TOWARD GAUSS-BONNET-CHERN INEQUALITIES AND ISOPERIMETRIC DEFICITS FOR CONFORMAL METRICS ON $\mathbb{R}^n, n \geq 3$ †

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Abstract. The aim of this paper is to establish the Gauss-Bonnet-Chern integral inequalities and isoperimetric deficit formulas for complete conformal metrics on $\mathbb{R}^n$, $n \geq 3$ with scalar curvature being nonnegative near infinity and Q-curvature being absolutely convergent.

1. Introduction

To begin with, let us agree to some basic conventions. We employ the symbols Δ and ∇ to denote the Laplace operator $\sum_{k=1}^n \partial^2/\partial x_k^2$ and the gradient vector $(\partial/\partial x_1, ..., \partial/\partial x_n)$ over the Euclidean space $\mathbb{R}^n$, $n \geq 2$. For notational convenience we use $X \lesssim Y$ as $X \leq CY$ for a constant $C > 0$. We always assume that u is a smooth real-valued function on $\mathbb{R}^n$, written $u \in C^\infty(\mathbb{R}^n)$, and then it generates a conformal metric $g = e^{2u} g_0$ which is indeed a conformal deformation of the standard Euclidean metric $g_0 = \sum_{k=1}^n dx_k^2$. The volume and surface area elements of the metric g are given by

$$dv_g = e^{nu} d\mathcal{H}^n \quad \text{and} \quad ds_g = e^{(n-1)u} d\mathcal{H}^{n-1}$$

where $\mathcal{H}^k$ stands for the k-dimensional Hausdorff measure. So, the volume and surface area of the open ball $B_r(x)$ and its boundary $\partial B_r(x)$ with radius $r > 0$ and center $x \in \mathbb{R}^n$ have the following values:

$$v_g\big(B_r(x)\big) = \int_{B_r(x)} e^{nu}\, d\mathcal{H}^n \quad \text{and} \quad s_g\big(\partial B_r(x)\big) = \int_{\partial B_r(x)} e^{(n-1)u}\, d\mathcal{H}^{n-1}.$$

More importantly, this metric takes two kinds of nonlinear operators as the simplest ways of describing the curvature of the Riemannian manifold $(\mathbb{R}^n, g)$. One is the scalar curvature (or Ricci scalar) function

$$S_{g,n}(x) = -2(n-1)e^{-2u(x)}\Big(\Delta u(x) + \frac{n-2}{2}|\nabla u(x)|^2\Big).$$

The other is the so-called Q-curvature (or Paneitz curvature) function which, according to Fefferman-Graham [6] (see also Ndiaye [11] and Xu [14]), can be determined by

$$Q_{g,n}(x) = e^{-nu(x)}(-\Delta)^{n/2} u(x),$$

whose even and odd cases $Q_{g,2m}$ and $Q_{g,2m-1}$ are regarded respectively as differential and pseudo-differential operators.

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Note that both the scalar curvature $2^{-1}S_{g,2}$ and the Q-curvature $Q_{g,2}$ are equal to the classical Gaussian curvature K which completely characterizes the curvature of the two-dimensional Riemannian manifold $(\mathbb{R}^2, g)$. So it is quite natural to recall a fundamental inequality of Gauss-Bonnet integral in the theory of complete surfaces of totally finite Gaussian curvature. According to Cohn-Vossen [4] and Finn [7] respectively, we see that

$$\int_{\mathbb{R}^2} |K| dv_g = \int_{\mathbb{R}^2} |(-\Delta u)| d\mathcal{H}^2 < \infty$$

yields the following Gauss-Bonnet integral inequality

$$\int_{\mathbb{R}^2} K\, dv_g = \int_{\mathbb{R}^2} (-\Delta u)\, d\mathcal{H}^2 \le 2\pi \tag{1.1}$$

and Finn's isoperimetric deficit formula

$$1 - \frac{1}{2\pi}\int_{\mathbb{R}^2} K\, dv_g = 1 - \frac{1}{2\pi}\int_{\mathbb{R}^2} (-\Delta u) d\mathcal{H}^2 = \lim_{r\to\infty} \frac{\big(s_g\big(\partial B_r(0)\big)\big)^2}{4\pi v_g\big(B_r(0)\big)}. \tag{1.2}$$

In their 2000 paper [2] (see also its follow-up [3]), Chang, Qing and Yang extends the above results (1.1) and (1.2) to $\mathbb{R}^4$ in terms of the scalar and Q curvatures. More precisely, if $g = e^{2u}g_0$ is complete and its scalar curvature $S_{g,4}$ is nonnegative near infinity, then

$$\int_{\mathbb{R}^4} |Q_{g,4}|\, dv_g = \int_{\mathbb{R}^4} |(-\Delta)^2 u|\, d\mathcal{H}^4 < \infty$$

implies the following Chang-Qing-Yang's integral inequality of Gauss-Bonnet-Chern type

$$\int_{\mathbb{R}^4} Q_{g,4}\, dv_g = \int_{\mathbb{R}^4} (-\Delta)^2 u\, d\mathcal{H}^4 \le 4\pi^2 \tag{1.3}$$

and Chang-Qing-Yang's isoperimetric deficit formula

$$1 - \frac{1}{4\pi^2}\int_{\mathbb{R}^4} Q_{g,4}\, dv_g = 1 - \frac{1}{4\pi^2}\int_{\mathbb{R}^4} (-\Delta)^2 u\, d\mathcal{H}^4 = \lim_{r\to\infty} \frac{\big(s_g\big(\partial B_r(0)\big)\big)^{4/3}}{4(2\pi^2)^{1/3} v_g\big(B_r(0)\big)}. \tag{1.4}$$

In his 2005 paper [5], Fang generalizes (1.3) but not (1.4) to the even-dimensional space $\mathbb{R}^n$. Explicitly speaking, suppose $n \ge 4$ is even, if $g = e^{2u}g_0$ is complete and its scalar curvature $S_{g,n}$ is nonnegative near infinity, then

$$\int_{\mathbb{R}^n} |Q_{g,n}|\, dv_g = \int_{\mathbb{R}^n} |(-\Delta)^{n/2} u|\, d\mathcal{H}^n < \infty$$

yields the following Fang's integral inequality of Gauss-Bonnet-Chern type

$$\int_{\mathbb{R}^n} Q_{g,n}\, dv_g = \int_{\mathbb{R}^n} (-\Delta)^{n/2} u\, d\mathcal{H}^n \le 2^{n-1}(n/2-1)!\pi^{n/2}. \tag{1.5}$$

In our current paper, we establish the odd-dimensional version of (1.5) (covering (1.3)) and the any-dimensional extension of (1.4). Actually, our main assertion is stated in such a way that can cover all Riemannian manifolds $(\mathbb{R}^n, g)$ with $n \ge 3$.

Theorem 1.1. *Given an integer $n \ge 3$ and a function $u \in C^\infty(\mathbb{R}^n)$, let $g = e^{2u}g_0$ be complete with*

$$\liminf_{|x|\to\infty} S_{g,n}(x) \ge 0 \quad \text{and} \quad \int_{\mathbb{R}^n} |Q_{g,n}|\, dv_g < \infty.$$

Then

$$\int_{\mathbb{R}^n} Q_{g,n}\,dv_g = \int_{\mathbb{R}^n} (-\Delta)^{n/2} u\, d\mathcal{H}^n \le 2^{n-1}\Gamma(n/2)\pi^{n/2} \tag{1.6}$$

and

$$1 - \frac{\int_{\mathbb{R}^n} Q_{g,n}\,dv_g}{2^{n-1}\Gamma(n/2)\pi^{n/2}} = 1 - \frac{\int_{\mathbb{R}^n} (-\Delta)^{n/2} u\, d\mathcal{H}^n}{2^{n-1}\Gamma(n/2)\pi^{n/2}} = \lim_{r\to\infty} \frac{\big(s_g\big(\partial B_r(0)\big)\big)^{n/(n-1)}}{(n\omega_n^{1/n})^{n/(n-1)} v_g\big(B_r(0)\big)}, \tag{1.7}$$

where $\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$ *is the Lebesgue measure of the unit ball* $B_1(0)$*.*

Perhaps it is appropriate to make two remarks. The first one is that (1.6) can be used to confirm that [1, Theorem 1.3] has an odd-dimensional analogue – that is – for each odd number $n \ge 3$ there is a dimensional constant $L_n \ge 1$ such that every manifold $(\mathbb{R}^n, g)$ is L_n-biLipschitz equivalent to the background manifold $(\mathbb{R}^n, g_0)$ provided that $u \in C^\infty(\mathbb{R}^n)$ satisfies:

$$u(x) = \text{constant} + \big(2^{n-1}\Gamma(n/2)\pi^{n/2}\big)^{-1} \int_{\mathbb{R}^n} \big(\log|y|/|x-y|\big)(-\Delta)^{n/2} u(y)\, d\mathcal{H}^n(y)$$

and

$$\big(2^{n-1}\Gamma(n/2)\pi^{n/2}\big)^{-1} \int_{\mathbb{R}^n} |(-\Delta)^{n/2} u|\, d\mathcal{H}^n < n2^{-(7+4n)} e^{-4n(n-1)} 3^{-2n} < 1.$$

The second one is that (1.7) (which is unknown until now except $n = 2, 4$) has suggested a geometric meaning of the so called Q-curvature of any $3 \le n$-dimensional manifold $(\mathbb{R}^n, g)$ – see also [12] for Chang's question on the geometric content of Q-curvature as well as Yang's study plan on Q-curvature in odd-dimensions. To deeply understand this suggestion, a dedicated investigation of: (a) The version of (1.7) over a complete $3 \le n$-manifold with only finitely many conformally flat simple ends (extending two/four-dimensional results in [7]/[3] and settling the equality issue for the even-dimensional Gauss-Bonnet-Chern inequality in [5, Theorem 1.1]); (b) The sharp isoperimetric inequality and the comparison principle on the Green's function for the n-Laplace operator $-div(|\nabla u|^{n-2}\nabla u)$ on the manifold $(\mathbb{R}^n, g)$ (generalizing the corresponding two-dimensional results in Huber [8] and Xiao [13]), is worth being carried out in the future.

The proof of Theorem 1.1 is allocated to the forthcoming four sections. Our argument techniques and methods (working for all dimensions bigger than or equal to three) come mainly from harmonic analysis based on the radially symmetric integral estimates and calculations – for example in Proposition 2.1 (i)-(ii)-(iii) (for $\mathbb{R}^n$, $n \ge 3$) of this paper there is no need to solve some induced ordinary differential equations such as ones treated in [2, pp.526-531] (for $\mathbb{R}^4$) and [5, p.478] (for $\mathbb{R}^{2m}$) – this direct approach makes our work be initially like no theirs. Here we want to acknowledge several interesting communications with M. Bonk, H. Fang, R. Graham, J. Li and X. Xu.

2. Proof of (1.6) – Special Case

In this section we provide a proof of (1.6) for the smooth radially symmetric function.

