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Foliations of asymptotically flat manifolds by surfaces of Willmore type

Tobias Lamm · Jan Metzger · Felix Schulze

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Abstract The goal of this paper is to establish the existence of a foliation of the asymptotic region of an asymptotically flat manifold with positive mass by surfaces which are critical points of the Willmore functional subject to an area constraint. Equivalently these surfaces are critical points of the Geroch–Hawking mass. Thus our result has applications in the theory of general relativity.

1 Introduction

In this paper we study foliations of asymptotically flat manifolds by surfaces of Willmore type. This means that we are interested in constructing embedded spheres Σ in a three dimensional Riemannian manifold (M, g) which satisfy the equation

$$-\Delta H - H|\mathring{A}|^2 - {}^M\mathrm{Rc}(\nu, \nu)H = \lambda H. \tag{1}$$

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T. Lamm (✉)
Department of Mathematics, University of British Columbia,
1984 Mathematics Road, Vancouver, BC V6T 1Z2, Canada
e-mail: tlamm@math.ubc.ca

J. Metzger
Institut für Mathematik, Universität Potsdam,
Am Neuen Palais 10, 14469 Potsdam, Germany
e-mail: jan@metweb.de

F. Schulze
Institut für Mathematik und Informatik, Freie Universität Berlin,
Arnimallee 3, 14195 Berlin, Germany
e-mail: Felix.Schulze@math.fu-berlin.de

Here H is the mean curvature of Σ, $\mathring{A}$ is the traceless part of the second fundamental form A of Σ in M, that is $\mathring{A} = A - \frac{1}{2}H\gamma$, and γ is the induced metric on Σ. Moreover ${}^{M}\mathrm{Rc}$ is the Ricci curvature of M and Δ the Laplace-Beltrami operator on Σ.

Equation 1 is the Euler–Lagrange equation of the functional

$$\mathcal{W}(\Sigma) = \frac{1}{2}\int_{\Sigma} H^2 \,\mathrm{d}\mu \tag{2}$$

subject to the constraint that $|\Sigma|$ be fixed. Then λ becomes the Lagrange parameter.

In mathematics this functional is known as the Willmore functional, at least in flat space, whereas for curved ambient manifolds the literature [23] also considers the functional

$$\mathcal{U}(\Sigma) = \int_{\Sigma} |\mathring{A}|^2 \,\mathrm{d}\mu.$$

In flat space these two functionals only differ by a topological constant. However, the second functional is conformally invariant and thus translation invariant in all conformally flat manifolds. Since our model space, the spatial Schwarzschild metric $g^S_m = \phi^4_m g^e$, with $\phi = 1 + \frac{m}{2r}$, g^e the Euclidean metric and $m > 0$ a mass parameter, is conformally flat, we could not hope to find unique surfaces minimizing the corresponding constrained problem.

Furthermore, the functional (2) appears naturally in general relativity in form of the Hawking mass $m_H(\Sigma)$ of a surface Σ, defined as

$$m_H(\Sigma) = \frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\left(16\pi - 2\mathcal{W}(\Sigma)\right).$$

This quantity is used to measure the mass of a region enclosed by Σ. Due to the area constraint, Eq. 1 also appears as the Euler–Lagrange equation when maximizing $m_H(\Sigma)$ subject to fixed area $|\Sigma|$.

Foliations of asymptotically flat manifolds using constant mean curvature surfaces have been considered in [6,9,24]. The uniqueness of such foliations was considered in [18]. In [9] these foliations have been used to define a center of mass for initial data sets for isolated gravitating systems in general relativity. Such data sets are three dimensional asymptotically flat manifolds. We argue here that, due to its relation to the Hawking mass, Eq. 1 is the most natural equation to consider when defining a geometric center of the Hawking mass. In fact, surfaces maximizing the Hawking mass are the optimal surfaces to calculate the Hawking mass. This intuition is backed by our observation that along the foliation we construct, the Hawking mass is non-negative and non-decreasing in the outward direction, provided the scalar curvature ${}^{M}\mathrm{Sc} \geq 0$ is non-negative, cf. Theorem 4. We remark that on stable surfaces of constant mean curvature the Hawking mass is also non-negative as was shown by Christodoulou and Yau [2].

Moreover, we wish to mention here that in [7] Huisken argues in the other direction and introduces a definition of quasi-local mass with the constant mean curvature equation as Euler–Lagrange equation for the optimal surfaces at a given enclosed volume. This then fits together with the center of mass definition by CMC spheres.

CMC foliations have also been studied in other contexts, in particular with asymptotically hyperbolic background in [13,16,17]. This setting is also relevant in general relativity when studying data sets which are asymptotically light-like. We expect that our results extend to the asymptotically hyperbolic case.

In $\mathbf{R}^3$, minima of functional (2) are round spheres, and since the functional is scale and translation invariant, we get an (at least) four dimensional transformation group. In particular, we can not expect solutions of (1) to be unique. The existence of surfaces $\Sigma \subset \mathbf{R}^n$ of higher genus which minimize the Willmore functional and in particular satisfy (1) with $\lambda = 0$ has been shown by Simon [21] and Bauer and Kuwert [1].

This changes when we take the background M not to be $\mathbf{R}^3$ but the exterior region of an asymptotically flat manifold. That is $M = \mathbf{R}^3 \backslash B_\sigma(0)$ and the metric on M is asymptotic to the spatial Schwarzschild metric g^S_m. This metric is the spatial part of the Schwarzschild metric which describes a single, static black hole of mass m. Thus m has the interpretation of a mass parameter.

In the g^S-metric we no longer have translation and scaling invariance. In fact we will show that solutions of (1) which are close enough to large centered round spheres are in fact equal to centered round spheres. The radius of the sphere is then uniquely determined by λ, provided $\lambda \in (0, \lambda_0)$ is small enough. If the metric on M is asymptotic to g^S with appropriate decay conditions, we can show that solutions to (1) behave accordingly and form a smooth foliation of the asymptotic region of (M, g).

To be precise, we consider metrics g on $\mathbf{R}^3 \backslash B_\sigma(0)$ with the following asymptotics

$$\sup_{\mathbf{R}^3 \backslash B_\sigma(0)} \Big(r^2 |g - g^S_m| + r^3 |\nabla - \nabla^S_m| + r^4 |\operatorname{Ric} - \operatorname{Ric}^S_m| + r^5 |\nabla \operatorname{Ric} - \nabla^S \operatorname{Ric}^S_m| \Big) \le \eta,$$

where g^S_m is the spatial Schwarzschild metric of mass $m > 0$, ∇^S_m its Levi-Civita connection and Ric^S_m its Ricci-curvature. Correspondingly, ∇ and Ric, are the connection and curvature of g. Furthermore, r is the Euclidean radius function on $\mathbf{R}^3 \backslash B_\sigma(0)$. Such metrics shall be called (m, η, σ)-*asymptotically Schwarzschild.*

In this setting, we will prove the following theorem.

Theorem 1 *For all $m > 0$ and σ there exists $\eta_0 > 0$, $\lambda_0 > 0$ and $C < \infty$ depending only on m and σ such that the following holds.*

Let (M, g) be an (m, η, σ)-asymptotically flat manifold with $\eta < \eta_0$ and

$$|{}^M\mathrm{Sc}| \le \eta r^{-5}$$

then for each $\lambda \in (0, \lambda_0)$ there exists a surface Σ_λ satisfying Eq. 1.

In Euclidean coordinates this surface is $W^{2,2}$-close to a Euclidean sphere $S_{R_\lambda}(a_\lambda)$ with radius R_λ and center a_λ such that

$$|a_\lambda| + \left| R_\lambda - (\lambda/2m)^{-1/3} \right| \le C\eta.$$

Moreover, there exists a compact set $K \subset M$ such that $M \backslash K$ is foliated by the surfaces $\{\Sigma_\lambda\}_{\lambda \in (0,\lambda_0)}$.

For an arbitrary surface $\Sigma \subset \mathbf{R}^3$ we can define a best matching sphere by introducing the geometric area radius and the center of gravity, both with respect to the Euclidean background:

$$R^e(\Sigma) = \sqrt{\frac{|\Sigma|_e}{4\pi}} \quad \text{and} \quad a_e(\Sigma) = |\Sigma|_e^{-1} \int\limits_{\Sigma} x \, \mathrm{d}\mu^e$$

where in the second integral, the integrand is the position vector. Then we define the scale-invariant translation parameter

$$\tau(\Sigma) = a_e(\Sigma)/R_e(\Sigma)$$

and we can state the uniqueness theorem

Theorem 2 *Let $m > 0$ and σ be given. Then there exists $\eta_0 > 0$, $\tau_0 > 0$, $\varepsilon > 0$ and $r_0 < \infty$ depending only on m and σ such that the following holds.*

If (M, g) is an (m, η, σ)-asymptotically flat manifold with $\eta < \eta_0$ and

$$|{}^M\mathrm{Sc}| \leq \eta r^{-5}$$

then every spherical surface $\Sigma \subset M$ with $r_{\min} := \min_\Sigma r > r_0$, $\tau(\Sigma) < \tau_0$, $R_e \leq \varepsilon r_{\min}^2$ and $H > 0$ satisfying Eq. 1 for some $\lambda > 0$ equals one of the surfaces Σ_λ constructed in Theorem 1. In particular $\lambda \in (0, \lambda_0)$.

The outline of the paper and the proof of the above theorems is as follows. After setting the stage by presenting some preliminary material in Sect. 2, we calculate the first and second variation of (2), to arrive at (1) and its linearization. This is done in Sect. 3.

In Sect. 4 we prove a priori estimates for solutions to (1) under the assumption that $H > 0$ and $\lambda > 0$. These estimates in particular show that with increasing area also the Hawking mass of the Σ_λ increases.

Section 5 is devoted to a technical improvement of the curvature estimates in Sect. 4, under the additional assumption that the surface in question is not too far off center in the sense that the translation parameter above is not too large.

This allows us to break the translation invariance in Sect. 6, where we prove position estimates. These estimates are at the heart of the uniqueness and are quite delicate. In this section we also state the final version of our a priori estimates. These estimates allow to control both the position and the shape of solutions to (1) in a very precise way.

In Sect. 7 we analyze the linearization of Eq. 1 and use the previous a priori estimates to show that this operator is invertible. The reason why we are able to do this, is that the estimates in Sect. 6 allow to compare the linearization of (1) to the corresponding operator on a centered sphere in Schwarzschild. The latter operator is invertible and thus invertability of the former one follows.

This is used in Sect. 8 to prove the existence and uniqueness of Theorem 1 and Theorem 2 using an argument based on the implicit function theorem.

2 Preliminaries

2.1 Geometric equations

We will consider three dimensional Riemannian manifolds (M, g), where g is the metric tensor, which we write as g_{ij} in coordinates. Its inverse is denoted by g^{ij}, its Levi-Civita connection by ∇. For the Riemanninan curvature tensor we use the convention

$$(\nabla_i \nabla_j - \nabla_j \nabla_i)\partial_k = {}^M\mathrm{Rm}_{ijkl} g^{lm} \partial_m.$$

Here we use the Einstein summation convention and sum over repeated indices. Then the Ricci-curvature is given by

$${}^M\mathrm{Rc}_{il} = g^{jk\,M}\mathrm{Rm}_{ijkl}$$

and the scalar curvature by ${}^M\mathrm{Sc} = g^{ij\,M}\mathrm{Rc}_{ij}$.

Our sign convention implies that commuting derivatives on a 2-tensor T_{ab} gives

$$\nabla_a \nabla_b T_{cd} = \nabla_b \nabla_a T_{cd} - {}^M\mathrm{Rm}_{abce} g^{ef} T_{fd} - {}^M\mathrm{Rm}_{abde} g^{ef} T_{cf}.$$

For a three dimensional manifold the Riemannian curvature tensor can be expressed in terms of the Ricci curvature as follows

$$\begin{aligned} {}^M\mathrm{Rm}_{ijkl} = {}^M\mathrm{Rc}_{il} g_{jk} - {}^M\mathrm{Rc}_{ik} g_{jl} - {}^M\mathrm{Rc}_{jl} g_{ik} + {}^M\mathrm{Rc}_{jk} g_{il} \\ - \frac{1}{2} {}^M\mathrm{Sc}(g_{il} g_{jk} - g_{ik} g_{jl}). \end{aligned} \tag{3}$$

If $\Sigma \subset M$ ia a surface we denote by γ the induced metric and by ν its normal. The second fundamental form of Σ is denoted by A and its mean curvature by H. The Riemannian curvature tensor ${}^\Sigma\mathrm{Rm}$ of Σ is given by the Gauss equation

$${}^\Sigma\mathrm{Rm}_{ijkl} = {}^M\mathrm{Rm}_{ijkl} + A_{il} A_{jk} - A_{ik} A_{jl}. \tag{4}$$

Taking the trace twice implies

$${}^\Sigma\mathrm{Sc} = {}^M\mathrm{Sc} - 2\,{}^M\mathrm{Rc}(\nu, \nu) + H^2 - |A|^2. \tag{5}$$

Furthermore, we have the Codazzi equation

$$\nabla_k A_{ij} = \nabla_i A_{kj} + {}^M\mathrm{Rm}_{kiaj} \nu^a. \tag{6}$$

Denote by $\omega := \mathrm{Ric}(\nu, \cdot)^T$ the tangential projection of the 1-form $\mathrm{Ric}(\nu, \cdot)$ to Σ. Then using the Gauss equation (4), the Codazzi equation (6) and equation (3), the Simons identity [22] becomes

$$\begin{aligned}\Delta A_{ij} &= \nabla_i \nabla_j H + H A_i^k A_{kj} - |A|^2 A_{ij} \\ &\quad A_j^k \gamma^{lm\,M}\mathrm{Rm}_{likm} + A^{kl} R_{ikjl} + 2\nabla_i \omega_j - \mathrm{div}\,\omega \gamma_{ij}. \end{aligned} \tag{7}$$

For any two-tensor T, we denote the traceless part by T^0, that is $T^0_{ij} = T_{ij} - \frac{1}{2}(\mathrm{tr}\,T)\gamma_{ij}$. In particular we have

$$\mathring{A}_{ij} = A_{ij} - \frac{1}{2} H \gamma_{ij}.$$

This implies that

$$|\mathring{A}|^2 + \frac{1}{2} H^2 = |A|^2.$$

With the help of these facts we get from Simons' identity that

$$\begin{aligned}\Delta \mathring{A}_{ij} &= (\nabla^2 H)^0_{ij} + H \mathring{A}_i^k \mathring{A}_{kj} + \frac{1}{2} H^2 \mathring{A}_{ij} - |\mathring{A}|^2 \mathring{A}_{ij} - \frac{1}{2} H |\mathring{A}|^2 \gamma_{ij} \\ &\quad + \mathring{A}_j^k \gamma^{lm\,M}\mathrm{Rm}_{likm} + \mathring{A}^{kl\,M}\mathrm{Rm}_{ikjl} + 2\nabla_i \omega_j - \mathrm{div}\,\omega \gamma_{ij}, \end{aligned} \tag{8}$$

and therefore

$$\begin{aligned}\mathring{A}^{ij} \Delta \mathring{A}_{ij} &= \langle \mathring{A}, \nabla^2 H \rangle + \frac{1}{2} H^2 |\mathring{A}|^2 - |\mathring{A}|^4 \\ &\quad - |\mathring{A}|^{2\,M}\mathrm{Rc}(\nu, \nu) + 2 \mathring{A}^{ij} \mathring{A}_j^{l\,M}\mathrm{Rc}_{il} + 2\langle \mathring{A}, \nabla \omega \rangle. \end{aligned} \tag{9}$$

2.2 Asymptotically Schwarzschild manifolds

Let g^S_m be the spatial, conformally flat Schwarzschild metric on $\mathbf{R}^3 \backslash \{0\}$ of mass m. That is $g^S_m = \phi_m^4 g^e$, where $\phi_m = 1 + \frac{m}{2r}$, g^e is the Euclidean metric on $\mathbf{R}^3$ and r the distance to the origin in $\mathbf{R}^3$. We will suppress the dependence of g^S_m and ϕ_m on m and denote the metric simply by g^S and ϕ_m by ϕ. The following lemma summarizes the relationship of the geometry of g^S and g^e.

Lemma 1 1. *The Ricci curvature of g^S is given by*

$$\mathrm{Ric}^S_{ij} = \frac{m}{r^3} \phi^{-2} \left(g^e_{ij} - 3\rho_i \rho_j \right), \tag{10}$$

where ρ_a is the 1-form dual to the vector $\frac{\partial}{\partial r}$ on $\mathbf{R}^3$. In particular, the scalar curvature of g^S vanishes.

2. *If $\Sigma \subset \mathbf{R}^3\backslash 0$ is a surface, we denote by ν^e the normal of Σ with respect to g^e and by ν^S the normal of Σ with respect to g^S. Analogously $\mathrm{d}\mu^e$, $\mathrm{d}\mu^S$ denote the respective volume forms, $\mathring{A}^e$, $\mathring{A}^S$ the respective traceless second fundamental forms and H^e and H^S the mean curvatures. We find the following relations:*

$$\nu^S = \phi^{-2}\nu^e, \tag{11}$$

$$\mathrm{d}\mu^S = \phi^4\,\mathrm{d}\mu^e, \tag{12}$$

$$\mathring{A}^S = \phi^{-2}\,\mathring{A}^e, \quad \textit{and} \tag{13}$$

$$H^S = \phi^{-2}H^e + 4\phi^{-3}\partial_{\nu^e}\phi. \tag{14}$$

Definition 1 We say that (M, g) is (m, η, σ)-asymptotically Schwarzschild if there exists a compact set $B \subset M$, and a diffeomorphism $x : M\backslash B \to \mathbf{R}^3\backslash B_\sigma(0)$, such that in these coordinates

$$\sup_{\mathbf{R}^3\backslash B_\sigma(0)}\Big(r^2|g-g^S|+r^3|\nabla^g-\nabla^S|+r^4|\operatorname{Ric}^g-\operatorname{Ric}^S|+r^5|\nabla\operatorname{Ric}^g-\nabla^S\operatorname{Ric}^S|\Big) \le \eta,$$

where g^S is the metric for mass m.

For brevity we will subsequently refer to Ric^g simply by Ric or by ${}^M\mathrm{Rc}$.

In the next lemma we relate geometric quantities with respect to g to quantities with respect to g^S.

Lemma 2 *If (M, g) is (m, η, σ) asymptotically Schwarzschild and if $\Sigma \subset \mathbf{R}^3\backslash B_\sigma(0)$ is a surface, we have the following relation between the normals ν with respect to g and ν^S with respect to g^S*

$$r^2|\nu - \nu^S| \le C\eta.$$

Furthermore, the area elements $\mathrm{d}\mu$ and $\mathrm{d}\mu^S$ satisfy $\mathrm{d}\mu - \mathrm{d}\mu^S = h\,\mathrm{d}\mu$ with

$$r^2|h| \le C\eta,$$

The second fundamental forms A and A^S satisfy

$$\begin{aligned}|A - A^S| &\le C\eta(r^{-3} + r^{-2}|A|)\\ |\nabla A - \nabla A^S| &\le C\eta(r^{-4} + r^{-3}|A| + r^{-2}|\nabla A|).\end{aligned}$$

To estimate integrals of decaying quantities we use the variant of [9, Lemma 5.2] as stated in [14, Lemma 2.3].

Lemma 3 *Let (M, g) be (m, η, σ)-asymptotically Schwarzschild, and let $p_0 > 2$ be fixed. Then there exists $c(p_0)$ and $r_0 = r_0(m, \eta, \sigma)$, such that for every surface $\Sigma \subset \mathbf{R}^3\backslash B_{r_0}(0)$, and every $p > p_0$, the following estimate holds*

$$\int_{\Sigma} r^{-p} \,\mathrm{d}\mu \le c(p_0) r_{\min}^{2-p} \int_{\Sigma} H^2 \,\mathrm{d}\mu.$$

Here $r_{\min} := \min_{\Sigma} r$, where r is the Euclidean radius.

In the sequel we will also need decay properties of volume integrals.

Lemma 4 *Let Ω be an exterior domain with compact interior boundary Σ. Then for all $p > 3$ there exists a constant $C(p)$ and r_0 such that if $r_{\min} > r_0$ we have*

$$\int_{\Omega} r^{-p} dV \le C(p) r_{\min}^{3-p} \int_{\Sigma} H^2 \,\mathrm{d}\mu.$$

Proof Let ρ be the Euclidean radial direction, and let $X = r^{-p+1}\rho$. With respect to g we have

$$\operatorname{div} X = (3-p) r^{-p} + O(r^{-p-1}).$$

Choose r_0 so large that the error term is dominated by the main term in this equation, that is

$$(p-3-\varepsilon) r^{-p} \le -\operatorname{div} X,$$

where ε is such that $p-3-\varepsilon > 0$. Integrating this relation over Ω and partially integrating on the right hand side yields the estimate

$$\int_{\Omega} r^{-p} dV \le \frac{1}{p-3-\varepsilon} \int_{\Sigma} \langle X, \nu \rangle.$$

Note that the boundary integral at infinity vanishes as the surface integrand decays like r^{-p+1}. The claim then follows from Lemma 3. □

Using the conformal invariance of $\|\mathring{A}\|_{L^2(\Sigma)}$, which can be seen via Lemma 1, we derive:

Lemma 5 *Let (M, g) be (m, η, σ)-asymptotically Schwarzschild. Then there exists $r_0 = r_0(\eta, \sigma)$ such that for every surface $\Sigma \subset \mathbf{R}^3 \backslash B_{r_0}(0)$ we have*

$$\begin{aligned} &\left| \|\mathring{A}^e\|^2_{L^2(\Sigma, g^e)} - \|\mathring{A}\|^2_{L^2(\Sigma, g)} \right| \\ &\quad \le C\eta r_{\min}^{-2} \left(\|\mathring{A}\|^2_{L^2(\Sigma,g)} + \|H\|_{L^2(\Sigma,g)} \|\mathring{A}\|_{L^2(\Sigma,g)} + \eta r_{\min}^{-2} \|H\|^2_{L^2(\Sigma,g)} \right). \end{aligned}$$

Corollary 1 *Let (M, g), r_0 and Σ be as in the previous lemma. Assume in addition that $\|H\|_{L^2(\Sigma)} \le C'$, then*

$$\|\mathring{A}^e\|_{L^2(\Sigma, g^e)} \le C(r_0) \|\mathring{A}\|_{L^2(\Sigma, g)} + C(r_0, C') \eta r_{\min}^{-2}.$$

We need the following variant of the Michael–Simon Sobolev inequality [15] as stated in [9, Proposition 5.4].

Proposition 1 *Let (M, g) be (m, η, σ)-asymptotically Schwarzschild. Then there is $r_0 = r_0(m, \eta, \sigma)$ and an absolute constant C_s such that for each surface $\Sigma \subset M \backslash B_{r_0}(0)$ and each Lipschitz function f on Σ we have the estimate*

$$\left(\int_\Sigma |f|^2 \, \mathrm{d}\mu \right)^{1/2} \leq C_s \int_\Sigma |\nabla f| + |Hf| \, \mathrm{d}\mu. \tag{15}$$

Via Hölder's inequality, this implies that for all $q \geq 2$

$$\left(\int_\Sigma |f|^q \, \mathrm{d}\mu \right)^{\frac{2}{2+q}} \leq C_s \int_\Sigma |\nabla f|^{\frac{2q}{2+q}} + |Hf|^{\frac{2q}{2+q}} \, \mathrm{d}\mu, \tag{16}$$

and for all $p \geq 1$,

$$\left(\int_\Sigma |f|^{2p} \, \mathrm{d}\mu \right)^{1/p} \leq C_s p^2 |\mathrm{supp} f|^{1/p} \int_\Sigma |\nabla f|^2 + H^2 f^2 \, \mathrm{d}\mu. \tag{17}$$

2.3 Almost umbilical surfaces in Euclidean space

To conclude that the surfaces we consider are close to spheres, we use the following theorem for surfaces in Euclidean space. This is proved in [3, Theorem 1] and [4, Theorem 2].

Theorem 3 *There exists a universal constant c such that for each compact connected surface without boundary $\Sigma \subset \mathbf{R}^3$ with area $|\Sigma| = 4\pi$, the following estimate holds*

$$\|A^e - \gamma^e\|_{L^2(\Sigma, \gamma^e)} \leq c \|\mathring{A}^e\|_{L^2(\Sigma, \gamma^e)}.$$

If in addition $\|\mathring{A}^e\|_{L^2(\Sigma, \gamma^e)} \leq 8\pi$, then Σ is topologically a sphere, and there exists a conformal map $\psi : S^2 \to \Sigma \subset \mathbf{R}^3$ such that

$$\|\psi - (a + \mathrm{id}_{S^2})\|_{W^{2,2}(S^2)} \leq c \|\mathring{A}^e\|_{L^2(\Sigma, \gamma^e)},$$

where id_{S^2} is the standard embedding of S^2 onto the sphere $S_1(0)$ in $\mathbf{R}^3$, and

$$a = |\Sigma|_e^{-1} \int_\Sigma \mathrm{id}_\Sigma \, \mathrm{d}\mu^e$$

is the center of gravity of Σ. The conformal factor h of the embedding ψ, that is $\psi^*\gamma^e = h^2\gamma_{S^2}$, satisfies

$$\|h-1\|_{W^{1,2}(S^2)} + \sup_{S^2}|h-1| \le c\|\mathring{A}^e\|_{L^2(\Sigma,\gamma^e)}.$$

The normal ν^e of Σ satisfies

$$\|N-\nu^e\circ\psi\|_{W^{1,2}(S^2)} \le c\|\mathring{A}^e\|_{L^2(\Sigma,\gamma^e)},$$

where N is the normal of $S_1(a)$.

To get the scale-invariant form of these estimates, we proceed as follows. For a surface Σ with arbitrary area $|\Sigma|_e$ let $R_e = \sqrt{|\Sigma|_e/4\pi}$. Then the first part of Theorem 3 implies that

$$\|A - R_e^{-1}\gamma^e\|_{L^2(\Sigma,\gamma^e)} \le c\|\mathring{A}^e\|_{L^2(\Sigma,\gamma^e)}.$$

Again let a_e denote the center of gravity of Σ,

$$a_e := \frac{1}{4\pi R_e^2}\int_\Sigma \mathrm{id}_\Sigma \; \mathrm{d}\mu^e \in \mathbf{R}^3.$$

Then if $\|\mathring{A}^e\|_{L^2(\Sigma,\gamma^e)} \le 8\pi$, the second part of Theorem 3 gives that there exists a conformal parametrization $\psi : S_{R_e}(a_e) \to \Sigma$. The estimates from Theorem 3 imply together with the Sobolev-embedding theorems on S^2, that the following estimates hold

$$\sup_{S_{R_e}(a_e)}\left|\psi - \mathrm{id}_{S_{R_e}(a_e)}\right| \le CR_e\|\mathring{A}^e\|_{L^2(\Sigma,\gamma^e)}, \tag{18}$$

$$\|N\circ \mathrm{id}_{S_{R_e}(a_e)} - \nu^e\circ\psi\|_{L^2(S)} \le CR_e\|\mathring{A}^e\|_{L^2(\Sigma,\gamma^e)}. \tag{19}$$

and

$$\sup_{S_{R_e}(a_e)} |h^2-1| \le C\|\mathring{A}^e\|_{L^2(\Sigma,\gamma^e)}. \tag{20}$$

Here, as before, h denotes the conformal factor of the map ψ and N is the normal of $S_{R_e}(a_e)$.

3 First and second variation

In this section we calculate the first and second variation of the Willmore functional subject to an area constraint.

