1}}^{a}\check{J}_{\check{2}}^{b}\mathcal{A}^{(0)}\big)(\check{J}_{1}^{a}\check{J}_{2}^{b}\mathcal{A}^{(0)})\\&+\tfrac{1}{4}\left(\frac{i}{4\pi^2 C_{\rm A}}\right)\int (d^{4|4}\Lambda)^2 \Delta_{12}^{\epsilon}(\check{J}_{\check{1}}^{a}\check{J}_{\check{2}}^{b}\mathcal{A}^{(0)})\big((\mathrm{G}_{1}^{\rm free}+\mathrm{G}_{2}^{\rm free})\check{J}_{1}^{a}\check{J}_{2}^{b}\mathcal{A}^{(0)}\big).\end{aligned}\tag{3.9}$$

Effectively this is the action of the free superconformal generator G on the internal particles of the cut integral. If the integral had been finite and if there had been no measure factor, the integral would have been perfectly superconformally invariant and the above expression would have vanished. Now one can convince oneself that for all free generators in (2.13) this is equivalent upon integration by parts to the generator acting on the one-loop measure factor

$$-\tfrac{1}{4}\left(\frac{i}{4\pi^2 C_{\rm A}}\right)\int (d^{4|4}\Lambda)^2 (\check{J}_{\check{1}}^{a}\check{J}_{\check{2}}^{b}\mathcal{A}^{(0)})\big[\mathrm{G}_{1}^{\rm free}+\mathrm{G}_{2}^{\rm free},\Delta_{12}^{\epsilon}\big](\check{J}_{1}^{a}\check{J}_{2}^{b}\mathcal{A}^{(0)}).\tag{3.10}$$

The derivation depends on the number of derivatives in the generator G, but the result is always the same. In particular there is no anomaly for those generators under which the one-loop measure is invariant, i.e. the super-Poincaré generators L, $\bar{\mathrm{L}}$, R, Q, $\bar{\mathrm{Q}}$ and P. The extra generators D, S, $\bar{\mathrm{S}}$ and K in the superconformal algebra are anomalous. The anomaly of K follows from the one of S, $\bar{\mathrm{S}}$ plus the algebra so we will not consider it separately.

The commutator gives rise to an overall factor of ϵ and so, as the actions of the generators are finite, we can focus on the IR-divergent part of the phase space integral. Divergent contributions arise only if one of the two subamplitudes has four legs. They originate within this subamplitude and they are localised where the ingoing legs are both collinear with the outgoing ones. For the purpose of computing the divergent part, we can therefore replace the loop momenta in the second subamplitude by the external momenta of the first (for a discussion of this see e.g. [44]). Importantly, the n-point tree-level amplitude can be pulled out of the integral so that we see that the action of the anomaly is diagonal which is to say it takes two legs and gives two legs back. Using the Schoutens identity and its cyclic symmetries the full four leg subamplitude, as opposed to just a single colour ordering (see (2.1)), can be written as

$$\mathcal{A}_4 = -\tfrac{1}{12} f^{abe} f^{cde}\int (d^{4|4}\Lambda)^4\, J_1^a J_2^b J_3^c J_4^d\, A_4, \qquad A_4 = \frac{\delta^4(P)\,\delta^8(Q)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\ .\tag{3.11}$$

We must at this point also address the definition of the measure factor Δ_{12}^{ϵ} which regulates the IR divergences. In the calculation of amplitudes it is common to use dimensional regularisation, however it is difficult to define the action of the super-conformal generators away from four dimensions. So we choose a regulator which can be written in four dimensional spinor variables

but which reproduces the answers of dimensional regularisation that is to say[7]

$$\Delta_{12}^{\epsilon} = c_\epsilon \left(\frac{\langle 14\rangle\langle 23\rangle[12]}{\mu^2\langle 34\rangle} \right)^{-\epsilon} \theta(-s_{12}) \ , \tag{3.12}$$

where 1 and 2 label the internal momenta and 3 and 4 label the external legs. The constant $c_\epsilon = 1 + \mathcal{O}(\epsilon)$ defined in (F.2) contains some unphysical artifacts of dimensional regularisation. This factor vanishes when external leg 1 becomes collinear with leg 4 or leg 2 becomes collinear with leg 3 thus softening the divergence in the loop integral that occurs for this configuration. With this definition it can be shown that for the generators of interest,

$$\delta^{(8)}(Q) \left[\mathrm{G}_1^{\text{free}} + \mathrm{G}_2^{\text{free}}, \log \frac{\langle 14\rangle\langle 23\rangle[12]}{\mu^2\langle 34\rangle} \right] = \delta^{(8)}(Q) \left[\mathrm{G}_3^{\text{free}} + \mathrm{G}_4^{\text{free}}, \log \frac{s_{34}}{-\mu^2} \right] . \tag{3.13}$$

This is obvious for the dilatation generator and requires only a little more effort for the fermionic special conformal generators. Using this expression when one of the amplitudes has only four legs, and using (3.11) such that

$$\check{J}_{\bar{1}}^a \check{J}_{\bar{2}}^b \mathcal{A}_4 = f^{ade} f^{bce} \int d^{4|4}\Lambda_3 d^{4|4}\Lambda_4 \, J_3^c J_4^d A_{\bar{1}\bar{2}34} \ , \tag{3.14}$$

the anomaly can be written as

$$\begin{aligned} \frac{\epsilon}{4} \left(\frac{i}{4\pi^2 C_{\mathrm{A}}} \right) \int \, d^{4|4}\Lambda_3 d^{4|4}\Lambda_4 \, \left(\int d^{4|4}\Lambda_1 d^{4|4}\Lambda_2 \Delta_{12}^{\epsilon} A_{\bar{1}\bar{2}34} \right) \\ \times \left[\mathrm{G}_3^{\text{free}} + \mathrm{G}_4^{\text{free}}, \log \frac{s_{34}}{-\mu^2} \right] f^{ace} f^{bde} (J_3^c J_4^d)(\check{J}_3^a \check{J}_4^b \mathcal{A}^{(0)}). \end{aligned} \tag{3.15}$$

We can use the fermionic delta-function to perform the Graßmann integrations over legs 1 and 2 of the four point amplitude. We can use the fact that we have chosen our regulator such that it reproduces the answers of dimensional regularisation to write

$$\begin{aligned} \int d^4\lambda_1 d^4\lambda_2 \Delta_{12}^{\epsilon} \frac{\langle 12\rangle^4 \, \delta^4(P)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} &= (2\pi)^2 \int d^D\mathrm{LIPS}(\lambda_1, \lambda_2) \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \\ &= 2i(2\pi)^2 \, \mathrm{disc} \, \frac{c_\epsilon}{\epsilon^2} \left(\frac{s_{34}}{-\mu^2} \right)^{-\epsilon} . \end{aligned} \tag{3.16}$$

We substitute this into the above anomaly, relabel the indices and obtain

$$\frac{-1}{2C_{\mathrm{A}}} \int (d^{4|4}\Lambda)^2 \left(\mathrm{disc} \, \frac{c_\epsilon}{\epsilon} \left(\frac{s_{12}}{-\mu^2} \right)^{-\epsilon} \right) \left[\mathrm{G}^{\text{free}}, \log \frac{s_{12}}{-\mu^2} \right] f^{ace} f^{bde} (J_1^c J_2^d)(\check{J}_1^a \check{J}_2^b \mathcal{A}^{(0)}). \tag{3.17}$$

[7]This is essentially the same regulator, after using momentum conservation across the two-particle cut, as was used in [58].

We have now obtained the anomaly of the cut arising from the measure factor. It is finite as $\epsilon \to 0$ and rational. However we are interested in the anomaly of the loop integral, not just its cuts. In principle we should perform a dispersion integral to obtain the loop anomaly, but here the result is obvious due to finiteness and rationality. For the generators of interest we have

$$\left(\text{disc } \frac{c_\epsilon}{\epsilon}\left(\frac{s_{12}}{-\mu^2}\right)^{-\epsilon}\right)\left[\mathrm{G}^{\text{free}}, \log\frac{s_{12}}{-\mu^2}\right] = \text{disc }\left[\mathrm{G}^{\text{free}}, -\frac{c_\epsilon}{\epsilon^2}\left(\frac{s_{12}}{-\mu^2}\right)^{-\epsilon}\right] \tag{3.18}$$

and we then simply remove the discontinuity operator from inside the anomaly. This is equivalent to multiplying by the logarithm and adding a constant, though divergent term. The multiplication by the logarithm clearly reproduces the correct discontinuity and the addition of the divergent constant corresponds to the usual ambiguity involved in reconstructing a function from its cuts. However as we shall see in the comparison to the explicit answers for the amplitudes in Sec. 4 this is the correct procedure.

Altogether the loop anomaly due to the measure reads

$$-\frac{1}{2C_{\mathrm{A}}}\int (d^{4|4}\Lambda)^2\left[\mathrm{G}^{\text{free}}, \frac{c_\epsilon}{\epsilon^2}\left(\frac{s_{12}}{-\mu^2}\right)^{-\epsilon}\right] f^{ace}f^{bde}J_1^c J_2^d \check{J}_1^a \check{J}_2^b \mathcal{A}^{(0)}. \tag{3.19}$$

3.3 Collinearities in Loops

We now turn to contributions of the second kind where the anomaly vertex sits with two legs inside the loop integral. Again we write it in a symmetric fashion

$$\begin{aligned}&\tfrac{1}{2}\left(\frac{i}{4\pi^2 C_{\mathrm{A}}}\right)\int (d^{4|4}\Lambda)^4 \Delta_{12}^\epsilon \,\mathrm{sign}(E_4E_2) J_4^d G^{dbc}_{42\bar{3}}(\check{J}_{\bar{1}}^a \check{J}_{\bar{2}}^b \mathcal{A}^{(0)})(\check{J}_1^a \check{J}_3^c \mathcal{A}^{(0)})\\ &+\tfrac{1}{2}\left(\frac{i}{4\pi^2 C_{\mathrm{A}}}\right)\int (d^{4|4}\Lambda)^4 \Delta_{12}^\epsilon \,\mathrm{sign}(E_4E_{\bar{2}}) J_4^d G^{dbc}_{4\bar{2}3}(\check{J}_{\bar{1}}^a \check{J}_{\bar{3}}^c \mathcal{A}^{(0)})(\check{J}_1^a \check{J}_2^b \mathcal{A}^{(0)}).\end{aligned} \tag{3.20}$$

We relabel particles 2 and 3 in the second term, and the two terms combine noting that on the cut the energies have equal signs

$$\tfrac{1}{2}\left(\frac{i}{4\pi^2 C_{\mathrm{A}}}\right)\int (d^{4|4}\Lambda)^4 \big(\Delta_{12}^\epsilon - \Delta_{13}^\epsilon\big)\,\mathrm{sign}(s_{12}-s_{13}) J_4^d G^{dbc}_{42\bar{3}}(\check{J}_{\bar{1}}^a \check{J}_{\bar{2}}^b \mathcal{A}^{(0)})(\check{J}_1^a \check{J}_3^c \mathcal{A}^{(0)}). \tag{3.21}$$

We have furthermore used $\mathrm{sign}(E_1E_4) = \mathrm{sign}(-s_{14}) = \mathrm{sign}(s_{12}-s_{13})$. Counting the delta functions in the integrands — in the amplitudes and in the anomaly vertex (2.23) (see also App. A) — we see that the three phase space integrals are completely localised: The loop momentum yields four degrees of freedom while the on-shell connections in the triangle (Fig. 4) contribute one constraint each. Collinearity in the anomaly vertex provides the final constraint which localises the integral. Alternatively one can argue that the anomaly vertex offers one degree of freedom corresponding to the momentum fraction. It is used up by forcing the third side of the triangle on shell. Thus this cut anomaly is a finite and rational function of the kinematic variables.

Rationality and finiteness ensure that discontinuities originate only from the original cuts. As we use dimensional regularisation (represented through the measure $\Delta^{\epsilon}_{j,k}$ here), it makes sense to consider D-dimensional cuts with the discontinuity

$$-2\pi i\,\Delta^{\epsilon}_{j,k} = -\frac{c_\epsilon}{\epsilon}\,\mathrm{disc}\left(\frac{s_{j,k}}{-\mu^2}\right)^{-\epsilon}. \tag{3.22}$$

Note that the factor $\mathrm{sign}(s_{12}-s_{13})$ does not lead to a discontinuity because it compensates a sign originating from the on-shell integration over $\Lambda_{1,2,3}$. Dropping the discontinuity operator we find the following expression for the loop anomaly

$$\begin{aligned}\tfrac{1}{2}\left(\frac{1}{16\pi^3 C_{\mathrm{A}}}\right)\int (d^{4|4}\Lambda)^4\,\mathrm{sign}(s_{12}-s_{13})J_4^d G^{dbc}_{42\bar{3}}(\check{J}^a_{\bar{1}}\check{J}^b_{\bar{2}}\mathcal{A}^{(0)})(\check{J}^a_1\check{J}^c_3\mathcal{A}^{(0)})\\ \cdot\frac{c_\epsilon}{\epsilon}\left(\left(\frac{s_{12}}{-\mu^2}\right)^{-\epsilon}-\left(\frac{s_{13}}{-\mu^2}\right)^{-\epsilon}\right).\end{aligned} \tag{3.23}$$

As this expression is finite, we are free to expand the bracket in ϵ

$$-\left(\frac{1}{16\pi^3 C_{\mathrm{A}}}\right)\int (d^{4|4}\Lambda)^4\,\mathrm{sign}(s_{12}-s_{13})J_4^d G^{dbc}_{42\bar{3}}(\check{J}^a_{\bar{1}}\check{J}^b_{\bar{2}}\mathcal{A}^{(0)})(\check{J}^a_1\check{J}^c_3\mathcal{A}^{(0)})\log\frac{s_{12}}{s_{13}}\,. \tag{3.24}$$

This result actually follows directly by replacing $\Delta^{\epsilon}_{j,k}$ by step functions originating from the discontinuity of a logarithm

$$-2\pi i\,\theta(-s_{j,k}) = \mathrm{disc}\log\frac{s_{j,k}}{-\mu^2}\,. \tag{3.25}$$

3.4 One-Loop Splitting

We now turn to the third type of anomaly in Fig. 4, which occurs when the anomaly vertex sits entirely inside the loop integral

$$\tfrac{1}{4}\left(\frac{i}{4\pi^2 C_{\mathrm{A}}}\right)\int (d^{4|4}\Lambda)^3\Delta^{\epsilon}_{12}\,\mathrm{sign}(E_1E_2)G^{abc}_{12\bar{3}}\Big[(\check{J}^a_{\bar{1}}\check{J}^b_{\bar{2}}\mathcal{A}^{(0)})(\check{J}^c_3\mathcal{A}^{(0)})+(\check{J}^c_3\mathcal{A}^{(0)})(\check{J}^a_{\bar{1}}\check{J}^b_{\bar{2}}\mathcal{A}^{(0)})\Big]\,. \tag{3.26}$$

Recalling the structure of the anomaly vertex we can see that this contributes when the internal momenta, labelled 1 and 2, become collinear and proportional to 3. Let us consider the case where we have a four point amplitude on the one side of the cut. Using the expression for the four-point amplitude, (3.11), we can write

$$G^{abc}_{12\bar{3}}(\check{J}^a_{\bar{1}}\check{J}^b_{\bar{2}}\mathcal{A}^{(0)}_4) = if^{abc}f^{adg}f^{beg}G_3\int d^{4|4}\Lambda_3 d^{4|4}\Lambda_4 J_3^e J_4^d A_{\bar{1}\bar{2}34}\,. \tag{3.27}$$

Now, with the aid of the Jacobi identity and the relation between the dual Coxeter number, $f_{abc}f_d{}^{bc} = C_{\mathrm{V}}\delta_{ad}$, and the quadratic Casimir, C_{A}, we see that the colour structures combine to produce an overall factor of $C_{\mathrm{A}}f^{ade}$ which cancels the C_{A} in the prefactor.

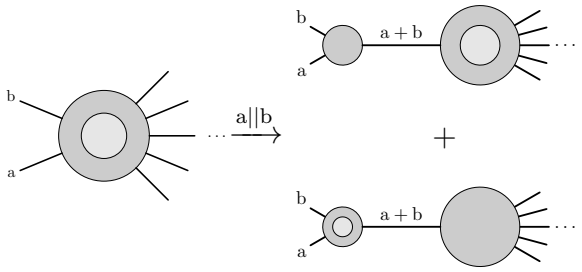

Figure 5: The two contributions to the collinear limit of the one-loop amplitude.

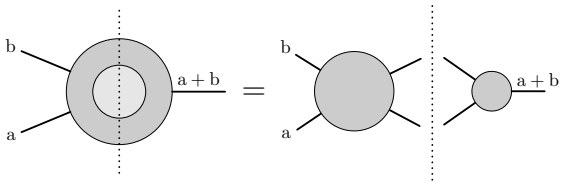

Figure 6: Unitarity cut of the one-loop splitting function.

As the two internal legs become collinear the remaining, external, two legs of the four-point amplitude also become collinear. We see that this contribution arises from the limit of the n-point amplitude as two external legs become collinear and so is related to the splitting function. There are several subtleties involved in taking this limit; for example the kinematic invariant for this channel, s_{12}, is actually zero. It is therefore useful to recall some salient facts about the one-loop splitting function; we will closely follow the discussion in [32].

The one-loop amplitude, in the limit where two momenta become collinear $p_{\rm a} \to z(p_{\rm a} + p_{\rm b})$, $p_{\rm b} \to (1-z)(p_{\rm a} + p_{\rm b})$, has two contributions, Fig. 5, which are given by

$$\begin{aligned} A_n^{(1)}(1,\ldots,{\rm a},{\rm b},\ldots,n) \quad &\xrightarrow{\rm a||b} {\rm Split}^{(0)}({\rm a},{\rm b}) A_{n-1}^{(1)}(1,\ldots,({\rm a}+{\rm b}),\ldots,n) \\ &+ {\rm Split}^{(1)}({\rm a},{\rm b}) A_{n-1}^{(0)}(1,\ldots,({\rm a}+{\rm b}),\ldots,n). \end{aligned} \tag{3.28}$$

The first term is the tree-level splitting function which scales as $s_{\rm ab}^{-1/2}$ and the effects of which have already been accounted for by the tree-level deformation of the generators. In terms of cuts they are captured by n-particle cuts, $n > 2$, where the anomalous contribution forces two external legs to become collinear. The second set of contributions correspond to the one loop splitting function

$$ {\rm Split}^{(1)}({\rm a},{\rm b}) = {\rm Split}^{(0)}({\rm a},{\rm b}) r_S(z, s_{\rm ab}) \tag{3.29}$$

where $r_{\rm S}(z, s_{\rm ab})$ is independent of the flavour or helicity of the particles labelled a and b.

This term is captured by the two-particle, "singular" channel, where on one side of the cut we have a four point amplitude. As discussed in [32], the four point function is singular in this limit, having a pole in $s_{\rm ab}$ rather than a square root singularity. However as momentum conservation also forces the loop momenta to become collinear we can use the factorisation of the n-particle amplitude on the other side of the cut to rewrite the expression as the cut of the function ${\rm Split}^{(1)}({\rm a},{\rm b})$, see Fig. 6. By evaluating this cut or, alternatively, by taking the limit on

the scalar box functions which define the one-loop amplitude one can find an explicit expression for $r_{\rm S}(s_{\rm ab}, z)$,

$$r_{\rm S}(s_{\rm ab}, z) = -\left(\frac{s_{\rm ab}}{-\mu^2}\right)^{-\epsilon}\left[\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\log z(1-z) + \log^2\frac{1-z}{z} + \frac{\pi^2}{6}\right] . \tag{3.30}$$

In this definition there is an ambiguity depending on the order in which one takes the limits $\epsilon \to 0$ and $s_{\rm ab} \to 0$. One prescription is to take the singular momentum to zero before going to four dimensions this results in $r_{\rm S}(0, z) = 0$. However one can imagine different prescriptions such as setting $s_{\rm ab} \to -\mu^2$ and then taking $\epsilon \to 0$ defining $r_{\rm S}(-\mu^2, z)$.

One can write the anomaly as a one-loop anomaly vertex, schematically, $G^{(1)} \sim r_{\rm S} G^{(0)}$ where $G^{(0)}$ denotes the tree-level anomaly vertex. More concretely for $\bar{\rm S}$,

$$\begin{aligned}\left(\bar{\rm S}^{(1)}\right)^{abc}_{123} = -2if^{abc}\int d^{4|4}\Lambda'\,\delta^4(\lambda')\,\varepsilon_{\dot\alpha\dot\gamma}\tilde\lambda_3^{\dot\gamma}\eta'^B\int d\alpha\,d\varphi\,d\vartheta\,e^{i\varphi+i\vartheta}r_{\rm S}(\alpha)\\ \cdot\,\delta^{4|4}(e^{-i\varphi}\Lambda_3\sin\alpha + e^{i\vartheta}\Lambda'\cos\alpha - \bar\Lambda_1)\\ \cdot\,\delta^{4|4}(e^{-i\vartheta}\Lambda_3\cos\alpha - e^{i\varphi}\Lambda'\sin\alpha - \bar\Lambda_2) + \begin{array}{c}\text{two cyclic}\\ \text{images}\end{array} ,\end{aligned} \tag{3.31}$$

where if we use the second prescription for the splitting function we have ($z = \sin\alpha$)

$$r_{\rm S}(\alpha) = -\left[\frac{1}{\epsilon^2} - \frac{2}{\epsilon}\log(\cos\alpha\sin\alpha) + 4\log^2\cot\alpha + \frac{\pi^2}{6}\right] . \tag{3.32}$$

Thus the anomalous contribution of the third kind is explicitly

$$\tfrac{1}{2}\int (d^{4|4}\Lambda)^3\,\mathrm{sign}(E_1E_2)(G^{(1)})^{abc}_{12\bar3}J_1^aJ_2^b\check J_3^c . \tag{3.33}$$

In principle we have to additionally consider the cases where we have more than two external legs on both sides of the cut. These terms correspond to one-loop corrections to multi-particle factorisation. For example, the contribution of the cut with a five-point amplitude one side is supported on the region of kinematical space where the sum of three external particles becomes null. However for our definition of the amplitudes, discussed further in Sec. 7 but essentially taking the principle part, there are no anomalous contributions to S or $\bar{\rm S}$ on this support.

