NO MASS DROP FOR MEAN CURVATURE FLOW OF MEAN CONVEX HYPERSURFACES

JAN METZGER AND FELIX SCHULZE

Abstract

A possible evolution of a compact hypersurface in $\mathbb{R}^{n+1}$ *by mean curvature past singularities is defined via the level set flow. In the case where the initial hypersurface has positive mean curvature, we show that the Brakke flow associated to the level set flow is actually a Brakke flow with equality. As a consequence, we obtain the fact that no mass drop can occur along such a flow. A further application of the techniques used above is to give a new variational formulation for mean curvature flow of mean convex hypersurfaces.*

1. Introduction

Let $M \subset \mathbb{R}^{n+1}$ be a smooth, compact n-dimensional submanifold without boundary, and let $(M_t)_{t\in[0,T)}$ be the maximal smooth evolution of M by mean curvature flow. Since M is compact, the maximal time of existence T is finite, and in general, the flow develops singularities before the surfaces vanish. One way to define a weak solution past singularities is the level set flow of Chen, Giga, and Goto [2] and Evans and Spruck [5]. Let us briefly recall one way of defining the level set flow. It uses the so-called avoidance principle: if two smooth mean curvature flows (where at least one of them is compact) are disjoint at time t_0, then they remain so for all times $t > t_0$. A weak mean curvature flow, generated by M, is a closed subset $\mathscr{M}$ of space time $\mathbb{R}^{n+1} \times \mathbb{R}^+$ such that for

$$M_t := \{x \mid (x,t) \in \mathscr{M}\},$$

we have $M_0 = M$, and the family of sets $(M_t)_{t\geqslant 0}$ satisfy the above avoidance principle with respect to any smooth mean curvature flow. The level set flow of M is then characterized as the unique maximal weak mean curvature flow generated by M. Assume now that M has nonnegative mean curvature. Following [15] and [13], the level set flow $\mathscr{M}$, generated by M, has further properties. Let $K \subset \mathbb{R}^{n+1} \times \mathbb{R}^+$ be the compact set enclosed by the level set flow $\mathscr{M}$, so that $\partial K = \mathscr{M}$. Then $M_t = \partial K_t$,

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where $K_t = \{x \in \mathbb{R}^{n+1} \mid (x, t) \in K\}$, and the family of Radon measures

$$\mu_t := \mathscr{H}^n \llcorner \partial^* K_t \tag{1}$$

constitutes a Brakke flow. It has further regularity properties for almost every t: $M_t = \partial^* K_t$ up to $\mathscr{H}^n$-measure zero, and M_t is a unit density, n-rectifiable varifold that carries a weak mean curvature $\vec{H}$. The fact that $(\mu_t)_{t \geqslant 0}$ is a Brakke flow can be characterized as follows. Given any $\phi \in C^2_c(\mathbb{R}^{n+1}; \mathbb{R}^+)$, the following inequality holds for every $t > 0$:

$$\bar{D}_t \mu_t(\phi) \leqslant \int -\phi |\vec{H}|^2 + \langle \nabla \phi, \vec{H} \rangle \, d\mu_t, \tag{2}$$

where $\bar{D}_t$ denotes the upper derivative at time t, and we take the left-hand side to be $-\infty$ if μ_t is not n-rectifiable or does not carry a weak mean curvature. Note that in the case where M_t moves smoothly by mean curvature, $\bar{D}_t$ is just the usual derivative, and we have equality in (2). We can now state our main result.

THEOREM 1.1
Let Ω be an open and bounded subset of $\mathbb{R}^{n+1}$. Assume further that $M = \partial\Omega$ is a closed submanifold of $\mathbb{R}^{n+1}$ of class C^1, carrying a nonnegative weak mean curvature in L^2. Then the family of Radon measures $(\mu_t)_{t \geqslant 0}$ associated to the level set flow of M is a Brakke flow with equality in the sense that

$$\mu_{t_2}(\phi) - \mu_{t_1}(\phi) = \int_{t_1}^{t_2} \int -\phi |\vec{H}|^2 + \langle \nabla \phi, \vec{H} \rangle \, d\mu_t \, dt \tag{3}$$

for $0 \leqslant t_1 \leqslant t_2$ and any $\phi \in C^2_c(\mathbb{R}^{n+1})$.

This implies the following property, which is known as *no mass drop*.

COROLLARY 1.2
The family of Radon measures $(\mu_t)_{0 \leqslant t \leqslant T}$, where T is the maximal time of existence of the level set flow, is continuous in time. Furthermore, $\lim_{t \nearrow T} \mu_t(\phi) = 0$ for any $\phi \in C^2_c(\mathbb{R}^{n+1}; \mathbb{R}^+)$.

In our proof, we use the method of elliptic regularization to obtain an approximation of the level set flow of Ω by a sequence of smooth mean curvature flows in one dimension higher. The key ingredients to analyze the finer properties of this approximation are the estimates of White [15] on the size of the singular set in mean curvature flow of mean-convex sets and Brakke's regularity theorem in [1]. As a further application of the techniques used in the proof, we give a variational formulation for mean curvature flow of mean convex surfaces which is similar to the variational principle applied by Huisken and Ilmanen in [10] to define weak solutions to the inverse mean curvature flow.

Since $\partial\Omega$ has nonnegative mean curvature and Ω is compact, by the strong maximum principle, the mean curvature of the evolving surfaces becomes immediately strictly positive and remains so. The surfaces can then be given as level sets of a continuous function $u : \bar{\Omega} \to \mathbb{R}^+$, $u = 0$, on $\partial\Omega$ via

$$\partial\big\{x \in \Omega \,\big|\, u(x) > t\big\},$$

and u satisfies the degenerate elliptic equation

$$\operatorname{div}\Big(\frac{Du}{|Du|}\Big) = -\frac{1}{|Du|}. \qquad (\star)$$

Note that if u is smooth at a point $x \in \Omega$ with $Du(x) \neq 0$, this equation just states that the level sets of u near x are flowing smoothly by mean curvature. To give this equation a variational structure, we proceed as follows. Given a function $w \in C^{0,1}(\bar{\Omega})$ such that $|Dw|^{-1} \in L^1(\Omega)$, we define the functional

$$J_w(v) := \int_\Omega |Dv| - \frac{v}{|Dw|}\, dx$$

for any Lipschitz continuous function v on Ω such that $\{w \neq v\} \Subset \Omega$. We then say that such a function w is a weak solution to $(\star)$ on Ω if

$$J_w(w) \leqslant J_w(v)$$

for any such v as above, and w fulfills the boundary conditions

$$w > 0 \quad \text{on } \Omega \qquad \text{and} \qquad \{w = 0\} = \partial\Omega. \qquad (4)$$

For an Ω satisfying the conditions of Theorem 1.1, we show that the level set flow of $\partial\Omega$ can be described as the level sets of a function u satisfying (4) with $|Du|^{-1} \in L^1(\Omega)$.

THEOREM 1.3
Let Ω be as in Theorem 1.1. Then the level set flow $u : \bar{\Omega} \to \mathbb{R}$ is the unique weak solution to $(\star)$ on Ω.

Mean curvature flow in the varifold setting was pioneered by Brakke [1]. Aside from the classical PDE setting (see, e.g., [7], [9]), the level set approach using viscosity techniques proved to be fruitful (see [2], [5]). The level set flow approach has the advantage of being able to define the evolution by mean curvature for any closed subset of $\mathbb{R}^{n+1}$. An alternative approach using geometric measure theory together with an approximative variational functional to yield Brakke flow solutions is given in [13]. In [6] or, equivalently, [13], a connection between the level set flow and the varifold setting is drawn: for a given initial function u_0 on $\mathbb{R}^{n+1}$, it is shown that the level set flow of almost every level set of u_0 constitutes a Brakke flow. For a compact initial hypersurface that has nonnegative mean curvature, this implies that its level set flow constitutes a Brakke flow. In our work here, we partly use these techniques to show that in this case, one actually has equality in the Brakke flow definition.

Outline

In §2, we recall a way of defining the level set flow by Evans and Spruck in [5] for initial hypersurfaces with positive mean curvature. We work out some geometric consequences of approximation by elliptic regularization. The approximating solutions u^ε have the important geometric property that, scaled appropriately, they constitute a smooth, graphical, translating solution to mean curvature flow in $\Omega \times \mathbb{R}$. Writing these translating graphs again as level sets of a function U^ε on $\Omega \times \mathbb{R}$, this yields an approximation of the level set flow U, where U is the constant extension of u in the e_{n+2}-direction, by smooth level set flows. In §3, we show that the obstacle to U being a Brakke flow with equality can be characterized by the possible existence of a nonnegative Radon measure γ such that for a subsequence $\varepsilon_i \to 0$,

$$|DU^{\epsilon_i}|^{-1} \rightharpoonup |DU|^{-1} + \gamma \tag{5}$$

on $\Omega \times \mathbb{R}$. Using the regularity results of White in [15], we can, furthermore, deduce that the 1-capacity of the support of γ has to vanish. Since the limit flow is still a kind of Brakke flow with equality, where this incorporates the defect measure γ, we can apply this limit equation to show that γ actually has to vanish entirely.

To be able to state Theorem 1.1 not only for boundaries $\partial\Omega$ that are smooth with positive mean curvature, we show that the level set flow of any boundary $\partial\Omega$ as in Theorem 1.1 is actually smooth for positive times close enough to zero and has positive mean curvature.

In §4, we employ the fact that (5) holds with $\gamma \equiv 0$ and use the approximation by smooth level set flows one dimension higher to show that u also is a weak solution to ($\star$). Since the variational principle implies that a weak solution constitutes a Brakke flow with equality, we can apply the avoidance principle for Brakke flows to show uniqueness of such a weak solution.

2. Preliminaries

In the case where our initial hypersurface M is the smooth boundary of an open and bounded set $\Omega \subset \mathbb{R}^{n+1}$, and M has mean curvature $H > 0$, the level set flow $\mathscr{M}$ generated by M can be written as the graph of a continuous function $u : \Omega \to \mathbb{R}^+$. In

[5], it is shown that u is the unique continuous viscosity solution of

$$\Big(\delta^{ij} - \frac{D^i u D^j u}{|Du|^2}\Big) D_i D_j u = -1 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{6}$$

Note that if u is smooth at a point $x \in \Omega$ with $Du(x) \neq 0$, equation (6) is identical to ($\star$). To prove the existence of a solution to (6), Evans and Spruck [5] use the method of elliptic regularization. Since ($\star$) is degenerate everywhere on Ω and singular at points where $Du = 0$, one replaces it by the following nondegenerate PDE:

$$\operatorname{div}\Big(\frac{Du^\varepsilon}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}}\Big) = -\frac{1}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}} \quad \text{in } \Omega, \qquad u^\varepsilon = 0 \quad \text{on } \partial\Omega, \tag{$\star_\varepsilon$}$$

for some small $\varepsilon > 0$. Since the mean curvature of the boundary $\partial\Omega$ is strictly positive, one can construct barriers at the boundary, which yield, together with a maximum principle for the gradient, uniform a priori gradient bounds for solutions u^ε, provided that $0 < \varepsilon < 1$. Applying De Giorgi–Nash-Moser and Schauder estimates, together with a continuity argument, then gives existence of solutions to ($\star_\varepsilon$) for small $\varepsilon > 0$. Furthermore, by the Arzelà-Ascoli theorem, there is a sequence $\varepsilon_i \to 0$ such that $u^{\varepsilon_i} \to \tilde{u}$, and $\tilde{u} \in C^{0,1}(\Omega)$ is a solution to (6). By uniqueness, $\tilde{u} = u$, and $\lim_{\varepsilon\to 0} u^\varepsilon = u$. Aside from the ε-independent gradient estimate, there is also a uniform integral estimate for the right-hand side of ($\star_\varepsilon$).

