

# Nonperturbative $SL(2,Z)$ $(p,q)$ -strings manifestly realized on the quantum M2

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February 6, 2008

## Abstract

The  $SL(2,Z)$  duality symmetry of  $IIB$  superstring is naturally realized on the  $D = 11$  supermembrane restricted to have central charges arising from a nontrivial wrapping. This supermembrane is minimally immersed on the target space (MIM2). The hamiltonian of the MIM2 has a discrete quantum spectrum. It is manifestly invariant under the  $SL(2,Z)$  symmetry associated to the conformal symmetry on the base manifold and under a  $SL(2,Z)$  symmetry on the moduli of the target space. The mass contribution of the string states on the MIM2 is obtained by freezing the remaining degrees of freedom. It exactly agrees with the perturbative spectrum of the  $(p,q)$   $IIB$  and  $IIA$  superstring compactified on a circle. We also construct a MIM2 in terms of a dual target space, then a  $(p,q)$  set of non-perturbative states associated to the  $IIA$  superstring is obtained.

**Keywords:**  $SL(2,Z)$ , S-duality, Membranes, Stringy Dyons.

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## 1 Introduction

In 1981 the  $SL(2, R)$  symmetry of the *IIB* supergravity was found [1]. A discrete subgroup of this symmetry  $SL(2, Z)$  is conjectured to be the exact symmetry of the full quantum type IIB theory, [2]. In fact this relation between classical and quantum symmetries seems to be a generic one [2]. A circumstance that gave strength to this argument is the fact that the string coupling constant transform in a non-trivial way under this symmetry.

In [3] a major step towards understanding this symmetry was obtained at the level of supergravity analysis of type II strings. A  $SL(2, Z)$  multiplet of string solutions of type *IIB* theory were found. It was also argued that this symmetry should have an origin in the 11D supermembrane. These solutions are called (p,q)-strings, where  $p, q$  corresponds to coprime integers associated to the electrical charge of RR and NSNS 3-form field strengths,  $F_3, H_3$  respectively. These solutions are stable [4], they do not decay into other string

states since it would violate charge conservation. Moreover, the tension of the bound state is less than the tension of two individual strings. This  $(p,q)$  strings may be seen as bound-states between fundamental string and D-string [5]. The elementary type *IIB* superstring is a source of the NSNS B-field but not of the RR fields [5]. It corresponds to a  $(1,0)$  string and through a  $SL(2, Z)$  can be mapped to a generic  $(p, q)$  string. On the other hand a D-string has charges  $(0, 1)$  since carry RR charge but not NSNS one. The  $(NS5, D5)$  also form a  $SL(2, Z)$  multiplet with the NSNS 5-form [6]. An exhaustive study of these  $SL(2, Z)$  bound-states in 9D supergravity was done in [7]. These results were extended for  $D < 10$  by [8][10], and the explicit tensions and charges of more general bound states related through T-duality,  $(F1, Dp)$  and  $(NS5, Dp)$  with  $p$  even and odd were explicitly computed [9]. Since these string states are BPS saturated, this gave strong evidence to think that the compactified theory also has an exact  $SL(2, Z)$  invariance. The qualitative results are seen to be preserved. In less than  $D < 10$  the symmetry group is enhanced and conjectured to be the U-group which contains the T-symmetry group and the S-duality [2]. Due to the fact that this symmetry is nonperturbative not many results have been obtained so far. In the case of the IIA theory these bound states do not appear at the level of the perturbative spectrum. Since IIA and IIB theories when compactified on a circle share the same massless spectrum (although they are different theories as pointed out in [13]), it was conjectured by [5] and [2] that in 10D an array of D0 branes should appear non perturbatively carrying the information of those  $D = 9$   $(p, q)$  states. We will report about these states along this paper.

The origin of a 12 dimensional IIB theory in terms of an eleven dimensional theory M-theory at first sight can be surprising. In [11],[12] this aspect was analyzed. They found that the 12th dimension, instead of a time coordinate, could admit an interpretation as the M2 tension, which can be regarded as the flux of a three-form worldvolume field strength. In [13] the precise meaning of T duality between *IIA* and *IIB* superstring theories compactified on circles of radius  $R$  and  $1/R$  was emphasized by considering  $N = 2$  space-time SUSY in nine dimensions. Moreover duality properties between M-theory compactified on  $T^2$  and *IIB* on  $S^1$  were derived from  $N = 2$   $D = 9$  supergravity. These duality properties are in complete agreement with the more general analysis we will present in terms of the  $D = 11$  supermembrane. Also in [14, 15] the identification of the  $\tau$  parameter and the moduli of the  $(p, q)$  string tension was obtained.

In this paper we consider the  $D = 11$  supermembrane restricted by a topological condition which may be interpreted as if the supermembrane has a central charge generated by an irreducible winding (M2MI). The theory is completely consistent since the topological constraint commutes with all constraints of the  $D = 11$  supermembrane, it determines a very special harmonic map from the base manifold to the target. It is an holomorphic map and hence a minimal immersion of the base manifold onto the compactified sector of the target. The main property of the MIM2 is that its hamiltonian has a discrete spectrum. It does allow a direct analysis of its symmetries at the quantum level. These symmetries which we will determine explicitly are related as expected to the  $SL(2, Z)$  IIB duality symmetry. In particular they ensure that some assumptions in [3] concerning the supermembrane winding modes are valid as well as explain why the supermembrane can be wrapped on a torus any number of times but only one on a circle. Moreover by considering appropriate matching-level conditions on the MIM2 and by freezing pure membrane states we can explicitly obtain all  $(p, q)$  string configurations within the MIM2. We will obtain the complete  $mass^2$  contributions of those configurations and they match exactly with the  $(p, q)$  analysis in [3]. We will also consider some decompactifications limit of interest, and introduce the T-dual MIM2 theory. From it we will obtain a set of  $(p, q)$  multiplet of non-perturbative states dual to the IIB  $(p, q)$  states corresponding to IIA theory.

The paper outline is the following: In section 2 we will briefly summarized the properties of the MIM2, a sector of the theory corresponding to nontrivial irreducible wrapping of the M2. In section 3, we show that the MIM2 on a  $T^2$  posseses two different  $SL(2, Z)$  symmetries, one associated to the conformal symmetries of the Riemann basis and other associated to a symmetry of the target space. The simultaneous action of these two symmetries generate a transformation rule for the axion-dilaton of the IIB, the radius of the target space and the gauge fields. In section 4 we reproduce the perturbative mass spectrum of the IIB theory covariant under the  $SL(2, Z)$  and IIA when compactified on a circle. In section 5 we detail the string states of the MIM2 obtaining also the natural candidates to nonperturbative  $(p, q)$ -string states of the IIA conjectured long time ago. We show the decompactification limits in which ordinary type IIA and type IIB strings are obtained. Finally we clarify an open puzzle with respect to the difference between compactifying the M2 on a circle (which can be consistently made only for winding  $n_1 = 1$ )

and compactifying on a torus with an arbitrary wrapping  $n$ . In section 6 we summarize and discuss the main properties of these results and conclude.

## 2 The $D = 11$ Supermembrane restricted by the topological constraint (MIM2).

The  $D = 11$  Supermembrane action [16] is

$$S = -T \int_{M_3} d\xi^3 \left[ \frac{1}{2} \sqrt{-g} g^{ij} \Pi_i^M \Pi_{jM} - \frac{1}{2} \sqrt{-g} + \epsilon^{ijk} \Pi_i^A \Pi_j^B \Pi_k^C B_{CBA} \right] \quad (1)$$

where  $A = (M, \alpha)$  and

$$\begin{aligned} \Pi_i^M &= \partial_i X^M - i \bar{\Psi} \Gamma^M \partial_i \Psi \\ \Pi_i^\alpha &= \partial_i \Psi^\alpha \end{aligned} \quad (2)$$

$X^M$ ,  $M = 0, \dots, 10$ , and  $\Psi$ , a Majorana spinor on the target space, are scalars under diffeomorphisms on  $M_3$ .