To compute the first variation of $\mathcal{W}$ let $\Sigma \subset M$ be a surface and let $F : \Sigma \times (-\varepsilon, \varepsilon) \to M$ be a variation of Σ with $F(\Sigma, s) = \Sigma_s$ and lapse $\frac{\partial F}{\partial s}\big|_{s=0} = \alpha\nu$. Recall the following well known evolution equations for deformations of hypersurfaces (see for example [8]). Here and in the following we will understand that all s-derivatives are evaluated at $s = 0$, and will not further denote this explicitly:

$$
\begin{aligned}
\frac{\partial}{\partial s}\gamma_{ij} &= 2\alpha A_{ij}, \\
\frac{\partial}{\partial s}\,\mathrm{d}\mu &= \alpha H\,\mathrm{d}\mu, \\
\frac{\partial}{\partial s}\gamma^{ij} &= -2\alpha A^{ij}, \\
\frac{\partial}{\partial s}\nu &= -\nabla\alpha, \\
\frac{\partial}{\partial s}A_{ij} &= -\nabla_i\nabla_j\alpha + \alpha\left(A_{ik}A_j^k - T_{ij}\right), \\
\frac{\partial}{\partial s}H &= L\alpha,
\end{aligned}
$$

where

$$
Lf = -\Delta f - f\left(|A|^2 + {}^M\mathrm{Rc}(\nu,\nu)\right) \tag{21}
$$

is the well known Jacobi operator for minimal surfaces,

$$
T_{ij} = {}^M\mathrm{Rm}(\partial_i, \nu, \nu, \partial_j) = {}^M\mathrm{Rc}^T_{ij} + G(\nu,\nu)\gamma_{ij}
$$

and $G = {}^M\mathrm{Rc} - \frac{1}{2}{}^M\mathrm{Sc}\cdot g$ is the Einstein tensor.

The first variation of $\mathcal{W}$ can then be computed as

$$
0 = \frac{d}{ds}\bigg|_{s=0}\mathcal{W}[\Sigma_s] = \int_\Sigma HL\alpha + \frac{1}{2}H^3\alpha\,\mathrm{d}\mu = \int_\Sigma \left(LH + \frac{1}{2}H^3\right)\alpha\,\mathrm{d}\mu. \tag{22}
$$

A critical point for $\mathcal{W}$ therefore satisfies the Euler–Lagrange equation

$$
LH + \frac{1}{2}H^3 = 0. \tag{23}
$$

To compute the second variation of $\mathcal{W}$, note that by (22)

$$
\begin{aligned}
\frac{d^2}{ds^2}\bigg|_{s=0}\mathcal{W}[\Sigma_s] = &\int_\Sigma \frac{\partial}{\partial s}\left(-\Delta H - H|A|^2 - H\,{}^M\mathrm{Rc}(\nu,\nu) + \frac{1}{2}H^3\right)\alpha\,\mathrm{d}\mu\bigg|_{s=0} \\
&+ \int_\Sigma \left(LH + \frac{1}{2}H^3\right)\left(\frac{\partial\alpha}{\partial s} + H\alpha^2\right)\bigg|_{s=0}\mathrm{d}\mu.
\end{aligned} \tag{24}
$$

Thus we have to compute the linearization of the Willmore operator defined as follows

$$W\alpha := \frac{d}{ds}\Big|_{s=0}\left(-\Delta H - H|A|^2 - H\,{}^M\mathrm{Rc}(\nu,\nu) + \frac{1}{2}H^3\right)$$
$$= -\left[\frac{\partial}{\partial s}, \Delta\right] H - H\frac{\partial}{\partial s}|A|^2 - H\frac{\partial}{\partial s}{}^M\mathrm{Rc}(\nu,\nu) + LL\alpha + \frac{3}{2}H^2 L\alpha. \quad (25)$$

Using the above formula for the variations of the metric and the second fundamental form we compute

$$\frac{\partial}{\partial s}A^{ij} = -3\alpha A^{ik}A_k^j - \nabla^i\nabla^j\alpha - \alpha T^{ij}$$

and therefore

$$\frac{\partial}{\partial s}|A|^2 = \frac{\partial}{\partial s}(A^{ij}A_{ij}) = -2\alpha\,\mathrm{tr}\,A^3 - 2A_{ij}\nabla^i\nabla^j\alpha - 2\alpha A^{ij}T_{ij}. \quad (26)$$

The next term we compute is $\frac{\partial}{\partial s}{}^M\mathrm{Rc}(\nu,\nu)$, yielding

$$\frac{\partial}{\partial s}{}^M\mathrm{Rc}(\nu,\nu) = \alpha\nabla_\nu{}^M\mathrm{Rc}(\nu,\nu) - 2\,{}^M\mathrm{Rc}(\nabla\alpha,\nu). \quad (27)$$

We turn to computing the commutator $\left[\frac{\partial}{\partial s}, \Delta\right]$. We write $\Delta = \mathrm{div}\,\nabla$ and we compute the commutator of $\left[\frac{\partial}{\partial s}, \mathrm{div}\right]$ and $\left[\frac{\partial}{\partial s}, \nabla\right]$ individually. First note that since $\nabla^k\phi = \gamma^{kl}\frac{\partial\phi}{\partial x^l}$ we have

$$\frac{\partial}{\partial s}\left(\nabla^k\phi\right) = -2\alpha A^{kl}\frac{\partial\phi}{\partial x^l} + \gamma^{kl}\frac{\partial}{\partial x^l}\frac{\partial}{\partial s}\phi,$$

and hence

$$\left[\frac{\partial}{\partial s}, \nabla\right]\phi = -2\alpha A_l^k\nabla^l\phi = -2\alpha S(\nabla\phi). \quad (28)$$

Here S is the shape operator, that is the tensor defined by

$$\gamma(S(X), Y) = A(X, Y)$$

for all $X, Y \in \mathcal{X}(\Sigma)$. Now we turn to the computation of $\left[\frac{\partial}{\partial s}, \mathrm{div}\right]$, operating on vector fields. Let $X, Y \in \mathcal{X}(\Sigma)$ be vector fields. We compute

$$\gamma(\nabla_X Y, X) = \gamma(\nabla_Y X, X) + \gamma([X, Y], X) = \frac{1}{2}Y(\gamma(X, X)) + \gamma(X, [X, Y]). \quad (29)$$

We choose a local orthonormal frame $\{e_i\}$ and propagate it using the ODE

$$\frac{\partial}{\partial s} e_i = -\alpha S(e_i).$$

Then the $\{e_i\}$ remain orthonormal under the evolution. Plugging $X = e_i$ into Eq. 29 yields

$$\gamma(\nabla_{e_i} Y, e_i) = \gamma(e_i, [e_i, Y]).$$

Differentiating this equation and using the above formulas we get by a fairly standard computation

$$\begin{aligned}
\frac{\partial}{\partial s}\gamma(\nabla_{e_i} Y, e_i) &= 2\alpha A(e_i, [e_i, Y]) - \gamma(\alpha S(e_i), [e_i, Y]) - \gamma(e_i, [\alpha S(e_i), Y]) \\
&= \alpha A(e_i, \nabla_{e_i} Y) - \alpha A(e_i, \nabla_Y e_i) - \alpha\gamma(e_i, \nabla_{S(e_i)} Y) \\
&\quad + \alpha Y(\gamma(e_i, S(e_i))) - \alpha\gamma(\nabla_Y e_i, S(e_i)) + Y(\alpha) A(e_i, e_i) \\
&= \alpha A(e_i, \nabla_{e_i} Y) - \alpha\gamma(e_i, \nabla_{S(e_i)} Y) + \alpha\nabla_Y A(e_i, e_i) + Y(\alpha) A(e_i, e_i).
\end{aligned}$$

If we now choose $\{e_i\}$ to be an orthogonal system of eigenvectors for S, that is $S(e_i) = \lambda_i e_i$, then we see that the first two terms cancel, and after summation over i we infer

$$\left[\frac{\partial}{\partial s}, \operatorname{div}\right] Y = \sum_i \alpha\nabla_Y A(e_i, e_i) + Y(\alpha) A(e_i, e_i) = \nabla_Y(\alpha H). \tag{30}$$

We combine Eqs. 30 and 28 and get, using $\Delta = \operatorname{div}\nabla$,

$$\left[\frac{\partial}{\partial s}, \Delta\right]\phi = \langle\nabla\phi, \nabla(\alpha H)\rangle - 2A(\nabla\alpha, \nabla\phi) - 2\alpha \operatorname{div}\left(S(\nabla\phi)\right). \tag{31}$$

Using an ON frame $\{e_i\}$, we compute further that

$$\operatorname{div}\left(S(\nabla\phi)\right) = \sum_i \nabla_{e_i} A(\nabla\phi, e_i) + A(\nabla_{e_i}\nabla\phi, e_i)$$

and in view of the Codazzi equation this yields

$$\begin{aligned}
\operatorname{div}\left(S(\nabla\phi)\right) &= \langle\nabla\phi, \nabla H\rangle + \sum_i {}^M\mathrm{Rm}(e_i, \nabla\phi, \nu, e_i) + A(\nabla_{e_i}\nabla\phi, e_i) \\
&= \langle\nabla\phi, \nabla H\rangle + {}^M\mathrm{Rc}(\nabla\phi, \nu) + \langle A, \nabla^2\phi\rangle.
\end{aligned}$$

Plugging this formula into (31) gives

$$\begin{aligned}
\left[\frac{\partial}{\partial s}, \Delta\right]\phi &= H\langle\nabla\alpha, \nabla\phi\rangle - \alpha\langle\nabla\phi, \nabla H\rangle - 2A(\nabla\alpha, \nabla\phi) \\
&\quad - 2\alpha\, {}^M\mathrm{Rc}(\nabla\phi, \nu) - 2\alpha\langle A, \nabla^2\phi\rangle.
\end{aligned} \tag{32}$$

Finally we substitute the results (26), (27) and (32) into (25) to obtain

$$\begin{aligned} W\alpha = &\, LL\alpha + \frac{3}{2}H^2 L\alpha - H\langle\nabla\alpha, \nabla H\rangle + \alpha|\nabla H|^2 \\ &+ 2A(\nabla\alpha, \nabla H) + 2\alpha\, {}^M\mathrm{Rc}(\nabla H, \nu) + 2\alpha\langle A, \nabla^2 H\rangle \\ &+ 2\alpha H \operatorname{tr} A^3 + 2H\langle A, \nabla^2\alpha\rangle + 2\alpha H\langle A, T\rangle \\ &- \alpha H \nabla_\nu {}^M\mathrm{Rc}(\nu, \nu) + 2H\, {}^M\mathrm{Rc}(\nabla\alpha, \nu). \end{aligned} \tag{33}$$

We rewrite Eq. 33 in dimension two, as it somewhat simplifies. We split $A = \mathring{A} + \frac{1}{2}H\gamma$ in the following terms

$$\begin{aligned} \langle A, \nabla^2\alpha\rangle &= \langle \mathring{A}, \nabla^2\alpha\rangle + \frac{1}{2}H\Delta\alpha, \\ A(\nabla\alpha, \nabla H) &= \mathring{A}(\nabla\alpha, \nabla H) + \frac{1}{2}H\langle\nabla\alpha, \nabla H\rangle, \\ \langle\nabla^2 H, A\rangle &= \frac{1}{2}H\Delta H + \langle\mathring{A}, \nabla^2 H\rangle, \\ \operatorname{tr} A^3 &= \operatorname{tr} \mathring{A}^3 + H|\mathring{A}|^2 + \frac{1}{2}H|A|^2 = H|\mathring{A}|^2 + \frac{1}{2}H|A|^2, \\ \langle A, T\rangle &= \frac{1}{2}H\, {}^M\mathrm{Rc}(\nu, \nu) + \langle\mathring{A}, T\rangle. \end{aligned}$$

Plugging these into (33), and setting $\omega = \mathrm{Ric}(\nu, \cdot)^T$ yields

$$\begin{aligned} W\alpha = &\, LL\alpha + \frac{1}{2}H^2 L\alpha + 2H\langle\mathring{A}, \nabla^2\alpha\rangle + 2H\omega(\nabla\alpha) + 2\mathring{A}(\nabla\alpha, \nabla H) \\ &+ \alpha\Big(|\nabla H|^2 + 2\omega(\nabla H) + H\Delta H + 2\langle\nabla^2 H, \mathring{A}\rangle \\ &+ 2H^2|\mathring{A}|^2 + 2H\langle\mathring{A}, T\rangle - H\nabla_\nu {}^M\mathrm{Rc}(\nu, \nu)\Big). \end{aligned} \tag{34}$$

To demonstrate that W is L^2-self adjoint we compute, with $D = |A|^2 + \mathrm{Ric}(\nu, \nu)$,

$$\begin{aligned} \int_\Sigma \beta H^2 L\alpha \, \mathrm{d}\mu &= \int_\Sigma \beta H^2(-\Delta\alpha - \alpha D)\, \mathrm{d}\mu \\ &= \int_\Sigma H^2\langle\nabla\alpha, \nabla\beta\rangle + 2H\beta\langle\nabla H, \nabla\alpha\rangle - \alpha\beta H^2 D \, \mathrm{d}\mu, \end{aligned}$$

and, using $\operatorname{div} \mathring{A} = \frac{1}{2}\nabla H + \omega$,

$$\begin{aligned} &\int_\Sigma \beta H\langle\mathring{A}, \nabla^2\alpha\rangle\, \mathrm{d}\mu \\ &\quad = -\int_\Sigma \beta\mathring{A}(\nabla\alpha, \nabla H) + H\mathring{A}(\nabla\alpha, \nabla\beta) + \frac{1}{2}\beta H\langle\nabla\alpha, \nabla H\rangle + H\beta\omega(\nabla\alpha)\, \mathrm{d}\mu. \end{aligned}$$

Thus

$$\int_\Sigma \beta W\alpha \,\mathrm{d}\mu = \int_\Sigma L\alpha L\beta + \frac{1}{2}H^2\langle\nabla\alpha,\nabla\beta\rangle - 2H\,\mathring{A}(\nabla\alpha,\nabla\beta)$$
$$+\alpha\beta\Big(|\nabla H|^2 + 2\omega(\nabla H) + H\Delta H + 2\langle\nabla^2 H,\,\mathring{A}\rangle + 2H^2|\mathring{A}|^2$$
$$+\;2H\langle\mathring{A},T\rangle - H\nabla_\nu{}^M\mathrm{Rc}(\nu,\nu) - \frac{1}{2}H^2|A|^2 - \frac{1}{2}H^{2M}\mathrm{Rc}(\nu,\nu)\Big). \qquad (35)$$

and from this representation it is obvious that the bilinear form associated to W is symmetric, and hence W is L^2-self adjoint.

Recall that the goal is to find a critical point of the Willmore energy in the class of surfaces with given area. From (23) we get that for a critical point of this problem we have

$$0 = \int_\Sigma \Big(LH + \frac{1}{2}H^3\Big)\alpha\,\mathrm{d}\mu \qquad (36)$$

for all α which respect the constraint $\int_\Sigma \alpha H\,\mathrm{d}\mu = 0$. We thus find the Euler–Lagrange equation

$$LH + \frac{1}{2}H^3 = \lambda H, \qquad (37)$$

where λ is a constant. Let us turn to the computation of the second variation

$$\frac{\partial^2}{\partial s^2}\Big|_{s=0}\mathcal{W}[\Sigma_s] = \int_\Sigma \alpha W\alpha + \Big(LH + \frac{1}{2}H^3\Big)\Big(\frac{\partial\alpha}{\partial s} + H\alpha^2\Big)\,\mathrm{d}\mu. \qquad (38)$$

At this point we only consider variations that leave the area constant up to second order. This gives

$$0 = \frac{\partial^2}{\partial s^2}\Big|_{s=0}|\Sigma_s| = \frac{\partial}{\partial s}\Big|_{s=0}\int_{\Sigma_s}\alpha H\,\mathrm{d}\mu = \int_\Sigma \frac{\partial\alpha}{\partial s}H + \alpha L\alpha + \alpha^2 H^2\,\mathrm{d}\mu. \qquad (39)$$

Thus we can compute

$$\int_\Sigma (LH + \frac{1}{2}H^3)\Big(\frac{\partial\alpha}{\partial s} + H\alpha^2\Big)\,\mathrm{d}\mu = \int_\Sigma \lambda H\Big(\frac{\partial\alpha}{\partial s} + H\alpha^2\Big)\,\mathrm{d}\mu$$
$$= -\lambda\int_\Sigma \alpha L\alpha. \qquad (40)$$

Plugging this into (38) yields that the second variation of $\mathcal{W}$ on a stationary surface Σ is given by

$$\delta^2\mathcal{W}(\alpha,\alpha) = \int_\Sigma \alpha W\alpha - \lambda\alpha L\alpha \,\mathrm{d}\mu, \tag{41}$$

for all valid test functions $\alpha \in C^\infty(\Sigma)$ satisfying $\int_\Sigma \alpha H \,\mathrm{d}\mu = 0$.

4 Integral curvature estimates

In this section we derive a priori bounds on the curvature of surfaces which are solutions of the Eq. 1. We will later make the assumption that both $H > 0$ and $\lambda > 0$ on these surfaces. Without the assumption on λ we can derive the following lemma.

Lemma 6 *If a surface Σ satisfies Eq. 1 with $H > 0$, then*

$$\lambda|\Sigma| + \int_\Sigma |\nabla \log H|^2 + \frac{1}{4}H^2 + \frac{1}{2}|\mathring{A}|^2 \,\mathrm{d}\mu \le 4\pi - \int_\Sigma \frac{1}{2}{}^M\mathrm{Sc}\,\mathrm{d}\mu.$$

If ${}^M\mathrm{Sc} \ge 0$ we have that

$$4\lambda|\Sigma| + \int_\Sigma H^2 \,\mathrm{d}\mu \le 16\pi.$$

Proof Multiply Eq. 1 by H^{-1} and integrate the first term by parts. This yields

$$\lambda|\Sigma| + \int_\Sigma |\nabla \log H|^2 + |\mathring{A}|^2 + {}^M\mathrm{Rc}(\nu,\nu)\,\mathrm{d}\mu = 0. \tag{42}$$

We can now use the Gauss equation (5) and the Gauss-Bonnet formula to get

$$\lambda|\Sigma| + \int_\Sigma |\nabla \log H|^2 + \frac{1}{4}H^2 + \frac{1}{2}|\mathring{A}|^2 \,\mathrm{d}\mu \le 4\pi - \int_\Sigma \frac{1}{2}{}^M\mathrm{Sc}\,\mathrm{d}\mu.$$

□

The above lemma already implies that the Hawking mass is positive on such surfaces.

Theorem 4 *If (M, g) satisfies ${}^M\mathrm{Sc} \ge 0$ and if Σ is a compact spherical surface satisfying Eq. 1 with $H > 0$, then $m_H(\Sigma) \ge 0$ if $\lambda \ge 0$.*

Furthermore if $F : \Sigma\times[0,\varepsilon) \to M$ is a variation with initial velocity $\frac{\partial F}{\partial s}\big|_{s=0} = \alpha\nu$ and $\int_\Sigma \alpha H \,\mathrm{d}\mu \ge 0$, then

$$\frac{d}{ds} m_H(F(\Sigma, s)) \ge 0.$$

Note that the condition on α means that the area is increasing along the variation.

Proof Non-negativity of the Hawking-mass is obvious from Lemma 6. To observe monotonicity, we compute the variation of the Hawking-mass. We denote $F(\Sigma, s) = \Sigma_s$.

$$(16\pi)^{3/2} \left.\frac{d}{ds}\right|_{s=0} m_H(\Sigma_s)$$
$$= \frac{1}{2|\Sigma|^{1/2}} \left(\int_\Sigma \alpha H \, \mathrm{d}\mu\right) \left(16\pi - \int_\Sigma H^2 \, \mathrm{d}\mu\right) - 2|\Sigma|^{1/2} \int_\Sigma \lambda\alpha H \, \mathrm{d}\mu$$

as Eq. 1 implies that the variation of $\int_\Sigma H^2 \, \mathrm{d}\mu$ is given by $2\lambda H$. This yields

$$(16\pi)^{3/2} \left.\frac{d}{ds}\right|_{s=0} m_H = \frac{1}{2|\Sigma|^{1/2}} \left(\int_\Sigma \alpha H \, \mathrm{d}\mu\right) \left(16\pi - 4\lambda|\Sigma| - \int_\Sigma H^2 \, \mathrm{d}\mu\right).$$

Lemma 6 implies non-negativity of the right hand side. □

Subsequently we assume that the manifold (M, g) is (m, η, σ)-asymptotically Schwarzschild for some $\eta < \eta_0$, where η_0 is fixed. Furthermore $\Sigma \subset M$ is a surface with $r_{\min} \geq r_0$ large enough. The particular r_0 will only depend on m, η_0 and σ, and we will no longer explicitly denote the dependence on these quantities. Similarly, constants denoted with a capital C are understood to depend on m, η_0 and σ, in addition to quantities explicitly mentioned. In contrast, constants denoted by c will not have any implicit dependency. We no longer require the condition ${}^M\mathrm{Sc} \geq 0$.

Lemma 7 *Let (M, g) be (m, η, σ)-asymptotically Schwarzschild. Then there exists $r_0 = r_0(m, \eta, \sigma)$ and a constant $C = C(m, \eta, \sigma)$ such that for all topologically spherical surfaces $\Sigma \subset M \backslash B_{r_0}(0)$ satisfying Eq. 1 with $\lambda > 0$ and $H > 0$, we have the following estimates.*

$$\int_\Sigma |\mathring{A}|^2 + |\nabla \log H|^2 \, \mathrm{d}\mu \leq C r_{\min}^{-1},$$

$$\left| \int_\Sigma H^2 \, \mathrm{d}\mu - 16\pi \right| \leq C r_{\min}^{-1},$$

and

$$\lambda |\Sigma| \leq C r_{\min}^{-1}.$$

Proof From Lemma 6 we get the bound

$$\int_\Sigma H^2 \, \mathrm{d}\mu \leq 16\pi - 2 \int_\Sigma {}^M\mathrm{Sc} \, \mathrm{d}\mu$$

As $|{}^M\mathrm{Sc}| \leq C(\eta) r^{-4}$ we find that in view of Lemma 3

$$\int_\Sigma H^2 \,\mathrm{d}\mu \leq 16\pi + C r_{\min}^{-2} \int_\Sigma H^2 \,\mathrm{d}\mu.$$

So if $r_{\min}$ is large enough, eventually

$$\int_\Sigma H^2 \,\mathrm{d}\mu \leq 16\pi + C r_{\min}^{-2}.$$

We can write the Gauss equation (5) in the following form

$$\frac{1}{2}{}^\Sigma\mathrm{Sc} \leq \frac{1}{2}{}^\Sigma\mathrm{Sc} + \frac{1}{2}|\mathring{A}|^2 = \frac{1}{4}H^2 + \frac{1}{2}{}^M\mathrm{Sc} - {}^M\mathrm{Rc}(\nu,\nu).$$

Integrating and using Lemma 3 gives

$$16\pi \leq \int_\Sigma H^2 \,\mathrm{d}\mu + C r_{\min}^{-1}.$$

The remaining claims now follow from Lemma 6. □

The initial bound on $\mathring{A}$ derived above is crucial for higher curvature estimates on Σ. We vary on the strategy outlined in [10, Sect. 2]. The estimates there were derived in flat ambient space and therefore we review them here for the readers convenience. More importantly, we can use the fact that $H > 0$, which improves the estimates, as the absolute error is slightly better behaved.

Lemma 8 *Under the assumptions of Lemma 7 we have*

$$\int_\Sigma \frac{|\Delta H|^2}{H^2} \,\mathrm{d}\mu \leq 2\int_\Sigma |\mathring{A}|^4 \,\mathrm{d}\mu + 2\int_\Sigma \left({}^M\mathrm{Rc}(\nu,\nu) + \lambda\right)^2 \,\mathrm{d}\mu.$$

Proof We use Eq. 1, divided by H, which gives

$$\begin{aligned}\int_\Sigma \frac{|\Delta H|^2}{H^2} \,\mathrm{d}\mu &= \int_\Sigma \left(|\mathring{A}|^2 + {}^M\mathrm{Rc}(\nu,\nu) + \lambda\right)^2 \,\mathrm{d}\mu \\ &\leq 2\int_\Sigma |\mathring{A}|^4 \,\mathrm{d}\mu + 2\int_\Sigma \left({}^M\mathrm{Rc}(\nu,\nu) + \lambda\right)^2 \,\mathrm{d}\mu.\end{aligned}$$

□

Lemma 9 *Under the assumptions of Lemma 7 we have*

$$\int_\Sigma \frac{|\nabla^2 H|^2}{H^2}\,\mathrm{d}\mu + \frac{1}{2}|\nabla H|^2\,\mathrm{d}\mu$$
$$\le C r_{\min}^{-3}\int_\Sigma |\nabla \log H|^2 + \int_\Sigma ({}^M\mathrm{Rc}(\nu,\nu)+\lambda)^2 + |\mathring{A}|^4 + |\nabla \log H|^4\,\mathrm{d}\mu.$$

Proof

$$\begin{aligned}\int_\Sigma \frac{|\nabla^2 H|^2}{H^2}\,\mathrm{d}\mu &= \int_\Sigma -H^{-2}\nabla_i\nabla_j\nabla_i H\nabla_j H + 2H^{-3}\nabla^2 H(\nabla H,\nabla H)\,\mathrm{d}\mu\\ &= \int_\Sigma -H^{-2}\nabla_j \Delta H\nabla_j H - H^{-2\,\Sigma}\mathrm{Rm}_{ijki}\nabla_j H\nabla_k H\,\mathrm{d}\mu\\ &\quad + \int_\Sigma 2H^{-3}\nabla^2 H(\nabla H,\nabla H)\,\mathrm{d}\mu\\ &= \int_\Sigma \frac{|\Delta H|^2}{H^2} - H^{-2\,\Sigma}\mathrm{Rm}_{ijki}\nabla_j H\nabla_k H\,\mathrm{d}\mu\\ &\quad + \int_\Sigma 2H^{-3}\nabla^2 H(\nabla H,\nabla H) - 2H^{-3}|\nabla H|^2\Delta H\,\mathrm{d}\mu. \end{aligned} \tag{43}$$

In view of the Gauss equation (4) the curvature term yields

$$\begin{aligned}{}^\Sigma\mathrm{Rm}_{ijki}\nabla_j H\nabla_k H &= \left({}^M\mathrm{Rm}_{ijki} + \frac{1}{4}H^2\gamma_{jk} - \mathring{A}_{ik}\mathring{A}_{ij}\right)\nabla_j H\nabla_k H\\ &= \frac{1}{4}H^2|\nabla H|^2 + {}^M\mathrm{Rm}_{ijki}\nabla_j H\nabla_k H - \mathring{A}_{ik}\mathring{A}_{ij}\nabla_j H\nabla_k H.\end{aligned}$$

Furthermore, we estimate

$$\begin{aligned}&\int_\Sigma 2H^{-3}\nabla^2 H(\nabla H,\nabla H) - 2H^{-3}|\nabla H|^2\Delta H\,\mathrm{d}\mu\\ &\quad\le \int_\Sigma \frac{1}{2}\frac{|\nabla^2 H|^2}{H^2} + c|\nabla \log H|^4\,\mathrm{d}\mu.\end{aligned}$$

The first term can be absorbed to the right hand side of Eq. 43. We infer

$$\begin{aligned}&\int_\Sigma \frac{1}{2}\frac{|\nabla^2 H|^2}{H^2}+\frac{1}{4}|\nabla H|^2\,\mathrm{d}\mu\\&\quad\le \int_\Sigma \frac{|\Delta H|^2}{H^2}+c|\nabla\log H|^4+c|\mathring{A}|^4+c|{}^M\mathrm{Rm}||\nabla\log H|^2\,\mathrm{d}\mu.\end{aligned}$$

We use $|{}^M\mathrm{Rm}|\le Cr_{\min}^{-3}$ and Lemma 8 to conclude the claimed inequality. □

Lemma 10 *Under the assumptions of Lemma 7 we have*

$$\int_\Sigma |\nabla\mathring{A}|^2\,\mathrm{d}\mu+\frac{1}{2}H^2|\mathring{A}|^2\,\mathrm{d}\mu\le\int_\Sigma|\omega|^2+Cr_{\min}^{-3}\int_\Sigma|\mathring{A}|^2\,\mathrm{d}\mu+\int_\Sigma|\nabla H|^2+|\mathring{A}|^4\,\mathrm{d}\mu.$$

Proof Integrate equation (9), and use integration by parts on the left hand side, and on the first and the last term on the right hand side to conclude

$$\begin{aligned}\int_\Sigma|\nabla\mathring{A}|^2\,\mathrm{d}\mu+\frac{1}{2}\int_\Sigma H^2|\mathring{A}|^2\,\mathrm{d}\mu=&\int_\Sigma 2\langle\operatorname{div}\mathring{A},\frac{1}{2}\nabla H+\omega\rangle+|\mathring{A}|^4\\&+|\mathring{A}|^{2\,M}\mathrm{Rc}(\nu,\nu)-2\mathring{A}^{ij}\mathring{A}^{l}_{j}\,{}^M\mathrm{Rc}_{il}\,\mathrm{d}\mu.\end{aligned}$$

From the Codazzi equation we conclude that $\operatorname{div}\mathring{A}=\frac{1}{2}\nabla H+\omega$, and hence

$$\int_\Sigma|\nabla\mathring{A}|^2\,\mathrm{d}\mu+\frac{1}{2}\int_\Sigma H^2|\mathring{A}|^2\,\mathrm{d}\mu\le\int_\Sigma|\nabla H|^2+|\mathring{A}|^4+4|\omega|^2+c|\mathring{A}|^2|{}^M\mathrm{Rm}|\,\mathrm{d}\mu.$$

In view of $|{}^M\mathrm{Rm}|\le Cr_{\min}^{-3}$ the claimed estimate follows. □

Combining Lemma 9 and Lemma 10, we infer the following estimate.

