

# A PTAS for $\ell_p$ -Low Rank Approximation

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## Abstract

A number of recent works have studied algorithms for entrywise  $\ell_p$ -low rank approximation, namely algorithms which given an  $n \times d$  matrix  $A$  (with  $n \geq d$ ), output a rank- $k$  matrix  $B$  minimizing  $\|A - B\|_p^p = \sum_{i,j} |A_{i,j} - B_{i,j}|^p$  when  $p > 0$ ; and  $\|A - B\|_0 = \sum_{i,j} [A_{i,j} \neq B_{i,j}]$  for  $p = 0$ , where  $[\cdot]$  is the Iverson bracket, that is,  $\|A - B\|_0$  denotes the number of entries  $(i, j)$  for which  $A_{i,j} \neq B_{i,j}$ . For  $p = 1$ , this is often considered more robust than the SVD, while for  $p = 0$  this corresponds to minimizing the number of disagreements, or robust PCA. This problem is known to be NP-hard for  $p \in \{0, 1\}$ , already for  $k = 1$ , and while there are polynomial time approximation algorithms, their approximation factor is at best  $\text{poly}(k)$ . It was left open if there was a polynomial-time approximation scheme (PTAS) for  $\ell_p$ -approximation for any  $p \geq 0$ . We show the following:

1. On the algorithmic side, for  $p \in (0, 2)$ , we give the first  $n^{\text{poly}(k/\varepsilon)}$  time  $(1 + \varepsilon)$ -approximation algorithm. For  $p = 0$ , there are various problem formulations, a common one being the binary setting in which  $A \in \{0, 1\}^{n \times d}$  and  $B = U \cdot V$ , where  $U \in \{0, 1\}^{n \times k}$  and  $V \in \{0, 1\}^{k \times d}$ . There are also various notions of multiplication  $U \cdot V$ , such as a matrix product over the reals, over a finite field, or over a Boolean semiring. We give the first almost-linear time approximation scheme for what we call the Generalized Binary  $\ell_0$ -Rank- $k$  problem, for which these variants are special cases. Our algorithm computes  $(1 + \varepsilon)$ -approximation in time  $(1/\varepsilon)^{2^{O(k)/\varepsilon^2}} \cdot nd^{1+o(1)}$ , where  $o(1)$  hides a factor  $(\log \log d)^{1.1} / \log d$ . In addition, for the case of finite fields of constant size, we obtain an alternate PTAS running in time  $n \cdot d^{\text{poly}(k/\varepsilon)}$ .
2. On the hardness front, for  $p \in (1, 2)$ , we show under the Small Set Expansion Hypothesis and Exponential Time Hypothesis (ETH), there is no constant factor approximation algorithm running in time  $2^{k^\delta}$  for a constant  $\delta > 0$ , showing an exponential dependence on  $k$  is necessary. For  $p = 0$ , we observe that there is no approximation algorithm for the Generalized Binary  $\ell_0$ -Rank- $k$  problem running in time  $2^{2^{\delta k}}$  for a constant  $\delta > 0$ . We also show for finite fields of constant size, under the ETH, that any fixed constant factor approximation algorithm requires  $2^{k^\delta}$  time for a constant  $\delta > 0$ .

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# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	Our Results . . . . .	2
1.1.1	Algorithms . . . . .	2
1.1.2	Hardness . . . . .	3
1.1.3	Additional Results . . . . .	3
1.2	Our Techniques . . . . .	4
1.2.1	Algorithms for $0 < p < 2$ . . . . .	4
1.2.2	Algorithms for $p = 0$ . . . . .	7
1.2.3	Hardness . . . . .	9
<b>2</b>	<b>Preliminaries</b>	<b>11</b>
<b>3</b>	<b><math>\ell_p</math>-Approximation Algorithms</b>	<b>15</b>
3.1	$\ell_1$ -Approximation Algorithm . . . . .	15
3.2	$1 < p < 2$ . . . . .	19
3.3	$0 < p < 1$ . . . . .	21
3.4	$p > 2$ . . . . .	22
3.5	Finite Fields . . . . .	23
<b>4</b>	<b>Generalized Binary Approximation</b>	<b>29</b>
4.1	Setup - A Sampling Theorem . . . . .	30
4.2	Simple PTAS . . . . .	31
4.3	Proof of the Sampling Theorem 16 . . . . .	31
4.3.1	Case 1: Small Difference . . . . .	34
4.3.2	Case 2: Large Difference . . . . .	34
4.3.3	Finishing the Proof . . . . .	35
4.4	Efficient Sampling Algorithm . . . . .	35
4.4.1	Existence of a $(U, V, \varepsilon)$ -Clusterable Solution . . . . .	35
4.4.2	The Algorithm Sample . . . . .	36
4.4.3	Correctness of Algorithm Sample . . . . .	37
4.4.4	Running Time Analysis of Algorithm Sample . . . . .	41
4.4.5	The Complete PTAS . . . . .	42
<b>5</b>	<b>Hardness</b>	<b>43</b>
5.1	$\ell_p$ -Low Rank Approximation and $\min_{p^* \rightarrow p}(A)$ . . . . .	44
5.2	Reducing $\ \cdot\ _{2 \rightarrow p^*}$ to $\min_{p^* \rightarrow p}(\cdot)$ . . . . .	44
5.3	Hardness of $2 \rightarrow q$ norm for all $q \in (2, \infty)$ . . . . .	45
5.4	Hardness of $\min_{p^* \rightarrow p}(\cdot)$ . . . . .	49
5.5	Hardness for Finite Fields . . . . .	52
<b>6</b>	<b>Additional Results</b>	<b>53</b>
6.1	Bicriteria Algorithm . . . . .	53
6.2	Weighted Low Rank Approximation . . . . .	53

# 1 Introduction

Low rank approximation is a common way of compressing a matrix via dimensionality reduction. The goal is to replace a given  $n \times d$  matrix  $A$  by a rank- $k$  matrix  $A'$  that approximates  $A$  well, in the sense that  $\|A - A'\|$  is small for some measure  $\|\cdot\|$ . Since we can write the rank- $k$  matrix  $A'$  as  $U \cdot V$ , where  $U$  is  $n \times k$  and  $V$  is  $k \times d$ , it suffices to store the  $k(n + d)$  entries of  $U$  and  $V$ , which is a significant reduction compared to the  $\mathbf{d}$  entries of  $A$ . Furthermore, computing  $A'x = U(Vx)$  takes time  $O(k(n + d))$ , which is much less than the time  $O(\mathbf{d})$  for computing  $Ax$ .

Low rank approximation is extremely well studied, see the surveys [36, 46, 71] and the many references therein. In this paper, we study the following two variants of entrywise  $\ell_p$ -low rank approximation. Given a matrix  $A$  and an integer  $k$ , one seeks to find a rank- $k$  matrix  $A'$ , minimizing  $\|A - A'\|_p^p = \sum_{i,j} |A_{i,j} - A'_{i,j}|^p$  when  $p > 0$ ; and  $\|A - A'\|_0 = \sum_{i,j} [A_{i,j} \neq A'_{i,j}]$  for  $p = 0$ , where  $[\cdot]$  is the Iverson bracket, that is,  $\|A - A'\|_0$  denotes the number of entries  $(i,j)$  for which  $A_{i,j} \neq A'_{i,j}$ .

When  $p = 2$ , this coincides with the Frobenius norm error measure, which can be solved in polynomial time using the singular value decomposition (SVD); see also [71] for a survey of more efficient algorithms based on the technique of linear sketching.

Recently there has been considerable interest in obtaining algorithms for  $p \neq 2$ . For  $0 \leq p < 2$ , this error measure is often considered more robust than the SVD, since one pays less attention to noisy entries as one does not square the differences, but instead raises the difference to a smaller power. Conversely, for  $p > 2$ , this error measure pays more attention to outliers, and  $p = \infty$  corresponds to a guarantee on each entry. This problem was shown to be NP-hard for  $p \in \{0, 1\}$  [21, 27, 53].

*$\ell_p$ -Low Rank Approximation for  $p > 0$ .* A number of initial algorithms for  $\ell_1$ -low rank approximation were given in [12–14, 37, 38, 41, 45, 48–51, 58, 75]. There is also related work on robust PCA [15, 17, 55, 56, 72, 74] and measures which minimize the sum of Euclidean norms of rows [20, 23–25, 65], though neither directly gives an algorithm for  $\ell_1$ -low rank approximation. Song et al. [67] gave the first approximation algorithms with provable guarantees for entrywise  $\ell_p$ -low rank approximation for  $p \in [1, 2)$ . Their algorithm provides a  $\text{poly}(k \log n)$  approximation and runs in polynomial time, that is, the algorithm outputs a matrix  $B$  for which  $\|A - B\|_p \leq \text{poly}(k \log n) \min_{\text{rank-}k A'} \|A - A'\|_p$ . This was generalized by Chierichetti et al. [18] to  $\ell_p$ -low rank approximation, for every  $p \geq 1$ , where the authors also obtained a  $\text{poly}(k \log n)$  approximation in polynomial time.

In Song et al. [67] it is also shown that if  $A$  has entries bounded by  $\text{poly}(n)$  then an  $O(1)$  approximation can be achieved, albeit in  $n^{\text{poly}(k)}$  time. This algorithm depends inherently on the triangle inequality and as a result the constant factor of approximation is greater than 3. Improving this constant of approximation requires techniques that break this triangle inequality barrier. This is a real barrier, since the algorithm of [67] is based on a row subset selection algorithm, and there exist matrices for which any subset of rows contains at best a  $2(1 - \Theta(1/n))$ -approximation (Theorem G.8 of [67]), which we discuss more below.

*$\ell_0$ -Low Rank Approximation.* When  $p = 0$ , one seeks a rank- $k$  matrix  $A'$  for which  $\|A - A'\|_0$  is as small as possible, where for a matrix  $C$ ,  $\|C\|_0$  denotes the number of non-zero entries of  $C$ . Thus, in this case, we are trying to minimize the number of disagreements between  $A$  and  $A'$ . Since  $A'$  has rank  $k$ , we can write it as  $U \cdot V$  and we seek to minimize  $\|A - U \cdot V\|_0$ . This was studied by Bringmann et al. [11] when  $A, U$ , and  $V$  are matrices over the reals and  $U \cdot V$  denotes the standard matrix product, and the work of [11] provides a  $\text{poly}(k \log n)$  bicriteria approximation algorithm. See also earlier work for  $k = 1$  giving a 2-approximation [34, 64].  $\ell_0$ -low rank approximation is also well-studied when  $A, U$ , and  $V$  are each required to be binary matrices. In this case, there are a number of choices for the ground field (or, more generally, semiring). Specifically, for  $A' = U \cdot V$  we can write the entry  $A'_{i,j}$  as the inner product of the  $i$ -th row of  $U$  with the  $j$ -th column of  $V$  – and the specific inner product function  $\langle \cdot, \cdot \rangle$  depends on the ground field. We consider both (1) the ground field is  $\mathbb{F}_2$  with inner product  $\langle xy \rangle = \bigoplus_{i=1}^k x_i \cdot y_i \in \{0, 1\}$  [21, 30, 57, 73], and (2) the Boolean semiring  $\{0, 1, \wedge, \vee\}$  in which the inner product becomes  $\langle xy \rangle = \bigvee_{i=1}^k x_i \wedge y_i = 1 - \prod_{i=1}^k (1 - x_i \cdot y_i) \in \{0, 1\}$  [8, 21, 54, 63, 66, 70]. Besides the abovementioned upper bounds, which coincide with all of these models when  $k = 1$ , the only other algorithm we are aware of is by Dan et al. [21], who for arbitrary  $k$  presented an  $n^{O(k)}$ -time  $O(k)$ -approximation over  $\mathbb{F}_2$ , and an  $n^{O(k)}$ -time  $O(2^k)$ -approximation over the

Boolean semiring.

Although  $\ell_p$ -low rank approximation is NP-hard for  $p \in \{0, 1\}$ , a central open question is if  $(1 + \varepsilon)$ -approximation is possible, namely: *Does  $\ell_p$ -low rank approximation have a polynomial time approximation scheme (PTAS) for any constant  $k$  and  $\varepsilon$ ?*

## 1.1 Our Results

We give the first PTAS for  $\ell_p$ -low rank approximation for  $0 \leq p < 2$  in the unit cost RAM model of computation. For  $p = 0$  our algorithms work for both finite fields and the Boolean semiring models. We also give time lower bounds, assuming the Exponential Time Hypothesis (ETH) [33] and in some cases the Small Set Expansion Hypothesis [59], providing evidence that an exponential dependence on  $k$ , for  $p \gg 0$ , and a doubly-exponential dependence on  $k$ , for  $p = 0$ , may be necessary.