We will consider  $M_3 = R \times \Sigma$ ,  $\Sigma$  a compact Riemann surface of genus  $g$ . The local coordinates are denoted  $\tau$ , a time coordinate on  $R$ , and  $\sigma^a$ ,  $a = 1, 2$ , on  $\Sigma$ . The action is invariant under diffeomorphisms on  $M_3$ , space-time supersymmetry and  $k$ -symmetry. The phase space is restricted by first class constraints generating the above symmetries and by fermionic second class constraints. We will consider, in this work, the target space to be a 9 dim Minkowski space  $M_9$  times a 2-dim flat torus  $T^2$ . An extension to other compactified sectors on the target space were considered in [17],[18][19]. We impose an additional constraint to the supermembrane action (1), a topological condition on the map:  $\Sigma \rightarrow T^2$ . We denote  $X^r$ ,  $r = 1, 2$  the maps:  $R \times \Sigma \rightarrow T^2$  and the  $X^m$   $m = 0, \dots, 9$  the maps;  $R \times \Sigma \rightarrow M^9$ . The topological condition we impose is the following [20],[21]:

$$\int_{\Sigma} dX^r \wedge dX^s = n \cdot \text{Area}_{\Sigma} \cdot \epsilon^{rs} \quad r, s = 1, 2, \quad n \neq 0 \quad (3)$$

This global constraint commutes with all first class constraints. This is so, because any variation  $\delta X$  generated by one of the symmetries of the theory is single-valued on  $\Sigma$  and hence

$$\int_{\Sigma} d(\delta X^r) \wedge dX^s = 0, \quad (4)$$

since  $\delta X^r dX^s$  is a well defined one-form over  $\Sigma$ . Consequently by adding the constraint (3) we do not change any of the original symmetries of the supermembrane. We are only selecting a sector of the theory.

The global restriction (3) ensures that all configurations map a torus  $\Sigma$  into another torus  $T^2$ , without any degeneration of the mapping. The left hand member of (3) corresponds to the central charge in the SUSY algebra of the  $D = 11$  Supermembrane [28]. The central charge is induced by the irreducible winding condition [20]. We may also interpret the left hand side of (3) as the integral over  $\Sigma$  of the closed two-form  $F = \epsilon_{rs} dX^r \wedge dX^s$ . Condition (3) ensures that  $F$  is the curvature of a connection on a nontrivial  $U(1)$  principle bundle over  $\Sigma$ . In fact, the  $U(1)$  principle bundles are classified by  $n$ .

We may now fix the Light Cone Gauge, (LCG), since symmetries have not been altered by the imposition of (3). The gauge fixing procedure is the usual one, we impose

$$X^+ = T^{-2/3} P^{0+\tau} = -T^{-2/3} P_-^0 \tau, \quad P_- = P_-^0 \sqrt{W}, \quad \Gamma^+ \Psi = 0 \quad (5)$$

$\sqrt{W}$  is a time independent density introduced in order to preserve the density behavior of  $P_-$  [22]. We eliminate  $X^-$ ,  $P_+$  from the constraints and solve the fermionic second class constraints in the usual way. We end up with the lagrangian density:

$$\mathcal{L} = P_m \dot{X}^m - \mathcal{H} \quad (6)$$

where the physical hamiltonian in the LCG is given by

$$H = \int_{\Sigma} \mathcal{H} = \int_{\Sigma} T^{-2/3} \sqrt{W} \left[ \frac{1}{2} \left( \frac{P_m}{\sqrt{W}} \right)^2 + \frac{1}{2} \left( \frac{P_r}{\sqrt{W}} \right)^2 + \frac{T^2}{2} \{X^r, X^m\}^2 \right. \\ \left. + \frac{T^2}{4} \{X^r, X^s\}^2 + \frac{T^2}{4} \{X^m, X^n\}^2 \right] + \text{fermionic terms} \quad (7)$$

subject to the constraints

$$d(P_r dX^r + P_m dX^m) = 0 \quad (8)$$

$$\oint_{\mathcal{C}_s} (P_r dX^r + P_m dX^m) = 0 \quad (9)$$

and the global restriction (3).  $\mathcal{C}_s$  is a canonical basis of homology on  $\Sigma$ . The constraints (8),(9) are the generators of area preserving diffeomorphisms

homotopic to the identity. the bracket in (7) is given by

$$\{X^m, X^n\} = \frac{\epsilon^{ab}}{\sqrt{W}} \partial_a X^m \partial_b X^n, \quad (10)$$

it is the symplectic bracket constructed from the non-degenerate two-form

$$\sqrt{W} \epsilon_{ab} d\sigma^a \wedge d\sigma^b \quad (11)$$

over  $\Sigma$ . In 2-dim the area preserving diffeomorphisms are the same as the symplectomorphisms. There is a natural election for  $W$  on the geometrical picture we have defined. We consider the  $2g$  dimensional space of harmonic one-forms on  $\Sigma$ . We denote  $dX^r$ ,  $r = 1, 2$ , the normalized harmonic one-forms with respect to  $\mathcal{C}_s$ ,  $s = 1, 2$ , a canonical basis of homology on  $\Sigma$ :

$$\oint_{\mathcal{C}_s} d\hat{X}^r = \delta_s^r. \quad (12)$$

We define

$$\sqrt{W} = \frac{1}{2} \epsilon_{rs} \partial_a \hat{X}^r \partial_b \hat{X}^s \epsilon^{ab}, \quad (13)$$

it is a regular density globally defined over  $\Sigma$ . It is invariant under a change of the canonical basis of homology.

The action of the new theory, a sector of the original supermembrane, is now completely defined. We will denote it MIM2. The interesting property of the MIM2 is that its spectrum is discrete. Moreover the resolvent of its regularized hamiltonian operator is compact. This was proven in [25] and also in a different way using properties of the heat kernel in [26]. The discreteness of the spectrum arises in first place from the absence of string-like spikes that are present in the original supermembrane case [27], [28]. This characteristic of the MIM2 is a consequence of restriction (3), and secondly from the linear dependence of the interacting bosonic-fermionic terms on the bosonic configuration variables, since it is a supersymmetric action in flat space.

Another very remarkable fact is that although we still do not know the explicit eigenvalues of the complete spectrum of the MIM2, since it is a theory highly non-linear -in fact it is equivalent to find the eigenvalues of a symplectic nonperturbative SYM - however we know that each eigenvalue of

the complete spectrum can be bounded by below up to a perturbation in an operatorial sense [25][26], [18] by its semiclassical value. The semiclassical spectrum of the MIM2 was discussed in [30]. It corresponds to the eigenvalues of a supersymmetric harmonic oscillator whose frequency is given by

$$\lambda_\Omega = \sum_{A \in \Omega} \omega_A; \quad \omega_A = \pi^2 \sqrt{(R^1 l^1 a_2)^2 + (R^2 l^2 a_1)^2} + \text{susy}. \quad (14)$$

where  $\Omega = \mathcal{N} \times \mathcal{N}$  is a set of excited modes  $l^1, l^2$  are the winding numbers and  $a_1, a_2$  are integers characterizing the excitation of the harmonic oscillator. This result hold for the infinite dimensional theory and also for a  $N$  regularized version of it [23]. In the latest the eigenvalues have in addition a sinusoidal dependence on the parameters and in the large  $N$  limit converge to the exact ones [18]. We finally emphasize that the results on this work are obtained from the complete theory without approximations.