Proposition 2.1. *Let $u \in C^\infty(\mathbb{R}^n)$ be radially symmetric and satisfy the hypotheses of Theorem 1.1. If*

$$v(x) = \frac{1}{2^{n-1}\Gamma(n/2)\pi^{n/2}} \int_{\mathbb{R}^n} \log\left|\frac{y}{x-y}\right| (-\Delta)^{n/2} u(y)\, d\mathcal{H}^n(y),$$

then:
(i)

$$\sup_{0<|x|,|y|<\infty} \frac{1}{\mathcal{H}^{n-1}\big(\partial B_{|x|}(0)\big)} \int_{\partial B_{|x|}(0)} \frac{\big||z|^2-|y|^2\big|}{|z-y|^2} d\mathcal{H}^{n-1}(z) < \infty.$$

(ii) *v is also radially symmetric and enjoys*

$$\lim_{r\to 0} r\frac{dv(r)}{dr} = 0 \quad \textit{and} \quad \lim_{r\to\infty} r\frac{dv(r)}{dr} = -\frac{1}{2^{n-1}\Gamma(n/2)\pi^{n/2}} \int_{\mathbb{R}^n} (-\Delta)^{n/2} u\, d\mathcal{H}^n.$$

(iii) $\limsup_{|x|\to\infty} |x||\nabla v(x)| < \infty$ *and* $\limsup_{|x|\to\infty} |x|^2|\Delta v(x)| < \infty$.
(iv) *In the sense of distribution,*

$$(-\Delta)^{n/2}(-\log|x-y|) = 2^{n-1}\Gamma(n/2)\pi^{n/2}\delta_y(x),$$

where $\delta_y(\cdot)$ is the Dirac measure at y.
(v) *There is a constant c such that $u(x) = v(x) + c$ for all $x \in \mathbb{R}^n$.*
(vi) *(1.6) holds.*

Proof. (i) Given $x, y \in \mathbb{R}^n$, for simplicity we not only assume

$$I(|x|,|y|) = \frac{1}{\mathcal{H}^{n-1}\big(\partial B_{|x|}(0)\big)} \int_{\partial B_{|x|}(0)} \frac{\big||z|^2-|y|^2\big|}{|z-y|^2} d\mathcal{H}^{n-1}(z),$$

but also split $\partial B_{|x|}(0)$ into two disjoint parts P_1 and P_2, where

$$P_1 = \big\{z \in \partial B_{|x|}(0) :\ |x|^2 + |y|^2 \le |z-y|^2\big\}$$

and

$$P_2 = \big\{z \in \partial B_{|x|}(0) :\ |x|^2 + |y|^2 > |z-y|^2\big\}.$$

Due to the structure of P_2, we further write P_2 as the union of countable disjoint sets as follows:

$$P_2 = \bigcup_{k\ge 0} \Big\{z \in \partial B_{|x|}(0) :\ 2^{-k-1} \le \frac{|z-y|}{\sqrt{|x|^2+|y|^2}} < 2^{-k}\Big\}.$$

Based on the spherical coordinate system on P_2 and the law of cosines for the triangle formed by vectors $z \in P_2$, y and $z-y$, we define

$$\phi_k = \arccos\frac{(1-2^{-2k})(|x|^2+|y|^2)}{2|x||y|}, \quad k = 0,1,2,..., \left[\frac{\log\frac{(|x|-|y|)^2}{|x|^2+|y|^2}}{\log\frac{1}{4}}\right].$$

After the above technical treatment, we now need to deal with two cases $n=3$ and $n>3$ respectively.

Case 1: $n=3$. Under this case, we put

$$H(|x|,|y|) = \frac{1}{\mathcal{H}^2\big(\partial B_{|x|}(0)\big)} \int_{\partial B_{|x|}(0)} |z-y|^{-1}\, d\mathcal{H}^2(z)$$

and then prove

$$\sup_{x,y\in\mathbb{R}^3} (|x|+|y|)H(|x|,|y|) < \infty$$

through handling two subcases.

Subcase 1: $|y| \le |x|$. When $|y| \le |x|/2$, we obviously have that $|z-y| \ge |z|-|y| \ge |x|/2$ as $z \in \partial B_{|x|}(0)$ and so that

$$(|x|+|y|)H(|x|,|y|) \lesssim |x|^{-1} \int_{\partial B_{|x|}(0)} |z-y|^{-1}\, d\mathcal{H}^2(z) \lesssim 1.$$

Suppose now $|x|/2 < |y| \le |x|$. Then we use $\partial B_{|x|}(0) = P_1 \cup P_2$ and ϕ_k to estimate

$$\begin{aligned}
&(|x|+|y|)H(|x|,|y|)\\
&\lesssim |x|^{-1}\left(\int_{P_1} |z-y|^{-1}\, d\mathcal{H}^2(z) + \int_{P_2} |z-y|^{-1}\, d\mathcal{H}^2(z)\right)\\
&\lesssim 1 + |x|^{-1}\sum_{k\ge 0}\int_{\{z\in\partial B_{|x|}(0):\ 2^{-k-1}\le\frac{|z-y|}{\sqrt{|x|^2+|y|^2}}<2^{-k}\}} |z-y|^{-1}\, d\mathcal{H}^2(z)\\
&\lesssim 1 + \sum_{k\ge 0} 2^k \int_{\phi_{k+1}}^{\phi_k} \sin\phi\, d\phi\\
&\lesssim 1 + \sum_{k\ge 0} 2^{-k}.
\end{aligned}$$

Subcase 2: $|y| > |x|$. When $|y| \ge 2|x|$, we similarly have that $|z-y| \ge |y|-|z| \ge |y|/2$ as $z \in \partial B_{|x|}(0)$, and so that

$$(|x|+|y|)H(|x|,|y|) \lesssim \frac{|y|}{|x|^2} \int_{\partial B_{|x|}(0)} |z-y|^{-1}\, d\mathcal{H}^2(z) \lesssim 1.$$

If $|x| < |y| < 2|x|$, then

$$\begin{aligned}
&(|x|+|y|)H(|x|,|y|)\\
&\lesssim |y|^{-1}\left(\int_{P_1} |z-y|^{-1}\, d\mathcal{H}^2(z) + \int_{P_2} |z-y|^{-1}\, d\mathcal{H}^2(z)\right)\\
&\lesssim 1 + |y|^{-1}\sum_{k\ge 0}\int_{\{z\in\partial B_{|x|}(0):\ 2^{-k-1}\le\frac{|z-y|}{\sqrt{|x|^2+|y|^2}}<2^{-k}\}} |z-y|^{-1}\, d\mathcal{H}^2(z)\\
&\lesssim 1 + \sum_{k\ge 0} 2^k \int_{\phi_{k+1}}^{\phi_k} \sin\phi\, d\phi\\
&\lesssim 1 + \sum_{k\ge 0} 2^{-k}.
\end{aligned}$$

The previous consideration of two subcases, plus the inequality

$$\big||z|^2-|y|^2\big|/|z-y|^2 \le (|z|+|y|)/|z-y|,$$

leads to

$$\sup_{0<|x|,|y|<\infty} I(|x|,|y|) \lesssim \sup_{x,y\in\mathbb{R}^3} (|x|+|y|)H(|x|,|y|) < \infty.$$

Case 2: $n > 3$. Under this case, we set

$$J(|x|,|y|) = \frac{1}{\mathcal{H}^{n-1}\big(\partial B_{|x|}(0)\big)} \int_{\partial B_{|x|}(0)} |z-y|^{-2} d\mathcal{H}^{n-1}(z)$$

and are about to show

$$\sup_{x,y\in\mathbb{R}^n}(|x|^2+|y|^2)J(|x|,|y|)<\infty$$

via handling two more subcases.

Subcase 1: $|y|\le|x|$. When $|y|\le|x|/2$, we have that $z\in\partial B_{|x|}(0)$ implies $|z-y|\ge|z|-|y|\ge|x|/2$ and consequently,

$$(|x|^2+|y|^2)J(|x|,|y|)\lesssim|x|^2J(|x|,|y|)\le|x|^{1-n}\int_{\partial B_{|x|}(0)}\frac{|x|^2}{|z-y|^2}d\mathcal{H}^{n-1}(z)\lesssim 1.$$

When $|x|/2<|y|\le|x|$, we continue using the spherical coordinate system to produce

$$\begin{aligned}
&(|x|^2+|y|^2)J(|x|,|y|)\\
&\lesssim|x|^2J(|x|,|y|)\\
&\lesssim|x|^{1-n}\left(\int_{P_1}\frac{|x|^2}{|z-y|^2}\,d\mathcal{H}^{n-1}(z)+\int_{P_2}\frac{|x|^2}{|z-y|^2}\,d\mathcal{H}^{n-1}(z)\right)\\
&\lesssim 1+|x|^{1-n}\int_{P_2}\frac{|x|^2}{|z-y|^2}d\mathcal{H}^{n-1}(z)\\
&\lesssim 1+|x|^{3-n}\sum_{k\ge0}\int_{\{z\in\partial B_{|x|}(0):\ 2^{-k-1}\le\frac{|z-y|}{\sqrt{|x|^2+|y|^2}}<2^{-k}\}}|z-y|^{-2}\,d\mathcal{H}^{n-1}(z)\\
&\lesssim 1+|x|^{3-n}\sum_{k\ge0}\int_{\phi_{k+1}}^{\phi_k}(|x|^2+|y|^2)^{-1}2^{2k}|x|^{n-1}\sin^{n-2}\phi\,d\phi\\
&\lesssim 1+\sum_{k\ge0}2^{2k}\int_{\phi_{k+1}}^{\phi_k}\sin^{n-2}\phi\,d\phi\\
&\lesssim 1+\sum_{k\ge0}2^{2k}\int_{\phi_{k+1}}^{\phi_k}(1-\cos^2\phi)^{(n-3)/2}\,d(-\cos\phi)\\
&\lesssim 1+\sum_{k\ge0}2^{2k}\int_{\frac{(1-2^{-2k})(|x|^2+|y|^2)}{2|x||y|}}^{\frac{(1-2^{-2(k+1)})(|x|^2+|y|^2)}{2|x||y|}}(1-t^2)^{(n-3)/2}dt\\
&\lesssim 1+\sum_{k\ge0}2^{2k}\left(1-\Big(\frac{(1-2^{-2k})(|x|^2+|y|^2)}{2|x||y|}\Big)^2\right)^{(n-3)/2}\Big(\frac{2^{-2k}(|x|^2+|y|^2)}{|x||y|}\Big)\\
&\lesssim 1+\sum_{k=0}^{\infty}2^{-k(n-3)}.
\end{aligned}$$