### 1.1.1 Algorithms

We first formally define the problem we consider for  $0 < p < 2$ . We may assume w.l.o.g. that  $n \geq d$ , and thus the input size is  $O(n)$ .

**Definition 1. (Entrywise  $\ell_p$ -Rank- $k$  Approximation)** *Given an  $n \times d$  matrix  $A$  with integer entries bounded in absolute value by  $\text{poly}(n)$ , and a positive integer  $k$ , output matrices  $U \in \mathbb{R}^{n \times k}$  and  $V \in \mathbb{R}^{k \times d}$  minimizing  $\|A - UV\|_p^p := \sum_{i=1, \dots, n, j=1, \dots, d} |A_{i,j} - (U \cdot V)_{i,j}|^p$ . An algorithm for Entrywise  $\ell_p$ -Rank- $k$  Approximation is an  $\alpha$ -approximation if it outputs  $U$  and  $V$  for which*

$$\|A - UV\|_p^p \leq \alpha \cdot \min_{U' \in \mathbb{R}^{n \times k}, V' \in \mathbb{R}^{k \times d}} \|A - U'V'\|_p^p.$$

Our main result for  $0 < p < 2$  is as follows.

**Theorem 1** (PTAS for  $0 < p < 2$ ). *For any  $p \in (0, 2)$  and  $\varepsilon \in (0, 1)$ , there is a  $(1 + \varepsilon)$ -approximation algorithm to Entrywise  $\ell_p$ -Rank- $k$  Approximation running in time  $n^{\text{poly}(k/\varepsilon)}$ .*

For any constants  $k \in \mathbb{N}$  and  $\varepsilon > 0$ , Theorem 1 computes in polynomial time a  $(1 + \varepsilon)$ -approximate solution to Entrywise  $\ell_p$ -Rank- $k$  Approximation. This significantly strengthens the approximation guarantees in [18, 67].

We next consider the case  $p = 0$ . In order to study the  $\mathbb{F}_2$  and Boolean semiring settings in a unified way, we introduce the following more general problem.

**Definition 2. (Generalized Binary  $\ell_0$ -Rank- $k$ )** *Given a matrix  $A \in \{0, 1\}^{n \times d}$  with  $n \geq d$ , an integer  $k$ , and an inner product function  $\langle \cdot, \cdot \rangle: \{0, 1\}^k \times \{0, 1\}^k \rightarrow \mathbb{R}$ , compute matrices  $U \in \{0, 1\}^{n \times k}$  and  $V \in \{0, 1\}^{k \times d}$  minimizing  $\|A - UV\|_0$ , where the product  $UV$  uses  $\langle \cdot, \cdot \rangle$ . An algorithm for the Generalized Binary  $\ell_0$ -Rank- $k$  problem is an  $\alpha$ -approximation, if it outputs matrices  $U \in \{0, 1\}^{n \times k}$  and  $V \in \{0, 1\}^{k \times d}$  such that*

$$\|A - UV\|_0 \leq \alpha \cdot \min_{U' \in \{0, 1\}^{n \times k}, V' \in \{0, 1\}^{k \times d}} \|A - U'V'\|_0.$$

Our first result for  $p = 0$  is as follows.

**Theorem 2** (PTAS for  $p = 0$ ). *For any  $\varepsilon \in (0, \frac{1}{2})$ , there is a  $(1 + \varepsilon)$ -approximation algorithm for the Generalized Binary  $\ell_0$ -Rank- $k$  problem running in time  $(1/\varepsilon)^{2^{O(k)/\varepsilon^2}} \cdot \mathbf{d}^{1+o(1)}$  and succeeds with constant probability<sup>1</sup>, where  $o(1)$  hides a factor  $(\log \log d)^{1.1} / \log d$ .*

Hence, we obtain the first almost-linear time approximation scheme for the Generalized Binary  $\ell_0$ -Rank- $k$  problem, for any constant  $k$ . In particular, this yields the first polynomial time  $(1 + \varepsilon)$ -approximation for constant  $k$  for  $\ell_0$ -low rank approximation of binary matrices when the underlying field is  $\mathbb{F}_2$  or the Boolean semiring. Even for  $k = 1$ , no PTAS was known before.

Theorem 2 is doubly-exponential in  $k$ , and we show below that this is necessary for any approximation algorithm for Generalized Binary  $\ell_0$ -Rank- $k$ . However, in the special case when the base field is  $\mathbb{F}_2$ , or more generally  $\mathbb{F}_q$  and  $A, U$ , and  $V$  have entries belonging to  $\mathbb{F}_q$ , it is possible to obtain an algorithm running in time  $n \cdot d^{\text{poly}(k/\varepsilon)}$ , which is an improvement for certain super-constant values of  $k$  and  $\varepsilon$ . We formally define the problem and state our result next.

<sup>1</sup> The success probability can be further amplified to  $1 - \delta$  for any  $\delta > 0$  by running  $O(\log(1/\delta))$  independent trials of the preceding algorithm.

**Definition 3.** (*Entrywise  $\ell_0$ -Rank- $k$  Approximation over  $\mathbb{F}_q$* ) Given an  $n \times d$  matrix  $A$  with entries that are in  $\mathbb{F}_q$  for any constant  $q$ , and a positive integer  $k$ , output matrices  $U \in \mathbb{F}_q^{n \times k}$  and  $V \in \mathbb{F}_q^{k \times d}$  minimizing  $\|A - UV\|_0$ . An algorithm for Entrywise  $\ell_0$ -Rank- $k$  Approximation over  $\mathbb{F}_q$  is an  $\alpha$ -approximation if it outputs matrices  $U$  and  $V$  such that

$$\|A - UV\|_0 \leq \alpha \cdot \min_{U' \in \mathbb{F}_q^{n \times k}, V' \in \mathbb{F}_q^{k \times d}} \|A - U'V'\|_0.$$

Our main result for Entrywise  $\ell_0$ -Rank- $k$  Approximation over  $\mathbb{F}_q$  is the following:

**Theorem 3** (Alternate  $\mathbb{F}_q$  PTAS for  $p = 0$ ). *For  $\varepsilon \in (0, 1)$  there is a  $(1 + \varepsilon)$ -approximation algorithm to Entrywise  $\ell_0$ -Rank- $k$  Approximation over  $\mathbb{F}_q$  running in time  $n \cdot d^{\text{poly}(k/\varepsilon)}$ .*

### 1.1.2 Hardness

We first obtain conditional time lower bounds for Entrywise  $\ell_p$ -Rank- $k$  Approximation for  $p \in (1, 2)$ . Our results assume the Small Set Expansion Hypothesis (SSEH). Originally conjectured by Raghavendra and Sturer [59], it is still the only assumption that implies strong hardness results for various graph problems such as Uniform Sparsest Cut [61] and Bipartite Clique [47]. Assuming this hypothesis, we rule out *any* constant factor approximation  $\alpha$ .

**Theorem 4** (Hardness for Entrywise  $\ell_p$ -Rank- $k$  Approximation). *Fix  $p \in (1, 2)$  and  $\alpha > 1$ . Assuming the Small Set Expansion Hypothesis, there is no  $\alpha$ -approximation algorithm for Entrywise  $\ell_p$ -Rank- $k$  Approximation that runs in time  $\text{poly}(n)$ .*

*Consequently, additionally assuming the Exponential Time Hypothesis, there exists  $\delta := \delta(p, \alpha) > 0$  such that there is no  $\alpha$ -approximation algorithm for Entrywise  $\ell_p$ -Rank- $k$  Approximation that runs in time  $2^{k^\delta}$ .*

This shows that assuming the SSEH and the ETH, any constant factor approximation algorithm needs at least a subexponential dependence on  $k$ . We also prove hardness of approximation results for  $p \in (2, \infty)$  (see Theorem 19) without the SSEH. They are the first hardness results for Entrywise  $\ell_p$ -Rank- $k$  Approximation other than  $p = 0, 1$ .

We next show that our running time for Generalized Binary  $\ell_0$ -Rank- $k$  is close to optimal, in the sense that the running time of any PTAS for Generalized Binary  $\ell_0$ -Rank- $k$  must depend exponentially on  $1/\varepsilon$  and doubly exponentially on  $k$ , assuming the Exponential Time Hypothesis.

**Theorem 5** (Hardness for Generalized Binary  $\ell_0$ -Rank- $k$ ). *Assuming the Exponential Time Hypothesis, Generalized Binary  $\ell_0$ -Rank- $k$  has no  $(1 + \varepsilon)$ -approximation algorithm in time  $2^{1/\varepsilon^{o(1)}}$ .  $2^{n^{o(1)}}$ . Further, for any  $\varepsilon \geq 0$ , Generalized Binary  $\ell_0$ -Rank- $k$  has no  $(1 + \varepsilon)$ -approximation algorithm in time  $2^{2^{o(k)}} \cdot 2^{n^{o(1)}}$ .*

Next we obtain conditional lower bounds for Entrywise  $\ell_0$ -Rank- $k$  Approximation over  $\mathbb{F}_q$  for any fixed  $q$ :

**Theorem 6** (Hardness for Entrywise  $\ell_0$ -Rank- $k$  Approximation over  $\mathbb{F}_q$ ). *Let  $\mathbb{F}_q$  be a finite field and  $\alpha > 1$ . Assuming  $P \neq NP$ , there is no  $\alpha$ -approximation algorithm for Entrywise  $\ell_0$ -Rank- $k$  Approximation over  $\mathbb{F}_q$  that runs in time  $\mathfrak{h}(n)$ .*

*Consequently, assuming the Exponential Time Hypothesis, there exists  $\delta := \delta(\alpha) > 0$  such that there is no  $\alpha$ -approximation algorithm for Entrywise  $\ell_0$ -Rank- $k$  Approximation over  $\mathbb{F}_q$  that runs in time  $2^{k^\delta}$ .*

This shows that assuming the ETH, any constant factor approximation algorithm needs at least a subexponential dependence on  $k$ .

### 1.1.3 Additional Results

We obtain several additional results on  $\ell_p$ -low rank approximation. We summarize our results below and defer the details to Section 6.

**$\ell_p$ -low rank approximation for  $p \gg 2$**  Let  $g$  be a standard Gaussian random variable and let  $\gamma_p := \mathbb{E}_g[|g|^p]^{1/p}$ . We note that  $\gamma_p > 1$ , for any  $p \gg 2$ . Then, under ETH no  $(\gamma_p^p - \varepsilon)$ -approximation algorithm runs in time  $O(2^{k^\delta})$ . On the algorithmic side, we give a simple  $(3 + \varepsilon)$ -approximation algorithm running in time  $n^{\text{poly}(k/\varepsilon)}$ .

**Weighted  $\ell_p$ -low rank approximation for  $0 < p < 2$**  We also generalize Theorem 1 to the following weighted setting. Given a matrix  $A$ , an integer  $k$  and a rank- $r$  matrix  $W$ , we seek to find a rank- $k$  matrix  $A'$  such that

$$\|W \circ (A - A')\|_p^p \leq (1 + \varepsilon) \min_{\text{rank-}k \ A_k} \|W \circ (A - A_k)\|_p^p.$$

Our algorithm runs in time  $n^{r \cdot \text{poly}(k/\varepsilon)}$ . We defer the details to Theorem 25.

## Related Work

Our results, in particular Theorem 2 and Theorem 5, had been in submission as of April 2018. Shortly after posting this manuscript [6] to arXiv on 16 July 2018, we became aware that in an unpublished work Fomin et al. have independently obtained a very similar PTAS for Binary  $\ell_0$ -Rank- $k$ . Their manuscript [26] was posted to arXiv on 18 July 2018. Interestingly, [6, 26] have independently discovered i) a reduction between the Binary  $\ell_0$ -Rank- $k$  problem and a clustering problem with constrained centers; ii) a structural sampling theorem extending [2] which yields a simple but inefficient deterministic PTAS; and iii) an efficient sampling procedure, building on ideas from [1, 3, 43], which gives an efficient randomized PTAS. Notably, by establishing an additional structural result, Fomin et al. [26] design a faster sampling procedure which yields a randomized PTAS for the Binary  $\ell_0$ -Rank- $k$  problem that runs in linear time  $(1/\varepsilon)^{2^{O(k)}/\varepsilon^2} \cdot \mathbf{d}$ .