LEMMA 2.1
For any solution u^ε of ($\star_\varepsilon$), we have the bound

$$\int_\Omega \frac{1}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}}\, dx \leqslant |\partial\Omega|. \tag{7}$$

Proof
Choose a smooth function φ, $0 \leqslant \varphi \leqslant 1$, such that $\varphi = 1$ on $\Omega_\delta := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \delta\}$, $\varphi = 0$ on $\partial\Omega$, and $|D\varphi| \leqslant \gamma/\delta$ for some $\gamma > 1, \delta > 0$. Multiplying ($\star_\varepsilon$) with φ and integrating by parts, we obtain

$$\int_{\Omega_\delta} \frac{1}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}}\, dx \leqslant \int_{\Omega\setminus\Omega_\delta} |D\varphi| \frac{|Du^\varepsilon|}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}}\, dx \leqslant \frac{\gamma}{\delta} |\Omega \setminus \Omega_\delta|.$$

Then taking the limit, $\gamma \to 1$, and $\delta \to 0$ gives the estimate. □

The sequence u^{ε_i} is bounded in $C^1(\overline{\Omega})$ and converges uniformly to u, which thus is in $C^{0,1}(\overline{\Omega})$. Thus, also,

$$Du^{\varepsilon_i} \rightharpoonup Du \quad \text{weakly-}* \text{ in } L^\infty(\Omega;\mathbb{R}^{n+1}).$$

Together with estimate (7), we can then apply [6, Theorems 3.1, 3.3] (by a slight modification of the proof there since here the functions u^ε are not defined on all of $\mathbb{R}^{n+1}$) to obtain the following.

PROPOSITION 2.2 (see Evans and Spruck [6])
We have the following convergence:

$$|Du^{\varepsilon_i}| \rightharpoonup |Du| \quad \textit{weakly-}* \textit{ in } L^\infty(\Omega), \tag{8}$$

and

$$\frac{Du^{\varepsilon_i}}{\sqrt{\varepsilon_i^2 + |Du^{\varepsilon_i}|^2}} \to \frac{Du}{|Du|} \quad \textit{in } L^p\big(\{|Du| \neq 0\} \cap \Omega; \mathbb{R}^{n+1}\big) \tag{9}$$

for any $p \geqslant 1$.

Lemma 2.3 and Proposition 2.4 are a direct consequence of this strengthened convergence. In fact, they are a variation of [6, Lemma 4.2] and step 4 in the proof of [6, Theorem 5.2].

LEMMA 2.3
We have

$$\mathscr{H}^{n+1}\big(\{x \in \Omega \,\big|\, Du(x) = 0\}\big) = 0.$$

Proof
Let $A := \{Du = 0\} \subset \Omega$. By Lemma 2.1, we have

$$\begin{aligned}
\int_A 1\,dx &= \lim_{\varepsilon_i \to 0} \int_A \sqrt{\varepsilon_i^2 + |Du^{\varepsilon_i}|^2}^{\,-1/2} \sqrt{\varepsilon_i^2 + |Du^{\varepsilon_i}|^2}^{\,1/2}\,dx \\
&\leqslant \limsup_{\varepsilon_i \to 0} \Big(\int_A \sqrt{\varepsilon_i^2 + |Du^{\varepsilon_i}|^2}^{\,-1}\,dx\Big)^{1/2} \cdot \Big(\int_A \sqrt{\varepsilon_i^2 + |Du^{\varepsilon_i}|^2}\,dx\Big)^{1/2} \\
&\leqslant C \limsup_{\varepsilon_i \to 0} \Big(\int_A \sqrt{\varepsilon_i^2 + |Du^{\varepsilon_i}|^2}\,dx\Big)^{1/2}.
\end{aligned}$$

Since $\left|\sqrt{\varepsilon_i^2+|Du^{\varepsilon_i}|^2}-|Du^{\varepsilon_i}|\right|\leqslant\varepsilon_i$, we have, by (8),

$$\mathscr{H}^{n+1}(A)\leqslant C\limsup_{\varepsilon_i\to 0}\Big(\int_A|Du^{\varepsilon_i}|\,dx\Big)^{1/2}=0. \qquad \square$$

Thus, by (9),

$$\frac{Du^{\varepsilon_i}}{\sqrt{\varepsilon_i^2+|Du^{\varepsilon_i}|^2}}\to\frac{Du}{|Du|} \tag{10}$$

in $L^p(\Omega)$ for any $p\geqslant 1$.

PROPOSITION 2.4
For $\phi\in L^\infty(\Omega)$, $\phi\geqslant 0$, we have

$$\int_\Omega\frac{\phi}{|Du|}\,dx\;\leqslant\;\liminf_{\varepsilon_i\to 0}\int_\Omega\frac{\phi}{\sqrt{\varepsilon_i^2+|Du^{\varepsilon_i}|^2}}\,dx.$$

Proof
Let $\phi,\psi\in L^\infty(\Omega)$, $\phi,\psi\geqslant 0$. One obtains

$$\begin{aligned}\int_\Omega\phi\psi\,dx&=\lim_{\varepsilon_i\to 0}\int_\Omega\Big(\phi^{1/2}\psi\sqrt{\varepsilon_i^2+|Du^{\varepsilon_i}|^2}^{\,1/2}\Big)\Big(\phi^{1/2}\sqrt{\varepsilon_i^2+|Du^{\varepsilon_i}|^2}^{\,-1/2}\Big)dx\\&\leqslant\liminf_{\varepsilon_i\to 0}\Big(\int_\Omega\phi\psi^2\sqrt{\varepsilon_i^2+|Du^{\varepsilon_i}|^2}\,dx\Big)^{1/2}\Big(\int_\Omega\phi\sqrt{\varepsilon_i^2+|Du^{\varepsilon_i}|^2}^{\,-1}dx\Big)^{1/2}\\&=\Big(\int_\Omega\phi\psi^2|Du|\,dx\Big)^{1/2}\liminf_{\varepsilon_i\to 0}\Big(\int_\Omega\phi\sqrt{\varepsilon_i^2+|Du^{\varepsilon_i}|^2}^{\,-1}dx\Big)^{1/2}.\end{aligned}$$

Now, choose $\psi:=\varphi_m(|Du|^{-1})$ with $\varphi_m:\mathbb{R}\to\mathbb{R}$:

$$\varphi_m(z)=\begin{cases}m, & z\geqslant m,\\ z, & z\in[-m,m],\\ -m, & z\leqslant -m.\end{cases}$$

Since $\psi\leqslant|Du|^{-1}$, we obtain, by the above calculation,

$$\Big(\int_\Omega\phi\psi\,dx\Big)^{1/2}\leqslant\liminf_{\varepsilon_i\to 0}\Big(\int_\Omega\phi\sqrt{\varepsilon_i^2+|Du^{\varepsilon_i}|^2}^{\,-1}dx\Big)^{1/2},$$

which, by the monotone convergence theorem for $m\to\infty$, proves the claim. $\square$

The equation ($\star_\varepsilon$) also has a geometric interpretation. It implies that the downward translating graphs

$$N_t^\varepsilon := \operatorname{graph}\Big(\frac{u^\varepsilon(x)}{\varepsilon} - \frac{t}{\varepsilon}\Big), \quad -\infty < t < \infty,$$

are hypersurfaces in $\Omega \times \mathbb{R}$ flowing smoothly by mean curvature flow. To verify this, define the function

$$U^\varepsilon(x,z) := u^\varepsilon(x) - \varepsilon z \quad \text{for}\,(x,z) \in \Omega \times \mathbb{R}, \tag{11}$$

so that $\{U^\varepsilon = t\} = N_t^\varepsilon$. It is easily checked that U^ε satisfies ($\star$) on $\Omega \times \mathbb{R}$ if and only if u^ε satisfies ($\star_\varepsilon$) on Ω. Lemma 2.3 implies nonfattening (see also [13]); that is,

$$\mathscr{H}^{n+1}(\{u = t\}) = 0$$

for all $t \geqslant 0$. Since, by definition of the level set flow,

$$K_t = \big\{x \in \Omega \,\big|\, u(x) \geqslant t\big\},$$

this implies that

$$\partial^* K_t = \partial^*\{u > t\} \quad \text{for all}\, t \geqslant 0.$$

Note that u is Lipschitz continuous and thus also a BV-function. By comparing the coarea formula for Lipschitz functions and for BV-functions, we see that

$$\partial^*\{u > t\} = \{u = t\}$$

up to $\mathscr{H}^n$-measure zero for almost every t. Then, define the family $\tilde{\mu}_t$ of $(n+1)$-rectifiable Radon measures on $\Omega \times \mathbb{R}$ by

$$\tilde{\mu}_t := \mathscr{H}^{n+1} \llcorner (\partial^* K_t \times \mathbb{R});$$

that is, $\tilde{\mu}_t = \mu_t \otimes \mathscr{L}^1$, where μ_t is the Brakke flow associated to the level set flow u. We can now make precise in what sense the translating graphs N_t^ε approximate the Brakke flow μ_t. For a proof of the following proposition, see [14, §5].

PROPOSITION 2.5
Let $\varepsilon_i \to 0$. Then for almost all $t \geqslant 0$, we have

$$\mathscr{H}^{n+1} \llcorner N_t^{\varepsilon_i} \to \tilde{\mu}_t \tag{12}$$

in the sense of Radon measures. Even more, for almost every $t \geqslant 0$, there is a subsequence $\{\varepsilon_j\}$, depending on t, such that

$$N_t^{\varepsilon_j} \to \partial^* K_t \times \mathbb{R} \tag{13}$$

in the sense of varifolds. Furthermore, the rectifiable sets $\partial^* K_t$, seen as unit density n-rectifiable varifolds, carry for almost every $t \in [0, T)$ a weak mean curvature $\vec{H}_t \in L^2(\partial^* K_t, \mathscr{H}^n)$.

3. The argument

Throughout this section, we work with the level set flows of U^{ε_i} in $\tilde{\Omega} := \Omega \times [0, 1]$. We denote $U(x, z) = u(x)$. Furthermore, let ν_t and $\nu_t^{\varepsilon_i}$ be the normal vectors to the level set flows U and U^{ε_i}, respectively, and denote $\Gamma_t := \partial^* K_t$ and $\tilde{\Gamma}_t := \Gamma_t \times \mathbb{R}$. From Lemmas 2.1 and 2.3 and Proposition 2.2, we derive the following facts.