### 3 The $SL(2, Z)$ symmetries of the MIM2

Let us define explicitly the torus  $T^2$  in the target space. We consider the lattice  $\mathcal{Z}$  on the complex plane  $C$ ,

$$\mathcal{Z} : z \rightarrow z + 2\pi R(l + m\tau) \quad (15)$$

where  $l, m$  are integers,  $R$  is a real parameter  $R > 0$  and  $\tau$  a complex parameter  $\tau = Re\tau + iIm\tau$ ,  $Im\tau > 0$ ,  $T^2$  is defined by  $C/\mathcal{Z}$ .  $\tau$  is is the coordinate of the Teichmüller space, for  $g = 1$  the upper half plane. The Teichmüller space is a covering of the moduli space of Riemann surfaces. It is in general a  $2g - 1$  complex analytic simply connected manifold. Any flat torus is diffeomorphic to  $T^2$ . The conformally equivalent torus are identified by  $\tau$  modulo the modular group. The parameter  $R$  is irrelevant in the identification of conformally equivalent classes of compact tori.

Without losing generality, we may decompose the closed one-forms  $dX^r$  into

$$dX^r = M_s^r d\widehat{X}^s + dA^r \quad r = 1, 2 \quad (16)$$

where  $d\widehat{X}^s, s = 1, 2$  is the basis of harmonic one-forms we have already introduced,  $dA^r$  are exact one-forms and  $M_s^r$  are constant coefficients. We

also define

$$\begin{aligned} dX &= dX^1 + idX^2 \\ dA &= dA^1 + idA^2 \end{aligned} \quad (17)$$

$X^r, r = 1, 2$ , are maps:  $\Sigma \rightarrow T^2$  if and only if they satisfy

$$\oint_{\mathcal{C}_s} dX = 2\pi R(l_s + m_s\tau) \quad (18)$$

$l_s, m_s, s = 1, 2$ , are integers. This condition is satisfied provided

$$M_s^1 + iM_s^2 = 2\pi R(l_s + m_s\tau) \quad (19)$$

Consequently, the most general expression for the maps  $X^r, r = 1, 2$ , is

$$dX = 2\pi R(l_s + m_s\tau)d\widehat{X}^s + dA, \quad (20)$$

$l_s, m_s, s = 1, 2$ , arbitrary integers. We now impose condition (3). We obtain after replacing (20) into (3),

$$n = \det \begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix} \quad (21)$$

and

$$\text{Area}_\Sigma = (2\pi R)^2 \text{Im}\tau. \quad (22)$$

The topological constraint (3) only restricts the integral numbers  $l_s, m_s, s = 1, 2$ , to satisfy (21) with  $n \neq 0$ . All configurations (20) satisfying condition (21) are then admissible.

In our previous works we considered  $\text{Re}\tau = 0$  and used the notation  $R_1 = R$  and  $R_2 = \text{Re}\tau$ , that is a 2-parameter torus on the target, besides this point everything is exactly the same. The MIM2 is invariant under conformal maps homotopic to the identity (biholomorphic maps). Those are diffeomorphisms which preserve  $d\widehat{X}^r, r = 1, 2$ , the harmonic one-forms.  $W$  is then invariant:

$$W'(\sigma) = W(\sigma). \quad (23)$$

Moreover MIM2 is invariant under diffeomorphisms changing the homology basis, and consequently the normalized harmonic one-forms, by a modular transformation on the Teichmüller space of the base torus  $\Sigma$ . In fact, if

$$d\widehat{X}^{r'}(\sigma) = S_s^r d\widehat{X}^r(\sigma) \quad (24)$$

then

$$\sqrt{W'(\sigma)}d\sigma^1 \wedge d\sigma^2 = \frac{1}{2}\epsilon_{rs}d\widehat{X}^{r'} \wedge d\widehat{X}^{s'} = \frac{1}{2}\epsilon_{rs}d\widehat{X}^r \wedge d\widehat{X}^s = \sqrt{W(\sigma)}d\sigma^1 \wedge d\sigma^2 \quad (25)$$

provided

$$\epsilon_{rs}S_t^r S_u^s = \epsilon_{tu} \quad (26)$$

that is  $S \in Sp(2, Z) \equiv SL(2, Z)$ . We then conclude that the MIM2 has an additional symmetry with respect to the  $D = 11$  Supermembrane. All conformal transformations on  $\Sigma$  are symmetries of the MIM2 [26],[17],[24],[18][19]. We notice that under (24)

$$dX \rightarrow 2\pi R(l_s + m_s\tau)S_r^s d\widehat{X}^r + dA' \quad (27)$$

where  $A'(\sigma') = A(\sigma)$  is the transformation law of a scalar. That is,

$$\begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix} \rightarrow \begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix} \begin{pmatrix} S_1^1 & S_2^1 \\ S_1^2 & S_2^2 \end{pmatrix} \quad (28)$$

$Sp(2, Z)$  acts from the right. Moreover, MIM2 is also invariant under the following transformation on the target torus  $T^2$ :

$$\begin{aligned} \tau &\rightarrow \frac{a\tau + b}{c\tau + d} \\ R &\rightarrow R|c\tau + d| \\ A &\rightarrow Ae^{i\varphi_\tau} \\ \begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix} &\rightarrow \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix} \end{aligned} \quad (29)$$

where  $c\tau + d = |c\tau + d|e^{-i\varphi_\tau}$  and  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2, Z)$ . In fact, the potential itself is invariant under (29). The invariance of the term  $\{X^r, X^m\}^2$  may be seen as follows. We rewrite it as<sup>1</sup>

$$\{X^1, X^m\}^2 + \{X^2, X^m\}^2 = \{2\pi R(l_s + m_s\tau)\widehat{X}^s + A, X^m\}.c.c \quad (30)$$

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<sup>1</sup>c.c denotes complex conjugation.

We then obtain directly

$$\begin{aligned} & \{2\pi R'(l'_s + m'_s \tau') \widehat{X}^s + A', X^m\}.cc = \\ & \{2\pi R[(l'_s d + m'_s b) + (l'_s c + m'_s a)\tau] \widehat{X}^s + A, X^m\}.cc \end{aligned} \quad (31)$$

where

$$\begin{pmatrix} l'_1 & l'_2 \\ m'_1 & m'_2 \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix} \quad (32)$$

In the same way it follows that  $\{X^r, X^s\}^2$  is invariant under (29). The hamiltonian density of the MIM2 is then invariant under (29). The  $Sp(2, Z)$  matrix now acts from the left of the matrix  $\begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix}$ . The two actions from the left and from the right by  $Sp(2, Z)$  matrices are not equivalent, they are complementary. The following remarks are valid.

### 3.1 Remark 1

Given  $\begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix}$  with determinant  $n \neq 0$ , there always exists a matrix  $\in SL(2, Z)$  such that its action from the right yields

$$\begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix} \begin{pmatrix} S_1^1 & S_2^1 \\ S_1^2 & S_2^2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ \rho & \lambda_2 \end{pmatrix} \quad (33)$$

where (of course)  $\lambda_1 \lambda_2 = n$ , and  $|\rho| < |\lambda_2|$ . There is an analogous result when the matrix  $\in SL(2, Z)$  acts from the left. In this case  $|\rho| < |\lambda_1|$ .  $\rho$  is in general different from zero. One cannot, in general, reduce to a diagonal matrix by acting solely from the left or from the right.

### 3.2 Remark 2

Given  $\begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix}$  with determinant  $n \neq 0$  there always exist matrices  $\in SL(2, Z)$  such that their action from the left and from the right yields

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix} \begin{pmatrix} S_1^1 & S_2^1 \\ S_1^2 & S_2^2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (34)$$

where (of course)  $\lambda_1 \lambda_2 = n$ .