Lemma 11 *Under the assumptions of Lemma 7 we have*

$$\begin{aligned}&\int_\Sigma\frac{|\nabla^2H|^2}{H^2}+|\nabla A|^2+|A|^2|\mathring{A}|^2\,\mathrm{d}\mu\\&\quad\le c\int_\Sigma|\omega|^2+\left({}^M\mathrm{Rc}(\nu,\nu)+\lambda\right)^2\mathrm{d}\mu+c\int_\Sigma|\mathring{A}|^4+|\nabla\log H|^4\,\mathrm{d}\mu\\&\qquad+Cr_{\min}^{-3}\int_\Sigma|\nabla\log H|^2+|\mathring{A}|^2\,\mathrm{d}\mu.\end{aligned}$$

At this point we need a variation on the multiplicative Sobolev inequality from [10, Lemma 2.5].

Lemma 12 *Under the assumptions of Lemma 7 we have*

$$
\begin{aligned}
&\int_\Sigma |\mathring{A}|^4 + |\nabla \log H|^4 \,\mathrm{d}\mu \\
&\quad \le c \left(\int_\Sigma |\mathring{A}|^2 + |\nabla \log H|^2 \,\mathrm{d}\mu \right) \\
&\qquad \cdot \left(\int_\Sigma \frac{|\nabla^2 H|^2}{H^2} + |\nabla A|^2 + |\nabla \log H|^4 + H^2 |\mathring{A}|^2 \,\mathrm{d}\mu \right).
\end{aligned}
$$

Proof We use the Michael–Simon–Sobolev inequality from Proposition 1 and Hölder's inequality to estimate

$$
\begin{aligned}
&\left(\int_\Sigma \left(|\nabla \log H|^2 \right)^2 \,\mathrm{d}\mu \right)^{1/2} \\
&\le c \int_\Sigma \frac{|\nabla^2 H|}{H} |\nabla \log H| + |\nabla \log H|^3 + H |\nabla \log H|^2 \,\mathrm{d}\mu \\
&\le c \left(\int_\Sigma |\nabla \log H|^2 \,\mathrm{d}\mu \right)^{1/2} \left(\int_\Sigma \frac{|\nabla^2 H|^2}{H^2} + |\nabla \log H|^4 + |\nabla H|^2 \,\mathrm{d}\mu \right)^{1/2}.
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
\left(\int_\Sigma |\mathring{A}|^4 \,\mathrm{d}\mu \right)^{1/2} &\le c \int_\Sigma |\mathring{A}| |\nabla \mathring{A}| + H |\mathring{A}|^2 \,\mathrm{d}\mu \\
&\le c \left(\int_\Sigma |\mathring{A}|^2 \,\mathrm{d}\mu \right)^{1/2} \left(\int_\Sigma |\nabla \mathring{A}|^2 + H^2 |\mathring{A}|^2 \,\mathrm{d}\mu \right)^{1/2}.
\end{aligned}
$$

Combining both inequalities yields the claim. □

The estimates above yield the initial curvature estimates.

Theorem 5 *For every m, η, σ there exist constants $r_0 = r_0(m, \eta, \sigma)$ and $C = C(m, \eta, \sigma)$ with the following properties:*

If (M, g) is (m, η, σ)-asymptotically Schwarzschild and $\Sigma \subset M \backslash B_{r_0}$ satisfies Eq. 1 with $H > 0$ and $\lambda > 0$, then Σ satisfies the estimate

$$\int_\Sigma \frac{|\nabla^2 H|^2}{H^2} + |\nabla A|^2 + |\nabla \log H|^4 + |A|^2 |\mathring{A}|^2 \, \mathrm{d}\mu$$
$$\leq c \int_\Sigma |\omega|^2 + ({}^M\mathrm{Rc}(\nu, \nu) + \lambda)^2 \, \mathrm{d}\mu + C r_{\min}^{-3} \int_\Sigma |\nabla \log H|^2 + |\mathring{A}|^2 \, \mathrm{d}\mu$$

Proof This is a consequence of Lemma 7, Lemma 11 and Lemma 12. □

Corollary 2 *Under the assumptions of Theorem 5 we have the estimate*

$$\int_\Sigma \frac{|\nabla^2 H|^2}{H^2} + |\nabla A|^2 + |\nabla \log H|^4 + |A|^2 |\mathring{A}|^2 \, \mathrm{d}\mu \leq C r_{\min}^{-4} + C r_{\min}^{-2} |\Sigma|^{-1}$$

Proof The claim follows in view of $|\omega| + |\operatorname{Ric}| \leq C r^{-3}$, Lemma 3 and the estimates from Lemma 7. □

Corollary 3 *Under the assumptions of Theorem 5, we have the estimate*

$$\int_\Sigma |\mathring{A}|^2 + |\nabla \log H|^2 \, \mathrm{d}\mu \leq C r_{\min}^{-4} |\Sigma| + C r_{\min}^{-2}.$$

Proof This follows from the Michael–Simon–Sobolev inequality and Kato's inequality. For example

$$\left(\int_\Sigma |\mathring{A}|^2 \, \mathrm{d}\mu \right)^{1/2} \leq C_s \int_\Sigma \Big| \nabla |\mathring{A}| \Big| + H |\mathring{A}| \, \mathrm{d}\mu$$
$$\leq c C_s |\Sigma|^{1/2} \left(\int_\Sigma |\nabla \mathring{A}|^2 + H^2 |\mathring{A}|^2 \, \mathrm{d}\mu \right)^{1/2}$$

Using Corollary 2 the claimed inequality for $\int |\mathring{A}|^2 \, \mathrm{d}\mu$ follows. The proof for $\int |\nabla \log H|^2 \, \mathrm{d}\mu$ is similar. □

5 Improved curvature estimates

Before we can approach the position estimates, we discuss how the decay rates in the curvature estimates in Sect. 4 can be improved. First we note that the estimates in Sect. 4 and Theorem 3 imply that solutions to Eq. 1 are close to spheres.

Proposition 2 *Let R_e be the geometric area radius of Σ with respect to the Euclidean metric, i.e. $\int_\Sigma \mathrm{d}\mu^e = 4\pi R_e^2$, and let a_e be the Euclidean center of gravity of Σ, that is*

$$a_e = \frac{\int_\Sigma \mathrm{id}_\Sigma \ \mathrm{d}\mu^e}{\int_\Sigma \mathrm{d}\mu^e}.$$

Let $S := S_{R_e}(a_e)$ be the sphere of radius R_e centered at a_e and let N be the Euclidean normal of S. Then there exists a conformal parameterization $\psi : S \to (\Sigma, \gamma^e)$ with conformal factor h^2 satisfying the following estimates.

$$\sup_S |\psi - \mathrm{id}_S| \le C R_e \left(\| \mathring{A} \|_{L^2} + \eta r_{\min}^{-2} \right) \tag{44}$$

$$\| N \circ \mathrm{id}_S - \nu^e \circ \psi \|_{L^2(S)} \le C R_e \left(\| \mathring{A} \|_{L^2} + \eta r_{\min}^{-2} \right) \tag{45}$$

$$\sup_S |h^2 - 1| \le C \left(\| \mathring{A} \|_{L^2} + \eta r_{\min}^{-2} \right) \tag{46}$$

Proof This follows immediately from Corollary 1, Theorem 3 and Corollary 2. □

In the sequel, an essential quantity will be the ratio between the center of mass and the radius of the approximating sphere. We denote it by

$$\tau := \frac{|a_e|}{R_e}, \tag{47}$$

where a_e and R_e are as in Proposition 2. Note that by Corollary 2 and (44) we have

$$\begin{aligned} r_{\min} &\ge R_e - |a_e| - C R_e (|| \mathring{A} ||_{L^2} + \eta r_{\min}^{-2}) \\ &\ge R_e (1 - \tau) - C_1 R_e (R_e r_{\min}^{-2} + r_{\min}^{-1} + \eta r_{\min}^{-2}). \end{aligned} \tag{48}$$

Analogously we can estimate $r_{\min}$ from above. If we now assume that

$$\tau \le (1 - \varepsilon) \quad \text{and} \quad R_e \le \frac{\varepsilon}{4 C_1} r_{\min}^2 \tag{49}$$

for some arbitrary $\varepsilon > 0$, we get for $r_{\min}$ large enough

$$C_1 (R_e r_{\min}^{-2} + r_{\min}^{-1} + \eta r_{\min}^{-2}) \le \frac{\varepsilon}{2}$$

and this shows that

$$C^{-1} r_{\min} \le R_e \le C r_{\min}.$$

Hence R_e and $r_{\min}$ are comparable to each other and therefore we will not distinguish between them any more and we phrase the estimates only in terms of $r_{\min}$. Constants C in this section will also depend on ε.

We can use the fact that Σ is well approximated by a round sphere to compute a precise expression for λ.

Proposition 3 *If* (M, g) *and* Σ *are as in Theorem 5, then if assumption* (49) *holds, we have*

$$\left|\lambda - \frac{2m}{R_S^3}\right| \le Cr_{\min}^{-2}\left(\|\mathring{A}\|_{L^2}^2 + \|\nabla \log H\|_{L^2}^2\right) + Cr_{\min}^{-4}\left(\tau + r_{\min}\|\mathring{A}\|_{L^2} + \eta r_{\min}^{-1}\right)$$

Here we set $R_S := \bar{\phi}^2 R_e$ *where* $\bar{\phi} = \phi(R_e) = 1 + \frac{m}{2R_e}$.

Proof Recall that from (42) we have

$$\left|\lambda|\Sigma| + \int_\Sigma \mathrm{Ric}(\nu, \nu)\,\mathrm{d}\mu\right| \le \int_\Sigma |\mathring{A}|^2 + |\nabla \log H|^2\,\mathrm{d}\mu \tag{50}$$

The goal is now to calculate the integral on the left. We start by estimating the error to the respective integral in Schwarzschild.

$$\begin{aligned}&\left|\int_\Sigma \mathrm{Ric}(\nu, \nu)\,\mathrm{d}\mu - \int_\Sigma \mathrm{Ric}^S(\nu^S, \nu^S)\,\mathrm{d}\mu^S\right|\\ &\quad\le c\int_\Sigma |\mathrm{Ric} - \mathrm{Ric}^S| + |\mathrm{Ric}||\nu - \nu^S| + |\mathrm{Ric}||\,\mathrm{d}\mu - \mathrm{d}\mu^S|\,\mathrm{d}\mu \le C\eta r_{\min}^{-2}.\end{aligned}$$

We furthermore replace ν^S and $\mathrm{d}\mu^S$ by the respective Euclidean quantities. This introduces some factors of ϕ which all cancel, and we therefore get no further error in the following step:

$$\left|\int_\Sigma \mathrm{Ric}(\nu, \nu)\,\mathrm{d}\mu - \int_\Sigma \mathrm{Ric}^S(\nu^e, \nu^e)\,\mathrm{d}\mu^e\right| \le C\eta r_{\min}^{-2}.$$

The second integral on the left can be replaced by an integration over the sphere $S = S_{R_e}(a_e)$ from Proposition 2, introducing only acceptable error terms. This technique was used extensively in [14]. To see how this works, we use the parameterization

$\psi : S \to \Sigma$ from Proposition 2 to calculate

$$
\begin{aligned}
&\left| \int_{\Sigma} \mathrm{Ric}^S(\nu^e, \nu^e)\, \mathrm{d}\mu^e - \int_S \mathrm{Ric}^S(N, N)\, \mathrm{d}\mu^e \right| \\
&\quad = \left| \int_S (\mathrm{Ric}^S \circ \psi) \left(\nu^e \circ \psi, \nu^e \circ \psi\right) h^2 - \mathrm{Ric}^S(N, N)\, \mathrm{d}\mu^e \right| \\
&\quad \le c \int_S |\, \mathrm{Ric}^S \circ \psi - \mathrm{Ric}^S \,| + |\, \mathrm{Ric}^S \,||h^2 - 1| + |\, \mathrm{Ric}^S \,||\nu^e \circ \psi - N|\, \mathrm{d}\mu^e \\
&\quad \le c\| \nabla^e \mathrm{Ric}^S \|_{L^\infty} \| \psi - \mathrm{Id} \|_{L^\infty} |\Sigma| + c\| \mathrm{Ric}^S \|_{L^1} \|h^2 - 1\|_{L^\infty} \\
&\qquad + c\| \mathrm{Ric}^S \|_{L^2} \|\nu^e \circ \psi - N\|_{L^2} \\
&\quad \le C r_{\min}^{-2} (r_{\min} \| \mathring{A} \|_{L^2} + \eta r_{\min}^{-1})
\end{aligned}
$$

Now use coordinates φ, ϑ on $S_R(a)$ such that $\cos\varphi = g^e\left(\frac{a_e}{|a_e|}, N\right)$. Then the representation $\mathrm{Ric}^S(N, N) = \phi^{-2} \frac{m}{r^3}(1 - 3g^e(\rho, N)^2)$ together with $\rho = r^{-1}(R_e N + a_e)$, implies that

$$
\begin{aligned}
&\int_S \mathrm{Ric}^S(N, N)\, \mathrm{d}\mu^e \\
&\quad = m \int_S \phi^{-2} \left(\frac{1}{r^3} - 3R_e^2 \frac{1}{r^5} - 6R_e|a_e| \frac{\cos\varphi}{r^5} - 3|a_e|^2 \frac{\cos^2\varphi}{r^5} \right) \mathrm{d}\mu^e
\end{aligned}
$$

Letting $\bar{\phi} := 1 + \frac{m}{2R_e}$ we can use the expression $r^2 = R_e^2 + 2R_e|a_e|\cos\varphi + |a_e|^2$ to estimate that

$$
\sup_S |\phi - \bar{\phi}| \le C\tau r_{\min}^{-1}
$$

which renders

$$
\begin{aligned}
&\int_S \mathrm{Ric}^S(N, N)\, \mathrm{d}\mu^e \\
&= \frac{m}{\bar{\phi}^2} \int_S \left(\frac{1}{r^3} - 3R_e^2 \frac{1}{r^5} - 6R_e|a_e| \frac{\cos\varphi}{r^5} - 3|a_e|^2 \frac{\cos^2\varphi}{r^5} \right) \mathrm{d}\mu^e + O(\tau r_{\min}^{-2})
\end{aligned}
$$

Integrals of this type can be computed explicitly as follows. First write

$$
\int_S \frac{\cos^l \varphi}{r^k}\, \mathrm{d}\mu^e = 2\pi R_e^2 \int_0^\pi \sin\varphi \frac{\cos^l\varphi}{r^k} d\varphi.
$$

We have $x = R_e N + a_e$, and hence $r = \sqrt{R_e^2 + 2R_e|a_e|\cos\varphi + |a_e|^2}$. Thus $\frac{d\varphi}{dr} = -\frac{r}{R_e|a_e|\sin\varphi}$, and $\cos\varphi = \frac{r^2 - R_e^2 - |a_e|^2}{2R_e|a_e|}$. Substituting this into the integral yields

$$2\pi R_e^2 \int_0^\pi \sin\varphi \frac{\cos^l\varphi}{r^k} d\varphi. = \frac{2\pi R_e}{|a_e|}(2R_e|a_e|)^{-l} \int_{|R_e - |a_e||}^{R_e + |a_e|} r^{1-k}(r^2 - R_e^2 - |a_e|^2)^l dr.$$

Thus we can compute (see Appendix A.1), if $|a_e| < R_e$,

$$\int_S \mathrm{Ric}^S(N, N)\, \mathrm{d}\mu^e = -\bar{\phi}^{-2}\frac{8\pi m}{R_e} + O\left(\tau r_{\min}^{-2}\right)$$

Collecting all error terms we introduced, this yields that

$$\left|\int_\Sigma \mathrm{Ric}(\nu, \nu)\, \mathrm{d}\mu + \frac{8\pi m}{R_S}\right| \le C r_{\min}^{-2}\left(\tau + r_{\min}\|\mathring{A}\|_{L^2} + \eta r_{\min}^{-1}\right)$$

The next step is to calculate the area of Σ. Similar to the above argument we estimate

$$\left|\int_\Sigma 1\, \mathrm{d}\mu - \int_\Sigma 1\, \mathrm{d}\mu^S\right| \le C\eta.$$

From Lemma 1 we get

$$\int_\Sigma 1\, \mathrm{d}\mu^S = \int_\Sigma \phi^4\, \mathrm{d}\mu^e.$$

We now replace ϕ by $\bar{\phi}$ in this integral. This yields an error of the following form

$$\left|\int_\Sigma \phi^4\, \mathrm{d}\mu^e - \int_\Sigma \bar{\phi}^4\, \mathrm{d}\mu^e\right| \le C r_{\min}\left(\tau + r_{\min}\|\mathring{A}\|_{L^2} + \eta r_{\min}^{-1}\right)$$

In conclusion we find that

$$\left||\Sigma| - 4\pi R_S^2\right| \le C r_{\min}\left(\tau + r_{\min}\|\mathring{A}\|_{L^2} + \eta r_{\min}^{-1}\right)$$

Using Lemma 7 we get

$$\left|\lambda|\Sigma| - 4\pi R_S^2\lambda\right| \le C r_{\min}^{-2}\left(\tau + r_{\min}\|\mathring{A}\|_{L^2} + \eta r_{\min}^{-1}\right)$$

Plugging this expression into Eq. 50, we arrive at the estimate

$$\left|\lambda - \frac{2m}{R_S^3}\right| \leq Cr_{\min}^{-2}\left(\|\mathring{A}\|_{L^2}^2 + \|\nabla \log H\|_{L^2}^2\right) + Cr_{\min}^{-4}\left(\tau + r_{\min}\|\mathring{A}\|_{L^2} + \eta r_{\min}^{-1}\right) \tag{51}$$

This yields the claim. □

If τ behaves as above, we have more control over the curvature terms which did not allow us to increase the decay rates in Sect. 4. In particular,

Proposition 4 *Under the assumptions of Theorem 5, if conditions (49) hold, then*

$$\begin{aligned}
\|\nu - \phi^{-2}\rho\|_{L^2(\Sigma)}^2 &\leq Cr_{\min}^2\left(\tau^2 + \|\mathring{A}\|_{L^2}^2 + \eta r_{\min}^{-2}\right)\\
\|\operatorname{Ric}(\nu,\nu) - \phi^{-4}\operatorname{Ric}^S(\rho,\rho)\|_{L^2(\Sigma)}^2 &\leq Cr_{\min}^{-4}\left(\tau^2 + \|\mathring{A}\|_{L^2}^2 + \eta r_{\min}^{-2}\right)\\
\|\omega\|_{L^2(\Sigma)}^2 &\leq Cr_{\min}^{-4}\left(\tau^2 + \|\mathring{A}\|_{L^2}^2 + \eta r_{\min}^{-2}\right)\\
\|\operatorname{Ric}^T - P_{\phi^{-2}\rho}^S\operatorname{Ric}^S\|_{L^2(\Sigma)}^2 &\leq Cr_{\min}^{-4}\left(\tau^2 + \|\mathring{A}\|_{L^2}^2 + \eta r_{\min}^{-2}\right)
\end{aligned}$$

Here, $P_{\phi^{-2}\rho}^S \operatorname{Ric}^S$ denotes the g^S-orthogonal projection of Ric^S to the subspace perpendicular to $\phi^{-2}\rho$.

Proof The proof is the similar to [14, Proposition 4.6]. However the claimed estimate here is somewhat more precise, so we briefly sketch the argument. To show the first assertion we first replace the quantities in the integral by the respective quantities computed with respect to the Schwarzschild metric

$$\left|\int_\Sigma g(\nu - \phi^{-2}\rho, \nu - \phi^{-2}\rho)\,\mathrm{d}\mu - \int_\Sigma g^S(\nu^S - \phi^{-2}\rho, \nu^S - \phi^{-2}\rho)\,\mathrm{d}\mu^S\right| \leq C\eta.$$

Then we note that

$$\int_\Sigma g^S(\nu^S - \phi^{-2}\rho, \nu^S - \phi^{-2}\rho)\,\mathrm{d}\mu^S = \int_\Sigma g^e(\nu^e - \rho, \nu^e - \rho)\,\mathrm{d}\mu^e.$$

We now parameterize again by ψ and calculate the difference to the respective quantity on S. We obtain

$$\begin{aligned}
&\left|\int_\Sigma g^e(\nu^e - \rho, \nu^e - \rho)\,\mathrm{d}\mu^e - \int_S g^e(N - \rho, N - \rho)\,\mathrm{d}\mu^e\right|\\
&\quad\leq C\left(\tau r_{\min}^2\|\mathring{A}\|_{L^2} + r_{\min}^2\|\mathring{A}\|_{L^2}^2 + \tau^2 + \tau\eta + \eta^2 r_{\min}^{-2}\right)
\end{aligned}$$

Since

$$\int_S g^e(N-\rho, N-\rho)\,\mathrm{d}\mu^e \le C\int_S r^{-2}(|r-R_e|^2+|a_e|^2)\,\mathrm{d}\mu^e$$
$$\le Cr_{\min}^2\left(\tau^2+\|\mathring{A}\|_{L^2}^2+\eta r_{\min}^{-2}\right),$$

where we used (48), we obtain the first inequality.

The other inequalities are then a consequence of the first, since they basically follow from expressing the quantities in terms of the respective quantities in Schwarzschild. □

This proposition can be used to improve the mean value estimate we obtained in Proposition 3 to the following L^2-estimate.

Proposition 5 *Under the assumptions of Theorem 5, if conditions (49) hold, we have*

$$\|\lambda+\mathrm{Ric}(\nu,\nu)\|_{L^2(\Sigma)} \le Cr_{\min}^{-2}\left(\tau+\|\mathring{A}\|_{L^2}+\|\nabla\log H\|_{L^2}+\eta r_{\min}^{-1}\right)$$

Proof We use the second estimate of Proposition 4 to express $\mathrm{Ric}(\nu,\nu)$ in terms of $\phi^{-4}\,\mathrm{Ric}^S(\rho,\rho)$ plus error. Then we use that up to second order $\phi^{-4}\,\mathrm{Ric}^S(\rho,\rho)=-\frac{2m}{R_S^3}$ plus error. In combination with Proposition 3 this yields the estimate. □

Propositions 4 and 5 give more precise estimates of the terms on the right hand side of Theorem 5. In combination with the initial estimate for $\|\mathring{A}\|_{L^2}$ we thus infer the following improved curvature estimates.

Theorem 6 *Under the assumptions of Theorem 5, if conditions (49) hold, then*

$$\int_\Sigma \frac{|\nabla^2 H|^2}{H^2}+|\nabla A|^2+|\nabla\log H|^4+|A|^2|\mathring{A}|^2\,\mathrm{d}\mu \le Cr_{\min}^{-4}\left(\tau^2+\eta r_{\min}^{-2}\right)$$

and furthermore

$$\int_\Sigma |\mathring{A}|^2+|\nabla\log H|^2\,\mathrm{d}\mu \le Cr_{\min}^{-2}\left(\tau^2+\eta r_{\min}^{-2}\right)$$

Proof First of all note that by the calculation in Corollary 3 we can estimate

$$\int_\Sigma |\mathring{A}|^2+|\nabla\log H|^2\,\mathrm{d}\mu \le C|\Sigma|\int_\Sigma \frac{|\nabla^2 H|^2}{H^2}+|\nabla A|^2+|\nabla\log H|^4$$
$$+|A|^2|\mathring{A}|^2\,\mathrm{d}\mu. \tag{52}$$

Since, under assumption (49) we have that $|\Sigma| r_{\min}^{-3} \to 0$, we can eventually absorb the second term on the right in Theorem 5 to the left hand side. In combination with Proposition 4 and Proposition 5 this yields that

$$\begin{aligned}\int_\Sigma \frac{|\nabla^2 H|^2}{H^2} + |\nabla A|^2 + |\nabla \log H|^4 + |A|^2 |\mathring{A}|^2 \, \mathrm{d}\mu \\ \leq C r_{\min}^{-4} \left(\tau^2 + \eta r_{\min}^{-2} + \| \mathring{A} \|_{L^2}^2 + \| \nabla \log H \|_{L^2}^2 \right)\end{aligned} \tag{53}$$

together with (52) we infer

$$\int_\Sigma |\mathring{A}|^2 + |\nabla \log H|^2 \, \mathrm{d}\mu \leq C r_{\min}^{-2} \left(\tau^2 + \eta r_{\min}^{-2} + \| \mathring{A} \|_{L^2}^2 + \| \nabla \log H \|_{L^2}^2 \right)$$

We absorb $\| \mathring{A} \|_{L^2}^2 + \| \nabla \log H \|_{L^2}^2$ to the left and obtain the second estimate. The first estimate follows from (53) and this estimate. □

Using this estimate, we also get a better control on derivatives of ω. In particular, we have the following

Proposition 6 *Under the assumptions of Theorem 5, if conditions (49) hold, then*

$$\| \nabla \omega \|_{L^2(\Sigma)}^2 \leq C r_{\min}^{-6} \left(\tau^2 + \eta r_{\min}^{-2} \right),$$

and

$$\| \nabla \operatorname{Ric}(\nu, \nu) \|_{L^2(\Sigma)}^2 \leq C r_{\min}^{-6} \left(\tau^2 + \eta r_{\min}^{-2} \right),$$

Proof To prove the first estimate calculate for $\{e_i\}$ a ON-frame on Σ that

$$\begin{aligned}\nabla_{e_i} \omega(e_k) &= e_i ({}^M\mathrm{Rc}(\nu, e_k)) - {}^M\mathrm{Rc}(\nu, {}^\Sigma\nabla_{e_i} e_k) \\ &= {}^M\nabla_{e_i} {}^M\mathrm{Rc}(\nu, e_k) + \frac{1}{2} H {}^M\mathrm{Rc}(e_i, e_k) - \frac{1}{2} H {}^M\mathrm{Rc}(\nu, \nu) \\ &\quad + {}^M\mathrm{Rc}(e_l, e_k) \mathring{A}_{il} - {}^M\mathrm{Rc}(\nu, \nu) \mathring{A}_{ik}.\end{aligned} \tag{54}$$

The last two terms including $\mathring{A}$ have the claimed decay, so we focus on the first three terms.

In Schwarzschild we have that on the centered spheres $\nabla^S \omega^S$ vanishes as ω^S vanishes, so we find that on centered spheres for a ON-frame $\{e_i^S\}$ tangent to the centered spheres

$$\begin{aligned}0 = \nabla_{e_i^S} \omega^S &= \nabla_{e_i^S}^S \operatorname{Ric}^S(\phi^{-2}\rho, e_k^S) + \frac{1}{2} H^S \operatorname{Ric}^S(e_i^S, e_k^S) \\ &\quad - \frac{1}{2} H \operatorname{Ric}^S(\phi^{-2}\rho, \phi^{-2}\rho).\end{aligned} \tag{55}$$

Following Proposition 4 we get that the first three terms of (55) equal the right hand side of (54) up to an error with L^2-norm bounded by $C\tau r_{\min}^{-3}$. This yields the first estimate. The second one is proved similarly. □

In the sequel we will use the improved integral estimates to derive improved pointwise estimates of the second fundamental form and its derivatives. Before doing this we need the following Lemma which is due to Kuwert and Schätzle [10] in the case that $M = \mathbf{R}^n$.