LEMMA 3.1
We have

$$\int_{\tilde{\Omega}} |DU^{\varepsilon_i}|^{-1}\, dx \leqslant |\partial\Omega|, \tag{14}$$

$$|DU^{\varepsilon_i}| \rightharpoonup |DU| \quad \textit{weakly-}* \textit{ in } L^\infty(\tilde{\Omega}), \tag{15}$$

and

$$\nu_t^{\varepsilon_i} \to \nu_t \quad \textit{in } L^1(\tilde{\Omega}, \mathbb{R}^{n+2}). \tag{16}$$

We now introduce several Radon measures on $\tilde{\Omega}$ which are central in the proof of Theorem 1.1. First, denote

$$\alpha^{\varepsilon_i} := |DU^{\varepsilon_i}|^{-1}\, dx^{n+2} \qquad \text{and} \qquad \alpha := |DU|^{-1}\, dx^{n+2}.$$

Since, for all $K \Subset \tilde{\Omega}$,

$$\alpha^{\varepsilon_i}(K) = \int_K |DU^{\varepsilon_i}|^{-1}\, dx \leqslant C(K),$$

by equation (14) we know that the α^{ε_i} have a convergent subsequence; that is, we can assume that

$$\alpha^{\varepsilon_i} \rightharpoonup \beta$$

in the sense of Radon measures. We clarify the relation of α and β subsequently. Note that in view of Proposition 2.4, we find that $\beta \geqslant \alpha$. We denote the defect measure, the difference of α and β, by

$$\gamma = \beta - \alpha,$$

which is a nonnegative Radon measure.

Before attempting the proof of Theorem 1.1, we collect some observations about the level set flow U.

LEMMA 3.2
Let $\vec{H}_t$ denote the mean curvature vector of the level set flow U. Then for all smooth vector fields X with compact support in $\tilde{\Omega}$,

$$\int_{\tilde{\Omega}} \langle \vec{H}_t, X\rangle |DU|\, dx = \int_{\tilde{\Omega}} \Big\langle \frac{DU}{|DU|}, X\Big\rangle\, dx;$$

that is, $\vec{H}_t$ agrees with $DU/|DU|^2$ almost everywhere with respect to the measure $|DU|\,dx$.

Proof
Let X be a smooth vector field with compact support in $\tilde{\Omega}$. Then for all $\varepsilon_i > 0$, there exists $T > 0$ such that

$$\operatorname{supp}(X) \cap N_t^{\varepsilon_i} = \emptyset \quad \text{for all } t \notin [-T, T].$$

Denote by $\nu_t^{\varepsilon_i}$ and $\vec{H}_t^{\varepsilon_i}$ the downward normal and mean curvature vector of $N_t^{\varepsilon_i}$. Using the fact that the $N_t^{\varepsilon_i}$ are smooth and the coarea formula for the level set function U^{ε_i}, we compute

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{N_t^{\varepsilon_i}} \langle \vec{H}_t^{\varepsilon_i}, X\rangle\, d\mu_{\varepsilon_i,t}\, dt &= -\int_{-T}^{T}\int_{N_t^{\varepsilon_i}} \operatorname{div}_{N_t^{\varepsilon_i}}(X)\, d\mu_{\varepsilon_i,t}\, dt \\ &= -\int_{\tilde{\Omega}} (\operatorname{div}_{\mathbb{R}^{n+2}} X - \langle D_{\nu_t^{\varepsilon_i}} X, \nu_t^{\varepsilon_i}\rangle)|DU^{\varepsilon_i}|\, dx. \qquad (17)\end{aligned}$$

We claim that

$$\int_{\tilde{\Omega}} \langle D_{\nu_t^{\varepsilon_i}} X, \nu_t^{\varepsilon_i}\rangle |DU^{\varepsilon_i}|\, dx \to \int_{\tilde{\Omega}} \langle D_{\nu_t} X, \nu_t\rangle |DU|\, dx \qquad (18)$$

as $i \to \infty$. Indeed, we can write

$$\begin{aligned}&\int_{\tilde{\Omega}} \langle D_{\nu_t^{\varepsilon_i}} X, \nu_t^{\varepsilon_i}\rangle |DU^{\varepsilon_i}|\, dx \\ &\quad = \int_{\tilde{\Omega}} \langle D_{\nu_t^{\varepsilon_i}-\nu_t} X, \nu_t^{\varepsilon_i}\rangle |DU^{\varepsilon_i}| + \langle D_{\nu_t} X, \nu_t^{\varepsilon_i} - \nu_t\rangle |DU^{\varepsilon_i}|\, dx \\ &\qquad + \int_{\tilde{\Omega}} \langle D_{\nu_t} X, \nu_t\rangle |DU^{\varepsilon_i}|\, dx.\end{aligned}$$

From the fact that $\langle D_{\nu_t} X, \nu_t \rangle \in L^1(\tilde{\Omega})$ and $|DU^{\varepsilon_i}| \rightharpoonup |DU|$ weakly-$*$ in $L^\infty(\tilde{\Omega})$, we infer that the last term in the above equation converges to $\int_{\tilde{\Omega}} \langle D_{\nu_t} X, \nu_t \rangle |DU|\, dx$. The first two terms go to zero since $\nu_t^{\varepsilon_i} \to \nu_t$ in $L^1(\tilde{\Omega})$ and the respective factors are bounded. Thus we have established (18). Subsequently, we use the coarea formula in the form

$$\int_{\mathbb{R}^{n+2}} f\, dx = \int_{\mathbb{R}} \int_{\{U=t\}} f |DU|^{-1}\, d\mathscr{H}^{n+1}\, dt$$

for $f \in L^\infty$. This is justified by [4, Theorem 2, p. 117] since, in view of Lemma 2.1 and Proposition 2.4, we have $\int_{\tilde{\Omega}} |DU|^{-1}\, dx < \infty$. Thus we can compute

$$\begin{aligned} -\int_{\tilde{\Omega}} (\operatorname{div}_{\mathbb{R}^{n+2}} X - \langle D_{\nu_t} X, \nu_t \rangle)|DU|\, dx &= -\int_{\tilde{\Omega}} \operatorname{div}_{\tilde{\Gamma}_t} X |DU|\, dx \\ &= -\int_{\{t>0\}} \int_{\tilde{\Gamma}_t} \operatorname{div}_{\tilde{\Gamma}_t} X\, d\tilde{\mu}_t\, dt \\ &= \int_{\{t>0\}} \int_{\tilde{\Gamma}_t} \langle \vec{H}_t, X \rangle\, d\tilde{\mu}_t\, dt. \end{aligned} \tag{19}$$

On the other hand, since the $N_t^{\varepsilon_i}$ constitute a smooth level set flow, we have

$$\begin{aligned} \int_{-T}^{T} \int_{N_t^{\varepsilon_i}} \langle \vec{H}_t^{\varepsilon_i}, X \rangle\, d\mu_{\varepsilon_i,t}\, dt &= \int_{-T}^{T} \int_{N_t^{\varepsilon_i}} \langle \nu_t^{\varepsilon_i}, X \rangle |DU^{\varepsilon_i}|^{-1}\, d\mu_{\varepsilon_i,t}\, dt \\ &= \int_{\tilde{\Omega}} \langle \nu_t^{\varepsilon_i}, X \rangle\, dx. \end{aligned}$$

As $\nu_t^{\varepsilon_i} \to \nu_t$ in L^1, we thus find

$$\begin{aligned} \int_{-T}^{T} \int_{N_t^{\varepsilon_i}} \langle \vec{H}_t^{\varepsilon_i}, X \rangle\, d\mu_{\varepsilon_i,t}\, dt \to & \int_{\tilde{\Omega}} \langle \nu_t, X \rangle\, dx \\ &= \int_{\{t>0\}} \int_{\tilde{\Gamma}_t} \langle \nu, X \rangle |DU|^{-1}\, d\tilde{\mu}_t\, dt. \end{aligned}$$

Combining this equation with (17)–(19), we find that for all X,

$$\int_{\{t>0\}} \int_{\tilde{\Gamma}_t} \langle \vec{H}_t, X \rangle\, d\tilde{\mu}_t\, dt = \int_{\{t>0\}} \int_{\tilde{\Gamma}_t} \langle \nu_t, X \rangle |DU|^{-1}\, d\tilde{\mu}_t\, dt.$$

An application of the coarea formula yields the claimed identity. □

We are now set up to perform the central computation. Fix $\phi \in C_c^\infty(\tilde{\Omega})$, and let $0 < t_1 < t_2$. We adopt the convention that if $t > \sup_\Omega u$, then $\tilde{\Gamma}_t = \emptyset$. Note also that $N_t^{\varepsilon_i} \cap \tilde{\Omega} = \emptyset$ if $t \notin [-T, T]$, provided that T is large enough.

Since $N_t^{\varepsilon_i}$ is a smooth level set flow, we know that

$$\begin{aligned}
&\int_{N_{t_2}^{\varepsilon_i}} \phi\, d\mu_{\varepsilon_i,t_2} - \int_{N_{t_1}^{\varepsilon_i}} \phi\, d\mu_{\varepsilon_i,t_1}\\
&\quad = \int_{t_1}^{t_2} \int_{N_t^{\varepsilon_i}} \langle \nabla\phi, \vec{H}_t^{\varepsilon_i}\rangle - \phi|\vec{H}_t^{\varepsilon_i}|^2\, d\mu_{\varepsilon_i,t}\, dt\\
&\quad = \int_{t_1}^{t_2} \int_{N_t^{\varepsilon_i}} -\mathrm{div}_{N_t^{\varepsilon_i}}(\nabla\phi) - \phi|\vec{H}_t^{\varepsilon_i}|^2\, d\mu_{\varepsilon_i,t}\, dt\\
&\quad = -\int_{\tilde{\Omega}\cap\{t_1\leqslant U^{\varepsilon_i}\leqslant t_2\}} \big(\mathrm{div}_{\mathbb{R}^{n+2}}(\nabla\phi) - \langle D_{\nu_t^{\varepsilon_i}}\nabla\phi, \nu_t^{\varepsilon_i}\rangle\big)|DU^{\varepsilon_i}|\, dx\\
&\qquad - \int_{\tilde{\Omega}\cap\{t_1\leqslant U^{\varepsilon_i}\leqslant t_2\}} \phi|DU^{\varepsilon_i}|^{-1}\, dx. \qquad (20)
\end{aligned}$$

To take this computation to the limit as $i \to \infty$, observe that nonfattening implies that

$$\chi_{\{t_1\leqslant U^{\varepsilon_i}\leqslant t_2\}} \to \chi_{\{t_1\leqslant U\leqslant t_2\}} \quad \text{in } L^1(\tilde{\Omega}),$$

and thus the first term on the right-hand side converges. Furthermore, for every $\delta > 0$, consider the open set

$$S_\delta = \big\{U \in (t_1-\delta, t_1+\delta)\big\} \cup \big\{U \in (t_2-\delta, t_2+\delta)\big\}.$$