Subcase 2: $|y|>|x|$. When $|y|\ge2|x|$, we clearly see that $z\in\partial B_{|x|}(0)$ implies $|z-y|\ge|y|-|z|\ge|y|/2$ and consequently,

$$(|x|^2+|y|^2)J(|x|,|y|)\lesssim|y|^2J(|x|,|y|)\lesssim|x|^{1-n}\int_{\partial B_{|x|}(0)}\frac{|y|^2}{|z-y|^2}d\mathcal{H}^{n-1}(z)\lesssim 1.$$

When $|x|<|y|<2|x|$, we analogously derive

$$
\begin{aligned}
&(|x|^2+|y|^2)J(|x|,|y|)\\
&\lesssim |y|^2 J(|x|,|y|)\\
&\lesssim |x|^{1-n}\left(\int_{P_1}\frac{|y|^2}{|z-y|^2}d\mathcal{H}^{n-1}(z)+\int_{P_2}\frac{|y|^2}{|z-y|^2}d\mathcal{H}^{n-1}(z)\right)\\
&\lesssim 1+|x|^{1-n}\sum_{k\geq 0}\int_{\{z\in\partial B_{|x|}(0):\ 2^{-(k+1)}\leq\frac{|z-y|}{\sqrt{|x|^2+|y|^2}}<2^{-k}\}}\frac{|y|^2}{|z-y|^2}\,d\mathcal{H}^{n-1}(z)\\
&\lesssim 1+\sum_{k\geq 0}\frac{2^{2k}|y|^2}{|x|^2+|y|^2}\int_{\phi_{k+1}}^{\phi_k}\sin^{n-2}\phi\,d\phi\\
&\lesssim 1+\sum_{k\geq 0}2^{-k(n-3)}.
\end{aligned}
$$

Taking the foregoing inequalities for $J(|x|,|y|)$ into account, we get the desired finiteness:

$$\sup_{0<|x|,|y|<\infty} I(|x|,|y|)\leq \sup_{x,y\in\mathbb{R}^n}(|x|^2+|y|^2)J(|x|,|y|)<\infty.$$

(ii) The radial symmetry of v follows easily from the assumption that u is radially symmetric. Using $|x|=r>0$ we calculate

$$\frac{d}{dr}\log\Big|\frac{y}{x-y}\Big|=\frac{-\frac{d}{dr}|x-y|^2}{2|x-y|^2}=\frac{|y|^2-|x|^2-|x-y|^2}{2|x||x-y|^2}$$

and then employ the radial symmetry of u to obtain

$$
\begin{aligned}
&-2^n\Gamma(n/2)\pi^{n/2}\Big(r\frac{dv(r)}{dr}\Big)\\
&=\int_{\mathbb{R}^n}\Big(\frac{|x|^2-|y|^2+|x-y|^2}{|x-y|^2}\Big)(-\Delta)^{n/2}u(y)\,d\mathcal{H}^n(y)\\
&=\int_{\mathbb{R}^n}\left(1+\frac{\int_{\partial B_{|x|}(0)}\frac{|z|^2-|y|^2}{|z-y|^2}\,d\mathcal{H}^{n-1}(z)}{\mathcal{H}^{n-1}\big(\partial B_{|x|}(0)\big)}\right)(-\Delta)^{n/2}u(y)\,d\mathcal{H}^n(y).
\end{aligned}
$$

Because both (i) and

$$\int_{\mathbb{R}^n}|(-\Delta)^{n/2}u|d\mathcal{H}^n=\int_{\mathbb{R}^n}|Q_{g,n}|dv_g<\infty$$

guarantee

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\left|\left(1+\frac{\int_{\partial B_{|x|}(0)}\frac{|z|^2-|y|^2}{|z-y|^2}\,d\mathcal{H}^{n-1}(z)}{\mathcal{H}^{n-1}\big(\partial B_{|x|}(0)\big)}\right)(-\Delta)^{n/2}u(y)\right|\,d\mathcal{H}^n(y)\\
&\lesssim\int_{\mathbb{R}^n}\Big(1+\sup_{0<|x|,|y|<\infty}I(|x|,|y|)\Big)\big|(-\Delta)^{n/2}u(y)\big|\,d\mathcal{H}^n(y)\\
&\lesssim\int_{\mathbb{R}^n}\big|(-\Delta)^{n/2}u\big|d\mathcal{H}^n<\infty,
\end{aligned}
$$

we apply the dominated convergence theorem to derive

$$\begin{aligned}
&\lim_{r\to 0} r\frac{dv(r)}{dr}\\
&= -\frac{1}{2^n\Gamma(n/2)\pi^{n/2}}\int_{\mathbb{R}^n}\lim_{|x|\to 0}\Big(\frac{|x|^2-|y|^2+|x-y|^2}{|x-y|^2}\Big)(-\Delta)^{n/2}u(y)\,d\mathcal{H}^n(y)\\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{r\to \infty} r\frac{dv(r)}{dr}\\
&= -\frac{1}{2^n\Gamma(n/2)\pi^{n/2}}\int_{\mathbb{R}^n}\lim_{|x|\to \infty}\Big(\frac{|x|^2-|y|^2+|x-y|^2}{|x-y|^2}\Big)(-\Delta)^{n/2}u(y)\,d\mathcal{H}^n(y)\\
&= -\frac{1}{2^{n-1}\Gamma(n/2)\pi^{n/2}}\int_{\mathbb{R}^n}(-\Delta)^{n/2}u\,d\mathcal{H}^n,
\end{aligned}$$

as required.

(iii) The first finiteness follows from (ii) right away since ∇ can rewritten as $(d/dr, r^{-1}\nabla_\sigma)$ under the spherical coordinate system where ∇_σ is the gradient operator on the unit sphere $\partial B_1(0)$. To verify the second finiteness, we observe (via an easy computation)

$$\Delta v(x) = \frac{n-2}{2^{n-1}\Gamma(n/2)\pi^{n/2}}\int_{\mathbb{R}^n}|x-y|^{-2}(-\Delta)^{n/2}u(y)\,d\mathcal{H}^n(y),$$

and then handle two cases.

Case 1: $n=3$. From the hypotheses and the spherical coordinate system it follows that

$$\begin{aligned}
&|x|^2|\Delta v(x)|\\
&\lesssim \left(\int_{\{y\in\mathbb{R}^3:\ |y-x|\ge|x|/2\}}+\int_{\{y\in\mathbb{R}^3:\ |y-x|<|x|/2\}}\right)\Big|\frac{x}{x-y}\Big|^2|(-\Delta)^{3/2}u(y)|\,d\mathcal{H}^3(y)\\
&\lesssim \int_{\mathbb{R}^3}|(-\Delta)^{3/2}u|\,d\mathcal{H}^3+\int_{\{y\in\mathbb{R}^3:\ |y|\ge|x|/2\}}\Big|\frac{x}{x-y}\Big|^2|(-\Delta)^{3/2}u(y)|\,d\mathcal{H}^3(y)\\
&\lesssim \int_{\mathbb{R}^3}|(-\Delta)^{3/2}u|\,d\mathcal{H}^3+\int_{\{y\in\mathbb{R}^3:\ 2|x|\ge|y|\ge|x|/2\}}\Big|\frac{x}{x-y}\Big|^2|(-\Delta)^{3/2}u(y)|\,d\mathcal{H}^3(y).
\end{aligned}$$

Furthermore, via the spherical coordinate system we deduce

$$\begin{aligned}
&|x|^2\int_{\{y\in\mathbb{R}^3:\ |x|\ge|y|\ge|x|/2\}}|x-y|^{-2}|(-\Delta)^{3/2}u(y)|\,d\mathcal{H}^3(y)\\
&\lesssim |x|^2\int_{|x|/2}^{|x|}|(-\Delta)^{3/2}u(t)|t^2\left(\int_0^{\pi/2}\frac{\sin\phi}{|x|^2-2|x|t\cos\phi+t^2}\,d\phi\right)dt\\
&\lesssim |x|^2\int_{|x|/2}^{|x|}|(-\Delta)^{3/2}u(t)|t^2\left(\int_0^{1}\frac{ds}{|x|^2-2|x|ts+t^2}\right)dt\\
&\lesssim \int_{\mathbb{R}^3}|(-\Delta)^{3/2}u|d\mathcal{H}^3+\int_{|x|/2}^{|x|}|(-\Delta)^{3/2}u(t)|t^2\left(\log\frac{|x|^2+t^2}{(|x|-t)^2}\right)dt.
\end{aligned}$$

Suppose now $U(t)=\int_0^t |(-\Delta)^{3/2}u(s)|s^2 ds$ for $t>0$. Integration by part and change of variables give

$$\begin{aligned}
&\int_{|x|/2}^{|x|} |(-\Delta)^{3/2}u(t)|t^2 \left(\log\frac{|x|}{|x|-t}\right)\, dt\\
&= -\int_0^{1/2} (\log s)\, d\big(U(|x|)-U(|x|-|x|s)\big)\\
&= (\log 2)\big(U(|x|)-U(|x|/2)\big) + \int_0^{1/2} \big(U(|x|)-U(|x|-|x|s)\big)s^{-1}ds.
\end{aligned}$$

Note that

$$U(\infty)=\lim_{t\to\infty}U(t)\lesssim \int_{\mathbb{R}^3}|(-\Delta)^{3/2}u|\, d\mathcal{H}^3<\infty$$

implies $\lim_{|x|\to\infty}\big(U(|x|)-U(|x|/2)\big)=0$. So Fatou's lemma yields

$$\begin{aligned}
0&\le \limsup_{|x|\to\infty}\int_0^{1/2}\big(U(|x|)-U(|x|-|x|s)\big)s^{-1}ds\\
&\le \int_0^{1/2}\limsup_{|x|\to\infty}\big(U(|x|)-U(|x|-|x|s)\big)s^{-1}ds=0.
\end{aligned}$$

As a result, we get

$$\limsup_{|x|\to\infty}|x|^2\int_{|x|\ge|y|\ge|x|/2}|x-y|^{-2}|(-\Delta)^{3/2}u(y)|\, d\mathcal{H}^3(y)=0.$$

In a similar manner, we can also obtain

$$\limsup_{|x|\to\infty}|x|^2\int_{\{y\in\mathbb{R}^3:\ |x|\le|y|\le2|x|\}}|x-y|^{-2}|(-\Delta)^{3/2}u(y)|\, d\mathcal{H}^3(y)=0,$$

thereby reaching

$$\limsup_{|x|\to\infty}|x|^2|\Delta v(x)|\lesssim\int_{\mathbb{R}^3}|(-\Delta)^{3/2}u|\, d\mathcal{H}^3<\infty.$$

Case 2: $n>3$. Since $(-\Delta)^{n/2}u$ is radially symmetric, it follows from the estimates on $J(|x|,|y|)$ that

$$\begin{aligned}
&|x|^2|\Delta v(x)|\\
&\lesssim |x|^2\int_{\mathbb{R}^n}\left(\frac{1}{\mathcal{H}^{n-1}\big(\partial B_{|x|}(0)\big)}\int_{\partial B_{|x|}(0)}\frac{d\mathcal{H}^{n-1}(z)}{|z-y|^2}\right)|(-\Delta)^{n/2}u(y)|d\mathcal{H}^n(y)\\
&\lesssim \int_{\mathbb{R}^n}|x|^2J(|x|,|y|)|(-\Delta)^{n/2}u(y)|d\mathcal{H}^n(y)\\
&\lesssim \int_{\mathbb{R}^n}|(-\Delta)^{n/2}u|\, d\mathcal{H}^n,
\end{aligned}$$

and so that

$$\limsup_{|x|\to\infty}|x|^2|\Delta v(x)|\lesssim\int_{\mathbb{R}^n}|(-\Delta)^{n/2}u|\, d\mathcal{H}^n<\infty.$$

(iv) In the even case this result may be found in [1]. A proof of this case and odd one is provided below. Of course, it suffices to verify the formula for $y=0$.

