

Yangian symmetry of long-range $\mathfrak{gl}(N)$ integrable spin chains

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Abstract. An interesting type of spin chain has appeared in the context of the planar AdS/CFT correspondence: it is based on an integrable nearest-neighbor spin chain, and it is perturbatively deformed by long-range interactions which apparently preserve the integrable structure. Similar models can be constructed by demanding the existence of merely one conserved local charge. Although the latter is not a sufficient integrability condition in general, the models often display convincing signs of full integrability.

Here we consider a class of long-range spin chains with spins transforming in the fundamental representation of $\mathfrak{gl}(N)$. For the most general such model with one conserved local charge we construct a conserved Yangian generator and show that it obeys the Serre relations. We thus provide a formal proof of integrability for this class of models.

Keywords: algebraic structures of integrable models, integrable spin chains (vertex models), quantum integrability (Bethe ansatz), symmetries of integrable models

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1. Introduction

Among integrable spin chains, the models with pairwise interactions of spins at adjacent sites are understood best. Their integrable structure is based on an R -matrix from which many interesting quantities and their relations can be derived. For instance, the R -matrix gives rise to commuting transfer matrices from which a Hamiltonian and a set of commuting conserved local charges follows. The existence of the latter is one way to define the integrability property of a Hamiltonian.

Let us consider the simplest type of model with manifest Lie algebra symmetry. The R -matrix of such models typically is rational and it is well known that it possesses an additional hidden symmetry: the Yangian. The Yangian is a Hopf algebra which extends the Lie symmetry to an infinite-dimensional algebra [1] (see [2] for reviews and further references). Formally, the Yangian is a deformation of the universal enveloping algebra

for the (half) loop algebra of the Lie symmetry. The Lie generators, \mathcal{J} , reside at level-0 of the loop algebra and we shall call the level-1 generators, \mathcal{Y} , the Yangian generators. Repeated commutators of the latter supply us with all the higher levels of the loop algebra and therefore there is no need to consider them explicitly. Merely, the Yangian algebra imposes a constraint on triple products of generators: the Serre relations.

Just as the Lie generators, the Yangian generators have a representation on the spin chain. Whereas the Lie generators act locally along the chain,

$$\mathcal{J}^A = \sum_k \mathcal{J}_k^A, \quad (1.1)$$

the Yangian generators act as bi-local products of Lie generator insertions,

$$\mathcal{Y}^A = \sum_{k < l} \frac{1}{2} F_{BC}^A \mathcal{J}_k^B \mathcal{J}_l^C. \quad (1.2)$$

Here \mathcal{J}_k^A is the Lie generator \mathcal{J}^A acting on site k of the chain and F_{BC}^A are the structure constants of the Lie algebra. The Yangian typically is not an exact symmetry of the Hamiltonian; it merely commutes up to contributions at the ends of the spin chain. Alternatively one can say that the boundary conditions normally break Yangian symmetry. Nevertheless, for an infinite spin chain the boundary contributions are dropped and the Yangian also becomes an exact symmetry of the Hamiltonian. Yangian symmetry (up to boundary terms) is thus another way to define the integrability of a Hamiltonian.

The integrable structures for spin chains with long-range interactions are less well understood. Among these, the best known class of models is the Haldane–Shastry chains [3]. The Hamiltonian of these models consists of interactions between two spins at a distance, and it is actually exactly invariant under a Yangian algebra. Less clear are the Inozemtsev spin chain models, which are a generalization of the Haldane–Shastry models, but whose Hamiltonian commutes with the Yangian only up to boundary contributions.

A much wider class of models [4]–[7] was found in the context of integrability in the so-called AdS/CFT correspondence (string/gauge duality) [8], cf [4, 9, 10] and [11, 12] for reviews, and in the plane-wave matrix model [13], cf [14, 15]. Here the Hamiltonian consists of interactions of arbitrarily many neighboring spins. An ordering principle for the interactions is provided by a global coupling constant λ , which is assumed to be small. The Hamiltonian is given as a series expansion in this coupling:

$$\mathcal{H}(\lambda) = \sum_{\ell=0}^{\infty} \lambda^\ell \mathcal{H}^{(\ell)}. \quad (1.3)$$

The leading-order Hamiltonian $\mathcal{H}^{(0)}$ is of nearest-neighbor type and the range of interactions increases by one site per order in λ : $\mathcal{H}^{(\ell)}$ consists of interactions among $\ell + 2$ sites. These models were called *long-range* spin chains because for finite values of the coupling the range of interactions would be infinite. On the other hand, one might also call them *short-range* models because up to a given perturbative order (as much as can be constructed explicitly in practice) the interactions are local and the range is bounded. Note that the Inozemtsev models [16] are effectively within this class [17], but for our models the interactions are more general and can be between more than two spins at the same time. Also the Hubbard model can effectively be described by a short-range spin chain in a certain limit [18].

The investigation of short-range integrable spin chains was initiated in [4] and continued in [5, 19, 6]. An obvious complication for this class of models is that there exists no general framework to construct and to prove the integrable structures. Unlike the case of nearest-neighbor models one cannot rely on an R -matrix because it could generate only interactions between two spins at a time and not between arbitrarily many. Instead one had to explicitly construct some commuting local charges and rely on good faith that infinitely many further charges exist. Intriguingly it turned out that in all cases considered the existence of merely one commuting local charge was sufficient to ensure the existence of as many further charges as could be constructed in practice.

The next immediate challenge was to determine the spectrum of short-range spin chains. Based on the special case of Inozemtsev models [16, 17], a proposal for the Bethe equations for a special short-range model was made in [20]. Furthermore, certain deformations of the Bethe equations [21] could be identified with certain deformations of the Hamiltonian [5]. These deformations are important for an accurate description of the AdS/CFT integrable model [7, 22] and a sector of the plane-wave matrix model [15]. Finally, a thorough derivation of Bethe equations for short-range models based on the asymptotic coordinate Bethe ansatz [19] was proposed in [6]. These Bethe equations have the curious feature that they can be written down for arbitrary finite values of the coupling λ while the Hamiltonian was only known at some perturbative order. One should note though that the Bethe equations are merely asymptotically correct and not exactly valid as for nearest-neighbor models. This means that the Bethe equations predict the spectrum of a chain of length L correctly only at the perturbative order $\ell = L - 2$. The currently most advanced study for short-range models with $\mathfrak{gl}(N)$ symmetry and spins in the fundamental representation [6] constructed the most general third-order Hamiltonian and made a proposal for the complete moduli space at arbitrary order and the corresponding asymptotic Bethe equations.

Nevertheless, the class of short-range integrable spin chains is still in its infancy. For example, we still have no proof of integrability beyond next-to-leading order; we merely have indications due to the construction of one or more conserved charges. Yangian symmetry would be one way to produce a formal proof of integrability at a given perturbative order. It is conceivable that short-range Hamiltonians display such a symmetry (up to boundary terms). Indeed there are indications that this is true in some cases [23, 17, 24, 25]. However, many of the models considered in these works are equivalent to Inozemtsev models at the given accuracy, and therefore we cannot tell if the most general deformation investigated in [6] still has Yangian symmetry. In fact, the most general models have scattering matrices [19, 26] which are not functions of the difference of some rapidity variables. This may or may not be an obstacle for Yangian symmetry which commonly leads to scattering matrices of difference form.

Consequently, we would like to see if general short-range models do display Yangian symmetry. This might also teach us more about the algebraic structures underlying short-range integrability. In this work we shall focus on the models with $\mathfrak{gl}(N)$ symmetry and spins in the fundamental representation.

We start in section 2 with a review of short-range spin chains; more specifically, we review the most general perturbative deformation with $\mathfrak{gl}(N)$ symmetry studied in [6] along with the asymptotic Bethe equations for such a spin chain. In section 3 we review Yangian symmetry of spin chains, which is used later to prove integrability. The Yangian

for the Inozemtsev chain is discussed to motivate the deformations of the Yangian in our spin chain. In section 4 we discuss the construction of the Yangian. With all the machinery ready we proceed to the calculation. We compare three different methods of constructing the system; these methods differ only by the order in which the constraints are applied. We then discuss the implications our findings. In appendix A we present some of the simplifications used in carrying out the calculation and in appendix B we explicitly perform the procedure to $\mathcal{O}(\lambda)$ for concreteness.

2. Short-range spin chains

Here we introduce our perturbatively integrable short-range spin chain models with $\mathfrak{gl}(N)$ symmetry introduced in [4]. We merely review the setup and some results; further details of the calculation can be found in the literature [4, 6].

2.1. Setup

We consider a spin chain with spins transforming in the fundamental representation of $\mathfrak{gl}(N)$. Thus a spin chain state of length L is an element of $(\mathbb{C}^N)^{\otimes L}$. A basis for such states is given by

$$|a_1, \dots, a_L\rangle \quad (2.1)$$

with $a_k = 1, \dots, N$.

We write the Hamiltonian as a power series in the coupling constant λ as in (1.3). At leading order the Hamiltonian $\mathcal{H}^{(0)}$ is a nearest-neighbor interaction as for conventional spin chain models. For each perturbative order the range of the interactions may grow by one site. Thus the order λ^k correction $\mathcal{H}^{(k)}$ consists of interactions of range $k + 2$.

This class of models is quite general. As a special case it includes the Inozemtsev hyperbolic chain [16] when the parameter is expanded around the point where the Hamiltonian becomes a nearest-neighbor interaction [17]. Furthermore, the short-range spin chain which arises in the $\mathfrak{su}(2)$ sector of planar $\mathcal{N} = 4$ supersymmetric Yang–Mills theory [4, 6] is of this kind.

The higher local conserved charges of the integrable model are denoted by \mathcal{Q}_r . They have a similar perturbation series as the Hamiltonian

$$\mathcal{Q}_r = \sum_{k=0}^{\infty} \lambda^k \mathcal{Q}_r^{(k)}, \quad (2.2)$$

but the range is increased by $r - 2$ units. In other words, $\mathcal{Q}_r^{(k)}$ consists of interactions of range $k + r$. For an integrable system all the charges have to commute with each other:

$$[\mathcal{Q}_r, \mathcal{Q}_s] = 0. \quad (2.3)$$

The Hamiltonian is defined to be the lowest charge $\mathcal{H} := \mathcal{Q}_2$. In this paper we will only consider the next higher conserved charge and denote it by $\mathcal{Q} := \mathcal{Q}_3$ for simplicity. Consequently, the defining relation for the integrable system is $[\mathcal{H}, \mathcal{Q}] = 0$.

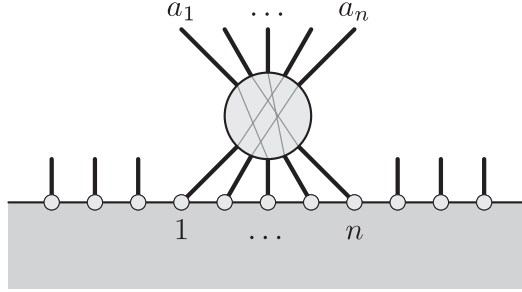


Figure 1. Graphical representation of a local invariant interaction of range n .