As $\beta(\{U = t\}) = 0$ for almost every t, also for almost every t_1, t_2,

$$\beta(\{U = t_1\} \cup \{U = t_2\}) = 0.$$

Since β is a Radon measure, $\lim_{\delta\to 0}\beta(S_\delta) = 0$. Hence, for every $\eta > 0$, there exists $\delta > 0$ such that

$$\beta(S_\delta) < \frac{\eta}{2}.$$

Therefore, there exists N such that

$$\int_{S_\delta} |DU^{\varepsilon_i}|^{-1}\, dx \leqslant \eta \quad \text{for all } i \geqslant N.$$

In other words, as $i \to \infty$, eventually, $\alpha^{\varepsilon_i}(S_\delta) \leqslant \eta$. Thus

$$\begin{aligned}
&\Big|\int_{\tilde{\Omega}} \phi\chi_{\{t_1\leqslant U^{\varepsilon_i}\leqslant t_2\}}\, d\alpha^{\varepsilon_i} - \int_{\tilde{\Omega}} \phi\chi_{\{t_1\leqslant U\leqslant t_2\}}\, d\beta\Big|\\
&\quad \leqslant \Big|\int_{\tilde{\Omega}} \phi\chi_{\{t_1\leqslant U^{\varepsilon_i}\leqslant t_2\}}(1-\chi_{S_\delta})\, d\alpha^{\varepsilon_i} - \int_{\tilde{\Omega}} \phi\chi_{\{t_1\leqslant U\leqslant t_2\}}(1-\chi_{S_\delta})\, d\beta\Big|\\
&\qquad + \Big|\int_{\tilde{\Omega}} \phi\chi_{\{t_1\leqslant U^{\varepsilon_i}\leqslant t_2\}}\chi_{S_\delta}\, d\alpha^{\varepsilon_i} - \int_{\tilde{\Omega}} \phi\chi_{\{t_1\leqslant U\leqslant t_2\}}\chi_{S_\delta}\, d\beta\Big|.
\end{aligned}$$

As $U^{\varepsilon_i} \to U$ uniformly, if i is big enough,

$$\chi_{\{t_1 \leqslant U^{\varepsilon_i} \leqslant t_2\}}(1-\chi_{S_\delta}) = \chi_{\{t_1 \leqslant U \leqslant t_2\}}(1-\chi_{S_\delta}),$$

which implies that the first term in the above computation goes to zero in view of the definition of β. The second term can be estimated as

$$\begin{aligned}\Big| \int_{\tilde{\Omega}} \phi \chi_{\{t_1 \leqslant U^{\varepsilon_i} \leqslant t_2\}} \chi_{S_\delta}\, d\alpha^{\varepsilon_i} - \int_{\tilde{\Omega}} \phi \chi_{\{t_1 \leqslant U \leqslant t_2\}} \chi_{S_\delta}\, d\beta \Big| \\ \leqslant \max|\phi| \big(\alpha^{\varepsilon_i}(S_\delta) + \beta(S_\delta)\big) \leqslant 2\eta \max|\phi|.\end{aligned}$$

In combination, we find

$$\Big| \int_{\tilde{\Omega}} \phi \chi_{\{t_1 \leqslant U^{\varepsilon_i} \leqslant t_2\}}\, d\alpha^{\varepsilon_i} - \int_{\tilde{\Omega}} \phi \chi_{\{t_1 \leqslant U \leqslant t_2\}}\, d\beta \Big| \to 0$$

as $i \to \infty$. By virtue of Proposition 2.5, we can assume that $N^{\varepsilon_i}_{t_j} \to \tilde{\Gamma}_{t_j}$ for $j = 1, 2$ in the sense of Radon measures. Then the above reasoning shows that equation (20) implies that

$$\begin{aligned}&\int_{\tilde{\Gamma}_{t_2}} \phi \, d\tilde{\mu}_{t_2} - \int_{\tilde{\Gamma}_{t_1}} \phi \, d\tilde{\mu}_{t_1} \\ &\quad = - \int_{\tilde{\Omega} \cap \{t_1 \leqslant U \leqslant t_2\}} \big(\operatorname{div}_{\mathbb{R}^{n+2}}(\nabla\phi) - \langle D_{\nu_t} \nabla\phi, \nu_t \rangle\big) |DU|\, dx - \int_{\tilde{\Omega} \cap \{t_1 \leqslant U \leqslant t_2\}} \phi\, d\beta \\ &\quad = \int_{\tilde{\Omega} \cap \{t_1 \leqslant U \leqslant t_2\}} \langle \nabla\phi, \vec{H}_t \rangle |DU|\, dx - \int_{\tilde{\Omega} \cap \{t_1 \leqslant U \leqslant t_2\}} \phi\, d\beta.\end{aligned}$$

In view of Lemma 3.2 and the definition of the defect measure γ, this yields

$$\begin{aligned}&\int_{\tilde{\Gamma}_{t_2}} \phi \, d\tilde{\mu}_{t_2} - \int_{\tilde{\Gamma}_{t_1}} \phi \, d\tilde{\mu}_{t_1} \\ &\quad = \int_{\tilde{\Omega} \cap \{t_1 \leqslant U \leqslant t_2\}} \Big\langle \nabla\phi, \frac{DU}{|DU|} \Big\rangle dx - \int_{\tilde{\Omega} \cap \{t_1 \leqslant U \leqslant t_2\}} \phi\, d\alpha - \int_{\tilde{\Omega} \cap \{t_1 \leqslant U \leqslant t_2\}} \phi\, d\gamma. \qquad (21)\end{aligned}$$

We now argue that the support of the defect measure γ is very small. To do this, we introduce the notion of capacity (see [4]). For a closed set $A \subset \mathbb{R}^n$, the 1-capacity, $\operatorname{Cap}_1(A)$, is defined as

$$\operatorname{Cap}_1(A) = \inf \Big\{ \int_{\mathbb{R}^n} |Df|\, dx : f \geqslant 0,\ f \in C^\infty_c,\ A \subset \{f \geqslant 1\}^\circ \Big\}.$$

Replacing f by $\min(f, 1)$ and mollification, we can assume that $0 \leqslant f \leqslant 1$. It turns out that the 1-capacity of $\operatorname{supp}(\gamma)$ vanishes.

LEMMA 3.3
We have

$$\operatorname{Cap}_1\big(\operatorname{supp}(\gamma) \cap \tilde{\Omega}\big) = 0.$$

Proof
From [15], we know that there is a closed singular set $\tilde{S} \subset \operatorname{graph} u$ of parabolic Hausdorff dimension at most $n-1$ such that outside of $\tilde{S}$, the sets $\{u = t\}$ constitute a smooth level set flow. From dimensionality, we know that $\mathscr{H}^n_{\text{par}}(\tilde{S}) = 0$, where $\mathscr{H}^n_{\text{par}}$ denotes the n-dimensional parabolic Hausdorff measure. If we let $S := \Pi(\tilde{S})$ be the projection of $\tilde{S} \subset \operatorname{graph} u$ to Ω, then we find that $\mathscr{H}^n(S) = 0$. In particular, S is closed.

Let $x_0 \in \Omega \setminus S$. Then there exists a neighborhood $B = B_\delta(x_0)$ of x_0 such that $\operatorname{graph} u\big|_B$ is a smooth mean curvature flow. Thus, by Brakke's regularity theorem (see [13]), the $N_t^{\varepsilon_i}\big|_{B\times\mathbb{R}}$ converge smoothly on compact subsets to $\tilde{\Gamma}_t \cap B \times \mathbb{R}$.

For $\phi \in C_c^\infty(B \times [0,1])$, we therefore conclude that

$$\begin{aligned}\int_{\tilde{\Omega}} \phi |DU^{\varepsilon_i}|^{-1}\,dx &= \int_{\mathbb{R}} \int_{N_t^{\varepsilon_i} \cap B \times [0,1]} H^2 \phi \, d\mu_{\varepsilon_i,t}\, dt \\ &\to \int_{\mathbb{R}} \int_{\tilde{\Gamma}_t \cap B \times [0,1]} H^2 \phi\, d\tilde{\mu}_t\, dt = \int_{\Omega \times [0,1]} \phi |DU|^{-1}\,dx\end{aligned}$$

as $i \to \infty$. Thus $x_0 \notin \operatorname{supp}(\gamma)$, which yields $\operatorname{supp}(\gamma) \subset S \times [0,1]$. Since $\mathscr{H}^n(S) = 0$, we find that $\mathscr{H}^{n+1}(S \times \mathbb{R}) = 0$. Hence [4, §4.7, Theorem 2], implies that $\operatorname{Cap}_1(S \times [0,1]) = 0$. □

LEMMA 3.4
For almost every $0 < t_1 < t_2$ *and any* $\phi \in C_c^\infty(\Omega \times \mathbb{R})$, *we have*

$$\int_{\tilde{\Gamma}_{t_2}} \phi\, d\tilde{\mu}_{t_2} - \int_{\tilde{\Gamma}_{t_1}} \phi\, d\tilde{\mu}_{t_1} = \int_{t_1}^{t_2} \int_{\tilde{\Gamma}_t} \langle \nabla\phi, \vec{H}_t\rangle\, d\tilde{\mu}_t\, dt - \int_{t_1}^{t_2} \int_{\tilde{\Gamma}_t} \phi |\vec{H}_t|^2\, d\tilde{\mu}_t\, dt.$$

Proof
Without loss of generality, we can assume that $\operatorname{supp}(\phi) \subset \Omega \times [0,1]$. Let $S = \operatorname{supp}(\gamma)$. Since $\operatorname{Cap}_1(S) = 0$ by Lemma 3.3, we can find functions $\eta_k \in C^\infty(\Omega \times \mathbb{R})$, $0 \leqslant \eta_k \leqslant 1$, so that $S \subset \{\eta_k = 1\}^\circ$ and $\|\eta_k\|_{W^{1,1}(\mathbb{R}^{n+2})} \to 0$ as $k \to \infty$. We can assume that the functions η_k converge $\mathscr{L}^{n+2}$ almost everywhere to $\eta \equiv 0$.