### 3.3 Remark 3

If  $\lambda_1$  and  $\lambda_2$  are not relatively primes, the common factor may be absorbed consistently by the parameter  $R$ . We then have the final result. If  $\lambda_1$  and  $\lambda_2$  are relatively primes there always exists matrices  $\in SL(2, Z)$  such that their action from the left and from the right yields

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} S_1^1 & S_2^1 \\ S_1^2 & S_2^2 \end{pmatrix} = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} \quad (35)$$

This proves that although we may have arbitrary winding numbers  $\begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix}$  the symmetries of the MIM2 allow to reduce everything to the central charge integer  $n$  (3). The MIM2 satisfies then the requirement raised in [3], footnote 6. Consequently, there is a canonical formula for the harmonic sector of the maps. The general expression for the  $dX$  maps is then

$$dX = 2\pi R(nd\widehat{X}^1 + \tau d\widehat{X}^2) + dA \quad (36)$$

or in components

$$\begin{aligned} dX^1 &= 2\pi R(ndX^1 + Re\tau d\widehat{X}^2) + dA^1 \\ dX^2 &= 2\pi R(Im\tau d\widehat{X}^2) + dA^2 \end{aligned} \quad (37)$$

There is also a formulation in which  $n$  is attached to  $d\widehat{X}^2$  instead of  $d\widehat{X}^1$ .

We may now determine the subgroup of (29) which together with the  $SL(2, Z)$  transformations on the base manifold, yield a covariant transformation law for the harmonic part of  $dX$ ,

$$dX_h = 2\pi R(nd\widehat{X}^1 + \tau d\widehat{X}^2). \quad (38)$$

It turns out to be the subgroup  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z)$  where  $b = nb_1$  and  $b_1$  an integral number. A fundamental region  $F$  on the upper half plane associated to those transformations is obtained from the canonical fundamental region associated to the  $SL(2, Z)$  by taking discrete translations on the real direction of values  $\pm 1, \pm 2, \dots, \pm(n-1)$ .  $F$  is contained in the union set of the fundamental region and its translations. This subgroup of transformations preserve the form (38) and leaves invariant the hamiltonian.

The harmonic map (38) is a minimal immersion from  $\Sigma$  to  $T^2$  on the target,

moreover it is directly related to a holomorphic immersion of  $\Sigma$  onto  $T^2$ . The extension of the theory of supermembranes restricted by the topological constraint to more general compact sectors in the target space is directly related to the existence of those holomorphic immersions, because of this property we denote this sector of the  $D = 11$  supermembrane MIM2.

We have shown that the MIM2 has two  $SL(2, Z)$  symmetries. These symmetries explain the necessary identification mentioned in [3], footnote 6. One is associated to the conformal invariance on the base manifold. It allows to identify the winding modes. The other  $SL(2, Z)$  symmetry (29) acts on the Teichmüller coordinate  $\tau$ . However since the other parameter  $R$  is also involved in the transformation, the equivalence classes of tori under this transformation are not the conformally equivalent classes. We will show in the following sections, using these two  $SL(2, Z)$  symmetries, that the mass contribution of the string states in the MIM2 exactly agree with the perturbative mass spectrum of  $(p, q)$  *IIB* and *IIA* superstring.

## 4 The irreducible winding and KK contribution to the mass<sup>2</sup>

We start writing explicitly all the contributions to the  $mass^2$  arising from the MIM2 theory. We consider  $mass^2 = -P_\mu P^\mu$ . The first term to evaluate is the pure harmonic contribution in (37) to the  $mass^2$ . It arises from the term in (7)

$$T^{-2/3} \int_{\Sigma} \sqrt{W} \frac{T^2}{4} \{X_h^r, X_h^s\}^2 \quad (39)$$

where  $X_h^r$  denotes the harmonic part in (37). The factor  $T^{-2/3}$  cancels an inverse factor from the  $2P_+P_-$  term. The result is

$$mass^2 = T^2((2\pi R)^2 n I m \tau)^2 + \dots \quad (40)$$

The dots represent additional terms which we will shortly write explicitly. This contribution is usually obtained by considering the tension  $T$  as the  $mass/area$ , times the total area  $nA$ ,  $A = (2\pi R)^2 I m \tau$ . This is correct provided the contribution of all winding modes resumes in the factor  $n$ , which we have shown explicitly to be the case for the MIM2.

The harmonic sector is related by the condition (3) to a nontrivial  $U(1)$  principal bundle, classified by the winding number  $n$ . This is the contribution of these nontrivial complex bundles to the  $mass^2$ . We will interpret the KK quantization condition in a similar way.

Suppose we are compactifying on  $S^1 \times S^1$  and the associated momenta are  $p_r$   $r = 1, 2$ . Then for  $r = 1, 2$  we have, we do not use the index explicitly

$$P = \int_{\Sigma} p d\sigma^1 \wedge d\sigma^2. \quad (41)$$

where  $p$  is a scalar density. It may always be expressed as the dual to a two-form  $F$ ,

$$Rp = \epsilon^{ab} F_{ab} \quad (42)$$

$R$  is the radius of the circle. (41) may be rewritten

$$RP = \int_{\Sigma} F, \quad (43)$$

we take  $P$  to have dimensions of mass, and  $F$  to be dimensionless. The quantization condition is to set

$$\int_{\Sigma} F = 2\pi m. \quad (44)$$

This is exactly the Weil condition on a closed two form, in dimension 2  $F$  is closed, to ensure the existence of a  $U(1)$  principle bundle and a connection whose curvature is  $F$ . We then have for each mode

$$R \frac{p}{\sqrt{W}} = \frac{\epsilon^{ab} F_{ab}}{\sqrt{W}} = F^* = m, \quad (45)$$

and the associate conjugate coordinate  $T^{-2/3}p = \dot{X}$ ,

$$X = T^{-2/3} \frac{m}{R} \sqrt{W} t. \quad (46)$$

That is, there is a geometrical global contribution from those configurations. In the case of the MIM2 we first have to change to a frame where the maps are onto circles. We have from (37)

$$\frac{1}{2\pi R} \oint_{\mathcal{C}_s} \mathbb{M}^{-1} \begin{pmatrix} dX^1 \\ dX^2 \end{pmatrix} = \oint_{\mathcal{C}_s} \begin{pmatrix} nd\hat{X}^1 \\ d\hat{X}^2 \end{pmatrix} \quad (47)$$

where  $\mathbb{M} = \begin{pmatrix} 1 & Re\tau \\ 0 & Im\tau \end{pmatrix}$ . The associated maps are then onto circles. The corresponding momenta are  $p_s M_r^s$ . The quantization condition is then

$$R \int_{\Sigma} p_s M_r^s d\sigma^1 \wedge d\sigma^2 = 2\pi m_r \quad r = 1, 2 \quad (48)$$

$m_r$  integers. We then have  $p_s = p_s^0 \sqrt{w}$  where

$$p_s^0 = \frac{m_r}{R} (M^{-1})_s^r, \quad (49)$$

that is

$$\begin{aligned} p_1^0 &= \frac{m_1}{R} \\ p_2^0 &= -\frac{m_1 Re\tau + m_2}{RI m\tau} \end{aligned} \quad (50)$$

We now observe that the only contribution to the total momenta in those directions arises solely from the nontrivial line bundles. There is no contribution from the local degrees of freedom. In fact we may always impose the gauge condition

$$\int_{\Sigma} A^r \sqrt{W} d\sigma^1 \wedge d\sigma^2 = 0 \quad (51)$$

In fact

$$\delta X^r = \{\xi, X_h^r\} + \{\xi, A^r\}, \quad (52)$$

and

$$\int_{\Sigma} \{\xi, A^r\} \sqrt{W} d\sigma^1 \wedge d\sigma^2 = 0 \quad (53)$$

since  $A^r$  is single-valued on  $\Sigma$ . However

$$\int_{\Sigma} \{\xi_h, X_h^r\} \sqrt{W} d\sigma^1 \wedge d\sigma^2 \neq 0 \quad (54)$$

provides a non zero contribution. Moreover since  $\xi_h = a_r(t) \hat{X}^r$ , the two parameters  $a_r(t)$  allow to impose the gauge fixing condition (51). Hence the

contribution of  $\dot{A}^r$  to the overall momenta is zero. In (54) the parameter  $\xi_h$  is associated to the generator (9),  $d\xi_h$  is a harmonic one-form. the bracket in (54) is constructed from  $d\xi$ . We then have