The $\mathfrak{gl}(N)$ generators will be denoted by \mathcal{J}_b^a . They act on a spin chain as follows:

$$\mathcal{J}_b^a |c_1, \dots, c_L\rangle = \sum_{k=1}^L \delta_b^{c_k} |c_1, \dots, c_{k-1}, a, c_{k+1}, \dots, c_L\rangle. \quad (2.4)$$

The Hamiltonian and the charges are assumed to be invariant under $\mathfrak{gl}(N)$:

$$[\mathcal{J}_b^a, \mathcal{H}] = [\mathcal{J}_b^a, \mathcal{Q}_r] = 0. \quad (2.5)$$

2.2. Notation

We use the symbol $[a_1, a_2, \dots, a_n]$ to describe a local $\mathfrak{gl}(N)$ -invariant interaction of range n . The numbers a_1, a_2, \dots, a_n are distinct integers between 1 and n which describe the permutation π with $\pi(k) = a_k$. The interaction sums over all sets of n adjacent sites of the spin chain and permutes them according to π :

$$[a_1, \dots, a_n] |b_1, \dots, b_L\rangle = \sum_k |b_1, \dots, b_k, b_{k+a_1}, \dots, b_{k+a_n}, b_{k+n+1}, \dots, b_L\rangle. \quad (2.6)$$

For example, $[1, 2, 3, \dots, n]$ (for any n) acts identically on the state and essentially counts the number of n adjacent sites within the chain: this number equals the length L on a periodic chain. The symbol $[2, 1]$ sums over nearest-neighboring spins and permutes them. This is the essential contribution of the Hamiltonian of a $\mathfrak{gl}(N)$ nearest-neighbor spin chain. A general local operator can be visualized as in figure 1.

Note that there is an implicit sum in this notation since the position of the first leg is summed over all sites of the spin chain. With this implicit sum in mind we see that many of the operators are equivalent:

$$[a_1, a_2, \dots, a_n, n + 1] = [a_1, a_2, \dots, a_n] = [1, a_1 + 1, a_2 + 1, \dots, a_n + 1], \quad (2.7)$$

since the last or first leg, respectively, acts trivially.

The interactions $[a_1, \dots, a_n]$ are all invariant under $\mathfrak{gl}(N)$, and they form a complete basis of invariant interactions of range n . However, due to the identification in (2.7) a minimal basis for range n consists of the interaction of range $m \leq n$ with all leading leg $a_1 \neq 1$ and trailing leg $a_n \neq n$. It can be shown that there are $n! - (n - 1)! + 1$ inequivalent such interactions of range n .

Table 1. Free parameters.

	\pm	λ^0	λ^1	λ^2	λ^3
Ansatz for Hamiltonian \mathcal{H}		2	5	19	97
Ansatz for conserved charge \mathcal{Q}	+	5	19	97	601
Constraint from consistency	-	2	16	102	666
Undetermined coefficients	=	5	8	14	32
$\gamma_{2,s}$ (mixing of Hamiltonian)	-	2	3	4	5
$\gamma_{3,s}$ (mixing of conserved charge)	-	3	4	5	6
Intrinsic parameters α, β, ϵ	=	0	1	5	21

2.3. Construction

The most general integrable Hamiltonians with the above properties were constructed in [6]; let us review the results. At the leading order, the Hamiltonian and the charges will take the standard form: a nearest-neighbor integrable spin chain with $\mathfrak{gl}(N)$ symmetry. At a fixed order in perturbation theory we will make the ansatz that the Hamiltonian $\mathcal{H}^{(k)}$ and the conserved charges $\mathcal{Q}^{(k)}$ are linear combinations of all the local invariant interactions of the aforementioned range. The linear combinations are given by a set of *a priori* undetermined coefficients. We then impose that \mathcal{H} and \mathcal{Q} commute at every order in perturbation theory:

$$\sum_{\ell=0}^k [\mathcal{H}^{(\ell)}, \mathcal{Q}^{(k-\ell)}] = 0. \quad (2.8)$$

At any given order this system of equations is linear in the coefficients of $\mathcal{H}^{(k)}$ and $\mathcal{Q}^{(k)}$. It turns out that, conveniently, it suffices to solve for these coefficients only. The contributions at intermediate orders, $0 < \ell < k$, serve as inhomogeneities for the system. The solution to the system has several free coefficients $\alpha, \beta, \gamma, \epsilon$; their counting is summarized in table 1. These correspond to solutions $\delta\mathcal{H}^{(k)}, \delta\mathcal{Q}^{(k)}$ of the homogeneous system

$$[\delta\mathcal{H}^{(k)}, \mathcal{Q}^{(0)}] + [\mathcal{H}^{(0)}, \delta\mathcal{Q}^{(k)}] = 0. \quad (2.9)$$

The main task addressed in [6] was to understand the origin of the various coefficients. While most coefficients appear in both $\delta\mathcal{H}^{(k)}$ and $\delta\mathcal{Q}^{(k)}$, some coefficients can be attributed to only one of them. These stem from solutions of the stronger equations $[\delta\mathcal{H}^{(k)}, \mathcal{Q}^{(0)}] = 0$ or $[\mathcal{H}^{(0)}, \delta\mathcal{Q}^{(k)}] = 0$. Obviously the solutions are the leading-order conserved charges. In the full perturbative setup they correspond to the mixing of conserved charges

$$\mathcal{H} = \gamma_{2,\mathcal{L}} \mathcal{L} + \sum_{s=2}^{\infty} \gamma_{2,s} \bar{\mathcal{Q}}_s, \quad \mathcal{Q} = \gamma_{3,\mathcal{L}} \mathcal{L} + \sum_{s=3}^{\infty} \gamma_{3,s} \bar{\mathcal{Q}}_s, \quad (2.10)$$

where $\bar{\mathcal{Q}}_s$ are some bare commuting charges and where \mathcal{L} is the length operator $\mathcal{L} = [1]$. Clearly the Hamiltonian \mathcal{H} and the charge \mathcal{Q} obey the same properties as the $\bar{\mathcal{Q}}_s$ provided that the coefficients start at the proper order:

$$\gamma_{r,\mathcal{L}} = \mathcal{O}(\lambda^0), \quad \gamma_{r,s} = \mathcal{O}(\lambda^{\max(0,s-r)}). \quad (2.11)$$

After subtracting these degrees of freedom, the remaining coefficients are strictly shared between \mathcal{H} and \mathcal{Q} . We will call them intrinsic coefficients of the integrable system and discuss them in the next subsection.

The final results for the bare charges $\bar{\mathcal{Q}}_2, \bar{\mathcal{Q}}_3$ were already obtained and presented in earlier work [6], however in a different notation. We present them in our notation up to second order in λ at the end of the paper in tables 6 and 8. For the convenience of the unexperienced reader we perform this process explicitly in appendix B up to $\mathcal{O}(\lambda)$. The higher-order calculation is done in precisely the same way but with significantly more terms.

2.4. Bethe equations

The intrinsic parameters $\alpha_k, \beta_{r,s}, \epsilon_{k,l}$ in the operator tables 6 and 8 can be understood best by considering the Bethe equations. The asymptotic Bethe equations (taking the assumption of integrability for granted) were derived in [6]. They read

$$\begin{aligned}
 1 &= \left(\frac{x(u_{1,k} - i/2)}{x(u_{1,k} + i/2)} \right)^L \prod_{\substack{j=1 \\ j \neq k}}^{K_1} \left(\frac{u_{1,k} - u_{1,j} + i}{u_{1,k} - u_{1,j} - i} \exp(2i\theta(u_{1,k}, u_{1,j})) \right) \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} - i/2}{u_{1,k} - u_{2,j} + i/2} \\
 &\quad \dots \\
 1 &= \prod_{j=1}^{K_{\ell-1}} \frac{u_{\ell,k} - u_{\ell-1,j} - i/2}{u_{\ell,k} - u_{\ell-1,j} + i/2} \prod_{\substack{j=1 \\ j \neq k}}^{K_{\ell}} \frac{u_{\ell,k} - u_{\ell,j} + i}{u_{\ell,k} - u_{\ell,j} - i} \prod_{j=1}^{K_{\ell+1}} \frac{u_{\ell,k} - u_{\ell+1,j} - i/2}{u_{\ell,k} - u_{\ell+1,j} + i/2} \\
 &\quad \dots \\
 1 &= \prod_{j=1}^{K_{n-2}} \frac{u_{n-1,k} - u_{n-2,j} - i/2}{u_{n-1,k} - u_{n-2,j} + i/2} \prod_{\substack{j=1 \\ j \neq k}}^{K_{n-1}} \frac{u_{n-1,k} - u_{n-1,j} + i}{u_{n-1,k} - u_{n-1,j} - i}.
 \end{aligned} \tag{2.12}$$

We now explain the rapidity map $x(u)$ and the dressing phase $\theta(u, v)$.

The *rapidity map* $x(u)$ is defined implicitly through its inverse,

$$u(x) = x + \sum_{\ell=0}^{\infty} \frac{\alpha_{\ell}(\lambda)}{x^{\ell+1}}. \tag{2.13}$$

The inverse map from the u -plane to the x -plane has the following form:

$$x(u) = \frac{u}{2} + \frac{u}{2} \sqrt{1 - 4 \sum_{\ell=0}^{\infty} \frac{\tilde{\alpha}_{\ell}(\lambda)}{u^{\ell+2}}}. \tag{2.14}$$

The parameters $\tilde{\alpha}_{\ell}(\lambda)$ are fixed uniquely by the components of $\alpha_k(\lambda)$. The coefficients $\alpha_{\ell}(\lambda)$ govern the propagation of spin flips in the ferromagnetic vacuum (magnons).