Lemma 13 *Under the assumptions of Theorem 5 we have for every smooth form φ along Σ*

$$\|\varphi\|^4_{L^\infty(\Sigma)} \le C\|\varphi\|^2_{L^2(\Sigma)} \int\limits_\Sigma (|\nabla^2\varphi|^2 + |H|^4|\varphi|^2)\,\mathrm{d}\mu. \tag{56}$$

Proof The proof of lemma 2.8 in [10] can be carried over to our situation since we saw in Proposition 1 that the Michael–Simon Sobolev inequality remains unchanged if (M, g) is (m, η, σ)-asymptotically Schwarzschild. □

In the next lemma we derive an L^2-estimate for $\nabla^2 H$.

Lemma 14 *Under the assumptions of Theorem 5, if conditions (49) hold, then*

$$\int\limits_\Sigma |\nabla^2 H|^2\,\mathrm{d}\mu \le C r_{\min}^{-4}\left(\|H\|^2_{L^\infty} + r_{\min}^{-2}\right)\left(\tau^2 + \eta r_{\min}^{-2}\right). \tag{57}$$

Proof We multiply Eq. 1 with ΔH and integrate to get

$$\begin{aligned}\int\limits_\Sigma |\Delta H|^2\,\mathrm{d}\mu &= -\int\limits_\Sigma H\Delta H(|\mathring{A}|^2 + {}^M\mathrm{Rc}(\nu,\nu) + \lambda)\,\mathrm{d}\mu\\ &\le \frac{1}{2}\int\limits_\Sigma |\Delta H|^2\,\mathrm{d}\mu + c\int\limits_\Sigma H^2|\mathring{A}|^4 + H^2\left({}^M\mathrm{Rc}(\nu,\nu) + \lambda\right)\,\mathrm{d}\mu\end{aligned} \tag{58}$$

Defining $f = |\mathring{A}|^2|H|$ and applying Proposition 1 we get

$$\begin{aligned}\left(\int\limits_\Sigma |\mathring{A}|^4 H^2\,\mathrm{d}\mu\right)^{1/2} &\le C\int\limits_\Sigma (|A||\mathring{A}||\nabla A| + |\mathring{A}|^2 H^2)\,\mathrm{d}\mu\\ &\le C\left(\int\limits_\Sigma |A|^2|\mathring{A}|^2\,\mathrm{d}\mu\right)^{1/2}\left(\int\limits_\Sigma |\nabla A|^2 + H^2|\mathring{A}|^2\,\mathrm{d}\mu\right)^{1/2}\end{aligned}$$

In combination, we infer

$$\int_\Sigma |\Delta H|^2 \,\mathrm{d}\mu \le \int_\Sigma H^2 \left({}^M\mathrm{Rc}(\nu,\nu)+\lambda\right)^2 \,\mathrm{d}\mu + C\left(\int_\Sigma |A|^2|\mathring{A}|^2 \,\mathrm{d}\mu\right)\left(\int_\Sigma |\nabla A|^2 + H^2|\mathring{A}|^2 \,\mathrm{d}\mu\right)$$

This implies the claim, since the first term is estimated in view of Proposition 5 and the second one in view of Theorem 6. Using the Bochner identity as in the proof of Lemma 9 finishes the proof. □

Now we are in a position to prove a pointwise estimate for H.

Proposition 7 *Let $S = S_{R_e}(a_e)$ be the approximating sphere for Σ from Proposition 2. As in Proposition 3 we let $\bar\phi = 1 + \frac{m}{2R_e}$ and define*

$$\bar H^S = \bar\phi^{-2}\frac{2}{R_e} - 2\bar\phi^{-3}\frac{m}{R_e^2}$$

Under the assumptions of Theorem 5, if conditions (49) hold, we have that

$$\|H - \bar H^S\|_{L^\infty(\Sigma)} \le C r_{\min}^{-2}\left(\tau + \sqrt{\eta}\, r_{\min}^{-1}\right). \tag{59}$$

Proof Since

$$\|H - H^S\|^2_{L^2(\Sigma)} \le C\eta^2 r_{\min}^{-4}$$

and $H^S = \phi^{-2}H^e - \frac{2m}{r^2}\phi^{-3}g^e(\rho,\nu^e)$ by Lemma 1, we can estimate using Propositions 2, 4 and Theorem 6 that

$$\begin{aligned}
\|H^S - \bar H^S\|^2_{L^2(\Sigma)} &\le C\Bigg(\left\|\phi^{-2}\left(H^e - \frac{2}{R_e}\right)\right\|^2_{L^2(\Sigma)} + \left\|(\phi^{-2}-\bar\phi^{-2})\frac{2}{R_e}\right\|^2_{L^2(\Sigma)} \\
&\quad + \left\|(\phi^{-3}-\bar\phi^{-3})\frac{2m}{R_e^2}\right\|^2_{L^2(\Sigma)} \\
&\quad + \left\|\phi^{-3}\left(\frac{2m}{r^2}g^e(\rho,\nu^e) - \frac{2m}{R_e^2}\right)\right\|^2_{L^2(\Sigma)}\Bigg) \\
&\le C\|\mathring{A}\|^2_{L^2(\Sigma)} + C\tau^2 r_{\min}^{-2} + C\eta r_{\min}^{-4} \\
&\le C r_{\min}^{-2}\left(\tau^2 + \eta r_{\min}^{-2}\right).
\end{aligned}$$

Combining these two estimates we conclude

$$\|H - \bar{H}^S\|^2_{L^2(\Sigma)} \leq C r_{\min}^{-2} \left(\tau^2 + \eta r_{\min}^{-2}\right).$$

We apply Lemma 13 to $\varphi = H - \bar{H}^S$ and get

$$\begin{aligned}\|H - \bar{H}^S\|^4_{L^\infty(\Sigma)} &\leq C\|H - \bar{H}^S\|^2_{L^2(\Sigma)} \left(\int_\Sigma (|\nabla^2 H|^2 + H^4|H - \bar{H}^S|^2 \, \mathrm{d}\mu\right)\\ &= I + II. \end{aligned} \tag{60}$$

Now we estimate term by term. We use Lemma 14 and the fact that $\|H\|_{L^\infty(\Sigma)} \leq \|\bar{H}^S\|_{L^\infty(\Sigma)} + \|H - \bar{H}^S\|_{L^\infty(\Sigma)}$ to get

$$\begin{aligned} I &\leq C r_{\min}^{-4} \left(\|H\|^2_{L^\infty(\Sigma)} + r_{\min}^{-2}\right)\left(\tau^2 + \eta r_{\min}^{-2}\right)\|H - \bar{H}^S\|^2_{L^2(\Sigma)}\\ &\leq C r_{\min}^{-2}\|H - \bar{H}^S\|^4_{L^\infty(\Sigma)} + C r_{\min}^{-8}\left(\tau^2 + \eta r_{\min}^{-2}\right)^2 \end{aligned}$$

where we also used the above estimate for $\|H - \bar{H}^S\|^2_{L^2(\Sigma)}$. Next we note that

$$\begin{aligned} \int_\Sigma H^4|H - \bar{H}^S|^2 \, \mathrm{d}\mu &\leq C \int_\Sigma H^2 \left((\bar{H}^S)^2|H - \bar{H}^S|^2 + |H - \bar{H}^S|^4\right) \mathrm{d}\mu\\ &\leq C(\bar{H}^S)^4 \int_\Sigma |H - \bar{H}^S|^2 \, \mathrm{d}\mu + C\|H - \bar{H}^S\|^4_{L^\infty(\Sigma)}. \end{aligned}$$

Hence we get

$$II \leq C r_{\min}^{-2}\|H - \bar{H}^S\|^4_{L^\infty(\Sigma)} + C r_{\min}^{-8}\left(\tau^2 + \eta r_{\min}^{-2}\right)^2.$$

Inserting these two estimates into (60) we conclude

$$\|H - \bar{H}^S\|^4_{L^\infty(\Sigma)} \leq C r_{\min}^{-2}\|H - \bar{H}^S\|^4_{L^\infty(\Sigma)} + C r_{\min}^{-8}\left(\tau^2 + \eta r_{\min}^{-2}\right)^2$$

and therefore, by choosing r_0 large enough we can absorb the first term on the right hand side and this finishes the proof of the proposition. □

In the next lemma we derive pointwise estimates for higher derivatives of the curvature.

Lemma 15 *Under the assumptions of Theorem 5, if conditions (49) hold, we have that*

$$r_{\min}\|\nabla H\|_{L^\infty(\Sigma)} + \|\mathring{A}\|_{L^\infty(\Sigma)} \leq C r_{\min}^{-2}\left(\tau + \sqrt{\eta} r_{\min}^{-1}\right) \tag{61}$$

Proof Using (8) we estimate

$$
\begin{aligned}
\|\Delta \mathring{A}\|_{L^2} &\le c(\|\nabla^2 H\|_{L^2} + \|H\|_{L^\infty}\|\mathring{A}\|_{L^4}^2 + \|H\|_{L^\infty}^2\|\mathring{A}\|_{L^2} + \|\mathring{A}\|_{L^2}\|\mathring{A}\|_{L^\infty}^2 \\
&\quad + \|{}^M\mathrm{Rm}\|_{L^\infty}\|\mathring{A}\|_{L^2} + \|\nabla\omega\|_{L^2}) \\
&\le C r_{\min}^{-3}\left(\tau + \sqrt{\eta} r_{\min}^{-1}\right) + C\|\mathring{A}\|_{L^2}\|\mathring{A}\|_{L^\infty}^2,
\end{aligned}
$$

where we used Theorem 6, Definition 1, Corollary 7 and Propositions 6 and 14. Using an integration by parts argument as in the proof of Lemma 9 we get

$$
\|\nabla^2 \mathring{A}\|_{L^2(\Sigma)} \le C r_{\min}^{-3}\left(\tau + \sqrt{\eta} r_{\min}^{-1}\right) + C\|\mathring{A}\|_{L^2(\Sigma)}\|\mathring{A}\|_{L^\infty(\Sigma)}^2,
$$

Hence we can apply Lemma 13 and get

$$
\begin{aligned}
\|\mathring{A}\|_{L^\infty(\Sigma)}^4 &\le c\|\mathring{A}\|_{L^2(\Sigma)}^2(\|\nabla^2 \mathring{A}\|_{L^2(\Sigma)}^2 + \|H\|_{L^\infty(\Sigma)}^4\|\mathring{A}\|_{L^2(\Sigma)}^2) \\
&\le C r_{\min}^{-8}\left(\tau^2 + \eta r_{\min}^{-2}\right)^2 + C r_{\min}^{-4}\|\mathring{A}\|_{L^\infty(\Sigma)}^4,
\end{aligned}
$$

where we used the above estimate for $\nabla^2 \mathring{A}$ and Theorem 6. Absorbing the last term on the right hand side into the term on the left hand side finishes the proof of the L^∞-estimate for $\mathring{A}$. For the estimate of ∇H we differentiate (1) and get

$$
\begin{aligned}
\|\nabla \Delta H\|_{L^2(\Sigma)} &\le c\big(\lambda\|\nabla H\|_{L^2(\Sigma)} + \|\mathring{A}\|_{L^\infty(\Sigma)}^2\|\nabla H\|_{L^2(\Sigma)} \\
&\quad + \|H\|_{L^\infty(\Sigma)}\|\mathring{A}\|_{L^\infty(\Sigma)}\|\nabla \mathring{A}\|_{L^2(\Sigma)} \\
&\quad + \|\mathrm{Ric}(\nu,\nu)\|_{L^\infty(\Sigma)}\|\nabla H\|_{L^2(\Sigma)} \\
&\quad + \|\mathrm{Ric}^T(\cdot,\nu)\|_{L^2(\Sigma)}\|A\|_{L^\infty(\Sigma)}^2 \\
&\quad + \|H\|_{L^\infty(\Sigma)}\|\nabla \mathrm{Ric}(\nu,\nu)\|_{L^2(\Sigma)}\big) \\
&\le C r_{\min}^{-4}\left(\tau + \sqrt{\eta} r_{\min}^{-1}\right).
\end{aligned}
$$

Hence by interchanging derivatives and integration by parts we get as before

$$
\|\nabla^3 H\|_{L^2(\Sigma)} \le C r_{\min}^{-4}\left(\tau + \sqrt{\eta} r_{\min}^{-1}\right).
$$

Applying Theorem 6 and Lemma 13 once more, we conclude

$$
\begin{aligned}
\|\nabla H\|_{L^\infty(\Sigma)}^4 &\le C\|\nabla H\|_{L^2(\Sigma)}^2\left(\|\nabla^3 H\|_{L^2(\Sigma)}^2 + r_{\min}^{-4}\|\nabla H\|_{L^2(\Sigma)}^2\right) \\
&\le C r_{\min}^{-12}\left(\tau^2 + \eta r_{\min}^{-2}\right)^2.
\end{aligned}
$$

This finishes the proof of the Lemma. □

6 Position estimates

To get estimates on the position of the approximating sphere, we exploit the translation sensitivity of surfaces satisfying

$$LH + \frac{1}{2}H^3 = \lambda H. \tag{62}$$

As it turns out, this position estimate is a delicate matter. The goal is to obtain an estimate for $\tau = |a_e|/R_e$ where a_e and R_e are the center and radius of the approximating sphere constructed in Proposition 2. In fact, we subsequently prove the following theorem

Theorem 7 *For all $m > 0$, η_0 and σ there exist $r_0 < \infty$, $\tau_0 > 0$ and $\varepsilon > 0$ with the following properties. Assume that (M, g) is (m, η, σ)-asymptotically Schwarzschild with $\eta \leq \eta_0$ and*

$$|{}^M\mathrm{Sc}| \leq \eta r^{-5}.$$

Then if Σ is a surface satisfying Eq. 1 with $H > 0$, $\lambda > 0$, $r_{\min} > r_0$ and

$$\tau \leq \tau_0 \quad \textit{and} \quad R_e \leq \varepsilon r_{\min}^2,$$

then

$$\tau \leq C\sqrt{\eta} r_{\min}^{-1}.$$

Note that the assumptions of Theorem 7 imply the assumptions (49). We will therefore take r_0 large enough to be able to apply the estimates derived in Sect. 5.

Theorem 7 follows from Proposition 8, which states that under the assumptions of Theorem 7 we have in fact

$$\tau \leq C\left(\tau^2 + \sqrt{\eta} r_{\min}^{-1}\right),$$

for some constant C depending only on m, η_0 and σ, whenever r_0 is large enough. Assuming that $\tau_0 < 1/(2C)$ yields the claim.

The crucial ingredients for this estimate are the quadratic structure of certain error terms, the translation invariance of the functional $\mathcal{U}$ with respect to the Schwarzschild background, the Pohozaev identity, and the contribution of the Schwarzschild geometry to break the translation invariance. We split the proof of the theorem into the following subsections.

6.1 Splitting

Integrating the Gauss equation on Σ yields

$$8\pi(1 - q(\Sigma)) = \mathcal{W}(\Sigma) - \mathcal{U}(\Sigma) - \mathcal{V}(\Sigma),$$

where $q(\Sigma)$ is the genus of Σ and

$$\mathcal{U}(\Sigma) := \int_\Sigma |\mathring{A}|^2 \,\mathrm{d}\mu,$$

$$\mathcal{V}(\Sigma) := 2\int_\Sigma G(\nu,\nu)\,\mathrm{d}\mu,$$

where $G = {}^M\mathrm{Rc} - \frac{1}{2}{}^M\mathrm{Sc}g$ is the Einstein tensor of M. Denoting by δ_f the variation induced by a normal variation of Σ with normal velocity f, we infer from the above relation that

$$\delta_f\mathcal{W}(\Sigma) = \delta_f\mathcal{U}(\Sigma) + \delta_f\mathcal{V}(\Sigma).$$

By assumption we have

$$\delta_f\mathcal{W}(\Sigma) = \lambda\int_\Sigma Hf\,\mathrm{d}\mu,$$

hence

$$\lambda\int_\Sigma Hf\,\mathrm{d}\mu = \delta_f\mathcal{U}(\Sigma) + \delta_f\mathcal{V}(\Sigma). \tag{63}$$

By a fairly straightforward computation (given all the expressions in Sect. 3), we find

$$\delta_f\mathcal{U}(\Sigma) = -\int_\Sigma 2\mathring{A}^{ij}\nabla^2_{ij}f + 2f\mathring{A}^{ij\,M}\mathrm{Rc}^T_{ij} + fH|\mathring{A}|^2\,\mathrm{d}\mu. \tag{64}$$

6.2 The variations of $\mathcal{U}$ in g and g^S

Here we compute the difference of the variation of $\mathcal{U}$ with respect to g and to g^S, that is the error when changing the metric.

To do this, we restrict to the special case where

$$f = \frac{g(\nu, b)}{H},$$

and $b = \frac{a_e}{|a_e|}$, where a_e is as in Proposition 2 and ν is the normal of Σ with respect to g. Thus, up to the factor of H^{-1}, the function f is the normal velocity induced by translating Σ in the direction of b. We also define

$$f^S = \frac{g^S(\nu^S, b)}{\bar{H}^s},$$

where $\bar{H}^S$ is as in Proposition 7. As $|\nu - \nu^S| \leq C\eta r^{-2}$ and $||H - \bar{H}^S||_{L^\infty(\Sigma)} \leq Cr_{\min}^{-2}(\tau + \sqrt{\eta}r_{\min}^{-1})$ we find that

$$|f - f^S| \leq C(\tau + \sqrt{\eta}r_{\min}^{-1}).$$

Before we proceed, we compute the first and second derivative of f.

$$\nabla_i f = H^{-1}\left(g(\nabla_i b, \nu) + g(b, A_i^j e_j)\right) - H^{-2}\nabla_i H g(b, \nu), \tag{65}$$

and hence, as $|\nabla b| \leq Cr^{-2}$, we find that

$$\int_\Sigma |\nabla f|^2 \, \mathrm{d}\mu \leq C \int_\Sigma \left(r^{-2} + \frac{|A|^2}{H^2} + \frac{|\nabla H|^2}{H^4}\right) \mathrm{d}\mu \leq Cr_{\min}^2.$$

The second derivative of f is given by

$$\begin{aligned}
&\nabla_i \nabla_j f \\
&\quad = -A_i^k A_{jk} f + 2H^{-3}\nabla_i H \nabla_j H g(b, \nu) - H^{-2}\nabla_{i,j}^2 H g(b, \nu) \\
&\qquad + H^{-1}\left(g(\nabla_i \nabla_j b, \nu) + g(\nabla_i b, e_k)A_j^k + g(\nabla_j b, e_k)A_i^k + \nabla_j A_i^k g(b, e_k)\right) \\
&\qquad - H^{-2}\left(\nabla_i H(g(\nabla_j b, \nu) + g(b, e_k)A_j^k) + \nabla_j H(g(\nabla_i b, \nu) + g(b, e_k)A_i^k)\right).
\end{aligned}$$

In view of our estimates and the rapid decay of ∇b, $\nabla^2 b$, ∇H and $\nabla^2 H$, the first term on the right hand side of this equation is one magnitude larger than the other ones. However, the main contribution is in the trace of $\nabla^2 f$. We will not have to consider the trace part, as $\nabla^2 f$ is contracted with the traceless $\mathring{A}$ in Eq. 64. The traceless part $(\nabla^2 f)^0$ can be estimated as follows

$$\int_\Sigma |(\nabla^2 f)^0|^2 \, \mathrm{d}\mu \leq C \int_\Sigma r^{-4} \, \mathrm{d}\mu \leq Cr_{\min}^{-2}. \tag{66}$$

Note the jump in decay rates compared to the L^2-norm of $|\nabla f|$. Finally we need to calculate the second derivative of f^S

$$\begin{aligned}
\nabla_i^S \nabla_j^S f^S = (\bar{H}^S)^{-1}\Big(&g^S(\nabla_i^S \nabla_j^S b, \nu^S) + g^S(\nabla_i^S b, e_k)(A^S)_j^k + g^S(\nabla_j^S b, e_k)(A^S)_i^k \\
&+ \nabla_j^S (A^S)_i^k g^S(b, e_k)\Big) - (A^S)_i^k (A^S)_{jk} f^S.
\end{aligned}$$

We are now in the position to examine

$$|\delta_f \mathcal{U}(\Sigma) - \delta_{f^S} \mathcal{U}^S(\Sigma)|.$$

We will do this in detail, as this requires some care. First, consider the first term in Eq. 64:

$$
\begin{aligned}
E_1 &= \left| \int_\Sigma g(\mathring{A}, (\nabla^2 f)^0)\,\mathrm{d}\mu - \int_\Sigma g^S(\mathring{A}^S, ((\nabla^2)^S f)^0)\,\mathrm{d}\mu^S \right| \\
&\leq \left| \int_\Sigma (g - g^S)(\mathring{A}, (\nabla^2 f)^0)\,\mathrm{d}\mu \right| + \left| \int_\Sigma g^S(\mathring{A} - \mathring{A}^S, (\nabla^2 f)^0)\,\mathrm{d}\mu \right| \\
&\quad + \left| \int_\Sigma g^S(\mathring{A}^S, (\nabla^2 f)^0)(\mathrm{d}\mu - \mathrm{d}\mu^S) \right| \\
&\quad + \left| \int_\Sigma g^S(\mathring{A}^S, (\nabla^2 f - (\nabla^S)^2 f^S)^0)\,\mathrm{d}\mu^S \right| .
\end{aligned}
$$

The first three terms can be estimated using the asymptotics of g and the curvature estimates from Theorems 5 and 3.

$$
\begin{aligned}
E_1^a &\leq C\eta r_{\min}^{-2} \int_\Sigma |\mathring{A}||(\nabla^2 f)^0)| + (r^{-1} + |A|)|(\nabla^2 f)^0)| + |\mathring{A}^S||(\nabla^2 f)^0)|\,\mathrm{d}\mu \\
&\leq C\eta r_{\min}^{-2} \|(\nabla^2 f)^0\|_{L^2(\Sigma)} \left(\|A\|_{L^2} + r_{\min}^{-1}|\Sigma|^{1/2} \right) \\
&\leq C\eta r_{\min}^{-3}.
\end{aligned}
$$

Using again the fact that we are contracting with the traceless second fundamental form and the above equations for the second derivatives of f and f^S we see that we can estimate the last term for E_1, denoted by E_1^b, by

$$
\begin{aligned}
E_1^b \leq\; & C \int_\Sigma |\mathring{A}^S| H^{-2} \left(|\nabla H||\nabla b| + |\nabla H||A| + |\nabla^2 H| + H^{-1}|\nabla H|^2 \right) \mathrm{d}\mu^S \\
&+ C \int_\Sigma \frac{|\mathring{A}^S||H - \bar{H}^S|}{H\bar{H}^S} \left(|\nabla^2 b| + |\nabla b||A| + |\nabla A| + H|\mathring{A}| \right) \mathrm{d}\mu^S \\
&+ C\bar{H}_S^{-1} \int_\Sigma |\mathring{A}^S||g(\nabla_i \nabla_j b, \nu) - g^S(\nabla_i^S \nabla_j^S b, \nu^S)| \\
&+ |\mathring{A}^S||g(\nabla_i b, e_k) A_j^k - g^S(\nabla_i^S b, e_k)(A^S)_j^k| \\
&+ |\mathring{A}^S||g(\nabla_j b, e_k) A_i^k - g^S(\nabla_j^S b, e_k)(A^S)_i^k|
\end{aligned}
$$

$$+ |\mathring{A}^S||\nabla_j A_i^k g(b, e_k) - \nabla_j^S (A^S)_i^k g^S(b, e_k)| \, \mathrm{d}\mu^S$$
$$+ C \int_\Sigma |\mathring{A}^S||(A_i^k A_{jk})^0 f - ((A^S)_i^k A_{jk}^S)^0 f^S| \, \mathrm{d}\mu^S. \quad (67)$$

By the curvature estimates from Sect. 5 the terms on the first two lines in Eq. 67 are estimated by

$$\begin{aligned} &C\|\mathring{A}^S\|_{L^2(\Sigma)} \big(r_{\min}\|\nabla A\|_{L^2(\Sigma)} + r_{\min}^2\|\nabla^2 H\|_{L^2(\Sigma)} + r_{\min}\|\nabla(\log H)\|_{L^4(\Sigma)}^2 \\ &\quad + \|\nabla^2 b\|_{L^2(\Sigma)} + r_{\min}^{-2}\|A\|_{L^2} + r_{\min}^{-1}\|\mathring{A}\|_{L^2(\Sigma)} \\ &\le C r_{\min}^{-2} \left(\tau^2 + \tau r_{\min}^{-1} + \sqrt{\eta} r_{\min}^{-1}\right). \end{aligned}$$

We estimate the terms on the last five lines of Eq. 67 separately. The third line yields

$$\begin{aligned} &\int_\Sigma |\mathring{A}^S||g(\nabla_i\nabla_j b, \nu) - g^S(\nabla_i^S\nabla_j^S b, \nu^S)| \, \mathrm{d}\mu^S \\ &\quad \le \int_\Sigma |\mathring{A}^S|\big(|(g - g^S)(\nabla_i\nabla_j b, \nu)| + |g^S((\nabla_i\nabla_j - \nabla_i^S\nabla_j^S)b, \nu^S)| \\ &\qquad + |g^S(\nabla_i\nabla_j b, \nu - \nu^S)|\big) \, \mathrm{d}\mu^S \\ &\quad \le C\eta r_{\min}^{-5}. \end{aligned}$$

The fourth and fifth line of (67) are estimated as follows

$$\begin{aligned} &\int_\Sigma |\mathring{A}^S||g(\nabla_i b, e_k)A_j^k - g^S(\nabla_i^S b, e_k)(A^S)_j^k| \, \mathrm{d}\mu^S \\ &\quad \le \int_\Sigma |\mathring{A}^S| \Big(|(g - g^S)(\nabla_i b, e_k)A_j^k| + |g^S((\nabla_i - \nabla_i^S)b, e_k)(A^S)_j^k| \\ &\qquad + |g^S(\nabla_i b, e_k)(A_j^k - (A^S)_j^k)|\Big) \, \mathrm{d}\mu^S \\ &\quad \le C\eta r_{\min}^{-4}. \end{aligned}$$

For the sixth line of (67) we get

$$
\begin{aligned}
&\int_\Sigma |\mathring{A}^S||g(b,e_k)\nabla_i A_j^k - g^S(b,e_k)\nabla_i^S (A^S)_j^k|\,\mathrm{d}\mu^S \\
&\quad\le \int_\Sigma |\mathring{A}^S| \Big(|(g-g^S)(b,e_k)\nabla_i A_j^k| + |g^S(b,e_k)\nabla_i^S (A_j^k - (A^S)_j^k)| \\
&\qquad + |g^S(b,e_k)(\nabla_i - \nabla_i^S)A_j^k| \Big)\,\mathrm{d}\mu^S \\
&\quad\le C\eta r_{\min}^{-4}.
\end{aligned}
$$

It remains to estimate the last line of (67)

$$
\begin{aligned}
&\int_\Sigma |\mathring{A}^S||\mathring{A}_i^k A_{jk} f - (\mathring{A}^S)_i^k A_{jk}^S f^S|\,\mathrm{d}\mu^S \\
&\quad\le C\int_\Sigma |\mathring{A}^S| \Big(|\mathring{A}_i^k A_{jk}||f-f^S| + |A-A^S||A||f^S| \Big)\,\mathrm{d}\mu^S \\
&\quad\le C r_{\min}^{-3}\left(\tau + \sqrt{\eta}\, r_{\min}^{-1}\right).
\end{aligned}
$$

Combining all these estimates we arrive at the estimate for the first error term

$$
E_1 \le C r_{\min}^{-2}\left(\tau^2 + \tau r_{\min}^{-1} + \sqrt{\eta}\, r_{\min}^{-1}\right).
$$