Rewriting Δ in terms of the spherical coordinate: $x = r\sigma$; $r > 0$, $\sigma \in \partial B_1(0)$, we have

$$\Delta = \frac{d^2}{dr^2} + \Big(\frac{n-1}{r}\Big)\frac{d}{dr} + \frac{\Delta_\sigma}{r^2},$$

where Δ_σ is the Laplacian on $\partial B_1(0)$. Since $\log|x|$ is radially symmetric, if $n = 2m$ is an even number, then a simple calculation with the basic equation (see also [10, p. 156, (1)])

$$(-\Delta)|x|^{2-n} = \frac{2(n-2)\pi^{n/2}}{\Gamma(n/2)}\delta_0(x)$$

gives

$$\begin{aligned}&(-\Delta)^{n/2}(-\log|x|)\\&= 2\cdot 4\cdots 2(m-2)(2-n)(4-n)\cdots(2m-2-n)(-\Delta)|x|^{2-n}\\&= 2^{n-1}\Gamma(n/2)\pi^{n/2}\delta_0(x).\end{aligned}$$

In the case that $n = 2m-1$ is an odd number, a similar computation yields

$$(-\Delta)^{m-1}(-\log|x|) = 2\cdot 4\cdots 2(m-2)(2-n)(4-n)\cdots(2m-2-n)|x|^{-2(m-1)}.$$

According to [9, p. 128, (2.10.1) & (2.10.8)] and [10, p. 132, (3)], we find

$$\begin{aligned}&(-\Delta)^{-1/2}|x|^{-2(m-1)}\\&= \frac{\Gamma(n/2-1/2)}{2\pi^{(n+1)/2}}\int_{\mathbb{R}^n}|x-y|^{1-n}|y|^{2-2m}d\mathcal{H}^n(y)\\&= \Big(\frac{\sqrt{\pi}\Gamma(n/2-1)}{2\Gamma(n/2-1/2)}\Big)|x|^{2-n},\end{aligned}$$

whence obtaining (via the above-mentioned basic equation)

$$\begin{aligned}&(-\Delta)^{n/2}(-\log|x|)\\&= (-\Delta)(-\Delta)^{-1/2}(-\Delta)^{m-1}(-\log|x|)\\&= 2\cdot 4\cdots(n-3)(2-n)(4-n)\cdots(-1)\Big(\frac{\sqrt{\pi}\Gamma(n/2-1)}{2\Gamma(n/2-1/2)}\Big)(-\Delta)|x|^{2-n}\\&= 2^{n-1}\Gamma(n/2)\pi^{n/2}\delta_0(x).\end{aligned}$$

(v) From (iv) we see immediately that $(-\Delta)^{n/2}v = (-\Delta)^{n/2}u$. To further get a constant c such that $u = v + c$, we consider two situations.

Situation 1: $n = 2m-1$ is an odd integer. Then $(-\Delta)^{n/2} = (-\Delta)^{1/2}(-\Delta)^{m-1}$ and hence $(-\Delta)^{m-1}(v-u) = 0$. Since $u - v$ is radially symmetric, we are required to seek the radially symmetric solutions to $(-\Delta)^{m-1}w = 0$. Under the spherical coordinate system the last equation becomes a linear ordinary differential equation (in the radius $r = |x|$) of order $2(m-1)$. It is plain to check that $2(m-1)$ functions $1, \log r, r^{\pm 2}, ..., r^{\pm(m-2)}$ satisfy the equation but also are linearly independent. Thus there are $2(m-1)$ constants $c_0, c_1, c_{\pm 2}, ..., c_{\pm(m-2)}$ such that

$$v - u = c_0 + c_1\log r + \sum_{k=1}^{m-2}(c_{2k}r^{2k} + c_{-2k}r^{-2k}).$$

Thanks to the smoothness of u and the first limit established in (ii), we find $\lim_{r\to 0} rd(v-u)(r)/dr = 0$ and consequently $c_1 = 0$ as well as $c_{-2k} = 0$ for

$k=1,...,m-2$. On the other hand, suppose N is the largest integer amongst $\{1,...,m-2\}$ such that c_{2N} is nonzero. Then

$$v-u=c_0+\sum_{k=1}^{N}c_{2k}r^{2k},$$

and hence according to (iii) we have

$$\limsup_{|x|\to\infty}\Big(|x|^2\Delta u(x)+(n/2-1)\big(|x||\nabla u(x)|\big)^2\Big)=(n/2-1)c_{2N}^2\limsup_{r\to\infty}r^{4N}=\infty,$$

But, nevertheless the hypothesis

$$0\le\liminf_{|x|\to\infty}S_{g,n}(x)=-2(n-1)\limsup_{|x|\to\infty}e^{-2u}\Big(\Delta u+(n/2-1)|\nabla u|^2\Big)$$

amounts to

$$\limsup_{|x|\to\infty}e^{-2u}\Big(\Delta u+(n/2-1)|\nabla u|^2\Big)\le 0.$$

With the above analysis, we reach a contradiction:

$$\infty=\limsup_{|x|\to\infty}|x|^2\Big(\Delta u+(n/2-1)|\nabla u|^2\Big)\le 0.$$

Therefore $c_{2k}=0$ for all $k=1,...,m-2$. Consequently, $u=v-c_0$.

Situation 2: $n=2m$ is an even integer. Then

$$(-\Delta)^{n/2}(v-u)=(-\Delta)^m(v-u)=0,$$

and hence the previous argument for $n=2m-1$ can be employed to deduce the result; see also [5].

(vi) Using (v) and the second limit in (ii) we obtain

$$\lim_{r\to\infty}r\frac{du(r)}{dr}=\lim_{r\to\infty}r\frac{dv(r)}{dr}=-\frac{1}{2^{n-1}\Gamma(n/2)\pi^{n/2}}\int_{\mathbb{R}^n}(-\Delta)^{n/2}u\,d\mathcal{H}^n,$$

whence having

$$\exp\big(u(r)\big)=r^{-\frac{1}{2^{n-1}\Gamma(n/2)\pi^{n/2}}\int_{\mathbb{R}^n}(-\Delta)^{n/2}u\,d\mathcal{H}^n+o(1)}\quad\text{as}\quad r\to\infty.$$

This last assertion, plus the hypothesis that $g=e^{2u}g_0$ is complete, ensures

$$-\frac{1}{2^{n-1}\Gamma(n/2)\pi^{n/2}}\int_{\mathbb{R}^n}(-\Delta)^{n/2}u\,d\mathcal{H}^n\ge -1,$$

thereby implying (1.6).

□

3. Proof of (1.6) – General Case

In this section we prove (1.6) through the radial symmetrization and Proposition 2.1. Although our argument ideas are similar to ones explored in [2] and [5], for the paper's completeness and the reader's convenience we feel that it is worth detailing the key steps of the proof.

Proposition 3.1. *Let $u\in C^\infty(\mathbb{R}^n)$ satisfy the hypotheses of Theorem 1.1. If*

$$\bar u(x)=\frac{1}{\mathcal{H}^{n-1}(\partial B_{|x|}(0))}\int_{\partial B_{|x|}(0)}u\,d\mathcal{H}^{n-1},$$

then:

(i) *There is a constant c such that*

$$u(x)=c+\frac{1}{2^{n-1}\Gamma(n/2)\pi^{n/2}}\int_{\mathbb{R}^n}\log\Big|\frac{y}{x-y}\Big|(-\Delta)^{n/2}u(y)\,d\mathcal{H}^n(y).$$

(ii) *For any $p>0$ one has*

$$\lim_{|x|\to\infty}e^{-p\bar{u}(x)}\left(\frac{1}{\mathcal{H}^{n-1}\big(\partial B_{|x|}(0)\big)}\int_{\partial B_{|x|}(0)}e^{pu}\,d\mathcal{H}^{n-1}\right)=1.$$

(iii) *$\bar{g}=e^{2\bar{u}}g_0$ is not only complete but also satisfies*

$$\liminf_{|x|\to\infty}S_{\bar{g},n}(x)\geq 0\quad\textit{and}\quad\int_{\mathbb{R}^n}|Q_{\bar{g},n}|dv_{\bar{g}}<\infty.$$

(iv) *(1.6) holds.*

Proof. (i) Continuing the use of v defined in Proposition 2.1, we get from Proposition 2.1 (iv) that $(-\Delta)^{n/2}(u-v)=0$. To reach the desired result, we fix a point $x_0\in\mathbb{R}^n$ and consider the radially symmetric versions of u, v and $u-v$ about x_0 as follows:

$$\begin{cases}\bar{u}(x;x_0)=\frac{\int_{\partial B_{|x-x_0|}(x_0)}u\,d\mathcal{H}^{n-1}}{\mathcal{H}^{n-1}\big(\partial B_{|x-x_0|}(x_0)\big)},\\ \bar{v}(x;x_0)=\frac{\int_{\partial B_{|x-x_0|}(x_0)}v\,d\mathcal{H}^{n-1}}{\mathcal{H}^{n-1}\big(\partial B_{|x-x_0|}(x_0)\big)},\\ \overline{u-v}(x;x_0)=\frac{\int_{\partial B_{|x-x_0|}(x_0)}(u-v)\,d\mathcal{H}^{n-1}}{\mathcal{H}^{n-1}\big(\partial B_{|x-x_0|}(x_0)\big)}.\end{cases}$$

Owing to

$$\begin{aligned}v(x)=&\frac{1}{2^{n-1}\Gamma(n/2)\pi^{n/2}}\int_{\mathbb{R}^n}\log\Big|\frac{y-x_0}{x-y}\Big|(-\Delta)^{n/2}u(y)\,d\mathcal{H}^n(y)\\ &+\frac{1}{2^{n-1}\Gamma(n/2)\pi^{n/2}}\int_{\mathbb{R}^n}\log\Big|\frac{y}{y-x_0}\Big|(-\Delta)^{n/2}u(y)\,d\mathcal{H}^n(y),\end{aligned}$$

we see from the proof of the forthcoming (iii) that the conformal metric $g_{x_0}=e^{2\bar{u}(\cdot;x_0)}g_0$ ensures

$$\liminf_{|x|\to\infty}S_{g_{x_0},n}(x)\geq 0\quad\text{and}\quad\int_{\mathbb{R}^n}|Q_{g_{x_0},n}|dv_{g_{x_0}}<\infty.$$