Next we present the *dressing phase*:

$$\theta(u, v) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} \beta_{r,s}(\lambda) (q_r(u) q_s(v) - q_s(u) q_r(v)). \tag{2.15}$$

These functions govern the scattering of two magnons. For (2.15) we also define the *elementary magnon charges* as

$$q_r(u) = \frac{1}{r-1} \left(\frac{i}{x(u+i/2)^{r-1}} - \frac{i}{x(u-i/2)^{r-1}} \right). \quad (2.16)$$

The solutions to the above asymptotic Bethe equations define the set of periodic eigenstates of the system. Finally, we have to specify the eigenvalues of the Hamiltonian and higher charges in terms of the rapidities $u_{\ell,k}$. The eigenvalue of the shift operator $\exp(i\mathcal{P})$ is given by

$$\exp(iP) = \prod_{k=1}^{K_1} \frac{x(u_{1,k} + i/2)}{x(u_{1,k} - i/2)}. \quad (2.17)$$

In particular, cyclic states obey the zero-momentum condition $\exp(iP) = 1$. The eigenvalues of the bare spin chain charges are determined by the formula

$$\bar{Q}_r = \sum_{k=1}^{K_1} q_r(u_{1,k}). \quad (2.18)$$

In the perturbative scheme the functions $\alpha_\ell(\lambda)$ and $\beta_{r,s}(\lambda)$ are series in λ starting at the orders

$$\alpha_\ell(\lambda) = \mathcal{O}(\lambda^{\ell+1}), \quad \beta_{r,s}(\lambda) = \mathcal{O}(\lambda^{s-1}). \quad (2.19)$$

For the coefficients $\tilde{\alpha}_k(\lambda)$ this implies that $\tilde{\alpha}_0(\lambda) = \mathcal{O}(\lambda)$ and $\tilde{\alpha}_{\ell \geq 1}(\lambda) = \mathcal{O}(\lambda^{[\ell/2]+2})$.

The coefficients $\epsilon_{k,l}$ do not appear in the Bethe equations and therefore they can have no impact on the spectrum. Instead they correspond to *perturbative similarity transformations* of all the operators:

$$\bar{Q}_r = \mathcal{T} \tilde{Q}_r \mathcal{T}^{-1}, \quad \mathcal{T} = 1 + \sum_{k=1}^{\infty} \lambda^k \mathcal{T}^{(k)}, \quad (2.20)$$

where $\mathcal{T}^{(k)}$ is an arbitrary interaction of range $k+1$ parameterized by $\epsilon_{k,l}$. Note that contributions to \mathcal{T} which are linear combinations of the commuting charges do not alter the charges. Thus for counting purposes one has to remove these *trivial similarity transformations*. The total counting of intrinsic coefficients is collected in table 2. The table shows that up to $\mathcal{O}(\lambda^3)$ all the intrinsic parameters can be accounted for and we have thus fully understood the moduli space of short-range integrable spin chain models with $\mathfrak{gl}(N)$ symmetry.

In fact we merely made the integrability of the model plausible, but have not explicitly proved it. The two known signs of integrability for this model are: one conserved charge exists, and the construction of further charges (as far as it has been tested) has in no case led to additional constraints on the Hamiltonian. The spectrum derived from the Bethe equations (as far as it has been tested) agrees precisely with the diagonalization of the Hamiltonian; see [6]. In the next section we aim for a proof of perturbative integrability by means of Yangian symmetry.

Table 2. Intrinsic parameters of the short-range chain.

	\pm	λ^0	λ^1	λ^2	λ^3
Intrinsic parameters		0	1	5	21
α_k (rapidity map)	–	0	1	2	3
$\beta_{r,s}$ (dressing factor)	–	0	0	1	3
$\epsilon_{k,l}$ (similarity transformations)	=	0	0	2	15
Trivial similarity transformations	+	1	2	3	4
All similarity transformations	=	1	2	5	19

3. Yangian symmetry

For integrability we need one conserved quantity per degree of freedom, which seems hard to show explicitly for the model introduced in the previous section. Instead we can show that the spin chain Hamiltonian commutes with a Yangian algebra (up to boundary terms). The algebra implies the existence of an infinite tower of conserved generators.

3.1. Yangian algebra

We will deal with only the lowest two levels of this algebra: the lowest level contains the Lie symmetry generators \mathcal{J} , in our case those of $\mathfrak{g} = \mathfrak{gl}(N)$. We shall call generators at the next level the Yangian generators and we shall denote them by \mathcal{Y} . All the generators at higher levels can be constructed iteratively from commutators of \mathcal{J}, \mathcal{Y} provided that the generators obey the Serre relations of the Yangian. Since both \mathcal{J}, \mathcal{Y} commute with \mathcal{H} , the higher generators will also commute and we have an infinite tower of conserved (non-local) charges. See [1] for a derivation of the Serre relations and the Yangian algebra. In order to prove integrability of the spin chain Hamiltonian it thus suffices to show:

- Conservation of a Lie-type symmetry \mathfrak{g} , i.e. $[\mathcal{H}, \mathcal{J}] = 0$.
- Conservation of a \mathfrak{g} -adjoint generator \mathcal{Y} , i.e. $[\mathcal{H}, \mathcal{Y}] = 0$.
- The two sets of generators \mathcal{J}, \mathcal{Y} must obey the Serre relations.

The generators \mathcal{J}, \mathcal{Y} can be written in several equivalent bases. The three most useful ones are the adjoint, bi-fundamental and auxiliary site bases. We present and compare them in what follows.

3.2. Adjoint basis

In the adjoint basis the Lie and Yangian generators are denoted by \mathcal{J}^A and \mathcal{Y}^A , respectively, where $A = 1, \dots, N^2$ for $\mathfrak{g} = \mathfrak{gl}(N)$. The indices A transform in the adjoint representation of $\mathfrak{gl}(N)$:

$$\begin{aligned} [\mathcal{J}^A, \mathcal{J}^B] &= F_C^{AB} \mathcal{J}^C, \\ [\mathcal{J}^A, \mathcal{Y}^B] &= F_C^{AB} \mathcal{Y}^C. \end{aligned} \quad (3.1)$$

The Serre relations are given by

$$[\mathcal{Y}^A, [\mathcal{J}^B, \mathcal{Y}^C]] + [\mathcal{Y}^B, [\mathcal{J}^C, \mathcal{Y}^A]] + [\mathcal{Y}^C, [\mathcal{J}^A, \mathcal{Y}^B]] = \frac{1}{6} A_{DEF}^{ABC} \{\mathcal{J}^D, \mathcal{J}^E, \mathcal{J}^F\}, \quad (3.2)$$

where $\{\dots\}$ is the totally symmetric product (with unit weight) and

$$A_{DEF}^{ABC} = \frac{1}{4} F_D^{AG} F_E^{BH} F_F^{CJ} F_{GHJ}. \quad (3.3)$$

Here indices of the structure constant are raised and lowered by the (inverse) Cartan metric C_{AB}, C^{AB} . For the case of $\mathfrak{g} = \mathfrak{gl}(2)$ the above cubic Serre relations are trivially satisfied and have to be supplemented by the quartic Serre relations

$$[[\mathcal{Y}^A, \mathcal{Y}^B], [\mathcal{J}^C, \mathcal{Y}^D]] + [[\mathcal{Y}^C, \mathcal{Y}^D], [\mathcal{J}^A, \mathcal{Y}^B]] = (A_{EFG}^{ABH} F_H^{CD} + A_{EFG}^{CDH} F_H^{AB}) \{\mathcal{J}^E, \mathcal{J}^F, \mathcal{Y}^G\}. \quad (3.4)$$

In all other cases the quartic relations follow from the cubic relations.

We can use the latter property to unify the computation for all $\mathfrak{gl}(N)$ with $N = 2$ and $N > 2$: in our case the proof of the cubic relation will make no reference to N whatsoever, explicitly or implicitly. Neither would a similar direct proof of the quartic relation make reference to N . Therefore, if the quartic relation holds for any one value of N , it will also hold for $N = 2$. Now because the above property tells us that the quartic relation follows from the cubic one for any $N > 2$, it is sufficient to prove the cubic relation (for all N or for any $N > 2$).

3.3. Bi-fundamental basis

In the bi-fundamental basis for $\mathfrak{g} = \mathfrak{gl}(N)$ the generators are denoted by $\mathcal{J}_b^a, \mathcal{Y}_b^a$, where $a, b = 1, \dots, N$. The adjoint transformation rule reads here

$$\begin{aligned} [\mathcal{J}_b^a, \mathcal{J}_d^c] &= \delta_b^c \mathcal{J}_d^a - \delta_d^a \mathcal{J}_b^c, \\ [\mathcal{J}_b^a, \mathcal{Y}_d^c] &= \delta_b^c \mathcal{Y}_d^a - \delta_d^a \mathcal{Y}_b^c. \end{aligned} \quad (3.5)$$

The commutator of two Yangian generators can be written explicitly; it yields the generator \mathcal{Y}' at the next level of the Yangian algebra plus a cubic combination of Lie generators:

$$[\mathcal{Y}_b^a, \mathcal{Y}_d^c] = \delta_b^c (\mathcal{Y}'_d)^a - \delta_d^a (\mathcal{Y}'_b)^c + \mathcal{J}_d^a \mathcal{J}_e^c \mathcal{J}_b^e - \mathcal{J}_e^a \mathcal{J}_d^e \mathcal{J}_b^c. \quad (3.6)$$

Likewise the higher-level generators (along with polynomials in the lower generators) can be obtained from repeated commutators of \mathcal{Y} . In the cubic Serre relations the \mathcal{Y}' terms drop out:

$$\begin{aligned} [\mathcal{J}_b^a, [\mathcal{Y}_d^c, \mathcal{Y}_f^e]] - [\mathcal{Y}_b^a, [\mathcal{J}_d^c, \mathcal{Y}_f^e]] &= [\mathcal{J}_b^a, \mathcal{J}_f^c \mathcal{J}_g^e \mathcal{J}_d^g - \mathcal{J}_g^c \mathcal{J}_f^g \mathcal{J}_d^e] \\ &\quad - \delta_d^e (\mathcal{J}_f^a \mathcal{J}_g^c \mathcal{J}_b^g - \mathcal{J}_g^a \mathcal{J}_f^g \mathcal{J}_b^c) \\ &\quad + \delta_f^c (\mathcal{J}_d^a \mathcal{J}_g^e \mathcal{J}_b^g - \mathcal{J}_g^a \mathcal{J}_d^g \mathcal{J}_b^e). \end{aligned} \quad (3.7)$$

The quartic Serre relation takes a similar form which we will not need explicitly here.

To convert between adjoint and bi-fundamental basis one employs some tensors $(t_A)_c^b$ and $(t_c^b)^A$ such that

$$\mathcal{J}_c^b = (t_A)_c^b \mathcal{J}^A, \quad \mathcal{J}^A = (t^A)_b^c \mathcal{J}_c^b, \quad (3.8)$$

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and similarly for \mathcal{Y} . The two tensors are inversely related by $(t_A)_c^b(t^B)_b^c = \delta_A^B$ and $(t^A)_c^b(t_A)_e^d = \delta_e^b\delta_c^d$.