Similarly, the second term in Eq. 64 gives the error

$$
\begin{aligned}
E_2 &:= \left| \int_\Sigma f\langle \mathring{A}, \mathrm{Ric}^T\rangle\,\mathrm{d}\mu - \int_\Sigma f^S\langle \mathring{A}^S, (\mathrm{Ric}^S)^T\rangle\,\mathrm{d}\mu^S \right| \\
&\le \int_\Sigma |\mathring{A} - \mathring{A}^S||\,\mathrm{Ric}^T\,||f|\,\mathrm{d}\mu + \int_\Sigma |\mathring{A}^S||\,\mathrm{Ric}^T - (\mathrm{Ric}^S)^T\,||f|\,\mathrm{d}\mu \\
&\quad + \int_\Sigma |\mathring{A}^S||(\mathrm{Ric}^S)^T\,||f||\,\mathrm{d}\mu - \mathrm{d}\mu^S| + \int_\Sigma |\mathring{A}^S||(\mathrm{Ric}^S)^T\,||f-f^S|\,\mathrm{d}\mu^S \\
&\le C r_{\min}^{-3}\left(\tau + \sqrt{\eta}\right).
\end{aligned}
$$

And the third term in Eq. 64 contributes

$$
\begin{aligned}
E_3 &:= \left| \int_\Sigma f H |\mathring{A}|^2 \, \mathrm{d}\mu - \int_\Sigma f^S H^S |\mathring{A}^S|^2 \, \mathrm{d}\mu^S \right| \\
&\le C \int_\Sigma |\mathring{A} - \mathring{A}^S||\mathring{A}| \, \mathrm{d}\mu + C \int_\Sigma |\mathring{A}^S|^2 | \mathrm{d}\mu - \mathrm{d}\mu^S | \, \mathrm{d}\mu \\
&\quad + C \int_\Sigma |f^S H^S - g^S(b, \nu^S)||\mathring{A}^S|^2 \, \mathrm{d}\mu^S \\
&\le C r_{\min}^{-3} \left(\tau + \sqrt{\eta} \right).
\end{aligned}
$$

In summary, we find that

$$
|\delta_f \mathcal{U}(\Sigma) - \delta_{f^S} \mathcal{U}^S(\Sigma)| \le C r_{\min}^{-2} \left(\tau^2 + \tau r_{\min}^{-1} + \sqrt{\eta} r_{\min}^{-1} \right). \tag{68}
$$

As the functional $\mathcal{U}^S$ is translation invariant, due to conformal invariance and conformal flatness of g^S, we find that

$$
\delta_{f^S} \mathcal{U}^S(\Sigma) = 0
$$

and hence

$$
|\delta_f \mathcal{U}(\Sigma)| \le C r_{\min}^{-2} \left(\tau^2 + \tau r_{\min}^{-1} + \sqrt{\eta} r_{\min}^{-1} \right). \tag{69}
$$

6.3 The left hand side of (63)

Here we estimate the left hand side of Eq. 63. By our choice of test function this becomes (omitting λ for now).

$$
\int_\Sigma g(b, \nu) \, \mathrm{d}\mu.
$$

First, we estimate the error when we take all quantities with respect to the metric g^S.

$$
\begin{aligned}
&\left| \int_\Sigma g(b, \nu) \, \mathrm{d}\mu - \int_\Sigma g^S(b, \nu^S) \, \mathrm{d}\mu^S \right| \\
&\quad \le \int_\Sigma |g - g^S| \, \mathrm{d}\mu + \int_\Sigma |\nu - \nu^S| \, \mathrm{d}\mu + \int_\Sigma | \mathrm{d}\mu - \mathrm{d}\mu^S | \, \mathrm{d}\mu \le C\eta.
\end{aligned}
$$

Then we insert the relations from Lemma 1 to compute

$$\begin{aligned}\int_\Sigma g^S(b,\nu^S)\,\mathrm{d}\mu^S &= \int_\Sigma \phi^6 g^e(b,\nu^e)\,\mathrm{d}\mu^e\\ &= \int_\Sigma \left(1+\frac{3m}{r}+\text{lower order}\right) g^e(b,\nu^e)\,\mathrm{d}\mu^e.\end{aligned}$$

We deal with the highest order term first. Note that by translation invariance of the volume enclosed by Σ in Euclidean space, we find

$$\int_\Sigma g^e(b,\nu^e)\,\mathrm{d}\mu^e = 0, \tag{70}$$

and hence

$$\int_\Sigma g^S(b,\nu^S)\,\mathrm{d}\mu^S = \int_\Sigma \left(\frac{3m}{r}+\text{lower order}\right) g^e(b,\nu^e)\,\mathrm{d}\mu^e.$$

The lower order terms are of the form $c_k r^{-k}$ where c_k depends only on m and $k = 2,\dots,6$. We can replace r by R_e in these integrals, and in view of Proposition 2 and Theorem 6 we find that

$$\left|r^{-k}-R_e^{-k}\right| \le C r_{\min}^{-k}\left(\tau+\sqrt{\eta}\,r_{\min}^{-1}\right).$$

Since $k \ge 2$, we can estimate all resulting error terms by

$$\sum_{k=2}^{6}\int_\Sigma \left|\frac{c_k}{r^k}-\frac{c_k}{R_e^k}\right|\mathrm{d}\mu \le C\left(\tau+\sqrt{\eta}\,r_{\min}^{-1}\right).$$

The remaining integrals satisfy

$$\int_\Sigma \frac{c_k}{R_e^k} g^e(b,\nu^e)\,\mathrm{d}\mu^e = 0$$

due to relation (70). Combining the above calculations, we find that

$$\left|\int_\Sigma g(b,\nu)\,\mathrm{d}\mu - \int_\Sigma \frac{3m}{r} g^e(b,\nu^e)\,\mathrm{d}\mu^e\right| \le C\left(\tau+\sqrt{\eta}\,r_{\min}^{-1}+\eta\right).$$

The estimate on $\|\mathring{A}\|_{L^2(\Sigma)}$ allows us to change the domain of integration to the round sphere $S := S_{R_e}(a_e)$, and change ν^e to N, the normal of S while introducing only an error estimated by $C(\tau + \sqrt{\eta} r_{\min}^{-1})$. The corresponding integral on the sphere can be computed using the methods introduced in the proof of Proposition 3. The result is (see Appendix A.2)

$$\int_S \frac{3m}{r} g^e(b, N)\,\mathrm{d}\mu^e = -4\pi m |a_e|.$$

Hence, collecting the error terms acquired on the way, we find

$$\left| \int_\Sigma Hf\,\mathrm{d}\mu + 4\pi m |a_e| \right| \le C\left(\tau + \sqrt{\eta} r_{\min}^{-1} + \eta\right). \tag{71}$$

recall that $\left|\lambda - \frac{2m}{R_S^3}\right| \le C\left(r_{\min}^{-4}(\tau + \sqrt{\eta} r_{\min}^{-1})\right)$, whence

$$\left| \lambda \int_\Sigma Hf\,\mathrm{d}\mu + \frac{8\pi m^2 \tau}{\bar{\phi}^2 R_s^2} \right| \le C r_{\min}^{-3}\left(\tau + \sqrt{\eta} r_{\min}^{-1} + \eta\right), \tag{72}$$

where $\bar{\phi} = 1 + \frac{m}{2R_e}$, $R_S = \bar{\phi}^2 R_e$ as in Proposition 3 and we used the definition $\tau = |a_e|/R_e$.

6.4 The Pohozaev identity

Before we study the variation of $\mathcal{V}$, we recall the (geometric) Pohozaev identity. To this end we denote the conformal Killing operator by

$$\mathcal{D}X := \mathcal{L}_X g - \frac{1}{3}\operatorname{tr}(\mathcal{L}_X g) g$$

where X is a vector field on M and $\mathcal{L}_X g$ denotes the Lie derivative of g with respect to X. Let $\Omega \subset M$ be a smooth domain with boundary Σ and let dV be the volume form of M. Then the Pohozaev identity[1] can be stated as

$$\frac{1}{2}\int_\Omega \langle G, \mathcal{D}X\rangle dV - \frac{1}{6}\int_\Omega {}^M\mathrm{Sc}\,\operatorname{div} X dV = \int_\Sigma G(X, \nu)\,\mathrm{d}\mu. \tag{73}$$

This identity can be seen as follows: In local coordinates we have

$$(\mathcal{D}X)_{kl} = \nabla_k X_l + \nabla_l X_k - \frac{2}{3}\operatorname{div} X g_{kl}$$

[1] In the literature (see for example [19]) the Pohozaev identity is usually stated for the trace-free Ricci tensor, not for the Einstein tensor. For our purposes however, it is more convenient to write it in terms of G.

and therefore

$$\begin{aligned}\frac{1}{2}\int_{\Omega}\langle G,\mathcal{D}X\rangle dV &= \frac{1}{2}\int_{\Omega}\left(g^{ik}g^{jl}G_{ij}(\nabla_k X_l+\nabla_l X_k)-\frac{2}{3}G_{ii}\operatorname{div}X\right)dV\\ &= -\int_{\Omega}\langle \operatorname{div}G, X\rangle dV+\frac{1}{6}\int_{\Omega}{}^{M}\mathrm{Sc}\operatorname{div}X dV\\ &\quad +\int_{\Sigma}G(X,\nu)\,\mathrm{d}\mu,\end{aligned}$$

which proves (73) since G is divergence free.

Lemma 16 *Let Σ be a surface as in Theorem 7 which bounds an exterior domain Ω, and let $b\in\mathbf{R}^3$ be a constant vector. Then*

$$\left|\int_{\Sigma}G(b,\nu)\,\mathrm{d}\mu\right|\le C\eta r_{\min}^{-3}.$$

Proof Consider the vector field b, where $b\in\mathbf{R}^3$ is constant. Then b is a Killing vector field in flat $\mathbf{R}^3$ and hence a conformal Killing vector field with respect to g^S. Denoting by $\mathcal{D}^S$ the conformal Killing operator with respect to g^S, we thus find

$$\mathcal{D}^S b=0.$$

With respect to the general metric g, this implies the decay rate

$$|\mathcal{D}b|\le C\eta r^{-3},$$

since $|\nabla-\nabla^S|\le C\eta r^{-3}$. The other terms in Eq. 73 have decay $|G|\le Cr^{-3}$, $|{}^{M}\mathrm{Sc}|\le C\eta r^{-4}$, and $|\operatorname{div}b|\le Cr^{-2}$.

Let S_σ be a coordinate sphere of radius σ outside of Σ and let Ω_σ be the domain bounded by Σ and S_σ. The contribution of S_σ to the boundary integral in Eq. 73 decays like σ^{-1} and thus we infer that

$$\int_{\Sigma}G(b,\nu)\,\mathrm{d}\mu=\lim_{\sigma\to\infty}\left(-\frac{1}{2}\int_{\Omega_\sigma}\langle G,\mathcal{D}b\rangle dV+\frac{1}{6}\int_{\Omega_\sigma}{}^{M}\mathrm{Sc}\operatorname{div}b\,dV\right).\qquad(74)$$

The sign of the right hand side is different to (73), as our conventions are that ν is the outward pointing normal to Σ which points into Ω.

The integrand in the volume integral decays like $C\eta r^{-6}$, which implies via Lemma 4 that the integral can be estimated by $C\eta r_{\min}^{-3}$ as claimed. □

6.5 The variation of $\mathcal{V}(\Sigma)$

The variation of $\mathcal{V}$ can be computed to be

$$\frac{1}{2}\delta_f \mathcal{V}(\Sigma) = \int_\Sigma f\big(\nabla_\nu G(\nu,\nu) + HG(\nu,\nu)\big) - 2G(\nu,\nabla f)\,\mathrm{d}\mu. \tag{75}$$

Since G is divergence-free we calculate

$$\begin{aligned}\nabla_\nu G(\nu,\nu) &= \operatorname{div} G(\nu) - \nabla_{e_i} G(\nu, e_i) = -\nabla_{e_i} G(\nu, e_i)\\
&= -\nabla_{e_i} {}^M\mathrm{Rc}(\nu, e_i)\\
&= -{}^\Sigma\mathrm{div}\omega + {}^M\mathrm{Rc}(h_{ik}e_k, e_i) - H\,{}^M\mathrm{Rc}(\nu,\nu)\\
&= -{}^\Sigma\mathrm{div}\omega - H\,{}^M\mathrm{Rc}(\nu,\nu) + \mathring{A}_{ik}\,{}^M\mathrm{Rc}_{ik}\\
&\quad + \frac{1}{2}H({}^M\mathrm{Sc} - {}^M\mathrm{Rc}(\nu,\nu))\\
&= -{}^\Sigma\mathrm{div}\omega + \langle \mathring{A}, G^T\rangle - \frac{1}{4}H\,{}^M\mathrm{Sc} - \frac{3}{2}HG(\nu,\nu),\end{aligned} \tag{76}$$

where, as usual, $\omega = {}^M\mathrm{Rc}(\nu,\cdot)^T = G(\nu,\cdot)^T$. Inserting this into (75), we find that

$$\begin{aligned}\frac{1}{2}\delta_f \mathcal{V}(\Sigma) &= \int_\Sigma f\langle \mathring{A}, G^T\rangle - f\,{}^\Sigma\mathrm{div}\omega - \frac{1}{2}fHG(\nu,\nu) - \frac{1}{4}fH\,{}^M\mathrm{Sc} - 2\omega(\nabla f)\,\mathrm{d}\mu\\
&= \int_\Sigma -\frac{1}{2}fHG(\nu,\nu) - \frac{1}{4}fH\,{}^M\mathrm{Sc} + f\langle \mathring{A}, G^T\rangle - \omega(\nabla f)\,\mathrm{d}\mu.\end{aligned}$$

We specialize again to the test function

$$f = \frac{g(b,\nu)}{H}$$

for a fixed vector $b \in \mathbf{R}^3$. In the expression (65) for ∇f we can split $A = \mathring{A} + \frac{1}{2}H\gamma$ and obtain

$$\nabla_i f = H^{-1}\left(g(\nabla_i b, \nu) + g(b, e_j)\mathring{A}_i^j - \nabla_i \log H g(b,\nu)\right) + \frac{1}{2}g(b, e_i). \tag{77}$$

Inserting this into Eq. 75, we find that

$$\begin{aligned}\frac{1}{2}\delta_f \mathcal{V}(\Sigma) &= \int_\Sigma -\frac{1}{2} f H G(\nu,\nu) - \frac{1}{4} f H\, {}^M\mathrm{Sc} + f\langle \mathring{A}, G^T\rangle - \frac{1}{2} G(\nu, b^T) \\ &\quad - H^{-1}\omega(e_i)\left(g(\nabla_i b,\nu) + g(b,e_j)\,\mathring{A}_i^j - \nabla_i \log H g(b,\nu)\right) \mathrm{d}\mu \\ &= \int_\Sigma -\frac{1}{2} G(b,\nu) - \frac{1}{4} g(b,\nu)\, {}^M\mathrm{Sc} + H^{-1} g(b,\nu)\langle \mathring{A}, G^T\rangle \\ &\quad - H^{-1}\omega(e_i)\left(g(\nabla_i b,\nu) + g(b,e_j)\,\mathring{A}_i^j - \nabla_i \log H g(b,\nu)\right) \mathrm{d}\mu. \qquad (78)\end{aligned}$$

It is this expression for $\delta_f \mathcal{V}$ which will give rise to the position estimates. We will thus spend some time on understanding the error terms. Because of Propositions 4 and 6 we have the estimate

$$\int_\Sigma H^{-1}\left(|\langle \mathring{A}, G^T\rangle| + |\omega||\mathring{A}| + |\omega||\nabla \log H|\right) \mathrm{d}\mu \le C r_{\min}^{-2}\left(\tau^2 + \eta r_{\min}^{-2}\right).$$

Note that Proposition 4 implies that $\|(G^T)^\circ\|^2_{L^2(\Sigma)} \le C r_{\min}^{-4}\left(\tau^2 + \eta r_{\min}^{-2}\right)$. Assuming that $|{}^M\mathrm{Sc}| \le \eta r^{-5}$ we find that

$$\left|\int_\Sigma {}^M\mathrm{Sc}\,\mathrm{d}\mu\right| \le C\eta r_{\min}^{-3}.$$

Lemma 16 implies that the first term on the right hand side of (78) is also estimated by $C\eta r_{\min}^{-3}$, so that the only term which yields a contribution of order $r_{\min}^{-2}$ is

$$\int_\Sigma H^{-1}\omega(e_i) g(\nabla_{e_i} b, \nu)\,\mathrm{d}\mu.$$

We will explicitly evaluate this term. To this end note that

$$\begin{aligned}&\left|\int_\Sigma H^{-1}\omega(e_i) g(\nabla_{e_i} b,\nu)\,\mathrm{d}\mu - \int_\Sigma (\bar{H}^S)^{-1}\,\mathrm{Ric}^S(e_i^S,\nu^S) g^S(\nabla^S_{e_i^S} b, \nu^S)\,\mathrm{d}\mu^S\right| \\ &\quad \le C r_{\min}^{-3}(\tau + \sqrt{\eta})\end{aligned}$$

where $\bar{H}^S$ is the quantity from Corollary 7 and e_i^S constitute a tangential ON-frame with respect to the metric induced by g^S. This estimate follows since the integrand scales like r^{-4} and the transition errors to Schwarzschild decay at least one order faster and have factor η. Furthermore, the replacement of H by $\bar{H}^S$ introduces an extra error term of the form $C r_{\min}^{-3}(\tau + \sqrt{\eta} r_{\min}^{-1})$. We calculate, using that $D^e b \equiv 0$

and the transformation properties of the Christoffel symbols under a conformal change of the metric (see for example [20]),

$$\nabla^S_{e_i^S} b = 2\phi^{-1}\left(e_i^S(\phi)b + b(\phi)e_i^S - D^e\phi g^e(b, e_i^S)\right),$$

which implies that

$$g^S(\nabla^S_{e_i^S} b, \nu^S) = \phi^{-1}\frac{m}{r^2}\left(g^e(\rho, \nu^e)g^e(b, e_i^e) - g^e(\rho, e_i^e)g^e(b, \nu^e)\right).$$

Here $e_i^e = \phi^2 e_i^S$ is a tangential ON-frame with respect to the metric induced by g^e. Furthermore, the formula from Lemma 1 yields that

$$\mathrm{Ric}^S(\nu^S, e_i^S) = -3\frac{m}{r^3}\phi^{-6}g^e(\rho, \nu^e)g^e(\rho, e_i^e).$$

Multiplying these terms gives (note that we sum over $i = 1, 2$)

$$\begin{aligned}
&\mathrm{Ric}^S(\nu^S, e_i^S)g^S(\nabla^s_{e_i^S} b, \nu^S)\\
&= 3\frac{m^2}{r^5}\phi^{-7}\left(|\rho^T|^2 g^e(\rho, \nu^e)g^e(b, \nu^e) - |\rho^\perp|^2 g^e(\rho^T, b^T)\right)\\
&= 3\frac{m^2}{r^5}\phi^{-7}g^e(\rho, \nu^e)\left(g^e(b, \nu^e) - g^e(\rho, \nu^e)g^e(b, \rho)\right).
\end{aligned}$$

As in the proof of Proposition 3, we replace the integral over Σ by an integral over $S = S_{R_e}(a_e)$ while introducing error terms of one order lower. This implies that

$$\begin{aligned}
&\left|\frac{3m^2}{\bar\phi^7\bar H^S}\int_S \frac{1}{r^5}g^e(\rho, N)\left(g^e(b, N) - g^e(\rho, N)g^e(b, \rho)\right)\,\mathrm{d}\mu^e\right.\\
&\left.\quad - \int_\Sigma H^{-1}\omega(e_i)g(\nabla_{e_i} b, \nu)\,\mathrm{d}\mu\right| \le Cr_{\min}^{-3}(\tau + \sqrt{\eta}),
\end{aligned}$$

where N is the Euclidean normal vector to S and $\bar\phi = 1 + \frac{m}{2R_e}$ the quantity introduced in Proposition 3. The first integral can be evaluated explicitly, where we again introduce coordinates ϑ, φ in which $g^e(b, N) = \cos\varphi$. As $\rho = r^{-1}(R_e N + a_e)$ we can express this integral by

$$
\begin{aligned}
Q(|a|, R) &:= \frac{3m^2}{\bar\phi^7 \bar H^S} \int\limits_S \frac{1}{r^5} g^e(\rho, N) \left(g^e(b, N) - g^e(\rho, N) g^e(b, \rho) \right) \, \mathrm{d}\mu^e \\
&= \frac{3m^2}{\bar\phi^7 \bar H^S} \int\limits_S \Bigg(R_e \frac{\cos\varphi}{r^6} + |a_e| \frac{\cos^2\varphi}{r^6} - |a_e| R_e^2 \frac{1}{r^8} - (R_e^3 + 2|a_e|^2 R_e) \frac{\cos\varphi}{r^8} \\
&\quad - \left(|a_e|^3 + 2|a_e| R_e^2 \right) \frac{\cos^2\varphi}{r^8} - |a_e|^2 R_e \frac{\cos^3\varphi}{r^8} \Bigg) \, \mathrm{d}\mu^e.
\end{aligned}
$$

Explicitly evaluating these terms (see Appendix A.3), we obtain the following expression for Q. We already substituted $\tau := |a_e|/R_e$:

$$
Q(\tau, R_e) = \frac{m^2 \pi}{4 \bar\phi^7 \bar H^S R_e^3} \frac{3(\tau^6 - 3\tau^4 + 3\tau^2 - 1) \ln \frac{1-\tau}{1+\tau} + 6\tau^5 - 16\tau^3 - 6\tau}{\tau^2 (1+\tau)^3 (1-\tau)^3}.
$$

To analyze this expression we set

$$
f(\tau) = \frac{3(\tau^6 - 3\tau^4 + 3\tau^2 - 1) \ln \frac{1-\tau}{1+\tau} + 6\tau^5 - 16\tau^3 - 6\tau}{\tau^2 (1+\tau)^3 (1-\tau)^3}. \tag{79}
$$

Recall the Taylor expansion of the function $\ln \frac{1-\tau}{1+\tau}$:

$$
\ln \frac{1-\tau}{1+\tau} = -2\tau - \frac{2}{3}\tau^3 + O(\tau^4),
$$

for small τ. Thus we find that the numerator in Eq. 79 is

$$
3(3\tau^2 - 1) \left(-2\tau - \frac{2}{3}\tau^3 \right) - 16\tau^3 - 6\tau + O(\tau^4) = -32\tau^3 + O(\tau^4).
$$

Hence we get that

$$
Q(\tau, R) = -\frac{8\pi m^2 \tau}{\bar\phi^7 \bar H^S R_e^3} + \frac{O(\tau^2)}{R_e^2}
$$

for small τ. In summary, the above computation implies the following estimate

$$
\left| \delta_f \mathcal{V}(\Sigma) - \frac{16\pi m^2 \tau}{\bar\phi^7 \bar H^S R_e^3} \right| \le C r_{\min}^{-2} \left(\tau^2 + \tau r_{\min}^{-1} + \sqrt{\eta} r_{\min}^{-1} \right). \tag{80}
$$

6.6 Position estimates

Theorem 7 is a consequence from an iterative application of the following proposition.

Proposition 8 *If (M, g) and Σ are as in Theorem 7, then*

$$\tau \leq C\left(\tau^2 + \tau r_{\min}^{-1} + \sqrt{\eta} r_{\min}^{-1}\right),$$

Proof We computed in Sect. 6.3 that (cf. (72)),

$$\left|\lambda \int_\Sigma H f \, \mathrm{d}\mu + \frac{8\pi m^2 \tau}{\bar{\phi}^2 R_S^2}\right| \leq C r_{\min}^{-3}\left(\tau + \sqrt{\eta}\right),$$

in Sect. 6.2 that (cf. (69)),

$$|\delta_f \mathcal{U}(\Sigma)| \leq C r_{\min}^{-2}\left(\tau^2 + \tau r_{\min}^{-1} + \sqrt{\eta} r_{\min}^{-1}\right),$$

and in Sect. 6.5 that (cf. (80))

$$\left|\delta_f \mathcal{V}(\Sigma) - \frac{16\pi m^2 \tau}{\bar{H}^S R_S^3}\right| \leq C r_{\min}^{-2}\left(\tau^2 + \tau r_{\min}^{-1} + \sqrt{\eta} r_{\min}^{-1}\right).$$

Inserting these equations into Eq. 63 we find, after absorbing the lower order terms on the left into the error terms, that

$$24\pi m^2 \tau \leq C\left(\tau^2 + \tau r_{\min}^{-1} + \sqrt{\eta} r_{\min}^{-1}\right),$$

which is the claimed estimate. □

6.7 Final version of the curvature estimates

In this subsection we state our final version of the previous curvature estimates.

Theorem 8 *For all $m > 0$, η_0 and σ there exist $r_0 < \infty$, $\tau_0 > 0$, $\varepsilon > 0$, and C depending only on m, σ and η_0 with the following properties.*

Assume that (M, g) is (m, η, σ)-asymptotically Schwarzschild with $\eta \leq \eta_0$ and

$$|{}^M\mathrm{Sc}| \leq \eta r^{-5}.$$

Then if Σ is a surface satisfying Eq. 1 with $H > 0$, $\lambda > 0$, $r_{\min} > r_0$ and

$$\tau \leq \tau_0 \quad \text{and} \quad R_e \leq \varepsilon r_{\min}^2,$$

where R_e and τ are as in Sect. 5, we have the following estimates

$$\|H - \bar{H}^S\|_{L^\infty} + \|\mathring{A}\|_{L^\infty} + r_{\min}\|\nabla H\|_{L^\infty} \leq C\sqrt{\eta} r_{\min}^{-3}. \tag{81}$$

Here $\bar{H}^S = \frac{2}{R_S} - \bar{\phi}\frac{2m}{R_S^2}$ with $R_S = \bar{\phi}^2 R_e$ and $\bar{\phi} = 1 + \frac{m}{2R_e}$. Furthermore, we have that

$$\|\nu - \phi^{-2}\rho\|_{L^\infty} \leq C\sqrt{\eta} r_{\min}^{-1}. \tag{82}$$

This implies,

$$\begin{aligned} \|\lambda + \operatorname{Ric}(\nu,\nu)\|_{L^\infty} + \|\operatorname{Ric}(\nu,\nu) + 2mR_S^{-3}\|_{L^\infty} &\leq C\sqrt{\eta} r_{\min}^{-4}. \\ \|\omega\|_{L^\infty} + r_{\min}\|\nabla\omega\|_{L^\infty} &\leq C\sqrt{\eta} r_{\min}^{-4}. \end{aligned} \tag{83}$$

Proof The estimates in (81) are straight-forward consequences of the estimates in Sect. 5 and the position estimate 7. The estimate for the gradient of the traceless second fundamental form is proven similarly as in Lemma 15. To prove (82) note that we can calculate the gradient of $\nu - \phi^{-2}\rho$ as follows. We let e_i be a vector tangent to Σ and calculate

$$\nabla_{e_i}\nu = \frac{1}{2}He_i + \mathring{A}(e_i, \cdot).$$

Since $\phi^{-2}\rho$ is the normal to $S_r(0)$ in the Schwarzschild metric, and $S_r(0)$ is umbilical in this metric, we find that

$$\nabla^S_{e_i}(\phi^{-2}\rho) = \frac{1}{2}H_S(r)\left(e_i - g^S(e_i,\rho)\rho\right)$$

for $H_S(r) = \phi^{-2}\frac{2}{r} - 2\phi^{-3}\frac{m}{r^2}$. We calculate further and find

$$g^S(e_i,\rho) = (g^S - g)(e_i,\rho) + g(e_i, \rho - \phi^2\nu) + g(e_i, \phi^2\nu).$$

Note that the last term vanishes. In view of the estimates in (81) and Definition 1 we thus have

$$\begin{aligned} \left|\nabla(\nu - \phi^{-2}\rho)\right| &\leq C\Big(r_{\min}^{-1}|g - g^S| + |\nabla - \nabla^S| + |\mathring{A}| + |H - \bar{H}_S| \\ &\qquad + |\bar{H}_S - H_S(r)| + r_{\min}^{-1}|\nu - \phi^{-2}\rho|\Big) \\ &\leq C\sqrt{\eta} r_{\min}^{-3} + Cr_{\min}^{-1}|\nu - \phi^{-2}\rho|. \end{aligned} \tag{84}$$

Proposition 4 then yields that

$$\|\nabla(\nu - \phi^{-2}\rho)\|_{L^2} \leq C\sqrt{\eta} r_{\min}^{-1}.$$

We can now use the Michael–Simon–Sobolev inequality, Proposition 1, to get L^4-estimates

$$\|\nu - \phi^{-2}\rho\|_{L^4} \le C\sqrt{\eta}r_{\min}^{-1/2}.$$

Together with Eq. 84, this implies L^4-bounds for the derivative of $\nu - \phi^{-2}\rho$. Thus an obvious modification of Theorem 5.6 in [11] then yields the desired L^∞-estimate:

$$\|\nu - \phi^{-2}\rho\|_{L^\infty} \le C\sqrt{\eta}r_{\min}^{-1}.$$

The estimates in (83) easily follow from (82). □

7 Estimates for the linearized operator

In this section we show that the linearized operator $W_\lambda = W - \lambda L$ is invertible.