So, by Proposition 2.1 (iv) and (v) we find that $\overline{u-v}(x;x_0)$ equals a constant – this especially derives $\Delta(u-v)(x_0)=\Delta(u-v)(x_0;x_0)=0$. Since x_0 is arbitrarily chosen, one has $\Delta(u-v)=0$, i.e., $u-v$ is a harmonic function on $\mathbb{R}^n$ and consequently, $\partial(u-v)/\partial x_k$ (for each $k=1,...,n$) is harmonic. A combined application of the mean-value-theorem, Cauchy-Schwarz's inequality and the representation of v yields

$$\begin{aligned}&\Big|\frac{\partial(u-v)}{\partial x_k}(x_0)\Big|^2\\&=\left|\frac{1}{\mathcal{H}^{n-1}\big(\partial B_r(x_0)\big)}\int_{\partial B_r(x_0)}\frac{\partial(u-v)}{\partial x_k}\,d\mathcal{H}^{n-1}\right|^2\\&\le\left|\frac{1}{\mathcal{H}^{n-1}\big(\partial B_r(x_0)\big)}\int_{\partial B_r(x_0)}|\nabla(u-v)|\,d\mathcal{H}^{n-1}\right|^2\\&\lesssim\frac{1}{\mathcal{H}^{n-1}\big(\partial B_r(x_0)\big)}\int_{\partial B_r(x_0)}(|\nabla u|^2+|\nabla v|^2)\,d\mathcal{H}^{n-1}.\end{aligned}$$

Now, the representation of v, the Cauchy-Schwarz inequality, Fubini's theorem and the proof of Proposition 2.1 (iii) produce

$$\begin{aligned}&\limsup_{r\to\infty}r^2\left(\frac{1}{\mathcal{H}^{n-1}\big(\partial B_r(x_0)\big)}\int_{\partial B_r(x_0)}|\nabla v|^2\,d\mathcal{H}^{n-1}\right)\\&\lesssim\limsup_{r\to\infty}r^2\left(\frac{\int_{\partial B_r(x_0)}\Big(\int_{\mathbb{R}^n}\frac{|(-\Delta)^{n/2}u(y)|}{|z-y|^2}\,d\mathcal{H}^n(y)\Big)\,d\mathcal{H}^{n-1}(z)}{\mathcal{H}^{n-1}\big(\partial B_r(x_0)\big)\Big(\int_{\mathbb{R}^n}|(-\Delta)^{n/2}u|\,d\mathcal{H}^n\Big)^{-1}}\right)\\&<\infty.\end{aligned}$$

In the meantime, the formula of $S_{g,n}$, $\liminf_{|z|\to\infty}S_{g,n}(z)\ge 0$ and $\Delta u=\Delta v$ ensure that if $r\to\infty$ then

$$\begin{aligned}&\frac{1}{\mathcal{H}^{n-1}\big(\partial B_r(x_0)\big)}\int_{\partial B_r(x_0)}|\nabla u|^2\,d\mathcal{H}^{n-1}\\&=\Big(\frac{2}{2-n}\Big)\frac{1}{\mathcal{H}^{n-1}\big(\partial B_r(x_0)\big)}\int_{\partial B_r(x_0)}\Big(\Delta u+\frac{e^{2u}S_{g,n}}{2(n-1)}\Big)\,d\mathcal{H}^{n-1}\\&\le\Big(\frac{2}{2-n}\Big)\frac{1}{\mathcal{H}^{n-1}\big(\partial B_r(x_0)\big)}\int_{\partial B_r(x_0)}\Delta v\,d\mathcal{H}^{n-1}\\&=-\frac{(2^{n-2}\Gamma(n/2)\pi^{n/2})^{-1}}{\mathcal{H}^{n-1}\big(\partial B_r(x_0)\big)}\int_{\partial B_r(x_0)}\left(\int_{\mathbb{R}^n}\frac{(-\Delta)^{n/2}u(y)}{|z-y|^2}\,d\mathcal{H}^n(y)\right)\,d\mathcal{H}^{n-1}(z),\end{aligned}$$

and hence by Fubini's theorem and the proof of Proposition 2.1 (iii),

$$\limsup_{r\to\infty}r^2\left(\frac{1}{\mathcal{H}^{n-1}\big(\partial B_r(x_0)\big)}\int_{\partial B_r(x_0)}|\nabla u|^2\,d\mathcal{H}^{n-1}\right)<\infty.$$

Therefore, $\limsup_{r\to\infty}r^2\Big|\frac{\partial(u-v)}{\partial x_k}(x_0)\Big|<\infty$. This forces that $u-v$ is a constant.

(ii) The argument comes from a non-essential adaption of the proof of [2, Lemma 3.2]. According to the just-established (i), we write $u=u_1+u_2$ where

$$u_1(x)=c+\frac{1}{2^{n-1}\Gamma(n/2)\pi^{n/2}}\int_{B_{|x|/2}(0)}\log\Big|\frac{y}{x-y}\Big|(-\Delta)^{n/2}u(y)\,d\mathcal{H}^n(y)$$

and

$$u_2(x)=\frac{1}{2^{n-1}\Gamma(n/2)\pi^{n/2}}\int_{\mathbb{R}^n\setminus B_{|x|/2}(0)}\log\Big|\frac{y}{x-y}\Big|(-\Delta)^{n/2}u(y)\,d\mathcal{H}^n(y).$$

If u_1 is further split into two pieces

$$u_{11}(x)=\frac{1}{2^{n-1}\Gamma(n/2)\pi^{n/2}}\int_{B_{|x|/2}(0)}\log\left|\frac{y}{x}\right|(-\Delta)^{n/2}u(y)\,d\mathcal{H}^n(y)$$

and

$$u_{12}(x)=\frac{1}{2^{n-1}\Gamma(n/2)\pi^{n/2}}\int_{B_{|x|/2}(0)}\log\left|\frac{x}{x-y}\right|(-\Delta)^{n/2}u(y)\,d\mathcal{H}^n(y),$$

then $u_1=u_{11}+u_{12}$, $u_{11}(x)=u_{11}(|x|)$, and $\lim_{|x|\to\infty}u_{12}(x)=0$ – this is because

$$u_{12}(x)\lesssim\log(1-\epsilon)^{-1}+\int_{B_{|x|/2}(0)\setminus B_{\epsilon|x|}(0)}\left|(-\Delta)^{n/2}u\right|d\mathcal{H}^n\to 0$$

when $\epsilon\to 0$ is taken so that $\epsilon|x|\to\infty$ as $|x|\to\infty$. As a result, we find

$$\begin{aligned}&\frac{p}{\mathcal{H}^{n-1}\big(\partial B_{|x|}(0)\big)}\int_{\partial B_{|x|}(0)}(u-u_2)\,d\mathcal{H}^{n-1}\\ &=\log\Big(\frac{1}{\mathcal{H}^{n-1}\big(\partial B_{|x|}(0)\big)}\int_{\partial B_{|x|}(0)}\exp\big(p(u-u_2)\big)\,d\mathcal{H}^{n-1}\Big)+o(1).\end{aligned}$$

On the one hand, we can make the following estimates for any $r\in(0,\infty)$ and suitably small $\theta\in(0,1/2)$:

$$\begin{aligned}&\left|\frac{1}{\mathcal{H}^{n-1}\big(\partial B_r(0)\big)}\int_{\partial B_r(0)}u_2\,d\mathcal{H}^{n-1}\right|\\ &\lesssim\left|\int_{\mathbb{R}^n\setminus B_{r/2}(0)}\left(r^{1-n}\int_{\partial B_r(0)}\log\left|\frac{y}{x-y}\right|d\mathcal{H}^{n-1}(x)\right)(-\Delta)^{n/2}u(y)\,d\mathcal{H}^n(y)\right|\\ &\lesssim\int_{\mathbb{R}^n\setminus B_{r/2}(0)}\left(r^{1-n}\int_{\partial B_r(0)}\left|\log\left|\frac{y}{x-y}\right|\right|d\mathcal{H}^{n-1}(x)\right)\left|(-\Delta)^{n/2}u(y)\right|d\mathcal{H}^n(y)\\ &\lesssim\int_{\mathbb{R}^n\setminus B_{r/2}(0)}r^{1-n}\left(\int_{\partial B_r(0)\setminus\{x\in\mathbb{R}^n:\ |x-y|\le\theta|y|\}}+\int_{\partial B_r(0)\cap\{x\in\mathbb{R}^n:\ |x-y|\le\theta|y|\}}\right)\\ &\quad\left|\log\left|\frac{y}{x-y}\right|\right|d\mathcal{H}^{n-1}\left|(-\Delta)^{n/2}u(y)\right|d\mathcal{H}^n(y)\\ &\lesssim\int_{\mathbb{R}^n\setminus B_{r/2}(0)}\left|(-\Delta)^{n/2}u\right|d\mathcal{H}^n\to 0\quad\text{as}\quad r\to\infty.\end{aligned}$$

On the other hand, we have that if $|r\sigma-y|\le|y|/3$, $|y|\ge r/2$ and $\sigma\in\partial B_1(0)$ then

$$\left|\log\left|\frac{y}{r\sigma-y}\right|\right|\le\log\frac{3}{2}+\left|\log\left|\sigma-\frac{y}{r}\right|\right|,$$

and consequently, if $E_t=\{\sigma\in\partial B_1(0):\ |u_2(r\sigma)|>t\}$ for $t>0$ then

$$
\begin{aligned}
&t\mathcal{H}^{n-1}(E_t)\\
&\le \int_{E_t} |u_2|\, d\mathcal{H}^{n-1}\\
&\lesssim \int_{\mathbb{R}^n\setminus B_{r/2}(0)} \Big(\int_{E_t} \Big|\log\Big|\frac{y}{r\sigma-y}\Big|\Big|\, d\mathcal{H}^{n-1}\Big) \big|(-\Delta)^{n/2}u(y)\big|\, d\mathcal{H}^n(y)\\
&\lesssim \int_{\mathbb{R}^n\setminus B_{r/2}(0)} \Big(\int_{E_t\setminus\{\sigma\in\partial B_1(0):\ |r\sigma-y|\le|y|/3\}} + \int_{E_t\cap\{\sigma\in\partial B_1(0):\ |r\sigma-y|\le|y|/3\}}\Big)\\
&\quad \Big|\log\Big|\frac{y}{r\sigma-y}\Big|\Big|\, d\mathcal{H}^{n-1}\big|(-\Delta)^{n/2}u(y)\big|\, d\mathcal{H}^n(y)\\
&\lesssim \mathcal{H}^{n-1}(E_t)\Big(\int_{\mathbb{R}^n\setminus B_{r/2}(0)} |(-\Delta)^{n/2}u|\, d\mathcal{H}^n\Big)\Big(1-\log\big(\mathcal{H}^{n-1}(E_t)\big)\Big),
\end{aligned}
$$

and hence

$$
\mathcal{H}^{n-1}(E_t) \lesssim \exp\Big(-\frac{t}{o(1)}\Big) \quad \text{as} \quad r\to\infty.
$$

