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SYMMETRIES, SINGULARITIES AND THE DE-EMERGENCE OF SPACE*

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Recent work has revealed intriguing connections between a Belinsky–Khalatnikov–Lifshitz-type analysis of spacelike singularities in general relativity and certain infinite-dimensional Lie algebras, particularly the "maximally extended" hyperbolic Kac–Moody algebra E_{10}. In this essay we argue that these results may lead to an entirely new understanding of the (quantum) nature of space(–time) at the Planck scale, and hence — via an effective "de-emergence" of space near the singularity — to a novel mechanism for achieving background independence in quantum gravity.

Keywords: Cosmology, singularities; quantum gravity; Kac–Moody algebras.

1. Introduction

A key challenge for a future theory of quantum gravity is the need to explain the fate of space–time singularities, where classical general relativity breaks down, and space and time "come to an end." This challenge concerns in particular spacelike (cosmological) singularities, the most prominent example of which is the big bang singularity that gave birth to our universe. At issue here is not so much the question of whether and how quantum effects might "*resolve*" the singularity, but the very meaning of the term "singularity resolution" itself. The latter hinges essentially on what the correct theory is, and will almost certainly require new concepts that go beyond established notions of space and time.

A naive extension of quantum mechanics would suggest that singularity resolution works essentially in the same way for quantum general relativity as it does

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for the hydrogen atom. There, as is well known, the expected classical "collapse" of an electron toward the $1/r$ singularity of the Coulomb potential is resolved by the Heisenberg uncertainty principle and the quantum-mechanical smearing of the electron wave function, which allows the electron to stay in a stable bound state around $r = 0$. This possibility is usually invoked in canonical approaches to quantum gravity, where one would thus hope to be able to replace the classical time evolution of the spatial geometry, described as a "trajectory" in the space of 3-geometries (i.e. Wheeler–DeWitt superspace), by a quantum-mechanical description in terms of a wave functional which "smears" the 3-geometries over the singular classical trajectories. This line of thought has been extensively pursued in the simplified context of the mini-superspace approximation, with varying results: while models derived from (or motivated by) loop quantum gravity generally tend to predict a "bounce" providing a quantum-mechanical bridge between two classical universes,[1] the more conventional quantum-geometrodynamical treatment of the mini-superspace Wheeler–DeWitt equation shows no such evidence.[2]

In this essay we would like to outline a very different proposal, motivated by recent work,[3,4] where the singularity is "resolved" by the effective "disappearance" of space, and the replacement of the dynamical fields, most notably the spatial metric $g_{ij}(t, \mathbf{x})$, by a single dynamical variable $\mathcal{V}(t)$ belonging to an *infinite-dimensional* coset space and depending only on time. Our proposal is based on the discovery of a profound relation between an analysis *à la* Belinsky–Khalatnikov–Lifshitz (BKL) of spacelike singularities[5,6] on the one hand, and the theory of indefinite Kac–Moody algebras on the other[7,8] (see Ref. 9 for an introduction to the theory of Kac–Moody algebras). More specifically, the main conjecture of Ref. 3, formulated in the context of the maximally extended $D = 11$ supergravity,[10] relates a BKL-type expansion in spatial gradients at a given spatial point to a Lie-algebraic expansion in the height of certain roots of the "maximally extended" hyperbolic Kac–Moody algebra E_{10}. Thereby the time evolution of 10-dimensional geometric data is mapped onto an *effectively one-dimensional dynamics*, namely a (constrained) null geodesic motion in the infinite-dimensional coset space $E_{10}/K(E_{10})$, which is formally defined as the quotient of the group E_{10} by its maximal compact subgroup $K(E_{10})$.[a] The appearance of E_{10} in this context is both unexpected and remarkable, because E_{10} enjoys a similarly distinguished status among the infinite-dimensional Lie groups[b] as the exceptional algebra E_8 does in the Cartan–Killing classification of the simple finite-dimensional Lie groups.[13]

2. BKL and Cosmobilliards

We start by summarizing the BKL-type analysis of the *near-spacelike singularity limit*, i.e. the asymptotic behavior of various fields, and in particular the (spatial)

[a]The emergence of E_{10} in the dimensional reduction of maximal supergravity to one dimension had already been conjectured long ago.[11] See also Ref. 12 for a conceptually very different proposal based on E_{11}.