3.4. Auxiliary site basis

The space of $N \times N$ matrices equals the space of $\mathfrak{gl}(N)$ generators which equals the space of linear operators acting on an N -dimensional vector space. Therefore we can use the latter as a basis for $\mathfrak{gl}(N)$ -adjoint generators. More explicitly, we define

$$\mathcal{J}_x = (\mathcal{J}_x)^A C_{AB} \mathcal{J}^B = (\mathcal{J}_x)_a^b \mathcal{J}_b^a. \tag{3.9}$$

Here $(\mathcal{J}_x)^A$ and $(\mathcal{J}_x)_a^b$ are $\mathfrak{gl}(N)$ generators acting on an auxiliary fundamental site labeled x . Consequently \mathcal{J}_x is a linear operator that acts on the site x as well as on the spin chain (via \mathcal{J}^B or \mathcal{J}_b^a). The adjoint transformation rules read

$$\begin{aligned} [\mathcal{J}_x, \mathcal{J}_y] &= [\mathcal{P}_{xy}, \mathcal{J}_x], \\ [\mathcal{J}_x, \mathcal{Y}_y] &= [\mathcal{P}_{xy}, \mathcal{Y}_x], \end{aligned} \tag{3.10}$$

where \mathcal{P}_{xy} is the permutation operator. The cubic Serre relations are given by

$$[\mathcal{J}_x, [\mathcal{Y}_y, \mathcal{Y}_z]] - [\mathcal{Y}_x, [\mathcal{J}_y, \mathcal{Y}_z]] = [\mathcal{J}_x, \mathcal{J}_z[\mathcal{P}_{yz}, \mathcal{J}_z]\mathcal{J}_z] + [\mathcal{P}_{yz}, \mathcal{J}_z[\mathcal{P}_{xz}, \mathcal{J}_z]\mathcal{J}_z]. \tag{3.11}$$

This basis has the advantage that one does not have to deal with explicit indices.

3.5. Short-range chains

In order to understand the structure of the Yangian for short-range chains, let us consider a specific known example: the Inozemtsev chain [16, 17]. There, the generator \mathcal{Y}^A has the following simple structure:

$$\mathcal{Y}^A = \sum_{k < \ell} \frac{1}{2} c_{\ell-k}(\lambda) F_{BC}^A \mathcal{J}_k^B \mathcal{J}_\ell^C. \tag{3.12}$$

Here $c_k(\lambda)$ are some coefficients whose precise expression will not be relevant. The coupling constant λ is the coupling constant of the Inozemtsev model, which is chosen such that at $\lambda = 0$ the model reduces to the Heisenberg chain. For the Yangian generator this implies $c_k(0) = 1$. If one considers small values of λ one finds that $c_k(\lambda) = 1 + \mathcal{O}(\lambda^k)$. The generator \mathcal{J}^A written as a series thus reads [17, 24]

$$\begin{aligned} \mathcal{Y}^A &= \sum_{k < \ell} \frac{1}{2} F_{BC}^A \mathcal{J}_k^B \mathcal{J}_\ell^C + \sum_{k < \ell} \sum_{m=\ell-k}^{\infty} \frac{1}{2} c_{\ell-k}^{(m)} \lambda^m F_{BC}^A \mathcal{J}_k^B \mathcal{J}_\ell^C \\ &= \sum_{k < \ell} \frac{1}{2} F_{BC}^A \mathcal{J}_k^B \mathcal{J}_\ell^C + \sum_{m=1}^{\infty} \lambda^m \sum_k \sum_{\ell=1}^m \frac{1}{2} c_\ell^{(m)} F_{BC}^A \mathcal{J}_k^B \mathcal{J}_{k+\ell}^C. \end{aligned} \tag{3.13}$$

In general we can write the above expression as

$$\mathcal{Y} = \mathcal{Y}_{\text{bi}} + \sum_{k=1}^{\infty} \lambda^k \mathcal{Y}^{(k)}, \tag{3.14}$$

where $\mathcal{Y}^{(k)}$ is a (local) operator which acts on no more than $k + 1$ adjacent sites of the chain and \mathcal{Y}_{bi} is the bi-local operator

$$\begin{aligned} \mathcal{Y}_{\text{bi}}^A &= \sum_{k < l} \frac{1}{2} F_{BC}^A \mathcal{J}_k^B \mathcal{J}_l^C, \\ (\mathcal{Y}_{\text{bi}})_{\text{b}}^a &= \sum_{k < l} [(\mathcal{J}_l)_c^a, (\mathcal{J}_k)_b^c], \\ (\mathcal{Y}_{\text{bi}})_x &= \sum_{k < l} [(\mathcal{J}_l)_x, (\mathcal{J}_k)_x]. \end{aligned} \tag{3.15}$$

We expect that the Yangian generator for general models of section 2 is also of the form (3.14). The bi-local piece \mathcal{Y}_{bi} in (3.15) is universal, while the details of the model will be encoded in the precise expressions for the local pieces $\mathcal{Y}^{(k)}$. Our expectation agrees with the generic asymptotic expression for the Yangian generator in perturbative short-range chains:

$$\mathcal{Y}^A \approx \sum_{k \ll l} \frac{1}{2} F_{BC}^A \mathcal{J}_k^B \mathcal{J}_l^C. \tag{3.16}$$

This expression explains why there should be no perturbative corrections to the bi-local piece \mathcal{Y}_{bi} in (3.15): the Lie symmetry generators \mathcal{J} are exact; they do not receive corrections in λ , and therefore their bi-local product is asymptotically exact. In a chain with perturbatively modified Lie generators, such as [25], the bi-local pieces also receive perturbative corrections, albeit given through the corrections of the Lie generators. The local terms $\mathcal{Y}^{(k)}$ can be understood as a proper regularization of the above expression when the two Lie symmetry generators \mathcal{J}^A are close, i.e. within the range of the Hamiltonian. These certainly depend on the details of the Hamiltonian if they are to commute with it.

4. Construction of the Yangian

In this section we will construct the Yangian generators for the short-range spin chain model introduced in section 2. In their construction we will require that the Yangian generators commute with the local charges (including the Hamiltonian) and that they satisfy the Serre relations. The existence of this conserved Yangian shows that the associated Hamiltonian describes an integrable system and suffices to show perturbative integrability. In the construction of the local charges and the Yangian, we will assume a general form with many undetermined coefficients which are fixed by imposing the above constraints.

4.1. Structure of the perturbative Yangian

The symbol $\mathcal{Y}^{(k)}$ contains the order λ^k perturbations which are local adjoint interactions of range $k + 1$. Note that we explicitly include contributions which act trivially on the

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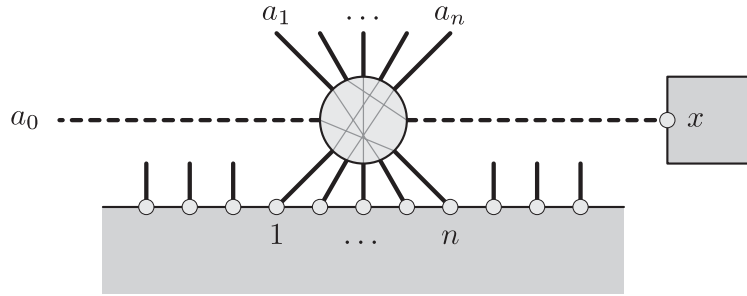


Figure 2. Graphical representation of a local adjoint interaction with one auxiliary site and n spin chain sites.

auxiliary space (invariant interactions) as well as terms proportional to the Lie generator, \mathcal{J} . In principle we are free to drop these contributions: local invariant interactions which commute with the Hamiltonian define the local charges. Perturbing the Yangian generator by a local charge or by \mathcal{J} does not alter its adjoint transformation or the Serre relations. We nevertheless would like to keep these degrees of freedom explicit, because it simplifies the counting.

This means that the Yangian generator can be written as some bare generator $\bar{\mathcal{Y}}$ plus some interactions proportional to the Lie generator or the conserved charges:

$$\mathcal{Y} = \bar{\mathcal{Y}} + \gamma_{\mathcal{Y},\mathcal{J}} \mathcal{J} + \gamma_{\mathcal{Y},\mathcal{L}} \mathcal{L} + \sum_{s=2}^{\infty} \gamma_{\mathcal{Y},s} \bar{\mathcal{Q}}_s. \quad (4.1)$$

In order to match the perturbative interaction range we have to demand that

$$\gamma_{\mathcal{Y},\mathcal{J}} = \mathcal{O}(\lambda^0), \quad \gamma_{\mathcal{Y},\mathcal{L}} = \mathcal{O}(\lambda^0), \quad \gamma_{\mathcal{Y},s} = \mathcal{O}(\lambda^{s-1}). \quad (4.2)$$

4.2. Notation

As explained in section 3.4, the Yangian is an adjoint operator which can be thought of as acting on the spin chain and an auxiliary site which is not on the spin chain. The notation introduced in section 2.2 does not suffice to write these operators, so here we extend the notation to account for adjoint operators and bi-local operators.

We denote an adjoint local operator which acts locally on n adjacent sites of the chain by the symbol $[a_0|a_1, a_2, \dots, a_n]$. A graphical representation of this operator is given in figure 2. Here the labels a_0, a_1, \dots, a_n are distinct integers from 1 to n and one of them is the symbol x , representing the auxiliary site. The label a_0 denotes the site which is mapped to the auxiliary site and the other n labels describe which sites are mapped to the sites on the chain. As before there is an implicit sum over all n adjacent sites of the chain. For example an operator $[x|\dots]$ acts identically on the auxiliary site and therefore is equivalent to the invariant operator $[\dots]$. The operator $[1|x]$ would interchange a site on the spin chain with the auxiliary site. In fact this is just how the $\mathfrak{gl}(N)$ symmetry generator acts in (2.4) and we have found a way to represent it in our notation:

$$\mathcal{J}_x = [1|x]. \quad (4.3)$$

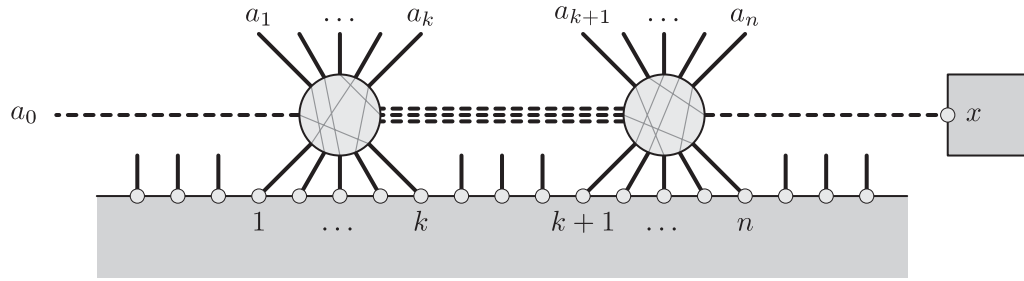


Figure 3. Graphical representation of a bi-local adjoint operator.

Simplifications as in (2.7) also apply to adjoint operators:

$$[a_0||a_1, a_2, \dots, a_n, n+1] = [a_0||a_1, a_2, \dots, a_n] = [a_0+1||1, a_1+1, a_2+1, \dots, a_n+1], \quad (4.4)$$

where in the last term the auxiliary label x is clearly not shifted by 1. It can be shown that there are $(n+1)! - n! + 1$ local adjoint operators of range n . This number coincides with the number of local invariant operators of range $n+1$.