$$p_r = p_r^0 \sqrt{w} + \Pi_r, \quad (55)$$

where

$$\int_{\Sigma} \Pi_r d\sigma^1 \wedge d\sigma^2 = 0. \quad (56)$$

The contribution of the KK modes to the  $mass^2$  arises from the substitution of (55) into the  $p_r^2$  term in (7), and the use of (56). We obtain

$$mass^2 = T^2((2\pi R)^2 n Im\tau)^2 + \frac{1}{R^2}((m_1^2 + (\frac{m_2 - m_1 Re\tau}{Im\tau})^2)) + T^{2/3} H \quad (57)$$

where the  $H$  is the hamiltonian of the MIM2. It was first obtained and analyzed as a symplectic non commutative theory in [21] and proved to have discrete spectrum in [25],[26]. It has the same expression (7) without of course the global modes that already have being separated. We will discuss  $H$  in the next section. The two explicit terms on the right hand side of (57) are exactly the ones in [3] which allow the identification of  $\tau$  to the moduli of the  $(p, q)$  class of *IIB* superstring solutions. In the next section we will obtain the complete identification of the mass formulas beyond these two terms.

The KK term in (57) can be written as

$$\left(\frac{|m_1\tau - m_2|}{RIm\tau}\right)^2 = \left(\frac{m|q_1\tau - q_2|}{RIm\tau}\right)^2 \quad (58)$$

Where  $q_1$  and  $q_2$  are relatively prime integral numbers. Under (29)

$$\begin{aligned} \tau &\rightarrow \tau' = \frac{a\tau + b}{c\tau + d} \\ R &\rightarrow R' = R|c\tau + d| \end{aligned} \quad (59)$$

and we add now,

$$(q_1, -q_2) \rightarrow (q'_1, -q'_2) \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad (60)$$

We then have

$$\frac{|q'_1\tau' - q'_2|}{R'Im\tau'} = \frac{|q_1\tau - q_2|}{RIm\tau}, \quad (61)$$

we have thus completed the transformation law (29).

## 5 String states in the MIM2 theory

We will consider in this section the states associated to configurations which depend on one local coordinate. Instead of considering local coordinates  $\sigma^a$   $a = 1, 2$ , we will work with  $\widehat{X}^r$   $r = 1, 2$ . The Jacobian of the transformation is  $\frac{1}{2}\epsilon^{ab}\partial_a\widehat{X}^r\partial_b\widehat{X}^s\epsilon_{rs}$  which is nonsingular over  $\Sigma$ . The local coordinates

$$\sigma^1, \sigma^2 \rightarrow \widehat{X}^1, \widehat{X}^2 \quad (62)$$

on single valued scalar fields is then well defined. We then consider within the physical configurations of the MIM2, the string configurations

$$X^m = X^m(\tau, q_1\widehat{X}^1 + q_2\widehat{X}^2), \quad A^r = A^r(\tau, q_1\widehat{X}^1 + q_2\widehat{X}^2) \quad (63)$$

where  $q_1, q_2$  are relative prime integral numbers.  $X^m, A^r$  are scalar fields on the torus  $\Sigma$ , a compact Riemann surface, hence they may always be expanded on a Fourier basis in term of a double periodic variable of that form. The restriction of  $q_1, q_2$  to be relatively prime integral numbers arises from the global periodicity condition. On that configurations all the brackets

$$\{X^m, X^n\} = \{X^m, A^r\} = \{A^r, A^s\} = 0 \quad (64)$$

vanish. The hamiltonian of the string configurations becomes

$$H|_{SC} = T^{-2/3} \int_{\sqrt{W}} \left[ \frac{1}{2} \left( \frac{P_m}{\sqrt{W}} \right)^2 + \frac{1}{2} \left( \frac{\Pi_r}{\sqrt{W}} \right)^2 + \frac{T^2}{2} \{X_h^r, X^m\}^2 + \frac{T^2}{2} \{X_h^r, A^s\}^2 \right] \quad (65)$$

where  $X_h^r$ ,  $r = 1, 2$ , denote the harmonic sector of the one-form  $dX^r$ ,  $r = 1, 2$ , subject to the constraints

$$\left\{ X_h^r, \frac{\Pi_r}{\sqrt{W}} \right\} = 0 \quad \oint_{\mathcal{C}_s} \left( \frac{P_r dX^r}{\sqrt{W}} + \frac{P_M dX^M}{\sqrt{W}} \right) = 0 \quad (66)$$

where  $\mathcal{C}_s$  is the basis of homology associated to the harmonic basis  $\widehat{X}^r$ . The local constraint may be explicitly solved:

$$\frac{\Pi_r}{\sqrt{W}} = T^{2/3} \{X_h^s, \frac{\Pi}{\sqrt{W}}\} \epsilon_{rs} \quad (67)$$

in terms of an unconstrained field  $\Pi$ . We then have for the kinetic term

$$\begin{aligned} \int_{\Sigma} p_r \partial_t X^r &= \int_{\Sigma} \Pi_r \partial_t A_r + \int_{\Sigma} \partial_t (\sqrt{W} p_r^0 A^r) = \\ &= \int_{\Sigma} (T^{-2/3} \dot{\Pi}) (T \{X_h^s, A^r\} \epsilon_{rs}) + \int_{\Sigma} \partial_t (\sqrt{w} p_r^0 A^r). \end{aligned} \quad (68)$$

The total time derivative may be eliminated from the Hamiltonian formulation. This property is valid in general for the MIM2 hamiltonian. We have then solved the local constraint and canonically reduced the theory to a locally unconstrained theory where the new conjugate pair is:

$$(T^{-2/3}\Pi), \quad T\{X_h^s, A^r\}_{\epsilon_{rs}}. \quad (69)$$

We then have

$$H|_{SC} = T^{-2/3} \int_{\Sigma} \sqrt{W} \left[ \frac{1}{2} \left( \frac{P_M}{\sqrt{W}} \right)^2 + \frac{T^2}{2} \{X_h^r, X^m\}^2 \right] + \text{fermionic terms} \quad (70)$$

where now  $M = 1, \dots, 8$ . It is a SUSY harmonic oscillator. It has the same number of bosonic and fermionic creation and annihilation operators, with cancelation of the zero point energy contribution [30]. The potential has the explicit form

$$(2\pi R) \frac{T^2}{2} |\{n\hat{X}^1 + \tau\hat{X}^2, X^M\}|^2 \quad (71)$$

We now perform a change on the canonical basis of homology and a corresponding change of the basis of harmonic one-forms:

$$\begin{aligned} d\tilde{X}^1 &= q_1 d\hat{X}^1 + q_2 d\hat{X}^2 \\ d\tilde{X}^2 &= nq_3 d\hat{X}^1 + q_4 d\hat{X}^2, \end{aligned} \quad (72)$$

given  $q_1$  relatively prime to  $q_2$  and to  $n$ , there always exist  $q_3$  and  $q_4$  such that expression  $\begin{pmatrix} q_1 & q_2 \\ nq_3 & q_4 \end{pmatrix} \in SL(2, Z)$ .

If  $q_1$  and  $n$  have a common factor we attached it to  $\hat{X}^1$  in (71),(72) and proceed with the remaining factors  $\tilde{q}$  and  $\tilde{n}$ .