Replace ϕ by $(1-\eta_k)\phi$ in equation (21). Since $(1-\eta_k)=0$ on S, the term containing the defect measure γ drops out, and we conclude that

$$
\begin{aligned}
&\int_{\tilde{\Gamma}_{t_2}} (1-\eta_k)\phi \, d\tilde{\mu}_{t_2} - \int_{\tilde{\Gamma}_{t_1}} (1-\eta_k)\phi \, d\tilde{\mu}_{t_1} \\
&\quad = \int_{\tilde{\Omega}\cap\{t_1 \leqslant U \leqslant t_2\}} \Big\langle \nabla\big((1-\eta_k)\phi\big), \frac{DU}{|DU|} \Big\rangle dx - \int_{\tilde{\Omega}\cap\{t_1 \leqslant U \leqslant t_2\}} (1-\eta_k)\phi \, d\alpha \\
&\quad = \int_{\tilde{\Omega}\cap\{t_1 \leqslant U \leqslant t_2\}} (1-\eta_k) \Big\langle \nabla\phi, \frac{DU}{|DU|} \Big\rangle dx \\
&\qquad - \int_{\tilde{\Omega}\cap\{t_1 \leqslant U \leqslant t_2\}} (1-\eta_k)|DU|^{-1}\phi \, dx - \int_{\tilde{\Omega}\cap\{t_1 \leqslant U \leqslant t_2\}} \phi \Big\langle \nabla\eta_k, \frac{DU}{|DU|} \Big\rangle dx. \qquad (22)
\end{aligned}
$$

As $|DU|$ is bounded and $\eta_k \to 0$ in $L^1(\tilde{\Omega})$, we find

$$\int_{\{t>0\}} \int_{\tilde{\Gamma}_t} |\eta_k \phi| \, d\tilde{\mu}_t \, dt = \int_{\tilde{\Omega}} |\eta_k \phi| \, |DU| \, dx \to 0$$

as $k \to \infty$. Hence

$$\int_{\tilde{\Gamma}_t} \eta_k \phi \, d\tilde{\mu}_t \to 0 \quad \text{for a.e. } t.$$

Thus for almost every $0 < t_1 < t_2$ as $t \to \infty$, the left-hand side of (22) converges to

$$\int_{\tilde{\Gamma}_{t_2}} (1-\eta_k)\phi \, d\tilde{\mu}_{t_2} - \int_{\tilde{\Gamma}_{t_1}} (1-\eta_k)\phi \, d\tilde{\mu}_{t_1} \to \int_{\tilde{\Gamma}_{t_2}} \phi \, d\tilde{\mu}_{t_2} - \int_{\tilde{\Gamma}_{t_1}} \phi \, d\tilde{\mu}_{t_1}.$$

To deal with the right-hand side of (22), note that as $\eta_k \to 0$ almost everywhere and $\phi|DU|^{-1}$ is integrable, the second integrand converges,

$$\int_{\tilde{\Omega}\cap\{t_1 \leqslant U \leqslant t_2\}} (1-\eta_k)|DU|^{-1}\phi \, dx \to \int_{\tilde{\Omega}\cap\{t_1 \leqslant U \leqslant t_2\}} |DU|^{-1}\phi \, dx,$$

in view of the dominated convergence theorem. The other integrands converge in view of $\eta_k \to 0$ in $W^{1,1}(\mathbb{R}^{n+2})$.

Therefore, in the limit as $k \to \infty$, equation (22) turns into the claimed identity. □

As a corollary of the proof of Lemma 3.4, we find that the defect measure γ is, in fact, zero, and we have convergence $\alpha^{\varepsilon_i} \to \alpha$.

COROLLARY 3.5
$|DU^{\varepsilon_i}|^{-1}\, dx \to |DU|^{-1}\, dx$ in the sense of Radon measures.

The next lemma removes the extra dimension from Corollary 3.5.

LEMMA 3.6
For almost every $0 < t_1 < t_2$ *and any* $\phi \in C_c^\infty(\Omega)$*, we have*

$$\int_{\Gamma_{t_2}} \phi \, d\mu_{t_2} - \int_{\Gamma_{t_1}} \phi \, d\mu_{t_1} = \int_{t_1}^{t_2} \int_{\Gamma_t} \langle \nabla\phi, \vec{H}_t \rangle \, d\mu_t \, dt - \int_{t_1}^{t_2} \int_{\Gamma_t} \phi |\vec{H}_t|^2 \, d\mu_t \, dt.$$

Proof
Let $\phi \in C_c^\infty(\Omega)$, and pick a function $\zeta : \mathbb{R} \to \mathbb{R}$ with compact support and $\int_0^1 \zeta \, dz = 1$. Define

$$\tilde{\phi} : \Omega \times \mathbb{R} : (x, z) \to \phi(x)\zeta(z).$$

Since $\vec{H}_t$ is tangent to Ω, we conclude that

$$\langle \nabla\tilde{\phi}, \vec{H}_t \rangle = \zeta \langle \nabla\phi, \vec{H}_t \rangle.$$

Plug $\tilde{\phi}$ into the statement of Lemma 3.4. As $\tilde{\mu} = \mu \otimes \mathscr{L}^1$ is a product and $\tilde{\phi}$ is adapted to the product structure, using Fubini's theorem, we can take out the integration of ζ, as in the following example:

$$\int_{\tilde{\Gamma}_t} \tilde{\phi} \, d\tilde{\mu}_t = \int_{\mathbb{R}} \int_{\Gamma_t} \zeta(z)\phi(x) \, d\mu_t \, dz = \Big(\int_{\mathbb{R}} \zeta \, dz \Big) \Big(\int_{\Gamma_t} \phi \, d\mu_t \Big) = \int_{\Gamma_t} \phi \, d\mu_t.$$

This yields the claim. □

We are almost done with the proof of the main theorem, Theorem 1.1; the only things that remain to be shown are that the statement of Lemma 3.6 holds for all $0 < t_1 < t_2$ and that we can well approximate our initial conditions.

Proof of Theorem 1.1
The general idea is to approximate arbitrary t_1, t_2 by sequences $t_1^j \geqslant t_1$ and $t_2^j \geqslant t_2$ for which, by Lemma 3.6, we have for $\phi \in C_c^\infty(\Omega)$,

$$\int_{\Gamma_{t_2^j}} \phi \, d\mu_{t_2^j} - \int_{\Gamma_{t_1^j}} \phi \, d\mu_{t_1^j} = \int_{t_1^j}^{t_2^j} \int_{\Gamma_t} \langle \nabla\phi, \vec{H}_t \rangle \, d\mu_t \, dt - \int_{t_1^j}^{t_2^j} \int_{\Gamma_t} \phi |\vec{H}_t|^2 \, d\mu_t \, dt. \quad (23)$$

Then we argue that this statement can be taken to the limit. As the function

$$t \mapsto \int_{\Gamma_t} \langle \nabla\phi, \vec{H}_t \rangle - |\vec{H}_t|^2 \, d\mu_t$$

is integrable in t, it is clear that the right-hand side of (23) converges to

$$\int_{t_1}^{t_2} \int_{\Gamma_t} \langle \nabla\phi, \vec{H}_t \rangle \, d\mu_t \, dt - \int_{t_1}^{t_2} \int_{\Gamma_t} \phi |\vec{H}_t|^2 \, d\mu_t \, dt$$

for any sequence $t_1^j \to t_1$ and $t_2^j \to t_2$. The left-hand side requires a little more argument. To this end, note that as by the Brakke flow inequality (2), we have, as $t_1 \leqslant t_1^j$,

$$|\Gamma_{t_1^j}| \leqslant |\Gamma_{t_1}|.$$

As the characteristic functions χ_{K_t} are BV-functions, and since $\chi_{K_{t_1^j}} \to \chi_{K_{t_1}}$ in L^1, the lower semicontinuity of the total variation of BV-functions implies that

$$|\Gamma_{t_1}| \leqslant \liminf_{j\to\infty} |\Gamma_{t_1^j}|.$$

Hence

$$|\Gamma_{t_1}| \leqslant \liminf_{j\to\infty} |\Gamma_{t_1^j}| \leqslant \limsup_{j\to\infty} |\Gamma_{t_1^j}| \leqslant |\Gamma_{t_1}|,$$

and we conclude that $|\Gamma_{t_1^j}| \to |\Gamma_{t_1}|$, as well as $|\Gamma_{t_2^j}| \to |\Gamma_{t_2}|$. Now, we appeal to Lemma 3.7 and infer that the left-hand side of (23) also converges.

Now, given an $\Omega \subset \mathbb{R}^{n+1}$ such that $\partial\Omega =: M_0$ is only C^1 and carries a nonnegative weak mean curvature in L^2, we use the fact that by Lemma 3.8, there is a smooth evolution by mean curvature M_t, $0 < t < \gamma$, such that $H_t > 0$. We also show in this lemma that the level set flow of $\partial\Omega$ coincides with the smooth evolution as long as the latter exists. Thus we can do the whole argument by replacing Ω with Ω_t, where Ω_t is the respective open set bounded by M_t for some $t \in (0, \gamma)$. Using the fact that, initially, the level set flow is smooth and a suitable cutoff function, we see that (3) holds for all $0 \leqslant t_1 \leqslant t_2$ and all $\phi \in C_c^2(\mathbb{R}^{n+1})$. □

LEMMA 3.7

Suppose that $E_j \subset \Omega$, $j \geqslant 1$ and $E \subset \Omega$ are Caccioppoli sets such that $|D\chi_E|(\Omega) < \infty$ and $\chi_{E_j} \to \chi_E$ in $L^1(\Omega)$, and suppose that

$$\lim_{j\to\infty} |D\chi_{E_j}|(\Omega) = |D\chi_E|(\Omega).$$

Then, for all $\phi \in C_c(\Omega)$,

$$\lim_{j\to\infty} \int_\Omega \phi\, |D\chi_{E_j}| = \int_\Omega \phi\, |D\chi_E|.$$

Proof

We denote $\mu_j = |D\chi_{E_j}|$ and $\mu = |D\chi_E|$. From [8, Proposition 1.13], we conclude that for every open set $A \subset \Omega$ with $\mu(\partial A \cap \Omega) = 0$, we have

$$\lim_{j\to\infty} \mu_j(A) = \mu(A).$$

Let $A_t := \{\phi > t\}$; then $\partial A_t \subset \{\phi = t\}$, whence $\mu(\partial A_t \cap \Omega) = 0$ for almost every t. Fix $\varepsilon > 0$, and choose $-T = t_0 < t_1 < \cdots < t_{N_\varepsilon} = T$ so that $|t_{i-1} - t_i| < \varepsilon$ for $i = 1, \ldots, N_\varepsilon$ and $|D\chi_E|(\partial A_{t_i} \cap \Omega) = 0$ for $i = 0, \ldots, N_\varepsilon$. Define the step function

$$\phi_\varepsilon = t_0 + \sum_{i=1}^{N_\varepsilon} (t_i - t_{i-1}) \chi_{A_{t_i}}.$$

It satisfies

$$\sup_\Omega |\phi_\varepsilon - \phi| < \varepsilon,$$

and thus

$$\Big| \int_\Omega \phi \, d\mu - \int_\Omega \phi_\varepsilon \, d\mu \Big| \leqslant \varepsilon \mu(\Omega),$$

and

$$\Big| \int_\Omega \phi \, d\mu_j - \int_\Omega \phi_\varepsilon \, d\mu_j \Big| \leqslant \varepsilon \mu_j(\Omega).$$

Furthermore,

$$\begin{aligned} \lim_{j\to\infty} \int_\Omega \phi_\varepsilon \, d\mu_j &= \lim_{j\to\infty} \Big(t_0 \mu_j(\Omega) + \sum_{i=1}^{N_\varepsilon} (t_i - t_{i-1}) \mu_j(A_{t_i}) \Big) \\ &= t_0 \mu(\Omega) + \sum_{i=1}^{N_\varepsilon} (t_i - t_{i-1}) \mu(A_{t_i}) = \int_\Omega \phi_\varepsilon \, d\mu. \end{aligned}$$

Thus, by letting $\varepsilon \to 0$, we infer the claim. □

In the last lemma, we present a slightly stronger version of [11, Lemma 2.6].