3.5 Deformation of the Representation

In summary of (3.19), (3.24) and (3.33) we find the total one-loop anomaly

$$\begin{aligned}{\rm G}^{(0)}\mathcal{A}^{(1)} = \frac{-1}{2C_{\rm A}}\int(d^{4|4}\Lambda)^2\left[{\rm G}^{\rm free}, \frac{c_\epsilon}{\epsilon^2}\left(\frac{s_{12}}{-\mu^2}\right)^{-\epsilon}\right]f^{ace}f^{bde}J_1^cJ_2^d\check J_1^a\check J_2^b\mathcal{A}^{(0)}\\ -\left(\frac{1}{16\pi^3C_{\rm A}}\right)\int(d^{4|4}\Lambda)^4\,\mathrm{sign}(s_{12}-s_{13})J_4^dG^{dbc}_{42\bar3}(\check J_{\bar1}^a\check J_{\bar2}^b\mathcal{A}^{(0)})(\check J_1^a\check J_3^c\mathcal{A}^{(0)})\log\frac{s_{12}}{s_{13}}\\ -\tfrac{1}{2}\int(d^{4|4}\Lambda)^3\,\mathrm{sign}(E_1E_2)G^{(1)abc}_{12\bar3}J_1^aJ_2^b\check J_3^c\mathcal{A}^{(0)}.\end{aligned} \tag{3.34}$$

Importantly all of these terms can be written as some variation acting on the tree amplitude $\mathrm{G}^{(0)}\mathcal{A}^{(1)} \sim \mathcal{A}^{(0)}$. Thus we can cancel the anomaly easily $\mathrm{G}^{(0)}\mathcal{A}^{(1)} + \mathrm{G}^{(1)}\mathcal{A}^{(0)} = 0$ by introducing a corresponding one-loop deformation of the superconformal representation

$$\mathrm{G}^{(1)} = \mathrm{G}^{(1)}_{2\to 2} + \sum_{k=3}^{\infty} \mathrm{G}^{(1)}_{2\to k} + \mathrm{G}^{(1)}_{1\to 2}. \tag{3.35}$$

The first term is meant to cancel the contribution due to the measure anomaly. Here we first introduce an operator $\hat{Z}^{(1)}_{2\to 2}$ which captures the IR singularities in a one-loop amplitude

$$\hat{Z}^{(1)}_{2\to 2} = -\frac{1}{2C_{\mathrm{A}}} \int (d^{4|4}\Lambda)^2 \, \frac{c_\epsilon}{\epsilon^2} \left(\frac{s_{12}}{-\mu^2}\right)^{-\epsilon} f^{ace} f^{bde} J^c_1 J^d_2 \check{J}^a_1 \check{J}^b_2. \tag{3.36}$$

Note that the momenta of the particles $1, 2$ are not changed by this operator, it merely acts non-trivially on the colour-structure and multiplies by a divergent function of the two-particle invariant s_{12}. In particular, this operator allows to split a one-loop amplitude into IR-divergent contributions and a finite remainder $\tilde{\mathcal{A}}^{(1)}$ [22–25]

$$\mathcal{A}^{(1)} = \hat{Z}^{(1)}_{2\to 2} \mathcal{A}^{(0)} + \tilde{\mathcal{A}}^{(1)}. \tag{3.37}$$

According to (3.34) the generator deformation can now be written in the form of the commutator

$$\mathrm{G}^{(1)}_{2\to 2} = \big[\hat{Z}^{(1)}_{2\to 2}, \mathrm{G}^{(0)}_{1\to 1}\big]. \tag{3.38}$$

It is obvious that this type of deformation respects the superconformal algebra because it merely consists in a perturbative similarity transformation of the free generators.

The second term cancels the anomaly due to collinearities in the loop

$$\mathrm{G}^{(1)}_{2\to k} = \left(\frac{1}{16\pi^3 C_{\mathrm{A}}}\right) \int (d^{4|4}\Lambda)^4 \, \mathrm{sign}(s_{12} - s_{13}) J^d_4 G^{dbc}_{42\bar{3}} (\check{J}^a_1 \check{J}^c_3 \mathcal{A}^{(0)}_{k+1}) \log \frac{s_{12}}{s_{13}} \, \check{J}^a_{\bar{1}} \check{J}^b_{\bar{2}}. \tag{3.39}$$

Note that this term is not uniquely determined because we only know its action on tree amplitudes $\mathcal{A}^{(0)}$ and not on generic functions. The point is that the expression already contains a tree amplitude $\mathcal{A}^{(0)}$ and thus when it acts on tree amplitudes it will automatically symmetrises the two. We could thus, alternatively, drop one of the terms of the logarithm involving the invariants $s_1 2$ or $s_1 3$ in (3.39) and multiply by two. It has the same effect on tree amplitudes, but it is a different deformation of the representation. The third term removes the one-loop collinear anomaly

$$\mathrm{G}^{(1)}_{1\to 2} = \tfrac{1}{2} \int (d^{4|4}\Lambda)^3 \, \mathrm{sign}(E_1 E_2) G^{(1)abc}_{12\bar{3}} J^a_1 J^b_2 \check{J}^c_{\bar{3}}. \tag{3.40}$$

See Fig. 7 for an illustration of the one-loop deformations $\mathrm{G}^{(1)}$ acting on an amplitude.

We will see in Sec. 4 that exactly these deformations are required to make planar MHV amplitudes superconformally invariant.

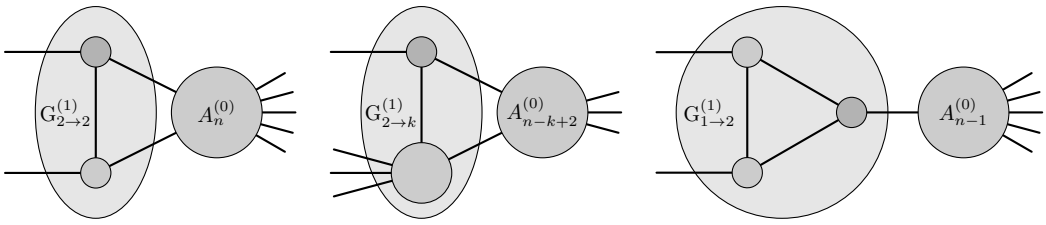

Figure 7: Structure of the deformations at one loop.

3.6 Planar Representation

The above anomaly and the corresponding deformation of the representation hold for arbitrary gauge groups of finite rank. It is often convenient to restrict to the planar limit in a $\mathrm{U}(N_\mathrm{c})$ gauge group where most expressions simplify. Let us therefore formulate the deformation in the planar limit. We act with the generators on a functional $\mathcal{X}[J]$ which is based on a colour-ordered function X according to (2.10). In the following we shall use a notation where the indices in some quantity X^k_j denote a range of k adjacent particles starting at particle j. For example we introduce the partial momentum, supermomentum and Lorentz invariants

$$P^k_j := \sum_{i=0}^{k-1} p_{j+i}, \qquad Q^k_j := \sum_{i=0}^{k-1} q_{j+i}, \qquad t^k_j := (P^k_j)^2. \tag{3.41}$$

The generator $\mathrm{G}^{(2)}_{2\to 2}$ is given in terms of the IR-singularity operator $\hat{Z}^{(1)}_{2\to 2}$ in (3.36). In the planar limit, the colour structure forces this operator to act on two adjacent legs of the amplitude. Hence we can write its action in the following form

$$\hat{Z}^{(1)}_{2\to 2} = \sum_{i=1}^{n} (\hat{Z}^{(1)}_{2\to 2})^2_i, \qquad (\hat{Z}^{(1)}_{2\to 2})^2_i = -\frac{c_\epsilon}{\epsilon^2}\left(\frac{t_i^2}{-\mu^2}\right)^{-\epsilon}. \tag{3.42}$$

Now the correction term $\mathrm{G}^{(1)}_{2\to 2}$ in (3.38) has a particularly simple structure in the planar limit

$$\mathrm{G}^{(1)}_{2\to 2} = \big[\hat{Z}^{(1)}_{2\to 2}, \mathrm{G}^{(0)}_{1\to 1}\big] = \sum_{i=1}^{n} (\mathrm{G}^{(1)}_{2\to 2})^2_i, \qquad (\mathrm{G}^{(1)}_{2\to 2})^2_i = \big[(\hat{Z}^{(1)}_{2\to 2})^2_i, \mathrm{G}^{\text{free}}_i + \mathrm{G}^{\text{free}}_{i+1}\big]. \tag{3.43}$$

This generator acts on two adjacent particles $i, i+1$. As noted above this type of deformation applies to the generators D, S, $\bar{\mathrm{S}}$ and K. The latter can always be expressed through commutators and hence we list an explicit result only for the first three

$$\begin{aligned}
(\mathrm{D}^{(1)}_{2\to 2})^2_i &= -\frac{2c_\epsilon}{\epsilon}\left(\frac{t_i^2}{-\mu^2}\right)^{-\epsilon}, \\
((\mathrm{S}^{(1)}_{2\to 2})^2_i)_{\alpha B} &= -\frac{c_\epsilon}{\epsilon}\left(\frac{t_i^2}{-\mu^2}\right)^{-\epsilon} \frac{\varepsilon_{\alpha\gamma}}{\langle i, i+1\rangle}\left(\lambda^\gamma_{i+1}\partial_{i,B} - \lambda^\gamma_i \partial_{i+1,B}\right), \\
((\bar{\mathrm{S}}^{(1)}_{2\to 2})^2_i)^B_{\dot\alpha} &= -\frac{c_\epsilon}{\epsilon}\left(\frac{t_i^2}{-\mu^2}\right)^{-\epsilon} \frac{\varepsilon_{\dot\alpha\dot\gamma}}{[i, i+1]}\left(\tilde\lambda^{\dot\gamma}_{i+1}\eta^B_i - \tilde\lambda^{\dot\gamma}_i \eta^B_{i+1}\right).
\end{aligned} \tag{3.44}$$

Let us now act on some colour-ordered amplitude function X_n with the generator $\mathrm{G}^{(1)}_{2\to k}$. Due to the inhomogeneous nature of the generator $\mathrm{G}^{(1)}_{2\to k}$ it will return some amplitude function Y_{n+k-2} with $(n+k-2)$ legs

$$\mathrm{G}^{(1)}_{2\to k} X_n = Y_{n+k-2}. \tag{3.45}$$

As before the two variations in (3.39) must hit adjacent sources, so we can introduce a generator $(\mathrm{G}^{(1)}_{2\to k})^2_j$ which acts on legs $j, j+1$ of X_n and replaces them by $(k-2)$ new legs

$$\mathrm{G}^{(1)}_{2\to k} = \sum_{j=1}^{n+k-2} (\mathrm{G}^{(1)}_{2\to k})^k_j, \qquad Y_{n+k-2} = \sum_{j=1}^{n+k-2} (Y_{n+k-2})^k_j = \sum_{j=1}^{n+k-2} (\mathrm{G}^{(1)}_{2\to k})^k_j X_n. \tag{3.46}$$

The range of the sum may seem surprising at first sight, but it is the only way the resulting expression can make sense: Both X_n and Y_{n+k-2} must be cyclic functions. Consequently, it does not matter which pair of legs of X_n is chosen. The problem is that the contribution $(Y_{n+k-2})^k_j$ is not cyclic. Cyclicity of Y_{n+k-2} is only restored in a sum over all cyclic permutations of its $(n+k-2)$ legs. Evaluating the colour structures in (3.39) we can write the planar action as

$$\begin{aligned}(\mathrm{G}^{(1)}_{2\to k})^k_j X_n = -\frac{1}{16\pi^3} \int d^{4|4}\Lambda_{\mathrm{a}}\, d^{4|4}\Lambda_{\mathrm{b}}\, d^{4|4}\Lambda_{\mathrm{c}}\, X_n(1,\dots,j-1,\mathrm{b},\mathrm{c},j+k,\dots,n+k-2)\\ \cdot \Big[\operatorname{sign}(t_j^k - t_{j+1}^{k-1}) G_3(\bar{\mathrm{a}},\bar{\mathrm{b}},j) A_k^{(0)}(\mathrm{a},j+1,\dots,j+k-1,\bar{\mathrm{c}}) \log \frac{t_j^k}{t_{j+1}^{k-1}}\\ - \operatorname{sign}(t_j^k - t_j^{k-1}) A_k^{(0)}(\bar{\mathrm{b}},j,\dots,j+k-2,\mathrm{a}) G_3(\bar{\mathrm{c}},\bar{\mathrm{a}},j+k-1) \log \frac{t_j^k}{t_j^{k-1}} \Big].\end{aligned} \tag{3.47}$$

Similarly, the planar action of the third generator reads

$$\begin{aligned}(\mathrm{G}^{(1)}_{1\to 2})^2_j X_n = \int d^{4|4}\Lambda_{\mathrm{a}}\, \operatorname{sign}(E_j E_{j+1})\, G_3^{(1)}(j,j+1,\bar{\mathrm{a}})\\ \cdot X_n(1,\dots,j-1,\mathrm{a},j+2,\dots,n+1).\end{aligned} \tag{3.48}$$

4 Superconformal Symmetry of MHV Amplitudes

Having established the general framework for superconformal symmetry of one-loop amplitudes, we will confirm it using the simple set of planar MHV amplitudes A_{MHV}. To avoid clutter, we shall drop the label MHV from the amplitude functions A and functionals $\mathcal{A}$ throughout this section.

4.1 One-Loop Correction

We summarise the construction and the properties of one-loop MHV amplitudes in App. B in order to focus on the one-loop anomalies here. For MHV amplitudes the helicity-dependence is

fully constrained by the symmetry. It forces the exact amplitude to equal the tree result times a function of the particle momenta

$$A_n^{(\ell)} = A_n^{(0)} M_n^{(\ell)}, \qquad A_n^{(0)} = \frac{\delta^4(P)\,\delta^8(Q)}{\langle 12\rangle \dots \langle n1\rangle}, \quad M_n^{(0)} = 1. \tag{4.1}$$

The one-loop amplitude in dimensional reduction reads [37]

$$\begin{aligned} M_n^{(1)} = &- \sum_{j=1}^{n} \frac{c_\epsilon}{\epsilon^2} \left(\frac{t_j^2}{-\mu^2}\right)^{-\epsilon} + \tfrac{1}{6} n\pi^2 - \tfrac{1}{2} \sum_{k=3}^{n-3} \sum_{j=1}^{n} \mathrm{Li}_2\left(1 - \frac{t_{j+1}^{k-1} t_j^{k+1}}{t_j^k t_{j+1}^k}\right) \\ &- \tfrac{1}{2} \sum_{k=2}^{n-3} \sum_{j=1}^{n} \log^2 \frac{t_j^k}{t_j^{k+1}} + \tfrac{1}{4} \sum_{k=2}^{n-2} \sum_{j=1}^{n} \log^2 \frac{t_j^k}{t_{j+1}^k}\,. \end{aligned} \tag{4.2}$$

The loop function depends on invariants t_j^k associated to the overall momentum of k consecutive particles starting at particle j introduced in (3.41). Furthermore $c_\epsilon = 1 + \mathcal{O}(\epsilon)$ is some function of the dimensional reduction parameter ϵ and μ is the regularisation scale. It is perhaps worth nothing that this formula is chosen to reproduce only the most "complicated" part of the loop integral. That is to say, it does not reproduce the imaginary parts of the logarithm and dilogarithms. In order to define the function one must specify the appropriate Riemann sheet for all values of the kinematic variables.

4.2 Measure Anomaly

First we will consider symmetry generators G which are anomaly-free at tree level, i.e. they act as in the free theory $\mathrm{G}^{(0)} = \mathrm{G}^{(0)}_{1\to 1} = \mathrm{G}^{\text{free}}$. In particular, the dilatation generator D and effectively also the superconformal boost S when acting on MHV amplitudes are of this form. The proposed one-loop deformation (3.43) is simple: It acts on nearest neighbouring particles only with a simple commutator form for the pairwise action in terms of the free generator G^{free}

$$\mathrm{G}^{(1)} = \mathrm{G}^{(1)}_{2\to 2} = \sum_{j=1}^{n} (\mathrm{G}^{(1)}_{2\to 2})^2_j, \qquad (\mathrm{G}^{(1)}_{2\to 2})^2_j = \left[\mathrm{G}^{\text{free}}, \frac{c_\epsilon}{\epsilon^2}\left(\frac{t_j^2}{-\mu^2}\right)^{-\epsilon}\right]. \tag{4.3}$$

The simplest non-trivial anomaly is the one of the generator of scale transformations D. The free representation (2.13) and the one-loop deformation (3.44) read

$$\mathrm{D}_j^{\text{free}} = 1 + \tfrac{1}{2}\lambda_j^\alpha \partial_{j,\alpha} + \tfrac{1}{2}\tilde{\lambda}_j^{\dot\alpha} \tilde{\partial}_{j,\dot\alpha}, \qquad (\mathrm{D}^{(1)}_{2\to 2})^2_j = -\frac{2c_\epsilon}{\epsilon}\left(\frac{t_j^2}{-\mu^2}\right)^{-\epsilon}. \tag{4.4}$$

The only term in (4.2) violating scaling invariance is the one containing the regularisation scale μ. The scaling anomaly reads

$$\mathrm{D}^{(0)}_{1\to 1} A_n^{(1)} = A_n^{(0)} \sum_{j=1}^{n} \frac{2c_\epsilon}{\epsilon}\left(\frac{t_j^2}{-\mu^2}\right)^{-\epsilon}. \tag{4.5}$$

As anticipated the anomaly depends on the momenta of two adjacent particles only. Obviously, the one-loop deformation cancels precisely the anomaly and makes the one-loop amplitude exactly invariant under (deformed) scaling transformations

$$\mathrm{D}^{(0)}_{1\to 1}A^{(1)}_n + \mathrm{D}^{(1)}_{2\to 2}A^{(0)}_n = 0. \tag{4.6}$$

Next we consider the superconformal boost generator S given in (2.13) and (3.44)

$$(\mathrm{S}^{\text{free}}_j)_{\alpha B} = \partial_{j,B}\partial_{j,\alpha}, \qquad ((\mathrm{S}^{(1)}_{2\to 2})^2_j)_{\alpha B} = -\frac{c_\epsilon}{\epsilon}\left(\frac{t_j^2}{-\mu^2}\right)^{-\epsilon}\frac{\varepsilon_{\alpha\delta}}{\langle j,j+1\rangle}\left(\lambda^\delta_{j+1}\partial_{j,B} - \lambda^\delta_j\partial_{j+1,B}\right). \tag{4.7}$$

In applying the free generator to $A^{(1)}_n$, the fermionic derivative will act on the $\delta^8(Q)$ in $A^{(0)}_n$ because it is the only piece depending on the η's. The bosonic derivative must act on the loop function $M^{(1)}_n$ because the tree-level amplitude $A^{(0)}_n$ is invariant

$$(\mathrm{S}^{(0)}_{1\to 1})_{\alpha B}A^{(1)}_n = \sum_{j=1}^n(\partial_{j,B}A^{(0)}_n)\,(\partial_{j,\alpha}M^{(1)}_n) = \sum_{j=1}^n\frac{\partial A^{(0)}_n}{\partial Q^{\gamma B}}\,\lambda^\gamma_j\partial_{j,\alpha}M^{(1)}_n. \tag{4.8}$$

The second form uses the identity $\partial_{j,B}Q^{\alpha C} = \delta^C_B\lambda^\alpha_j$, cf. (2.4). The combination $\lambda^\gamma_j\partial_{j,\alpha}$, summed over all sites, equals the Lorentz generator $\mathrm{L}^\gamma{}_\alpha$ up to its trace. The function $M^{(1)}_n$ is a Lorentz invariant, and hence it is annihilated by $\mathrm{L}^\gamma{}_\alpha$. Furthermore the trace contribution measures the weight in λ's which is the same as the scaling weight for the invariants t^k_j. The superconformal boost anomaly is thus very similar to the scaling anomaly:

$$(\mathrm{S}^{(0)}_{1\to 1})_{\alpha B}A^{(1)}_n = \frac{\partial A^{(0)}_n}{\partial Q^{\gamma B}}\left(\mathrm{L}^\gamma{}_\alpha + \tfrac{1}{2}\delta^\gamma_\alpha\sum_{j=1}^n\lambda^\delta_j\partial_{j,\delta}\right)M^{(1)}_n = \frac{\partial A^{(0)}_n}{\partial Q^{\alpha B}}\sum_{j=1}^n\frac{c_\epsilon}{\epsilon}\left(\frac{t_j^2}{-\mu^2}\right)^{-\epsilon}. \tag{4.9}$$

We should compare this expression to the one-loop deformation. As above we make use of the fact that the fermionic derivative $\partial_{j,B}$ only hits the fermionic delta function $\delta^8(Q)$ and that $\partial_{j,B}Q^{\alpha C} = \delta^C_B\lambda^\alpha_j$

$$\begin{aligned}(\mathrm{S}^{(0)}_{2\to 2})_{\alpha B}A^{(0)}_n &= -\sum_{j=1}^n\frac{c_\epsilon}{\epsilon}\left(\frac{t_j^2}{-\mu^2}\right)^{-\epsilon}\frac{\varepsilon_{\alpha\delta}}{\langle j,j+1\rangle}\left(\lambda^\delta_{j+1}\partial_{j,B} - \lambda^\delta_j\partial_{j+1,B}\right)A^{(0)}_n\\ &= -\sum_{j=1}^n\frac{c_\epsilon}{\epsilon}\left(\frac{t_j^2}{-\mu^2}\right)^{-\epsilon}\varepsilon_{\alpha\delta}\frac{\lambda^\gamma_j\lambda^\delta_{j+1} - \lambda^\delta_j\lambda^\gamma_{j+1}}{\langle j,j+1\rangle}\frac{\partial A^{(0)}_n}{\partial Q^{\gamma B}}\\ &= -\sum_{j=1}^n\frac{c_\epsilon}{\epsilon}\left(\frac{t_j^2}{-\mu^2}\right)^{-\epsilon}\frac{\partial A^{(0)}_n}{\partial Q^{\alpha B}}\,.\end{aligned} \tag{4.10}$$

Altogether we obtain the invariance condition

$$\mathrm{S}^{(0)}_{1\to 1}A^{(1)}_n + \mathrm{S}^{(1)}_{2\to 2}A^{(0)}_n = 0. \tag{4.11}$$

Note that S is not anomaly-free at tree level [35]. In general one therefore expects the correction term $\mathrm{S}^{(0)}_{1\to 2}$ at tree level [35] and further corrections $\mathrm{S}^{(1)}_{2\to k}$ loop level. This anomaly however does not apply to MHV amplitudes which is why the treatment of S was relatively simple.