The harmonic sector when expressed in the new harmonic basis satisfies

$$2\pi RT |nd\hat{X}^1 + \tau d\hat{X}^2| = 2\pi \tilde{R}T |nd\tilde{X}^1 + \tilde{\tau} d\tilde{X}^2| \quad (73)$$

where as in (29)

$$\tilde{R} = R|q_4 - \tau q_3|, \quad \tilde{\tau} = \frac{-nq_2 + \tau q_1}{q_4 - \tau q_3}. \quad (74)$$

It is then in its canonical form. In the case when  $n$  and  $q_1$  have a common factor the formulas are the same with  $n$  and  $q_1$  replaced by  $\tilde{n}$  and  $\tilde{q}_1$ . We

now consider the global constraint. We take the homology basis to be the corresponding one to the new harmonic basis. We are allowed to do that since the global constraint has to be imposed on any basis of homology. The constraint associated to the element of the basis  $\mathcal{C}_2$  does not have contributions from the local degrees of freedom since they depend only on  $\tilde{X}^1$ . We then obtain  $m_2 = 0$ . The constraint associated to expression  $\mathcal{C}_1$  yields

$$m_1 n = N_R - N_L \quad (75)$$

the level-matching condition and

$$(P_1^0)^2 + (P_2^0)^2 = \left( \frac{|\tilde{\tau}|}{\tilde{R}Im\tilde{\tau}} \right)^2. \quad (76)$$

There are also non-stringy states contributing to the matching level conditions. We are freezing those contributions and considering only the string contributions. Let us go on the final step of the argument. We started in (71) with  $\tau$  in the fundamental region  $F$  described at the end of section 3.  $\tilde{\tau}$  in (74) is exactly a general transformation with the subgroup of  $SL(2, Z)$  associated to  $F$ .  $\tilde{\tau}$  is then a generic point on the upper half plane. But any point on the upper half plane may be obtain from the fundamental region of  $SL(2, Z)$  by a Möbius transformation

$$\tilde{\tau} = \frac{q\tau - p}{Q\tau + P} \quad (77)$$

where  $\begin{pmatrix} q & -p \\ Q & P \end{pmatrix} \in SL(2, Z)$ . (the "-" is only conventional). We now notice that the reduced expression of the hamiltonian is covariant under that more general transformation. We then obtain the final expression for the mass contribution of the string states:

$$M_{11}^2|_{SC} = (nT_{11}(2\pi R_{11}Im\tau)^2 + \left( \frac{m|q\tau - p|}{R_{11}Im\tau} \right)^2 + 8\pi^2 R_{11}T_{11}|q\tau - p|(N_L + N_R))(78)$$

where  $(p, q)$  are relatively prime. We have denoted  $R_{11}, \tau$  the transformed variables, and  $T_{11} = T$  the eleven dimensional tension. We notice that  $(p, q)$  may be interpreted as the wrapping of the membrane around the two cycles of the target torus. In fact, we can always re-express the KK-term in terms of  $\tilde{\tau} = \frac{q\tau - p}{Q\tau + P}$ , as

$$\frac{m|\tilde{\tau}|}{\tilde{R}Im\tilde{\tau}} \quad (79)$$

using(29). The corresponding change in the harmonic sector is

$$dX_h = (qmd\tilde{X}^1 + pd\tilde{X}^2) + \tilde{\tau}(-Qnd\tilde{X}^1 + Pd\tilde{X}^2), \quad (80)$$

the hamiltonian is invariant under that change.  $p, q$  and  $Q, P$  are now the winding numbers of the supermembrane. Given  $p, q$  there always exist  $Q$  and  $P$  with the above property, although the correspondance is not unique. The  $(p, q)$  type IIB strings may indeed be interpreted as different wrappings of the MIM2. This nice interpretation was first given in [3] from a heuristic point of view.

We may now compare the mass formula for the string states in the MIM2 with respect to the mass formula of the  $(p, q)$  IIB and IIA strings.

The  $(p, q)$  IIB string compactified on a circle of radius  $R_B$  has tension[3]

$$T_{(p,q)}^2 = \frac{|q\lambda_0 - p|^2}{Im\lambda_0} T^2 \quad (81)$$

where  $\lambda_0 = \xi_0 + ie^{-i\phi_0}$ .  $\xi$  and  $\phi$  can be identified to the scalar fields of the type IIB theory,  $\phi$  corresponds to the dilaton fields.  $\lambda_0$  is the asymptotic value of  $\lambda$  -the axion-dilaton of the type IIB theory- specifying the vacuum of the theory. The perturbative spectrum of the  $(p, q)$  IIB string is [3],

$$M_B^2 = \left(\frac{n}{R_B}\right)^2 + (2\pi R_B m T)^2 + T_{(p,q)} 4\pi(N_L + N_R) \quad (82)$$

If we use following [3] a factor  $\beta^2$  to identify term by term both mass formulas, since there were obtained using different metrics, one gets

$$\begin{aligned} \tau &= \lambda_0 \\ \beta^2 &= \frac{T_{11} A_{11}^{1/2}}{T} \\ R_B^{-2} &= T T_{11} A_{11}^{3/2} \end{aligned} \quad (83)$$

where  $A_{11} = (2\pi R_{11})^2 Im\tau$  is the area of the torus. These relations were obtained in [3] by comparing the first two terms on the right hand side of (78) and (82). They were obtained by counting modes under some assumptions on the supermembrane wrapping modes, as mentioned on one of the footnotes [3]. Here we have derived the expressions from a consistent definition of the MIM2. In addition the comparison of the third terms in (78) and (82) is

completely consistent with relations (83).

We notice that  $A_{11}, \beta$  and  $R_B$  are invariant under (29),  $T_{p,q}$  is also invariant provided  $(p, q)$  transforms under an associated  $SL(2, Z)$  matrix.

The identification of (78) to the mass formula of IIA string compactified on a circle of radius  $R_A$  and tension  $T_A$  may also be performed. In order to have a consistent identification one has to take  $Re\tau = 0$ ,  $p = 1$  and hence  $q = 0$  in (78). The mass formula for the perturbative spectrum of type IIA is

$$M_A^2 = \left(\frac{m}{R_A}\right)^2 + (2\pi R_A n T_A)^2 + T_A 4\pi(N_L + N_R) \quad (84)$$

Identification after the limit process explained in section 6 of the winding contributions and KK ones using a factor  $(\beta\gamma)$  to compare the  $mass^2$  formulas, since they are obtained using different metrics, yields

$$\begin{aligned} R_A &= \beta\gamma R_{11} \\ T_A &= \gamma^{-2}(Im\tau)^{1/2}T \end{aligned} \quad (85)$$

which imply

$$(2\pi R_A R_B) = \left(\frac{1}{T_A T Im\tau^{1/2}}\right)^{1/2} \quad (86)$$

We have thus obtained the  $(p, q)$  IIB and IIA perturbative spectrum, when compactified on circles  $R_B$  and  $R_A$  respectively, from the string states on the MIM2.

## 5.1 $SL(2, Z)$ $(\mathbf{p}, \mathbf{q})$ -strings on IIA

We introduce now the dual tori  $\tilde{T}^2$ .  $T^2$  was defined in terms of the moduli  $(R, \tau)$ . The moduli of  $\tilde{T}^2$ ,  $(\tilde{R}, \tilde{\tau})$  are defined by the relation

$$z\tilde{z} = 1 \quad (87)$$

where the dimensionless variables  $z, \tilde{z}$  are

$$z = T_{11} A_{11} \tilde{Y} \quad (88)$$

$$\tilde{z} = T_{11} \tilde{A}_{11} Y \quad (89)$$

$A_{11} = (2\pi R)^2 Im\tau$ ,  $\tilde{A}_{11} = (2\pi\tilde{R})^2 Im\tilde{\tau}$  are the area of the  $T^2$  and  $\tilde{T}^2$  and

$$Y = \frac{RI m\tau}{|\tau|} \quad \tilde{Y} = \frac{\tilde{R}Im\tilde{\tau}}{|\tilde{\tau}|}, \quad (90)$$

We now look for the stringy states of the MIM2 wrapping on a  $\tilde{T}^2$  torus. Their contribution to the mass formula is given by

$$\tilde{M}_{11}^2 = (mT(2\pi\tilde{R})^2 Im\tilde{\tau})^2 + \left(\frac{n|\tilde{\tau}|}{\tilde{R}Im\tilde{\tau}}\right)^2 + 2\pi\tilde{R}T|\tilde{\tau}|4\pi(N_L + N_R) \quad (91)$$

We may now match the winding term of  $M_{11}^2$  with the KK term of the  $\tilde{M}_{11}^2$  and viceversa. We obtain only one relation, the other two are identically satisfied, which gives the proportionality constant  $\alpha$  between  $M_{11}$  and  $\tilde{M}_{11}$ :

$$\alpha^2 = \frac{\tilde{z}}{z} \quad (92)$$

It defines a T-duality relation on MIM2. We may take  $\tilde{\tau} \rightarrow \frac{a\tau+b}{c\tau+d}$ ,  $R \rightarrow R|c\tau + d|$  and obtain the corresponding  $\tilde{M}_{11}$  formula. For the  $(p,q)$  string states contribution to  $M_{11}^2$  there is associated a multiplet of infinite states on the MIM2 wrapping on  $\tilde{T}^2$ , which should correspond to D-brane bound states in lower dimensions. The precise relation has not yet been determined. There is an interesting nonlinear relation between the  $SL(2, Z)$  on  $T^2$  and the  $SL(2, Z)$  on  $\tilde{T}^2$ . We will report on it elsewhere.