## 1.2 Our Techniques

We give an overview of our techniques, separating them into those for our algorithms for  $0 < p < 2$ , those for our algorithms for  $p = 0$ , and those for our hardness proofs.

### 1.2.1 Algorithms for $0 < p < 2$

We illustrate the techniques for  $p = 1$ ; the algorithms for other  $p \in (0, 2)$  follow similarly. Consider a target rank  $k$ . One of the surprising aspects of our  $(1 + \varepsilon)$ -approximation result is that for  $p = 1$ , it breaks a potential lower bound from [67]. Indeed, in Theorem G.8, they construct  $(n - 1) \times n$  matrices  $A$  such that the closest rank- $k$  matrix  $B$  in the row span of  $A$  provides at best a  $2(1 - \Theta(1/n))$ -approximation to  $A$ !

This should be contrasted with  $p = 2$ , for which it is well-known that for any  $A$  there exists a subset of  $k\varepsilon$  rows of  $A$  containing a  $k$ -dimensional subspace in its span which is a  $(1 + \varepsilon)$ -approximation (these are called column subset selection algorithms; see [71] for a survey). In fact, for  $p = 1$ , all known algorithms [18, 67] find a best  $k$ -dimensional subspace in either the span of the rows or of the columns of  $A$ , and thus provably cannot give better than a 2-approximation. To bypass this, we therefore critically need to leave the row space and column space of  $A$ .

Our starting point is the “guess a sketch” technique of [62], which was used in the context of weighted low rank approximation. Let us consider the optimization problem  $\min_V \|U^*V - A\|_1$ , where  $U^*$  is a left factor of an optimal  $\ell_1$ -low rank approximation for  $A$ . Suppose we could choose a *sketching matrix*  $S$  with a small number  $r$  of rows for which  $\|SU^*V - SA\|_1 = (1 \pm \varepsilon)\|U^*V - A\|_1$  for all  $V$ . Then, if we somehow knew  $U^*$ , we could optimize for  $V$  in the sketched space to find a good right factor  $V$ .

Of course we do not know  $U^*$ , but if  $S$  had a small number  $r$  of rows, then we could consider instead the  $\|\cdot\|_{1,2}$ -norm optimization problem  $\min_V \|SU^*V - SA\|_{1,2}$ , where for a matrix  $C$ ,  $\|C\|_{1,2}$  is defined as  $\sum_{i=1}^d \|C_{:,i}\|_2$ , the sum of the  $\|\cdot\|_2$ -norms of its columns. The solution  $V$  to  $\min_V \|SU^*V - SA\|_{1,2}$  is a  $\sqrt{r}$ -approximation to the original problem  $\min_V \|SU^*V - SA\|_1$ .

In the  $\|\cdot\|_{1,2}$  norm, the solution  $V$  can be written in terms of the so-called normal equations for regression, namely,  $V = (SU^*)^\dagger SA$ , where  $C^\dagger$  denotes the Moore-Penrose pseudoinverse of  $C$ . The key property exploited in [67] is then that although we do not know  $U^*$ ,  $(SU^*)^\dagger SA$  is a  $k$ -dimensional subspace in the row span of  $SA$  providing a  $\sqrt{r}$ -approximation, and one does know  $SA$ . This line of reasoning ultimately leads to a  $\text{poly}(k)$ -approximation.

The approach above fails to give a  $(1 + \varepsilon)$ -approximation for multiple reasons: (1) we may not be able to find a  $(1 + \varepsilon)$ -approximation from the row span of  $A$ , and (2) we lose a  $\sqrt{r}$  factor when we switch to the  $\|\cdot\|_{1,2}$  norm.

Instead, suppose we were instead just to guess all the values of  $SU^*$ . These values might be arbitrary real numbers, but observe that we can assume there is an optimal solution  $U^*V^*$  for

which  $V^*$  is a so-called  $\ell_1$ -well conditioned basis, which loosely speaking means that  $\|y^*\|_1 \approx \|y\|_1$  for any row vector  $y$ . Also, we can show that if  $U^*V^* \neq A$ , then  $\|U^*V^* - A\|_1 \geq n^{-\Theta(k)}$ . Furthermore, we can assume that the entries of  $A$  are bounded by  $\text{poly}(n)$ . These three facts allow us to round the entries of  $U^*$  to an integer multiple of  $n^{-\Theta(k)}$  of absolute value at most  $n^{O(k)}$ . Now suppose we could also discretize the entries of  $S$  to multiples of  $n^{-\Theta(k)}$  and of absolute value at most  $n^{O(k)}$ . Then we would actually be able to guess the correct  $SU^*$  after  $n^{\Theta(k^2r)}$  tries, where recall  $r$  is the number of rows of  $S$ . We will show below that  $r$  can be  $\text{poly}(1/\epsilon)$ , so this will be within our desired running time.

In general, if  $\mathcal{A}(Ux) = (1 \pm \epsilon)\|Ux\|$  for all  $x$ , then we say that  $\mathcal{A}$  defines a subspace embedding. At this point, we can use the triangle inequality to get a constant factor approximation. If  $S$  is a subspace embedding, then

$$\|U^*V - A\|_1 \leq \|U^*(V - V^*)\|_1 + \|U^*V^* - A\|_1 \leq (1 + O(\epsilon))\|SU^*(V - V^*)\|_1 + \|U^*V^* - A\|_1$$

and

$$\|SU^*(V - V^*)\|_1 \leq \|SU^*V - SA\|_1 + \|SU^*V^* - SA\|_1$$

so by taking  $V$  to be a minimizer for  $\|SU^*V - SA\|_1$  we can get an approximation factor close to 3. The triangle inequality was useful here because  $S$  had a small distortion on the subspace defined by  $U^*$ . To improve this result, we would need a mapping that has small distortion on the *affine space* defined by  $U^*V - A$ , as  $V$  varies.

Given  $SU^*$  and  $SA$ , if in fact  $S$  has the property that  $\|SU^*V - SA\|_1 = (1 \pm \epsilon)\|U^*V - A\|_1$  for all  $V$ , then we will be in good shape. At this point we can solve for the optimal  $V$  to  $\min_V \|SU^*V - SA\|_1$  by solving an  $\ell_1$ -regression problem using linear programming. Notice that unlike [62], the approach described above does not create “unknowns” to represent the entries of  $SU^*$  and set up a polynomial system of inequalities. For Frobenius norm error, this approach is feasible because  $\|SU^*V - SA\|_F^2 = \sum_{i=1}^n \|SU^*V_{:,i} - SA_{:,i}\|_F^2$  can be minimized over each column  $V_{:,i}$  using the normal equations for regression. However, we do not know how to set up a polynomial system of inequalities for  $\ell_1$ -error (which define  $V$  in terms of the  $SU^*$  variables).

Unfortunately the approach above is fatally flawed; there is no known sketching matrix  $S$  with a small number  $r$  of rows for which  $\|SU^*V - SA\|_1 = (1 \pm \epsilon)\|U^*V - A\|_1$  for all  $V$ . Instead, we adapt a “median-based” embedding with a non-standard subspace embedding analysis that appeared in the context of sparse recovery [5]. In Lemma F.1 of that paper, it is shown that if  $L$  is a  $d$ -dimensional subspace of  $\mathbb{R}^n$ , and  $S$  is an  $r \times n$  matrix of i.i.d. standard Cauchy random variables for  $r = O(d\epsilon^{-2} \log(d/\epsilon))$ , then with constant probability,  $(1 - \epsilon)\|x\|_1 \leq \text{med}(Sx) \leq (1 + \epsilon)\|x\|_1$  simultaneously for all  $x \in L$ . Here for a vector  $y$ ,  $\text{med}(y)$  denotes the median of absolute values of its entries. For a matrix  $M$ ,  $\text{med}(M)$  denotes the sum of the medians of its columns  $\sum_i \text{med}(M_{:,i})$ .

In our context, this gives us that for a fixed column  $A_{:,i}$  of  $A$  and  $i$ -th column  $V_{:,i}$  of  $V$ , if  $S$  is an i.i.d. Cauchy matrix with  $O(k^{-2} \log(1/\epsilon))$  rows, then with constant probability  $\text{med}(SU^*V_{:,i} - SA_{:,i}) = (1 \pm \epsilon)\|U^*V_{:,i} - A_{:,i}\|_1$  for all vectors  $V_{:,i}$ . Since  $V_{:,i}$  is only  $k$ -dimensional, and one can show that its entries can be taken to be integer multiples of  $n^{-\text{poly}(k)}$  bounded in absolute value by  $n^{\text{poly}(k)}$ , we can enumerate over all  $V_{:,i}$  and find the best solution. We need, however, to adapt the argument in [5] to argue that if rather than taking the median, we take a  $(1/2 \pm \epsilon)$ -quantile, we still obtain a subspace embedding. We do this in Lemma 7 and explain why this modification is crucial for the argument below.

Unfortunately, this still does not work. The issue is that  $S$  succeeds only with constant probability in achieving  $\text{med}(SU^*V_{:,i} - SA_{:,i}) = (1 \pm \epsilon)\|U^*V_{:,i} - A_{:,i}\|_1$  for all vectors  $V_{:,i}$ . Call this property, of an index  $i \in [n] := \{1, 2, \dots, n\}$ , *good*. A naïve amplification of the probability to  $1 - 1/n$  would allow us to union bound over all  $i$ , but this would require  $S$  to have  $\Omega(\log n)$  rows. At this point though, we would not obtain a PTAS since enumerating the entries of  $SU^*$  would take  $n^{\Omega(\log n)}$  time. Nor can we use different  $S$  for different columns of  $A$ , since we may guess different  $SU^*$  for different  $i$  and not obtain a consistent solution  $V$ .

Before proceeding, we first relax the requirement that  $\text{med}(SU^*V - SA) = (1 \pm \epsilon)\|U^*V - A\|_1$  for all  $V$ . We only need  $\text{med}(SU^*V - SA) \geq (1 - \epsilon)\|U^*V - A\|_1$  for all  $V$ , and  $\text{med}(SU^*V^* - SA) \leq (1 + \epsilon)\|U^*V^* - A\|_1$  for the fixed optimum  $U^*V^*$ . We can prove  $\text{med}(SU^*V^* - SA) \leq (1 + \epsilon)\min_V \|U^*V - A\|_1$  by using tail bounds for a Cauchy random variable; we do so in Lemma 6.

Moreover, we next argue that it suffices to have the properties: i) a  $(1 - \text{poly}(\varepsilon/k))$ -fraction of columns are good, and ii) the error introduced by bad columns is small. We can achieve (i) by increasing the number of rows of  $S$  by a  $\log(k/\varepsilon)$  factor, which still allows for an enumeration in time  $n^{\text{poly}(k/\varepsilon)}$ . The main issue is to control the error from bad columns. In particular, it is possible to have a matrix  $V$  and a column  $A_{:,i}$  such that  $\|U^*V_{:,i} - A_{:,i}\|_1$  is large and yet  $\text{med}(SU^*V_{:,i} - SA_{:,i})$  is small, which results in accepting a bad solution  $V$ . While for an average matrix  $V$ , the expected value of  $\sum_{i \text{ is bad}} \|U^*V_{:,i} - A_{:,i}\|_1$  is small, we need to argue that this holds for every matrix  $V$ .

In order to control the error from bad columns, we first show that  $\text{med}(SU^*V^* - SA) = (1 \pm \varepsilon)\|U^*V^* - A\|_1$  for the fixed matrix  $U^*V^* - A$ , and then we demonstrate that the total contribution to  $\|U^*V^* - A\|_1$  from bad columns, is small. We show the latter using Markov's bound for the fixed matrix  $U^*V^* - A$ . Combining this with the former, yields that the total contribution of  $\text{med}(SU^*V_{:,i}^* - SA_{:,i})$  to  $\|SU^*V^* - SA\|_1$  from bad columns (in the original, unsketched space) is small.