4.3 Collinearities in Loops

For generators G which are anomalous at tree level, $\mathrm{G}^{(0)} = \mathrm{G}^{(0)}_{1\to1} + \mathrm{G}^{(0)}_{1\to2}$, we have to work harder. The prototype example when acting on MHV amplitudes is $\bar{\mathrm{S}}$. In addition to the homogeneous $\mathrm{G}^{(1)}_{2\to2}$ corrections, there are inhomogeneous terms $\mathrm{G}^{(1)}_{2\to k}$, $k \geq 3$

$$\mathrm{G}^{(1)} = \mathrm{G}^{(1)}_{2\to2} + \sum_{k=3}^{n-2} \mathrm{G}^{(1)}_{2\to k}, \qquad \mathrm{G}^{(1)}_{2\to k} = \sum_{j=1}^{n} (\mathrm{G}^{(1)}_{2\to k})^2_j \tag{4.12}$$

When acting on the $n-k+2$-particle amplitude $A^{(0)}_{n-k+2}$ they yield an n-particle function to cancel the anomaly. The action of $\mathrm{G}^{(1)}_{2\to k}$ on particles $j, j+1$ of $A^{(0)}_{n-k+2}$ defined in (3.47) uses the anomaly three-vertex G_3 in (2.23).

In principle there can be loop corrections $\mathrm{G}^{(1)}_{1\to2}$ to the collinear anomaly $\mathrm{G}^{(0)}_{1\to2}$ itself. In this section we assume for convenience that the particle momenta are in a general position and pairwise linearly independent. The case of collinear external momenta will be considered in the following section.

We now consider the conjugate superconformal generator $\bar{\mathrm{S}}$. We first act on $A^{(1)}_n$ with the free generator $(\bar{\mathrm{S}}^{\mathrm{free}}_j)^B_{\dot\alpha} = \eta^B_j \tilde\partial_{j,\dot\alpha}$ (2.13). The straight-forward variations will produce a lot of terms. Let us therefore first consider the general variation of the loop function $M^{(1)}_n$ under shifts of the invariants t^k_j, and simplify it as far as possible. A very convenient expression for the variation of (4.2) reads (see App. B.3 for some intermediate expressions)

$$\begin{aligned}\delta M^{(1)}_n = &\sum_{j=1}^{n} \left(\delta \log \frac{\langle j-1, j\rangle [j, j+1] \langle j+1, j+2\rangle}{-\mu^2 \langle j, j+1\rangle}\right) \frac{c_\epsilon}{\epsilon} \left(\frac{t^2_j}{-\mu^2}\right)^{-\epsilon} \\ &- \sum_{k=2}^{n-3} \sum_{j=1}^{n} \left(\delta \log \frac{\Upsilon^{k+1}_{j-1}}{\Upsilon^k_j}\right) \log \frac{t^{k+1}_j}{t^k_j}\,.\end{aligned} \tag{4.13}$$

The symbol Υ^k_j is defined as the following Lorentz invariant combination (see (3.41) for the definition of the fractional momentum P^k_j)

$$\Upsilon^k_j = \langle j | P^{k-1}_{j+1} | j+k] = \varepsilon_{\beta\delta}\varepsilon_{\dot\alpha\dot\gamma} \lambda^\beta_j (P^{k-1}_{j+1})^{\delta\dot\alpha} \tilde\lambda^{\dot\gamma}_{j+k}, \tag{4.14}$$

and it originates from the following combination occurring frequently in $\delta M^{(1)}_n$, see App. B.3,

$$t^k_j t^k_{j+1} - t^{k-1}_{j+1} t^{k+1}_j = -\Upsilon^k_j \Upsilon^{n-k}_{j+k}. \tag{4.15}$$

Note that this identity makes use of momentum conservation $P^k_j = -P^{n-k}_{j+k}$ and the fact that $D = 4$. In particular the latter is interesting because the above factorisation is crucial for conformal symmetry which is special to four dimensions.

For simplifying the anomaly arising from the first line in (4.13) we note that $\bar{\mathrm{S}}$ acts on conjugate spinors only. Thus for the purpose of $\bar{\mathrm{S}}^{(0)}$ we can replace the argument of the logarithm

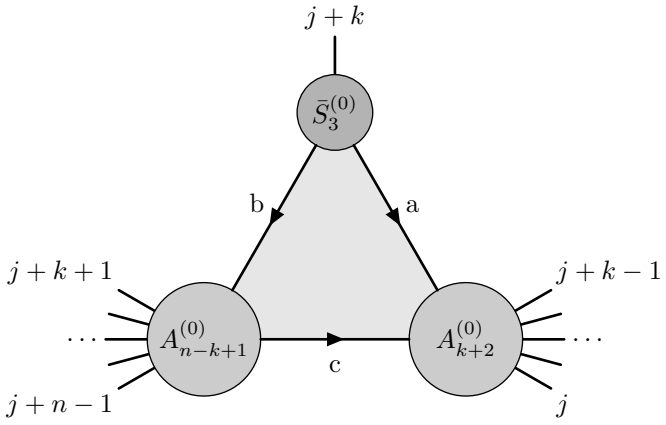

Figure 8: $\bar{\mathrm{S}}$ cut anomaly T_j^k.

by $t_j^2 = \langle j, j+1\rangle[j+1, j]$. This yields

$$(\mathrm{S}^{(0)}_{1\to 1})^B_{\dot\alpha} \log\frac{t_j^2}{-\mu^2} = \frac{\varepsilon_{\dot\alpha\dot\gamma}\big(\eta_j^B\tilde\lambda^{\dot\gamma}_{j+1} - \eta^B_{j+1}\tilde\lambda^{\dot\gamma}_j\big)}{[j, j+1]}\,. \tag{4.16}$$

The action in the brackets of the second line in (4.13) can be evaluated and simplified using spinor algebra:

$$(\bar{\mathrm{S}}^{(0)}_{1\to 1})^B_{\dot\alpha} \log\frac{\Upsilon^{k+1}_{j-1}}{\Upsilon^k_j} = \varepsilon_{\dot\alpha\dot\gamma}\tilde\lambda^{\dot\gamma}_{j+k}\frac{\langle j-1, j\rangle}{\Upsilon^k_j\Upsilon^{k+1}_{j-1}}\left(\tilde\lambda^{\dot\epsilon}_{j+k}\varepsilon_{\dot\epsilon\dot\kappa}(P^k_j)^{\lambda\dot\kappa}\varepsilon_{\lambda\delta}(Q^k_j)^{\delta B} - t^k_j\eta^B_{j+k}\right). \tag{4.17}$$

Here Q^k_j is a fractional supermomentum defined in (3.41). Altogether the conjugate superconformal boost anomaly reads[8]

$$\begin{aligned}(\bar{\mathrm{S}}^{(0)}_{1\to 1})^B_{\dot\alpha} A^{(1)}_n = {}& A^{(0)}_n\sum_{j=1}^{n}\left((\bar{\mathrm{S}}^{(0)}_{1\to 1})^B_{\dot\alpha}\log\frac{t_j^2}{-\mu^2}\right)\frac{c_\epsilon}{\epsilon}\left(\frac{t_j^2}{-\mu^2}\right)^{-\epsilon}\\ & - A^{(0)}_n\sum_{k=2}^{n-3}\sum_{j=1}^{n}\left((\bar{\mathrm{S}}^{(0)}_{1\to 1})^B_{\dot\alpha}\log\frac{\Upsilon^{k+1}_{j-1}}{\Upsilon^k_j}\right)\log\frac{t_j^{k+1}}{t_j^k}\,.\end{aligned} \tag{4.18}$$

The anomaly on the first line is obviously cancelled by the $\bar{\mathrm{S}}^{(1)}_{2\to 2}$ correction (3.43). The remaining anomaly from the terms on the second line should be cancelled by terms from $\mathrm{S}^{(1)}_{2\to k}$ in (3.47). We write this as

$$((\bar{\mathrm{S}}^{(1)}_{2\to n-k})^k_j)^B_{\dot\alpha} A^{(0)}_{k+2} = -\tfrac{1}{2}(T^k_{j-k})^B_{\dot\alpha}\log\frac{t_j^{n-k}}{t_{j+1}^{n-k-1}} + \tfrac{1}{2}(T^{n-k-1}_j)^B_{\dot\alpha}\log\frac{t_j^{n-k}}{t_j^{n-k-1}} \tag{4.19}$$

[8]Essentially identical formulae were found by [36] and [43] in their analysis of the $\bar{\mathrm{S}}$ (or correspondingly dual $\bar{\mathrm{Q}}$) anomaly of the cuts of MHV and NMHV amplitudes.

where T_j^k is the on-shell triangle integral, see Fig. 8 for the configuration of momenta,

$$(T_j^k)_{\dot\alpha}^B = \frac{1}{8\pi^3}\int d^{4|4}\Lambda_{\rm a}\, d^{4|4}\Lambda_{\rm b}\, d^{4|4}\Lambda_{\rm c}\ \mathrm{sign}(t_j^k - t_j^{k+1})\,\bar S_3(\bar{\rm b},\bar{\rm a}, j+k)_{\dot\alpha}^B \\ \cdot A_{k+2}^{(0)}({\rm a},{\rm c},j,\ldots,j+k-1)A_{n-k+1}^{(0)}(\bar{\rm c},{\rm b},j+k+1,\ldots,j+n-1). \tag{4.20}$$

We carry out the lengthy calculation for T in App. C, the final result can be related to the quantity in (4.17)

$$\begin{aligned}(T_j^k)_{\dot\alpha}^B &= A_n^{(0)}\varepsilon_{\dot\alpha\dot\gamma}\tilde\lambda_{j+k}^{\dot\gamma}\big(\tilde\lambda_{j+k}^{\dot\kappa}\varepsilon_{\dot\kappa\dot\lambda}(P_j^k)^{\delta\dot\lambda}\varepsilon_{\delta\epsilon}(Q_j^k)^{\epsilon B} - t_j^k\eta_{j+k}^B\big)\frac{\langle j-1,j\rangle}{\Upsilon_j^k\Upsilon_{j-1}^{k+1}}\\ &= A_n^{(0)}(\bar{\rm S}_{1\to1}^{(0)})_{\dot\alpha}^B\log\frac{\Upsilon_j^k}{\Upsilon_{j-1}^{k+1}}\,.\end{aligned} \tag{4.21}$$

Summing over all contributions, expanding the logarithms and reordering some of the sums, we find the contribution of the remaining deformation

$$\begin{aligned}\sum_{k=2}^{n-3}(\bar{\rm S}_{2\to n-k}^{(1)})_{\dot\alpha}^B A_{k+2}^{(0)} &= +\sum_{k=2}^{n-3}\sum_{j=1}^{n}(T_j^k)_{\dot\alpha}^B\log\frac{t_j^{k+1}}{t_j^k}\\ &= A_n^{(0)}\sum_{k=2}^{n-3}\sum_{j=1}^{n}\left((\bar{\rm S}_{1\to1}^{(0)})_{\dot\alpha}^B\log\frac{\Upsilon_j^k}{\Upsilon_{j-1}^{k+1}}\right)\log\frac{t_j^{k+1}}{t_j^k}\,.\end{aligned} \tag{4.22}$$

By comparison to the above results we find proper invariance under deformed conjugate superconformal boosts

$$\bar{\rm S}_{1\to1}^{(0)}A_n^{(1)} + \bar{\rm S}_{2\to2}^{(1)}A_n^{(0)} + \sum_{k=2}^{n-3}\bar{\rm S}_{2\to n-k}^{(1)}A_{k+2}^{(0)} = 0. \tag{4.23}$$

4.4 Splitting Anomaly

Finally we consider the terms arising when the generator acts on the tree-level prefactor. As described previously, when the conjugate superconformal generator is applied to the tree-level MHV amplitude, it will see the holomorphic poles as anomalies. This produces a delta-function which forces certain momenta in the loop integral to become collinear, effectively setting some $t_k^2 = (p_k + p_{k+1})^2$ to zero. This in turn corresponds to considering the collinear limits of the one-loop amplitude which is known to be governed by the one-loop splitting function, $r_{\rm S}$ [37],

$$\begin{aligned}\bar{\rm S}^{(0)}A_n^{(1)} &= \big(\bar{\rm S}^{(0)}A_n^{(0)}\big)\,M_n^{(1)} + \ldots\\ &\propto \delta^{(2)}(\langle k,k+1\rangle)A_{n-1}^{(0)}M_n^{(1)} + \ldots\\ &= \delta^{(2)}(\langle k,k+1\rangle)A_{n-1}^{(0)}\Big(M_{n-1}^{(1)} + r_{\rm S}\Big) + \ldots\ .\end{aligned} \tag{4.24}$$

For the appropriate definitions of the collinear limit $r_{\rm S}$ is the same function as was found in the analysis of the unitarity cuts, Sec. 3.4. The first term is simply cancelled by the tree level

deformation $\bar{\mathrm{S}}^{(0)}_{1\to 2}$ while the second are cancelled by $\bar{\mathrm{S}}^{(1)}_{1\to 2}$. Combining this with the previous terms we find the full deformation of the conjugate superconformal generator at one loop

$$\bar{\mathrm{S}}^{(0)}_{1\to 1}A^{(1)}_n + \bar{\mathrm{S}}^{(0)}_{1\to 2}A^{(1)}_{n-1} + \bar{\mathrm{S}}^{(1)}_{1\to 2}A^{(0)}_{n-1} + \bar{\mathrm{S}}^{(1)}_{2\to 2}A^{(0)}_n + \sum_{k=2}^{n-3}\bar{\mathrm{S}}^{(1)}_{2\to n-k}A^{(0)}_{k+2} = 0. \tag{4.25}$$

As was discussed in Sec. 3.4, and further in App. B.4, the exact definition of the collinear is subtle. There is an inherent ambiguity related to the order in which one takes the $\epsilon \to 0$ and collinear limits. Different prescriptions will give different results, however as long as we are consistent this is accounted for by the appropriate definition of r_{S} that appears in both the deformed generator and the collinear limit of the amplitude.

5 Invariance of the Six-Point NMHV Amplitude

The invariance of one-loop amplitudes with respect to the deformed generators, as described in Sec. 3, is designed to apply to generic amplitudes not just MHV. To check that this is indeed the case and to test the specifics of the proposal we examine the simplest non-trivial NMHV amplitude i.e. the six-point one-loop NMHV amplitude, $A^{(1)}_{6;\mathrm{NMHV}}$.

A convenient, manifestly supersymmetric, expression for this amplitude was given in [13] (see [38] for earlier calculations of the component amplitudes). It is written in terms of the dual-superconformal invariants, R_{rst}, which are polynomials in the η's of order four,

$$A^{(1)}_{6;\mathrm{NMHV}} = A^{(0)}_{6;\mathrm{MHV}}\Big[R_{146}F^{[1]}_6 + \mathrm{cyclic}\Big]\ . \tag{5.1}$$

The functions $F^{[i]}_6$ depend on the kinematic invariants t^k_j and are combinations of two-mass hard and one-mass scalar box functions. We give explicit expression for the R_{rst}'s and $F^{[i]}_6$'s in App. D.

For simplicity we will focus on the variation of this amplitude with respect to the generator $\bar{\mathrm{S}}$; for 6 legs the action of S follows by conjugation. As in previous sections we will show that the non-trivial action of the tree level $\bar{\mathrm{S}}^{(0)}_{1\to 1}$ is cancelled by the deformations constructed previously. That is to say,

$$\bar{\mathrm{S}}^{(0)}_{1\to 1}A^{(1)}_{6;\mathrm{NMHV}} + \bar{\mathrm{S}}^{(1)}_{2\to 2}A^{(0)}_{6;\mathrm{NMHV}} + \sum_{k=2}^{3}\bar{\mathrm{S}}^{(1)}_{2\to n-k}A^{(0)}_{k+2;\mathrm{NMHV}} = 0\ . \tag{5.2}$$

In this equation we are ignoring the anomaly in the tree-level amplitudes; these terms are cancelled by the deformations, $\bar{\mathrm{S}}^{(0)}_{1\to 2}$ and $\bar{\mathrm{S}}^{(1)}_{1\to 2}$, corresponding to the splitting function. We will thus focus on the variation of the loop integral portion of the amplitude.

5.1 Variation

Using the explicit expressions for the functions $F^{[i]}_6$, (D.6), one can straightforwardly calculate their variation. Let us first consider the terms in the variation with log cuts in the three-particle

channels, say t_1^3. These terms occur only multiplying the R_{146} and R_{413} structures, for example,

$$\bar{\mathrm{S}}^{(0)}_{1\to 1} A^{(1)}_{6;\mathrm{NMHV}} = A^{(0)}_{6;\mathrm{MHV}} R_{146} \log t_1^3 \left(\bar{\mathrm{S}}^{(0)}_{1\to 1} \log \left[\frac{t_1^2 t_2^2}{t_3^2 t_6^2} \right] \right) + \dots \ . \tag{5.3}$$

In principle one could completely expand in powers of the η's and consider the various terms independently. It turns out to be more convenient to leave the R's intact and to consider the coefficients of the various $\eta_A R_{rst}$ terms. However, the R's are not independent but rather satisfy various identities. For example, for the six-point amplitude one can use the identity (D.5) to remove one of the six R's and so one could expect cancellations between different terms.[9] Nonetheless, through a judicious choice we will be able to consider the coefficients of the $\eta_A R_{rst}$ terms separately. In fact, for the case of the three-particle channel, the coefficients of $\eta_A R_{146}$ and $\eta_A R_{413}$ are independent as can seen by noting that the first term will give rise to terms, once we include the overall fermionic supermomentum delta-function, such as $\eta_A \eta_4^4 \eta_5^4 \eta_6^4$ which cannot arise from the second term. Concretely, we focus on the coefficient of R_{146} and using the fermionic delta-functions to remove η_1, η_3 and η_5. We find

$$\begin{aligned}
A^{(0)}_{6;\mathrm{MHV}} R_{146} \bar{\mathrm{S}}^{(0)}_{1\to 1} \log \left[\frac{t_2^2}{t_3^2} \right] &= A^{(0)}_{6;\mathrm{MHV}} R_{146} \frac{\bar{\lambda}_3 \langle 1 | P_1^3 | 4]}{\langle 13 \rangle [23][34][46]} \left(\eta_2 [46] + \eta_4 [62] + \eta_6 [24] \right), \\
A^{(0)}_{6;\mathrm{MHV}} R_{146} \bar{\mathrm{S}}^{(0)}_{1\to 1} \log \left[\frac{t_1^2}{t_6^2} \right] &= A^{(0)}_{6;\mathrm{MHV}} R_{146} \frac{\bar{\lambda}_1 \langle 3 | P_3^3 | 6]}{\langle 13 \rangle [12][46][61]} \left(\eta_2 [46] + \eta_4 [62] + \eta_6 [24] \right) \ .
\end{aligned} \tag{5.4}$$

Combining these terms we find the complete variation in the three-particle channel t_1^3 with coefficient R_{146}. It is interesting to note that this calculation is essentially the same as that performed in [34] which used the holomorphic anomaly of the collinearity operator to fix the box coefficients of the split-helicity NMHV amplitudes. Indeed the coefficient of η_2 in $\bar{\mathrm{S}}$ is exactly the collinearity operator for particles 1, 2 and 3 used in [34]. All the other three particle cuts can be found using cyclicity.

Turning to the two-particle channel cuts of the variation, for example cuts in the variable t_1^2, we choose to remove R_{413} using (D.5). The resulting expression for the variation of the amplitude in terms of the remaining R's is

$$\begin{aligned}
\bar{\mathrm{S}}^{(0)}_{1\to 1} A^{(1)}_{6;\mathrm{NMHV}} = A^{(0)}_{6;\mathrm{MHV}} \Bigg(& (R_{146} + R_{362} + R_{524}) \ \bar{\mathrm{S}}^{(0)}_{1\to 1} \Big[- \frac{c_\epsilon}{\epsilon^2} \left(\frac{t_1^2}{-\mu^2} \right)^{-\epsilon} \Big] \\
& - \log \frac{t_1^2}{-\mu^2} R_{146} \, \bar{\mathrm{S}}^{(0)}_{1\to 1} \log \left[\frac{t_2^2}{t_3^2} \right] - \log \frac{t_1^2}{-\mu^2} R_{635} \, \bar{\mathrm{S}}^{(0)}_{1\to 1} \log \left[\frac{t_6^2}{t_5^2} \right] \Bigg) + \dots
\end{aligned}$$

[9] In fact, if we use the overall supermomentum delta-function and the fermionic delta-function in the R to remove three of the η_A's we have fifteen terms, $\eta_A R_{rst}$, in the variation. If we alternatively use only the supermomentum condition we get four η's times six R's. Now for each R we have an additional constraint between the η's – six constraints – while for each η we have a relation between the various R's – four constraints – giving a total of ten constraints and hence only fourteen independent terms. Thus if these constraints are not degenerate we find that there must be at least one additional relation between the various terms.

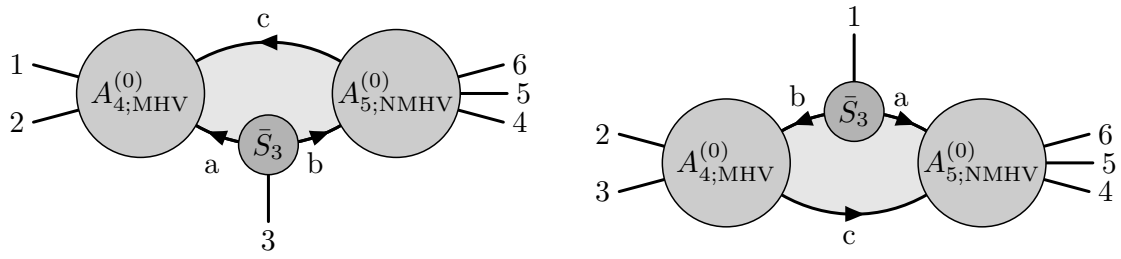

Figure 9: Anomaly for six-point NMHV with coefficient R_{146}.