7.1 Eigenvalues of the Jacobi operator

To fix the notation let ν_i be the i-th eigenvalue of the negative of the Laplace operator on $\mathbb{S}^2$, where we count the eigenvalues with multiplicities, i.e. $\nu_0 = 0,\ \nu_1 = \nu_2 = \nu_3 = 2$, $\nu_4 = \cdots = \nu_8 = 4$ and $\nu_i > 4$ for $i \ge 9$. We denote by γ_i^e the eigenvalues of the negative of the Laplace operator on Σ, with respect to the Euclidean metric. We will need the following estimate from [4, Corollary 1].

Theorem 9 *There exist constants C_i such that for every surface Σ as in Theorem 8 there holds*

$$|\gamma_i^e - R_e^{-2}\nu_i| \le C_i\sqrt{\eta}r_{min}^{-4}\ .$$

Proof Note that by Theorem 8 and Lemma 2 we have that

$$\|\mathring{A}^e\|^2_{L^2(\Sigma,g^e)} \le C\eta r_{\min}^{-4}$$

Scaling the estimate in [4, Corollary 1] gives the result. □

It can be checked from [4] that

$$C_i \le C\nu_i\ , \tag{85}$$

where C does not depend on i.

In the following we let $\bar{g}^S := \bar{\phi}^4 g^e$ be a uniform Schwarzschild-reference metric on Σ. Thus $\bar{\Delta}^S := \Delta^{\bar{g}^S} = \bar{\phi}^{-4}\Delta^{g^e}$ and we denote the eigenvalues of $-\bar{\Delta}^S$ by $\bar{\gamma}_i^S$.

Corollary 4 *For any surface Σ as in Theorem 8 we have the estimate*

$$|\bar{\gamma}_i^S - R_S^{-2}\nu_i| \le C_i\sqrt{\eta}r_{min}^{-4}.$$

To compute the eigenvalues of the Jacobi operator on Σ we aim to compare it with the operator

$$\bar{L}\alpha := -\bar{\Delta}^S\alpha - \left(\frac{1}{2}\left(\bar{H}^S\right)^2 - \lambda\right)\alpha. \tag{86}$$

Let the eigenvalues and eigenfunctions of L and $\bar{L}$ be denoted by μ_i, φ_i and $\bar{\mu}_i, \bar{\varphi}_i$, respectively. Note that

$$\bar{\mu}_i = \bar{\gamma}_i^S - \frac{1}{2}\left(\bar{H}^S\right)^2 + \lambda. \tag{87}$$

Lemma 17 *For any surface Σ as in Theorem 8 we have the estimate*

$$|\mu_i - \bar{\mu}_i| \le C\big(|\bar{\mu}_i| + r_{min}^{-2}\big)\sqrt{\eta} r_{min}^{-2}.$$

Proof We use the following characterization of the i-th eigenvalue

$$\mu_i = \inf_{\substack{V\subset W^{1,2}(\Sigma)\\ \dim(V)=i+1}} \sup_{\psi\in V} \frac{\int_\Sigma \psi L\psi\, d\mu}{\int_\Sigma \psi^2\, d\mu},$$

where V is any linear subspace of $W^{1,2}(\Sigma)$. Let $\bar{\varphi} \in W^{1,2}(\Sigma)$ with $\int \bar{\varphi}^2\, d\bar{\mu}^S = 1$. We estimate, using (81) and (83)

$$\begin{aligned}\int \bar{\varphi} L\bar{\varphi}\, d\mu &= \int |\nabla\bar{\varphi}|^2 - \bar{\varphi}^2\left(|\mathring{A}|^2 + \frac{1}{2}H^2 + {}^M\mathrm{Rc}(\nu,\nu)\right) d\mu \\ &\le \int |\nabla\bar{\varphi}|^2 - \bar{\varphi}^2\left(\frac{1}{2}(\bar{H}^S)^2 - \lambda\right) d\mu + C\sqrt{\eta} r_{\min}^{-4}.\end{aligned} \tag{88}$$

In the following we repeatedly use the estimates from Definition 1 and Lemma 2. We can estimate the first term on the right hand side by

$$\begin{aligned}\int |\nabla\bar{\varphi}|^2\, d\mu &\le \int |\nabla\bar{\varphi}|^2\, d\mu^S + C\eta r_{\min}^{-2}\int |\nabla\bar{\varphi}|^2_{g^S}\, d\mu^S \\ &\le \int |\nabla\bar{\varphi}|^2_{g^S}\, d\mu^S + C\eta r_{\min}^{-2}\int |\nabla\bar{\varphi}|^2_{g^S}\, d\mu^S \\ &\le \int |\nabla\bar{\varphi}|^2_{\bar{g}^S}\, d\bar{\mu}^S + C\eta r_{\min}^{-2}\int |\nabla\bar{\varphi}|^2_{\bar{g}^S}\, d\bar{\mu}^S,\end{aligned} \tag{89}$$

where we used the conformal invariance of the Dirichlet energy from the second to the third line. The second term on the right hand side is estimated similarly by

$$\begin{aligned}-\int \bar{\varphi}^2\left(\frac{1}{2}(\bar{H}^S)^2 - \lambda\right) d\mu &\le -\int \bar{\varphi}^2\left(\frac{1}{2}(\bar{H}^S)^2 - \lambda\right) d\mu^S \\ &\quad + C\eta r_{\min}^{-2}\int \bar{\varphi}^2\left|\frac{1}{2}(\bar{H}^S)^2 - \lambda\right| d\mu^S \\ &\le -\int \bar{\varphi}^2\left(\frac{1}{2}(\bar{H}^S)^2 - \lambda\right) d\bar{\mu}^S + C\sqrt{\eta} r_{\min}^{-4},\end{aligned} \tag{90}$$

where we used that $d\mu^S = (\phi/\bar{\phi})^4 d\bar{\mu}^S$ and

$$\left|\frac{\phi}{\bar{\phi}} - 1\right| \le Cm\left|\frac{1}{r} - \frac{1}{R_e}\right| \le \frac{C}{r_{\min}}\left(\tau + \frac{\sqrt{\eta}}{r_{\min}}\right) \le C\sqrt{\eta}r_{\min}^{-2},$$

by Proposition 2 and Theorem 7. Now

$$\begin{aligned}\int |\nabla\bar{\varphi}|^2_{\bar{g}^S}\, d\bar{\mu}^S &= \int \bar{\varphi}\bar{L}\bar{\varphi}\, d\bar{\mu}^S + \int \bar{\varphi}^2\left(\frac{1}{2}(\bar{H}^S)^2 - \lambda\right) d\bar{\mu}^S \\ &\le \int \bar{\varphi}\bar{L}\bar{\varphi}\, d\bar{\mu}^S + Cr_{\min}^{-2}. \end{aligned} \tag{91}$$

Combining (89), (90) and (91) we see that

$$\int \bar{\varphi}L\bar{\varphi}\, d\mu \le \left(1 + C\eta r_{\min}^{-2}\right)\int \bar{\varphi}\bar{L}\bar{\varphi}\, d\bar{\mu}^S + C\sqrt{\eta}r_{\min}^{-4}. \tag{92}$$

Moreover, by arguing as above, we have the estimate

$$\left|\int \bar{\varphi}^2\, d\mu - 1\right| = \left|\int \bar{\varphi}^2\, d\mu - \int \bar{\varphi}^2\, d\bar{\mu}^S\right| \le C\sqrt{\eta}r_{\min}^{-2}.$$

Combining this with (92) and the variational characterization of the eigenvalues, we see that

$$\mu_i \le \bar{\mu}_i + C\sqrt{\eta}r_{\min}^{-2}|\bar{\mu}_i| + C\sqrt{\eta}r_{\min}^{-4}.$$

The reverse inequality follows from a similar calculation, interchanging $\bar{L}$ and L. □

From Theorem 8, (87) and Lemma 17 we get the following

Corollary 5 *For any surface Σ as in Theorem 8 we have the estimate*

$$\left|\mu_i - \left(\frac{\nu_i - 2}{R_S^2} + 3\lambda\right)\right| \le C(1+\nu_i)\sqrt{\eta}r_{\min}^{-4} + Cr_{\min}^{-5} + C\nu_i\sqrt{\eta}r_{\min}^{-6}.$$

7.2 The linearized Willmore equation

In the following we aim at proving a positive lower bound for the first eigenvalue of the linearization of the Willmore equation with prescribed area. We start by recalling the expression (see (35))

$$
\begin{aligned}
\int_\Sigma \alpha W_\lambda \alpha \,\mathrm{d}\mu &= \int_\Sigma \alpha W\alpha - \lambda\alpha L\alpha \,\mathrm{d}\mu \\
&= \int_\Sigma (L\alpha)^2 - \lambda\alpha L\alpha + \frac{1}{2}H^2|\nabla\alpha|^2 - 2H\mathring{A}(\nabla\alpha, \nabla\alpha) \\
&\quad + \alpha^2 \Big(|\nabla H|^2 + 2\omega(\nabla H) + H\Delta H + 2\langle\nabla^2 H, \mathring{A}\rangle \\
&\quad + 2H^2|\mathring{A}|^2 + 2H\langle\mathring{A}, T\rangle - H\nabla_\nu{}^M\mathrm{Rc}(\nu,\nu) \\
&\quad - \frac{1}{2}H^2|A|^2 - \frac{1}{2}H^{2M}\mathrm{Rc}(\nu,\nu)\Big)\,\mathrm{d}\mu.
\end{aligned}
$$

Integration by parts of the third term on the right yields

$$
\frac{1}{2}\int_\Sigma H^2|\nabla\alpha|^2\,\mathrm{d}\mu = \frac{1}{2}\int_\Sigma \alpha^2\left(|\nabla H|^2 + H\Delta H\right) - H^2\alpha\Delta\alpha\,\mathrm{d}\mu.
$$

Together with $L\alpha = -\Delta\alpha - \alpha\left(|A|^2 + {}^M\mathrm{Rc}(\nu,\nu)\right)$ and (1) this yields

$$
\begin{aligned}
\int_\Sigma \alpha W_\lambda\alpha\,\mathrm{d}\mu &= \int_\Sigma (L\alpha)^2 + \frac{1}{2}H^2\alpha L\alpha - \lambda\alpha L\alpha - 2H\mathring{A}(\nabla\alpha,\nabla\alpha) \\
&+ \alpha^2\Big(\frac{3}{2}|\nabla H|^2 - \frac{3}{2}(H^2|\mathring{A}|^2 + H^{2M}\mathrm{Rc}(\nu,\nu) + \lambda H^2) + 2\omega(\nabla H) \\
&+ 2\langle\nabla^2 H, \mathring{A}\rangle + 2H^2|\mathring{A}|^2 + 2H\langle\mathring{A}, T\rangle - H\nabla_\nu{}^M\mathrm{Rc}(\nu,\nu)\Big)\,\mathrm{d}\mu.
\end{aligned}
$$

To understand the last term on the RHS above we recall that the Einstein tensor is divergence free and (76), which implies

$$
\begin{aligned}
\nabla_\nu{}^M\mathrm{Rc}(\nu,\nu) &= -\nabla_{e_i}G(\nu, e_i) - \frac{1}{2}\nabla_\nu{}^M\mathrm{Sc} \\
&= -{}^\Sigma\mathrm{div}\omega + \langle\mathring{A}, G^T\rangle + \frac{1}{2}H^M\mathrm{Sc} - \frac{3}{2}H^M\mathrm{Rc}(\nu,\nu) - \frac{1}{2}\nabla_\nu{}^M\mathrm{Sc}. \qquad (93)
\end{aligned}
$$

Note that

$$
\begin{aligned}
\frac{1}{2}\int_\Sigma H^2\alpha L\alpha\,\mathrm{d}\mu &= \frac{1}{2}\int_\Sigma (H\alpha)L(H\alpha) + \alpha^2 H\Delta H + H\langle\nabla H, \nabla(\alpha^2)\rangle\,\mathrm{d}\mu \\
&= \frac{1}{2}\int_\Sigma (H\alpha)L(H\alpha) - \alpha^2|\nabla H|^2\,\mathrm{d}\mu.
\end{aligned}
$$

Putting all together we arrive at

$$\begin{aligned}
&\int_\Sigma \alpha W_\lambda \alpha \,\mathrm{d}\mu \\
&\quad = \int_\Sigma L\alpha(L\alpha - 3\lambda\alpha) + \frac{1}{2}\Big((H\alpha)L(H\alpha) - 3\lambda(H\alpha)^2\Big) + 2\lambda\alpha L\alpha \\
&\qquad - 2H\mathring{A}(\nabla\alpha, \nabla\alpha) + \alpha^2\bigg(|\nabla H|^2 + \frac{1}{2}H^2|\mathring{A}|^2 + 2\langle\nabla^2 H, \mathring{A}\rangle \\
&\qquad + H\langle\mathring{A}, T\rangle - \frac{1}{2}H^{2\,M}\mathrm{Sc} + \frac{1}{2}H\nabla_\nu{}^M\mathrm{Sc} + H^\Sigma\mathrm{div}\omega + 2\omega(\nabla H)\bigg)\,\mathrm{d}\mu. \qquad (94)
\end{aligned}$$

We decompose $W^{2,2}(\Sigma)$ using the eigenspaces of L, more precisely consider the $L^2(\Sigma)$-orthonormal decomposition $W^{2,2}(\Sigma) = V_0 \oplus V_1 \oplus V_2$ where $V_0 = \mathrm{span}\{\varphi_0\}$, $V_1 = \mathrm{span}\{\varphi_1, \varphi_2, \varphi_3\}$, $V_2 = \mathrm{span}\{\varphi_4, \varphi_5, \ldots\}$. For any $\alpha \in W^{2,2}(\Sigma)$ let $\alpha_0, \alpha_1, \alpha_2$ be the respective orthogonal projections on these subspaces. Our aim is to show that $\int \alpha W_\lambda \alpha$ is positive on $V_0^\perp$.

Lemma 18 *For any surface Σ as in Theorem 8 we have the estimate*

$$\int_\Sigma L\alpha(L\alpha - 3\lambda\alpha) + 2\lambda\alpha L\alpha \,\mathrm{d}\mu \geq \Big(24m^2R_S^{-6} - C\sqrt{\eta}r_{\min}^{-7} - Cr_{\min}^{-8}\Big)\int_\Sigma \alpha^2\,\mathrm{d}\mu$$

for all $\alpha \in V_0^\perp$.

Proof This follows from the estimates on the eigenvalues of L in Corollary 5 and Theorem 8. ◻

Lemma 19 *For any surface Σ as in Theorem 8 we have the estimate*

$$\begin{aligned}
&\int_\Sigma (H\alpha)L(H\alpha) - 3\lambda(H\alpha)^2\,\mathrm{d}\mu \\
&\quad \geq -C\sqrt{\eta}r_{\min}^{-6}\int_\Sigma \alpha_1^2\,\mathrm{d}\mu + \frac{1}{4}r_{\min}^{-2}\int_\Sigma |\nabla\alpha_2|^2\,\mathrm{d}\mu + \frac{1}{4}r_{\min}^{-4}\int_\Sigma \alpha_2^2\,\mathrm{d}\mu
\end{aligned}$$

for all $\alpha \in V_0^\perp$.

Proof We can write

$$\begin{aligned}
\int_\Sigma (H\alpha)L(H\alpha) - 3\lambda(H\alpha)^2\,\mathrm{d}\mu &= \int_\Sigma (H\alpha_1)L(H\alpha_1) - 3\lambda(H\alpha_1)^2\,\mathrm{d}\mu \\
&\quad + 2\int_\Sigma (H\alpha_1)L(H\alpha_2) - 3\lambda(H\alpha_1)(H\alpha_2)\,\mathrm{d}\mu \\
&\quad + \int_\Sigma (H\alpha_2)L(H\alpha_2) - 3\lambda(H\alpha_2)^2\,\mathrm{d}\mu,
\end{aligned}$$

and we denote the terms on the RHS by (i), (ii) and (iii). Note that we can always estimate

$$\begin{aligned} \left|\langle H\alpha_i, \varphi_j\rangle_{L^2(\Sigma)}\right| &= \left|\int_\Sigma H\alpha_i\varphi_j \, \mathrm{d}\mu\right| \\ &\le \int_\Sigma |H - \bar{H}^S||\alpha_i|\,|\varphi_j| \, \mathrm{d}\mu \le C\sqrt{\eta}r_{\min}^{-3}\left(\int_\Sigma \alpha_i^2 \, \mathrm{d}\mu\right)^{1/2} \end{aligned} \tag{95}$$

for $i \neq j$. So we see

$$\begin{aligned} (i) \quad &\ge -|\mu_0 - 3\lambda| \int_\Sigma |(H\alpha_1)_0|^2 \, \mathrm{d}\mu - \max_{j=1,2,3} |\mu_j - 3\lambda| \int_\Sigma \left|(H\alpha_1)_{V_0^\perp}\right|^2 \mathrm{d}\mu \\ &\ge -C\eta r_{\min}^{-8} \int_\Sigma \alpha_1^2 \, \mathrm{d}\mu - C\sqrt{\eta}r_{\min}^{-4} \int_\Sigma \left|(H\alpha_1)_{V_0^\perp}\right|^2 \mathrm{d}\mu \\ &\ge -C\eta r_{\min}^{-8} \int_\Sigma \alpha_1^2 \, \mathrm{d}\mu - C\sqrt{\eta}r_{\min}^{-4} \left(\int_\Sigma (H\alpha_1)^2 \, \mathrm{d}\mu + C\eta r_{\min}^{-6} \int_\Sigma \alpha_1^2 \, \mathrm{d}\mu\right) \\ &\ge -C\eta r_{\min}^{-8} \int_\Sigma \alpha_1^2 \, \mathrm{d}\mu - C\sqrt{\eta}r_{\min}^{-6} \int_\Sigma \alpha_1^2 \, \mathrm{d}\mu. \end{aligned} \tag{96}$$

To estimate (ii) we write

$$\begin{aligned} (ii) \quad &\ge -2|\mu_0 - 3\lambda| \int_\Sigma |(H\alpha_1)_0|\,|(H\alpha_2)_0| \, \mathrm{d}\mu \\ &\quad - 2 \max_{j=1,2,3} |\mu_j - 3\lambda| \int_\Sigma |(H\alpha_1)|\,|(H\alpha_2)_1| \, \mathrm{d}\mu \\ &\quad + 2\int_\Sigma (H\alpha_1)_2(L - 3\lambda)(H\alpha_2) \, \mathrm{d}\mu \\ &\ge -C\eta r_{\min}^{-8} \left(\int_\Sigma \alpha_1^2 \, \mathrm{d}\mu\right)^{1/2} \left(\int_\Sigma \alpha_2^2 \, \mathrm{d}\mu\right)^{1/2} \\ &\quad + 2\int_\Sigma (H\alpha_1)_2(L - 3\lambda)(H\alpha_2) \, \mathrm{d}\mu. \end{aligned} \tag{97}$$

For the last term in (97) we write

$$(H\alpha_1)_2 = H\alpha_1 - \sum_{j=0}^{3} \langle H\alpha_1, \varphi_j\rangle \varphi_j = \sum_{j=0}^{3} \beta_j \varphi_j,$$

where $\beta_j = H\langle \alpha_1, \varphi_j\rangle - \langle H\alpha_1, \varphi_j\rangle$. Note that

$$|\nabla \beta_j| \le C\sqrt{\eta} r_{\min}^{-4} \left(\int_\Sigma \alpha_1^2 \, \mathrm{d}\mu \right)^{1/2}$$

and

$$|\beta_j| \le 2||H - \bar{H}^S||_{L^\infty} \sum_{j=0}^{3} |\langle \alpha_1, \varphi_j\rangle| \le C\sqrt{\eta} r_{\min}^{-3} \left(\int_\Sigma \alpha_1^2 \, \mathrm{d}\mu \right)^{1/2}.$$

Then

$$\begin{aligned}
&2 \int_\Sigma (H\alpha_1)_2 (L - 3\lambda)(H\alpha_2) \, \mathrm{d}\mu \\
&= 2 \int_\Sigma \left\langle \nabla \sum_{j=0}^{3} \beta_j \varphi_j, \nabla(H\alpha_2) \right\rangle - (H\alpha_1)_2 (H\alpha_2) \left(|A|^2 + {}^M\mathrm{Rc}(\nu, \nu) + 3\lambda \right) \mathrm{d}\mu \\
&\ge -C \left(\int_\Sigma \sum_{j=0}^{3} |\nabla \beta_j \varphi_j + \beta_j \nabla \varphi_j|^2 \, \mathrm{d}\mu \right)^{1/2} \left(\int_\Sigma |\nabla(H\alpha_2)|^2 \, \mathrm{d}\mu \right)^{1/2} \\
&\quad - C\sqrt{\eta} r_{\min}^{-6} \left(\int_\Sigma \alpha_1^2 \, \mathrm{d}\mu \right)^{1/2} \left(\int_\Sigma \alpha_2^2 \, \mathrm{d}\mu \right)^{1/2} \\
&\ge -C\sqrt{\eta} r_{\min}^{-4} \left(\int_\Sigma \alpha_1^2 \, \mathrm{d}\mu \right)^{1/2} \left(\int_\Sigma |\nabla(H\alpha_2)|^2 \, \mathrm{d}\mu \right)^{1/2} \\
&\quad - C\sqrt{\eta} r_{\min}^{-6} \left(\int_\Sigma \alpha_1^2 \, \mathrm{d}\mu \right)^{1/2} \left(\int_\Sigma \alpha_2^2 \, \mathrm{d}\mu \right)^{1/2},
\end{aligned} \tag{98}$$

where we used in the last step that $\int_\Sigma |\nabla \varphi_j|^2 \, \mathrm{d}\mu \le C r_{\min}^{-2}$ for $0 \le j \le 3$. This follows from

$$\begin{aligned}\int_\Sigma |\nabla\varphi_j|^2\,\mathrm{d}\mu &= \int_\Sigma \varphi_j L\varphi_j + \varphi_j^2\left(|A|^2 + {}^M\mathrm{Rc}(\nu,\nu)\right)\mathrm{d}\mu\\ &\le \left(\max_{j=0,1,2,3}|\mu_j| + Cr_{\min}^{-2}\right)\int_\Sigma \varphi_j^2\,\mathrm{d}\mu \le Cr_{\min}^{-2}.\end{aligned}\tag{99}$$

Putting (98) and (97) together we see

$$\begin{aligned}(ii)\quad &\ge -C\eta\varepsilon^{-1}r_{\min}^{-8}\int_\Sigma \alpha_1^2\,\mathrm{d}\mu - \varepsilon r_{\min}^{-4}\int_\Sigma \alpha_2^2\,\mathrm{d}\mu - \varepsilon\int_\Sigma |\nabla(H\alpha_2)|^2\,\mathrm{d}\mu\\ &\ge -C\eta\varepsilon^{-1}r_{\min}^{-8}\int_\Sigma \alpha_1^2\,\mathrm{d}\mu - \varepsilon r_{\min}^{-4}\int_\Sigma \alpha_2^2\,\mathrm{d}\mu - 2\varepsilon\int_\Sigma H^2|\nabla\alpha_2|^2\,\mathrm{d}\mu\\ &\quad - C\varepsilon\eta r_{\min}^{-8}\int_\Sigma \alpha_2^2\,\mathrm{d}\mu\end{aligned}\tag{100}$$

for an arbitrary $\varepsilon > 0$.

For the term (iii), we see

$$\begin{aligned}(iii)\quad &\ge -|\mu_0 - 3\lambda|\int_\Sigma |(H\alpha_2)_0|^2\,\mathrm{d}\mu - \max_{j=1,2,3}|\mu_j - 3\lambda|\int_\Sigma |(H\alpha_2)_1|^2\,\mathrm{d}\mu\\ &\quad + \int_\Sigma (H\alpha_2)_2(L-3\lambda)(H\alpha_2)_2\,\mathrm{d}\mu\\ &\ge -C\eta r_{\min}^{-8}\int_\Sigma \alpha_2^2\,\mathrm{d}\mu + \int_\Sigma (H\alpha_2)_2(L-3\lambda)(H\alpha_2)_2\,\mathrm{d}\mu.\end{aligned}$$

If $\beta \in V_2$ and $\delta > 0$ are arbitrary, we have the estimate

$$\begin{aligned}\int_\Sigma \beta(L-3\lambda)\beta\,\mathrm{d}\mu &= \int_\Sigma \delta|\nabla\beta|^2 + \beta(L+\delta\Delta-3\lambda)\beta\,\mathrm{d}\mu\\ &= \int_\Sigma \delta|\nabla\beta|^2 + (1-\delta)\beta(L-3\lambda)\beta\\ &\quad - \delta\beta^2\left(|A|^2 + {}^M\mathrm{Rc}(\nu,\nu) + 3\lambda\right)\mathrm{d}\mu\\ &\ge \delta\int_\Sigma |\nabla\beta|^2\,\mathrm{d}\mu + (1-\delta)\left((\nu_4-2)R_S^{-2} - C\sqrt{\eta}r_{\min}^{-4}\right)\int_\Sigma \beta^2\,\mathrm{d}\mu\end{aligned}$$

$$
\begin{aligned}
&- 3\delta r_{\min}^{-2} \int\limits_{\Sigma} \beta^2 \, \mathrm{d}\mu \\
&\geq \delta \int\limits_{\Sigma} |\nabla \beta|^2 \, \mathrm{d}\mu + (1 - 4\delta) r_{\min}^{-2} \int\limits_{\Sigma} \beta^2 \, \mathrm{d}\mu.
\end{aligned}
$$

With $\beta = (H\alpha_2)_2$ and $\delta = 1/5$ this yields

$$
\begin{aligned}
(iii) \quad &\geq -C\eta r_{\min}^{-8} \int\limits_{\Sigma} \alpha_2^2 \, \mathrm{d}\mu + \frac{1}{5} \int\limits_{\Sigma} |\nabla (H\alpha_2)_2|^2 \, \mathrm{d}\mu + \frac{1}{5} r_{\min}^{-2} \int\limits_{\Sigma} |(H\alpha_2)_2|^2 \, \mathrm{d}\mu \\
&\geq \frac{2}{5} r_{\min}^{-2} \int\limits_{\Sigma} |\nabla \alpha_2|^2 \, \mathrm{d}\mu + \frac{2}{5} r_{\min}^{-4} \int\limits_{\Sigma} \alpha_2^2 \, \mathrm{d}\mu,
\end{aligned} \tag{101}
$$

where we used that

$$
\begin{aligned}
\int\limits_{\Sigma} |(H\alpha_2)_2|^2 \, \mathrm{d}\mu &= \int\limits_{\Sigma} H^2 \alpha_2^2 \, \mathrm{d}\mu - \int\limits_{\Sigma} |(H\alpha_2)_0|^2 + |(H\alpha_2)_1|^2 \, \mathrm{d}\mu \\
&\geq \int\limits_{\Sigma} H^2 \alpha_2^2 \, \mathrm{d}\mu - C\eta r_{\min}^{-6} \int\limits_{\Sigma} \alpha_2^2 \, \mathrm{d}\mu \geq 3 r_{\min}^{-2} \int\limits_{\Sigma} \alpha_2^2 \, \mathrm{d}\mu,
\end{aligned}
$$

and

$$
\begin{aligned}
\int\limits_{\Sigma} |\nabla (H\alpha_2)_2|^2 \, \mathrm{d}\mu &= \int\limits_{\Sigma} |\nabla (H\alpha_2 - \sum_{j=0}^{3} \langle H\alpha_2, \varphi_j \rangle \varphi_j)|^2 \, \mathrm{d}\mu \\
&\geq \frac{3}{4} \int\limits_{\Sigma} |\nabla (H\alpha_2)|^2 \, \mathrm{d}\mu - C \sum_{j=0}^{3} |\langle H\alpha_2, \varphi_j \rangle|^2 \int\limits_{\Sigma} |\nabla \varphi_j|^2 \, \mathrm{d}\mu \\
&\geq \frac{2}{3} \int\limits_{\Sigma} H^2 |\nabla \alpha_2|^2 \, \mathrm{d}\mu - C \int\limits_{\Sigma} |\nabla H|^2 \alpha_2^2 \, \mathrm{d}\mu - C\eta r_{\min}^{-8} \int\limits_{\Sigma} \alpha_2^2 \, \mathrm{d}\mu \\
&\geq 2 r_{\min}^{-2} \int\limits_{\Sigma} |\nabla \alpha_2|^2 \, \mathrm{d}\mu - C\eta r_{\min}^{-8} \int\limits_{\Sigma} \alpha_2^2 \, \mathrm{d}\mu.
\end{aligned}
$$

Combining the estimates for (i), (ii) and (iii), and choosing $\varepsilon = 1/100$ and r_0 big enough we arrive at the claimed statement. □

Theorem 10 *In addition to the hypotheses of Theorem 8, there exists η_0 and r_0, depending only on m, σ and ε such that on such a surface Σ it holds*

$$\int_\Sigma \alpha W_\lambda \alpha \, \mathrm{d}\mu \geq 12 m^2 R_S^{-6} \int \alpha^2 \, \mathrm{d}\mu$$

for all $\alpha \in V_0^\perp$.