We convert the preceding argument for bad columns of the fixed matrix  $U^*V^* - A$ , into an argument for bad columns of a general matrix  $U^*V - A$ . Inspired by ideas for  $\|\cdot\|_{1,2}$  norm, established in [20], we partition the bad columns of a given matrix  $V$  into classes, using the following measurement, which differs substantially from [20]. We look at *quantiles* to handle the median operator, and we say that a bad column  $A_{:,i}$  is *large* if

$$\|U^*V_{:,i} - A_{:,i}\|_1 \geq \frac{1}{\varepsilon} \left( \|U^*V_{:,i}^* - A_{:,i}\|_1 + \frac{1}{1 - O(\varepsilon)} q_{1-\varepsilon/2}(S(U^*V^* - A)_{:,i}) \right), \quad (1)$$

where  $q_{1-\varepsilon/2}$  is the  $(1 - \varepsilon/2)$ -th quantile of coordinates of column  $S(U^*V^* - A)_{:,i}$  arranged in order of non-increasing absolute values. Otherwise, a bad column  $A_{:,i}$  is *small*.

We show that small bad columns can be handled by applying the preceding argument for the fixed matrix  $U^*V^* - A$ , since intuitively, the error they introduce is dominated by the contribution of the corresponding columns of matrix  $U^*V^* - A$ , and we can control this contribution.

Our analysis for the large bad columns uses a different approach, which we summarize in Claim 2. The key insight is to use the additivity of a sketch matrix  $S$ , and to write

$$S(U^*V - A)_{:,i} = S(U^*V - U^*V^*)_{:,i} + S(U^*V^* - A)_{:,i}. \quad (2)$$

Then, by applying our ‘‘robust’’ version (Lemma 5) of median-based subspace embedding [5], it follows that at least a  $(1/2 + \varepsilon)$ -fraction of the entries of column vector  $S(U^*V - U^*V^*)_{:,i}$  have absolute value at least

$$\begin{aligned} & (1 - O(\varepsilon)) \cdot \|U^*(V - V^*)_{:,i}\|_1 \\ & \stackrel{(a)}{\geq} (1 - O(\varepsilon)) \cdot \left( \|(U^*V - A)_{:,i}\|_1 - \|(U^*V^* - A)_{:,i}\|_1 \right) \\ & \stackrel{(b)}{\geq} (1 - O(\varepsilon)) \cdot \|(U^*V - A)_{:,i}\|_1 + q_{1-\varepsilon/2}(S(U^*V^* - A)_{:,i}), \end{aligned}$$

where (a) follows by triangle inequality, and (b) by (1) since the bad column  $A_{:,i}$  is *large*. Thus, at least a  $(1/2 + \varepsilon)$ -fraction of entries of  $S(U^*V - U^*V^*)_{:,i}$  have absolute value at least

$$(1 - O(\varepsilon)) \cdot \|(U^*V - A)_{:,i}\|_1 + q_{1-\varepsilon/2}(S(U^*V^* - A)_{:,i}). \quad (3)$$

Since at most an  $\varepsilon/2$  fraction of entries of  $S(U^*V^* - A)_{:,i}$  have absolute value at least  $q_{1-\varepsilon/2}(S(U^*V^* - A)_{:,i})$ , by definition of quantile, it follows by (3) that in equation (2) at most an  $\varepsilon/2$ -fraction of entries of  $S(U^*V - A)_{:,i}$  can have their absolute value reduced to less than  $(1 - O(\varepsilon)) \cdot \|(U^*V - A)_{:,i}\|_1$ . Furthermore, by (3) at least  $(1/2 + \varepsilon/2)$ -fraction of entries of  $S(U^*V - U^*V^*)$  have absolute value at least  $(1 - O(\varepsilon))\|(U^*V - A)_{:,i}\|_1$ . Therefore, the median of absolute value of the entries of  $S(U^*V - A)_{:,i}$  is at least  $(1 - O(\varepsilon))\|(U^*V - A)_{:,i}\|_1$ , as desired.

Our analysis for  $0 < p < 2$  uses similar arguments, but in contrast relies on  $p$ -stable random variables. In the case when  $0 < p < 1$ , special care is needed since the triangle inequality does not hold.



### 1.2.2 Algorithms for $p = 0$

In the case when  $p = 0$  and the entries of matrix  $A$  belong to a finite field  $\mathbb{F}_q$  for constant  $q$ , we use similar arguments as in the case for  $p = 1$ . Here, instead of  $p$ -stable random variables we apply a linear sketch for estimating the number of distinct elements, established in [35]. We show that it suffices to set the number of rows of the sketching matrix  $S$  to  $\text{poly}(k/\varepsilon) \cdot \log d$ . Further, since each entry of  $S$  has only  $q$  possible values, it is possible to guess matrix  $S$  by enumeration in time  $q^{\text{poly}(k/\varepsilon) \cdot \log d} = d^{\text{poly}(k/\varepsilon)}$ , which will lead to a total running time of  $n \cdot d^{\text{poly}(k/\varepsilon)}$ . This yields a PTAS for constant  $q$ . We defer the details to Section 3.

We now consider the binary setting, where both the input matrix  $A$  has entries in  $\{0, 1\}$  and any solution  $U, V$  is restricted to have entries in  $\{0, 1\}$ . In this case, the Generalized Binary  $\ell_0$ -Rank- $k$  problem can be rephrased as a clustering problem with constrained centers, whose goal is to choose a set of centers satisfying a certain system of linear equations, in order to minimize the total  $\ell_0$ -distance of all columns of  $A$  to their closest center. The main difference to usual clustering problems is that the centers cannot be chosen independently.

We view the choice of matrix  $U$  as picking a set of “cluster centers”  $S_U \stackrel{\text{def}}{=} \{U \cdot y \mid y \in \{0, 1\}^k\}$ . Observe that any column of  $U \cdot V$  is in  $S_U$ , and thus we view the choice of column  $V_{:,j}$  as picking one of the constrained centers in  $S_U$ . Formally, we rephrase the Generalized Binary  $\ell_0$ -Rank- $k$  problem as

$$\begin{aligned} \min_{U \in \{0,1\}^{n \times k}, V \in \{0,1\}^{k \times d}} \|A - U \cdot V\|_0 &= \min_{U \in \{0,1\}^{n \times k}} \sum_{j=1}^d \min_{V_{:,j} \in \{0,1\}^k} \|A_{:,j} - U \cdot V_{:,j}\|_0 \\ &= \min_{U \in \{0,1\}^{n \times k}} \sum_{j=1}^d \min_{s \in S_U} \|A_{:,j} - s\|_0. \end{aligned} \quad (4)$$

Any matrix  $V$  gives rise to a “clustering” as partitioning  $C_V = (C_y)_{y \in \{0,1\}^k}$  of the columns of  $V$  with  $C_y = \{j \in [d] \mid V_{:,j} = y\}$ . If we knew an optimal clustering  $C = C_V$ , for some optimal matrix  $V$ , we could compute an optimal matrix  $U$  as the best response to  $V$ . Note that

$$\min_{U \in \{0,1\}^{n \times k}} \sum_{y \in \{0,1\}^k} \sum_{j \in C_y} \|A_{:,j} - U \cdot y\|_0 = \sum_{i=1}^n \min_{U_{i,:} \in \{0,1\}^k} \sum_{y \in \{0,1\}^k} \sum_{j \in C_y} \|A_{i,j} - U_{i,:} \cdot y\|_0.$$

Therefore, given  $C$  we can compute independently for each  $i \in [n]$  the optimal row  $U_{i,:} \in \{0, 1\}^k$ , by enumerating over all possible binary vectors of dimension  $k$  and selecting the one that minimizes the summation

$$\sum_{y \in \{0,1\}^k} \sum_{j \in C_y} \|A_{i,j} - U_{i,:} \cdot y\|_0.$$

What if instead we could only *sample from*  $C$ ? That is, suppose that we are allowed to draw a constant number  $t = \text{poly}(2^k/\varepsilon)$  of samples from each of the optimal clusters  $C_y$  uniformly at random. Denote by  $\tilde{C}_y$  the samples drawn from  $C_y$ . A natural approach is to replace the exact cost above by the following unbiased estimator:

$$\tilde{E} \stackrel{\text{def}}{=} \sum_{y \in \{0,1\}^k} \frac{|C_y|}{|\tilde{C}_y|} \cdot \sum_{j \in \tilde{C}_y} \|A_{:,j} - U \cdot y\|_0.$$

We show that with good probability any matrix  $U = U(\tilde{C})$  minimizing the estimated cost  $\tilde{E}$  is close to an optimal solution. In particular, we prove for any matrix  $V \in \{0, 1\}^{k \times d}$  that

$$\mathbb{E}_{\tilde{C}}[\|A - U(\tilde{C}) \cdot V\|_0] \leq (1 + \varepsilon) \cdot \min_{U \in \{0,1\}^{n \times k}} \|A - U \cdot V\|_0. \quad (5)$$

The biggest issue in proving statement (5) is that the number of samples  $t = \text{poly}(2^k/\varepsilon)$  is independent of the ambient space dimension  $d$ . A key prior probabilistic result, established by Alon and Sudakov [2], gives an additive  $\pm \varepsilon \mathbf{d}$  approximation for the maximization version of a clustering problem with unconstrained centers, known as Hypercube Segmentation. Since the optimum value of this maximization problem is always at least  $\mathbf{d}/2$ , a multiplicative factor

$(1+\varepsilon)$ -approximation is obtained. Our contribution is twofold. First, we generalize their analysis to clustering problems with constrained centers, and second we prove a multiplicative factor  $(1+\varepsilon)$ -approximation for the minimization version. The proof of (5) takes a significant fraction of this chapter.

We combine the sampling result (5) with the following observations to obtain a deterministic polynomial time approximation scheme (PTAS) in time  $n \cdot d^{\text{poly}(2^k/\varepsilon)}$ . We later discuss how to further improve this running time. Let  $U, V$  be an optimal solution to the Generalized Binary  $\ell_0$ -Rank- $k$  problem.

- (1) To evaluate the estimated cost  $\tilde{E}$ , we need the sizes  $|C_y|$  of an optimal clustering  $C$ . We can guess these sizes with an  $d^{2^k}$  overhead in the running time. In fact, it suffices to know these cardinalities approximately, see Lemma 23, and thus this overhead <sup>2</sup> can be reduced to  $(t + \varepsilon^{-1} \cdot \log d)^{2^k}$ .
- (2) Using the (approximate) size  $|C_y|$  and the samples  $\tilde{C}_y$  drawn u.a.r. from  $C_y$ , for all  $y \in \{0, 1\}^k$ , we can compute in time  $2^{O(k)} \mathbf{d}$  a matrix  $U(\tilde{C})$  minimizing the estimated cost  $\tilde{E}$ , since the estimator  $\tilde{E}$  can be split into a sum over the rows of  $U(\tilde{C})$  and each row is chosen independently as a minimizer among all possible binary vectors of dimension  $k$ .
- (3) Given  $U(\tilde{C})$ , we can compute a best response matrix  $V(\tilde{C})$  which has cost  $\|A - U(\tilde{C}) \cdot V(\tilde{C})\|_0 \leq \|A - U(\tilde{C}) \cdot V\|_0$ , and thus by (5) the expected cost at most  $(1+\varepsilon)\text{OPT}$ .
- (4) The only remaining step is to draw samples  $\tilde{C}_y$  from the optimal clustering. However, in time  $O(d^{2^k t}) = d^{\text{poly}(2^k/\varepsilon)}$  we can enumerate all possible families  $(\tilde{C}_y)_{y \in \{0,1\}^k}$ , and the best such family yields a solution that is at least as good as a random sample. In total, we obtain a PTAS in time  $n \cdot d^{\text{poly}(2^k/\varepsilon)}$ .

The largest part of this chapter is devoted to make the above PTAS efficient, i.e., to reduce the running time from  $n \cdot d^{\text{poly}(2^k/\varepsilon)}$  to  $(2/\varepsilon)^{2^{O(k)}/\varepsilon^2} \cdot \mathbf{d}^{1+o(1)}$ , where  $o(1)$  hides a factor  $(\log \log d)^{1.1} / \log d$ . By the preceding outline, it suffices to speed up Steps (1) and (4), i.e., to design a fast algorithm that guesses approximate cluster sizes and samples from the optimal clusters.