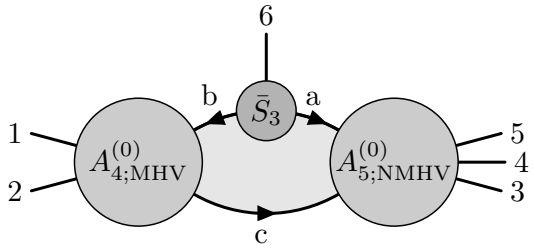

Figure 10: Anomaly for six-point NMHV with coefficient R_{635}.

$$
\begin{aligned}
&= A^{(0)}_{6;\mathrm{NMHV}} \bar{\mathrm{S}}^{(0)}_{1\to 1}\Big[-\frac{c_\epsilon}{\epsilon^2}\left(\frac{t_1^2}{-\mu^2}\right)^{-\epsilon}\Big] \\
&\quad - A^{(0)}_{6;\mathrm{MHV}} \log\frac{t_1^2}{-\mu^2}\left(R_{146}\,\bar{\mathrm{S}}^{(0)}_{1\to 1}\log\Big[\frac{t_2^2}{t_3^2}\Big] + R_{635}\,\bar{\mathrm{S}}^{(0)}_{1\to 1}\log\Big[\frac{t_6^2}{t_5^2}\Big]\right) + \dots \ .
\end{aligned}
\tag{5.5}
$$

Looking at this term we can immediately see that the first term is the same divergent structure as appeared for the MHV amplitudes. As was the case there, this term corresponds to the $\bar{\mathrm{S}}^{(1)}_{2\to 2}$ deformation due to the measure correction. The subsequent terms are given by (5.4) and its cyclic permutation. We expect these terms to be cancelled by $\bar{\mathrm{S}}^{(1)}_{2\to k}$ terms arising from collinear anomalies in loops and, as we will see, indeed this is the case.

5.2 Deformations

The deformation due to the measure factor, $\bar{\mathrm{S}}^{(1)}_{2\to 2}$, is straightforwardly seen to be the same for NMHV amplitudes as for the MHV ones. It cancels the divergent terms (and the finite parts grouped with them). Similarly, the contributions for the splitting function, at tree and one-loop level, are the same due to the universality of the splitting function and it is well known that the amplitudes have the correct collinear behaviour. Thus we move to consider the contributions from triangle diagrams where the anomaly sits inside the loop, the analogues of Fig. 8. These arise from the deformation

$$
\sum_{k=2}^{3} \bar{\mathrm{S}}^{(1)}_{2\to n-k} A^{(0)}_{k;\mathrm{MHV}} = \sum_{k=2}^{3}\sum_{j=1}^{6} T_j^k \log\frac{t_j^k}{t_j^{k+1}} \ , \tag{5.6}
$$

where, T_j^k, are on-shell triangle integrals. Let us first calculate the coefficient of the three-particle channel cuts in the variable t_1^3 which are proportional to R_{146} i.e. the terms needed to cancel (5.3). That is we wish to evaluate

$$(T_1^2 - T_1^3)\log t_1^3 \tag{5.7}$$

where the triangle integrals are represented in Fig. 9 and, for example,

$$T_1^2 = \frac{1}{(2\pi)^3}\int (d^{4|4}\Lambda)^3\,\mathrm{sign}(t_1^2 - t_1^3)\bar{S}_3(\bar{\mathrm{b}},\bar{\mathrm{a}},3)A^{(0)}_{4;\mathrm{MHV}}(\mathrm{a},\mathrm{c},1,2)A^{(0)}_{5;\mathrm{NMHV}}(\bar{\mathrm{c}},\mathrm{b},4,5,6). \tag{5.8}$$

As for the MHV case the delta-functions enable one to trivially evaluate these expressions and indeed the calculations are quite similar. It can then be shown that

$$\begin{aligned} T_1^2 &= A^{(0)}_{6;\mathrm{MHV}}R_{146}\frac{\langle 1|P_1^3|4]\bar{\lambda}_3}{\langle 13\rangle[23][34][46]}\left(\eta_2[46]+\eta_4[26]+\eta_6[24]\right), \\ -T_4^3 &= A^{(0)}_{6;\mathrm{MHV}}R_{146}\frac{\langle 3|P_3^3|6]\bar{\lambda}_1}{\langle 13\rangle[12][46][61]}\left(\eta_2[46]+\eta_4[26]+\eta_6[24]\right), \end{aligned} \tag{5.9}$$

which are identical to (5.4) and so cancel the variation (5.3).

Turning to the two-particle channels in the variable t_1^2 we see that they can only arise with coefficients R_{146}, from the first triangle diagram in Fig. 9, and R_{635}, from the triangle diagram Fig. 10. From (5.9), and its appropriate permutation, it is clear that these terms will indeed cancel the relevant terms in the variation (5.5). The remaining two-particle channels similarly follow by use of cyclicity. Thus we see that the deformations constructed previously also annihilate the six-point NMHV amplitude.

6 Yangian Symmetry at One Loop

In addition to the standard super-conformal symmetries it has recently become clear that planar scattering amplitudes in $\mathcal{N}=4$ transform covariantly under a "dual" superconformal algebra [13]. First hints towards this symmetry appeared at the level of loop integrals contributing to the amplitudes [9, 59]. This surprising additional symmetry of the planar theory has its origin in the ordinary conformal symmetry of the dual Wilson loop description of MHV amplitudes [11, 12]. Tree amplitudes have been proven to be covariant with respect to dual superconformal transformations [14, 15]. At loop level the dual conformal boost $\widetilde{\mathrm{K}}^{\alpha\dot{\alpha}}$ is naively broken and picks up an anomaly term whose form, however, was conjectured to be under control to all loop orders in [12, 13]. Recently it was shown that all one-loop $\mathcal{N}=4$ SYM scattering amplitudes indeed obey the dual conformal anomaly relation [44], following earlier results on MHV and NMHV amplitudes [13, 60, 14, 42, 61].

The dual superconformal symmetry can be made manifest by introducing new dual coordinates on which the dual symmetry generators act locally

$$\lambda_i^\beta\tilde{\lambda}_i^{\dot{\alpha}} = x_i^{\beta\dot{\alpha}} - x_{i+1}^{\beta\dot{\alpha}}\,, \qquad \lambda_i^\beta\eta_i^A = \theta_i^{\beta A} - \theta_{i+1}^{\beta A} \tag{6.1}$$

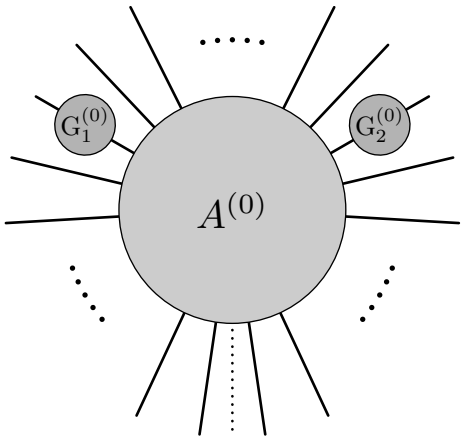

Figure 11: Action of the Yangian generator $\widehat{\mathrm{G}}$ on the colour-ordered amplitude A at tree level. The action is defined as the bi-local insertion of two superconformal generators G_1 and G_2, the former acting to the left of the latter. To that end one has to define an origin for the colour-ordered amplitude (dotted line).

and one makes the identifications $x_{n+1} := x_1$, $\theta_{n+1} := \theta_1$, [13]. At tree level it was shown in [20] that this additional symmetry can be understood to lift the conventional superconformal symmetry algebra $\mathfrak{psu}(2,2|4)$ to a Yangian symmetry $\mathrm{Y}[\mathfrak{psu}(2,2|4)]$. The level-one Yangian generators $\widehat{\mathrm{G}}$ are given by the standard coproduct rule for evaluation representations with homogeneous evaluation parameters, cf. Fig. 11,

$$\widehat{\mathrm{G}}_M = f_M^{KL} \sum_{1\leq k<\ell\leq n} \mathrm{G}_{k,K}\mathrm{G}_{\ell,L} \ . \tag{6.2}$$

The level-one generators $\widehat{\mathrm{G}}$ thereby satisfy

$$[\mathrm{G}_K, \widehat{\mathrm{G}}_L\} = f_{KL}^M \, \widehat{\mathrm{G}}_M \ . \tag{6.3}$$

Making use of the invariance of tree-amplitudes with respect to the locally acting central charge, C_i, it can be shown that this representation is compatible with the cyclicity of the amplitudes [20]. Furthermore by means of the Serre relations and the covariance of the tree-level amplitudes under the "dual" conformal generators one can show that the amplitudes are indeed invariant under the full Yangian algebra [20].

In the following we determine the one-loop deformation of the Yangian generator $\widehat{\mathrm{P}}^{\alpha\dot{\alpha}}$ and $\widehat{\mathrm{Q}}^{\alpha A}$.

6.1 Dual Conformal Boost alias Level-One Momentum

Indeed by virtue of (6.3) it suffices to construct the one-loop deformation of $\widehat{\mathrm{P}}_{\alpha\dot{\alpha}}$ as all the other level-one Yangian generators follow by commutation with level-zero ones.

To begin with, let us then quantify the relation between dual conformal boost

$$\widetilde{\mathrm{K}}^{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \left[x_i^{\alpha\dot{\beta}} x_i^{\dot{\alpha}\beta} \frac{\partial}{\partial x_i^{\beta\dot{\beta}}} + x_i^{\dot{\alpha}\beta} \theta_i^{\alpha B} \frac{\partial}{\partial \theta_i^{\beta\dot{\beta}}} + x_i^{\alpha\dot{\alpha}} \right] \tag{6.4}$$

and the level-one Yangian

$$(\widehat{\mathrm{P}}^{(0)})^{\alpha\dot{\alpha}} = \sum_{1\leq i<j\leq n} \left[\left(\mathrm{L}_i^{\alpha}{}_{\gamma}\, \delta_{\dot{\gamma}}^{\dot{\alpha}} + \bar{\mathrm{L}}_i^{\dot{\alpha}}{}_{\dot{\gamma}}\, \delta_{\gamma}^{\alpha} + \mathrm{D}_i^{(0)} \delta_{\gamma}^{\alpha} \delta_{\dot{\gamma}}^{\dot{\alpha}} \right) \mathrm{P}_j^{\gamma\dot{\gamma}} + \mathrm{Q}_j^{\alpha C} \bar{\mathrm{Q}}_{iC}^{\dot{\alpha}} - (i \leftrightarrow j) \right] \tag{6.5}$$

at tree-level [20]. Note that both $\widetilde{\mathrm{K}}^{\alpha\dot{\alpha}}$ and $\widehat{\mathrm{P}}^{\alpha\dot{\alpha}}$ annihilate $A_n^{(0)}$. Following [20] one may solve the new dual coordinates in terms of the original ones via (6.1)

$$x_i^{\alpha\dot{\alpha}} = x_1^{\alpha\dot{\alpha}} - \sum_{1\leq j<i} \lambda_j^{\alpha} \tilde{\lambda}_j^{\dot{\alpha}}\,, \qquad \theta_i^{\alpha A} = \theta_1^{\alpha A} - \sum_{1\leq j<i} \lambda_j^{\alpha} \eta_j^{A}\,, \tag{6.6}$$

and rewrite the dual conformal boost (6.4) as a differential operator acting in the original on-shell superspace coordinates $\{\lambda_i, \tilde{\lambda}_i, \eta_i\}$ to find

$$\begin{aligned} \widetilde{\mathrm{K}}^{\alpha\dot{\alpha}} = {} & \tfrac{1}{2}\widehat{\mathrm{P}}^{\alpha\dot{\alpha}} + \sum_i \left(\mathrm{P}_i^{\alpha\dot{\alpha}}\, \mathrm{C}_i \right) - \tfrac{1}{2}\mathrm{P}^{\alpha\dot{\alpha}} \\ & - \tfrac{1}{2}\mathrm{P}^{\gamma\dot{\gamma}} \left(\mathrm{L}^{\alpha}{}_{\gamma}\, \delta_{\dot{\gamma}}^{\dot{\alpha}} + \bar{\mathrm{L}}^{\dot{\alpha}}{}_{\dot{\gamma}}\, \delta_{\gamma}^{\alpha} + \mathrm{D}\, \delta_{\gamma}^{\alpha} \delta_{\dot{\gamma}}^{\dot{\alpha}} \right) - \tfrac{1}{2}\mathrm{Q}^{\alpha B} \bar{\mathrm{Q}}_B^{\dot{\alpha}} \\ & + x_1^{\gamma\dot{\gamma}} \left(\mathrm{L}^{\alpha}{}_{\gamma}\, \delta_{\dot{\gamma}}^{\dot{\alpha}} + \bar{\mathrm{L}}^{\dot{\alpha}}{}_{\dot{\gamma}}\, \delta_{\gamma}^{\alpha} + \mathrm{D}\, \delta_{\gamma}^{\alpha} \delta_{\dot{\gamma}}^{\dot{\alpha}} \right) \\ & + \theta_{\alpha}^{1B}\, \bar{\mathrm{Q}}_B^{\dot{\alpha}}\,, \end{aligned} \tag{6.7}$$

where the only non-local structure on the right-hand side resides in the level-one Yangian momentum generator. We now make use of the full one-loop anomaly relation [44] in our conventions

$$\begin{aligned} \widetilde{\mathrm{K}}^{\alpha\dot{\alpha}} A_n^{(1)} &= 2\, A_n^{(0)} \sum_{i=1}^{n} x_{i+1}^{\alpha\dot{\alpha}}\, \frac{c_\epsilon}{\epsilon} \left(\frac{x_{i,i+2}^2}{-\mu^2} \right)^{-\epsilon} \\ &= -2\, A_n^{(0)} \left[\sum_{1\leq j<i\leq n} p_j^{\alpha\dot{\alpha}}\, \frac{c_\epsilon}{\epsilon} \left(\frac{t_i^2}{-\mu^2} \right)^{-\epsilon} + \sum_{i=1}^{n} p_i^{\alpha\dot{\alpha}}\, \frac{c_\epsilon}{\epsilon} \left(\frac{t_i^2}{-\mu^2} \right)^{-\epsilon} \right. \\ &\qquad \left. - x_1^{\alpha\dot{\alpha}} \sum_{i=1}^{n} \frac{c_\epsilon}{\epsilon} \left(\frac{t_i^2}{-\mu^2} \right)^{-\epsilon} \right] \end{aligned} \tag{6.8}$$

where $x_{i,j} := x_i - x_j$. Now acting with (6.7) on the one-loop amplitude, taking into account that the tree-level generators $\{\mathrm{L}^{\alpha}{}_{\beta}, \bar{\mathrm{L}}^{\dot{\alpha}}{}_{\dot{\beta}}, \mathrm{Q}^{\alpha B}, \bar{\mathrm{Q}}_B^{\dot{\alpha}}\}$ all annihilate $A_n^{(1)}$ as well as the *local* generator $\mathrm{C}_i\, \widetilde{\mathrm{K}}^{\alpha\dot{\alpha}} = 0$ and that (3.44)

$$\mathrm{D}^{(0)}\, A_n^{(1)} = 2 A_n^{(0)} \sum_{j=1}^{n} \frac{c_\epsilon}{\epsilon} \left(\frac{t_j^2}{-\mu^2} \right)^{-\epsilon} \tag{6.9}$$

one easily computes the action of the tree-level Yangian generator on the one-loop amplitude

$$(\widehat{\mathrm{P}}^{(0)})^{\alpha\dot{\alpha}} A_n^{(1)} = -2\left[\sum_{1\le j<i\le n}\left(p_j^{\alpha\dot{\alpha}}\frac{c_\epsilon}{\epsilon}\left(\frac{t_i^2}{-\mu^2}\right)^{-\epsilon} - (i\leftrightarrow j)\right) + \sum_{i=1}^{n} p_i^{\alpha\dot{\alpha}}\frac{c_\epsilon}{\epsilon}\left(\frac{t_i^2}{-\mu^2}\right)^{-\epsilon}\right] A_n^{(0)}\,, \quad (6.10)$$

remarkably recovering a bi-local structure again. We hence conclude that the one-loop deformation of $\widehat{\mathrm{P}}^{\alpha\dot{\alpha}}$ takes the simple form

$$(\widehat{\mathrm{P}}^{(1)})^{\alpha\dot{\alpha}} = -\left[\sum_{1\le j<i\le n}\Big(\mathrm{P}_j^{\alpha\dot{\alpha}}\,(\mathrm{D}^{(1)}_{2\to 2})_i^2 - (i\leftrightarrow j)\Big) + \sum_{i=1}^{n}\mathrm{P}_i^{\alpha\dot{\alpha}}\,(\mathrm{D}^{(1)}_{2\to 2})_i^2\right]. \quad (6.11)$$

where we have inserted the the one-loop deformation of the dilatation operator of (3.44). Rewriting $\widehat{\mathrm{P}}^{(1)}_{\alpha\dot{\alpha}}$ in a fashion of keeping the bi-local terms free of contact terms we have

$$(\widehat{\mathrm{P}}^{(1)})^{\alpha\dot{\alpha}} = -\left[\sum_{1\le j<i\le n}\Big(\mathrm{P}_j^{\alpha\dot{\alpha}}\,(\mathrm{D}^{(1)}_{2\to 2})_i^2 - \mathrm{P}_i^{\alpha\dot{\alpha}}\,(\mathrm{D}^{(1)}_{2\to 2})_{j-1}^2\Big) + \sum_{i=1}^{n}(\mathrm{P}_i^{\alpha\dot{\alpha}} - \mathrm{P}_{i+1}^{\alpha\dot{\alpha}})\,(\mathrm{D}^{(1)}_{2\to 2})_i^2\right]. \quad (6.12)$$

Now the following curious picture emerges: Recalling the split of the one-loop amplitude into a finite and a divergent piece as done in (3.37) but here restricted to the planar limit (3.42)

$$A_n^{(1)} = \hat{Z}^{(1)}_{2\to 2} A_n^{(0)} + \tilde{A}_n^{(1)}\,, \quad \text{with} \quad \hat{Z}^{(1)}_{2\to 2} = -\sum_{j=1}^{n}\frac{c_\epsilon}{\epsilon^2}\left(\frac{t_j^2}{-\mu^2}\right)^{-\epsilon}, \quad (6.13)$$

we can identify the bi-local deformation of $\widehat{\mathrm{P}}^{(1)}_{\alpha\dot{\alpha}}$ in (6.12) as arising from the action of the tree-level generator on $\hat{Z}^{(1)}_{2\to 2}$

$$\begin{aligned}\big[(\widehat{\mathrm{P}}^{(0)})^{\alpha\dot{\alpha}}, \hat{Z}^{(1)}_{2\to 2}\big] &= \sum_{i=2}^{n-1}\left[\sum_{1\le j<i}\mathrm{P}_j\,(\mathrm{D}^{(1)}_{2\to 2})_i^2 - \sum_{i\le j\le n}\mathrm{P}_j\,(\mathrm{D}^{(1)}_{2\to 2})_{i-1}^2\right] - \big(\mathrm{P}_n^{\alpha\dot{\alpha}} - \mathrm{P}_1^{\alpha\dot{\alpha}}\big)\,(\mathrm{D}^{(1)}_{2\to 2})_n^2 \\ &= \sum_{i=1}^{n}\left[\sum_{1\le j<i}\mathrm{P}_j\,(\mathrm{D}^{(1)}_{2\to 2})_i^2 - \sum_{i\le j\le n}\mathrm{P}_j\,(\mathrm{D}^{(1)}_{2\to 2})_{i-1}^2\right]. \end{aligned} \quad (6.14)$$

Conversely, the last local term in (6.12) is nothing but the anomaly of $\tilde{A}_n^{(1)}$ once spelled out in the dual coordinates

$$(\widehat{\mathrm{P}}^{(0)})^{\alpha\dot{\alpha}}\,\tilde{A}_n^{(1)} = 2\sum_{i=1}^{n} x_{i,i+1}^{\alpha\dot{\alpha}}\,\log\left(\frac{x_{i,i+2}^2}{x_{i-1,i+1}^2}\right) A_n^{(0)} = -(\widehat{\mathrm{P}}^{(1)}_{2\to 2})^{\alpha\dot{\alpha}} A_n^{(0)}, \quad (6.15)$$

with

$$(\widehat{\mathrm{P}}^{(1)}_{2\to 2})^{\alpha\dot{\alpha}} = -\sum_{i=1}^{n}(\mathrm{P}_i^{\alpha\dot{\alpha}} - \mathrm{P}_{i+1}^{\alpha\dot{\alpha}})\,(\mathrm{D}^{(1)}_{2\to 2})_i^2. \quad (6.16)$$

In conclusion we can cleanly separate the deformation from the divergent measure and a genuine one-loop anomaly

$$(\widehat{\mathrm{P}}^{(1)})^{\alpha\dot\alpha} = \big[(\widehat{\mathrm{P}}^{(0)})^{\alpha\dot\alpha}, \hat{Z}^{(1)}_{2\to2}\big] + (\widehat{\mathrm{P}}^{(1)}_{2\to2})^{\alpha\dot\alpha}\,. \tag{6.17}$$

While perhaps a mere curiosity it is worth noting that one can find the anomaly from the action of a bi-local operator on $\hat{Z}^{(1)}_{2\to2}$,

$$(\widehat{\mathrm{P}}^{(1)}_{2\to2})^{\alpha\dot\alpha} = \sum_{1\le i<j\le n} \Big[2\left(\mathrm{L}_i^{\ \alpha}\,\delta^{\dot\alpha}_{\dot\gamma} + \bar{\mathrm{L}}_i^{\ \dot\alpha}\,\delta^{\alpha}_{\gamma}\right)\mathrm{P}_j^{\ \dot\gamma\gamma} - (i\leftrightarrow j), \hat{Z}^{(1)}_{2\to2}\Big]\,. \tag{6.18}$$

Effectively the first term in (6.17) is sufficient to reproduce the complete $\widehat{\mathrm{P}}^{(1)}$ if the sign of L and $\bar{\mathrm{L}}$ in $\widehat{\mathrm{P}}^{(0)}$, cf. (6.5), is flipped.