LEMMA 3.8
Let $F_0 : M^n \to \mathbb{R}^{n+1}$ be a closed, oriented hypersurface embedding of class C^1 with measurable nonnegative weak mean curvature in $L^2(M_0, \mathscr{H}^n)$. Then M_0 is of class $C^1 \cap W^{2,2}$, and there exists a smooth evolution by mean curvature $F : M^n \times (0, \varepsilon) \to \mathbb{R}^{n+1}$, $\varepsilon > 0$, such that $M_t \to M_0$ in $C^1 \cap W^{2,2}$ and $H_{M_t} > 0$ for all $t \in (0, \varepsilon)$. Furthermore, this smooth evolution coincides with the level set flow of $F_0(M)$ as long as the former exists.

Proof
Since $M_0 = F_0(M)$ is in C^1 and $H \in L^2$, a calculation similar to that in [8, Appendix C] shows that M_0 is in $W^{2,2}$. Thus, by the work of Hutchinson [12], M_0 carries a weak second fundamental form in L^2. Note that since M_0 is compact and in C^1 for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $p \in M_0$,

$$M_0 \cap B_\delta(p) = \operatorname{graph} u,$$

where $u : \Omega_p \subset \mathbb{R}^n \to \mathbb{R},\ \Omega_p$ open, such that

$$\sup_{\Omega_p}|Du| \leqslant \varepsilon \qquad \text{and} \qquad Du(x) = 0,$$

where $p = (x, u(x))$. By mollification, we can pick a sequence of smooth hypersurfaces M^i converging locally uniformly to M_0 in $C^1 \cap W^{2,2}$. The convergence in C^1 implies that we can choose, for given $\varepsilon > 0$, the above δ uniform in i. Now, consider standard mean curvature flow starting from the approximating surfaces $F^i : M^n \to \mathbb{R}^{n+1}$, $F^i_0(M) = M^i$. In view of the local gradient estimates for mean curvature flow in [3], the surfaces $M^i_t = F^i(\cdot, t)(M)$ exist on some fixed time interval $[0, \gamma)$, independent of i, and remain controlled graphs in the above family of coordinate systems relative to M_0. Even more, by the local interior estimates in [3, Theorems 2.1, 2.3], we have

$$\sup_{M^i_t}|A| \leqslant \frac{C}{t^{1/2}}, \quad t \in (0, \gamma), \tag{24}$$

independent of i, and also uniform estimates, interior in time, on all higher derivatives. Sending $i \to \infty$, we extract a limiting mean curvature flow M_t, $t \in (0, \gamma)$, satisfying the same estimates. Note that the uniform local gradient estimate and (24) imply that $M_t \to M_0$ in $C^{0,\alpha}$ as $t \to 0$. By the local interior gradient estimates in [3, Theorem 2.1], one checks that since M_0 is in C^1, the surfaces M_t are equibounded in C^1, and thus by the Arzelà-Ascoli theorem, $M_t \to M_0$ in C^1 as $t \to 0$.

Since $M^i \to M_0$ in C^1 and by the local interior gradient estimates, we can now choose for a given $\tilde{\varepsilon} > 0$ a smooth vector field X of unit length on an η-neighborhood U of M_0 so that, taking γ smaller, if necessary,

$$M^i_t \subset U, \qquad \big\langle \nu^i\big((p,t), XF^i(p,t)\big)\big\rangle \geqslant 1 - \tilde{\varepsilon}, \qquad \big|\big\langle v^i_{p,t}\,, X\big(F^i(p,t)\big)\big\rangle\big| \leqslant \tilde{\varepsilon}, \tag{25}$$

for all $v^i_{p,t} \in T_p M^i_t$ and for all $(p, t) \in M \times [0, \gamma)$, $i \geqslant i_0$. In a local adapted coordinate system, we can compute

$$\frac{d}{dt}\langle X, \nu\rangle = \Delta\langle X, \nu\rangle + |A|^2\langle X, \nu\rangle - 2h^{ij}\langle DX(e_i), e_j\rangle - \langle \Delta X, \nu\rangle - H\langle DX(\nu), \nu\rangle. \tag{26}$$

If we assume that $\tilde{\varepsilon} < 1/4$, we have $1/2 \leqslant \langle X, \nu\rangle - 1/4 \leqslant 3/2$, and we can define

$$v := \langle X, \nu\rangle - \frac{1}{4}, \qquad w := \frac{|A|^2}{v^2}.$$

Writing the evolution equation of w again in a local adapted coordinate system, we can estimate

$$\begin{aligned}
\frac{d}{dt}w &= \Delta w + \frac{4}{v^3}\langle \nabla |A|^2, \nabla v\rangle - 6\frac{|A|^2}{v^4}|\nabla v|^2 - \frac{2}{v^2}|\nabla A|^2 \\
&\quad + 2\frac{|A|^2}{v^3}\Big(-\frac{1}{4}|A|^2 + 2h^{ij}\langle DX(e_i), e_j\rangle + \langle \Delta X, \nu\rangle + H\langle DX(\nu), \nu\rangle\Big) \\
&\leqslant \Delta w + \frac{8|A|}{v^3}\big|\nabla |A|\big||\nabla v| - \frac{6|A|^2}{v^4}|\nabla v|^2 - \frac{2}{v^2}\big|\nabla |A|\big|^2 \\
&\quad + \frac{|A|^2}{v^3}\Big(-\frac{1}{4}|A|^2 + C(1+|A|)\Big) \\
&\leqslant \Delta w + \frac{2|A|^2}{v^4}|\nabla v|^2 + \frac{|A|^2}{v^3}\Big(-\frac{1}{8}|A|^2 + C\Big).
\end{aligned} \tag{27}$$

By (25), we can estimate

$$|\nabla_i v| = |\langle \nabla_i X, \nu\rangle + h_i^{\ j}\langle X, e_j\rangle| \leqslant C + \tilde{\varepsilon}|A|,$$

which yields, for $\tilde{\varepsilon}$ small enough,

$$\frac{d}{dt}w \leqslant \Delta w + Cw \qquad \text{and} \qquad \frac{d}{dt}\int_{M_t^i} w\, d\mu \leqslant C\int_{M_t^i} w\, d\mu\,.$$

Integrating this on $[0, t]$ for $t \leqslant \gamma$, we see

$$\int_{M_t^i} \frac{|A|^2}{v^2}\, d\mu \leqslant \exp(Ct)\int_{M_0^i} \frac{|A|^2}{v^2}\, d\mu.$$

Since $M_0^i \to M_0$ in $W^{2,2}$, this estimate also holds in the limit. By this estimate, $A_t \rightharpoonup A_0$ in $W^{2,2}$, and since $M_t \to M_0$ in C^1, we have

$$\lim_{t\to 0}\int_{M_t} |A|^2\, d\mu = \int_{M_0} |A|^2\, d\mu,$$

which implies full convergence: $M_t \to M_0$ in $W^{2,2}$. Thus $(H_t)_- = \min\{H_t, 0\} \to (H_0)_-$ strongly in L^2. We can then check that, similarly to the computation before, the quantity $f := H/v$ satisfies the evolution equation

$$\begin{aligned}
\frac{d}{dt}f &= \Delta f + \frac{2}{v}\langle \nabla v, \nabla f\rangle \\
&\quad + \frac{f}{v}\Big(-\frac{1}{4}|A|^2 + 2h^{ij}\langle DX(e_i), e_j\rangle + \langle \Delta X, \nu\rangle + H\langle DX(\nu), \nu\rangle\Big)
\end{aligned}$$

and deduce, as above, that

$$\frac{d}{dt}\int_{M_t}\frac{|H_-|^2}{v^2}\,d\mu \leqslant C\int_{M_t}\frac{|H_-|^2}{v^2}\,d\mu,$$

which implies, by Gronwall's lemma for $t \in (0, \gamma)$, that

$$\int_{M_t}\frac{|H_-|^2}{v^2}\,d\mu \leqslant \exp(Ct)\int_{M_0}\frac{|H_-|^2}{v^2}\,d\mu = 0,$$

proving that $H_t \geqslant 0$ for $0 < t < \gamma$. By the strong maximum principle and the compactness of $M_t \subset \mathbb{R}^{n+1}$, it follows that $H_t > 0$ for all $0 < t < \gamma$, as required.

To see that this smooth evolution coincides with the level set flow of M_0, we define a good coordinate system in a neighborhood of M_0. Again, take $\tilde{M}$ to be a smooth approximating hypersurface of M_0 which is still transverse to the vector field X. Let Φ_s be the flow generated by X. Now, define coordinates $\Psi : U \to \tilde{M} \times (-\eta, \eta)$, where $U = \bigcup_{-\eta<s<\eta} \Phi_s(\tilde{M})$ for $\eta > 0$ small enough, so that U is a neighborhood of $\tilde{M}$ as follows. We employ the flow Φ_s to "project" any point $p \in U$ onto $\tilde{M}$ to define the first n coordinates, and the parameter s to define the $(n+1)$-coordinate. We can assume that $M_0 \subset U$ and thus write M_0 in these coordinates as a "graph" over $\tilde{M}$. Now, let $M^s := \Phi_s(M_0)$ be the translates in "x_{n+1}-direction" in these coordinates, and let M_t^s be mean curvature flow with initial condition M^s. Since we have locally uniform gradient bounds in s, we can assume that these flows all exist on a common time interval, say, $[0, \varepsilon/2)$, and remain in U for $|s|$ small enough. Let $u^s : \tilde{M} \times [0, \varepsilon/2) \to (-\eta, \eta)$ be such that $\Psi^{-1}(M_t^s) = \mathrm{graph}(u^s(\cdot, t))$. Note that by the interior estimates for mean curvature flow in [3], we have $u \in C^1(\tilde{M} \times [0, \varepsilon/2)) \cap C^\infty(\tilde{M} \times (0, \varepsilon/2))$. Take $\tilde{g} := \Psi^* g$ to be the induced metric on $\tilde{M} \times (-\eta, \eta)$. It can then be checked that the functions u^s satisfy a parabolic PDE on $\tilde{M} \times (0, \varepsilon/2)$ of the form

$$D_t u^s = \bar{g}^{ij} D_{ij} u^s + f(x, u^s, Du^s),$$

where $\bar{g}^{ij}$ is the inverse of the metric induced on $\mathrm{graph}(u^s)$ by $\tilde{g}$ and f depends smoothly on x, u^s, Du^s. Note that $\bar{g}^{ij}$ depends smoothly on x, u^s, Du^s but not on D^2u^s. By (24) and the interior gradient estimates, we have

$$|Du^s| \leqslant C, \qquad |D^2u^s| \leqslant \frac{C}{\sqrt{t}},$$

independent of s for some $C > 0$ and all $t \in (0, \varepsilon/2)$. Thus interpolating between two solutions u^{s_1}, u^{s_2} and applying the maximum principle, we obtain

$$\sup_{p\in\tilde{M}}|u^{s_1}(p,t) - u^{s_2}(p,t)| \leqslant \exp(C\sqrt{t})\sup_{p\in\tilde{M}}|u^{s_1}(p,0) - u^{s_2}(p,0)|$$

for some constant $C > 0$ and all $t \in [0, \varepsilon/2)$. But note that since the level set flow has to avoid all smooth flows that are initially disjoint, this implies that for $t \in [0, \varepsilon/2)$, the level set flow of M_0 coincides with the smooth evolution M_t. Since, for $t > 0$, the surfaces M_t are smooth, it is well known that the level set flow coincides with the smooth evolution as long as the latter exists. □

4. The variational principle

As stated in the introduction, we give in this final section a variational formulation for mean curvature flow of mean convex surfaces. Let us define, for $K \subset \Omega$, K compact,

$$J_u(v) = J_u^K(v) := \int_K |Dv| - \frac{v}{|Du|}\,dx. \tag{28}$$

Definition 4.1

Let $u \in C^{0,1}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$, and let $|Du|^{-1} \in L^1_{\text{loc}}(\Omega)$. Then u is a weak subsolution (resp., supersolution) of $(\star)$ in Ω if

$$J_u^K(u) \leqslant J_u^K(v) \tag{29}$$

for every function $v \leqslant u$ (resp., $v \geqslant u$) which is locally Lipschitz continuous and satisfies $\{v \neq u\} \subset K \subset \Omega$, where K is compact.