The standard sampling approach for clustering problems such as  $k$ -means [43] is as follows. At least one of the clusters of the optimal solution is “large”, say  $|C_y| \geq d/2^k$ . Sample  $t$  columns uniformly at random from the set  $[d]$  of all columns. Then with probability at least  $(1/2^k)^t$  all samples lie in  $C_y$ , and in this case they form a uniform sample from this cluster. In the usual situation without restrictions on the cluster centers, the samples from  $C_y$  allow us to determine an approximate cluster center  $\tilde{s}^{(y)}$ . Do this as long as large clusters exist (recall that we have guessed approximate cluster sizes in Step (1), so we know which clusters are large). When all remaining clusters are small, remove the  $d/2$  columns that are closest to the approximate cluster centers  $\tilde{s}^{(y)}$  determined so far, and estimate the cost of these columns using the centers  $\tilde{s}^{(y)}$ . As there are no restrictions on the cluster centers, this yields a good cost estimation of the removed columns, and since the  $\ell_0$ -distance is additive the algorithm recurses on the remaining columns, i.e. on an instance of twice smaller size. We continue this process until each cluster is sampled. This approach has been used to obtain linear time approximation schemes for  $k$ -means and  $k$ -median in a variety of ambient spaces [1, 43, 44].

The issue in our situation is that we cannot fix a cluster center  $\tilde{s}^{(y)}$  by looking only at the samples  $\tilde{C}_y$ , since we have dependencies among cluster centers. We nevertheless make this approach work, by showing that a uniformly random column  $r^{(y)} \in [d]$  is a good “representative” of the cluster  $C_y$  with not-too-small probability. In the case when all remaining clusters are small, we then simply remove the  $d/2$  columns that are closest to the representatives  $r^{(y)}$  of the clusters that we already sampled from. Although these representatives can be far from the optimal cluster centers due to the linear restrictions on the latter, we show in Section 4.4 that nevertheless this algorithm yields samples from the optimal clusters.

We prove that the preceding algorithm succeeds with probability at least  $(\varepsilon/t)^{2^{O(k)} \cdot t}$ . Further, we show that the approximate cluster sizes  $|\tilde{C}_y|$  of an optimal clustering can be guessed with

<sup>2</sup> In Section 4.4, we establish an efficient sampling procedure, see Algorithm 3, that further reduces the total overhead for guessing the sizes  $|C_y|$  of an optimal clustering to  $(2^k/\varepsilon)^{2^{O(k)}} \cdot (\log d)^{(\log \log d)^{0.1}}$ .

an overhead of  $(2^k/\varepsilon)^{2^{O(k)}} \cdot (\log d)^{(\log \log d)^{0.1}}$ . In contrast to the standard clustering approach, the representatives  $r^{(y)}$  do not yield a good cost estimation of the removed columns. We overcome this issue by first collecting all samples  $\tilde{C}$  from the optimal clusters, and then computing approximate cluster centers that satisfy certain linear constraints, i.e. a matrix  $U(\tilde{C})$  and its best response matrix  $V(\tilde{C})$ . The latter computation runs in linear time  $2^{O(k)} \cdot \mathbf{d}$  in the size of the *original* instance, and this in combination with the guessing overhead, yields the total running time of  $(2/\varepsilon)^{2^{O(k)}/\varepsilon^2} \cdot \mathbf{d}^{1+o(1)}$ . For further details, we refer the reader to Algorithm 3 in Subsection 4.4.2.

Our algorithm achieves a substantial generalization of the standard clustering approach and applies to the situation with constrained centers. This yields the first randomized almost-linear time approximation scheme for the Generalized Binary  $\ell_0$ -Rank- $k$  problem.

### 1.2.3 Hardness

Our hardness results for the  $\ell_p$  norm for  $p \in (1, 2)$  in Theorem 4 and  $p \in (2, \infty)$  in Theorem 19 are established via a connection to the matrix  $p \rightarrow q$  norm problem and its variants. Given a matrix  $A \in \mathbb{R}^{n \times d}$ ,  $\|A\|_{p \rightarrow q}$  is defined to be  $\|A\|_{p \rightarrow q} := \max_{x \in \mathbb{R}^d, \|x\|_p=1} \|Ax\|_q$ .

Approximately computing this quantity for various values of  $p$  and  $q$  has been known to have applications to the Small Set Expansion Hypothesis [7], quantum information theory [31], robust optimization [68], and the Grothendieck problem [28]. After active research [7, 9, 10, 32], it is now known that computing the  $p \rightarrow q$  norm of a matrix is NP-hard to approximate within some constant  $c(pq) > 1$  except when  $p = q = 2$ ,  $p = 1$ , or  $q = \infty$ . (Hardness of the case  $p < q$  with  $2 \in [pq]$  is only known under stronger assumptions such as the Small Set Expansion Hypothesis or the Exponential Time Hypothesis.) See [10] for a survey of recent results on the approximability of these problems.

We also introduce the problem of computing the following quantity

$$\min_{p \rightarrow q}(A) := \min_{x \in \mathbb{R}^d, \|x\|_p=1} \|Ax\|_q$$

as an intermediate problem. Recall that  $p^* = p/(p-1)$  is the Hölder conjugate of  $p$  for which  $1/p + 1/p^* = 1$ . The following lemma shows that computing  $\ell_p$ -low rank approximation when  $k = d - 1$  is equivalent to computing  $\min_{p^* \rightarrow p}(\cdot)$ .

**Lemma 1.** *Let  $p \in (1, \infty)$ . Let  $A \in \mathbb{R}^{n \times d}$  with  $n \geq d$  and  $k = d - 1$ . Then*

$$\min_{U \in \mathbb{R}^{n \times k}, V \in \mathbb{R}^{k \times d}} \|UV - A\|_p = \min_{x \in \mathbb{R}^d, \|x\|_{p^*}=1} \|Ax\|_p = \min_{p^* \rightarrow p}(A).$$

A simple but crucial observation for the above lemma is that if we let  $a_1, \dots, a_n \in \mathbb{R}^d$  be the rows of  $A$ , computing the best  $(d-1)$ -rank approximation of  $A$  in the entrywise  $\ell_p$  norm is equivalent to computing the  $(d-1)$ -dimensional subspace  $S \subseteq \mathbb{R}^d$  (i.e.,  $\text{rowspace}(V) = S$ ) that minimizes  $\|(\rho_1, \dots, \rho_n)\|_p$ , where  $\rho_i := \min_{y \in S} \|y - a_i\|_p$  denotes the  $\ell_p$ -distance between  $S$  and  $a_i$ .

If  $x \in \mathbb{R}^d$  is a vector orthogonal to  $S$ , Hölder's inequality shows that

$$\rho_i = \min_{y \in S} \|y - a_i\|_p = \min_{\langle x, z + a_i \rangle = 0} \|z\|_p \geq \frac{|\langle x, a_i \rangle|}{\|x\|_{p^*}} = \frac{|\langle x, a_i \rangle|}{\|x\|_{p^*}}.$$

Taking  $z$  to be the Hölder dual of  $x$ , we can show that indeed  $\rho_i = |\langle x, a_i \rangle|/\|x\|_{p^*}$ . Then  $\|(\rho_1, \dots, \rho_n)\|_p = \|Ax\|_p/\|x\|_{p^*}$ , finishing the lemma.

This new connection allows us to prove a number of new hardness results for low rank approximation problems. Previously, even exact hardness results were known only for  $p = 0, 1$  and there was no APX-hardness result.

**$\ell_p$  norm with  $1 < p < 2$ .** For  $p \in (1, 2)$ , we reduce computing  $\|\cdot\|_{2 \rightarrow p^*}$  to computing  $\min_{p^* \rightarrow p}(\cdot)$ .

If  $A$  is an invertible matrix, then

$$\min_{p \rightarrow p^*}(A^{-1}) = \min_{x \neq 0} \frac{\|A^{-1}x\|_p}{\|x\|_{p^*}} = \left( \max_{x \neq 0} \frac{\|x\|_{p^*}}{\|A^{-1}x\|_p} \right)^{-1} = \left( \max_{y \neq 0} \frac{\|Ay\|_{p^*}}{\|y\|_p} \right)^{-1} = \frac{1}{\|A\|_{p \rightarrow p^*}},$$

and thus computing  $\min_{p^* \rightarrow p}(\cdot)$  is equivalent to computing  $\|\cdot\|_{p \rightarrow p^*}$ .

By appropriately perturbing and padding 0's, we can show that computing the latter can be reduced to computing the former modulo arbitrarily small error. Standard facts from Banach spaces additionally show that  $\|AA^T\|_{p \rightarrow p^*} = \|A\|_{2 \rightarrow p^*}^2$ , proving the following lemma.

**Lemma 2.** *For any  $\varepsilon > 0, p \in (1, \infty)$ , there is an algorithm that runs in  $\mathfrak{b} \quad (n \log(1/\varepsilon))$  time and on a non-zero input matrix  $A$ , computes a matrix  $B$  satisfying*

$$(1 - \varepsilon)\|A\|_{2 \rightarrow p^*}^{-2} \leq \min_{p^* \rightarrow p}(B) \leq (1 + \varepsilon)\|A\|_{2 \rightarrow p^*}^{-2}.$$

To finish Theorem 4 for  $\ell_p$ -low rank approximation for  $p \in (1, 2)$ , we use the hardness of approximating the  $2 \rightarrow q$  norm of a matrix proved by Barak et al. [7] assuming the Small Set Expansion Hypothesis when  $q = p^* > 2$ . Given a  $d$ -regular graph  $G = (V, E)$  and size bound  $\delta \in (0, 1/2)$ , the Small Set Expansion problem asks to find a subset  $U \subseteq V$  with  $|U|/|V| \leq \delta$  that minimizes  $\Phi(U) = \frac{|E(U, V \setminus U)|}{d|U|} = 1 - (1_U)^T A(1_U)$ , where  $A$  and  $1_U$  are the normalized adjacency matrix of  $G$  and the normalized indicator vector of  $U$ , respectively. Consequently, the problem is equivalent to finding a sparse indicator vector  $v$  with high Rayleigh quotient  $v^T A v$ , and one natural approach is to find a sparse vector in a subspace corresponding to large eigenvalues of  $A$ . For  $q > 2$ , since  $\|v\|_q / \|v\|_2$  is maximized when  $v$  is supported on only one coordinate and minimized when all entries of  $v$  are equal in magnitude,  $\|v\|_q / \|v\|_2$  is a natural analytic notion of sparsity, so if we let  $P$  be the orthogonal projection on to the subspace corresponding to large eigenvalues, a high  $\|P\|_{2 \rightarrow q}$  seems to indicate that  $G$  has a non-expanding small set. Barak et al. formalized this and proved the following theorem when  $q \geq 4$  is an even integer, but the same proof essentially works for  $q \in (2, \infty)$ . For completeness, we present the proof in Section 5.

**Theorem 7 ([7]).** *Assuming the Small Set Expansion Hypothesis, for any  $q \in (2, \infty)$  and  $r > 1$ , it is NP-hard to approximate the  $\|\cdot\|_{2 \rightarrow q}$  norm within a factor  $r$ .*

**$\ell_p$  norm with  $2 < p$ .** Our hardness results for  $p \in (2, \infty)$  are proved directly from the above intermediate problem. The following hardness result for  $\min_{p^* \rightarrow p}(\cdot)$  implies our hardness result for  $p \in (2, \infty)$ . It follows from a similar result by Guruswami et al. [29], which proves the same hardness for the  $\min_{2 \rightarrow p}(\cdot)$  norm, with some modifications that connect the 2 norm and the  $p^*$  norm. Recall that  $\gamma_p := \mathbb{E}_g[|g|^p]^{1/p}$  where  $g$  is a standard Gaussian, which is strictly greater than 1 for  $p > 2$ .