Now, the layer-cake representation theorem yields

$$
\begin{aligned}
&\left|\frac{1}{\mathcal{H}^{n-1}\big(\partial B_r(0)\big)}\int_{\partial B_r(0)} \Big(\exp\big(pu_2(x)\big)-1\Big)\, d\mathcal{H}^{n-1}\right|\\
&= \frac{p}{\mathcal{H}^{n-1}\big(\partial B_r(0)\big)}\int_0^\infty \mathcal{H}^{n-1}(E_t)\exp(pt)dt = o(1) \quad \text{as} \quad r\to\infty.
\end{aligned}
$$

The previously-established equalities and inequalities indicate that

$$
\begin{aligned}
&\frac{p}{\mathcal{H}^{n-1}\big(\partial B_r(0)\big)}\int_{\partial B_r(0)} u\, d\mathcal{H}^{n-1}\\
&= \frac{p}{\mathcal{H}^{n-1}\big(\partial B_r(0)\big)}\Big(\int_{\partial B_r(0)} (u-u_2)\, d\mathcal{H}^{n-1}\Big) + o(1)\\
&= \log\Big(\frac{1}{\mathcal{H}^{n-1}\big(\partial B_{|x|}(0)\big)}\int_{\partial B_{|x|}(0)} \exp(pu)\big(\exp(-pu_2)\big)\big)\, d\mathcal{H}^{n-1}\Big) + o(1)\\
&= \log\Big(\frac{1}{\mathcal{H}^{n-1}\big(\partial B_{|x|}(0)\big)}\int_{\partial B_{|x|}(0)} \exp(pu)\, d\mathcal{H}^{n-1}\Big) + o(1)
\end{aligned}
$$

holds whenever $r\to\infty$, as desired.

(iii) It is clear that $S_{g,n}\ge 0$ is equivalent to $\Delta u + (n/2-1)|\nabla u|^2 \le 0$. Since

$$
\Delta\bar{u} = \frac{1}{\mathcal{H}^{n-1}(\partial B_1(0))}\int_{\partial B_1(0)} \Delta u\, d\mathcal{H}^{n-1}
$$

and (thanks to Cauchy-Schwarz's inequality)

$$
\begin{aligned}
|\nabla\bar{u}|^2 &= \Big(\frac{1}{\mathcal{H}^{n-1}(\partial B_1(0))}\int_{\partial B_1(0)} \frac{du}{dr}\, d\mathcal{H}^{n-1}\Big)^2\\
&\le \frac{1}{\mathcal{H}^{n-1}(\partial B_{|x|}(0))}\int_{\partial B_{|x|}(0)} |\nabla u|^2\, d\mathcal{H}^{n-1},
\end{aligned}
$$

one gets that $S_{g,n}\ge 0$ implies $\Delta\bar u+(n/2-1)|\nabla\bar u|^2\le 0$ which in turn gives $S_{\bar g,n}\ge 0$.

Next, the fact that $Q_{\bar g,n}$ is absolutely integrable with respect to $dv_{\bar g}$ follows from the following estimate (via Fubini's theorem):

$$
\begin{aligned}
&\int_{\mathbb{R}^n}|Q_{\bar g,n}|dv_g=\int_{\mathbb{R}^n}|(-\Delta)^{n/2}\bar u|\,d\mathcal{H}^n\\
&=\int_{\mathbb{R}^n}\left|(-\Delta)^{n/2}\left(\frac{1}{\mathcal{H}^{n-1}(\partial B_{|x|}(0))}\int_{\partial B_{|x|}(0)}u\,d\mathcal{H}^{n-1}\right)\right|\,d\mathcal{H}^n\\
&=\int_{\mathbb{R}^n}\left|\frac{1}{\mathcal{H}^{n-1}(\partial B_{|x|}(0))}\int_{\partial B_{|x|}(0)}(-\Delta)^{n/2}u\,d\mathcal{H}^{n-1}\right|\,d\mathcal{H}^n\\
&\le\frac{1}{\mathcal{H}^{n-1}(\partial B_1(0))}\left(\int_{\partial B_1(0)}\int_{\mathbb{R}^n}|(-\Delta)^{n/2}u|\,d\mathcal{H}^n\right)\,d\mathcal{H}^{n-1}\\
&=\int_{\mathbb{R}^n}|(-\Delta)^{n/2}u|\,d\mathcal{H}^n=\int_{\mathbb{R}^n}|Q_{g,n}|dv_g<\infty.
\end{aligned}
$$

Note that (i) and (ii) in Proposition 3.1, together with the completeness of $g=e^{2u}g_0$, yield that $\int_0^\infty e^{u(r\sigma)}dr$ diverges for any given $\sigma\in\partial B_1(0)$ and so that

$$
\int_0^\infty e^{\bar u}dr=\frac{1}{\mathcal{H}^{n-1}\big(\partial B_1(0)\big)}\int_{\partial B_1(0)}\left(\int_0^\infty e^{u(r\sigma)}dr\right)\,d\mathcal{H}^{n-1}(\sigma)
$$

diverges. Therefore $\bar g=e^{2\bar u}g_0$ is complete.

(iv) Making a simple calculation with the spherical coordinate system and applying Proposition 2.1 (vi) to the conformal metric $\bar g$, we immediately obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^n}Q_{g,n}\,dv_g=\int_{\mathbb{R}^n}(-\Delta)^{n/2}u\,d\mathcal{H}^n=\int_{\mathbb{R}^n}(-\Delta)^{n/2}\bar u\,d\mathcal{H}^n\\
&=\int_{\mathbb{R}^n}Q_{\bar g,n}\,dv_{\bar g}=\int_{\mathbb{R}^n}(-\Delta)^{n/2}\bar u\,d\mathcal{H}^n\le 2^{n-1}\Gamma(n/2)\pi^{n/2},
\end{aligned}
$$

whence completing the argument.

□

4. Proof of (1.7) – Special Case

In this section we verify that (1.7) is true under the radial symmetry.

Proposition 4.1. *Let $u\in C^\infty(\mathbb{R}^n)$ be radially symmetric and satisfy the hypotheses of Theorem 1.1. If $w(s)=s+u(e^s)$ and*

$$
\begin{cases}
V_n(t)=\int_{B_{e^t}(0)}e^{nu}\,d\mathcal{H}^n=n\omega_n\int_{-\infty}^t e^{nw(s)}\,ds,\\
V_{n-1}(t)=\frac{1}{n}\int_{\partial B_{e^t}(0)}e^{(n-1)u}\,d\mathcal{H}^{n-1}=\omega_n e^{(n-1)w(t)},\\
V_{n-2}(t)=\frac{1}{n(n-1)}\int_{\partial B_{e^t}(0)}H_1e^{(n-1)u}\,d\mathcal{H}^{n-1}=\left(\frac{\omega_n}{n-1}\right)\frac{H_1(e^t)}{e^{(1-n)w(t)}},\\
V_{n-3}(t)=\frac{2}{n(n-1)(n-2)}\int_{\partial B_{e^t}(0)}H_2e^{(n-1)u}\,d\mathcal{H}^{n-1}=\left(\frac{2\omega_n}{(n-1)(n-2)}\right)\frac{H_2(e^t)}{e^{(1-n)w(t)}},
\end{cases}
$$

where H_k stands for the k-th symmetric form of the principle curvature of the boundary of a convex domain in $\mathbb{R}^n$, then:

(i)

$$\lim_{t\to\infty}\frac{\big(V_{n-3}(t)\big)^{\frac{n-2}{n-1}}}{\omega_n^{\frac{1}{n-1}}\big(V_{n-2}(t)\big)^{\frac{n-3}{n-1}}}=1-\frac{1}{2^{n-1}\Gamma(n/2)\pi^{n/2}}\int_{\mathbb{R}^n}(-\Delta)^{n/2}u\,d\mathcal{H}^n.$$

(ii) (1.7) *holds.*

Proof. (i) From (ii) and (v) of Proposition 2.1 we read off

$$\lim_{r\to\infty}r\frac{dv(r)}{dr}=-\frac{1}{2^{n-1}\Gamma(n/2)\pi^{n/2}}\int_{\mathbb{R}^n}(-\Delta)^{n/2}u\,d\mathcal{H}^n=\lim_{r\to\infty}r\frac{du(r)}{dr},$$

where v is given as in Proposition 2.1.

We now consider cylindrical coordinates $|x|=r=e^t$ and then use $w(t)=u(e^t)+t$ to get

$$\frac{dw}{dt}=r\frac{du}{dr}+1\quad\text{and}\quad\lim_{t\to\infty}\frac{dw}{dt}=\lim_{r\to\infty}r\frac{du}{dr}+1,$$

whence obtaining

$$\lim_{t\to\infty}\frac{dw}{dt}=1-\frac{1}{2^{n-1}\Gamma(n/2)\pi^{n/2}}\int_{\mathbb{R}^n}(-\Delta)^{n/2}u\,d\mathcal{H}^n.$$

On the other hand, from the rule of transformation of H_1 under conformal changes and the formula $n\omega_n=\mathcal{H}^{n-1}\big(\partial B_1(0)\big)$ we see

$$H_1(r)=(n-1)e^{-u(r)}\Big(\frac{1}{r}+\frac{du}{dr}\Big).$$

This equality, plus change of variables and the relation $2H_2(r)=H_1^2(r)-trL^2(r)$, easily implies

$$V_{n-2}(t)=\omega_n e^{(n-2)w(t)}\frac{dw}{dt}\quad\text{and}\quad V_{n-3}(t)=\omega_n e^{(n-3)w(t)}\Big(\frac{dw}{dt}\Big)^2.$$

Consequently, we find

$$\frac{\big(V_{n-3}(t)\big)^{\frac{n-2}{n-1}}}{\omega_n^{\frac{1}{n-1}}\big(V_{n-2}(t)\big)^{\frac{n-3}{n-1}}}=\frac{dw}{dt}$$

thereby establishing the required formula:

$$\lim_{t\to\infty}\frac{\big(V_{n-3}(t)\big)^{\frac{n-2}{n-1}}}{\omega_n^{\frac{1}{n-1}}\big(V_{n-2}(t)\big)^{\frac{n-3}{n-1}}}=1-\frac{1}{2^{n-1}\Gamma(n/2)\pi^{n/2}}\int_{\mathbb{R}^n}(-\Delta)^{n/2}u\,d\mathcal{H}^n.$$

(ii) Using the definitions of V_n and V_{n-1}, we conclude

$$\frac{\big(V_{n-1}(t)\big)^{\frac{n}{n-1}}}{\omega_n^{\frac{1}{n-1}}V_n(t)}=\frac{n^{-1}e^{nw(t)}}{\int_{-\infty}^t e^{nw(s)}ds}.$$