[b]For simplicity of notation, we denote the group and its Lie algebra by the same symbol.

metric, near a singular hypersurface, here taken to be "located" at $T = 0$ in proper time T. To this end, it is convenient[8] to decompose the $D = 11$ metric $g_{\mu\nu}$ into nondynamical and dynamical components, namely the lapse N and the shift vector N^m (set here to zero), and into "diagonal" and "off-diagonal" components $e^{-2\beta^a}$ and $\theta^a{}_i$, respectively, such that the line element becomes

$$ds^2 = -N^2dt^2 + \sum_{a=1}^{10} e^{-2\beta^a} \theta^a{}_i \theta^a{}_j dx^i dx^j. \tag{1}$$

Here, the "off-diagonal" components $\theta^a{}_i$ are *upper triangular matrices* with 1's on the diagonal and enter the Iwasawa decomposition of the spatial metric g_{ij}. We choose a gauge for N in terms of g_{ij} in such a way that $N \sim \mathcal{O}(T) \to 0$ when $T \to 0$. Thus $t \sim -\log T$ becomes a "Zeno-like" time coordinate with $t \to +\infty$ as $T \to 0$.

The Hamiltonian constraint, at a given spatial point, can be written as

$$\mathcal{H}(\beta^a, \pi_a, Q, P) = \tilde{N}\left[\frac{1}{2}G^{ab}\pi_a\pi_b + \sum_A c_A(Q, P, \partial\beta, \partial Q)\exp\left(-2w_A(\beta)\right)\right], \tag{2}$$

with the rescaled lapse $\tilde{N} \equiv N/\sqrt{g}$, where g is the determinant of the spatial metric. Here π_a $(a = 1, \ldots, 10)$ are the canonical momenta conjugate to the logarithmic scale factors β^a, and G^{ab} is the (Lorentzian) DeWitt "superspace" metric induced by the Einstein–Hilbert action. (Q, P) denote the remaining canonical degrees of freedom associated with the off-diagonal metric components $\theta^a{}_i$ and various matter degrees of freedom [such as the 3-form $A_{\mu\nu\lambda}(t, \mathbf{x})$ of $D = 11$ supergravity], as well as their respective conjugate momenta, and $(\partial\beta, \partial Q)$ are the *spatial* gradients of β and Q. The exponential terms in (2) involve the *linear forms* $w_A(\beta) \equiv w_{Aa}\beta^a$, where the specific coefficients w_{Aa} and the range of labels A depend on the model under consideration (see Ref. 8 for details).

The BKL limit $T \to 0$ amounts to considering the *large β limit* in Eq. (2), and is determined by the exponential "walls" $\propto \exp(-2w_A(\beta))$.[8] The latter can be ordered in "layers." The first layer, corresponding to the subset of "dominant walls" $w_{A'}(\beta)$ — whose coefficients $c_{A'}$ can be shown to be nonnegative — confines the motion in β space to a *fundamental billiard chamber* defined by the inequalities $w_{A'}(\beta) \geq 0$. The remaining (subdominant) exponential walls introduce fractional corrections into the chaotic motion of (β^a, π_a) within the fundamental billiard chamber. All the other dynamical variables (Q, P), together with their spatial gradients, "freeze" as $T \to 0$, and thus exhibit a very different behavior in this limit.

3. Coset Space Dynamics

Let us next consider an *a priori* very different dynamical system, namely null geodesic motion on the infinite-dimensional coset space $E_{10}/K(E_{10})$. A curve on this coset space can be parametrized by a time-dependent (but *space-independent*) element of the E_{10} group in upper triangular (Iwasawa) form: $\mathcal{V}(t) = \exp h(t) \exp \nu(t)$. Here, $h(t) = \beta^a(t)H_a$ belongs to the ten-dimensional Cartan subalgebra (= CSA) of E_{10}. Our use of the same notation as above is justified by the fact that we will eventually identify the ten CSA coordinates β^a of E_{10} with the logarithms of the ten "diagonal" components of the spatial metric g_{ij} introduced above. On the other hand, $\nu(t) = \sum_{\alpha>0} \nu^\alpha(t) E_\alpha$ belongs to a (Borel) subalgebra of E_{10} and has an infinite number of components labeled by positive roots α of E_{10}. The geodesic action is formally very simple; it reads

$$\int dt\, \mathcal{L}(t) = \int \frac{dt}{n(t)} \langle \mathcal{P}(t) | \mathcal{P}(t) \rangle, \tag{3}$$

where $\mathcal{P} := (1/2)\big[\dot{\mathcal{V}}\mathcal{V}^{-1} + (\dot{\mathcal{V}}\mathcal{V}^{-1})^T\big]$ is the "velocity"[c] $\langle .|. \rangle$ is the standard invariant bilinear form generalizing the finite-dimensional matrix trace,[9] and $n(t)$ is a one-dimensional "lapse" needed to ensure (time) reparametrization invariance of the action (3). The Zeno-like coordinate time t of the previous section is recovered upon identifying n with the rescaled lapse $\tilde{N}$ introduced after (2) and choosing the gauge $n(t) = 1$.