Proof By Lemma 18 and Lemma 19 we only have to check that the remaining terms in (94) have the right decay. First we note that by arguing as in the estimate (99) we get

$$\int_\Sigma |\nabla \alpha_1|^2 \, \mathrm{d}\mu \leq C r_{\min}^{-2} \int_\Sigma \alpha_1^2 \, \mathrm{d}\mu.$$

Thus we have

$$\begin{aligned}
\left| \int_\Sigma 2H \mathring{A}(\nabla\alpha, \nabla\alpha) \, \mathrm{d}\mu \right| &\leq C\sqrt{\eta} r_{\min}^{-4} \int_\Sigma |\nabla\alpha_1|^2 + |\nabla\alpha_2|^2 \, \mathrm{d}\mu \\
&\leq C\sqrt{\eta} r_{\min}^{-4} \left(r_{\min}^{-2} \int_\Sigma \alpha_1^2 \, \mathrm{d}\mu + \int_\Sigma |\nabla\alpha_2|^2 \, \mathrm{d}\mu \right).
\end{aligned}$$

We rewrite

$$\int_\Sigma 2\alpha^2 \langle \nabla^2 H, \mathring{A} \rangle \, \mathrm{d}\mu = -\int_\Sigma 4\alpha \nabla_i \alpha \nabla_j H \mathring{A}_{ij} + 2\alpha^2 \langle \nabla H, \operatorname{div} \mathring{A} \rangle \, \mathrm{d}\mu.$$

Furthermore

$$\left| \int_\Sigma 2\alpha^2 \langle \nabla H, \operatorname{div} \mathring{A} \rangle \, \mathrm{d}\mu \right| = \left| \int_\Sigma 2\alpha^2 \langle \nabla H, \frac{1}{2} \nabla H + \omega \rangle \, \mathrm{d}\mu \right| \leq C\eta r_{\min}^{-8} \int_\Sigma \alpha^2 \, \mathrm{d}\mu,$$

and

$$\begin{aligned}
\left| \int_\Sigma 4\alpha \nabla_i \alpha \nabla_j H \mathring{A}_{ij} \, \mathrm{d}\mu \right| &\leq C\eta r_{\min}^{-7} \left(\int_\Sigma \alpha^2 \, \mathrm{d}\mu \right)^{1/2} \left(\int_\Sigma |\nabla\alpha|^2 \, \mathrm{d}\mu \right)^{1/2} \\
&\leq C\eta r_{\min}^{-7} \left(\int_\Sigma \alpha^2 \, \mathrm{d}\mu \right)^{1/2}
\end{aligned}$$

$$\times\left(r_{\min}^{-2}\int\limits_{\Sigma}\alpha_1^2\,\mathrm{d}\mu+\int\limits_{\Sigma}|\nabla\alpha_2|^2\,\mathrm{d}\mu\right)^{1/2}$$
$$\leq C\eta r_{\min}^{-8}\int\limits_{\Sigma}\alpha^2\,\mathrm{d}\mu+C\eta r_{\min}^{-6}\int\limits_{\Sigma}|\nabla\alpha_2|^2\,\mathrm{d}\mu.$$

In view of the estimates of Theorem 8 we find

$$\Bigg|\int\limits_{\Sigma}\alpha^2\Big(|\nabla H|^2+\frac{1}{2}H^2|\mathring{A}|^2+H\langle\mathring{A},T\rangle-\frac{1}{2}H^{2\,M}\mathrm{Sc}+\frac{1}{2}H\nabla_\nu{}^M\mathrm{Sc}$$
$$+H^{\Sigma}\mathrm{div}\omega+2\omega(\nabla H)\Big)\,\mathrm{d}\mu\Bigg|\leq C\sqrt{\eta}r_{\min}^{-6}\int\limits_{\Sigma}\alpha^2\,\mathrm{d}\mu.$$

Altogether this finishes the proof of the theorem. □

7.3 Invertibility of the linearized operator

In this subsection we show that the linearized operator W_λ is invertible. In order to do this, we need good estimates for the projection of a function onto V_0. We start with a different calculation for the first eigenvalue μ_0 of L.

Lemma 20 *For any surface Σ as in Theorem 8 we have the estimate*

$$\left|\mu_0+|A|^2+{}^M\mathrm{Rc}(\nu,\nu)\right|\leq C\sqrt{\eta}r_{\min}^{-4}. \tag{102}$$

Proof From Theorem 8 we know that

$$\left|\frac{1}{2}\bar{H}_S^2-|A|^2\right|\leq C\sqrt{\eta}r_{\min}^{-4}$$

and

$$\left|\frac{1}{2}\bar{H}_S^2-\frac{2}{R_S^2}+\frac{4m}{R_S^3}\right|\leq Cr_{\min}^{-5}.$$

Combining these two estimates with Theorem 8 and Corollary 5 we get

$$\left|\mu_0+|A|^2+{}^M\mathrm{Rc}(\nu,\nu)\right|\leq\left|3\lambda-\frac{2}{R_S^2}+\frac{2}{R_S^2}-\frac{6m}{R_S^3}\right|+C\sqrt{\eta}r_{\min}^{-4}\leq\frac{C\sqrt{\eta}}{r_{\min}^4}.$$

□

Next we prove a $W^{2,2}$-estimate for the eigenfunction of L corresponding to the eigenvalue μ_0.

Lemma 21 *Let Σ be a surface as in Theorem 8 and let $u \in C^\infty(\Sigma)$ be a solution of $Lu = \mu_0 u$. Then we have*

$$\int_\Sigma |u-\bar{u}|^2 \,\mathrm{d}\mu + r_{\min}^2 \int_\Sigma |\nabla u|^2 \,\mathrm{d}\mu + r_{\min}^6 \int_\Sigma |\nabla^2 u|^2 \,\mathrm{d}\mu \leq C\sqrt{\eta} r_{\min}^{-2} \|u\|_{L^2(\Sigma)}^2, \tag{103}$$

where $\bar{u} = |\Sigma|^{-1} \int_\Sigma u \,\mathrm{d}\mu$. Moreover we have the pointwise estimate

$$\|u-\bar{u}\|_{L^\infty(\Sigma)} \leq C\eta^{1/4} r_{\min}^{-2} \|u\|_{L^2(\Sigma)}. \tag{104}$$

Proof By a scaling argument we see that we can assume without loss of generality that $\|u\|_{L^2(\Sigma)} = 1$. Using the definition of L and Lemma 20 we get

$$\begin{aligned}\int_\Sigma |\nabla u|^2 \,\mathrm{d}\mu &= \int_\Sigma uLu + u^2 \left(|A|^2 + {}^M\mathrm{Rc}(\nu,\nu)\right) \mathrm{d}\mu \\ &= \int_\Sigma u^2 \left(\mu_0 + |A|^2 + {}^M\mathrm{Rc}(\nu,\nu)\right) \mathrm{d}\mu \\ &\leq C\sqrt{\eta} r_{\min}^{-4}.\end{aligned}$$

In view of Theorem 9 there is a Poincaré inequality on Σ with constant close to the one on S_R^2. This yields

$$\int_\Sigma |u-\bar{u}|^2 \,\mathrm{d}\mu \leq cR_S^2 \|\nabla u\|_{L^2(\Sigma)}^2 \leq C\sqrt{\eta} r_{\min}^{-2}.$$

Similarly as above we calculate

$$\begin{aligned}\int_\Sigma |\Delta u|^2 \,\mathrm{d}\mu &= \int_\Sigma (Lu)^2 + 2uLu \left(|A|^2 + {}^M\mathrm{Rc}(\nu,\nu)\right) + u^2 \left(|A|^2 + {}^M\mathrm{Rc}(\nu,\nu)\right)^2 \mathrm{d}\mu \\ &= \int_\Sigma u^2 \left(\mu_0 + |A|^2 + {}^M\mathrm{Rc}(\nu,\nu)\right)^2 \mathrm{d}\mu.\end{aligned}$$

Hence, again by Lemma 20, we get the estimate

$$\int_\Sigma |\Delta u|^2 \,\mathrm{d}\mu \leq C\eta r_{\min}^{-8}.$$

Integrating by parts and interchanging derivatives as in (43) (note that by doing this we get an additional Gauss curvature term from which we now know that it is positive) we conclude

$$\int_\Sigma |\nabla^2 u|^2 \, d\mu \le \int_\Sigma |\Delta u|^2 \, d\mu \le C\sqrt{\eta} r_{\min}^{-8}.$$

Lemma 13 and the previous estimates now give

$$\|u-\bar{u}\|^4_{L^\infty(\Sigma)} \le C \int_\Sigma |u-\bar{u}|^2 \, d\mu \int_\Sigma |\nabla^2 u|^2 + H^4 |u-\bar{u}|^2 \, d\mu \le C\eta r_{\min}^{-8}.$$

This finishes the proof of the lemma. □

In the following lemma we show an L^2-estimate for solutions of $W_\lambda u = f$.

Lemma 22 *Let $\delta > 0$, let Σ be a surface as in Theorem 8 and let $u \in C^\infty(\Sigma)$ be a solution of $W_\lambda u = f$ with $\int_\Sigma (f - f_0)^2 \, d\mu \le \delta R_S^{-12} \|u\|^2_{L^2(\Sigma)}$, where f_0 and u_0 are the projections of f respectively u onto V_0. Then we have*

$$\|u-u_0\|_{L^2(\Sigma)} \le C(\sqrt{\delta} + \sqrt{\eta} + R_S^{-1}) \|u\|_{L^2(\Sigma)}. \tag{105}$$

Proof By a scaling argument we see that we can assume without loss of generality that $\|u\|_{L^2(\Sigma)} = 1$. Next we combine our assumption with Eq. 34 and the fact that $Lu_0 = \mu_0 u_0$ to get

$$\begin{aligned} W_\lambda(u-u_0) = f &- \mu_0 u_0 \left(\mu_0 + \frac{1}{2}H^2 - \lambda\right) + 2H\langle \mathring{A}, \nabla^2 u_0\rangle + 2H\omega(\nabla u_0) \\ &+ 2\mathring{A}(\nabla u_0, \nabla H) + u_0 \Big(|\nabla H|^2 + 2\omega(\nabla H) + H\Delta H \\ &+ \langle \nabla^2 H, \mathring{A}\rangle + 2H^2|\mathring{A}|^2 + 2H\langle \mathring{A}, T\rangle - H\nabla_\nu{}^M\mathrm{Rc}(\nu,\nu)\Big). \end{aligned} \tag{106}$$

With the help of Theorem 10 we conclude

$$\int_\Sigma (u-u_0) W_\lambda(u-u_0) \, d\mu \ge 12 m^2 R_S^{-6} \int_\Sigma (u-u_0)^2 \, d\mu.$$

To get an upper bound for this integral we multiply Eq. 106 by $(u-u_0)$ and estimate term by term. We start with the term involving f

$$\begin{aligned} \left| \int_\Sigma f(u-u_0) \, d\mu \right| &= \left| \int_\Sigma (f-f_0)(u-u_0) \, d\mu \right| \\ &\le m^{-2} R_S^6 \int_\Sigma (f-f_0)^2 \, d\mu + m^2 R_S^{-6} \int_\Sigma (u-u_0)^2 \, d\mu \\ &\le C\delta R_S^{-6} + m^2 R_S^{-6} \int_\Sigma (u-u_0)^2 \, d\mu. \end{aligned}$$

Next, using a variant of Lemma 20, we estimate

$$\begin{aligned}
&\left|\int_\Sigma \mu_0 u_0 \left(\mu_0 + \frac{1}{2}H^2 - \lambda\right)(u-u_0)\,\mathrm{d}\mu\right| \\
&\quad \le m^2 R_S^{-6}\int_\Sigma (u-u_0)^2\,\mathrm{d}\mu + m^{-2}R_S^2\int_\Sigma u_0^2(\mu_0 + \frac{1}{2}H^2-\lambda)^2\,\mathrm{d}\mu \\
&\quad \le m^2 R_S^{-6}\int_\Sigma (u-u_0)^2\,\mathrm{d}\mu + C\eta R_S^{-6}.
\end{aligned}$$

Now we estimate all terms containing derivatives of u_0. By arguing as before we see that we only have to bound the term

$$Cm^2R_S^6\int_\Sigma H^2|\mathring{A}|^2|\nabla^2 u_0|^2 + |\nabla u_0|^2(H^2|\omega|^2 + |\mathring{A}|^2|\nabla H|^2)\,\mathrm{d}\mu \le CR_S^{-8},$$

where we used Theorem 8 and Lemma 21. Finally we estimate the terms involving u_0. We start with

$$\begin{aligned}
&R_S^6\int_\Sigma u_0^2\left(|\nabla H|^4 + |\omega|^2|\nabla H|^2 + H^2|\Delta H|^2 + |\mathring{A}|^2|\nabla^2 H|^2 + H^4|\mathring{A}|^4\right)\mathrm{d}\mu \\
&\le CR_S^{-8} + cR_S^4\int_\Sigma u_0^2|\Delta H|^2\,\mathrm{d}\mu + C\eta\int_\Sigma u_0^2|\nabla^2 H|^2\,\mathrm{d}\mu \\
&\le CR_S^{-8} + CR_S^2\int_\Sigma u_0^2\left(|\mathring{A}|^4 + \lambda + {}^M\mathrm{Rc}(\nu,\nu)\right)^2\mathrm{d}\mu \\
&\le CR_S^{-8} + C\eta R_S^{-6},
\end{aligned}$$

where we used Lemma 14, Theorem 8 and Lemma 21. In the third term in the second line we can use Lemma 21 to replace u_0^2 by $\bar{u}_0^2$. Finally, we use (93) and Theorem 8 to get

$$\begin{aligned}
&\left|\int_\Sigma (u-u_0)u_0 H\nabla_\nu {}^M\mathrm{Rc}(\nu,\nu)\,\mathrm{d}\mu\right| \\
&\quad \le \frac{3}{2}\left|\int_\Sigma (u-u_0)u_0H^{2M}\mathrm{Rc}(\nu,\nu)\,\mathrm{d}\mu\right| + m^2R_S^{-6}\int_\Sigma (u-u_0)^2\,\mathrm{d}\mu + \frac{C\eta}{R_S^6}.
\end{aligned}$$

Now we use the L^2-orthogonality of u_0 and $u - u_0$ to estimate

$$\begin{aligned}
&\frac{3}{2}\left|\int_\Sigma (u-u_0)u_0 H^{2M}\mathrm{Rc}(\nu,\nu)\,\mathrm{d}\mu\right| \\
&\le \frac{3}{2}\left|\int_\Sigma (u-u_0)u_0 H^2\left({}^M\mathrm{Rc}(\nu,\nu)+\frac{2m}{R_S^3}\right)\mathrm{d}\mu\right| \\
&\quad + 3mR_S^{-3}\left|\int_\Sigma (u-u_0)u_0(H^2-4R_S^{-2})\,\mathrm{d}\mu\right| \\
&\le m^2R_S^{-6}\int_\Sigma (u-u_0)^2\,\mathrm{d}\mu + C\eta R_S^{-6}.
\end{aligned}$$

Combining all these estimates we get

$$3m^2R_S^{-6}\int_\Sigma (u-u_0)^2\,\mathrm{d}\mu \le CR_S^{-6}(\delta+\eta+R_S^{-2})$$

which finishes the proof of the lemma. □

From the proof of the lemma we directly obtain the following

Corollary 6 *Let $\delta > 0$, let Σ be a surface as in Theorem 8 and let $u \in C^\infty(\Sigma)$. Then we have*

$$\left|\int_\Sigma (u-u_0)W_\lambda u_0\,\mathrm{d}\mu\right| \le \frac{4m^2}{R_S^6}\|u-u_0\|_{L^2}^2 + \frac{c}{R_S^6}(\delta+\eta+R_S^{-2})\|u\|_{L^2(\Sigma)}^2. \tag{107}$$

Moreover, if u is a solution of $W_\lambda u = f$ with

$$\int_\Sigma (u-u_0)f\,\mathrm{d}\mu \le \delta R_S^{-6}\|u\|_{L^2(\Sigma)}\|u-u_0\|_{L^2(\Sigma)},$$

then we have

$$\|u-u_0\|_{L^2(\Sigma)} \le C(\sqrt{\delta}+\sqrt{\eta}+R_S^{-1})\|u\|_{L^2(\Sigma)}. \tag{108}$$

In the following lemma we prove L^2-estimates for the operator W_λ.

Lemma 23 *Let Σ be as in Theorem 8. Then we have*

$$\|\nabla^2 u\|_{L^2(\Sigma)}^2 + R_S^{-2}\|\nabla u\|_{L^2(\Sigma)}^2 \le CR_S^{-4}\|u\|_{L^2(\Sigma)}^2 + CR_S\left|\int_\Sigma uW_\lambda u\,\mathrm{d}\mu\right|.$$

Proof From (35), we get the following expression, after integration by parts of the term $u\Delta u(|A|^2 + {}^M\mathrm{Rc}(\nu,\nu))$ in $(Lu)^2$:

$$\begin{aligned}\int_\Sigma uW_\lambda u\,\mathrm{d}\mu = \int_\Sigma &(\Delta u)^2 + \left(\frac{1}{2}H^2 - \lambda - 2|A|^2 - 2\,{}^M\mathrm{Rc}(\nu,\nu)\right)|\nabla u|^2\\ &+ u^2\left(-\frac{1}{2}H^2|A|^2 - \frac{1}{2}H^{2\,M}\mathrm{Rc}(\nu,\nu) - H\nabla_\nu{}^M\mathrm{Rc}(\nu,\nu)\right.\\ &\left.+\lambda|A|^2 + |A|^4 + 2|A|^{2\,M}\mathrm{Rc}(\nu,\nu)\right)\\ &+ a(u,\nabla u) + bu^2 + u\nabla_k u A^{ij}\nabla^k A_{ij}\,\mathrm{d}\mu. \end{aligned} \tag{109}$$

Here $|a(u,\nabla u) + bu^2| \le CR_S^{-4}|\nabla u|^2 + CR_S^{-6}u^2$, where we integrated by parts and used Lemma 1, Definition 1 and Theorem 8. In particular we can estimate

$$\begin{aligned}\left|\nabla_{e_i}\left({}^M\mathrm{Rc}(\nu,\nu)\right)\right| &\le \left|\left(\nabla_{e_i}{}^M\mathrm{Rc}\right)(\nu,\nu) + 2h_i^k\omega_k\right|\\ &\le \left|\left(\nabla^S_{e_i}{}^M\mathrm{Rc}^S\right)(\nu,\nu)\right| + C\sqrt{\eta}r_{\min}^{-5}\\ &\le \left|\left(\nabla^S_{P_\rho^\perp(e_i)}{}^M\mathrm{Rc}^S\right)(\rho,\rho)\right| + C\sqrt{\eta}r_{\min}^{-5}\\ &\le C\sqrt{\eta}r_{\min}^{-5}, \end{aligned} \tag{110}$$

where we used the above mentioned theorems, and where $P_\rho^\perp$ is the projection onto the g^S-orthogonal subspace to ρ. In view of the Gauss equation, the Bochner formula [5, Chapter IV, Proposition 4.15] implies that

$$\int_\Sigma (\Delta u)^2\,\mathrm{d}\mu = \int_\Sigma 2|(\nabla^2 u)^\circ|^2 + \left({}^M\mathrm{Sc} - 2\,{}^M\mathrm{Rc}(\nu,\nu) + \frac{1}{2}H^2 - |\mathring{A}|^2\right)|\nabla u|^2\,\mathrm{d}\mu.$$

Together with (109) this yields

$$\begin{aligned}\int_\Sigma uW_\lambda u\,\mathrm{d}\mu = \int_\Sigma &2|(\nabla^2 u)^\circ|^2 + |\nabla u|^2\left(-4\,{}^M\mathrm{Rc}(\nu,\nu) - \lambda\right) + u\nabla_k u A^{ij}\nabla^k A_{ij}\\ &+ u^2\left(-\frac{1}{2}H^{2\,M}\mathrm{Rc}(\nu,\nu) - H\nabla_\nu{}^M\mathrm{Rc}(\nu,\nu) + \lambda|A|^2\right.\\ &\left.+ 2|A|^{2\,M}\mathrm{Rc}(\nu,\nu)\right) + a(u,\nabla u) + bu^2\,\mathrm{d}\mu. \end{aligned}$$

In combination with the estimate $|{}^M\mathrm{Rc}(\nu,\nu) + \lambda| \le CR_S^{-4}$ and the fact that

$$\begin{aligned}-\frac{1}{2}H^{2\,M}\mathrm{Rc}(\nu,\nu) - H\nabla_\nu{}^M\mathrm{Rc}(\nu,\nu) + \lambda|A|^2 + 2|A|^{2\,M}\mathrm{Rc}(\nu,\nu) = -\frac{3}{2}H^2\lambda\\ + O(R_S^{-6})\end{aligned}$$

we obtain the estimate

$$
\begin{aligned}
&2\|(\nabla^2 u)^\circ\|_{L^2}^2 + 2\lambda\|\nabla u\|_{L^2}^2 \\
&\quad \le C R_S^{-2}\lambda\|u\|_{L^2}^2 + C\left|\int_\Sigma u W_\lambda u \,\mathrm{d}\mu\right| + C\int_\Sigma |u||\nabla u||A||\nabla A|\,\mathrm{d}\mu. \qquad (111)
\end{aligned}
$$

To treat the last term, observe that

$$
\begin{aligned}
\int_\Sigma |u||\nabla u||A||\nabla A|\,\mathrm{d}\mu &\le \int_\Sigma \lambda|\nabla u|^2 + \frac{1}{4\lambda}|u|^2|A|^2|\nabla A|^2\,\mathrm{d}\mu \\
&\le \lambda\|\nabla u\|_{L^2}^2 + C R_S^{-5}\|u\|_{L^\infty}^2
\end{aligned}
$$

using Theorem 6, Theorem 7 and $\lambda = 2m/R_S^3 + O(R_S^{-4})$. In particular

$$
\|\nabla u\|_{L^2}^2 \le C R_S^{-2}\|u\|_{L^2}^2 + C R_S^3\left|\int_\Sigma u W_\lambda u\,\mathrm{d}\mu\right| + C R_S^{-2}\|u\|_{L^\infty}^2.
$$

Note that in view of this estimate (109) implies that

$$
\|\Delta u\|_{L^2}^2 \le C R_S^{-2}\|\nabla u\|_{L^2}^2 + C R_S^{-4}\|u\|_{L^2}^2 + C\left|\int_\Sigma u W_\lambda u\,\mathrm{d}\mu\right| + C R_S^{-5}\|u\|_{L^\infty}^2.
$$

Together with (111), we obtain that

$$
\|\nabla^2 u\|_{L^2}^2 + R_S^{-2}\|\nabla u\|_{L^2}^2 \le C R_S^{-4}\|u\|_{L^2}^2 + C R_S\left|\int_\Sigma u W_\lambda u\,\mathrm{d}\mu\right| + C R_S^{-4}\|u\|_{L^\infty}^2. \qquad (112)
$$

From Lemma 13 we conclude that in view of Theorem 8

$$
\|u\|_{L^\infty}^2 \le C R_S^{-2}\|u\|_{L^2}^2 + C\|u\|_{L^2}\|\nabla^2 u\|_{L^2}.
$$

Inserting this into Eq. 112, we get

$$
\begin{aligned}
&\|\nabla^2 u\|_{L^2}^2 + R_S^{-2}\|\nabla u\|_{L^2}^2 \\
&\quad \le C R_S^{-4}\|u\|_{L^2}^2 + C R_S\left|\int_\Sigma u W_\lambda u\,\mathrm{d}\mu\right| + C R_S^{-4}\|u\|_{L^2}\|\nabla^2 u\|_{L^2}. \qquad (113)
\end{aligned}
$$

For large enough R_S, we can therefore apply the Cauchy–Schwarz inequality and absorb the term containing second derivatives to the left. This yields the claimed estimate. □

With the help of the last two results we are able to show that certain solutions of $W_\lambda u = f$ are almost constant.

Lemma 24 *There exists $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$, all surfaces Σ as in Theorem 8 and all solutions $u \in C^\infty(\Sigma)$ of $W_\lambda u = f$ with*

$$\int_\Sigma (u - u_0) f \, \mathrm{d}\mu \leq \delta R_S^{-6} \|u\|_{L^2(\Sigma)} \|u - u_0\|_{L^2}$$

we have

$$\|u - \bar{u}_0\|_{L^\infty(\Sigma)} \leq C(\sqrt{\delta} + \eta^{1/4} + R_S^{-1})|\bar{u}_0|. \tag{114}$$

Proof We assume that $\|u\|_{L^2(\Sigma)} = 1$ and apply Corollary 6 to get

$$\|u - u_0\|_{L^2(\Sigma)} \leq C(\sqrt{\delta} + \sqrt{\eta} + R_S^{-1}).$$

Moreover, by Lemma 21, we have that

$$\|u_0 - \bar{u}_0\|_{L^\infty(\Sigma)} \leq C\eta^{1/4} R_S^{-2}.$$

Combining these two facts we get

$$\begin{aligned} \|u - \bar{u}_0\|_{L^2(\Sigma)} &\leq \|u - u_0\|_{L^2(\Sigma)} + C R_S \|u_0 - \bar{u}_0\|_{L^\infty(\Sigma)} \\ &\leq C(\sqrt{\delta} + \eta^{1/4} + R_S^{-1}). \end{aligned} \tag{115}$$

Using Lemma 23 (with u replaced by $u - u_0$) we get

$$\begin{aligned} \|\nabla^2(u - u_0)\|^2_{L^2(\Sigma)} &\leq C R_S^{-4} \|u - u_0\|^2_{L^2(\Sigma)} + c R_S \left| \int_\Sigma (u - u_0) W_\lambda (u - u_0) \, \mathrm{d}\mu \right| \\ &\leq C R_S^{-4} (\delta + \sqrt{\eta} + R_S^{-2}), \end{aligned}$$

where we used Corollary 6 and the assumption of the lemma. Combining this with Lemma 21 we have

$$\|\nabla^2(u - \bar{u}_0)\|^2_{L^2(\Sigma)} \leq C R_S^{-4} (\delta + \sqrt{\eta} + R_S^{-2})$$

and therefore, with the help of Lemma 13 and (115), we conclude

$$\|u - \bar{u}_0\|_{L^\infty(\Sigma)} \leq C R_S^{-1} (\sqrt{\delta} + \eta^{1/4} + R_S^{-1}). \tag{116}$$

Next we note that by orthogonality

$$0 \le 1 - \|u_0\|_{L^2}^2 = \|u - u_0\|_{L^2}^2$$

and from Theorem 10, (107) and the assumption of the lemma we get

$$\begin{aligned}\|u - u_0\|_{L^2}^2 &\le \frac{R_S^6}{12m^2} \int_\Sigma (u - u_0) W_\lambda (u - u_0)\, \mathrm{d}\mu \\ &\le C\sqrt{\delta}\|u - u_0\|_{L^2}^2 + \frac{1}{3}\|u - u_0\|_{L^2}^2 + C(\delta + \eta + R_S^{-2}) \\ &\le \frac{1}{2}\|u - u_0\|_{L^2}^2 + C(\delta + \eta + R_S^{-2}).\end{aligned}$$

Hence for δ, η small enough and R_S large enough we have

$$\|u_0\|_{L^2}^2 \ge \frac{1}{4}$$

and moreover, by Lemma 21, this implies that there exists a constant $c_1 > 0$ such that

$$c_1^{-1} R_S^{-1} \le |\bar{u}_0| \le c_1 R_S^{-1}.$$

Inserting this estimate into (116) we get

$$\|u - \bar{u}_0\|_{L^\infty(\Sigma)} \le C(\sqrt{\delta} + \sqrt{\eta} + R_S^{-1})|\bar{u}_0|.$$

□

Next we show that the above estimates yield the invertibility of the operator $W_\lambda : C^{4,\alpha}(\Sigma) \to C^{0,\alpha}(\Sigma)$.