To represent bi-local adjoint operators we use the symbol $[a_0||a_1, \dots, a_k|a_{k+1}, \dots, a_n]$. A bi-local operator acts on k and $n-k$ adjacent sites which can be at arbitrary distance on the spin chain. The first block of sites will always be left of the second block. Again a_0 denotes the site which is mapped to the auxiliary site. The labels a_1, \dots, a_k denote to which sites the first block of sites is mapped and a_{k+1}, \dots, a_n represents the second block. Again there are implicit sums over the positions of k and $n-k$ adjacent sites such that the first block is always towards the left of the second block. In figure 3 we show a generic bi-local operator. In fact one may think of the bi-local adjoint operator as the following sum of local operators:

$$[a_0||a_1, \dots, a_k|a_{k+1}, \dots, a_n] = \sum_{\ell=0}^{\infty} [a'_0||a'_1, \dots, a'_k, k+1, \dots, k+\ell, a'_{k+1}, \dots, a'_n]. \quad (4.5)$$

The primed labels equal $a'_j = a_j + \ell$ when $a_j > k$ and $a'_j = a_j$ otherwise. The trouble with this expression is that it contains infinitely many terms and cannot be represented easily in a computer algebra. Instead one should implement directly the original bi-local interaction symbols. In this notation the bi-local piece of the Yangian (3.15) is represented by

$$(\mathcal{Y}_{\text{bi}})_x = [1||2|x] - [2||x|1]. \quad (4.6)$$

The simplifications for bi-local operators are slightly more elaborate:

$$\begin{aligned} [a_0||a_1, \dots, a_k|a_{k+1}, \dots, a_n, n+1] &= [a_0||a_1, \dots, a_k|a_{k+1}, \dots, a_n], \\ [a_0+1||1, a_1+1, \dots, a_k+1|a_{k+1}+1, \dots, a_n+1] &= [a_0||a_1, \dots, a_k|a_{k+1}, \dots, a_n], \\ [a'_0||a'_1, \dots, a'_k, k+1|a'_{k+1}, \dots, a'_n] &= [a_0||a_1, \dots, a_k|a_{k+1}, \dots, a_n] \\ &\quad - [a_0||a_1, \dots, a_k, a_{k+1}, \dots, a_n], \\ [a'_0||a'_1, \dots, a'_k|k+1, a'_{k+1}, \dots, a'_n] &= [a_0||a_1, \dots, a_k|a_{k+1}, \dots, a_n] \\ &\quad - [a_0||a_1, \dots, a_k, a_{k+1}, \dots, a_n]. \end{aligned} \quad (4.7)$$

The latter two rules take into account that there is always at least one site in between the two blocks. The primed labels equal $a'_\ell = a_\ell + 1$ if $a_\ell > k+1$ and $a'_\ell = a_\ell$ otherwise.

Table 3. Number of degrees of freedom when starting with conserved local charges.

	\pm	λ^0	λ^1	λ^2	λ^3
Intrinsic parameters		0	1	5	21
Ansatz for Yangian \mathcal{Y}	+	2	5	19	97
Constraint from $[\mathcal{H}, \mathcal{Y}] = 0$	-	0	2	15	92
Undetermined coefficients	=	2	4	9	26
$\gamma_{\mathcal{Y},s}$ (mixing of Yangian)	-	2	3	4	5
Intrinsic parameters	=	0	1	5	21

4.3. Starting from conserved local charges

In the first approach we start with the results of section 2, i.e. with a Hamiltonian \mathcal{H} and a conserved charge \mathcal{Q} . We then assume the most general expression for the Yangian (3.14) with $(k+2)! - (k+1)! + 1$ undetermined coefficients for the local adjoint operators at $\mathcal{O}(\lambda^k)$. Finally, we require that the Yangian generator commutes with the Hamiltonian and the conserved charge, as well as that it satisfies the Serre relations. This calculation is done explicitly to first order in λ in appendix B; at higher orders it proceeds in a very similar but not very illuminating fashion. In table 3 we list the number of free parameters and the constraints placed on them at each perturbative order. The resulting expression for the bare Yangian generator $\bar{\mathcal{Y}}$ is presented at the end of the paper in table 7. We discuss it further below. In appendix B we construct this Yangian to $\mathcal{O}(\lambda)$ as a concrete example.

The first important comment is that the system of equations can be solved; thus we have found a perturbatively integrable system. The second outcome of this computation is that the requirement imposes *no further constraints* on the coefficients of the Hamiltonian or of the local charge, but only on the coefficients of the Yangian generator. Therefore the Hamiltonian found in section 2 is indeed integrable for the most general set of parameters. This confirms the observation [4, 6] that for the present type of short-range spin chain models it is sufficient to impose the existence of merely one conserved local charge, \mathcal{Q} , to obtain an integrable Hamiltonian. The surprising third comment is that all the constraints originate from the conservation of the Yangian generator; commutation with the conserved charge and the Serre relations follow automatically without further constraints on the coefficients. Finally, the Yangian is completely fixed up to the obvious deformations in (4.1).

4.4. Starting from conserved Yangian

We have seen that merely one conserved local charge already appears to be sufficient to restrict to an integrable model. A useful question is whether different subsets of the full set of integrability conditions already guarantee integrability. In other words, we will now impose the various integrability conditions in a different order.

Here we start with a conserved Yangian for an otherwise unconstrained Hamiltonian, i.e. we do not impose the existence of a conserved local charge \mathcal{Q} . We use the same expansion for the Hamiltonian and for the Yangian as before. By imposing $[\mathcal{H}, \mathcal{Y}] = 0$ we

Table 4. Number of degrees of freedom when starting with conserved Yangian. Note the same number of intrinsic parameters as in table 3.

	\pm	λ^0	λ^1	λ^2	λ^3
Ansatz for Hamiltonian \mathcal{H}		2	5	19	97
Ansatz for Yangian \mathcal{Y}	+	2	5	19	97
Constraint from $[\mathcal{H}, \mathcal{Y}] = 0$	-	0	3	25	163
Undetermined coefficients	=	4	7	13	31
$\gamma_{2,s}$ (mixing of Hamiltonian)	-	2	3	4	5
$\gamma_{\mathcal{Y},s}$ (mixing of Yangian)	-	2	3	4	5
Intrinsic parameters	=	0	1	5	21

fix some coefficients of the Hamiltonian and the Yangian. We keep track of the number of undetermined coefficients at each perturbative order in table 4. From the total number coefficients in the Hamiltonian and Yangian we have to subtract the number of constraints and the number of degrees of freedom of choosing \mathcal{H} and \mathcal{Y} as a linear combination of all local suitable local charges. The remaining number of undetermined coefficients for the integrable system coincides nicely with the number obtained in table 1 by taking all constraints into account. Thus, already demanding a conserved Yangian effectively constrains us to an integrable system. Note that the Yangian and Hamiltonian here are identical to those of the previous section; this is not just a coincidence of numbers.

We can now go on to fix the conserved charge by making it commute with the Yangian. This step does not introduce further constraints on the Yangian or the Hamiltonian obtained previously; it merely adjusts several coefficients of \mathcal{Q} . This charge then automatically commutes with the Hamiltonian.

4.5. Starting from Serre relations

Finally, we would like to see whether the Serre relations by themselves define the same integrable model. In fact, the Serre relations guarantee integrability, so we have to show that we can construct a suitable Hamiltonian which commutes with the Yangian. This procedure is quite interesting as it makes no reference to local charges at all. Consequently, the coefficients $\gamma_{r,s}$ governing the linear combinations of local charges will enter only at a later stage.

The Yangian is expanded as in (3.14). Only \mathcal{Y}_{bi} is specified; the perturbations given by adjoint operators are left unconstrained. We then require this form to satisfy the Serre relations in (3.11) to each order. Since \mathcal{J} receives no corrections the right-hand side is zero at subleading order so we can use simplified relations which only have commutators of \mathcal{Y} and \mathcal{P} ; see (A.2). In table 5 we keep track of the degrees of freedom. At each perturbative order we take the number of coefficients in the Yangian ansatz and subtract those fixed by the Serre relations. The undetermined degrees of freedom agree with those from the two other approaches above.

The existence of a Yangian proves integrability if we can find a local operator that commutes with it; we need to construct a suitable Hamiltonian. We make the same ansatz for the Hamiltonian as before. In counting the degrees of freedom we must again subtract the constraints and the degrees of freedom used in choosing \mathcal{H} as a linear combination of all suitable local charges. The constraint of $[\mathcal{H}, \mathcal{Y}] = 0$ completely specifies the Hamiltonian

Table 5. Number of degrees of freedom when starting with Serre relations. Note the intrinsic parameters are identical to those in tables 3 and 4.

	\pm	λ^0	λ^1	λ^2	λ^3
Ansatz for Yangian \mathcal{Y}		2	5	19	97
Constraint from Serre relations	−	0	1	10	71
Undetermined coefficients	=	2	4	9	26
$\gamma_{\mathcal{Y},s}$ (mixing of Yangian)	−	2	3	4	5
Intrinsic parameters	=	0	1	5	21
Ansatz for Hamiltonian \mathcal{H}	+	2	5	19	97
Constraint from $[\mathcal{H}, \mathcal{Y}] = 0$	−	0	2	15	92
Undetermined coefficients	=	2	4	9	26
$\gamma_{2,s}$ (eigenvalue Hamiltonian)	−	2	3	4	5
Intrinsic parameters	=	0	1	5	21

Table 6. Normalized Hamiltonian printed up to second order.

$$\begin{aligned}
 \bar{Q}_2(\lambda) = & ([1] - [2, 1]) \\
 & + \alpha_0(\lambda) (-3 [1] + 4 [2, 1] - [3, 2, 1]) \\
 & + \alpha_0(\lambda)^2 (20 [1] - 29 [2, 1] + 10 [3, 2, 1] - [4, 2, 3, 1] \\
 & \quad + [3, 1, 4, 2] + [2, 4, 1, 3] - [4, 1, 2, 3] - [2, 3, 4, 1]) \\
 & + \frac{1}{2} \alpha_1(\lambda) (6 [2, 3, 1] - 6 [3, 1, 2] + [4, 1, 3, 2] + [4, 2, 1, 3] - [2, 4, 3, 1] - [3, 2, 4, 1]) \\
 & + \frac{1}{2} \beta_{2,3}(\lambda) (-4 [1] + 8 [2, 1] - 2 [3, 1, 2] - 2 [2, 3, 1] \\
 & \quad - 2 [2, 1, 4, 3] - 2 [4, 1, 2, 3] - 2 [2, 3, 4, 1] \\
 & \quad + 2 [3, 1, 4, 2] + 2 [2, 4, 1, 3] - 2 [3, 4, 1, 2] \\
 & \quad + [4, 1, 3, 2] + [2, 4, 3, 1] + [4, 2, 1, 3] + [3, 2, 4, 1]) \\
 & + i\epsilon_{2,1}^+(\lambda) ([2, 4, 1, 3] - [3, 1, 4, 2]) \\
 & + i\epsilon_{2,2}^+(\lambda) ([4, 2, 1, 3] + [2, 4, 3, 1] - [4, 1, 3, 2] - [3, 2, 4, 1]) \\
 & + \mathcal{O}(\lambda^3)
 \end{aligned}$$

but gives no additional constraints on the Yangian, so we have the same number of undetermined coefficients as above. We can also construct a local charge \mathcal{Q} in the same way and obtain similar results, i.e. no further constraints on the Yangian generator. This charge automatically commutes with the Hamiltonian, as expected.