**Theorem 8.** *For any  $p \in (2, \infty)$  and  $\varepsilon > 0$ , it is NP-hard to approximate the  $\min_{p^* \rightarrow p}(\cdot)$  norm within a factor  $\gamma_p - \varepsilon$ .*

**Finite Fields.** Our hardness results for finite fields rely on the following lemma.

**Lemma 3.** *Let  $\mathbb{F}$  be a finite field and  $A \in \mathbb{F}^{n \times d}$  with  $n \geq d$  and  $k = d - 1$ . Then, we have*

$$\min_{U \in \mathbb{F}^{n \times k}, V \in \mathbb{F}^{k \times d}} \|UV - A\|_0 = \min_{x \in \mathbb{F}^d, x \neq 0} \|Ax\|_0.$$

The proof has a similar structure to Lemma 1 for the  $\ell_p$  norm in  $\mathbb{R}$ . We can still identify a subspace  $S \subseteq \mathbb{F}^d$  with codimension 1 with a vector  $x$  with  $\langle vx, \cdot \rangle = 0$  for every  $v \in S$ . In finite fields,  $x$  can be possibly in  $S$ , but it does not affect the proof. Then for each row  $a_i$  of  $A$ , if  $\langle a_i, x \rangle = 0$ , then  $a_i \in S$  and we incur no error on the  $i$ th row. If  $\langle a_i, x \rangle \neq 0$ , changing one entry of  $a_i$  will ensure that it will be contained in  $S$ , so the total number of errors given  $S$  is exactly  $\|Ax\|_0$ .

The quantity in the right-hand side,  $\min_{x \in \mathbb{F}^d, x \neq 0} \|Ax\|_0$ , is exactly the minimum Hamming weight of any non-zero codeword of the code that has  $A^T$  as a generator matrix, or the minimum distance of the code. Then Theorem 6 above immediately follows from the following theorem by Austrin and Khot [4].

**Theorem 9 ([4]).** *For any finite field  $\mathbb{F}$  and  $r > 1$ , unless  $P = NP$ , there is no  $r$ -approximation algorithm for computing the minimum distance of a given linear code in polynomial time.*

**Paper Outline:** In Section 2 we give preliminaries. In Section 3 we give our algorithms for  $\ell_p$ -low rank approximation,  $0 < p < 2$ , and since it is technically similar, our algorithm for  $p = 0$  over finite fields. In Section 4 we give our algorithm for Generalized Binary  $\ell_0$ -Rank- $k$ . In Section 5 we give all of our hardness results. In Section 6 we mention various additional results.

## 2 Preliminaries

For a matrix  $A$  we write  $A_{i,j}$  for its entry at position  $(i,j)$ ,  $A_{i,:}$  for its  $i$ -th row, and  $A_{:,i}$  for its  $i$ -th column.

For  $0 \leq p \leq \infty$ , we will let  $\|A\|_p$  denote the entrywise  $\ell_p$ -norm of  $A$ . That is,  $\|A\|_0$  equals the number of non-zero entries of  $A$ ,  $\|A\|_\infty = \max_{i,j} \|A_{i,j}\|$ , and  $\|A\|_p = (\sum_{i,j} A_{i,j}^p)^{1/p}$ .

For two matrices  $A, B$  the value  $\|A - B\|_0$  is a measure of similarity that is sometimes called their Hamming distance.

We will typically give the dimensions of a matrix  $A$  as  $n \times d$  when  $A$  has entries from a field such as  $\mathbb{R}$  or  $\mathbb{F}_q$ . When the entries of  $A$  are binary, we will typically give its dimensions as  $m \times n$ .

We first recall some basic results about Cauchy variables. These have the property that if  $x \in \mathbb{R}^n$  and  $Z, C_i$  are i.i.d standard Cauchy variables (for  $i = 1, \dots, n$ ) then it holds that  $\sum_{i=1}^n x_i C_i \sim \|x\|_1 Z$ .

**Fact 1.** *If  $C$  is a Cauchy variable with scale  $\gamma$ , then*

1. For  $\tau > 1$ ,  $\Pr[|C| > \tau\gamma] \leq \frac{1}{\tau}$
2. For small  $\varepsilon > 0$ ,  $\Pr[|C| > (1 + \varepsilon)\gamma] < \frac{1}{2} - \Theta(\varepsilon)$
3. For small  $\varepsilon > 0$ ,  $\Pr[|C| < (1 - \varepsilon)\gamma] < \frac{1}{2} - \Theta(\varepsilon)$

The following results are adapted from [5]. We want to analyze the quantiles of the entries of a vector after a dense Cauchy sketch is applied to it.

**Definition 4.** *Let  $0 < \alpha < 1$ . Let  $v \in \mathbb{R}^m$ . We let  $q_\alpha(v)$  denote the  $\frac{1}{\alpha}$ -quantile of  $|v_1|, |v_2|, \dots, |v_m|$ , or the minimum value greater than  $\lceil \alpha n \rceil$  of the values  $|v_1|, |v_2|, \dots, |v_m|$ . For  $M \in \mathbb{R}^{m \times n}$ , we let*

$$q_\alpha(M) \stackrel{\text{def}}{=} \sum_{i=1}^n q_\alpha(M_{:,i}).$$

We will be particularly interested in the median of the entries of a sketched vector.

**Definition 5.** *For  $v \in \mathbb{R}^n$ , we write  $\text{med}(v)$  as shorthand for  $q_{\frac{1}{2}}(v)$ . Further, for  $M \in \mathbb{R}^{m \times n}$ , we let*

$$\text{med}(M) \stackrel{\text{def}}{=} \sum_{i=1}^n \text{med}(M_{:,i}).$$

**Lemma 4.** *Let  $S \in \mathbb{R}^{m \times n}$  have entries that are i.i.d. standard Cauchy variables and let  $x \in \mathbb{R}^n$ . Then*

1.  $\Pr[q_{\frac{1}{2} - \Theta(\varepsilon)}(Sx) < (1 - \varepsilon)\|x\|_1] < \exp(-\Theta(\varepsilon^2)m)$
2.  $\Pr[q_{\frac{1}{2} + \Theta(\varepsilon)}(Sx) > (1 + \varepsilon)\|x\|_1] < \exp(-\Theta(\varepsilon^2)m)$
3. For  $M > 2$ ,  $\Pr[q_{1 - \frac{\varepsilon}{M}}(Sx) > \frac{M}{\varepsilon}\|x\|_1] < \exp(-\Theta(\varepsilon)Mm)$
4. For  $M > 2$ ,  $\Pr[\text{med}(Sx) > M\|x\|_1] < \exp(-\Theta(m)M)$

*Proof.* Note that for each  $1 \leq i \leq m$ ,  $(Sx)_i$  is distributed as a Cauchy variable with scale  $\|x\|_1$ . By Fact 1,  $\Pr[|(Sx)_i| < (1 - \varepsilon)\|x\|_1] < \frac{1}{2} - \Theta(\varepsilon)$ . We want to bound the probability that more than a  $\frac{1}{2} - \Theta(\varepsilon)$  fraction of the  $(Sx)_i$ 's are smaller than  $(1 - \varepsilon)\|x\|_1$ . The desired upper bound follows from Chernoff's bound as  $\exp(-\Theta(m)(\frac{1}{2} - \Theta(\varepsilon) - (\frac{1}{2} - \Theta(\varepsilon)))^2)$ , from which (i) follows. We can prove (ii) using a similar argument.

For (iii), we know from Fact 1 that  $\Pr[(Sx)_i > \frac{M}{\varepsilon}] < \frac{\varepsilon}{M}$ . Thus a Chernoff bound gives

$$\Pr[q_{1 - \frac{\varepsilon}{M}}(Sx) > \frac{M}{\varepsilon}\|x\|_1] < \exp(-\Theta(m)(\frac{\varepsilon}{2} - \frac{\varepsilon}{M})^2(\frac{\varepsilon}{M})^{-1})$$

and the result follows. For (iv), a similar proof holds using  $\Pr[(Sx)_i > M\|x\|_1] < \frac{1}{M}$ .  $\square$

**Lemma 5.** Let  $X \subset \mathbb{R}^n$  be a  $k$ -dimensional space and  $\varepsilon, \delta > 0$ . Let  $S$  have  $O(\frac{1}{\varepsilon^2} k \log \frac{k}{\varepsilon\delta})$  rows,  $n$  columns, and i.i.d. Cauchy entries with scale parameter  $\gamma = 1$ . Then with probability at least  $1 - \Theta(\delta)$ , for all  $x \in X$ ,

$$(1 - \Theta(\varepsilon))\|x\|_1 \leq q_{\frac{1}{2}-\varepsilon}(Sx) \leq q_{\frac{1}{2}+\varepsilon}(Sx) \leq (1 + O(\varepsilon))\|x\|_1$$

*Proof.* Let  $N$  be an  $\frac{\varepsilon\delta}{k^3}$ -net for the intersection of  $X$  and the unit  $\ell_1$  ball. Then  $|N| = \exp(O(k \log \frac{k}{\varepsilon\delta}))$

By Lemma 4,  $\Pr[q_{\frac{1}{2}-\Theta(\varepsilon)}(Sy) < (1 - \varepsilon)\|y\|_1] < \exp(-\Theta(k \log \frac{k}{\varepsilon\delta}))$ . Thus, for all  $y \in N$ ,  $q_{\frac{1}{2}-\Theta(\varepsilon)}(Sy) \geq 1 - \varepsilon$  holds with probability  $1 - \Theta(\delta)$  by a union bound.

Let  $X'$  be a matrix whose columns form an Auerbach basis ([52]) for the subspace  $X$ . That is, each column of  $X'$  has  $\ell_1$  norm 1 and  $\|z'\|_\infty \leq \|X'z'\|_1$  for all  $z'$ . By Fact 1, each entry of  $SX'$  is greater than  $\Theta(\frac{k^2}{\delta})$  with probability at most  $O(\frac{\delta}{k^2})$  because each column of  $X'$  has  $\ell_1$  norm 1. A union bound tells us that  $\|SX'\|_\infty \leq O(\frac{k^2}{\delta})$  with probability at least  $1 - \frac{\delta}{2}$ .

For arbitrary  $z \in X$ , we can write  $z = X'z'$ . Thus

$$\begin{aligned} \|Sz\|_\infty &= \|SX'z'\|_\infty \leq \|SX'\|_\infty \cdot \|z'\|_1 \leq O(k^2/\delta) \cdot k\|z'\|_\infty \\ &\leq O(k^3/\delta) \cdot \|X'z'\|_1 = O(k^3/\delta) \cdot \|z\|_1. \end{aligned}$$

Given any  $x$  in the intersection of the unit  $\ell_1$  ball and  $X$ , we can write  $x = y + z$  where  $y \in N$ ,  $z \in X$ , and  $\|z\|_1 \leq \frac{\varepsilon\delta}{k^3}$ . By the above argument, we know  $\|Sz\|_\infty \leq O(\frac{k^3}{\delta})\|z\|_1 \leq O(\varepsilon)$ . Since  $Sx = Sy + Sz$ , then  $(1 - \Theta(\varepsilon))\|x\|_1 \leq q_{\frac{1}{2}-\Theta(\varepsilon)}(Sx)$  for any unit  $x$ . We can scale  $x$  and  $\varepsilon$  by the appropriate constants to get the desired statement.

The RHS inequality follows from a similar argument.  $\square$

We immediately have the following corollary about medians of Cauchy sketches over subspaces.

**Corollary 1.** Let  $X \subset \mathbb{R}^n$  be a  $k$ -dimensional space and  $\varepsilon, \delta > 0$ . Let  $S$  have  $O(\frac{1}{\varepsilon^2} k \log \frac{k}{\varepsilon\delta})$  rows,  $n$  columns, and i.i.d. Cauchy entries with scale parameter  $\gamma = 1$ . With probability at least  $1 - \Theta(\delta)$ , for all  $x \in X$ ,

$$(1 - \varepsilon)\|x\|_1 \leq \text{med}(Sx) \leq (1 + \varepsilon)\|x\|_1$$

We can also bound the median and the  $(1 - \varepsilon/2)$ -quantile of a Cauchy sketch of a fixed matrix.

**Lemma 6.** Let  $S$  be an  $m \times n$  matrix ( $m = \Theta(1/\text{poly}(\varepsilon))$ ) with i.i.d. standard Cauchy entries and let  $M$  be an  $n \times d$  matrix. For  $\varepsilon > 0$ , with probability  $1 - 1/\Omega(1)$ ,

$$(1 - \varepsilon)\|M\|_1 \leq \text{med}(SM) \leq (1 + \varepsilon)\|M\|_1$$

*Proof.* Lemma 4 tells us that we can choose  $m$  so that  $\Pr[\text{med}(SM_{:,i}) = (1 \pm \varepsilon)\|M_{:,i}\|_1] \geq 1 - \Theta(\varepsilon)$  for each  $i$ . Say  $i$  is good if  $\text{med}(SM_{:,i}) \geq (1 - \varepsilon)\|M_{:,i}\|_1$  and bad otherwise. Then  $\mathbb{E}[\sum_{\text{bad } i} \|M_{:,i}\|_1] \leq \varepsilon\|M\|_1$  so Markov's inequality tells us  $\sum_{\text{bad } i} \|M_{:,i}\|_1 \leq O(\varepsilon)\|M\|_1$  with probability  $1 - 1/\Omega(1)$  and also  $\sum_{\text{good } i} \|M_{:,i}\|_1 \geq (1 - \Theta(\varepsilon))\|M\|_1$ .