Let $u : \bar\Omega \to [0, \infty)$, $u \in C^{0,1}(\bar\Omega)$, so that $\{x \in \bar\Omega \,|\, u(x) = 0\} = \partial\Omega$ and $|Du|^{-1} \in L^1(\Omega)$. Then we call u a weak solution to $(\star)$ if

$$J_u^K(u) \leqslant J_u^K(v)$$

for every locally Lipschitz continuous function v with $\{v \neq u\} \subset K \subset \Omega$, where K is compact.

Since

$$J_u\big(\min(v, w)\big) + J_u\big(\max(v, w)\big) = J_u(v) + J_u(w)$$

for $\{v \neq w\} \Subset \Omega$, it follows that u is a weak solution if and only if u is a weak subsolution and supersolution, provided that the boundary conditions are fulfilled. Note also that the requirement $|Du|^{-1} \in L^1_{\text{loc}}$ implies that u is nonfattening; that is, $\mathscr{H}^{n+1}(\{u = t\}) = 0$ for all t.

Equivalent formulation

Let $K \subset \Omega$ be compact, and let $F \subset \Omega$ be a Caccioppoli set in a neighborhood of K. For a Lipschitz continuous function u on Ω with $|Du|^{-1} \in L^1_{\text{loc}}(\Omega)$, we can define the functional

$$J_u^K(F) := |\partial^* F \cap K| - \int_{F\cap K} |Du|^{-1}\, dx. \tag{30}$$

We say that E minimizes J_u in a set A (from the outside, resp., from the inside) if

$$J_u^K(E) \leqslant J_u^K(F)$$

for all F with $F \triangle E \Subset A$ (with, resp., $F \supset E,\ F \subset E$), with a compact set K with $F \triangle E \subset K \subset A$.

By the general inequality

$$J_u(E \cup F) + J_u(E \cap F) \leqslant J_u(E) + J_u(F) \tag{31}$$

for $E \triangle F \Subset A$, it is clear that E minimizes J_u in A if it minimizes it from the inside and from the outside. As in [10], we can show the following.

LEMMA 4.2
Let $u \in C^{0,1}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$, and let $|Du|^{-1} \in L^1_{\mathrm{loc}}(\Omega)$. Then u is a weak subsolution (resp., supersolution) of ($\star$) in Ω if and only if, for every t, the sets $E_t := \{u > t\}$ minimize J_u in Ω from the inside (resp., from the outside).

Proof
(1) Let v be locally Lipschitz continuous with $\{v \neq u\} \subset K \subset \Omega$. For $F_t := \{v > t\}$, we have $F_t \triangle E_t \subset K$ for all t. Then, choose $a < b$ with $a < u, v < b$ on K. Using the coarea formula, one sees that

$$\begin{aligned} J_u^K(v) &= \int_K |Dv| - \frac{v}{|Du|}\, dx \\ &= \int_a^b |\partial^* F_t \cap K|\, dt - \int_K \int_a^b \chi_{F_t} |Du|^{-1}\, dt\, dx - a\int_K |Du|^{-1}\, dx \\ &= \int_a^b J_u^K(F_t)\, dt - a\int_K |Du|^{-1}\, dx. \end{aligned} \tag{32}$$

Thus, if every E_t minimizes J_u in Ω, then u also minimizes J_u. The same works for subsolutions and supersolutions.

(2) Now, let u be a subsolution of (28). Choose t_0 and F so that

$$F \subset E_{t_0} \qquad \text{and} \qquad E_{t_0} \setminus F \Subset \Omega.$$

We aim to show that $J_u(E_{t_0}) \leqslant J_u(F)$. Since J_u is lower semicontinuous with respect to L^1_{loc}-convergence, we can assume that

$$J_u(F) \leqslant J_u(G) \tag{33}$$

for all G with $G \triangle E_{t_0} \subset E_{t_0} \setminus F$, by minimizing among competing sets. Define

$$F_t := \begin{cases} F \cap E_t, & t \geqslant t_0, \\ E_t, & 0 \leqslant t < t_0. \end{cases}$$

By (33), $J_u(F) \leqslant J_u(E_t \cup F)$ for all $t \geqslant t_0$, and thus, by (31),

$$J_u(E_t \cap F) \leqslant J_u(E_t)$$

for $t \geqslant t_0$. Thus

$$J_u(F_t) \leqslant J_u(E_t) \quad \text{for all } t.$$

Now, define v by $v > t$ on F_t, which implies that $v \leqslant u$ and $\{v \neq u\} \Subset \Omega$. By construction, $v \in BV_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$, and $J_u(v)$ is well defined. Approximating v by smooth functions $v_i \to v$ with $|Dv_i| \to |Dv|$, we see that $J_u(u) \leqslant J_u(v)$ as u is a subsolution. Since, then, (32) also is true for v, we have

$$\int_a^b J_u(E_t)\,dt \leqslant \int_a^b J_u(F_t)\,dt,$$

which implies that $J_u(F_t) = J_u(E_t)$ for almost all t. With (31), it follows that

$$J_u(E_t \cup F) \leqslant J_u(F)$$

for almost all $t \geqslant t_0$. Taking the limit $t \searrow t_0$, we have, by lower semicontinuity,

$$J_u(E_{t_0}) \leqslant J_u(F).$$

(3) In the case where u is a supersolution, we choose, as in (2), t_0 and F with

$$E_{t_0} \subset F \qquad \text{and} \qquad F \setminus E_{t_0} \Subset \Omega.$$

As before, we can assume that

$$J_u(F) \leqslant J_u(G)$$

for all G with $G \triangle E_{t_0} \subset F \setminus E_{t_0}$. One defines again

$$F_t := \begin{cases} F \cup E_t, & 0 \leqslant t \leqslant t_0, \\ E_t, & t > t_0, \end{cases}$$

which leads, as above, to

$$J_u(E_t \cap F) \leqslant J_u(F)$$

for almost all $t \leqslant t_0$. Since $|Du|^{-1} \in L^1_{\mathrm{loc}}(\Omega)$, we have $\mathscr{H}^{n+1}(\{u = t_0\}) = 0$, and $E_t \to E_{t_0}$ for $t \nearrow t_0$. Especially, $E_t \cap F \to E_{t_0}$, which implies, by lower semicontinuity, that

$$J_u(E_{t_0}) \leqslant J_u(F). \qquad \square$$

Applying this equivalent formulation, we immediately see that all super level sets of a weak supersolution u minimize area from the outside in Ω.

COROLLARY 4.3
Let u be a weak supersolution on Ω. Then the sets E_t minimize area from the outside in Ω.

To prove uniqueness, we aim to show that a weak solution constitutes a Brakke flow. The idea, then, is to use the avoidance principle for Brakke flows to show that the level sets of two weak solutions have to avoid each other if they are initially disjoint. In a first step, we show that the mean curvature of almost every level set is given by $Du/|Du|^2$, as expected.

LEMMA 4.4
Let u be a weak solution on Ω. Then for almost every $t \in [0, T]$, $T = \sup_\Omega u$, the sets $\Gamma_t := \partial^*\{u > t\}$, seen as unit density n-rectifiable varifolds, carry a weak mean curvature $\vec{H}_t \in L^2(\Gamma_t; \mathscr{H}^n)$. Furthermore, for almost every t, it holds that

$$\vec{H}_t = \frac{Du}{|Du|^2}.$$

Proof
Take $X \in C_c^\infty(\Omega; \mathbb{R}^{n+1})$, and let Φ_s be the flow generated by X. We compute

$$\begin{aligned} 0 = \frac{d}{ds}\Big|_{s=0} J_u(u \circ \Phi_s) &= \frac{d}{ds}\Big|_{s=0} \int_{-\infty}^{+\infty} \mathscr{H}^n(\partial^*\{u \circ \Phi_s > t\})\, dt - \int_\Omega \frac{u \circ \Phi_s}{|Du|}\, dx \\ &= -\int_{-\infty}^{+\infty} \int_{\Gamma_t} \operatorname{div}_{\Gamma_t} X \, d\mathscr{H}^n \, dt - \int_\Omega \frac{\langle Du, X\rangle}{|Du|}\, dx. \end{aligned} \tag{34}$$

By approximation, the last expression still vanishes for any $X \in C_c^{0,1}(\Omega)$. Let $\Psi : \mathbb{R} \to \mathbb{R}$ be any smooth function, and replace X above by $\Psi(u)X$. Note that at any point $p \in \Omega$, where u is differentiable and $\Gamma_{u(p)}$ has a weak tangent space, we have $\operatorname{div}_{\Gamma_{u(p)}}(\Psi(u)X) = \Psi(u)\operatorname{div}_{\Gamma_{u(p)}}(X)$. This yields

$$\int_{-\infty}^{+\infty}\Psi(t)\int_{\Gamma_t}\operatorname{div}_{\Gamma_t}X\,d\mathscr{H}^n\,dt = -\int_{-\infty}^{+\infty}\Psi(t)\int_{\Gamma_t}\Big\langle\frac{Du}{|Du|^2},X\Big\rangle\,d\mathscr{H}^n\,dt.$$

Now, let A be a countable dense subset of $C^1_c(\Omega;\,\mathbb{R}^{n+1})$. By the reasoning above, there is a set $B \subset [0,T)$ of full measure such that

$$\int_{\Gamma_t}\operatorname{div}_{\Gamma_t}X\,d\mathscr{H}^n = -\int_{\Gamma_t}\Big\langle\frac{Du}{|Du|^2},X\Big\rangle\,d\mathscr{H}^n \tag{35}$$

for all $X \in A$. Since $|Du|^{-1} \in L^1(\Omega)$, we can furthermore assume that $|Du|^{-1} \in L^2(\Gamma_t,\,\mathscr{H}^n)$ and is well defined for all $t \in B$. Thus, by approximation, (35) holds for all $X \in C^1_c(\Omega;\,\mathbb{R}^{n+1})$ and $t \in B$. This proves the claim. □

PROPOSITION 4.5
Let $\Omega \subset \mathbb{R}^{n+1}$ be open and bounded with $\partial\Omega \in C^1$, and let u be a weak solution of $(\star)$ on Ω. Then u is a Brakke flow with equality in the sense of (3), where $\mu_t := \mathscr{H}^n \llcorner \Gamma_t$ for $t \geqslant 0$ and the mean curvature $\vec{H}_t$ of Γ_t is given as in Lemma 4.4.