4.6. Summary

In the first method, we first constructed the Hamiltonian, \mathcal{H} , and the conserved charge, \mathcal{Q} , by requiring that they commute: $[\mathcal{H}, \mathcal{Q}] = 0$. We then constructed the Yangian generator \mathcal{Y} by demanding that it is conserved: $[\mathcal{Y}, \mathcal{H}] = 0$. We finally checked that \mathcal{Y} satisfies the Serre relations and that it commutes with \mathcal{Q} . In the second approach, we started with generic \mathcal{Y}, \mathcal{H} and required $[\mathcal{Y}, \mathcal{H}] = 0$. We then showed that this Yangian generator satisfies the Serre relations. In the final approach we constructed a Yangian generator by requiring that it satisfies the Serre relations. We then constructed a Hamiltonian by requiring $[\mathcal{Y}, \mathcal{H}] = 0$.

All three methods turn out to give the same number of undetermined parameters. Moreover, the obtained Hamiltonian (table 6), conserved charge (table 8) and Yangian

Table 7. Normalized Yangian generator printed up to second order.

$$\begin{aligned}
 \bar{\mathcal{Y}}_x = & ([1||2|x] - [2||x|1]) \\
 & + 2\alpha_0(\lambda) ([1||2, x] - [2||x, 1]) \\
 & + 2\alpha_0(\lambda)^2 (2 [2||x, 1] - 2 [1||2, x] + [1||3, 2, x] - [3||x, 2, 1]) \\
 & + i\alpha_1(\lambda) (2 [1||2, 3, x] + 2 [3||x, 1, 2] \\
 & \quad - [1||3, x, 2] - [2||x, 3, 1] - [3||2, x, 1] - [2||3, 1, x]) \\
 & + \beta_{2,3}(\lambda) (2 [1||2, x] - 2 [2||x, 1] \\
 & \quad - [1||3, x, 2] + [2||x, 3, 1] + [3||2, x, 1] - [2||3, 1, x]) \\
 & + 2i\epsilon_{2,1}^+(\lambda) ([3||2, 1, x] - [1||x, 3, 2]) \\
 & + 2i\epsilon_{2,2}^+(\lambda) ([2||3, 1, x] + [3||2, x, 1] - [1||3, x, 2] - [2||x, 3, 1]) \\
 & + \mathcal{O}(\lambda^3)
 \end{aligned}$$

Table 8. Normalized conserved charge printed up to second order.

$$\begin{aligned}
 \bar{\mathcal{Q}}_3(\lambda) = & \frac{i}{2}([3, 1, 2] - [2, 3, 1]) \\
 & + \frac{i}{2}\alpha_0(\lambda) (6 [2, 3, 1] - 6 [3, 1, 2] + [4, 1, 3, 2] + [4, 2, 1, 3] - [2, 4, 3, 1] - [3, 2, 4, 1]) \\
 & + \frac{i}{2}\alpha_0(\lambda)^2 (46 [3, 1, 2] - 46 [2, 3, 1] \\
 & \quad + 12 [3, 2, 4, 1] + 12 [2, 4, 3, 1] - 12 [4, 2, 1, 3] - 12 [4, 1, 3, 2] \\
 & \quad + 2 [2, 4, 1, 5, 3] + 2 [2, 3, 5, 1, 4] + 2 [3, 1, 4, 5, 2] + 2 [5, 1, 2, 3, 4] \\
 & \quad - 2 [4, 1, 2, 5, 3] - 2 [3, 1, 5, 2, 4] - 2 [2, 5, 1, 3, 4] - 2 [2, 3, 4, 5, 1] \\
 & \quad + [5, 2, 3, 1, 4] + [5, 1, 3, 4, 2] + [5, 2, 1, 4, 3] \\
 & \quad - [2, 5, 3, 4, 1] - [4, 2, 3, 5, 1] - [3, 2, 5, 4, 1]) \\
 & + \frac{i}{4}\alpha_1(\lambda) (-20 [1] + 24 [2, 1] - 8 [4, 1, 2, 3] - 8 [2, 3, 4, 1] - 4 [4, 2, 3, 1] \\
 & \quad + 6 [3, 1, 4, 2] + 6 [2, 4, 1, 3] + 2 [4, 3, 1, 2] + 2 [3, 4, 2, 1] \\
 & \quad + 2 [5, 1, 3, 2, 4] + 2 [2, 4, 3, 5, 1] - 2 [2, 5, 3, 1, 4] - 2 [4, 1, 3, 5, 2] \\
 & \quad + [5, 2, 1, 3, 4] + [3, 2, 4, 5, 1] + [5, 1, 2, 4, 3] + [2, 3, 5, 4, 1] \\
 & \quad - [3, 1, 5, 4, 2] - [4, 2, 1, 5, 3] - [2, 5, 1, 4, 3] - [3, 2, 5, 1, 4]) \\
 & + \frac{i}{4}\beta_{2,3}(\lambda) (-4 [3, 1, 2] + 4 [2, 3, 1] + 4 [4, 1, 2, 3] - 4 [2, 3, 4, 1] - 2 [4, 2, 1, 3] \\
 & \quad - 2 [4, 1, 3, 2] - 2 [3, 4, 2, 1] + 2 [4, 3, 1, 2] + 2 [3, 2, 4, 1] + 2 [2, 4, 3, 1] \\
 & \quad - 4 [4, 1, 2, 5, 3] - 4 [3, 1, 5, 2, 4] - 4 [2, 3, 4, 5, 1] - 4 [2, 5, 1, 3, 4] \\
 & \quad + 4 [2, 4, 1, 5, 3] + 4 [2, 3, 5, 1, 4] + 4 [3, 1, 4, 5, 2] + 4 [5, 1, 2, 3, 4] \\
 & \quad - [5, 2, 1, 3, 4] - [3, 1, 5, 4, 2] - [5, 1, 2, 4, 3] - [3, 2, 5, 1, 4] \\
 & \quad + [2, 3, 5, 4, 1] + [4, 2, 1, 5, 3] + [3, 2, 4, 5, 1] + [2, 5, 1, 4, 3] \\
 & \quad + 2 [3, 1, 2, 5, 4] + 2 [2, 1, 5, 3, 4] - 2 [2, 1, 4, 5, 3] - 2 [2, 3, 1, 5, 4] \\
 & \quad + 2 [4, 1, 5, 2, 3] + 2 [3, 5, 1, 2, 4] - 2 [5, 1, 3, 2, 4] - 2 [3, 4, 1, 5, 2] \\
 & \quad + 2 [2, 4, 3, 5, 1] - 2 [2, 4, 5, 1, 3]) \\
 & + \frac{i}{2}\epsilon_{2,1}^+(\lambda) ([4, 2, 1, 3] + [3, 2, 4, 1] - [4, 1, 3, 2] - [2, 4, 3, 1] \\
 & \quad + [3, 1, 4, 5, 2] + [2, 5, 1, 3, 4] - [4, 1, 2, 5, 3] - [2, 3, 5, 1, 4]) \\
 & + \frac{i}{2}\epsilon_{2,2}^+(\lambda) ([5, 2, 1, 3, 4] + [3, 1, 5, 4, 2] + [3, 2, 4, 5, 1] + [2, 5, 1, 4, 3] \\
 & \quad - [4, 2, 1, 5, 3] - [5, 1, 2, 4, 3] - [3, 2, 5, 1, 4] - [2, 3, 5, 4, 1]) \\
 & + \mathcal{O}(\lambda^3)
 \end{aligned}$$

(table 7) fully agree in the three approaches. This means that, at least for this system, the three approaches are equivalent. Curiously, imposing a single integrability property such as

- the existence of *one* conserved charge,
- the existence of a conserved *Yangian-like* operator,
- a Yangian obeying the Serre relations,

seems to imply full integrability, at least within the current class of models. In particular, we observe in the three cases that

- *many* conserved charges [4, 6] and a conserved Yangian exist,
- the Yangian obeys the Serre relations,
- a local Hamiltonian which commutes with the Yangian exists,

even without imposing further constraints on the existing operators. This is clearly not a rigorous proof, but it supports the claim in [4, 6] that for the present class of short-range spin chains, one additional conserved local charge is sufficient to prove that a Hamiltonian is integrable.

5. Conclusions and outlook

In this paper we have proved the (third-order) perturbative integrability of the $\mathfrak{gl}(N)$ long-range spin chain Hamiltonian proposed in [6] by means of Yangian symmetry. Integrability has long been assumed in the use of the asymptotic Bethe ansatz for such systems which predominantly come to use in the planar AdS/CFT correspondence [4, 7]. This also supports the observation of [27] that merely one conserved local charge is often sufficient to constrain a spin chain system to be integrable.

We believe that a similar Yangian symmetry can be used for algebras other than $\mathfrak{gl}(N)$ and for different representations. Work has already been done to find the Yangian of a $\mathfrak{su}(2|1)$ long-range spin chain [25]. Studying larger algebras involving fermions, gauge fields and derivatives (in the AdS/CFT context) leads to complications on the spin chain side such as fluctuations of the length [28] for which new machinery will be required. In general it would be interesting to find out to what extent long-range integrable spin chains can be extended.

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Appendix A. Simplifications of commutators

In the construction of the Yangian we come across many different commutators. In order to make the problem tractable we utilize several cancellations which simplify these commutators. To see these cancellations it is useful to think of the operators diagrammatically as in figures 1 and 2. We will describe the first simplification in detail; the others can be done in a similar fashion.

When taking commutators we must consider the placements of the operator on the spin chain. As mentioned above, there are implicit sums for these operators. When we

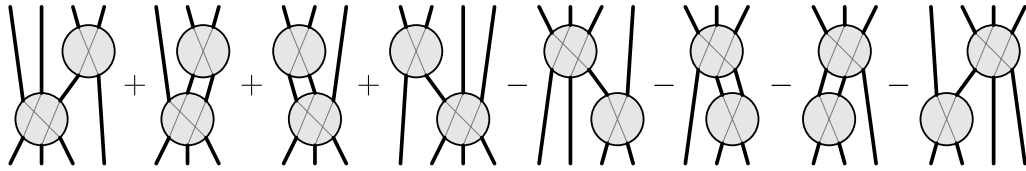


Figure A.1. Graphical representation of the commutator of two local invariant interactions; here: $[[2, 1], [3, 1, 2]]$.