## 5.2 The decompactification limits

We consider first the limit when  $R_B \rightarrow \infty$ ,  $\tau$  in the upper half plane and  $n \rightarrow \infty$  as defined below. If  $R_B \rightarrow \infty$  then  $A_{11} = (2\pi R_{11})^2 Im\tau \rightarrow 0$  and necessarily  $R_{11} \rightarrow 0$ . The KK term in the  $M_{11}^2$  of the MIM2 behaves as

$$\frac{m^2|q\tau - p|^2}{(R_{11}Im\tau)^2} \sim \frac{m^2}{R_{11}^2} \rightarrow \infty. \quad (93)$$

The winding term in  $M_{11}^2$  behaves as

$$(nA_{11}T_{11})^2 \sim n^2 R_{11}^4 \sim \left(\frac{1}{R_{11}}\right)^2 \rightarrow \infty \quad \text{if} \quad n \sim \frac{1}{R_{11}^3}, \quad (94)$$

we thus consider  $n \rightarrow \infty$  in the above way. The hamiltonian contribution, first in the semiclassical approximation, to the  $M_{11}^2$  in terms of  $p_1, p_2$  the two integers identifying the MIM2 modes, is

$$\sum_{p_1, p_2} 2\pi R_{11} T_{11} |np_2 + \tau p_1| \quad (95)$$

where  $R_{11} Im\tau \rightarrow 0$  while  $R_{11}n \sim \frac{1}{R_{11}^2} \rightarrow \infty$ . The hamiltonian contribution of the exact theory is bounded from below, up to perturbative (in the operatorial sense) terms by its semiclassical hamiltonian [25],[26],[18]. That means that eigenvalues of the exact theory are bounded from below by the semiclassical ones, up to small perturbations. These perturbative terms arise from the fermionic cubic interactions and were rigourously characterized in [25],[26].

We then conclude that the contribution to the energy of (95) is mainly from the  $p_1$  modes only. That is, given a fixed value of the energy  $E$ , for small enough values of  $R_{11}$  the eigenvalues less than  $E$  arise from the  $p_1$  modes only. The total energy is then

$$2\pi R_{11} T_{11} |\tau| 4\pi(N_L + N_R). \quad (96)$$

It may be expressed now in terms of  $\tilde{\tau}$  on the fundamental region and from (29) we obtain

$$2\pi \tilde{R}_{11} T_{11} |q\tilde{\tau} - p| 4\pi(N_L + N_R) \quad (97)$$

as expected for the energy spectrum of 10 dimensional  $(p, q)$  IIB strings, in the eleven dimensional metric.

Let us now consider the limit when  $R_A \rightarrow \infty$  and consequently  $A_{11} \rightarrow \infty$ . The limit is defined by taking  $Im\tau \rightarrow \infty$ ,  $Re\tau \rightarrow \infty$ , for any finite  $n$  and  $R_{11} \rightarrow 0$ . We then have

$$\begin{aligned} \frac{m^2 |q\tau - p|^2}{(R_{11} Im\tau)^2} &\sim \frac{m^2}{R_{11}^2} \rightarrow \infty \\ (nA_{11} T_{11})^2 &\sim R_{11}^4 Im\tau^2 \sim \frac{1}{R_{11}} \rightarrow \infty \end{aligned} \quad (98)$$

provided  $Im\tau \sim \frac{1}{R_{11}^3}$ , which we assume. The hamiltonian contribution to  $M_{11}^2$  is then

$$\begin{aligned} 2\pi R_{11} T_{11} Im\tau &\sim \frac{1}{R_{11}} \rightarrow \infty \\ 2\pi R_{11} T_{11} n &\rightarrow 0 \end{aligned} \quad (99)$$

By the same reason as before the contribution for the small  $R_{11}$  is mainly from the  $p_2$  modes only, with a tension independent of  $\tau$ . In both decompactified limits we have considered the right hand member of the topological constraint becomes

$$nA_{11} \sim \frac{1}{R_{11}} \rightarrow \infty, \quad (100)$$

they correspond to limit cases. We will comment on these limits in the conclusions.

### 5.3 The wrapping of the MIM2 on a circle

Let us consider now the wrapping of the MIM2 on a circle but with a finite value of the topological constraint. We will show that this is possible if the MIM2 wraps only once. It then answers, in the context of the MIM2, the puzzle raised in [3]: “Why the membrane can be wrapped on a torus any number of times but only one on a circle?” The limit we are going to consider was first discussed in [30]. It considers the limit when the radius of one of the circles on the target space goes to zero but the supermembrane wraps infinite number of times around it in a way that  $l_1 R_{11} = R_1$  finite and  $R_{11} Im\tau = R_2$  finite. It thus imply  $Im\tau \rightarrow \infty$ . We then have for the right hand member of the topological constraint

$$nA_{11} = l_2 A \quad (101)$$

where  $A = (4\pi^2 R_1 R_2)$  and  $l_2$  is the winding number around the circle of radius  $R_2$ . To analyze this limit it is better to consider an expression of the hamiltonian in terms of the windings  $l_1$  and  $l_2$ :  $l_1 l_2 = n$  explicitly. According to the remarks in section 3 we can always work in terms of  $n$  or equivalently in terms of  $l_1$  and  $l_2$ . All possible decompositions of  $n$  in terms of  $l_1$  and  $l_2$  are equivalent as a consequence of the symmetries of the MIM2. We have now for the winding contribution,

$$(nA_{11}T_{11})^2 = (l_2AT_{11})^2, \quad (102)$$

and for the KK contribution, it is finite only when  $q = 0, p = 1$ :

$$\left(\frac{m}{R_2}\right)^2. \quad (103)$$

That is, the only possibility is to have  $q = 0, p = 1$ . Finally the hamiltonian contribution at the semiclassical level is

$$\sum_{p_1, p_2} 2\pi R_2 T_{11} \left| i \frac{R_1}{R_2} p_2 + l_2 p_1 \right|, \quad (104)$$

which in the string limit we consider below becomes exact. We may now interpret  $\frac{R_2}{R_1}$  as  $Im\tau$  and obtain the canonical contribution of a MIM2 with  $Re\tau = 0$  and winding number  $l_2$ . We consider now together with  $R_{11} \rightarrow 0$  no dependence on  $\widehat{X}^1$ , that is on the  $p_1$  modes. We are considering the stringy states obtained from the above MIM2 by freezing the remaining membrane states. We then obtain exactly the nine dimensional  $mass^2$  contribution of the IIA superstring compactified on a circle of radius  $R_2$  with winding  $l_2$ , in the 11 dimensional metric. In fact, the matching level condition arising from the MIM2 constraints in the above limit is

$$ml_2 = N_R - N_L. \quad (105)$$

The wrapping of the supermembrane on a circle to obtain the type IIA theory is then only possible if  $p = 1, q = 0$ .