On the other hand, from Proposition 2.1 (v) with connection to $\lim_{t\to\infty}dw/dt$ we infer $\lim_{t\to\infty}dw/dt\ge 0$. Next we handle two cases:

Case 1: $\lim_{t\to\infty}dw/dt>0$. Under this condition, we clearly have

$$\lim_{t\to\infty}e^{nw(t)}=\lim_{t\to\infty}\int_{-\infty}^t e^{nw(s)}ds=\infty,$$

and thus use L'Hôpital's rule to get

$$\lim_{t\to\infty}\frac{\big(V_{n-1}(t)\big)^{\frac{n}{n-1}}}{\omega_n^{\frac{1}{n-1}}V_n(t)}=\lim_{t\to\infty}\frac{dw}{dt}.$$

Case 2: $\lim_{t\to\infty} dw/dt = 0$. When $\lim_{t\to\infty} V_n(t) = \infty$, we may have either $\lim_{t\to\infty} V_{n-1}(t) = \infty$ or $\sup_{t\in\mathbb{R}^n} V_{n-1}(t) < \infty$. For the former we can once again use L'Hôpital's rule to deduce

$$\lim_{t\to\infty}\frac{\big(V_{n-1}(t)\big)^{\frac{n}{n-1}}}{\omega_n^{\frac{1}{n-1}}V_n(t)}=\lim_{t\to\infty}\frac{dw}{dt}=0.$$

For the latter, we trivially get

$$\lim_{t\to\infty}\frac{\big(V_{n-1}(t)\big)^{\frac{n}{n-1}}}{\omega_n^{\frac{1}{n-1}}V_n(t)}=0=\lim_{t\to\infty}\frac{dw}{dt}.$$

On the other hand, when $\sup_{t\in\mathbb{R}} V_n(t) < \infty$, we have $\lim_{t\to\infty} e^{nw(t)} = 0$ which in turn yields

$$\lim_{t\to\infty}\frac{\big(V_{n-1}(t)\big)^{\frac{n}{n-1}}}{\omega_n^{\frac{1}{n-1}}V_n(t)}=0=\lim_{t\to\infty}\frac{dw}{dt}.$$

All in all, we arrive at

$$\lim_{t\to\infty}\frac{\big(V_{n-1}(t)\big)^{\frac{n}{n-1}}}{\omega_n^{\frac{1}{n-1}}V_n(t)}=\lim_{t\to\infty}\frac{dw}{dt}=1-\frac{1}{2^{n-1}\Gamma(n/2)\pi^{n/2}}\int_{\mathbb{R}^n}(-\Delta)^{n/2}u\,d\mathcal{H}^n,$$

as desired. □

5. Proof of (1.7) – General Case

In this section we handle the validity of (1.7) without the radially symmetric hypothesis.

Proposition 5.1. *Let $u \in C^\infty(\mathbb{R}^n)$ satisfy the hypotheses of Theorem 1.1. If*

$$\begin{cases} \mathsf{V}_n(r)=\int_{B_r(0)}e^{nu}\,d\mathcal{H}^n,\\ \mathsf{V}_{n-1}(r)=\frac{1}{n}\int_{\partial B_r(0)}e^{(n-1)u}\,d\mathcal{H}^{n-1},\\ \mathsf{V}_{n-2}(r)=\frac{1}{n(n-1)}\int_{\partial B_r(0)}H_1\frac{d\mathcal{H}^{n-1}}{e^{(1-n)u}}=\frac{1}{n}\int_{\partial B_r(0)}\left(\frac{1}{r}+\frac{\partial u}{\partial r}\right)\frac{d\mathcal{H}^{n-1}}{e^{(2-n)u}},\\ \mathsf{V}_{n-3}(r)=\frac{2}{n(n-1)(n-2)}\int_{\partial B_r(0)}H_2\frac{d\mathcal{H}^{n-1}}{e^{(1-n)u}}=\frac{1}{n}\int_{\partial B_r(0)}\left(\frac{1}{r}+\frac{\partial u}{\partial r}\right)^2\frac{d\mathcal{H}^{n-1}}{e^{(3-n)u}}, \end{cases}$$

where H_k still means the k-th symmetric form of the principle curvature of the boundary of a convex domain in $\mathbb{R}^n$, and if $\bar{\mathsf{V}}_n$, $\bar{\mathsf{V}}_{n-1}$, $\bar{\mathsf{V}}_{n-2}$ and $\bar{\mathsf{V}}_{n-3}$ denote the analogously-defined mixed volumes with respect to the conformal metric $e^{2\bar{u}}g_0$, where

$$\bar{u}(x)=\bar{u}(r)=\frac{1}{\mathcal{H}^{n-1}\big(\partial B_r(0)\big)}\int_{\partial B_r(0)}u\,d\mathcal{H}^{n-1},$$

then:
(i)

$$\frac{1}{\mathcal{H}^{n-1}\big(\partial B_r(0)\big)}\int_{\partial B_r(0)}\left(\frac{\partial u}{\partial r}\right)^k d\mathcal{H}^{n-1}=O\left(\frac{1}{r^k}\right)\quad \textit{for}\quad k=1,2,3,4,$$

and

$$\frac{1}{\mathcal{H}^{n-1}\big(\partial B_r(0)\big)}\int_{\partial B_r(0)}\left(\frac{\partial u}{\partial r}\right)^2 d\mathcal{H}^{n-1}=\left(\frac{\partial \bar{u}}{\partial r}\right)^2+o\left(\frac{1}{r^2}\right),$$

as $r\to\infty$.

(ii)

$$\frac{d\mathsf{V}_n(r)}{dr}=\frac{d\bar{\mathsf{V}}_n(r)}{dr}\big(1+o(1)\big)\quad \textit{and}\quad \mathsf{V}_{n-1}(r)=\bar{\mathsf{V}}_{n-1}(r)\big(1+o(1)\big)\quad \textit{as}\quad r\to\infty.$$

Moreover,

$$\mathsf{V}_{n-2}(r)=\bar{\mathsf{V}}_{n-2}(r)\big(1+o(1)\big)\quad \textit{and}\quad \mathsf{V}_{n-3}(r)=\bar{\mathsf{V}}_{n-3}(r)\big(1+o(1)\big)\quad \textit{as}\quad r\to\infty,$$

provided

$$\lim_{r\to\infty}\left(1+r\frac{\partial\bar{u}}{\partial r}\right)>0.$$

(iii)

$$\lim_{r\to\infty}\frac{\big(\mathsf{V}_{n-3}(r)\big)^{\frac{n-2}{n-1}}}{\omega_n^{\frac{1}{n-1}}\big(\mathsf{V}_{n-2}(r)\big)^{\frac{n-3}{n-1}}}=1-\frac{1}{2^{n-1}\Gamma(n/2)\pi^{n/2}}\int_{\mathbb{R}^n}(-\Delta)^{n/2}u\,d\mathcal{H}^n$$

whenever

$$1-\frac{1}{2^{n-1}\Gamma(n/2)\pi^{n/2}}\int_{\mathbb{R}^n}(-\Delta)^{n/2}u\,d\mathcal{H}^n>0.$$

(iv) *(1.7) holds.*

Proof. (i) The argument can be achieved via a slight modification of the proof of [2, Lemma 3.4] – the details are left for the interested readers.

(ii) The first two relations follow directly from Proposition 3.1 (ii). To prove the second two relations, we will bring the ideas used in proving [2, Lemma 3.5] into play.

For simplicity, in what follows, let us put $a=\frac{\partial\bar{u}}{\partial r}$ and $b=e^{\bar{u}}$. Then from the definition of $\bar{\mathsf{V}}_{n-2}$ and the easily-checked equation $\int_{\partial B_r(0)}\big(\frac{\partial u}{\partial r}-a\big)\,d\mathcal{H}^{n-1}=0$ we get

$$\bar{\mathsf{V}}_{n-2}(r)=\frac{1}{n}\int_{\partial B_r(0)}\left(\frac{1}{r}+\frac{\partial u}{\partial r}\right)e^{(n-2)u}\,d\mathcal{H}^{n-1}=\frac{1}{n}\mathcal{H}^{n-1}\big(\partial B_r(0)\big)\left(\frac{1}{r}+a\right)b^{n-2},$$

and consequently,

$$\begin{aligned}
&\mathsf{V}_{n-2}(r)-\bar{\mathsf{V}}_{n-2}(r)\\
&=\frac{1}{n}\left(\frac{1}{r}+a\right)\int_{\partial B_r(0)}\left(e^{(n-2)u}-b^{n-2}\right)d\mathcal{H}^{n-1}\\
&+\frac{1}{n}\int_{\partial B_r(0)}\left(\frac{\partial u}{\partial r}-a\right)e^{(n-2)u}\,d\mathcal{H}^{n-1}\\
&=\frac{1}{n}\left(\frac{1}{r}+a\right)\int_{\partial B_r(0)}\left(e^{(n-2)u}-b^{n-2}\right)d\mathcal{H}^{n-1}\\
&+\frac{1}{n}\int_{\partial B_r(0)}\left(\frac{\partial u}{\partial r}-a\right)\left(e^{(n-2)u}-b^{n-2}\right)d\mathcal{H}^{n-1}.
\end{aligned}$$

Now, using Proposition 3.1 (ii) and the above-established formula for $\bar{\mathsf{V}}_{n-2}$ we get

$$
\begin{aligned}
&\frac{1}{n}\left(\frac{1}{r}+a\right)\int_{\partial B_r(0)}\left(e^{(n-2)\bar{u}}-b^{n-2}\right)d\mathcal{H}^{n-1}\\
&=\frac{1}{n}\left(\frac{1}{r}+a\right)\mathcal{H}^{n-1}\big(\partial B_r(0)\big)\left(e^{o(1)}-1\right)b^{n-2}\,d\mathcal{H}^{n-1}\\
&=\bar{\mathsf{V}}_{n-2}(r)o(1).
\end{aligned}
$$