Varying (3) w.r.t. to the lapse n, we obtain the Hamiltonian constraint:

$$H(\beta^a, \pi_a, \nu, p) = n \left[\frac{1}{2} G^{ab} \pi_a \pi_b + \sum_{\alpha>0} \sum_{s=1}^{\mathrm{mult}(\alpha)} (\Pi_{\alpha,s}(\nu^\alpha, p_\alpha))^2 \exp\big(-2\alpha(\beta)\big) \right], \tag{4}$$

where π_a denotes the conjugate momenta of the ten diagonal CSA coordinates β^a, and p_α denotes the conjugate momenta of the "off-diagonal" coordinates ν^α parametrizing the Borel part of $\mathcal{V}$. The sum on the r.h.s. of (4) ranges over all positive roots α of E_{10} with their multiplicities [= mult(α)]. We recall that the roots α are *linear forms* on the CSA, i.e. we have $\alpha(\beta) \equiv \alpha_a \beta^a$ for the exponents in (4). Although the dynamics encapsulated in (4) is very complicated, a general feature is that in order to satisfy $H = 0$, we must always have $G^{ab}\pi_a\pi_b \leq 0$; this means that the coset null geodesics must always maintain a future-directed CSA velocity π^a, and hence *cannot bounce.*

4. Correspondence between BKL and Coset Space Dynamics

The formal similarity between the gravitational Hamiltonian (2) (considered at a given spatial point) and the coset Hamiltonian (4) is evident, but the correspondence between the two extends considerably farther. In particular, while the metric G^{ab} entering (4) is the restriction of the invariant bilinear form on E_{10} to its CSA, it happens to be *identical* with the DeWitt metric appearing in (2). This fact enables us to *identify the space of logarithmic scale factors with the Cartan subalgebra of* E_{10} (as anticipated by our notation). Moreover, one can analyze the asymptotic dynamics of the coset variables $(\beta^a, \pi_a, \nu^\alpha, p_\alpha)$ in the limit of large β's. At first order in an expansion in "height" of the simple roots of E_{10}, one finds that the CSA variables β

[c]Here the transpose operation T denotes the negative of the Chevalley (or "compact") involution ω.[9] The elements of the Lie subalgebra $K(E_{10})$ are thus "antisymmetric."

are confined to a chaotic billiard motion within the Weyl chamber of E_{10}. The latter is defined by the inequalities $\alpha_i(\beta) \leq 0$, where α_i $(i = 1, \ldots, 10)$ are the *simple roots* of E_{10}, and turns out to *coincide* with the fundamental BKL billiard chamber defined by the dominant potential walls $w_{A'}(\beta)$ for $D = 11$ supergravity. Consideration of the subleading exponential walls in both models now shows that one can actually identify the two dynamics *up to height 30*, i.e. much beyond the leading billiard dynamics (corresponding to height 1 only).[3] This result suggests the existence of a *hidden equivalence* between the two models, i.e. the existence of a map preserving the dynamics between the infinite tower of coset variables $(\beta^a, \pi_a, \nu^\alpha, p_\alpha)$, and the infinite sequence of spatial Taylor coefficients $(\beta, \pi, Q, P, \partial\beta, \partial Q, \partial^2\beta, \partial^2 Q, \ldots)$ formally describing the dynamics of the (super)gravity variables (β, π, Q, P) in the neighborhood of some given spatial point. In this way, a BKL-type expansion in spatial gradients (also termed a "small tension expansion" with space as a kind of elastic medium) gets related to a purely Lie-algebraic expansion in terms of heights of roots. While the full details of this correspondence (which is expected to be very nonlocal in the space–time fields) remain to be worked out, it has been possible recently to extend these results to the fermionic sector on both sides.[14–16]

Most importantly for our present proposal, certain (partially) known *quantum corrections* to the classical supergravity action can be shown to be compatible with specific terms, of very large height, present in the coset action.[4] For instance, the leading term quartic in the Weyl curvature,

$$\begin{aligned}\mathcal{L}^{(4)} = 192E(&-C^{ABCD}C_{AB}{}^{EF}C_{CE}{}^{GH}C_{DFGH} \\ &+4\,C^{ABCD}C_{A}{}^{E}{}_{C}{}^{F}C_{E}{}^{G}{}_{B}{}^{H}C_{FGDH}),\end{aligned} \tag{5}$$

is dominated, near the singularity, by an exponential term $\propto \exp[-2\alpha(\beta)]$ in the coset Hamiltonian (4) for a specific *imaginary* E_{10} root α of height (-115). Detailed study of the *sign* of the combination of curvature terms in (5) has established an inequality[4] confirming the *no-bounce* property exhibited by the coset dynamics, explained after (4).