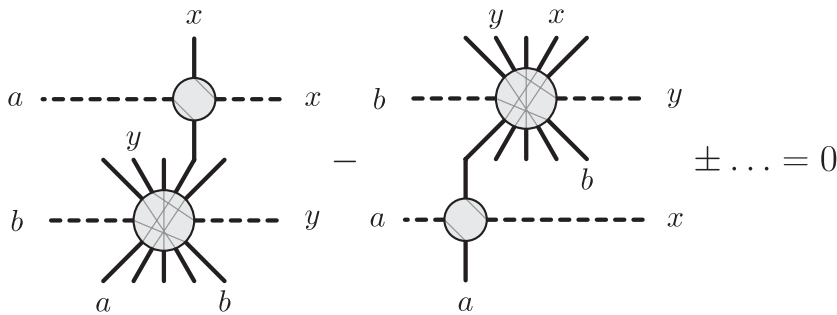


Figure A.2. Pairwise cancellation of contributions to $[\mathcal{J}_x, L_y]$. The cancellation extends to all terms when \mathcal{J}_x interacts with a spin chain site a which maps to/from another spin chain site (i.e. those which are not labeled by b or y).

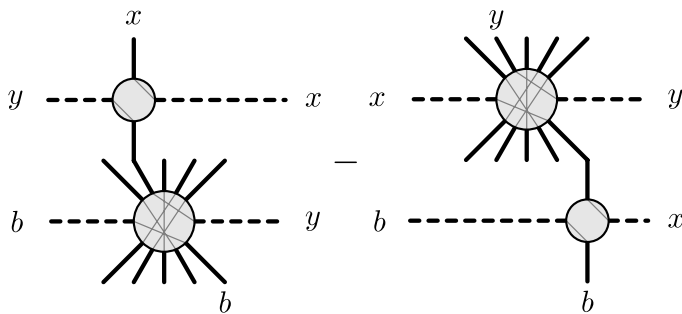


Figure A.3. The remaining contribution to $[\mathcal{J}_x, L_y]$.

take a commutator we need only consider the terms where the operators overlap as in figure A.1.

Commutators of local adjoint operators

The first commutator we will consider is $[\mathcal{J}_x, L_y]$, where L_y is an arbitrary adjoint local operator acting on the spin chain and on the auxiliary site y . The symmetry generator for $\mathfrak{gl}(N)$ is defined in (4.3). We should think of both \mathcal{J}_x and L_y diagrammatically as in figure 2, albeit \mathcal{J}_x acts on only one site on the spin chain. When we commute \mathcal{J}_x with L_y there are two possible contributions: \mathcal{J}_x interacts either with a site on L_y which does not map to/from the auxiliary site as in figure A.2 or with a site which does map to/from the auxiliary site as in figure A.3. We see there is no contribution from the former but there is a contribution from the latter.

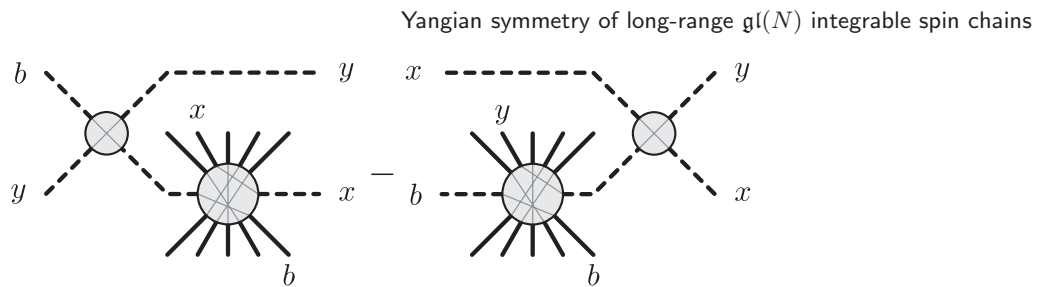


Figure A.4. The commutator $[\mathcal{P}_{xy}, L_x]$. We see that this is identical to figure A.3.

We can also consider $[\mathcal{P}_{xy}, L_x]$, as in figure A.4, and we see it is equivalent to $[\mathcal{J}_x, L_y]$. Hence we have the relation

$$[\mathcal{J}_x, L_y] = [\mathcal{P}_{xy}, L_x]. \tag{A.1}$$

Note that if we substitute L_y with \mathcal{J}_y or \mathcal{Y}_y in (A.1) we get the adjoint transformation rules, which justifies that we call L_y an adjoint operator.

Commutators involving Yangian generators

The simplification above is a simple one, but the same technique can be used on more complicated commutators involving bi-local Yangian generators in (3.14). In the construction of the Yangian generator we need to perform commutators of the sort $[(\mathcal{Y}_{bi})_x, L]$, where L is now a local invariant operator. When only one leg of $(\mathcal{Y}_{bi})_x$ interacts with L we get a cancelation similar to the one in figure A.2. This is also clear from (3.15) since the commutator is equivalent to $[\mathcal{J}_x, L]$, which is zero. Thus there is only a contribution when both legs of \mathcal{Y}_{bi} interact with L . This simplifies our calculations since the outcome of $[(\mathcal{Y}_{bi})_x, L]$ will always be a local adjoint operator (which is simpler than a bi-local adjoint operator).

Serre relations

To implement the cubic Serre relation (3.11) it is helpful to first simplify it for our purposes. We use the adjoint transformation (3.10) to convert some generators \mathcal{J} to permutations of auxiliary sites:

$$\begin{aligned} & -[\mathcal{P}_{xz}, [\mathcal{Y}_y, \mathcal{Y}_z]] - [\mathcal{P}_{xy}, [\mathcal{Y}_z, \mathcal{Y}_x]] - [\mathcal{P}_{yz}, [\mathcal{Y}_x, \mathcal{Y}_y]] \\ & = [\mathcal{J}_x, \mathcal{J}_z[\mathcal{P}_{yz}, \mathcal{J}_z]\mathcal{J}_z] + [\mathcal{P}_{yz}, \mathcal{J}_z[\mathcal{P}_{xz}, \mathcal{J}_z]\mathcal{J}_z]. \end{aligned} \tag{A.2}$$

The permutations are somewhat simpler because they do not involve sums over sites of the spin chain.

We can now expand (A.2) using the expansion of the Yangian generators in (3.14). At leading order we see that (A.2) involves commutators of two bi-local generators. However, it is clear that the equation holds at leading order because this is just the undeformed Yangian generator of [1]. At subleading orders the right-hand side of (A.2) is zero since \mathcal{J} for $\mathfrak{gl}(N)$ receives no perturbative corrections. We see that the subleading corrections of the left-hand side of (A.2) will include commutators of the form $[(\mathcal{Y}_{bi})_x, \mathcal{Y}_y^{(k)}]$ and $[\mathcal{Y}_x^{(j)}, \mathcal{Y}_y^{(k-j)}]$. Terms of the first type could give a bi-local contribution, but fortunately

all terms when only one leg of the bi-local interacts with the adjoint operator sum to zero in (A.2). In fact, this result holds for the commutator of \mathcal{Y}_{bi} with any adjoint operator, not just those corresponding to deformations of the Yangian. Thus there are contributions from $[(\mathcal{Y}_{\text{bi}})_x, \mathcal{Y}_y^{(k)}]$ when both legs of the bi-local interact with the adjoint operator and from $[\mathcal{Y}_x^{(j)}, \mathcal{Y}_y^{(k-j)}]$ which both give local contributions.

Appendix B. Sample calculation at next-to-leading order

Here we shall perform explicitly to first order in λ the calculation which is described in sections 2 and 4.3.

Local charges

We start with the most general Hamiltonian, \mathcal{H} , and conserved charge, \mathcal{Q} . At leading order \mathcal{H} has length 2 and \mathcal{Q} has length 3:

$$\begin{aligned}\mathcal{H}^{(0)} &= a_{0,1}[1] + a_{0,2}[2, 1], \\ \mathcal{Q}^{(0)} &= b_{0,1}[1] + b_{0,2}[2, 1] + b_{0,3}[2, 3, 1] + b_{0,4}[3, 1, 2] + b_{0,5}[3, 2, 1],\end{aligned}\tag{B.1}$$

where $a_{j,k}$ is the coefficient of the k^{th} term of $\mathcal{H}^{(j)}$ and likewise $b_{j,k}$ for $\mathcal{Q}^{(k)}$. We now impose the constraint $[\mathcal{H}^{(0)}, \mathcal{Q}^{(0)}] = 0$. This commutator is

$$\begin{aligned}[\mathcal{H}^{(0)}, \mathcal{Q}^{(0)}] &= a_{0,2}(b_{0,3} + b_{0,4})([2, 4, 1, 3] - [3, 1, 4, 2]) \\ &\quad + a_{0,2}b_{0,5}([2, 4, 3, 1] + [4, 2, 1, 3] - [3, 2, 4, 1] - [4, 1, 3, 2]).\end{aligned}\tag{B.2}$$

The constraints on the coefficients read

$$b_{0,3} = -b_{0,4}, \quad b_{0,5} = 0,\tag{B.3}$$

which leaves us with

$$\begin{aligned}\mathcal{H}^{(0)} &= a_{0,1}[1] + a_{0,2}[2, 1], \\ \mathcal{Q}^{(0)} &= b_{0,1}[1] + b_{0,2}[2, 1] + b_{0,3}[2, 3, 1] - b_{0,3}[3, 1, 2].\end{aligned}\tag{B.4}$$

Now let us count the degrees of freedom. There are 2 unique local operators of length 2 and 5 of length 3. There are 2 constraints from imposing $[\mathcal{H}^{(0)}, \mathcal{Q}^{(0)}] = 0$. According to (2.10), we can rewrite $\mathcal{H}^{(0)}$ and $\mathcal{Q}^{(0)}$ using linear combinations of the bare charges. This allows us to fully specify $\bar{\mathcal{Q}}_2^{(0)}$ and $\bar{\mathcal{Q}}_3^{(0)}$:

$$\begin{aligned}\mathcal{H}^{(0)} &= (a_{0,1} + a_{0,2})\mathcal{L} - a_{0,2}\bar{\mathcal{Q}}_2^{(0)}, \\ \mathcal{Q}^{(0)} &= (b_{0,1} + b_{0,2})\mathcal{L} - b_{0,2}\bar{\mathcal{Q}}_2^{(0)} + 2ib_{0,3}\bar{\mathcal{Q}}_3^{(0)}\end{aligned}\tag{B.5}$$

with

$$\begin{aligned}\mathcal{L} &= [1], \\ \bar{\mathcal{Q}}_2^{(0)} &= [1] - [2, 1], \\ \bar{\mathcal{Q}}_3^{(0)} &= -\frac{i}{2}([2, 3, 1] - [3, 1, 2]).\end{aligned}\tag{B.6}$$