This implies that

$$\text{med}(SM) \geq \sum_{\text{good } i} \text{med}(SM_{:,i}) \geq (1 - \varepsilon) \sum_{\text{good } i} \|M_{:,i}\|_1 \geq (1 - \varepsilon)(1 - \Theta(\varepsilon))\|M\|_1$$

which gives our first desired inequality.

Now say that column  $i$  is small if  $\text{med}(SM_{:,i}) < (1 + \varepsilon)\|M_{:,i}\|_1$  and (for  $k \geq 1$ )  $k$ -large if

$$(k + 1 + \varepsilon)\|M_{:,i}\|_1 > \text{med}(SM_{:,i}) \geq (k + \varepsilon)\|M_{:,i}\|_1.$$

For  $k \geq 3$ , we can bound

$$\begin{aligned} \mathbb{E} \left[ \sum_{k \geq 1} k \sum_{k\text{-large } i} \|M_{:,i}\|_1 \right] &\leq \Theta(\varepsilon)\|M\|_1 + 2\Theta(\varepsilon)\|M\|_1 + \sum_{k \geq 3} k \exp(-\Theta(m)k)\|M\|_1 \\ &\leq O(\varepsilon)\|M\|_1 \sum_{k \geq 3} \frac{k}{\exp(\Theta(m)k)} \leq O(\varepsilon)\|M\|_1 \end{aligned}$$

where the second inequality comes from Lemma 4 and the third inequality comes from choosing  $m = \Theta(1/\text{poly}(\varepsilon))$ .

For  $k = 1$  or  $k = 2$ , note that if  $i$  is  $k$ -large, then  $\text{med}(SM_{:,i}) \geq (1 + \varepsilon)\|M_{:,i}\|_1$  which occurs with probability at most  $\Theta(\varepsilon)$  as mentioned earlier.

This lets us bound

$$\begin{aligned} \mathbb{E} \left[ \sum_{k \geq 1} k \sum_{k\text{-large } i} \|M_{:,i}\|_1 \right] &\leq \Theta(\varepsilon)\|M\|_1 + 2\Theta(\varepsilon)\|M\|_1 + \sum_{k \geq 3} k \exp(-\Theta(m)k)\|M\|_1 \\ &\leq O(\varepsilon)\|M\|_1 \sum_{k \geq 3} \frac{k}{\exp(\Theta(m)k)} \leq O(\varepsilon)\|M\|_1 \end{aligned}$$

where the last inequality occurs because the given infinite series converges by the ratio test.

Therefore

$$\begin{aligned} \text{med}(SM) &= \sum_{\text{small } i} \text{med}(SM_{:,i}) + \sum_{k \geq 1} \sum_{k\text{-large } i} \text{med}(SM_{:,i}) \\ &\leq (1 + \varepsilon)\|M\|_1 + \sum_{k \geq 1} (k + 1 + \varepsilon) \sum_{k\text{-large } i} \|M_{:,i}\|_1 \\ &\leq (1 + \varepsilon)\|M\|_1 + \sum_{k \geq 1} 3k \sum_{k\text{-large } i} \|M_{:,i}\|_1 \\ &\leq (1 + O(\varepsilon)) \cdot \|M\|_1 \end{aligned}$$

where the first inequality holds by the definition of  $k$ -large and the third inequality holds with probability  $1 - 1/\Omega(1)$  by Markov's inequality.  $\square$

**Lemma 7.** *When  $S$  is an  $m \times n$  matrix with i.i.d Cauchy entries,  $m = \Theta(1/\text{poly}(\varepsilon))$ , and  $M$  is  $n \times d$ , then with probability  $1 - 1/\Omega(1)$ ,*

$$q_{1-\varepsilon/2}(SM) \leq O\left(\frac{1}{\varepsilon}\right) \|M\|_1$$

*Proof.* Say that column  $i$  is small if  $q_{1-\varepsilon/2}(SM_{:,i}) < \frac{3}{\varepsilon}\|M_{:,i}\|_1$  and (for  $k \geq 3$ )  $k$ -large if

$$\frac{k+1}{\varepsilon}\|M_{:,i}\|_1 > q_{1-\varepsilon/2}(SM_{:,i}) \geq \frac{k}{\varepsilon}\|M_{:,i}\|_1.$$

We can bound

$$\begin{aligned} \Pr[i \text{ is } k\text{-large}] &\leq \Pr[q_{1-\varepsilon/2}(SM_{:,i}) \geq \frac{k}{\varepsilon}\|M_{:,i}\|_1] \\ &< \exp(-\Theta(\varepsilon)\frac{k}{\varepsilon}m) < \exp(-\Theta(m)k), \end{aligned}$$

where the second inequality comes from Lemma 4.

This lets us bound

$$\begin{aligned} \mathbb{E} \left[ \sum_{k \geq 3} \frac{k}{\varepsilon} \sum_{k\text{-large } i} \|M_{:,i}\|_1 \right] &\leq \sum_{k \geq 3} \frac{k}{\varepsilon} \exp(-\Theta(m)k)\|M\|_1 \\ &\leq \frac{1}{\varepsilon}\|M\|_1 \sum_{k \geq 3} \frac{k}{\exp(\Theta(m)k)} \\ &\leq O\left(\frac{1}{\varepsilon}\right) \|M\|_1 \end{aligned}$$

where the last inequality occurs because the given infinite series converges by the ratio test.

Therefore

$$\begin{aligned}
q_{1-\varepsilon/2}(SM) &= \sum_{\text{small } i} q_{1-\varepsilon/2}(SM_{:,i}) + \sum_{k \geq 3} \sum_{k\text{-large } i} q_{1-\varepsilon/2}(SM_{:,i}) \\
&\leq \frac{3}{\varepsilon} \|M\|_1 + \sum_{k \geq 3} \frac{k+1}{\varepsilon} \sum_{k\text{-large } i} \|M_{:,i}\|_1 \\
&\leq \frac{3}{\varepsilon} \|M\|_1 + \sum_{k \geq 3} \frac{2k}{\varepsilon} \sum_{k\text{-large } i} \|M_{:,i}\|_1 \\
&\leq O\left(\frac{1}{\varepsilon}\right) \|M\|_1
\end{aligned}$$

where the first inequality holds by the definition of  $k$ -large and the third inequality holds with probability  $1 - 1/\Omega(1)$  by Markov's inequality.  $\square$

**Chebyshev's inequality.** We record some basic facts. Let  $Z_1, \dots, Z_n$  be independent Bernoulli random variables, with  $Z_i \sim \text{Ber}(p_i)$ . Let  $Z := Z_1 + \dots + Z_n$  and  $\mu := \mathbb{E}[Z]$ .

**Lemma 8.** For any  $\Delta > 0$ , we have  $\Pr[|Z - \mu| > \Delta] \leq \mu/\Delta^2$ .

*Proof.* By independence, we have

$$\text{Var}(Z) = \sum_{i=1}^n \text{Var}(Z_i) = \sum_{i=1}^n p_i(1-p_i) \leq \sum_{i=1}^n p_i = \mu.$$

By Chebyshev's inequality, for any  $\Delta > 0$  we have

$$\Pr[|Z - \mu| > \Delta] \leq \text{Var}(Z)/\Delta^2.$$

With  $\text{Var}(Z) \leq \mu$  we thus obtain the claim.  $\square$

**Lemma 9.** For any  $\Delta > 0$ , we have  $\Pr[|Z - \mu| > \Delta] \leq \sqrt{\mu} \Delta$ .

*Proof.* As in the previous lemma's proof, we have

$$\Pr[|Z - \mu| > \Delta] \leq \text{Var}(Z)/\Delta^2,$$

where  $\text{Var}(Z) \leq \mu \leq n$ , and thus

$$\Pr[|Z - \mu| > \Delta] \leq \sqrt{\mu} \Delta.$$

The statement follows since if  $\sqrt{\mu} \Delta < 1$  we have  $\sqrt{\mu} \Delta^2 \leq \sqrt{\mu} \Delta$ , and otherwise the inequality is trivial.  $\square$



### 3 $\ell_p$ -Approximation Algorithms

Recall that in the Entrywise  $\ell_p$ -Rank- $k$  Approximation problem (for  $0 < p < 2$ ) we are given an  $n \times d$  matrix  $A$  with integer entries bounded in absolute value by  $\text{poly}(n)$ , a positive integer  $k$ , and we want to output matrices  $U \in \mathbb{R}^{n \times k}$  and  $V \in \mathbb{R}^{k \times d}$  minimizing  $\|A - UV\|_p^p := \sum_{i=1, \dots, n, j=1, \dots, d} |A_{i,j} - (U \cdot V)_{i,j}|^p$ . In this section, we prove Theorem 1, restated here for convenience.

**Theorem 1** (PTAS for  $0 < p < 2$ ). *Let  $p \in (0, 2)$  and  $\varepsilon \in (0, 1)$ . There is a  $(1 + \varepsilon)$ -approximation algorithm to Entrywise  $\ell_p$ -Rank- $k$  Approximation running in  $n^{\text{poly}(k/\varepsilon)}$  time.*

In Subsections 3.1, we prove in Corollary 2 our core algorithm result which solves the Entrywise  $\ell_p$ -Rank- $k$  Approximation problem for  $p = 1$ . In Subsection 3.2, we give an algorithm for the case when  $1 < p < 2$ , and we prove its correctness in Corollary 4. In Subsection 3.3, we prove in Corollary 6 the correctness of our algorithm for  $0 < p < 1$ . Then, we conclude the proof of Theorem 1 by combining Corollary 2, Corollary 4 and Corollary 6.

In Subsection 3.4, we give a  $(3 + \varepsilon)$ -approximation algorithm for the Entrywise  $\ell_p$ -Rank- $k$  Approximation problem in the case when  $p > 2$ .

Recall that in the Entrywise  $\ell_0$ -Rank- $k$  Approximation over  $\mathbb{F}_q$  problem we are given an  $n \times d$  matrix  $A$  with entries in  $\mathbb{F}_q$ , a positive integer  $k$ , and we want to output matrices  $U \in \mathbb{F}_q^{n \times k}$  and  $V \in \mathbb{F}_q^{k \times d}$  minimizing  $\|A - UV\|_0$ . In Subsection 3.5, we prove Theorem 3 and for reader's convenience we restate here our result.

**Theorem 3** (Alternate  $\mathbb{F}_q$  PTAS for  $p = 0$ ). *For  $\varepsilon \in (0, 1)$  there is a  $(1 + \varepsilon)$ -approximation algorithm to Entrywise  $\ell_0$ -Rank- $k$  Approximation over  $\mathbb{F}_q$  running in  $n \cdot d^{\text{poly}(k/\varepsilon)}$  time.*

#### 3.1 $\ell_1$ -Approximation Algorithm

In this subsection,  $A_k$  will denote the rank  $k$  matrix closest to  $A$  in the entrywise  $\ell_1$ -norm. We will need a claim adapted from [19].

**Claim 1.** *If  $A$  is  $n$  by  $d$  and has integer entries bounded by  $\gamma = \text{poly}(n)$  and rank  $r > k$ , then we have*

$$\min_{\text{rank } k \ A_k} \|A - A_k\|_1 \geq \frac{1}{\text{poly}(n)^k}$$

*Proof.* Note that it suffices to lower bound  $\sigma_{k+1}$ , the  $k$ th singular value of  $A$ , because  $\|A - A_k\|_1 \geq \|A - A_k\|_F \geq \sigma_{k+1}$ .

Since  $A$  has integer entries, then so does  $A^T A$  and its characteristic polynomial has integer coefficients. Now  $A^T A$  has eigenvalues  $\sigma_i^2$  so its characteristic polynomial's last term is  $\prod_{i=1}^r \sigma_i^2$  which is at least 1 because it is a positive integer. For any  $j$ ,  $\sigma_j^2 \leq \|A\|_F^2 \leq \mathbf{d}\gamma^2$ .