5. The Cosmological Singularity: A New Paradigm?

The evidence summarized above suggests an entirely new picture of the (quantum) fate of space and time at the cosmological singularity. Namely, we here propose to take seriously the idea that near the singularity (i.e. when the curvature gets larger than the Planck scale) the description in terms of a spatial continuum and space–time based (quantum) field theory breaks down, and should be replaced by a much more abstract Lie-algebraic description. Thereby the information previously encoded in the spatial variation of the geometry and of the matter fields gets transferred to an infinite tower of Lie-algebraic variables depending only on "time." In other words, we are led to the conclusion that space — and thus, upon quantization, also space–time — actually *disappears* (or "de-emerges") as the singularity is approached.[d] There is no "quantum bounce" bridging the gap between an incoming collapsing and an outgoing expanding quasi-classical universe. Instead, "life continues" at the singularity for an *infinite affine time*, but with the understanding that (i) dynamics no longer "takes place" in space, and (ii) the infinite affine time interval [measured, say, by the Zeno-like time coordinate t of (3)] corresponds to a sub-Planckian interval $0 < T < T_{\text{Planck}}$ of geometrical proper time.

Upon quantization, the geodesic equations of motion following from (3) are replaced by a quantum version of the Hamiltonian constraint (4), analogous to the Wheeler–DeWitt equation, and acting on some "wave function of the universe" $\Psi = \Psi(\beta^a, \nu^\alpha)$ depending on the coset variables. This, then, is the step where time also "disappears": as in all canonical approaches to quantum gravity, the wave function (or functional) Ψ no longer depends on any extrinsic "time" (although one can, of course, choose a "clock field" from the coset variables so as to define an "operational" time, in terms of which the quantum dynamics of the remaining variables can be parametrized). The quantum constraint would take the form of a Klein–Gordon-like equation[e]

$$\Box\Psi(\beta^a, \nu^\alpha) = 0, \tag{6}$$

where $\Box$ is the (formal) Laplace–Beltrami operator on the infinite-dimensional (Lorentzian) coset manifold $E_{10}/K(E_{10})$. In analogy with the discretization of finite-dimensional duality symmetries upon quantization, this would then suggest that Ψ ultimately might turn out to be a "modular form" over the arithmetic group $E_{10}(\mathbb{Z})$.[17,18]

6. Outlook

If correct, the picture outlined here will not only affect our understanding of what "happens" at the cosmological singularity, but may also shed completely new light on the issue of background independence in quantum gravity. More succinctly, taking the quantum coset dynamics (6) as a guiding principle, the correct theory of quantum gravity may well turn out to be background-independent in the sense that near the singularity, the theory — rather than "quantizing" the spatial geometry, or some other spatially extended background structure — simply does away with the background altogether, whence the whole issue will become moot!

Let us also note some potentially important implications of this picture for the so-called "information loss paradox" in black hole physics. Indeed, the present ideas might also be applied to the case of a "localized" big crunch (as the one inside a black hole formed within an asymptotically flat space–time). It would then suggest that some of the information contained within the horizon might transmigrate to

[d]We have in mind here a "big crunch," i.e. where we move "back in time" *toward* the singularity. Conversely, and *mutatis mutandis*, we would say that space "appears" or "emerges" at the big bang.

[e]Or a "Dirac-like" (first order) constraint if fermions are included.[14–16]

the state of motion of a coset "baby particle" (whose dynamics describes physics at sub-Planckian scales near the big crunch). The Hawking evaporation of the black hole containing this localized big crunch poses interesting conceptual challenges with regard to an infinite affine "coset lifetime" near the singularity.

As is well known, symmetry concepts have been of central importance in the advancement of theoretical physics over the last century. They have been a key ingredient in the development of the two most successful theories of physics, namely general relativity (via the principle of general covariance) and the standard model of elementary particle physics (via gauge invariance and Yang–Mills quantum field theories). In view of its distinguished place among all Lie algebras, E_{10} is a most worthy candidate for symmetry of nature, deeply intertwining space–time with matter degrees of freedom, and thus necessarily implying a unification of gravity and matter. For this reason, one may also anticipate for it a key role in elucidating the quantum nature of space–time, and hence space–time singularities.

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