Proof
Note that $|Du|^{-1} \in L^1(\Omega)$ implies that $\mathscr{H}^{n+1}(\{x \in \Omega \mid Du(x) = 0\}) = 0$, whence $\mathscr{H}^{n+1}(\{u = t\}) = 0$ for all t and u is nonfattening. This yields

$$\liminf_{t\nearrow T} J_u(\{u > t\}) \geqslant 0.$$

Since, for every fixed $t \in (0,T)$, we have $J_u(\{u > \tau\}) \leqslant J_u(\{u > t\})$ for all $\tau \in (0,T)$, we obtain $J_u(\{u > t\}) \geqslant 0$ for all $t \in (0,T)$. Also, using again the equivalent formulation of the variational principle, we see that

$$J_u(\{u > t\}) \leqslant J_u\big(B_\rho(x)\big)$$

for all $t \in (0,T)$ and all $B_\rho(x) \Subset \Omega$. Since $J_u(B_\rho(x)) \to 0$ as $\rho \searrow 0$, we see that $J_u(\{u > t\}) \leqslant 0$ for all $t \in (0,T)$, and thus $J_u(\{u > t\}) = 0$ for all $t \in (0,T)$. Furthermore, this implies that $\mathscr{H}^n(\Gamma_t) \to 0$ as $t \nearrow T$. Using Lemma 3.7 and the fact that the sets $\{u > t\}$ minimize area from the outside in Ω, we see that the family of Radon measures $\mu_t := \mathscr{H}^n \llcorner \Gamma_t$ is continuous for $t \geqslant 0$.

Now, let $\phi \in C^\infty_c((0,\infty))$, let $\varphi \in C^1_c(\mathbb{R}^{n+1})$, and define a variation $v_s : \Omega \to \mathbb{R}$ of u by

$$v_s := u + s\,\phi(u)\,\varphi.$$

Note that $\phi(u)\varphi$ has compact support in Ω, and thus v_s is an admissible variation of u. We obtain

$$\begin{aligned} 0 = \frac{d}{ds}\Big|_{s=0} J_u(v_s) &= \frac{d}{ds}\Big|_{s=0} \int_\Omega |Dv_s| - \frac{v_s}{|Du|}\, dx \\ &= \int_\Omega \left(\phi'(u)\varphi + \phi(u)\Big(\Big\langle \frac{Du}{|Du|^2}, D\varphi\Big\rangle - \frac{\varphi}{|Du|^2}\Big)\right)|Du|\, dx \\ &= \int_0^\infty \phi'(t)\int_{\Gamma_t} \varphi\, d\mathscr{H}^n + \phi(t)\int_{\Gamma_t} \langle D\varphi, \vec{H}_t\rangle - \varphi|\vec{H}_t|^2\, d\mathscr{H}^n\, dt. \qquad (36) \end{aligned}$$

Now, take $t_1, t_2 \in [0,\infty)$, $t_1 < t_2$. Letting ϕ appropriately increase to the characteristic function of the interval $[t_1, t_2]$, we see from (36) that u is a Brakke flow with equality, as in (3). □

THEOREM 4.6
Let $\Omega \subset \mathbb{R}^{n+1}$ be open and bounded with $\partial\Omega \in C^1$, and let u be a weak solution of ($\star$) on Ω. Then u is unique.

Proof
Let u_1, u_2 be two weak solutions to ($\star$) on Ω. Since $u_1, u_2 > 0$ on Ω and $\{u_1 = 0\} = \{u_2 = 0\}$, we have $\{u_2 > \tau\} \Subset \{u_1 > 0\}$ for all $\tau > 0$. The avoidance principle for codimension-one Brakke flows, [13, Theorem 10.6], then implies that

$$\operatorname{dist}(\Gamma_t^{u_1}, \Gamma_{t+\tau}^{u_2})$$

is increasing in t for all $\tau > 0$. Note that Ilmanen's proof of the avoidance principle, [13, proof of Theorem 10.6], also works for the time-integrated version of a Brakke flow. Since u_1 and u_2 are continuous, this implies that $u_2 \leqslant u_1$. Repeating this argument with u_1 and u_2 interchanged, we arrive at the reverse inequality, which implies that $u_1 = u_2$. □

In the next lemma, we show that any smooth mean curvature flow is a weak subsolution and supersolution on the set that it sweeps out. To show that the level set flow is a weak solution on Ω, we later apply this lemma to the approximating flows N_t^ε on $\Omega \times \mathbb{R}$ and use Corollary 3.5 to pass to limits.

LEMMA 4.7
Let $(N_t)_{c\leqslant t\leqslant d}$ be a family of smooth hypersurfaces $\Omega \times \mathbb{R}$ with strictly positive, uniformly bounded mean curvature which flow by mean curvature flow. Let W be the set that is swept out by the flow $(N_t)_{c\leqslant t\leqslant d}$, and on W, let the function u be defined by $u = t$ on N_t with $E_t := \{u > t\}$. Then the sets E_t minimize J_u on W for all $t \in [c, d]$.

Proof

The outer unit normal, defined by $\nu_u := -Du/|Du|$, is a smooth vector field on W with $\operatorname{div}(\nu_u) = H_{N_t} = |Du|^{-1} > 0$. For a set F with $F \triangle E_t \subset K \Subset W$, we obtain, by the divergence theorem, using ν_u as a calibration,

$$
\begin{aligned}
&|\partial^* E_t \cap K| - \int_{E_t \cap K} |Du|^{-1}\, dx \\
&\quad = \int_{\partial E_t \cap K} \nu_{\partial E_t} \cdot \nu_u \, d\mathscr{H}^{n+1} - \int_{E_t \cap K} |Du|^{-1}\, dx \\
&\quad = \int_{\partial E_t \cap \overline{F}} \nu_{\partial E_t} \cdot \nu_u \, d\mathscr{H}^{n+1} + \int_{\partial E_t \setminus F} \nu_{\partial E_t} \cdot \nu_u \, d\mathscr{H}^{n+1} - \int_{E_t \cap K} |Du|^{-1}\, dx \\
&\quad = \int_{\partial^* F \setminus E_t} \nu_{\partial^* F} \cdot \nu_u \, d\mathscr{H}^{n+1} - \int_{F \setminus E_t} |Du|^{-1}\, dx + \int_{\partial^* F \cap \overline{E_t}} \nu_{\partial^* F} \cdot \nu_u \, d\mathscr{H}^{n+1} \\
&\qquad + \int_{E_t \setminus F} |Du|^{-1}\, dx - \int_{E_t \cap K} |Du|^{-1}\, dx \\
&\quad = \int_{\partial^* F \cap K} \nu_{\partial^* F} \cdot \nu_u \, d\mathscr{H}^{n+1} - \int_{F \cap K} |Du|^{-1}\, dx \leqslant |\partial^* F \cap K| - \int_{F \cap K} |Du|^{-1}\, dx.
\end{aligned}
$$

□

THEOREM 4.8

Let $\Omega \subset \mathbb{R}^{n+1}$ be open and bounded. Assume further that $\partial\Omega \in C^1$, carrying a nonnegative weak mean curvature in L^2. Then the level set flow $u : \bar{\Omega} \to \mathbb{R}$ of $\partial\Omega$ is a weak solution of ($\star$) on Ω.

Proof

We show that $U((x,z)) := u(x)$, defined on $\Omega \times \mathbb{R}$, is a weak subsolution and supersolution of ($\star$) on $\Omega \times \mathbb{R}$. That u then is also a weak subsolution and supersolution on Ω follows by a simple cutoff argument. Note that by Lemma 3.8, $u > 0$ on Ω, and so $\{u = 0\} = \partial\Omega$.

We first show that U is a weak supersolution on $\Omega \times \mathbb{R}$. So, take $V \geqslant U$, $\{U \neq V\} \Subset \Omega \times \mathbb{R}$, $V \in C^{0,1}_{\mathrm{loc}}(\Omega \times \mathbb{R})$. Let $K \subset \Omega \times \mathbb{R}$, so that K is compact with $\{U \neq V\} \subset K$, and

$$\delta_i := \max_K |U - U^{\varepsilon_i}|;$$

thus $\delta_i \to 0$ for $i \to \infty$. Let

$$V_i := \begin{cases} \max\{U^{\varepsilon_i}, V - 2\delta_i\} & \text{for } x \in K, \\ U^{\varepsilon_i} & \text{for } x \notin K. \end{cases}$$

We have $V_i \in C^{0,1}_{\mathrm{loc}}(\Omega \times \mathbb{R})$, $V_i \geqslant U^{\varepsilon_i}$, $\{V_i \neq U^{\varepsilon_i}\} \subset K$. Furthermore, $V_i \to V$ locally uniformly, $V_i = U_i$ on $\Omega \setminus \{V > U\}$, and

$$\mathscr{H}^{n+2}(\{DV_i \neq DV\} \cap \{V > U\}) \to 0. \tag{37}$$

By Lemma 4.7, we have

$$J^K_{U^{\varepsilon_i}}(U^{\varepsilon_i}) \leqslant J^K_{U^{\varepsilon_i}}(V_i),$$

which can be written as

$$\int_K |DU^{\varepsilon_i}| + (V_i - U^{\varepsilon_i})|DU^{\varepsilon_i}|^{-1}\,dx \leqslant \int_K |DV_i|\,dx.$$

Now, we have

$$|DU^{\varepsilon_i}|^{-1} \to |DU|^{-1}$$

in the sense of Radon measures. Since $V_i \to V$ and $U^{\varepsilon_i} \to U$ locally uniformly, we see that

$$\int_K (V-U)|DU|^{-1}\,dx = \lim_{i\to\infty} \int_K (V_i - U^{\varepsilon_i})|DU^{\varepsilon_i}|^{-1}\,dx.$$

The convergence of $|DU^{\varepsilon_i}| \rightharpoonup |DU|$ weakly-$*$ in $L^\infty(\Omega \times \mathbb{R})$ as well as (37) yields, together with the uniform Lipschitz bound of the V_i's,

$$\begin{aligned}&\left|\int_K |DV_i| - |DV|\,dx\right| \\ &\quad\leqslant \left|\int_{K\setminus\{V>U\}} |DU^{\varepsilon_i}| - |DU|\,dx\right| + C\mathscr{H}^{n+2}(\{DV_i \neq DV\} \cap \{V > U\}) \to 0.\end{aligned}$$

Putting this together, we have

$$\int_K |DU| + (V-U)|DU|^{-1}\,dx \leqslant \int_K |DV|\,dx.$$

The fact that U is also a weak subsolution follows analogously. □

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Metzger
Max-Planck-Institut für Gravitationsphysik, D-14476 Potsdam, Germany;
jan.metzger@aei.mpg.de

Schulze
Institut für Mathematik I, Freie Universität Berlin, D-14195 Berlin, Germany;
felix.schulze@math.fu-berlin.de