6.2 The All-Loop Form of D and $\widehat{\mathrm{P}}$

This insight together with the all-loop conjecture for the form of the dual conformal anomaly of Drummond, Henn, Korchemsky and Sokatchev [13] then leads to a transparent conjecture for the all-loop form of the dilatation and Yangian level-one momentum generators. Due to the exponentiation of IR singularities the all-loop expression for the planar n-point amplitudes takes the form

$$A_n(g^2) = \exp\big(\hat{Z}_{2\to2}(g^2,\epsilon)\big)\,\tilde{A}_n(g^2)\,, \tag{6.19}$$

with $\tilde{A}_n$ being the all-loop finite piece, and $\hat{Z}_{2\to2}$ is the logarithm of the all-loop IR divergences. The form of this functions is well studied with the leading term for planar amplitudes being given by[10]

$$\hat{Z}_{2\to2}(g^2,\epsilon) = \tfrac{1}{4}\Gamma(g^2,\epsilon)\,\hat{Z}^{(1)}_{2\to2}(\epsilon) + \mathcal{O}(\epsilon^0)\ , \tag{6.20}$$

where $\Gamma(g^2,\epsilon)$ contains the cusp dimension $\Gamma_{\rm cusp}(g^2) = g^2 + \mathcal{O}(g^4)$ as well as the collinear dimension $\Gamma_{\rm coll}(g^2) = \mathcal{O}(g^4)$

$$\Gamma(g^2,\epsilon) = \Gamma_{\rm cusp}(g^2) + \epsilon\Gamma_{\rm coll}(g^2) + \mathcal{O}(\epsilon^2). \tag{6.21}$$

The subleading in ϵ terms are scheme dependent. Acting on this with tree-level D yields

$$\mathrm{D}^{(0)}\,A_n = \tfrac{1}{4}\Gamma(g^2,\epsilon)\big[\mathrm{D}^{(0)}, \hat{Z}^{(1)}_{2\to2}\big]A_n = -\tfrac{1}{4}\Gamma(g^2,\epsilon)\,\mathrm{D}^{(1)}\,A_n \tag{6.22}$$

hence the all-loop *planar* dilatation operator is simply

$$\mathrm{D}(g^2) = \mathrm{D}^{(0)} + \tfrac{1}{4}\Gamma(g^2,\epsilon)\,\mathrm{D}^{(1)}, \tag{6.23}$$

such that $\mathrm{D}\,A_n = 0$.

[10] In the following we shall use a sloppy notation where we disregard the effect of the loop order on the ϵ-dependence. This dependence is very systematic and can be implemented easily by the rule $\epsilon \to \ell\epsilon$ at ℓ loops. For a review of the all-loop structure of $\mathcal{N}=4$ SYM amplitudes including their IR divergences see e.g. [45] and the many papers referenced therein.

Similarly, the all-loop form of the level-one Yangian generator $\widehat{\mathrm{P}}^{\alpha\dot\alpha}$ can be established. For this we observe that the action of the tree-level $(\widehat{\mathrm{P}}^{(0)})^{\alpha\dot\alpha}$ results in

$$(\widehat{\mathrm{P}}^{(0)})^{\alpha\dot\alpha}\,A_n = \tfrac{1}{4}\Gamma(g^2,\epsilon)\,\big[(\widehat{\mathrm{P}}^{(0)})^{\alpha\dot\alpha},\hat{Z}^{(1)}_{2\to2}\big]A_n + \exp\big(\hat{Z}_{2\to2}\big)(\widehat{\mathrm{P}}^{(0)})^{\alpha\dot\alpha}\,\tilde{A}_n\,. \tag{6.24}$$

Now by virtue of the conjectured form [13] of the all-loop dual-conformal anomaly we note

$$\begin{aligned}(\widehat{\mathrm{P}}^{(0)})^{\alpha\dot\alpha}\,\tilde{A}_n = 2\,\widetilde{\mathrm{K}}^{\alpha\dot\alpha}\,\tilde{A}_n &= \tfrac{1}{2}\Gamma(g^2,\epsilon)\sum_{i=1}^{n} x^{\alpha\dot\alpha}_{i,i+1}\,\log\left(\frac{x^2_{i,i+2}}{x^2_{i-1,i+1}}\right)\tilde{A}_n\\ &= -\tfrac{1}{4}\Gamma(g^2,\epsilon)\,(\widehat{\mathrm{P}}^{(1)}_{2\to2})^{\alpha\dot\alpha}\,\tilde{A}_n\,,\end{aligned} \tag{6.25}$$

where we have made use of the fact that the discrepancy terms between $(\widehat{\mathrm{P}}^{(0)})^{\alpha\dot\alpha}$ and $2\widetilde{\mathrm{K}}^{\alpha\dot\alpha}$ in (6.7) are inactive when acting on $\tilde{A}$. Hence we know all terms on the right-hand-side of (6.24). This enables us to also conjecture the all-loop form of the level-one Yangian generator $\widehat{\mathrm{P}}^{\alpha\dot\alpha}$

$$\widehat{\mathrm{P}}(g^2)^{\alpha\dot\alpha} = (\widehat{\mathrm{P}}^{(0)})^{\alpha\dot\alpha} + \tfrac{1}{4}\Gamma(g^2,\epsilon)\,(\widehat{\mathrm{P}}^{(1)})^{\alpha\dot\alpha}, \tag{6.26}$$

whose structure is dictated by the one-loop deformation, as is also the case for the dilatation operator (6.23).

6.3 Dual Superconformal Boosts and Bi-Local Generators

Invariance of the amplitude under the dual superconformal boosts $\widetilde{\mathrm{S}}$ alias the level-one supersymmetry generators $\widehat{\mathrm{Q}}$ follows straight-forwardly from invariance under $\widehat{\mathrm{P}}$ and $\bar{\mathrm{S}}$ using (6.3)

$$\big[\bar{\mathrm{S}}^A_{\dot\beta},\widehat{\mathrm{P}}^{\gamma\dot\delta}\big] = -\delta^{\dot\delta}_{\dot\beta}\widehat{\mathrm{Q}}^{\gamma A}. \tag{6.27}$$

Nevertheless, it is instructive to see how the one-loop deformation of $\widehat{\mathrm{Q}}$ acts qualitatively. In particular, we have seen in Sec. 6.2 that $\widehat{\mathrm{P}}$ receives only relatively simple corrections, but here length-changing interactions as discussed in Sec. 3.3 enter. This follows directly from the above commutator which introduces them via $\bar{\mathrm{S}}$.

Let us compare the structure of the one-loop deformation of $\widehat{\mathrm{P}}$ in (6.12) with the tree-level generator in (6.5): For the bi-local contribution in (6.12) one promotes all instances of D in (6.5) to its one-loop correction and drops the other terms which involve only super-Poincaré generators $\mathrm{L},\bar{\mathrm{L}},\mathrm{P},\mathrm{Q},\bar{\mathrm{Q}}$. In other words, the bi-local contribution is obtained from a perturbative generalisation of (6.2)

$$\widehat{\mathrm{G}}_M = f^{KL}_M \mathrm{G}_K\otimes\mathrm{G}_L + \widehat{\mathrm{G}}_{\mathrm{loc},M} := f^{KL}_M\sum_{k\ll\ell}\mathrm{G}_{k,K}\,\mathrm{G}_{\ell,L} + \sum_k \widehat{\mathrm{G}}_{\mathrm{loc},k,M}\ , \tag{6.28}$$

where G_K are the perturbative generalisations of the superconformal generators. The local contributions $\widehat{\mathrm{G}}_{\mathrm{loc}}$ are needed to specify the action of the bi-local generators when the constituent operators overlap. When expanded to one loop one obtains, cf. Fig. 12,

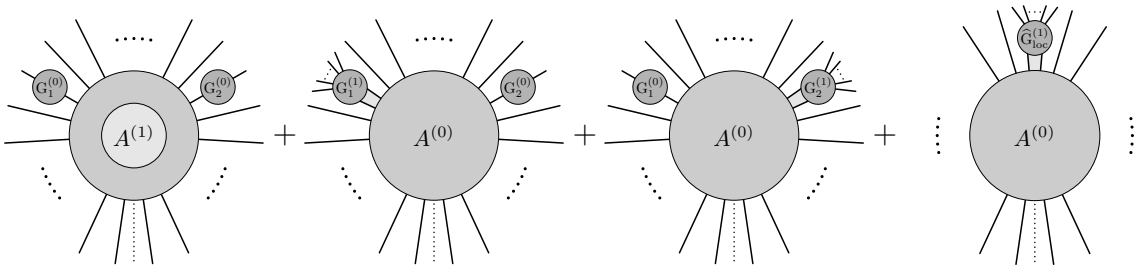

Figure 12: Action of the Yangian generator $\widehat{\mathrm{G}}$ on the colour-ordered planar amplitude A at one loop. The standard contribution of $\widehat{\mathrm{G}}^{(1)}$ is the same as at tree level where one constituent superconformal generators $\mathrm{G}_1, \mathrm{G}_2$ is replaced by its one-loop deformation. Furthermore $\widehat{\mathrm{G}}^{(1)}_{\mathrm{loc}}$ represents the action when the constituent generators overlap. Here, one may have to specify separately how $\widehat{\mathrm{G}}^{(1)}_{\mathrm{loc}}$ acts on the two particles separated by the origin.

$$\widehat{\mathrm{G}}^{(1)}_M = f^{KL}_M \mathrm{G}^{(1)}_K \otimes \mathrm{G}^{(0)}_L + f^{KL}_M \mathrm{G}^{(0)}_K \otimes \mathrm{G}^{(1)}_L + \widehat{\mathrm{G}}^{(1)}_{\mathrm{loc},M}. \tag{6.29}$$

In the case of $\widehat{\mathrm{P}}$ the only bi-local deformation originates from the dilatation generator D, and it is precisely of this form. Moreover, the anomaly $\widehat{\mathrm{P}}^{(1)}_{2\to 2}$ equals the local term $\widehat{\mathrm{P}}^{(1)}_{\mathrm{loc}}$.

The structure (6.29,6.28) is well-known for perturbative Yangians, cf. [62–64]. Importantly, the bi-local term is stable under the adjoint action (6.3) provided that the superconformal algebra is satisfied. The local terms serve as a regularisation and are more sensitive to the details of the algebra and its deformation.

For the correction to the dual superconformal boost $\widehat{\mathrm{Q}}$ these considerations imply the structure

$$(\widehat{\mathrm{Q}}^{(1)})^{A\beta} = \mathrm{P}^{\beta\dot\gamma} \wedge (\bar{\mathrm{S}}^{(1)})^A_{\dot\gamma} + \tfrac{1}{2}\mathrm{Q}^{A\beta} \wedge \mathrm{D}^{(1)} + \widehat{\mathrm{Q}}^{A\beta}_{\mathrm{loc}}, \tag{6.30}$$

where $A \wedge B := A \otimes B - B \otimes A$. This expression has length-preserving contributions from $\bar{\mathrm{S}}^{(1)}_{2\to 2}$, $\mathrm{D}^{(1)}_{2\to 2}$ and $\widehat{\mathrm{Q}}^{A\beta}_{\mathrm{loc},2\to 2}$, but there are also length-changing contributions from $\bar{\mathrm{S}}^{(1)}_{2\to k}$ and $\widehat{\mathrm{Q}}^{A\beta}_{\mathrm{loc},2\to k}$, cf. Sec. 3. The bi-local contributions follow directly from the correction to the superconformal generators, whereas the local terms follow from the commutator (6.27). We refrain from presenting the results of such a computation as the result is guaranteed to annihilate all amplitudes anyway.

Similar considerations apply for $\widehat{\bar{\mathrm{Q}}}$ which has a structure analogous to $\widehat{\mathrm{Q}}$ but does not derive from one of the dual superconformal generators. The remaining level-one generators follow the same pattern, but they all involve K which has a yet more complicated structure than S and $\bar{\mathrm{S}}$.

We have not yet addressed the issues regarding the algebra of the deformed generators and while we postpone a detailed consideration to future work a few comments are in order. Due to the choice of regulator all the deformations respect the manifest super-Poincaré algebra. For the special conformal and fermionic conformal generators the situation is much less clear. Of course, the algebra is trivially satisfied on the space of amplitudes however on larger spaces it is remains non-trivial to demonstrate closure. Already at tree-level the algebra is seen to close only up to field dependent gauge transformations [35] and demonstrating closure at one-loop remains an open problem. Furthermore, to establish the existence of a Yangian algebra one must additionally

show that the deformed generators satisfy the Serre relations. It is not clear that Yangian algebra will be satisfied, and it is very possible that this is a subtle issue. Note that some evidence in favour of this point of view was already found in a study of the perturbative Yangian [63]. As a point of comparison, at strong coupling the scattering amplitudes are described by open strings in the AdS geometry which, being essentially a coset sigma model, are known to possess an infinite family of non-local charges [7]. While the full quantum algebra of these charges is not currently known we point out that even at the classical level there is likely an inherent ambiguity in their algebra. That this is so was pointed out for the Yangian symmetries of the non-linear sigma model by [65] and discussed by e.g. [66]. Essentially, due to the non-ultralocal terms in the current algebra, one must define a regularisation for the charges as integrals over densities, taking particular care with the end points, and different regularisations can lead to different terms on the r.h.s. of (6.3).

7 Propagator Prescriptions

In this section we comment on the exact definition of the amplitudes we have analysed. This will allow us to clarify the relationship to a different proposal [43] of how to deform the symmetry generators at loop level. As it appears to be different already at tree level, we shall start there and later continue at loops.

For simplicity let us consider a generator G which has only deformations of the type $1 \to 2$ at tree level. In practice this can be one of the two superconformal boost generators S or $\bar{\mathrm{S}}$, but not the conformal boost K.

7.1 Tree Level

In [35] a deformation of the free representation was proposed in order to make all tree-level amplitudes exactly invariant. In [43] it was subsequently shown that additional terms are needed for exact invariance of the tree-level S-matrix. Here we shall illustrate the differences between the two proposals and show that they are in fact compatible.

In [35] it was shown that the amplitude generating functional $\mathcal{A}_{\mathrm{P}}$ is annihilated by the deformed representation of the generator $\mathrm{G} = \mathrm{G}_{1\to1} + \mathrm{G}_{1\to2}$ (see Fig. 13)

$$(\mathrm{G}_{1\to1} + \mathrm{G}_{1\to2})\mathcal{A}_{\mathrm{P}} = 0. \tag{7.1}$$

In [43] it was shown that additional deformations are needed, namely $\mathrm{G} = \mathrm{G}_{1\to1} + \mathrm{G}_{1\to2} + \mathrm{G}_{2\to1} + \mathrm{G}_{3\to0}$. The deformations $\mathrm{G}_{2\to1}$ and $\mathrm{G}_{3\to0}$ are almost the same as $\mathrm{G}_{1\to2}$, but they have a different distribution of in and out legs. Moreover, the connected amplitude $\mathcal{A}_{i\epsilon}$ itself is not invariant, but only its exponential

$$(\mathrm{G}_{1\to1} + \mathrm{G}_{1\to2} + \mathrm{G}_{2\to1} + \mathrm{G}_{3\to0}) \exp(i\mathcal{A}_{i\epsilon}) = 0. \tag{7.2}$$

While $\mathcal{A}_{i\epsilon}$ is a connected amplitude, the expansion of the exponential yields disconnected graphs. The additional generators $\mathrm{G}_{2\to1}$ and $\mathrm{G}_{3\to0}$ can now connect two or three subgraphs into a single component. The invariance equation is depicted in Fig. 14.

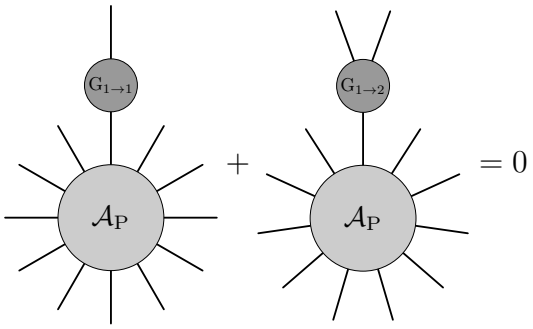

Figure 13: Invariance condition for tree amplitudes with principal part prescription.

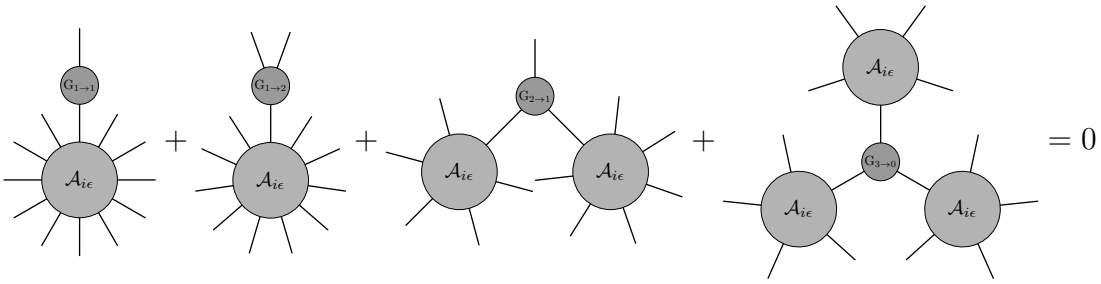

Figure 14: Invariance condition for tree amplitudes with $i\epsilon$ prescription.

It appears that the two invariance equations differ by terms which are non-zero in general and thus only one of them could be true. However, one has to be careful about the precise definition of the amplitudes considered in each case. The first equation (7.1) assumes a principal value prescription for all internal propagators of the tree amplitude $\mathcal{A}_{\rm P}$. Conversely, the second equation (7.2) requires use of an $i\epsilon$ prescription for the internal propagators of $\mathcal{A}_{i\epsilon}$. Indeed, this minor change is what accounts for the difference in the two equations.

Consider, e.g., an amplitude with 7 legs: It has poles corresponding to an internal propagator going on shell. The residue is given by the product of two subamplitudes with 4 and 5 legs, respectively. Now, the difference between a principal value and an $i\epsilon$ prescription is given by the same residue supported by a delta function forcing the internal particle on shell

$$\frac{1}{p^2 \mp i\epsilon} = \frac{1}{p^2} \pm i\pi\delta(p^2). \tag{7.3}$$

Consequently there is the relation $A_{i\epsilon,7} = A_{{\rm P},7} + iA_4\text{—}A_5$ which is illustrated in Fig. 15. The point is that the $A_{{\rm P},7}$ is exactly annihilated by $G_{1\to1} + G_{1\to2}$, but $A_4\text{—}A_5$ requires an extra contribution from $G_{2\to1}$ acting on $\frac{i}{2}(A_4)^2$, i.e. the third term in Fig. 14.

How do the extra anomalies in $\mathcal{A}_{i\epsilon}$ arise in practice? When the free representation $G_{1\to1}$ acts on a rational function, there can be delta-function contributions localised at the poles. Collinear singularities yield such anomalies which are subsequently cancelled by $G_{1\to2}$. Secondly, there are multi-particle singularities, but these do not cause anomalies, whether they are evaluated in principal value or in $i\epsilon$ prescriptions. The third type of singularity can cause anomalies, but it is

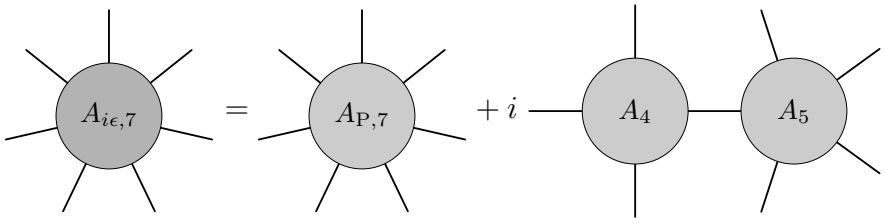

Figure 15: Reduction of tree amplitude with $i\epsilon$ prescription to combinations of amplitudes with principal value prescription.

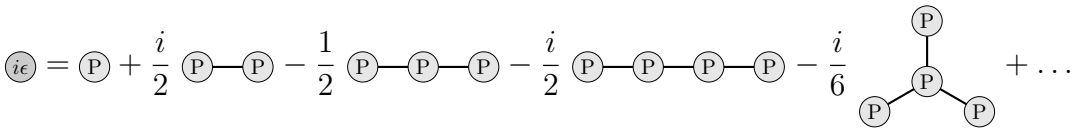

Figure 16: Relation between amplitudes with $i\epsilon$ prescription and amplitudes with principal part prescription at tree level.

spurious and cancels in the complete amplitude because the residues cancel. Almost! In the $i\epsilon$ prescription there are some left-over terms with delta function support. These are the anomalies to be cancelled by $\mathrm{G}_{2\to1}$ and $\mathrm{G}_{3\to0}$ acting on disconnected amplitudes.

It turns out that the two invariance equations are perfectly compatible, and they have the same physical and mathematical implications. Is one of the two preferable over the other? One the one hand, the amplitude $\mathcal{A}_{i\epsilon}$ with $i\epsilon$ prescription may appear to be a more natural and more physical object than the amplitude $\mathcal{A}_{\mathrm{P}}$ with principal value prescription. On the other hand, the symmetry relations for $\mathcal{A}_{\mathrm{P}}$ are simpler.

In the remainder of this section we will explore the formal relationship between the various quantities in order to gain a clearer understanding of amplitudes and their invariance conditions. Using the above relationship (7.3) between propagators it is clear that amplitudes with $i\epsilon$ prescription can be expanded in terms on-shell connections of amplitudes with principal part prescription, see Fig. 16. The prefactors are the natural symmetry factors associated to the graphs. Note that we have terminated the expansion at four powers of $\mathcal{A}_{\mathrm{P}}$ and at tree level. This relation can be formalised as follows

$$\mathcal{S}[J] = \exp\big(i\mathcal{A}_{i\epsilon}[J]\big) = \exp(\hat{C})\exp\big(i\mathcal{A}_{\mathrm{P}}[J]\big), \qquad \hat{C} = \frac{1}{4}\int d^{4|4}\Lambda\, \check{J}^a(\Lambda)\check{J}^a(\bar{\Lambda}). \tag{7.4}$$

In words it says that the S-matrix is given by a collection of arbitrarily many amplitudes $\mathcal{A}_{i\epsilon}$. Equivalently, it is given by a collection of arbitrarily many amplitudes $\mathcal{A}_{\mathrm{P}}$ with arbitrarily many on-shell connections $\hat{C}$ between the legs.

We start with the statement of superconformal symmetry at tree level [35][11]

$$(\mathrm{G}_{1\to1} + \mathrm{G}_{1\to2})\mathcal{A}_{\mathrm{P}}[J] = 0. \tag{7.5}$$

[11]Note that $\mathcal{A}_{\mathrm{P}}[J]$ uses principal part propagators. As loop corrections require at least one on-shell propagator, $\mathcal{A}_{\mathrm{P}}[J]$ terminates at tree level. Thus the following argument (which is analogous to the one used in [43]) *formally* applies to all loops when neglecting effects of regularisation.