Theorem 11 *There exists $\delta_0 > 0$ such that for every surface Σ as in Theorem 8 the operator $W_\lambda : C^{4,\alpha}(\Sigma) \to C^{0,\alpha}(\Sigma)$ is invertible for every $0 < \alpha < 1$. Its inverse $W_\lambda^{-1} : C^{0,\alpha}(\Sigma) \to C^{4,\alpha}(\Sigma)$ exists and is continuous. Moreover it satisfies the estimates*

$$\|W_\lambda^{-1} f\|_{L^2(\Sigma)} \le \frac{R_S^6}{\delta_0}\|f\|_{L^2(\Sigma)} \quad \text{for every} \quad f \in L^2(\Sigma) \quad \text{and} \tag{117}$$

$$\|W_\lambda^{-1} f\|_{C^{0,\alpha}(\Sigma)} \le \frac{cR_S^6}{\delta_0}\|f\|_{C^{4,\alpha}(\Sigma)} \quad \text{for every} \quad f \in C^{4,\alpha}(\Sigma). \tag{118}$$

Proof We argue by contradiction as in [14]. Namely we assume that there exists a smooth function u with $\|u\|_{L^2(\Sigma)} = 1$ and

$$\sup_{\|v\|_{L^2(\Sigma)}=1} \left| \int_\Sigma v W_\lambda u \, \mathrm{d}\mu \right| \le \delta_0 R_S^{-6}. \tag{119}$$

Choosing $v = u - u_0$, we conclude from Lemma 24 that $\bar{u}_0 \neq 0$ and therefore we can assume without loss of generality that $\bar{u}_0 > 0$. Again from Lemma 24 we then conclude that for δ_0, η small and R_S large enough we have for every $x \in \Sigma$ that $\frac{\bar{u}_0}{2} \leq u(x) \leq 2\bar{u}_0$. Arguing as in the proof of Lemma 24 we get $\frac{1}{2} \leq \|u_0\|_{L^2(\Sigma)} \leq 1$ and, with the help of Lemma 21, this implies

$$\frac{1}{2}|\Sigma|^{-1/2} \leq |\Sigma|^{-1/2}\|u_0\|_{L^2(\Sigma)} \leq \bar{u}_0 \leq |\Sigma|^{-1/2}\|u_0\|_{L^2(\Sigma)} \leq |\Sigma|^{-1/2}.$$

Moreover, by choosing $v = 1$ in (119), we get

$$\left|\int_\Sigma W_\lambda u \,\mathrm{d}\mu\right| \leq \delta_0 R_S^{-6}|\Sigma|^{1/2} \leq C\delta_0 R_S^{-5}. \tag{120}$$

On the other hand, by using (35) and the corresponding equation for the λL term, we get

$$\begin{aligned}\int_\Sigma W_\lambda u \,\mathrm{d}\mu = \int_\Sigma u \Big(&|A|^4 + 2|A|^{2\,M}\mathrm{Rc}(\nu,\nu) + ({}^M\mathrm{Rc}(\nu,\nu))^2 \\ &+ \Delta(|A|^2 + {}^M\mathrm{Rc}(\nu,\nu)) + \lambda\left(|A|^2 + {}^M\mathrm{Rc}(\nu,\nu)\right) + |\nabla H|^2 \\ &+ 2\omega(\nabla H) + H\Delta H + 2\langle\nabla^2 H, \mathring{A}\rangle + 2H^2|\mathring{A}|^2 + 2H\langle\mathring{A}, T\rangle \\ &- H\nabla_\nu{}^M\mathrm{Rc}(\nu,\nu) - \frac{1}{2}H^2|A|^2 - \frac{1}{2}H^{2\,M}\mathrm{Rc}(\nu,\nu)\Big)\,\mathrm{d}\mu.\end{aligned}$$

Now we calculate

$$\begin{aligned}&|A|^2\left(|A|^2 + 2^M\mathrm{Rc}(\nu,\nu) + \lambda\right) - H\nabla_\nu{}^M\mathrm{Rc}(\nu,\nu) - \frac{1}{2}H^2\left(|A|^2 + {}^M\mathrm{Rc}(\nu,\nu)\right) \\ &\quad= \frac{3}{2}H^{2\,M}\mathrm{Rc}(\nu,\nu) + O(R_S^{-6}).\end{aligned}$$

Moreover we estimate $\|\Delta\mathring{A}\|_{L^2(\Sigma)}$ as in the proof of Lemma 15, and using Lemma 14

$$\begin{aligned}\left|\int_\Sigma u\Delta|A|^2\,\mathrm{d}\mu\right| &= \left|\int_\Sigma u\Delta(|\mathring{A}|^2 + \frac{1}{2}H^2)\,\mathrm{d}\mu\right| \\ &\leq \left|\int_\Sigma uH\Delta H\,\mathrm{d}\mu\right| + CR_S^{-6} \leq CR_S^{-5}.\end{aligned}$$

Now we integrate by parts and use Proposition 6 and Lemma 23 to conclude

$$\left|\int_{\Sigma} u\Delta^{M}\mathrm{Rc}(\nu,\nu)\,\mathrm{d}\mu\right| \le CR_S^{-4}||\nabla u||_{L^2(\Sigma)} \le CR_S^{-5},$$

where we used in the last step that $|\int_{\Sigma} uW_{\lambda}u\,\mathrm{d}\mu| \le \delta_0 R_S^{-6}$, which follows from (119). We combine these estimates with the ones done previously in this section, (120) and Theorem 8 to conclude

$$-\int_{\Sigma} uH^{2M}\mathrm{Rc}(\nu,\nu)\,\mathrm{d}\mu \le C\left|\int_{\Sigma} W_{\lambda}u\,\mathrm{d}\mu\right| + CR_S^{-5} \le CR_S^{-5}.$$

The estimates $\bar{u}_0 \le 2u$ and $\frac{1}{\bar{u}_0} \le 2R_S$ imply

$$\begin{aligned}
2mR_S^{-3}\int_{\Sigma} H^2\,\mathrm{d}\mu &\le -\int_{\Sigma} H^{2M}\mathrm{Rc}(\nu,\nu)\,\mathrm{d}\mu + CR_S^{-4}\\
&\le -\frac{1}{2\bar{u}_0}\int_{\Sigma} uH^{2M}\mathrm{Rc}(\nu,\nu)\,\mathrm{d}\mu + CR_S^{-4}\\
&\le CR_S^{-4}.
\end{aligned}$$

This contradicts the estimate for $\int_{\Sigma} H^2\,\mathrm{d}\mu$ in Lemma 7. Hence the operator W_{λ} is injective. By the Fredholm alternative W_{λ} is also surjective. The rest of the statements in the theorem are then a consequence of standard elliptic theory. □

8 Existence and uniqueness of the foliation

In this last section we use the implicit function theorem to prove Theorem 1 and Theorem 2.

8.1 Uniqueness in Schwarzschild

In this subsection we show that in Schwarzschild the only surfaces satisfying the assumptions of Theorem 7 are the round spheres with center at the origin.

Theorem 12 *For all $m > 0$ there exist $r_0 < \infty$, $\tau_0 > 0$ and $\varepsilon > 0$ with the following properties.*
Assume that $(M, g) = (\mathbf{R}^3, g_m^S)$ and let Σ be a surface satisfying (1) with $H > 0$, $\lambda > 0$, $r_{\min} > r_0$ and

$$\tau \le \tau_0 \quad \textit{and} \quad R_e \le \varepsilon r_{\min}^2,$$

where R_e and τ are as in Sect. 5. Then $\Sigma = S_{R_e}(0)$.

Proof Since $(M, g) = (\mathbf{R}^3, g_m^S)$ we can apply Proposition 2, Theorem 7 and Theorem 8 with $\eta = 0$ to get $\tau = 0$, $\mathring{A}^S = 0$, and $\lambda = \frac{2m}{R_S^3}$. Since $\mathring{A}^S = 0$, we get that Σ is umbilical with respect to the Euclidean background metric, as $\mathring{A}^S = \phi^{-2}\mathring{A}^e$. Hence Σ is a sphere. Since $\tau = 0$ in fact $\Sigma = S_{R_e}(0)$ where $R_e = \phi^{-2}R_S$, or otherwise the expression for λ could not be true. □

8.2 Existence and uniqueness for the general case

The main goal in this subsection is to show that for any manifold which is (m, η, σ)-asymptotically Schwarzschild and all small enough Lagrange multipliers λ there exists a unique surface Σ_λ which solves the Eq. 1. More precisely we have the following theorem.

Theorem 13 *For all $m > 0$ and σ there exist $\eta_0 > 0$, $\lambda_0 > 0$ and C depending only on m and σ with the following properties.*

If (M, g) is (m, η, σ)-asymptotically Schwarzschild and satisfies

(1) $|{}^M\mathrm{Sc}| \le \eta r^{-5}$ *and*
(2) $\eta \le \eta_0$

then for all $0 < \lambda < \lambda_0$ there exists a surface Σ_λ which solves (1) for the given λ. Moreover the surface is well approximated in the C^3-norm by a coordinate sphere $S_{r_\lambda}(a_\lambda)$ with $|a_\lambda| \le C$.

Proof We define $g_\tau = (1 - \tau)g^S + \tau g$ and we note that (M, g_τ) is (m, η, σ)-asymptotically Schwarzschild. For (M, g^S) a standard calculation shows that all spheres $S_r(0)$ centered at the origin solve Eq. 1 with

$$\lambda(r) = \frac{2m}{r^3}\left(1 + \frac{m}{2r}\right)^{-6}.$$

This function is invertible for r large enough. Moreover this shows that we can solve Eq. 1 in (M, g^S) for any λ small enough. More precisely, for any small λ there exists a radius $r(\lambda)$ such that $S_{r(\lambda)}(0)$ solves (1) with the given λ. Next we want to use the implicit function theorem to get the existence of a family of such solutions for all $0 \le \tau \le 1$.

In order to do this we consider the following conditions on our surfaces

(A1) $H > 0$,

(A2) $\tau \le \tau_0$ and

(A3) $R_e \le \varepsilon r_{\min}^2$,

where τ_0 and ε are chosen such that we can apply the results from Sect. 6. From these results we then get that the above conditions hold with better constants on surfaces Σ with $r_{\min} > r_0$

(B1) $|H - 2R_S^{-1} + (1 + \frac{m}{2R_e})\, 2mR_S^{-2}| \le C\sqrt{\eta} r_{\min}^{-3}$,

(B2) $\tau \le C\sqrt{\eta} r_{\min}^{-1}$ and

(B3) $C^{-1} r_{\min} \le R_e \le C r_{\min}$.

Without loss of generality we can furthermore assume that the conditions (B1)-(B3) imply that the linearized operator W_λ is invertible. From (82) we also get that Σ is globally a graph over S^2.

Now we define the sets

$$
\begin{aligned}
S_1(\tau) &= \{\Sigma |\ r_{\min} > r_0 \ \text{ and } \ (A1) - (A3) \ \text{ hold w.r.t. } \ g_\tau\} \\
S_2(\tau) &= \{\Sigma |\ r_{\min} > 2r_0 \ \text{ and } \ (B1) - (B3) \ \text{ hold w.r.t. } \ g_\tau\}.
\end{aligned}
$$

We choose λ_2 so small that the centered spheres $S_r(0)$ which solve (1) with $0 < \lambda < \lambda_2$ are in $S_2(\tau)$. Finally (for λ_1 small) we let

$$
\begin{aligned}
\kappa : [0, 1] &\to (0, \lambda_1) \times [0, 1] \\
\kappa(t) &= (\lambda(t), \tau(t))
\end{aligned}
$$

be a continuous, piecewise smooth curve with $\tau(0) = 0$ and we define

$$
I_\kappa = \{t \in [0, 1] | \exists \ \ \Sigma(t) \in S_2(\tau(t)) \ \text{ satisfying } \ (1) \ \text{ with } \ \lambda = \lambda(t)\}.
$$

As in [14] we can show that I_κ is open and closed and since moreover $0 \in I_\kappa$ by our assumption we get $I_\kappa = [0, 1]$ and this finishes the proof of the Theorem. □

By reversing the process used in the above theorem as in the proof of theorem 6.5 in [14] we furthermore get a uniqueness result for solutions of (1).

Theorem 14 *Let $m > 0$ and σ be given. Then there exist $\eta_0 > 0$, τ_0, $r_0 < \infty$, and $\varepsilon > 0$ depending only on m and σ such that the following holds.*

Assume that (M, g) is (m, σ, η)-asymptotically Schwarzschild with

(1) $|{}^M\mathrm{Sc}| \le \eta r^{-5}$, *and*
(2) $\eta < \eta_0$.

Furthermore, let Σ be a surface with approximating sphere $S_{r_\lambda}(a_\lambda)$ as in Sect. 5, such that

(3) *Σ satisfies Eq. 1,*
(4) $H > 0$,
(5) $r_{\min} > r_0$, *and* $r_\lambda < \varepsilon r_{\min}^2$,
(6) $\tau_\lambda = r_\lambda / a_\lambda < \tau_0$,

then $\Sigma = \Sigma_\lambda$, where Σ_λ is the surface from Theorem 13. □

8.3 Foliation

Next we show that the surfaces obtained in Theorem 13 form a foliation.

Theorem 15 *For all $m > 0$ and σ there exists $\eta_0 > 0$ depending only on m and σ with the following properties.*

If (M, g) is (m, η, σ)-asymptotically Schwarzschild and satisfies

(1) $|{}^M\mathrm{Sc}| \leq \eta r^{-5}$ *and*
(2) $\eta \leq \eta_0$

then for all $0 < \lambda < \lambda_0$ the surfaces Σ_λ constructed in Theorem 13 form a foliation. In addition, there is a differentiable map

$$F : S^2 \times (0, \lambda_0) \times [0, 1] \to M$$

such that the surfaces $F(S^2, \lambda, \tau)$ satisfy (1) with respect to the metric $g_\tau = (1 - \tau)g^S + \tau g$ for the given λ. This foliation can therefore be obtained by deforming a piece of the foliation of $(\mathbf{R}^3, g^S)$ by centered spheres.

Proof The proof follows along the same lines as the one given in [14, Theorem 6.4]. Therefore we only sketch the main ideas of the argument.

For $0 < \lambda < \lambda_0$ we consider the curve $\kappa_\lambda(t) = (\lambda, t)$ and by using Theorem 13 we obtain a family of surfaces $\Sigma_{\lambda,t}$ which solve (1) for the given λ.

The map F can now be defined by $F(S^2, \lambda, t) = \Sigma_{\lambda,t}$ where we can choose the parametrization of $\Sigma_{\lambda,t}$ such that $\frac{\partial F}{\partial \lambda} \perp \Sigma_{\lambda,t}$. The differentiability of F with respect to $p \in S^2$ and τ follows from the construction of $\Sigma_{\lambda,t}$.

It remains to prove that the surfaces form a foliation. In order to show this we fix $\lambda_1 \in (0, \lambda_0)$ and we get from the above construction a surface $\Sigma_{\lambda_1,1}$. For $\lambda_2 < \lambda_1$ we define the curve $h_{\lambda_2}(t) = ((1 - t)\lambda_1 + t\lambda_2, 1)$. By combining the curves κ_{λ_1} and h_{λ_2} we get a family of surfaces $\Sigma'_{\lambda(t),1}$ which solve (1) with $\lambda(t) = (1 - t)\lambda_1 + t\lambda_2$ for $t \in [0, 1]$. Moreover we get a differentiable map $G : S^2 \times [\lambda_2, \lambda_1] \to M$ such that $G(S^2, \lambda(t)) = \Sigma'_{\lambda(t),1}$. From the local uniqueness statement in the implicit function theorem we get that $\Sigma'_{\lambda(t),1} = \Sigma_{\lambda(t),1} =: \Sigma_{\lambda(t)}$.

Now we let $\nu_{\lambda(t)}$ be the normal to $\Sigma_{\lambda(t)}$ in M and we let $\alpha_{\lambda(t)} = g\big(\nu_{\lambda(t)}, \frac{\partial G}{\partial \lambda}\big)$. We calculate

$$\begin{aligned} H(\lambda_1 - \lambda_2) &= \frac{d}{dt}\Big(-\Delta H - H|\mathring{A}|^2 - H\,{}^M\mathrm{Rc}(\nu, \nu)\Big) - \lambda(t)\frac{d}{dt}H \\ &= W_{\lambda(t)}\alpha_{\lambda(t)}(\lambda_1 - \lambda_2). \end{aligned}$$

Next we claim that

$$\int_\Sigma (\alpha_{\lambda(t)} - (\alpha_{\lambda(t)})_0)H \,\mathrm{d}\mu \leq \frac{C\eta^{1/4}}{R_S^6}\|\alpha_{\lambda(t)}\|_{L^2(\Sigma)}\|\alpha_{\lambda(t)} - (\alpha_0)_{\lambda(t)}\|_{L^2(\Sigma)}. \tag{121}$$

If we assume that this claim is true we see that for η small enough we can apply Lemma 24 and get that $\alpha_{\lambda(t)}$ does not change sign. Therefore the family $\Sigma_{\lambda(t)}$ is a foliation.

In order to prove (121) we let $W_{\lambda(t)} = W_\lambda$, $\alpha = \alpha_{\lambda(t)}$ and we note that we can argue as in the proof of Theorem 11 to get

$$\begin{aligned}\left|\int_\Sigma W_\lambda \alpha \,\mathrm{d}\mu\right| \le C R_S^{-6} \int_\Sigma |\alpha| \,\mathrm{d}\mu + \frac{3}{2}\left|\int_\Sigma \alpha H^{2M}\mathrm{Rc}(\nu,\nu)\,\mathrm{d}\mu\right| \\ + C\left|\int_\Sigma \alpha \Delta(|A|^2 + {}^M\mathrm{Rc}(\nu,\nu))\,\mathrm{d}\mu\right|.\end{aligned}$$

Using Theorem 8 we get

$$\frac{3}{2}\left|\int_\Sigma \alpha H^{2M}\mathrm{Rc}(\nu,\nu)\,\mathrm{d}\mu\right| \le 12 m R_S^{-5}\int_\Sigma |\alpha|\,\mathrm{d}\mu + C R_S^{-6}\int_\Sigma |\alpha|\,\mathrm{d}\mu.$$

Moreover, using integration by parts, Theorem 8, Lemma 23 and (110) we estimate

$$\begin{aligned}\left|\int_\Sigma \alpha \Delta(|A|^2 + {}^M\mathrm{Rc}(\nu,\nu))\,\mathrm{d}\mu\right| &\le C\sqrt{\eta} R_S^{-4}\|\nabla\alpha\|_{L^2(\Sigma)} \\ &\le C\sqrt{\eta}\left(R_S^{-5}\|\alpha\|_{L^2(\Sigma)} + R_S^{-5/2}\|\alpha\|^{1/2}_{L^2(\Sigma)}\right).\end{aligned}$$

Putting these estimates together we conclude

$$\left|\int_\Sigma W_\lambda \alpha\,\mathrm{d}\mu\right| \le C_1 R_S^{-4}\|\alpha\|_{L^2(\Sigma)} + C(R_S^{-5}\|\alpha\|_{L^2(\Sigma)} + R_S^{-5/2}\|\alpha\|^{1/2}_{L^2(\Sigma)}). \tag{122}$$

On the other hand we have, using again Theorem 8,

$$\begin{aligned}\left|\int_\Sigma W_\lambda \alpha\,\mathrm{d}\mu\right| &= \left|\int_\Sigma H\,\mathrm{d}\mu\right| \\ &\ge C_2 R_S - C R_S^{-1}.\end{aligned} \tag{123}$$

Combining the two estimates we get

$$C_2 R_S^5 - C R_S^3 \le C_1\|\alpha\|_{L^2(\Sigma)} + C(R_S^{-1}\|\alpha\|_{L^2(\Sigma)} + R_S^{3/2}\|\alpha\|^{1/2}_{L^2(\Sigma)}).$$

From this estimate we easily see that there exists a constant $C_3 > 0$ such that for R_S large enough we have

$$\|\alpha\|_{L^2(\Sigma)} \geq C_3 R_S^5. \tag{124}$$

Using Hölder's inequality we get

$$\int_\Sigma (\alpha - \alpha_0) H \, \mathrm{d}\mu \leq \|H - H_0\|_{L^2(\Sigma)} \|\alpha - \alpha_0\|_{L^2(\Sigma)}$$

and hence, combing this with (124), we see that (121) will be a consequence of the estimate

$$\|H - H_0\|_{L^2(\Sigma)} \leq C\eta^{1/4} R_S^{-1}. \tag{125}$$

We note that

$$LH = \left(\lambda - \frac{1}{2}H^2\right) H = \mu_0 H + \left(\lambda - \frac{1}{2}H^2 - \mu_0\right) H$$

and therefore we can estimate

$$\mu_0 \leq \frac{\int_\Sigma HLH \, \mathrm{d}\mu}{\int_\Sigma H^2 \, \mathrm{d}\mu} \leq \mu_0 + C\sqrt{\eta} r_{\min}^{-4},$$

where the first inequality follows from the Rayleigh quotient characterization of the eigenvalues of L and the second inequality follows from the above estimate and Lemma 20. Next we decompose $H = \sum_i \langle H, \varphi_i \rangle \varphi_i$ and we calculate

$$\begin{aligned} \frac{\int_\Sigma HLH \, \mathrm{d}\mu}{\int_\Sigma H^2 \, \mathrm{d}\mu} &= \frac{\sum_i \mu_i \int_\Sigma H_i^2 \, \mathrm{d}\mu}{\int_\Sigma H^2 \, \mathrm{d}\mu} \\ &= \mu_0 + \frac{\sum_i (\mu_i - \mu_0) \int_\Sigma H_i^2 \, \mathrm{d}\mu}{\int_\Sigma H^2 \, \mathrm{d}\mu}. \end{aligned}$$

Hence we get

$$0 \leq \sum_{i=1}^\infty (\mu_i - \mu_0) \int_\Sigma H_i^2 \, \mathrm{d}\mu \leq C\sqrt{\eta} r_{\min}^{-4}.$$

For every $i \in \mathbb{N}$ we have $(\mu_i - \mu_0) \geq 2R_S^{-2}$ (see Corollary 5) and therefore

$$\|H - H_0\|_{L^2(\Sigma)}^2 \leq C R_S^2 \sum_{i=1}^\infty (\mu_i - \mu_0) \int_\Sigma H_i^2 \, \mathrm{d}\mu \leq C\sqrt{\eta} r_{\min}^{-2},$$

which finishes the proof of (125) and therewith also the proof of the theorem. □

Appendix A: Maple scripts for the calculations

For the explicit calculations in the proof of Proposition 3, in Sect. 6.3 and in Sect. 6.5 we used Maple [12] to evaluate certain integrals. Here we present the scripts we used.

A.1: Proposition 3

Here it is necessary to evaluate the integral

$$E_1 := \int_S \left(\frac{1}{r^3} - 3R_e^2 \frac{1}{r^5} - 6R_e|a_e| \frac{\cos\varphi}{r^5} - 3|a_e|^2 \frac{\cos^2\varphi}{r^5} \right) \mathrm{d}\mu^e \tag{126}$$

where $S = S_{R_e}(a_e)$ is a fixed sphere with center a and radius R_e. The calculation is based on the formula

$$C_k^l := \int_S \frac{\cos^l \varphi}{r^k} \,\mathrm{d}\mu^e = \frac{2\pi R_e}{|a_e|} (2R_e|a_e|)^{-l} \int_{|R_e - |a_e||}^{R_e + |a_e|} r^{1-k} (r^2 - R_e^2 - |a_e|^2)^l \, dr.$$

which was derived in the proof of Proposition 3. Hence Eq. 126 can be written as

$$E_1 = C_3^0 - 3R_e^2 C_5^0 - 6R_e|a_e| C_5^1 - 3|a_e|^2 C_5^2.$$

This is evaluated using the following Maple script.

```
assume (R>0, a>0, R>a);
c0r3 := 2*PI*R/a *(2*R*a)^(0)
        * int(r^(-2)*(r^2 - R^2 - a^2 )^(0),r=R-a..R+a);
c0r5 := 2*PI*R/a *(2*R*a)^(0)
        * int(r^(-4)*(r^2 - R^2 - a^2)^(0),r=R-a..R+a);
c1r5 := 2*PI*R/a *(2*R*a)^(-1)
        * int(r^(-4)*(r^2 - R^2 - a^2 )^(1), r=R-a..R+a);
c2r5 := 2*PI*R/a *(2*R*a)^(-2)
        * int(r^(-4)*(r^2 - R^2 - a^2 )^(2), r=R-a..R+a);
E1   := c0r3 -3*R^2*c0r5 - 6*R*a*c1r5 - 3*a^2*c2r5;
simplify(%);
```

where we used `R` to denote R_e, `a` to denote $|a_e|$ and `clrk` to denote C_k^l.

A.2: Section 6.3

In Sect. 6.3 the integral to evaluate was

$$E_2 := \int_S \frac{\cos\phi}{r^3} = C_3^1$$

This is evaluated by the script

```
assume (R>0, a>0, R>a);
c1r3 := 2*PI*R/a *(2*R*a)^(-1)
         * int(r^(-2)*(r^2 - R^2 - a^2 )^(1),r=R-a..R+a);
E2   := c1r3;
simplify(%);
```

A.3: Section 6.5

The longest calculation is for the term

$$\bar{Q} := \int_S \left(R_e \frac{\cos\varphi}{r^6} + |a_e| \frac{\cos^2\varphi}{r^6} - |a_e| R_e^2 \frac{1}{r^8} - (R_e^3 + 2|a_e|^2 R_e) \frac{\cos\varphi}{r^8} \right.$$
$$\left. - \ (|a_e|^3 + 2|a_e| R_e^2) \frac{\cos^2\varphi}{r^8} - |a_e|^2 R_e \frac{\cos^3\varphi}{r^8} \right) \mathrm{d}\mu^e$$

from Sect. 6.5, where we omit certain fixed factors here. The following script evaluates this expression.

```
assume (R>0, a>0, R>a);
c1r6:=2*PI*R/a *(2*R*a)^(-1)
       * int(r^(-5)*(r^2 - R^2 - a^2)^(1),r=R-a..R+a);
c2r6:=2*PI*R/a *(2*R*a)^(-2)
       * int(r^(-5)*(r^2 - R^2 - a^2 )^(2),r=R-a..R+a);
c0r8:=2*PI*R/a *(2*R*a)^(0)
       * int(r^(-7)*(r^2 - R^2 - a^2 )^(0),r=R-a..R+a);
c1r8:=2*PI*R/a *(2*R*a)^(-1)
       * int(r^(-7)*(r^2 - R^2 - a^2 )^(1),r=R-a..R+a);
c2r8:=2*PI*R/a *(2*R*a)^(-2)
       * int(r^(-7)*(r^2 - R^2 - a^2 )^(2),r=R-a..R+a);
c3r8:=2*PI*R/a *(2*R*a)^(-3)
       * int(r^(-7)*(r^2 - R^2 - a^2 )^(3),r=R-a..R+a);
Q   := R * c1r6 + a * c2r6 - a*R^2*c0r8 - (R^3 + 2*a^2*R)*c1r8
        - (2*a*R^2 + a^3) *c2r8 - a^2 * R * c3r8);
subs(a = tau * R, Q);
simplify(%);
```

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