## 6 Discussion and conclusions

We obtained the quantum symmetries of the MIM2 theory, a sector of the  $D = 11$  supermembrane. There is a  $SL(2, Z)$  symmetry realized by the area preserving diffeomorphisms not homotopic to the identity. It acts as the modular group on the Teichmüller space associated to the base manifold. This symmetry transforms equivalent classes of maps, under area preserving diffeomorphisms homotopic to the identity (gauge transformations), into equivalent classes. The hamiltonian is invariant under these transformations. In addition the hamiltonian density is invariant under a  $SL(2, Z)$  transformations acting on the moduli of the target space. Although this transformation on the Teichmüller parameter  $\tau$  of the target torus is a Möbius transformation, the equivalence classes of tori under it are not the conformal classes. Both transformations are not equivalent and both are relevant in the analysis of the MIM2 hamiltonian. Consequently the usual statement that the  $SL(2, Z)$  duality symmetry of *IIB* superstrings is associated to the reparametrization invariance on the M-theory is not precise. We explicitly

showed that the duality symmetry of *IIB* superstrings has its origin in the supermembrane theory and it is related, at least in the MIM2 sectors to the space of holomorphic immersions, or more generally minimal immersions, from the base manifold to the target space. The space of holomorphic immersions is a very interesting one and was discussed in a different context in [29]. The holomorphic immersion may be constructed, in the case discussed in this work, in terms of the harmonic maps. The above symmetries allow to express the harmonic maps in a canonical form. The corresponding harmonic one-forms are covariant under a subgroup of  $SL(2, Z)$ : the matrices  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z)$  with  $b = nb_1$ , where  $b_1$  is an integral number and  $n$  the winding number of the MIM2.  $n$  is introduced in the theory through the topological constraint. The corresponding fundamental region on the upper half plane may be determined from the fundamental region of the modular group. In particular, the symmetries of the MIM2 allow to understand why the different winding modes of the supermembrane are equivalent. This point was left as an assumption in [3]. From the action of the MIM2 we obtained directly the winding and KK contributions to the nine dimensional  $mass^2$  formula, in exact agreement with [3]. We obtain all  $(p, q)$  type *IIB* string states within the MIM2. This is done by considering the matching level constraints of the MIM2 and the string configurations on it. By freezing the pure membrane states satisfying the matching level condition, the contribution of the stringy configurations to the  $mass^2$  may be determined. This can be done from the full theory without any approximation, nor any particular limit, using only its symmetries. This  $mass^2$  contribution is exactly the  $mass^2$  formula for the  $(p, q)$  type *IIB* superstring compactified on a circle, including the winding, KK and oscillator contributions. We also obtained the perturbative mass spectrum of *IIA* superstring compactified on a circle from the MIM2. This was done following [30] by taking the limit  $R_{11} \rightarrow 0$ ,  $Im\tau \rightarrow \infty$ , in such a way that the topological constraint has a finite right-hand member. This limit is only consistent if  $p = 1, q = 0$ , showing that the supermembrane can wrap a torus any number of times but a circle only once. This was a puzzle raised in [3]. The solution in [4] has the same conclusion although the limits taken there are different from ours.

We also discussed decompactifications limits, the  $R_B \rightarrow \infty$  one implies  $A_{11} \rightarrow 0$ . In that limit the spectrum of the MIM2 hamiltonian reduces to the ten-dimensional  $(p, q)$  type *IIB* spectrum. This is in agreement with [31],

where the  $A_{11} \rightarrow 0$  limit of the supermembrane was analysed.

Finally we introduced the MIM2 theory compactified on a T-dual torus. In the same way as the  $(p, q)$  IIB string states can be identified on the MIM2 there is multiplet of  $(p, q)$  non-perturbative states associated or the MIM2 with a corresponding T-dual  $SL(2, Z)$  symmetry. Although the relation of these states to lower dimensional bound states has not yet been determined, we expect this symmetry to be the conjectured type IIA  $SL(2, Z)$ .

The KK states of the supermembrane may be associated to a dynamical tension, from type IIB perspective it is seen as an additional dimension as pointed out by [11],[12]. The stability of the  $SL(2, Z)$  dyonic strings is a natural consequence of the quantum stability of the MIM2, since it is contained in its spectrum and it fully agrees with the results of stability of these bound states of strings founded in [4] and [8]. The full symmetry in fact encodes also the transformation of the radii, and the gauge field in a nontrivial way. This type of S-duality is suprising and may be constitute an indication of a putative lift of the degeneracy of the radii.

We are dealing with T-duality and S-duality at the non-perturbative level. The T-duality which we consider represents an extension of the Buscher rules. In [2] they conjectured in the context of supergravity the existence of some simmetries: the nonperturbative version of T-duality  $O(n, n, Z)$  and S-duality  $SL(2, Z)$  which in low dimensions could be immersed a unified discrete symmetry group  $U$ . Some cases like the  $N = 8, 4$  supergravity analysis were explicitly considered. The low effective action of the MIM2 should present this discrete symmetry  $U$  in lower dimensions. In fact, in 4D it has  $N = 1$  supersymmetries and so the particular discrete groups proposed do not hold on this analysis. However, it has been pointed out that the  $N = 1$  in 4D supergravity the discrete U-group is associated to the symplectic groups [32]. The action of the supermembrane in 4D with  $N = 1$  supersymmetry has been recently obtained and it contains as a symmetry group  $Sp(6, Z)$ . One could think that the  $Sp(6, Z)$  may be related to the full discrete group of U-duality for  $N = 1$  supersymmetry however further study to determine it would be needed.

Associated to the MIM2 theory there exists a  $SL(2, Z)$  multiplet of conserved charges  $(q, p)$  whose field strengths are the  $H_3, F_3$  on type IIB string theory. From the type IIB supergravity effective actions this may represent a nonperturbative origin of these set of fluxes, -a well known open question-. Fluxes, [33], have become very important in the search for realistic compact-

ifications. They are able to smooth singularities, give masses to moduli as well as allow to break supersymmetry in a controlled manner. At present their values are completely arbitrary, and so far there has been lacking an explanation of its origin at a noneffective theory. Our results indicate that at least a subset of them would be related to the winding of the MIM2 and consequently can acquire a more intrinsic meaning since for a particular theory with a fixed value of the central charge  $n$ , those values are specified. An analogous reasoning can be in principle applied to the subset of fluxes associated to the  $SL(2, Z)$  on the *IIA* supergravity effective theory.

At the level of D-brane interpretation, the supermembrane with central charges represents the M-theory lifting of some bound states. We have seen that the MIM2 contains the bound states of dyonic strings of *IIA* and *IIB* theories. In [34] it was found that the MIM2 in 9D corresponds to a bundle of D2-D0 states in the type IIA picture. Bound states of Dp-branes are associated to mixed boundary conditions on the strings attached to the (Dp,Dq) branes, [35]. In the case of (F,Dp) branes the boundary state formalism may also be applied [36]. It would be interesting to precise the relation between both types of bound states since they are examples of nonperturbative corrections in string theory with well-developed techniques of computation, see for example applications in relation with the  $SL(2, Z)$  duality [37].

## 7 Acknowledgements

We would like to thank for very fruitful conversations to M. Frau, O. Lechtenfeld, A. Lerda, P. Merlatti, H. Nicolai, T. Ortin, J. Rosseel and A. Uranga. M.P.G.M. is partially supported by Dipartimento di Fisica di Universita di Torino under European Community's Human Potential Programme and by the Italian MUR under contracts PRIN-2005023102 and PRIN-2005024045. I.M. is funded by Decanato de Investigaciones y Desarrollo (DID-USB). Proyecto G11, USB, Venezuela. The work of A. R. is supported by a grant from MPG, Albert Einstein Institute, Germany and I.M. and A.R. are also supported by PROSUL, under contract CNPq 490134/2006-08.

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