Now we would like to transform this equation to a statement for $\mathcal{S}$. The generators $\mathrm{G}_{1\to1}$ and $\mathrm{G}_{1\to2}$ are linear in derivatives and therefore $\exp(i\mathcal{A}_{\mathrm{P}}) = \exp(-\hat{C})\mathcal{S}$ is equally invariant. We conclude that $\mathcal{S}$ is invariant under the same generator conjugated by $\exp(\hat{C})$

$$\exp(\hat{C})\big(\mathrm{G}_{1\to1} + \mathrm{G}_{1\to2}\big)\exp(-\hat{C})\,\mathcal{S}[J] = 0. \tag{7.6}$$

This is precisely the claim of [43]. Namely, one can easily confirm that the free generator is invariant under conjugation

$$\mathrm{G}_{1\to1} = \exp(\hat{C})\mathrm{G}_{1\to1}\exp(-\hat{C}). \tag{7.7}$$

This translates to the statement that on-shell contractions respect free superconformal symmetry. Furthermore the deformations obey

$$\mathrm{G}_{1\to2} + \mathrm{G}_{2\to1} + \mathrm{G}_{3\to0} = \exp(\hat{C})\mathrm{G}_{1\to2}\exp(-\hat{C}). \tag{7.8}$$

In other words $\mathrm{G}_{2\to1} = [\hat{C}, \mathrm{G}_{1\to2}]$ and $\mathrm{G}_{3\to0} = \frac{1}{2}[\hat{C}, \mathrm{G}_{2\to1}]$ which is essentially how the additional deformations were derived in [43]. Starting with $\mathrm{G}_{1\to2}$ in (2.21) we find in agreement with [43]

$$\begin{aligned}
\mathrm{G}_{1\to2} &= \frac{1}{2}\int (d^{4|4}\Lambda)^3\ \mathrm{sign}(E_1E_2)\, G^{abc}_{123} J^a_1 J^b_2 \check{J}^c_3,\\
\mathrm{G}_{2\to1} &= \big[\hat{C}, \mathrm{G}_{1\to2}\big] = \frac{1}{2}\int (d^{4|4}\Lambda)^3\ \mathrm{sign}(E_1E_2)\, G^{abc}_{123} J^a_1 \check{J}^b_2 \check{J}^c_3\\
&= -\frac{1}{2}\int (d^{4|4}\Lambda)^3\, \theta(E_2E_3)\, G^{abc}_{123} J^a_1 \check{J}^b_2 \check{J}^c_3,\\
\mathrm{G}_{3\to0} &= \tfrac{1}{2}[\hat{C}, \mathrm{G}_{2\to1}] = -\frac{1}{8}\int (d^{4|4}\Lambda)^3\, \theta(E_2E_3)\, G^{abc}_{123} \check{J}^a_1 \check{J}^b_2 \check{J}^c_3\\
&= -\frac{1}{24}\int (d^{4|4}\Lambda)^3\, G^{abc}_{123} \check{J}^a_1 \check{J}^b_2 \check{J}^c_3.
\end{aligned} \tag{7.9}$$

The transformations between the sign and step factors makes use of permutation symmetries and momentum conservation. Thus the two proposals for superconformal symmetry agree.

Let us conclude with some remarks. One may wonder about the different structures of $\mathrm{G}_{1\to2}$, $\mathrm{G}_{2\to1}$ and $\mathrm{G}_{3\to0}$ concerning the signs of the particle energies [43]. For instance, all signatures compatible with energy-momentum conservation are permitted in $\mathrm{G}_{1\to2}$ and $\mathrm{G}_{3\to0}$. The difference is that $\mathrm{G}_{1\to2}$ makes explicit reference to the sign of energies while $\mathrm{G}_{3\to0}$ does not. Conversely, $\mathrm{G}_{2\to1}$ requires the two in-going particles to have equal signs. In a canonical quantisation framework this distinction between $\mathrm{G}_{1\to2}$ and $\mathrm{G}_{2\to1}$ actually makes sense. In such a picture, positive energy states are represented by creation operators and negative energy states by annihilation operators. The deformed symmetry generator would thus consist of two creation and one annihilation operator or vice versa. Now invariance of an operator means that it commutes with a symmetry generator. Commuting the deformed generator with some operator can connect the two objects by one (on-shell) Wick contraction (corresponding to $\mathrm{G}_{1\to2}$) or by two (corresponding to $\mathrm{G}_{2\to1}$).

In the case of two contractions, the energies automatically align in agreement with the structure of $\mathrm{G}_{2\to1}$.[12]

We would also like to remark that the structures $\mathrm{G}_{1\to2}$ and $\mathrm{G}_{2\to1}$ in (7.8) are reminiscent of the brackets and cobrackets in a classical double of a Lie bialgebra (cf. [67] for some explicit expressions). In this analogy, positive and negative energy states correspond to the two copies of the original bialgebra. Brackets are defined between elements of both copies, while the cobracket remains confined to each subalgebra. The deeper meaning of this observation remains obscure, but it may help to obtain a better understanding of the deformation.

7.2 One Loop

Having convinced ourselves of the equivalence of the two proposals [35] and [43] at tree level, we should now compare our one-loop results to [43].

The most obvious difference is that while [43] calculate the action of generators on amplitudes in the final analysis they are only concerned with the invariance of so-called IR-finite observables such as inclusive cross-sections, see for example [28]. These are derived from cross sections made from scattering amplitudes. Conversely, we formulate an invariance condition for the scattering amplitudes themselves. Therefore anomaly contributions due to the integration measure could be discarded in [43] while we have to take them into account. In particular, (7.7) does not hold strictly which eventually leads to the deformation $\mathrm{G}^{(1)}_{2\to2}$ derived in Sec. 3.2.

One could now wonder whether our remaining deformations $\mathrm{G}^{(1)}_{2\to k}$, $k\geq3$, match with the contribution from $\mathrm{G}_{2\to1}$ in the framework of [43]. This expectation is reasonable because the contributions have a similar structure, and in the case of MHV amplitudes they actually yield coincident contributions. There is however one conceptual problem with applying the generator $\mathrm{G}_{2\to1}$ to some tree amplitude $\mathcal{A}_{i\epsilon,n+1}$: The generator forces two legs of the amplitude to be strictly collinear. On the one hand, the amplitude diverges near collinear configurations. On the other hand, there are cancellations in the numerator which compensate the divergence. We give an explicit example in App. E. In practice the amplitude $\mathcal{A}_{i\epsilon,n+1}$ is split up into two subamplitudes $\mathcal{A}_{k+1}$ and $\mathcal{A}_{n-k+2}$ in [43]. The split is performed using the CSW rules, [56], which require one to go off shell or to violate momentum conservation by a tiny amount. Then the calculation can be completed in terms of the subamplitudes and one ends up with finite contributions to $\mathrm{G}_{2\to1}\mathcal{A}_{i\epsilon,n+1}$ (unless one of the two subamplitudes has only three legs).

In our approach we substitute the deformation $\mathrm{G}_{2\to1}$ by the set of generators $\mathrm{G}^{(1)}_{2\to k}$. While $\mathrm{G}_{2\to1}$ is an extremely simple generator, its action involves computing loop integrals which are complicated and potentially divergent. For our generators we have essentially already performed the regularised loop integrals. Consequently the generators are somewhat more complicated, but their action on amplitudes is straight-forward, and the IR divergences are manifest.

[12] One may wonder what about $\mathrm{G}_{3\to0}$ which apparently has no representation in a canonical quantisation framework. Consequently the S-matrix operator does not seem to commute with the deformed generator. Nevertheless there appears to exist a slightly deformed version of the S-matrix operator which does commute and which is unitary.

8 Conclusions

In this work we have analysed, in the context of $\mathcal{N} = 4$ SYM, the fate of the superconformal symmetries of generic scattering amplitudes and of the Yangian symmetries of planar amplitudes once radiative corrections are taken into account. Our central message is that the IR singularities pose no serious threat, as the symmetry generators can be deformed in such a fashion as to render the amplitudes, defined in a dimensional regularisation scheme, invariant to one-loop order. The key input was the inclusion of the deformation of the tree-level generators due to collinear terms which give rise to leg changing effects. Here it proved very advantageous to represent these terms arising from the holomorphic anomaly as an on-shell triangle graph. Acting with the tree level generators on the branch cuts of the one-loop scattering amplitudes then led to a natural one-loop deformation of the representation which moreover could be lifted off the cut by virtue of the cut-constructibility of the theory. Specialising to the planar theory, an analogous result was obtained for the level-one Yangian generator of momentum, which, along with the dilatation generator, could even be deformed to the all-loop level.

Importantly, we were able to represent a universal form of the deformation of any generator by putting the on-shell triangle anomaly inside the loop in all possible ways. This construction turned out to localise all integrals, so that effectively no loop integration had to be performed and only tree-level data, consisting of vertices and amplitudes, entered the construction.

It would be interesting to repeat the study of deformations with an alternative regulator. In particular the recently introduced Higgs IR regulator of [21] comes to mind, where the all-loop deformation of the dilatation and dual conformal generator are simple and have a natural five-dimensional holographic interpretation.

The most pressing question left open in our work is the closure of the algebra of the one-loop deformed generators. While it is obvious that this will be the case for the super-Poincaré and R-symmetry generators (by virtue of the regularisation procedure) this is not at all so for the super-conformal part. Acting on the amplitudes the algebra is of course trivially obeyed, but one would like to know if it closes for the generators. Note that this will depend on the functional space on which the generators are allowed to act: For instance, even at tree level the algebra closes only on gauge-invariant functions [35]. At one loop, extra constraints may become necessary, such as, perhaps, (super) Poincaré-invariance or cyclicity. The question of the algebra is also intimately connected with the existence of a deformed Yangian symmetry at loop level, as the algebra along with the explicit form of the level-one momentum generator gives rise to all further level-one generators. To establish the complete Yangian symmetry a check of the super-Serre relations for the deformed generators is also needed. However, it is possible that while one can find an infinite tower of charges that annihilate amplitudes the issue regarding their algebra may be subtle as is the case for non-ultralocal two-dimensional integrable field theories. We intend to address these questions in the future.

Recently, remarkable formulae have been proposed based on an integral over a certain Graßmannian with manifest superconformal invariance, which reproduce the $\mathcal{N} = 4$ SYM tree-level amplitudes and even integral coefficients of higher-loop integral topologies [68] thereby respecting Yangian symmetry [69]. It would be interesting to clarify the relation to our approach.

A further important question is whether the loop-deformed symmetries are constructive in the sense of determining the loop-level amplitudes completely. This was argued to be the case for the tree-level amplitudes upon the incorporation of the collinear terms in [36, 35]. We would certainly expect this to remain true at the loop level. One immediate question is what can the symmetries tell us about the remainder function, that is the difference between the ABDK/BDS ansatz of [70] and the true finite part of the amplitude, starting at the two-loop order? It is known that the naive dual conformal symmetries alone are insufficient to fix this function and, while there has already been significant numerical and analytic work [71], determining its exact form remains a challenging open problem.

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A Anomaly as a Three-Vertex

In this appendix we derive the form the anomaly vertex for superconformal boosts by acting on a 3-vertex by a superconformal boost generator. The result is expected to reproduce the superconformal boost deformation found in [35].

A.1 Three-Vertices

Consider the MHV 3-vertex

$$A_3 = \frac{\delta^4(P)\,\delta^8(Q)}{\langle 1,2\rangle\langle 2,3\rangle\langle 3,1\rangle}\,. \tag{A.1}$$

Momentum conservation does not allow for a proper phase space in $(3,1)$ signature. Let us therefore continue in $(2,2)$ signature where the full phase space exists. In this form, however, the anomaly cannot be seen easily, and we first recast the vertex into a different form. The two-component spinors λ and $\tilde\lambda$ are now independent and real. The amplitude can be represented in an alternative form which will be useful for further considerations: We note identities which allow to express spinors λ or $\tilde\lambda$ in a given basis of two different spinors μ, μ' or $\tilde\mu, \tilde\mu'$, respectively

$$\begin{aligned} 1 &= \big|\langle \mu, \mu'\rangle\big| \int dx\, dx'\, \delta^2(\lambda - x\mu - x'\mu'), \\ 1 &= \big|[\tilde\mu, \tilde\mu']\big| \int d\tilde x\, d\tilde x'\, \delta^2(\tilde\lambda - \tilde x\tilde\mu - \tilde x'\tilde\mu'). \end{aligned} \tag{A.2}$$

We use the latter to express $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ in a basis of $\tilde{\lambda}_3$ and some reference spinor $\tilde{\mu}$.

$$A_3 = \frac{[3,\tilde{\mu}]^2}{\langle 1,2\rangle\langle 2,3\rangle\langle 3,1\rangle}\int d\tilde{x}_1\, d\tilde{x}_1'\, d\tilde{x}_2\, d\tilde{x}_2'\, \delta^2(\tilde{\lambda}_1 - \tilde{x}_1\tilde{\lambda}_3 - \tilde{x}_1'\tilde{\mu})\, \delta^2(\tilde{\lambda}_2 - \tilde{x}_2\tilde{\lambda}_3 - \tilde{x}_2'\tilde{\mu}) \\ \cdot\, \delta^4\big((\tilde{x}_1\lambda_1 + \tilde{x}_2\lambda_2 + \lambda_3)\tilde{\lambda}_3 + (\tilde{x}_1'\lambda_1 + \tilde{x}_2'\lambda_2)\tilde{\mu}\big)\, \delta^8\big(\lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3\big). \tag{A.3}$$

A further identity converts a spinorial delta function into a product of two regular delta functions

$$\delta^2(a\lambda + b\mu) = \frac{\delta(a)\,\delta(b)}{\big|\langle\lambda,\mu\rangle\big|}\,, \qquad \delta^2(a\tilde{\lambda} + b\tilde{\mu}) = \frac{\delta(a)\,\delta(b)}{\big|[\tilde{\lambda},\tilde{\mu}]\big|}\,. \tag{A.4}$$

The momentum delta function implies $\lambda_3 = -\tilde{x}_1\lambda_1 - \tilde{x}_2\lambda_2$ which we can use to convert the supermomentum delta function to $\delta^8(\lambda_1(\eta_1 - \tilde{x}_1\eta_3) + \lambda_2(\eta_2 - \tilde{x}_2\eta_3))$. Again we can use the identity (A.4), but now the argument is fermionic and the measure factor $|\langle\lambda_1,\lambda_2\rangle|$ must be in the numerator instead of the denominator. We end up with A_3 given through a set of delta functions and a signum function

$$A_3 = \operatorname{sign}\big(\langle 1,2\rangle\big)\int \frac{d\tilde{x}_1}{\tilde{x}_1}\,\frac{d\tilde{x}_2}{\tilde{x}_2}\,\delta^2(\tilde{x}_1\lambda_1 + \tilde{x}_2\lambda_2 + \lambda_3) \\ \cdot\, \delta^2(\tilde{\lambda}_1 - \tilde{x}_1\tilde{\lambda}_3)\,\delta^2(\tilde{\lambda}_2 - \tilde{x}_2\tilde{\lambda}_3)\,\delta^4(\eta_1 - \tilde{x}_1\eta_3)\,\delta^4(\eta_2 - \tilde{x}_2\eta_3). \tag{A.5}$$

Similarly the conjugate MHV 3-vertex reads

$$\bar{A}_3 = \operatorname{sign}\big([1,2]\big)\int \frac{dx_1}{x_1}\,\frac{dx_2}{x_2}\,\delta^2(\lambda_1 - x_1\lambda_3)\,\delta^2(\lambda_2 - x_2\lambda_3) \\ \cdot\, \delta^2(x_1\tilde{\lambda}_1 + x_2\tilde{\lambda}_2 + \tilde{\lambda}_3)\delta^4(x_1\eta_1 + x_2\eta_2 + \eta_3). \tag{A.6}$$

A.2 Anomaly Three-Vertices

The collection of delta functions is superconformal and hence it is annihilated by the superconformal boosts S and $\bar{\mathrm{S}}$. Only the signum factor violates invariance under the conjugate superconformal boost $\bar{\mathrm{S}}^{\mathrm{free}}$. According to the identity $d\,\mathrm{sign}(x) = 2dx\,\delta(x)$ the derivative in the generator converts it to a delta function forcing $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ to be collinear. Expressing $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ in a basis of $\tilde{\lambda}_3$ and a reference spinor $\tilde{\mu}$ we arrive at

$$(\bar{\mathrm{S}}^{\mathrm{free}})^B_{\dot{\alpha}}\bar{A}_3 = 2\varepsilon_{\dot{\alpha}\dot{\gamma}}(\tilde{\lambda}_2^{\dot{\gamma}}\eta_1^B - \tilde{\lambda}_1^{\dot{\gamma}}\eta_2^B)\int \frac{dx_1}{x_1}\,\frac{dx_2}{x_2}\,d\tilde{x}_1\,d\tilde{x}_2\,\delta(1 + x_1\tilde{x}_1 + x_2\tilde{x}_2) \\ \cdot\, \delta^2(\lambda_1 - x_1\lambda_3)\,\delta^2(\lambda_2 - x_2\lambda_3) \\ \cdot\, \delta^2(\tilde{\lambda}_1 - \tilde{x}_1\tilde{\lambda}_3)\,\delta^2(\tilde{\lambda}_2 - \tilde{x}_2\tilde{\lambda}_3)\,\delta^4(x_1\eta_1 + x_2\eta_2 + \eta_3). \tag{A.7}$$

We can recast this expression into a different form which may be more convenient for some purposes. To that end we insert $1 = \int d^{4|4}\Lambda'\delta^2(\lambda')\delta^2(\tilde{\lambda}')\delta^4(\eta' - \tilde{x}_2\eta_1 + \tilde{x}_1\eta_2)$ and use an identity which holds when $1 + x_1\tilde{x}_1 + x_2\tilde{x}_2 = 0$

$$\delta^4(\eta' - \tilde{x}_2\eta_1 + \tilde{x}_1\eta_2)\,\delta^4(x_1\eta_1 + x_2\eta_2 + \eta_3) = \delta^4(\eta_1 - \tilde{x}_1\eta_3 + x_2\eta')\,\delta^4(\eta_2 - \tilde{x}_2\eta_3 - x_1\eta'). \tag{A.8}$$

Next we supplement $d^4\eta'$ by $d^4\lambda' = d^2\lambda' d^2\tilde{\lambda}'$ and the corresponding delta function $\delta^4(\lambda') = \delta^2(\lambda')\delta^2(\tilde{\lambda}')$ to $d^{4|4}\Lambda'\delta^4(\lambda')$. Subsequently we can add terms $\lambda', \tilde{\lambda}'$ to the delta function to make them appear more symmetric

$$\begin{aligned}(\bar{\mathrm{S}}^{\mathrm{free}})^B_{\dot\alpha}\bar{A}_3 = 2\int d^{4|4}\Lambda'\,\delta^4(\lambda')\,\varepsilon_{\dot\alpha\dot\gamma}\tilde{\lambda}_3^{\dot\gamma}\eta'^B \int \frac{dx_1}{x_1}\,\frac{dx_2}{x_2}\,d\tilde{x}_1\,d\tilde{x}_2\,\delta(1+x_1\tilde{x}_1+x_2\tilde{x}_2)\\ \cdot\,\delta^2(\lambda_1 - x_1\lambda_3 + \tilde{x}_2\lambda')\,\delta^2(\lambda_2 - x_2\lambda_3 - \tilde{x}_1\lambda')\\ \cdot\,\delta^2(\tilde{\lambda}_1 - \tilde{x}_1\tilde{\lambda}_3 + x_2\tilde{\lambda}')\,\delta^2(\tilde{\lambda}_2 - \tilde{x}_2\tilde{\lambda}_3 - x_1\tilde{\lambda}')\\ \cdot\,\delta^4(\eta_1 - \tilde{x}_1\eta_3 + x_2\eta')\,\delta^4(\eta_2 - \tilde{x}_2\eta_3 - x_1\eta').\end{aligned} \tag{A.9}$$

In order to convert the expression to the physical $(3,1)$ spacetime signature we perform a change of variables such that $\tilde{x}_{1,2} = \pm\bar{x}_{1,2}$. Here we must distinguish three different cases depending on the energy signatures of the particles: $(\pm\pm\mp)$, $(\mp\pm\pm)$ and $(\pm\mp\pm)$. They are achieved by the substitutions ($0 \le \alpha, \beta \le \frac{1}{2}\pi$, $0 \le \varphi, \vartheta < 2\pi$)

$$\begin{aligned}
&x_1 = e^{-i\varphi}\sin\alpha, && \tilde{x}_1 = -e^{i\varphi}\sin\alpha, && x_2 = e^{-i\vartheta}\cos\beta, && \tilde{x}_2 = -e^{i\vartheta}\cos\beta,\\
&x_1 = e^{-i\varphi+i\vartheta}\tan\alpha, && \tilde{x}_1 = +e^{i\varphi-i\vartheta}\tan\alpha, && x_2 = e^{i\vartheta}\sec\beta, && \tilde{x}_2 = -e^{-i\vartheta}\sec\beta,\\
&x_1 = e^{i\varphi}\sec\alpha, && \tilde{x}_1 = -e^{-i\varphi}\sec\alpha, && x_2 = e^{-i\vartheta+i\varphi}\tan\beta, && \tilde{x}_2 = +e^{i\vartheta-i\varphi}\tan\beta.
\end{aligned} \tag{A.10}$$

The delta function for the x's leads to $\beta = \alpha$. We combine the delta functions $\delta^{4|4}(\Lambda) = \delta^4(\lambda)\delta^4(\eta) = \delta^2(\lambda)\delta^2(\tilde{\lambda})\delta^4(\eta)$ and multiply with a suitable prefactor of $-1/2$ to obtain the anomaly vertex

$$\begin{aligned}(\bar{S}_3)^B_{\dot\alpha} &= -\tfrac{1}{2}(\bar{\mathrm{S}}^{\mathrm{free}})^B_{\dot\alpha}\bar{A}_3\\ &= -2\int d^{4|4}\Lambda'\,\delta^4(\lambda')\,\varepsilon_{\dot\alpha\dot\gamma}\tilde{\lambda}_3^{\dot\gamma}\eta'^B\int d\alpha\,d\varphi\,d\vartheta\,e^{i\varphi+i\vartheta}\\ &\qquad\cdot\,\delta^{4|4}(e^{-i\varphi}\bar{\Lambda}_3\sin\alpha + e^{i\vartheta}\bar{\Lambda}'\cos\alpha - \Lambda_1)\\ &\qquad\cdot\,\delta^{4|4}(e^{-i\vartheta}\bar{\Lambda}_3\cos\alpha - e^{i\varphi}\bar{\Lambda}'\sin\alpha - \Lambda_2)\\ &\quad + 2\text{ cyclic images.}\end{aligned} \tag{A.11}$$

An analogous construction leads to the anomaly vertex for the superconformal boost

$$\begin{aligned}(\mathrm{S}^{\mathrm{free}})_{\alpha B}A_3 = 2\varepsilon_{\alpha\gamma}(\lambda_2^\gamma\partial_{1,B} - \lambda_1^\gamma\partial_{2,B})\int\frac{d\tilde{x}_1}{\tilde{x}_1}\,\frac{d\tilde{x}_2}{\tilde{x}_2}\,dx_1\,dx_2\,\delta(1+x_1\tilde{x}_1+x_2\tilde{x}_2)\\ \cdot\,\delta^2(\lambda_1 - x_1\lambda_3)\,\delta^2(\lambda_2 - x_2\lambda_3)\\ \cdot\,\delta^2(\tilde{\lambda}_1 - \tilde{x}_1\tilde{\lambda}_3)\,\delta^2(\tilde{\lambda}_2 - \tilde{x}_2\tilde{\lambda}_3)\\ \cdot\,\delta^4(\eta_1 - \tilde{x}_1\eta_3)\,\delta^4(\eta_2 - \tilde{x}_2\eta_3).\end{aligned} \tag{A.12}$$

Here we insert $1 = \int d^{4|4}\Lambda'\delta^{4|4}(\Lambda')$, expand some of the delta function by terms in Λ' and finally

make the above replacements to convert to $(3,1)$ signature

$$\begin{aligned}(S_3)_{\alpha B} &= -\tfrac{1}{2}(\mathrm{S}^{\mathrm{free}})_{\alpha B} A_3 \\ &= -2\int d^{4|4}\Lambda'\,\delta^{4|4}(\Lambda')\,\varepsilon_{\alpha\gamma}\lambda_3^\gamma \partial'_B \int d\alpha\, d\varphi\, d\vartheta\, e^{-i\varphi - i\vartheta} \\ &\qquad \cdot \delta^{4|4}(e^{-i\varphi}\bar\Lambda_3 \sin\alpha + e^{i\vartheta}\bar\Lambda' \cos\alpha - \Lambda_1) \\ &\qquad \cdot \delta^{4|4}(e^{-i\vartheta}\bar\Lambda_3 \cos\alpha - e^{i\varphi}\bar\Lambda' \sin\alpha - \Lambda_2) \\ &\quad + 2 \text{ cyclic images.}\end{aligned} \tag{A.13}$$

A.3 Tree-Level Superconformal Anomaly

The above expressions agree (up to a conventional overall factor and phase redefinitions) with the superconformal boost deformations found in [35]. Let us repeat the calculation for the colour-ordered planar MHV amplitudes in order to fix the overall factors.