To $\mathcal{O}(\lambda)$ we increase the length of each operator by one. $\mathcal{H}^{(1)}$ has 5 possible terms and $\mathcal{Q}^{(1)}$ has 19 possible terms. We now require \mathcal{H} and \mathcal{Q} commute to $\mathcal{O}(\lambda)$, which imposes the constraint

$$[\mathcal{H}^{(0)}, \mathcal{Q}^{(1)}] + [\mathcal{H}^{(1)}, \mathcal{Q}^{(0)}] = 0.\tag{B.7}$$

This imposes 16 constraints, and gives

$$\begin{aligned}\mathcal{H}^{(1)} &= a_{1,1}[1] + a_{1,2}[2, 1] + a_{1,3}[2, 3, 1] - a_{1,3}[3, 1, 2] + a_{1,5}[3, 2, 1], \\ \mathcal{Q}^{(1)} &= b_{1,1}[1] + b_{1,2}[2, 1] + b_{1,3}[2, 3, 1] - b_{1,3}[3, 1, 2] + (b_{1,10} + a_{1,5}b_{0,2}/a_{0,2})[3, 2, 1] \\ &\quad - b_{1,10}[2, 3, 4, 1] + b_{1,10}[2, 4, 1, 3] + b_{1,10}[3, 1, 4, 2] - b_{1,10}[4, 1, 2, 3] \\ &\quad + (a_{1,5}b_{0,3}/a_{0,2})[2, 4, 3, 1] + (a_{1,5}b_{0,3}/a_{0,2})[3, 2, 4, 1] \\ &\quad - (a_{1,5}b_{0,3}/a_{0,2})[4, 1, 3, 2] - (a_{1,5}b_{0,3}/a_{0,2})[4, 2, 1, 3].\end{aligned}\tag{B.8}$$

As before, we can rewrite these charges in terms of linear combinations (2.10) of the bare charges:

$$\begin{aligned}\mathcal{H}^{(1)} &= (a_{1,1} + a_{1,2} + a_{1,5})\mathcal{L} + (-a_{1,2} - 4a_{1,5})\bar{\mathcal{Q}}_2^{(0)} - a_{0,2}\bar{\mathcal{Q}}_2^{(1)} + 2ia_{1,3}\bar{\mathcal{Q}}_3^{(0)}, \\ \mathcal{Q}^{(1)} &= (b_{1,1} + b_{1,2} + b_{1,10} + b_{0,2}a_{1,5}/a_{0,2})\mathcal{L} \\ &\quad + (-b_{1,2} - 2b_{1,10} - 4b_{0,2}a_{1,5}/a_{0,2})\bar{\mathcal{Q}}_2^{(0)} - b_{0,2}\bar{\mathcal{Q}}_2^{(1)} \\ &\quad + (2ib_{1,3} + 12ia_{1,5}b_{0,3}/a_{0,2})\bar{\mathcal{Q}}_3^{(0)} + 2ib_{0,3}\bar{\mathcal{Q}}_3^{(1)} - 3b_{1,10}\bar{\mathcal{Q}}_4^{(0)},\end{aligned}\tag{B.9}$$

where

$$\begin{aligned}\bar{\mathcal{Q}}_2^{(1)} &= (a_{1,5}/a_{0,2})(-3[1] + 4[2, 1] - [3, 2, 1]), \\ \bar{\mathcal{Q}}_3^{(1)} &= \frac{i}{2}(a_{1,5}/a_{0,2})(6[2, 3, 1] - 6[3, 1, 2] + [4, 1, 3, 2] + [4, 2, 1, 3] - [2, 4, 3, 1] - [3, 2, 4, 1]), \\ \bar{\mathcal{Q}}_4^{(0)} &= \frac{1}{3}(-[1] + 2[2, 1] - [3, 2, 1] + [2, 3, 4, 1] + [4, 1, 2, 3] - [2, 4, 1, 3] - [3, 1, 4, 2]).\end{aligned}\tag{B.10}$$

We see that there is one degree of freedom left in describing the bare charges: $a_{1,5}/a_{0,2}$. This counting corresponds well with table 1. The expressions (B.6) and (B.10) agree with the expansion of tables 6 and 8, and we find an expression for the function $\alpha_0(\lambda)$:

$$\alpha_0(\lambda) = \lambda(a_{1,5}/a_{0,2}) + \mathcal{O}(\lambda^2).\tag{B.11}$$

Note that the results in tables 6 and 8 use the coefficients described in section 2.4; these coefficients distinguish the physical degrees of freedom from the gage degrees of freedom.

For completeness we will give all the coefficients specified in (2.10) to order λ . Comparing (B.5) and (B.9) to (2.10) we get the following:

$$\begin{aligned}
 \gamma_{2,\mathcal{L}}(\lambda) &= a_{0,1} + a_{0,2} + \lambda(a_{1,1} + a_{1,2} + a_{1,5}) + \mathcal{O}(\lambda^2), \\
 \gamma_{2,2}(\lambda) &= -a_{0,2} + \lambda(-a_{1,2} - 4a_{1,5}) + \mathcal{O}(\lambda^2), \\
 \gamma_{2,3}(\lambda) &= \lambda(2ia_{1,3}) + \mathcal{O}(\lambda^2), \\
 \gamma_{3,\mathcal{L}}(\lambda) &= b_{0,1} + b_{0,2} + \lambda(b_{1,1} + b_{1,2} + b_{1,10} + b_{0,2}a_{1,5}/a_{0,2}) + \mathcal{O}(\lambda^2), \\
 \gamma_{3,2}(\lambda) &= -b_{0,2} + \lambda(-b_{1,2} - 2b_{1,10} - 4b_{0,2}a_{1,5}/a_{0,2}) + \mathcal{O}(\lambda^2), \\
 \gamma_{3,3}(\lambda) &= 2ib_{0,3} + \lambda(2ib_{1,3} + 12ia_{1,5}b_{0,2}/a_{0,2}) + \mathcal{O}(\lambda^2), \\
 \gamma_{3,4}(\lambda) &= \lambda(-3b_{1,10}) + \mathcal{O}(\lambda^2).
 \end{aligned} \tag{B.12}$$

Yangian generator

Now that we have \mathcal{H} and \mathcal{Q} to first order we can construct the Yangian, \mathcal{Y} . To leading order this is given by the bi-local operator plus invariant terms of length 2, which gives the Yangian ansatz 2 degrees of freedom.

$$(\mathcal{Y}^{(0)})_x = [1||2|x] - [2||x|1] + c_{0,1}[x||1] + c_{0,2}[1||x]. \tag{B.13}$$

We then consider the constraint

$$[(\mathcal{Y}^{(0)})_x, \mathcal{H}^{(0)}] = 0, \tag{B.14}$$

which is satisfied automatically. According to (4.1), we can rewrite $\mathcal{Y}^{(0)}$ as a linear combination of the bare Yangian, the bare charges and the symmetry generator:

$$(\mathcal{Y}^{(0)})_x = (\bar{\mathcal{Y}}^{(0)})_x + c_{0,1}\mathcal{L} + c_{0,2}\mathcal{J}_x, \tag{B.15}$$

where

$$(\bar{\mathcal{Y}}^{(0)})_x = [1||2|x] - [2||x|1]. \tag{B.16}$$

Thus $\bar{\mathcal{Y}}^{(0)}$ is specified exactly and there are no new constraints on $\mathcal{H}^{(0)}$.

To $\mathcal{O}(\lambda)$ we have the following most general ansatz:

$$(\mathcal{Y}^{(1)})_x = c_{1,1}[x||1] + c_{1,2}[1||x] + c_{1,3}[x||2, 1] + c_{1,4}[1||2, x] + c_{1,5}[2||x, 1]. \tag{B.17}$$

Requiring \mathcal{H} and \mathcal{Y} to commute to $\mathcal{O}(\lambda)$ is equivalent to the condition

$$\begin{aligned}
 [(\mathcal{Y}^{(0)})_x, \mathcal{H}^{(1)}] + [(\mathcal{Y}^{(1)})_x, \mathcal{H}^{(0)}] &= (a_{0,2}c_{1,4} - 2a_{1,5})([1||3, x, 2] - [2||3, 1, x]) \\
 &\quad + (a_{0,2}c_{1,5} + 2a_{1,5})([3||2, x, 1] - [2||x, 3, 1]) \\
 &= 0,
 \end{aligned} \tag{B.18}$$

which yields the constraints

$$c_{1,4} = 2a_{1,5}/a_{0,2}, \quad c_{1,5} = -2a_{1,5}/a_{0,2}. \tag{B.19}$$

Using (4.1) we now rewrite $\mathcal{Y}^{(1)}$ as a linear combination of the bare Yangian, bare charges and symmetry generator:

$$(\mathcal{Y}^{(1)})_x = (\bar{\mathcal{Y}}^{(1)})_x + (c_{1,1} + c_{1,3})\mathcal{L} - c_{1,3}\bar{\mathcal{Q}}_2^{(0)} + c_{1,2}\mathcal{J}_x, \tag{B.20}$$

with the next-to-leading order bare Yangian

$$(\bar{\mathcal{Y}}^{(1)})_x = 2(a_{1,5}/a_{0,2})([1||2, x] - [2||x, 1]). \quad (\text{B.21})$$

As with the leading-order case, $\bar{\mathcal{Y}}^{(1)}$ is specified exactly and there are no new constraints on $\mathcal{H}^{(1)}$. This counting is continued to $\mathcal{O}(\lambda^3)$ in table 3.

For completeness we once again give the coefficients specified in (4.1). Comparing (B.15) and (B.20) to (4.1) we get

$$\begin{aligned} \gamma_{\mathcal{Y}, \mathcal{J}}(\lambda) &= c_{0,2} + \lambda(c_{1,2}) + \mathcal{O}(\lambda^2), \\ \gamma_{\mathcal{Y}, \mathcal{L}}(\lambda) &= c_{0,1} + \lambda(c_{1,1} + c_{1,3}) + \mathcal{O}(\lambda^2), \\ \gamma_{\mathcal{Y}, 2}(\lambda) &= \lambda(-c_{1,3}) + \mathcal{O}(\lambda^2). \end{aligned} \quad (\text{B.22})$$

Serre relations

Finally, we have to show that this Yangian satisfies the cubic Serre relation (A.2). The details of this are not very illuminating, so they are omitted. This Yangian we have constructed also commutes with \mathcal{Q} automatically. So given the system formed by requiring one conserved local charge $[\mathcal{H}, \mathcal{Q}] = 0$, we can construct a Yangian which imposes no further constraints on the system but guarantees integrability.

We were only explicit here up to order λ but the higher orders follow in the same way. The other methods are also done in a similar fashion, albeit with the constraints imposed in a different order.

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