At the same time, a combined application of Cauchy-Schwarz's inequality, the binomial identity, the last-established (i) and Proposition 3.1 (ii) yields

$$
\begin{aligned}
&\int_{\partial B_r(0)}\left(\frac{\partial u}{\partial r}-a\right)\left(e^{(n-2)u}-b^{n-2}\right)d\mathcal{H}^{n-1}\\
&\le\left(\int_{\partial B_r(0)}\left(\frac{\partial u}{\partial r}-a\right)^2 d\mathcal{H}^{n-1}\right)^{\frac{1}{2}}\left(\int_{\partial B_r(0)}\left(e^{(n-2)u}-b^{n-2}\right)^2 d\mathcal{H}^{n-1}\right)^{\frac{1}{2}}\\
&=\left(\int_{\partial B_r(0)}\left(\left(\frac{\partial u}{\partial r}\right)^2-2a\frac{\partial u}{\partial r}+a^2\right)d\mathcal{H}^{n-1}\right)^{\frac{1}{2}}\\
&\times\left(\int_{\partial B_r(0)}\left(e^{(n-2)u}-b^{n-2}\right)^2 d\mathcal{H}^{n-1}\right)^{\frac{1}{2}}\\
&=\left(\mathcal{H}^{n-1}\big(\partial B_r(0)\big)o\left(\frac{1}{r^2}\right)\right)^{\frac{1}{2}}\left(\int_{\partial B_r(0)}\left(e^{(n-2)u}-b^{n-2}\right)^2 d\mathcal{H}^{n-1}\right)^{\frac{1}{2}}\\
&=\left(\mathcal{H}^{n-1}\big(\partial B_r(0)\big)o\left(\frac{1}{r^2}\right)\right)^{\frac{1}{2}}\left(\mathcal{H}^{n-1}\big(\partial B_r(0)\big)b^{2n-4}o(1)\right)^{\frac{1}{2}}\\
&=r^{-1}\mathcal{H}^{n-1}\big(\partial B_r(0)\big)b^{n-2}o(1).
\end{aligned}
$$

As a result, we find

$$
\begin{aligned}
&\mathsf{V}_{n-2}(r)-\bar{\mathsf{V}}_{n-2}(r)\\
&=\Big(\bar{\mathsf{V}}_{n-2}(r)+r^{-1}\mathcal{H}^{n-1}\big(\partial B_r(0)\big)b^{n-2}\Big)o(1)\\
&=\bar{\mathsf{V}}_{n-2}(r)\Big(1+\frac{n}{ar+1}\Big)o(1)\\
&=\bar{\mathsf{V}}_{n-2}(r)o(1),
\end{aligned}
$$

thanks to the assumption $\lim_{r\to\infty}(1+ar)>0$. This proves the third relation.

To prove the fourth one, we argue in a similar way. First of all, using the definition of $\bar{V}_{n-3}$, we get

$$
\bar{\mathsf{V}}_{n-3}(r)=\frac{1}{n}\int_{\partial B_r(0)}\left(\frac{1}{r}+\frac{\partial u}{\partial r}\right)^2 e^{(n-3)u}\,d\mathcal{H}^{n-1}=\frac{\mathcal{H}^{n-1}\big(\partial B_r(0)\big)}{nb^{3-n}}\left(\frac{1}{r}+a\right)^2,
$$

and then

$$
\begin{aligned}
&\mathsf{V}_{n-3}(r) - \bar{\mathsf{V}}_{n-3}(r)\\
&= \frac{1}{n}\int_{\partial B_r(0)} \left(\frac{1}{r}+a\right)^2 \left(e^{(n-3)u} - b^{n-3}\right) d\mathcal{H}^{n-1}\\
&+\frac{2}{rn}\int_{\partial B_r(0)} \left(\frac{\partial u}{\partial r} - a\right) e^{(n-3)u}\, d\mathcal{H}^{n-1}\\
&+\frac{1}{n}\int_{\partial B_r(0)} \left(\left(\frac{\partial u}{\partial r}\right)^2 - a^2\right) e^{(n-3)u}\, d\mathcal{H}^{n-1}.
\end{aligned}
$$

In the sequel, we control the three terms in the last formula. As in the proof of the third relation, using Proposition 3.1 (ii), we estimate the first term as follows:

$$
\frac{1}{n}\int_{\partial B_r} \left(\frac{1}{r}+a\right)^2 \left(e^{(n-3)u} - b^{n-3}\right) d\mathcal{H}^{n-1} = V_{n-3}(r)o(1).
$$

Next, still following the same argument based on Cauchy-Schwarz's inequality, Proposition 3.1 (ii) and Proposition 5.1 (i), we get the estimate for the second term:

$$
\frac{2}{rn}\int_{\partial B_r(0)} \left(\frac{\partial u}{\partial r} - a\right) e^{(n-3)u}\, d\mathcal{H}^{n-1} = r^{-2}\mathcal{H}^{n-1}\big(\partial B_r(0)\big)b^{n-3}o(1).
$$

Now, in order to estimate the third term, we firstly employ Young's inequality to get

$$
\begin{aligned}
&\left|\int_{\partial B_r(0)} \left(\frac{\partial u}{\partial r} - a\right) \frac{\partial u}{\partial r} e^{(n-3)u}\, d\mathcal{H}^{n-1}\right|\\
&\le \left(\int_{\partial B_r(0)} \left(\frac{\partial u}{\partial r} - a\right)^2 d\mathcal{H}^{n-1}\right)^{\frac{1}{2}}\\
&\times \left(\int_{\partial B_r(0)} \left(\frac{\partial u}{\partial r}\right)^4 d\mathcal{H}^{n-1}\right)^{\frac{1}{4}}\\
&\times \left(\int_{\partial B_r(0)} e^{4(n-3)u}\, d\mathcal{H}^{n-1}\right)^{\frac{1}{4}}.
\end{aligned}
$$

Secondly, we rewrite the third term and use Cauchy-Schwarz's inequality, Proposition 3.1 (ii) and Proposition 5.1 (i) and the finite limit $\lim_{r\to\infty}(1+ra) > 0$ to derive

$$
\begin{aligned}
&\int_{\partial B_r(0)} \left(\left(\frac{\partial u}{\partial r}\right)^2 - a^2\right) e^{(n-3)u}\, d\mathcal{H}^{n-1}\\
&= a\int_{\partial B_r(0)} \left(\frac{\partial u}{\partial r} - a\right) e^{(n-3)u}\, d\mathcal{H}^{n-1}\\
&+\int_{\partial B_r(0)} \left(\frac{\partial u}{\partial r} - a\right)\left(\frac{\partial u}{\partial r}\right) e^{(n-3)u}\, d\mathcal{H}^{n-1}\\
&\le r^{-2}\mathcal{H}^{n-1}\big(\partial B_r(0)\big)b^{n-3}o(1).
\end{aligned}
$$

With the help of the above estimates and the limit $\lim_{r\to\infty}(1+ar)>0$ we get

$$\mathsf{V}_{n-3}(r)-\bar{\mathsf{V}}_{n-3}(r)=\mathsf{V}_{n-3}(r)o(1)+r^{-2}\mathcal{H}^{n-1}\big(\partial B_r(0)\big)o(1)=\mathsf{V}_{n-3}(r)o(1),$$

completing the proof of the fourth relation.

(iii) Under the given assumption, the proofs of Propositions 2.1 (iv) and 3.1 (iv) yield

$$\lim_{r\to\infty}\Big(1+r\frac{\partial\bar{u}}{\partial r}\Big)=1-\frac{\int_{\mathbb{R}^n}(-\Delta)^{n/2}\bar{u}\,d\mathcal{H}^n}{2^{n-1}\Gamma(n/2)\pi^{n/2}}=1-\frac{\int_{\mathbb{R}^n}(-\Delta)^{n/2}u\,d\mathcal{H}^n}{2^{n-1}\Gamma(n/2)\pi^{n/2}}>0.$$

This fact, along with Proposition 4.1 (i) and Proposition 5.1 (ii), implies

$$1-\frac{\int_{\mathbb{R}^n}(-\Delta)^{n/2}u\,d\mathcal{H}^n}{2^{n-1}\Gamma(n/2)\pi^{n/2}}=\lim_{r\to\infty}\frac{\big(\bar{\mathsf{V}}_{n-3}(r)\big)^{\frac{n-2}{n-1}}}{\omega_n^{\frac{1}{n-1}}\big(\bar{\mathsf{V}}_{n-2}(r)\big)^{\frac{n-3}{n-1}}}=\lim_{r\to\infty}\frac{\big(\mathsf{V}_{n-3}(r)\big)^{\frac{n-2}{n-1}}}{\omega_n^{\frac{1}{n-1}}\big(\mathsf{V}_{n-2}(r)\big)^{\frac{n-3}{n-1}}},$$

as desired.

(iv) The formula (1.7) is demonstrated through the equalities

$$v_g\big(B_r(0)\big)=\mathsf{V}_n(r);\quad s_g\big(B_r(0)\big)=n\mathsf{V}_{n-1}(r)$$

and the forthcoming analysis. By Proposition 5.1 (ii), we have

$$\lim_{r\to\infty}\frac{\big(\mathsf{V}_{n-1}(r)\big)^{\frac{n}{n-1}}}{\omega_n^{\frac{1}{n-1}}\mathsf{V}_n(r)}=\lim_{r\to\infty}\frac{\big(\bar{\mathsf{V}}_{n-1}(r)\big)^{\frac{n}{n-1}}}{\omega_n^{\frac{1}{n-1}}\mathsf{V}_n(r)}.$$

Case 1: $1-\frac{\int_{\mathbb{R}^n}(-\Delta)^{n/2}u\,d\mathcal{H}^n}{2^{n-1}\Gamma(n/2)\pi^{n/2}}>0$. This condition implies

$$\lim_{r\to\infty}\bar{\mathsf{V}}_{n-1}(r)=\lim_{r\to\infty}\mathsf{V}_n(r)=\infty.$$

Consequently, a combined application of L'Höpital's rule and (ii)'s of Propositions 5.1 and 4.1 yields

$$\lim_{r\to\infty}\frac{\big(\mathsf{V}_{n-1}(r)\big)^{\frac{n}{n-1}}}{\omega_n^{\frac{1}{n-1}}\mathsf{V}_n(r)}=\lim_{r\to\infty}\frac{\frac{d}{dr}\big(\bar{\mathsf{V}}_{n-1}(r)\big)^{\frac{n}{n-1}}}{\omega_n^{\frac{1}{n-1}}\frac{d}{dr}\bar{\mathsf{V}}_n(r)}=1-\frac{\int_{\mathbb{R}^n}(-\Delta)^{n/2}u\,d\mathcal{H}^n}{2^{n-1}\Gamma(n/2)\pi^{n/2}}.$$

Case 2: $1-\frac{\int_{\mathbb{R}^n}(-\Delta)^{n/2}u\,d\mathcal{H}^n}{2^{n-1}\Gamma(n/2)\pi^{n/2}}=0$. With this hypothesis, we argue as in the radial case, and thus we have to consider the situation where V_n is bounded – under this boundedness we employ Proposition 5.1 (ii) to derive

$$\lim_{r\to\infty}\frac{d\mathsf{V}_n(r)}{dr}=0=\lim_{r\to\infty}\frac{d\bar{\mathsf{V}}_n(r)}{dr},$$

and $\lim_{r\to\infty}r^ne^{n\bar{u}}=0$. Hence we obtain

$$\lim_{r\to\infty}\frac{\big(\mathsf{V}_{n-1}(r)\big)^{\frac{n}{n-1}}}{\omega_n^{\frac{1}{n-1}}\mathsf{V}_n(r)}=0=1-\frac{\int_{\mathbb{R}^n}(-\Delta)^{n/2}u\,d\mathcal{H}^n}{2^{n-1}\Gamma(n/2)\pi^{n/2}}.$$

□

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