Consider the holomorphic anomaly for spinor variables (2.18). First we resolve the delta function in terms of an explicit relation between the spinors using an identity analogous to (A.2)

$$\delta^2\big(\langle\lambda,\mu\rangle\big) = \int_0^\infty 2r\,dr \int_0^{2\pi} d\varphi\,\delta^2(\lambda - re^{i\varphi}\mu)\big(\delta^2(\tilde\lambda - re^{-i\varphi}\tilde\mu) + \delta^2(\tilde\lambda + re^{-i\varphi}\tilde\mu)\big). \tag{A.14}$$

Now we can compute the anomaly of MHV amplitudes

$$\begin{aligned}(\bar{\mathrm{S}}^{\mathrm{free}})^B_{\dot\alpha} A_n = 4\pi\varepsilon_{\dot\alpha\dot\gamma}\sum_{k=1}^n \int_0^\infty r\,dr \int_0^{2\pi} d\varphi\, \frac{\big(\tilde\lambda^{\dot\gamma}_{k+1}\eta^B_k - \tilde\lambda^{\dot\gamma}_k\eta^B_{k+1}\big)\,\delta^4(P)\,\delta^8(Q)}{\langle 12\rangle\ldots\langle k,k+1\rangle^0\ldots\langle n1\rangle} \\ \cdot \delta^2(\lambda_k - re^{i\varphi}\lambda_{k+1})\,\big(\delta^2(\tilde\lambda_k - re^{-i\varphi}\tilde\lambda_{k+1}) - \delta^2(\tilde\lambda_k + re^{-i\varphi}\tilde\lambda_{k+1})\big).\end{aligned} \tag{A.15}$$

We compare this to the deformation $\bar{\mathrm{S}}_{1\to 2}$ of the representation which consists in inserting the anomaly three-vertex into the amplitude

$$\begin{aligned}(\bar{\mathrm{S}}_{1\to 2})^B_{\dot\alpha} A_{n-1} = \sum_{k=1}^n \int d^{4|4}\Lambda_{\bar{\mathrm{a}}}\ \mathrm{sign}(E_k E_{k+1}) \\ \cdot \bar S_3(k,k+1,\bar{\mathrm{a}})^B_{\dot\alpha} A_{n-1}(1,\ldots,k-1,\mathrm{a},k+2,\ldots,n).\end{aligned} \tag{A.16}$$

For the anomaly vertex we use the above result (A.11) where the prefactor was already chosen

correctly. The deformation of the representation yields the contribution

$$
\begin{aligned}
(\bar{\mathrm{S}}_{1\to 2})^B_{\dot\alpha} A_{n-1} = 4\pi \sum_{k=1}^{n} \varepsilon_{\dot\alpha\dot\gamma} \frac{\big(\tilde\lambda^{\dot\gamma}_{k+1}\eta^B_k - \tilde\lambda^{\dot\gamma}_k \eta^B_{k+1}\big)\,\delta^4(P)\,\delta^8(Q)}{\langle 12\rangle \dots \langle k,k+1\rangle^0 \dots \langle n1\rangle} \\
\cdot \Big[- \int_0^\infty r\,dr\,d\varphi\,\delta^2(\lambda_k - re^{i\varphi}\lambda_{k+1})\,\delta^2(\tilde\lambda_k - re^{-i\varphi}\tilde\lambda_{k+1}) \\
+ \int_0^1 r\,dr\,d\varphi\,\delta^2(\lambda_k - re^{i\varphi}\lambda_{k+1})\,\delta^2(\tilde\lambda_k + re^{-i\varphi}\tilde\lambda_{k+1}) \\
+ \int_1^\infty r\,dr\,d\varphi\,\delta^2(\lambda_k - re^{i\varphi}\lambda_{k+1})\,\delta^2(\tilde\lambda_k + re^{-i\varphi}\tilde\lambda_{k+1}) \Big],
\end{aligned}
\tag{A.17}
$$

where each of the three term originates from the above three components of (A.11). This shows that the prefactors for the three terms have to be chosen as in (A.11) in order for the anomaly to be cancelled.

B One-Loop MHV Amplitude

In this appendix we collect results and identities for the (planar) one-loop MHV amplitude and the underlying "2-mass easy" box integrals. The n-point MHV amplitude in $\mathcal{N} = 4$ SYM was found in [37] and the derivations of many of these results can be found there.

B.1 Box Integrals

The one-loop correction to any amplitude can be obtained as a linear combination of scalar box integrals

$$
I^{\square} = -i \int \frac{\mu^{2\epsilon} d^{4-2\epsilon}\ell}{(2\pi)^{4-2\epsilon}} \frac{1}{\ell^2 (\ell+p_1)^2 (\ell+p_1+p_2)^2 (\ell-p_4)^2}\,. \tag{B.1}
$$

For MHV amplitudes the only contributions come from special "2-mass easy" box integrals with light-like momentum inflow at two opposite corners, $p_2^2 = p_4^2 = 0$. It makes sense to split the integral $I^{\square}$ up into a dimensionless loop function F and a rational prefactor

$$
I^{\square} = \frac{F}{16\pi^2 \Delta}\,, \qquad \Delta = -\tfrac{1}{2}(st - uv). \tag{B.2}
$$

where the invariants s, t, u, v are defined as

$$
s = (p_1+p_2)^2, \quad t = (p_1+p_4)^2, \quad u = p_1^2, \quad v = p_3^2. \tag{B.3}
$$

In dimensional regularisation the F takes the following form[13]

$$F = -\frac{c_\epsilon}{\epsilon^2}\left(\frac{s}{-\mu^2}\right)^{-\epsilon} - \frac{c_\epsilon}{\epsilon^2}\left(\frac{t}{-\mu^2}\right)^{-\epsilon} + \frac{c_\epsilon}{\epsilon^2}\left(\frac{u}{-\mu^2}\right)^{-\epsilon} + \frac{c_\epsilon}{\epsilon^2}\left(\frac{v}{-\mu^2}\right)^{-\epsilon} + \tfrac{1}{2}\log^2\frac{s}{t}$$
$$+ \mathrm{Li}_2\left(1-\frac{u}{s}\right) + \mathrm{Li}_2\left(1-\frac{u}{t}\right) + \mathrm{Li}_2\left(1-\frac{v}{s}\right) + \mathrm{Li}_2\left(1-\frac{v}{t}\right) - \mathrm{Li}_2\left(1-\frac{uv}{st}\right). \quad \text{(B.4)}$$

It has been normalised such that the coefficients of the resulting Li_2 and $\log^2$ terms are ± 1 and $\pm\frac{1}{2}$, respectively. Here c_ϵ is a frequently occurring function of dimensional regularisation parameter ϵ

$$c_\epsilon = (4\pi)^\epsilon \frac{\Gamma(1+\epsilon)\,\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)} = 1 + \mathcal{O}(\epsilon). \quad \text{(B.5)}$$

The one-mass and massless box functions can be viewed as a special case with $u = 0$ or/and $v = 0$. Here the third or/and fourth terms in (B.4) are singular and the correct prescription in dimensional regularisation is to drop them altogether.

Note that the above expression is not meant to reproduce the physically correct imaginary part in all cases. One would have to pick the applicable Riemann sheet of the function for each physical situation. Here we have written it such that the function is real when all invariants s, t, u, v are negative.

B.2 BCF Construction

The coefficients of the scalar box integrals for any one-loop amplitude in $\mathcal{N} = 4$ SYM are determined through quadruple cuts [40]. Due to the tight supersymmetry constraints on MHV amplitudes, all coefficients must equal the tree-level amplitude. One can thus write

$$A_n^{(1)} = A_n^{(0)}\, M_n^{(1)}. \quad \text{(B.6)}$$

where $M^{(1)}$ is a sum over normalised 2-mass easy scalar box integrals, cf. Fig. 17,[14]

$$M_n^{(1)} = \tfrac{1}{2}\sum_{k=2}^{n-2}\sum_{j=1}^{n} F(t_j^k, t_{j+1}^k, t_{j+1}^{k-1}, t_j^{k+1}). \quad \text{(B.7)}$$

Note that the boundary terms $k = 2$ and $k = n-2$ in the sum a third leg of the box is light-like and the loop function is actually a one-mass box. As described above, in these cases one has to carefully drop singular terms from F in (B.4). Furthermore, many of the individual terms in

[13] Actually all terms — not just the first four, divergent ones — should be proportional to $\mu^{2\epsilon}$. For the finite terms this plays almost no role and hence such minute factors can be safely discarded. The only place where it does matter is in collinear limits.

[14] In this sum every term effectively appears twice, hence a factor of $\frac{1}{2}$. This way of writing the sum has the benefit that no distinction has to be made between even and odd n.

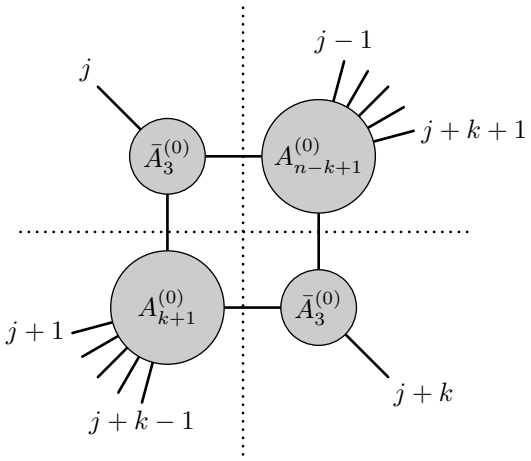

Figure 17: A box integral contribution to the one-loop MHV amplitude

$F^{\square 2\mathrm{m}}$ cancel in the sum $M_n^{(1)}$ for which we obtain the convenient final result

$$M_n^{(1)} = -\sum_{j=1}^{n} \frac{c_\epsilon}{\epsilon^2} \left(\frac{t_j^2}{-\mu^2}\right)^{-\epsilon} + \tfrac{1}{6} n\pi^2 - \tfrac{1}{2} \sum_{k=3}^{n-3} \sum_{j=1}^{n} \mathrm{Li}_2 \left(1 - \frac{t_{j+1}^{k-1} t_j^{k+1}}{t_j^k t_{j+1}^k}\right) - \tfrac{1}{2} \sum_{k=2}^{n-3} \sum_{j=1}^{n} \log^2 \frac{t_j^k}{t_j^{k+1}} + \tfrac{1}{4} \sum_{k=2}^{n-2} \sum_{j=1}^{n} \log^2 \frac{t_j^k}{t_{j+1}^k} \,. \tag{B.8}$$

B.3 Variations

For acting with superconformal symmetries we must take derivatives of loop function $M^{(1)}$ with respect to the external momenta. As these appear only within the Mandelstam invariants t_j^k it suffices to compute the variation w.r.t. them

$$\delta M_n^{(1)} = +\tfrac{1}{2} \sum_{j=1}^{n} \left(\delta \log \frac{t_{j-1}^2 t_{j+1}^2}{\mu^4}\right) \frac{c_\epsilon}{\epsilon} \left(\frac{t_j^2}{-\mu^2}\right)^{-\epsilon} + \tfrac{1}{2} \sum_{k=2}^{n-3} \sum_{j=1}^{n} \left(\delta \log \frac{\Delta_j^k}{\Delta_{j-1}^{k+1}}\right) \log \frac{t_j^{k+1}}{t_j^k} \,. \tag{B.9}$$

Here Δ_j^k is the following combination of invariants which arises from derivatives of the dilogs in (B.4)

$$\Delta_j^k = -\tfrac{1}{2}(t_j^k t_{j+1}^k - t_{j+1}^{k-1} t_j^{k+1}). \tag{B.10}$$

Reducing all invariants to t_{j+1}^{k-1} plus extra terms, this equals

$$\Delta_j^k = (p_k \cdot p_{j+k})(P_{j+1}^{k-1} \cdot P_{j+1}^{k-1}) - 2(p_j \cdot P_{j+1}^{k-1})(p_{j+k} \cdot P_{j+1}^{k-1}). \tag{B.11}$$

In four dimensions one can furthermore use spinor helicity variables to write $p_j \cdot p_k = \frac{1}{2}\langle j,k\rangle[k,j]$. Then Δ_j^k magically factorises into two terms $\Upsilon_j^k = \langle j|P_{j+1}^{k-1}|j+k]$

$$\begin{aligned}\Delta_j^k &= \tfrac{1}{4}\sum_{m,n=j+1}^{j+k-1}((p_k\cdot p_{j+k})(p_m\cdot p_n)-(p_j\cdot p_m)(p_{j+k}\cdot p_n)-(p_j\cdot p_n)(p_{j+k}\cdot p_m))\\ &= -\tfrac{1}{2}\sum_{m,n=j+1}^{j+k-1}\langle j,m\rangle[m,j+k]\langle j+k,n\rangle[n,j]\\ &= -\tfrac{1}{2}\langle j|P_{j+1}^{k-1}|j+k]\langle j+k|P_{j+1}^{k-1}|j] = \tfrac{1}{2}\Upsilon_j^k\Upsilon_{j+k}^{n-k}. \end{aligned} \tag{B.12}$$

After substituting Δ_j^k and using the identities

$$\begin{aligned}\Upsilon_{j-1}^2 &= \langle j-1,j\rangle[j,j+1],\\ \Upsilon_{j+2}^{n-2} &= -\langle j+1,j+2\rangle[j,j+1],\\ t_j^2 &= -\langle j,j+1\rangle[j,j+1]\end{aligned} \tag{B.13}$$

we end up with a convenient form for the variation of the loop function

$$\begin{aligned}\delta M_n^{(1)} = &\sum_{j=1}^{n}\left(\delta\log\frac{\langle j-1,j\rangle[j,j+1]\langle j+1,j+2\rangle}{-\mu^2\langle j,j+1\rangle}\right)\frac{c_\epsilon}{\epsilon}\left(\frac{t_j^2}{-\mu^2}\right)^{-\epsilon}\\ &-\sum_{k=2}^{n-3}\sum_{j=1}^{n}\left(\delta\log\frac{\Upsilon_{j-1}^{k+1}}{\Upsilon_j^k}\right)\log\frac{t_j^{k+1}}{t_j^k}\,.\end{aligned} \tag{B.14}$$

B.4 Collinear Configurations

Finally we would like to address the question what happens when two adjacent legs, say $n-1$ and n, are strictly collinear while evaluating the loop integral $M_n^{(1)}$.[15] Most terms of the sum (B.7) reduce to terms of $M_{n-1}^{(1)}$ when combining the two collinear momenta into one $p_{n-1}+p_n \to p_{n-1}$. This is because the function $M_n^{(1)}$ depends only on ranges of momenta P_j^k in t_j^k. The only exceptions arise when the range begins or ends between the collinear momenta.

Let us therefore analyse more carefully the differences between $M_n^{(1)}$ and $M_{n-1}^{(1)}$. Assume that the collinear momenta obey

$$p_{n-1} \to z p_{n-1}, \quad p_n \to \bar{z} p_{n-1}, \qquad z+\bar{z}=1. \tag{B.15}$$

[15]Note that one has to distinguish between the collinear limit of the loop function and its value when two momenta are collinear. This does not mean that the limit is not smooth, but it apparently does not commute with removing the regulator.

The Mandelstam invariants then reduce according to

$$t_j^k \to \begin{cases} t_j^k & \text{when } j < n-k, \\ z t_j^k + \bar{z} t_j^{k-1} & \text{when } j = n-k, \\ t_j^{k-1} & \text{when } n-k < j < n, \\ \bar{z} t_{j-1}^k + z t_j^{k-1} & \text{when } j = n, \\ t_{j-1}^k & \text{when } n < j, \end{cases} \tag{B.16}$$

To account for these different cases, we should split up the sum over j in (B.7) into the ranges $\{1, \ldots, n-k-2\}$, $\{n-k+1, \ldots, n-2\}$ and treat the four remaining values separately. It turns out that almost all terms combine as follows (see e.g. [72])

$$M_n^{(1)} \to M_{n-1}^{(1)} + F(0, \bar{z} t_{n-1}^2, t_{n-1}^2, 0) + F(z t_{n-2}^2, 0, 0, t_{n-2}^2). \tag{B.17}$$

In combining some terms we made use of a splitting identity for the box function

$$F(s,t,u,v) = F(s, zt + \bar{z}v, \bar{z}s + zu, v) + F(\bar{z}s + zu, t, u, zt + \bar{z}v). \tag{B.18}$$

It follows from two dilog identities ($x = u/s$, $y = v/t$)

$$\begin{aligned} 0 &= \mathrm{Li}_2(1-z) + \mathrm{Li}_2\left(1 - \frac{1}{z}\right) + \tfrac{1}{2}\log^2 z, \\ 0 &= +\mathrm{Li}_2\,(1-x) + \mathrm{Li}_2\,(1-y) - \mathrm{Li}_2(1-xy) \\ &\quad - \mathrm{Li}_2\big(z(1-x)\big) - \mathrm{Li}_2\big(\bar{z}(1-y)\big) - \log(\bar{z} + zx)\log(z + \bar{z}y) \\ &\quad - \mathrm{Li}_2\,\frac{z(1-y)}{z+\bar{z}y} - \mathrm{Li}_2\,\frac{\bar{z}(1-x)}{\bar{z}+zx} + \mathrm{Li}_2\,\frac{z(1-xy)}{z+\bar{z}y} + \mathrm{Li}_2\,\frac{\bar{z}(1-xy)}{\bar{z}+zx}\,. \end{aligned} \tag{B.19}$$

It is tricky to determine the value of $F(0,t,u,0)$. It originates form a one-mass box integral evaluated at $s = 0$. Unfortunately, the expression (B.4) is very singular at this point. One way to obtain a value is to consider a particular configuration of invariants and show that $F(zu, t, u, zt) = 0$. Then the limit $z \to 0$ suggests that $F(0,t,u,0) = 0$, but it is certainly not a smooth limit in general. Another indication in favour of this result is that the original box integral $I^{\square}(0,t,u,0)$ is finite. Furthermore, the multiplicative factor $\Delta = 0$ and thus $F(0,t,u,0) = 0$. So we are led to the conclusion that the MHV loop factor with two collinear momenta reduces exactly to the loop factor with the two collinear momenta replaced by their sum

$$M_n^{(1)}\big|_{n-1\|n} = M_{n-1}^{(1)}. \tag{B.20}$$

However, this is not the only way to define this limit and in general

$$M_n^{(1)}\big|_{n-1\|n} = M_{n-1}^{(1)} + r_{\mathrm{S}}\ , \tag{B.21}$$

with the function r_{S} being non-trivial. For example, in the prescription given by [32] and used widely in literature this function is at one-loop that given in Sec. 3.4.

C Computation of the On-Shell Triangle Anomaly

In this appendix we compute the on-shell triangle integral (4.20)

$$(T_j^k)_{\dot\alpha}^B = \frac{1}{8\pi^3}\int d^{4|4}\Lambda_{\rm a}\, d^{4|4}\Lambda_{\rm b}\, d^{4|4}\Lambda_{\rm c}\ \mathrm{sign}(t_j^k - t_j^{k+1})\ \bar S_3(\bar{\rm b},\bar{\rm a}, j+k)_{\dot\alpha}^B \\ \cdot A_{k+2}^{(0)}({\rm a},{\rm c},j,\ldots,j+k-1)A_{n-k+1}^{(0)}(\bar{\rm c},{\rm b},j+k+1,\ldots,j+n-1). \tag{C.1}$$

representing the superconformal anomaly. Substituting the anomaly vertex (2.23), performing the trivial phase integrals over φ and ϑ and pulling out an overall tree-level MHV amplitude we arrive at

$$(T_j^k)_{\dot\alpha}^B = \frac{1}{\pi}A_n^{(0)}\varepsilon_{\dot\alpha\dot\gamma}\tilde\lambda_{j+k}^{\dot\gamma}\,\mathrm{sign}(t_j^{k+1}-t_j^k)\int d\alpha\, d^4\lambda_{\rm c}\,\delta^4(p_{\rm a}+p_{\rm c}+P_j^k) \\ \int d^4\eta_{\rm c}\, d^4\eta'\,\eta'^B\,\delta^8(q_{\rm a}+q_{\rm c}+Q_j^k) \\ \frac{\langle j-1,j\rangle}{\langle j-1,{\rm c}\rangle\langle{\rm c},j\rangle}\,\frac{\langle j+k-1,j+k\rangle\langle j+k,j+k+1\rangle}{\langle j+k-1,{\rm a}\rangle\langle{\rm a},{\rm c}\rangle\langle{\rm c},{\rm b}\rangle\langle{\rm b},j+k+1\rangle}\,, \tag{C.2}$$

where the spinor helicity variables of two intermediate particles are determined through the momentum fraction angle α

$$\begin{aligned} \lambda_{\rm a} &= \lambda_{j+k}\cos\alpha, & \tilde\lambda_{\rm a} &= \tilde\lambda_{j+k}\cos\alpha, & \eta_{\rm a} &= \eta_{j+k}\cos\alpha - \eta'\sin\alpha, \\ \lambda_{\rm b} &= \lambda_{j+k}\sin\alpha, & \tilde\lambda_{\rm b} &= \tilde\lambda_{j+k}\sin\alpha, & \eta_{\rm b} &= \eta_{j+k}\sin\alpha + \eta'\cos\alpha. \end{aligned} \tag{C.3}$$

We now evaluate the three lines of the above expression in parts. The bosonic integral on the first line of (C.2) is of the form[16]

$$\int d^4\lambda_2\,\delta^4(p_1+p_2+P) = 2\pi\delta\big(\langle 1|P|1] + P^2\big)\,. \tag{C.4}$$

The spinor $\lambda_2, \tilde\lambda_2$ is fixed up to a phase

$$|2\rangle = xP|1], \qquad [2| = \tilde x\langle 1|P, \qquad x\tilde x = -\frac{1}{\langle 1|P|1]}, \qquad \tilde x = \pm x^*. \tag{C.5}$$

We then substitute the appropriate momenta and note that $\langle j+k|P_j^k|j+k] = t_j^{k+1} - t_j^k$. The resulting delta function subsequently localises the integral over α

$$\int d\alpha\, d^4\lambda_{\rm c}\,\delta^4(p_{\rm a}+p_{\rm c}+P_j^k) = 2\pi\int d\alpha\,\delta\big((t_j^{k+1}-t_j^k)\cos^2\alpha + t_j^k\big) \\ = \frac{\pi}{\big|t_j^{k+1}-t_j^k\big|\sin\alpha\cos\alpha}\,. \tag{C.6}$$

[16] We use a delta function for momenta P in spinor notation $\delta^4(P^{\beta\dot\alpha})$ rather than in vector notation $\delta^4(P^\mu) = 4\delta^4(P^{\beta\dot\alpha})$.

The variables are fixed at[17]

$$\lambda_{\rm c}^{\beta} = x(P_j^k)^{\beta\dot\alpha}\varepsilon_{\dot\alpha\dot\gamma}\tilde\lambda_{j+k}^{\dot\gamma}, \qquad \cos^2\alpha = \frac{t_j^k}{t_j^k - t_j^{k+1}}\,. \tag{C.7}$$

The fermionic integral on the second line of (C.2) can be evaluated by expressing Q in a basis of $\lambda_{\rm a}$ and $\lambda_{\rm c}$ and using (A.4) to split up the δ^8 into the product of two δ^4

$$\begin{aligned}&\int d^4\eta_{\rm c}\, d^4\eta'\,\eta'^B\delta^8(q_{\rm a}+q_{\rm c}+Q_j^k)\\ &= \big(\sin\alpha\cos\alpha\langle j+k,{\rm c}\rangle\big)^3\big(\cos^2\alpha\langle j+k,{\rm c}\rangle\eta_{j+k}^B - \lambda_{\rm c}^{\delta}\varepsilon_{\delta\epsilon}(Q_j^k)^{\epsilon B}\big).\end{aligned} \tag{C.8}$$

The rational spinor function on the third line of (C.2) reads

$$\begin{aligned}&\frac{\langle j-1,j\rangle}{\langle j-1,{\rm c}\rangle\langle {\rm c},j\rangle}\,\frac{\langle j+k-1,j+k\rangle\langle j+k,j+k+1\rangle}{\langle j+k-1,{\rm a}\rangle\langle {\rm a},{\rm c}\rangle\langle {\rm c},{\rm b}\rangle\langle {\rm b},j+k+1\rangle}\\ &= \frac{\langle j-1,j\rangle}{(\sin\alpha\cos\alpha\langle j+k,c\rangle)^2\langle j,{\rm c}\rangle\langle j-1,{\rm c}\rangle}\,.\end{aligned} \tag{C.9}$$

We assemble and simplify these expressions and altogether we find

$$(T_j^k)_{\dot\alpha}^B = A_n^{(0)}\varepsilon_{\dot\alpha\dot\gamma}\tilde\lambda_{j+k}^{\dot\gamma}\frac{\langle j-1,j\rangle\big(\tilde\lambda_{j+k}^{\dot\kappa}\varepsilon_{\dot\kappa\dot\lambda}(P_j^k)^{\delta\dot\lambda}\varepsilon_{\delta\epsilon}(Q_j^k)^{\epsilon B} - t_j^k\eta_{j+k}^B\big)}{\langle j|P_j^k|j+k]\,\langle j-1|P_j^k|j+k]}\,. \tag{C.10}$$

D Details of Six-Point NMHV Amplitude

Here we give some more details relevant to the six-point NMHV amplitude

$$A_{6;{\rm NMHV}}^{(1)} = A_{6;{\rm MHV}}^{(0)}\Big(R_{146}F_6^{[1]} + R_{251}F_6^{[2]} + R_{362}F_6^{[3]} + R_{413}F_6^{[1]} + R_{524}F_6^{[2]} + R_{635}F_6^{[3]}\Big)\,. \tag{D.1}$$

Explicit expressions of the R invariants can be found in [13]. For the six-point amplitudes they can be written as, e.g.

$$R_{146} = \frac{\langle 34\rangle\langle 56\rangle\langle 61\rangle\langle 45\rangle\delta^{(4)}(\zeta_{456})}{x_{14}^2\langle 1|P_1^3|4]\langle 3|x_{36}|6][45][56]} \tag{D.2}$$

where

$$\zeta_{456} = \eta_4[56] + \eta_5[64] + \eta_6[45])\,. \tag{D.3}$$

A general relation amongst the R structures which holds for any amplitude is

$$R_{r,r+2,s} = R_{r+2,s,r+1} \tag{D.4}$$

[17] In fact, for real $0 \le \alpha \le \pi/2$ we must assume $0 \le \cos^2\alpha \le 1$ implying an energy signature $(\pm\pm\mp)$ of the three particles. Here we also allow for the ranges $\cos^2\alpha < 0$ and $1 < \cos^2\alpha$. These additional ranges correspond to the energy signatures $(\mp\pm\pm)$ and $(\pm\mp\pm)$ of the three particles contributed by the 2 cyclic images in (2.23).

and an important relation which holds for the specific case of the six-point amplitude is

$$R_{146} + R_{135} + R_{136} = R_{624} + R_{625} + R_{635} \ . \tag{D.5}$$

Thus, using $R_{251} = R_{625}$, $R_{362} = R_{136}$, $R_{413} = R_{624}$, $R_{524} = R_{135}$, we could pick R_{146}, R_{625}, R_{136}, R_{624} and R_{135} as the independent structures (i.e. we remove R_{635} in terms of the others). A useful expression for the sum of box functions which occur in $A^{(1)}_{6;\mathrm{NMHV}}$, from which one can extract the variation, is:

$$\begin{aligned} F_6^{[1]} = &-\frac{c_\epsilon}{2\epsilon^2}\sum_{i=1}^{6}\left(\frac{-t_i^2}{\mu^2}\right)^{-\epsilon} \\ &- \Big[\log\frac{t_1^3}{t_1^2}\log\frac{t_1^3}{t_2^2} + \log\frac{t_1^3}{t_4^2}\log\frac{t_1^3}{t_5^2} - \log\frac{t_1^3}{t_3^2}\log\frac{t_1^3}{t_6^2}\Big] \\ &+ \frac{1}{2}\Big[\log\frac{t_1^2}{t_4^2}\log\frac{t_2^2}{t_5^2} + \log\frac{t_6^2}{t_1^2}\log\frac{t_2^2}{t_3^2} + \log\frac{t_3^2}{t_4^2}\log\frac{t_5^2}{t_6^2}\Big] + \frac{\pi^2}{3} \ . \end{aligned} \tag{D.6}$$

The remaining $F_6^{[i]}$ can be found by cyclicly permuting this expression. The variation can be written as

$$\begin{aligned} \bar{\mathrm{S}}^{(0)}_{1\to1}A^{(1)}_{6;\mathrm{NMHV}} = A^{(0)}_{6;\mathrm{MHV}}R_{146}\Bigg[&\sum_{i=1}^{6}\bar{\mathrm{S}}^{(0)}_{1\to1}\Big[-\frac{c_\epsilon}{2\epsilon^2}\left(\frac{t_i^2}{-\mu^2}\right)^{-\epsilon}\Big] + \log t_1^3\,\bar{\mathrm{S}}^{(0)}_{1\to1}\log\Big[\frac{t_1^2t_2^2}{t_3^2t_6^2}\Big] \\ &+\log\frac{t_1^2}{-\mu^2}\,\bar{\mathrm{S}}^{(0)}_{1\to1}\left(-\frac{1}{2}\log\Big[\frac{(t_2^2)^2}{t_3^2t_1^3}\Big]\right) + \log\frac{t_2^2}{-\mu^2}\,\bar{\mathrm{S}}^{(0)}_{1\to1}\left(-\frac{1}{2}\log\Big[\frac{(t_1^2)^2}{t_6^2t_1^3}\Big]\right) \\ &+\log\frac{t_3^2}{-\mu^2}\,\bar{\mathrm{S}}^{(0)}_{1\to1}\left(+\frac{1}{2}\log\Big[\frac{t_1^2}{t_1^3}\Big]\right) + \log\frac{t_4^2}{-\mu^2}\,\bar{\mathrm{S}}^{(0)}_{1\to1}\left(-\frac{1}{2}\log\Big[\frac{t_2^2}{t_6^2}\Big]\right) \\ &+\log\frac{t_5^2}{-\mu^2}\,\bar{\mathrm{S}}^{(0)}_{1\to1}\left(-\frac{1}{2}\log\Big[\frac{t_1^2}{t_3^2}\Big]\right) + \log\frac{t_6^2}{-\mu^2}\,\bar{\mathrm{S}}^{(0)}_{1\to1}\left(-\frac{1}{2}\log\Big[\frac{t_1^3}{t_2^2}\Big]\right)\Bigg] + \ldots \end{aligned} \tag{D.7}$$

where we have made use of the relations

$$R_{146}\bar{\mathrm{S}}^{(0)}_{1\to1}\log\frac{t_5^2}{t_1^3} = R_{146}\bar{\mathrm{S}}^{(0)}_{1\to1}\log\frac{t_4^2}{t_1^3} = 0 \ . \tag{D.8}$$

The remaining terms can again be found by cyclic permutations.

E One-Loop Anomaly of MHV-4 Amplitude

In this appendix we consider the one-loop superconformal invariance of amplitudes using the approach of [43]. Although the deformation of the superconformal generators itself does not make reference to the CSW rules [56], their application to amplitudes is hard to define properly without them. Here we perform an explicit calculation for the four-particle MHV amplitude pointing out an ambiguity and how it may be resolved.

In the proposal [43] the one-loop superconformal anomaly for n-particle MHV amplitudes is compensated by the action of $\bar{\mathrm{S}}_{2\to1}$ (7.9) on $(n+1)$-particle NMHV amplitudes. We now consider the simplest case of 4-particle MHV amplitudes. We apply $\bar{\mathrm{S}}_{2\to1}$ directly to the 5-particle NMHV amplitude without the use of CSW rules in order to understand the subtleties concerning collinear configurations. The 5-particle NMHV amplitude reads [60]

$$A_5^{\mathrm{NMHV}} = \frac{\delta^4(P)\,\delta^8(Q)\,\delta^4(\eta_3[45]+\eta_4[53]+\eta_5[34])}{[12][23][34][45][51]\langle12\rangle^4}\,. \tag{E.1}$$

We wish to act with $\bar{\mathrm{S}}_{2\to1}$ on legs 4 and 5. In order to gain access to the collinear divergence, we express $\tilde\lambda_4$ in a basis of $\tilde\lambda_3$ and $\tilde\lambda_5$ using (A.2) (for simplicity we shall work in $(2,2)$ signature where λ and $\tilde\lambda$ are independent)

$$A_5^{\mathrm{NMHV}} = \int \frac{d\tilde y}{\tilde y}\,\frac{d\tilde z}{\tilde z}\,\delta^2(\tilde z\tilde\lambda_3+\tilde y\tilde\lambda_5-\tilde\lambda_4)\,\delta^4(\tilde z\eta_3+\tilde y\eta_5-\eta_4)\,\frac{[35]^2\,\big|[35]\big|\,\delta^4(P)\,\delta^8(Q)}{[12][23][51]\langle12\rangle^4}\,. \tag{E.2}$$

We then multiply by the vertex $\bar S_3$ in the form of (A.7)

$$\begin{aligned}\bar S_{\dot a}^B = -\varepsilon_{\dot\alpha\dot\gamma}(\tilde\lambda_5^{\dot\gamma}\eta_4^B-\tilde\lambda_4^{\dot\gamma}\eta_5^B)\int\frac{dx_4}{x_4}\,\frac{dx_5}{x_5}\,d\tilde x_4\,d\tilde x_5\,\delta(1+x_4\tilde x_4+x_5\tilde x_5)\\ \cdot\,\delta^2(\lambda_4-x_4\lambda_{45})\,\delta^2(\lambda_5-x_5\lambda_{45})\\ \cdot\,\delta^2(\tilde\lambda_4+\tilde x_4\tilde\lambda_{45})\,\delta^2(\tilde\lambda_5+\tilde x_5\tilde\lambda_{45})\,\delta^4(x_4\eta_4+x_5\eta_5-\eta_{45}).\end{aligned} \tag{E.3}$$

and integrate out Λ_4 and Λ_5. Here we make sure that at first only the integrations over spinors are performed; the integrals over the auxiliary variables are left untouched. This yields

$$\begin{aligned}((\bar{\mathrm{S}}_{2\to1})_4^1)_{\dot\alpha}^B A_5^{\mathrm{NMHV}} = -\int dx_4\,dx_5\,d\tilde x_4\,d\tilde x_5\,d\tilde y\,d\tilde z\,\delta(1+x_4\tilde x_4+x_5\tilde x_5)\,\delta(\tilde z)\,\delta(\tilde y-\tilde x_4/\tilde x_5)\\ \cdot\,\frac{(x_4\tilde x_5\tilde y+x_5\tilde x_5)^3}{x_4x_5\tilde x_5\tilde y(x_4\tilde x_4+x_5\tilde x_5)^2}\,\frac{\delta^4(P-(1+x_4\tilde x_4+x_5\tilde x_5)p_4)\,\delta^8(Q)}{\langle12\rangle\langle23\rangle\langle34\rangle\langle41\rangle}\\ \cdot\,\frac{\varepsilon_{\dot\alpha\dot\gamma}\tilde\lambda_4^{\dot\gamma}}{[34]}\left(\frac{x_4\tilde x_4+x_5\tilde x_5}{\tilde x_5}\,\eta_3^B+\frac{\tilde y-\tilde x_4/\tilde x_5}{\tilde z}\,\eta_4^B\right).\end{aligned} \tag{E.4}$$

The second term in the brackets is undetermined because it equals $0/0$ on the support of the delta functions; let us replace it by some undetermined expression $*$. The remaining term is well-defined and we can now perform the integrals over the auxiliary variables

$$((\bar{\mathrm{S}}_{2\to1})_4^1)_{\dot\alpha}^B A_5^{\mathrm{NMHV}} = -\int\frac{dx_4}{x_4}\,\frac{dx_5}{x_5}\,\frac{d\tilde x_4}{\tilde x_4}\,\frac{d\tilde x_5}{\tilde x_5}\,\delta(1+x_4\tilde x_4+x_5\tilde x_5)\frac{\varepsilon_{\dot\alpha\dot\gamma}\tilde\lambda_4^{\dot\gamma}\,(\eta_3^B+*\eta_4^B)}{[34]}\,A_4^{\mathrm{MHV}}. \tag{E.5}$$

After performing the integrals over the x's in proper $(3,1)$ Minkowski signature making sure that the energies of particles 4 and 5 have equals signs in agreement with (7.9) we end up with

$$((\bar{\mathrm{S}}_{2\to1})_4^1)_{\dot\alpha}^B A_5^{\mathrm{NMHV}} = -16\pi^2\int_0^{\pi/2}\frac{d\alpha}{2\sin\alpha\cos\alpha}\,\frac{\varepsilon_{\dot\alpha\dot\gamma}\tilde\lambda_4^{\dot\gamma}\,(\eta_3^B+*\eta_4^B)}{[34]}\,A_4^{\mathrm{MHV}}. \tag{E.6}$$

The integral is clearly divergent. This divergence is of infrared type, and it is expected from [43]. In fact it looks similar to the action of $\bar{\mathrm{S}}^{(1)}_{2\to 2}$ defined in (3.44). Luckily, we can adjust the undetermined coefficient $*$ in order to match the structure precisely. Noting an identity for valid for physical four-particle configurations

$$\frac{\tilde{\lambda}_4\eta_3}{[34]} = \frac{\tilde{\lambda}_4\eta_1}{[14]} + \frac{\tilde{\lambda}_4\eta_4}{[14]}\frac{\langle 24\rangle}{\langle 21\rangle} = \frac{\tilde{\lambda}_4\eta_1}{[14]} - \frac{\tilde{\lambda}_4\eta_4}{[34]}\frac{\langle 24\rangle}{\langle 23\rangle} \tag{E.7}$$

we set $* \to \frac{1}{2}(\langle 24\rangle/\langle 23\rangle)$ so that the spinor structure becomes

$$\frac{\varepsilon_{\dot\alpha\dot\gamma}\tilde{\lambda}_4^{\dot\gamma}(\eta_3^B + \frac{1}{2}(\langle 24\rangle/\langle 23\rangle)\eta_4^B)}{[34]}\, A_4^{\mathrm{MHV}} = \left(\frac{\varepsilon_{\dot\alpha\dot\gamma}\tilde{\lambda}_4^{\dot\gamma}\eta_3^B}{2[34]} + \frac{\varepsilon_{\dot\alpha\dot\gamma}\tilde{\lambda}_4^{\dot\gamma}\eta_1^B}{2[14]}\right) A_4^{\mathrm{MHV}}. \tag{E.8}$$

The expression agrees with (3.44) up to cyclic permutations of the four particles. Even the prefactor appears to agree once one imposes some ad-hoc regulator.

In conclusion we see that the alternative proposal [43] does appear to give analogous results up to interpreting terms of the kind 0/0 in a suitable fashion. This is presumably achieved by the CSW rules. Regularising divergent terms is another (separate) issue. In our proposal, cf. Sec. 4, all expressions are well-defined in dimensional regularisation (or any other suitable scheme), however, at the cost of having a substantially more involved deformation than just $\bar{\mathrm{S}}_{2\to 1}$.

F Conventions and Identities

In this appendix we list a few of the basic conventions and identities used in this paper.

Coupling Constant. We define the coupling constant g which we use for the loop expansion:

$$g^2 = \frac{\lambda}{16\pi^2}\,, \qquad \lambda = g_{\mathrm{YM}}C_{\mathrm{A}}. \tag{F.1}$$

The 't Hooft coupling is written using the adjoint Casimir which equals $C_{\mathrm{A}} = N_{\mathrm{c}}$ for a $\mathrm{SU}(N_{\mathrm{c}})$ gauge group. We employ a dimensional regularisation scheme with straight minimal subtraction. The undesirable contributions of Euler's gamma constant and $\log 4\pi$'s are absorbed into a constant c_ϵ which typically dresses poles in ϵ

$$c_\epsilon = (4\pi)^\epsilon \frac{\Gamma(1+\epsilon)\,\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)} = \exp\big((\log 4\pi - \gamma)\epsilon - \tfrac{1}{12}\pi^2\epsilon^2 + \mathcal{O}(\epsilon^3)\big) = 1 + \mathcal{O}(\epsilon). \tag{F.2}$$

The precise form of c_ϵ has no physical significance whatsoever.

Complex Integrals. For performing integrals over the complex plane we use the convention that $z = (x + iy)/\sqrt{2}$. We can then write two-dimensional integrals as simple products of one-dimensional integrals as follows

$$d^2z = dz\, d\bar{z}. \tag{F.3}$$

This proves particularly useful when Wick rotating to two independent real coordinates $z, \bar{z}$. Similarly, the corresponding delta functions factorise and the holomorphic anomaly reads

$$\delta^2(z) = \delta(z)\delta(\bar{z}), \qquad \frac{\partial}{\partial\bar{z}}\frac{1}{z} = 2\pi\delta(z)\delta(\bar{z}). \tag{F.4}$$

Gauge Generators. The generators T^a and structure constants f^{abc} of the $\mathrm{U}(N_\mathrm{c})$ gauge group are normalised such that

$$[T^a, T^b] = if^{abc}T^c, \qquad \mathrm{Tr}(T^aT^b) = \delta^{ab}. \tag{F.5}$$

This leads to the following identities in traces

$$T^a X T^a = \mathrm{Tr}\, X, \qquad T^a\, \mathrm{Tr}(T^a X) = X. \tag{F.6}$$

Vectors and Spinors. For vectors we choose $(-,+,+,+)$ as the signature of the Minkowski metric, hence the mass shell condition for a massive particle is $p^2 = -m^2$.

The conversion between vector and spinor indices is normalised such that for two light-like momenta p_1, p_2

$$(p_1 + p_2)^2 = 2p_1 \cdot p_2 = p_1^{\delta\dot{\alpha}}\varepsilon_{\dot{\alpha}\dot{\gamma}}p_2^{\beta\dot{\gamma}}\varepsilon_{\beta\delta} = \mathrm{Tr}(\varepsilon p_1 \varepsilon p_2^{\mathrm{T}}) = [1,2]\langle 2,1\rangle. \tag{F.7}$$

Moreover for a generic momentum P one has $P\varepsilon P^{\mathrm{T}} = -P^2\varepsilon$.

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