We have

$$\sigma_{k+1}^{r-k} \geq \prod_{k < i \leq r} \sigma_i^2 \geq \frac{\prod_{1 \leq i \leq r} \sigma_i^2}{(\mathbf{d}\gamma^2)^k} \geq \frac{1}{(\mathbf{d}\gamma^2)^k}$$

so  $\sigma_{k+1} \geq \frac{1}{(nd\gamma^2)^k}$  because  $r - k \geq 1$ . □

We can now describe our  $(1 + \varepsilon)$ -approximation algorithm. For the rest of this section, let  $U^*$  and  $V^*$  be minimizers for  $\|UV - A\|_1$  with  $\text{OPT} = \|U^*V^* - A\|_1$ . The quantities  $\theta, \psi$  will be bounded above by  $\text{poly}(n)$ . The quantity  $q$  will be bounded above by  $\text{poly}(k)$ . The specifics of how these values are chosen will be described in the algorithm's proof of correctness. The validity of the specific sampling described in Step 1 of Algorithm 1 will be proved in Corollary 2.

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**Algorithm 1**  $(1 + \varepsilon)$ - $\ell_1$  low rank approximation

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**Input:** A  $n \times d$  matrix  $A$  with integer entries bounded by  $\gamma = \text{poly}(n)$ . An integer  $k \in [d]$  and a real  $\varepsilon \in (0, 1)$ .

**Output:** Matrices  $U \in \mathbb{R}^{n \times k}$  and  $V \in \mathbb{R}^{k \times d}$  satisfying  $\|UV - A\|_1 \leq (1 + \varepsilon)OPT$ .

1. **If**  $A$  has rank at most  $k$ , then **return** a rank  $k$  decomposition  $U, V$  of  $A$ .
  2. **Sample** an  $m \times n$  matrix  $S$  satisfying the conditions of Theorem 10 (e.g. by taking  $m = \text{poly}(k\varepsilon)$  and sampling each entry of  $S$  from a standard Cauchy distribution).
  3. **Round** the entries of  $S$  to the nearest multiple of  $\frac{\varepsilon^2}{(\theta\psi)^k}$  where  $\theta, \psi \leq \text{poly}(n)$  are chosen as described in the proof of Theorem 10.
  4. **Set**  $U$  and  $V$  to be zero matrices as a default.
  5. Exhaustively **guess** all possible values of  $SU^*$  with entries **rounded** to the nearest multiple of  $\frac{\varepsilon}{qnk\theta^k} \frac{\varepsilon^2}{(\theta\psi)^k}$ , where  $q \leq \text{poly}(k)$  is chosen as described in the proof of Theorem 10.
    6. For each guessed  $SU^*$ , **set**  $\tilde{V} = \arg \min_V \text{med}(SU^*V - SA)$  s.t.  $\|V\|_\infty \leq \frac{2nd\gamma qnk\theta^k}{\varepsilon}$ .
    7. For each  $\tilde{V}$ , **set**  $\tilde{U} = \arg \min_U \|U\tilde{V} - A\|_1$ .
    8. **If**  $\|\tilde{U}\tilde{V} - A\|_1 < \|UV - A\|_1$ , then **set**  $U = \tilde{U}, V = \tilde{V}$ .
  9. **Return**  $U, V$ .
- 

**Theorem 10.** Let  $A$  be an  $n \times d$  matrix with integer entries such that  $\|A\|_\infty$  is bounded by  $\gamma = \text{poly}(n)$ . Suppose  $S$  is an  $m \times n$  random matrix such that with probability  $1 - 1/\Omega(1)$ ,  $\text{med}(SU^*V - SA) \geq (1 - \varepsilon)\|U^*V - A\|_1$  for all  $V$  and for a fixed  $V^*$ ,  $\text{med}(SU^*V^* - SA) \leq (1 + \varepsilon)\|U^*V^* - A\|_1$  with probability  $1 - 1/\Omega(1)$ . Suppose further that  $\|S\|_\infty \leq \text{poly}(n)$ . Then Algorithm 1 is a  $(1 + \varepsilon)$ -approximation algorithm for rank  $k$  low rank approximation in the entrywise  $\ell_1$  norm and runs in time  $\text{poly}(n)^{mk}$ .

*Proof.* First, if  $A$  has rank at most  $k$ , then we can just use Gaussian elimination to deduce that its optimal low rank approximation has value 0. We will assume its rank is greater than  $k$ .

We can assume  $V^*$  is an  $\ell_1$  well-conditioned basis since we can replace  $U^*$  and  $V^*$  with  $U^*R$  and  $V^*R^{-1}$  respectively for an invertible  $R$ . Thus for all  $x$  we have  $\frac{\|x\|_1}{q'} \leq \|x^T V^*\|_1 \leq q\|x\|_1$  where  $q', q = \text{poly}(k)$ . Using this well-conditioned basis property we see that each entry of  $U^*$  is at most  $2d\gamma q' \leq \text{poly}(n)$  because otherwise  $\|U^*V^* - A\|_1 \geq 2d\gamma - \|A\|_1 \geq \|A\|_1$  and we could improve the  $\ell_1$  error by taking  $U^* = 0$ .

Claim 1 says that there exists  $\theta \leq \text{poly}(n)$  such that  $OPT \geq \frac{1}{\theta^k}$ . By using the well-conditioned basis property of  $V^*$  and Claim 1, we can also assume that each entry of  $U^*$  is rounded to nearest multiple of  $\frac{\varepsilon}{qnk\theta^k}$  as this will incur an additive error of at most  $\varepsilon OPT$ . Thus  $U^*$  has discretized and bounded entries. Note that there are at most  $\varepsilon^{-1} \text{poly}(n)^k$  possible values for each entry of  $U^*$ .

Since the entries of  $U^*$  are discretized by  $\frac{\varepsilon}{qnk\theta^k}$ , then the entries of  $V^*$  can be bounded above by  $\frac{2nd\gamma qnk\theta^k}{\varepsilon}$  because otherwise  $\|U^*V^* - A\|_1 \geq 2d\gamma - \|A\|_1 \geq \|A\|_1$  and we might as well have set  $V^* = 0$ .

Let  $V$  be arbitrary with  $\|V\|_\infty \leq \frac{2nd\gamma qnk\theta^k}{\varepsilon}$ . Then  $\|U^*V - A\|_1 \leq \varepsilon^{-1}\psi^k$  where  $\psi \leq \text{poly}(n)$ . We will round each entry of  $S$  to the nearest multiple of  $\frac{\varepsilon^2}{(\theta\psi)^k}$ , so we can write  $S = \tilde{S} + \Delta$  where  $\tilde{S}$  is discretized and  $\|\Delta\|_\infty \leq \frac{\varepsilon^2}{(\theta\psi)^k}$ . Note that  $\|\Delta(U^*V - A)\|_1 \leq \frac{\varepsilon}{\theta^k} \leq \varepsilon OPT$ .

Now we will prove the correctness of our algorithm. We can sample  $S = \tilde{S} + \Delta$ . Note that  $\tilde{S}U^*$  will have entries that are multiples of  $\frac{\varepsilon}{qnk\theta^k} \frac{\varepsilon^2}{(\theta\psi)^k} \geq \text{poly}(\frac{\varepsilon}{n})^k$  and bounded by  $\text{poly}(\frac{n}{\varepsilon})^k$  because  $\tilde{S}$  is discretized and bounded. Since  $\tilde{S}U^*$  is  $m \times n$ , then in  $\text{poly}(\frac{n}{\varepsilon})^k$  time we can exhaustively search through all possible values of  $\tilde{S}U^*$  and one of them will be correct.

For each guess of  $\tilde{S}U^*$  and each  $i$  we minimize  $\text{med}(\tilde{S}U^*V_{:,i} - \tilde{S}A_{:,i})$  over  $\|V_{:,i}\|_\infty \leq \frac{2nd\gamma qnk\theta^k}{\varepsilon}$  to get  $\tilde{V}_{:,i}$ . We have  $\text{med}(\tilde{S}U^*\tilde{V} - \tilde{S}A) \leq \text{med}(\tilde{S}U^*V^* - \tilde{S}A)$ .

---

<sup>3</sup>Observe that there are at most  $m!$  orderings of the entries of  $\tilde{S}U^*V_{:,i} - \tilde{S}A_{:,i}$  and we are minimizing a linear function over  $V_{:,i}$  subject to a linear constraint. This can be solved with linear programming, so it will be done within the  $\text{poly}(n)^{mk}$  runtime.

Now

$$\begin{aligned}
\text{med}(\tilde{S}U^*V^* - \tilde{S}A) &= \text{med}(S(U^*V^* - A) - \Delta(U^*V^* - A)) \\
&\leq \text{med}(S(U^*V^* - A)) + \varepsilon OPT \\
&\leq (1 + \varepsilon)\|U^*V^* - A\|_1 + \varepsilon OPT \\
&\leq (1 + O(\varepsilon))OPT.
\end{aligned}$$

We choose  $\tilde{U}$  to minimize  $\|\tilde{U}\tilde{V} - A\|_1$ , so

$$\begin{aligned}
\text{med}(\tilde{S}U^*\tilde{V} - \tilde{S}A) &= \text{med}(S(U^*\tilde{V} - A) - \Delta(U^*\tilde{V} - A)) \\
&\geq \text{med}(S(U^*\tilde{V} - A)) - \varepsilon OPT \\
&\geq (1 - \varepsilon)\|U^*\tilde{V} - A\|_1 - \varepsilon OPT \\
&\geq (1 - \varepsilon)\|\tilde{U}\tilde{V} - A\|_1 - \varepsilon OPT
\end{aligned}$$

It follows that the best  $\tilde{U}$  and  $\tilde{V}$  will satisfy  $\|\tilde{U}\tilde{V} - A\|_1 \leq (1 + O(\varepsilon))OPT$ .  $\square$

Note that if  $m = \Theta(\text{poly}(\frac{k \log d}{\varepsilon}))$ , then the above algorithm is a quasipolynomial time  $(1 + \varepsilon)$ -approximation scheme (treating  $k$  like a constant). This is because we can use Corollary 1 (with  $\delta = \text{poly}(1/d)$ ) to see that

$$\text{med}(S \begin{bmatrix} U^* & A_{:,i} \\ [V_{:,i} & 1]^T \end{bmatrix}) = (1 \pm \varepsilon)\| [U^* & A_{:,i}] [V_{:,i} & 1]^T \|_1$$

(when  $V_{:,i}$  is arbitrary) with probability at least  $1 - \text{poly}(1/d)$  for each  $i$ . By a union bound,  $\text{med}(S(U^*V - A)) = (1 \pm \varepsilon)\|U^*V - A\|_1$  for arbitrary  $V$  with probability  $1 - 1/\Omega(1)$ . Furthermore, Fact 1 tells us that  $\Pr[S_{i,j} \geq \text{poly}(n)] \leq \text{poly}(n)^{-1}$  so by a union bound, all entries of  $S$  are bounded by  $\text{poly}(n)$  with probability  $1 - 1/\Omega(1)$ .

Of course, if we could reduce  $m$  to  $\Theta(\text{poly}(\frac{k}{\varepsilon}))$ , then we would have a PTAS. With the target bound for  $m$ , we would still have a  $(1 \pm \varepsilon)$ -embedding for each column index  $i$  with probability  $1 - 1/\Omega(1)$ , but we need all  $d$  embeddings to be valid at once. We accomplish this in the next result which is a variant of Lemma 27 from [20].

**Theorem 11.** *Let  $U \in \mathbb{R}^{n \times k}$ ,  $A \in \mathbb{R}^{n \times d}$ . Let  $V^*$  be chosen to minimize  $\|UV^* - A\|_1$ . Suppose  $S$  is an  $m \times n$  matrix satisfying*

$$(i) \quad q_{\frac{1}{2}-\varepsilon}(SUx) \geq (1 - \Theta(\varepsilon))\|Ux\|_1$$

$$(ii) \quad \text{For each } i \text{ with probability at least } 1 - \varepsilon^3, \text{med}(S[U \ A_{:,i}]x) \geq (1 - \varepsilon^3)\|[U \ A_{:,i}]x\|_1 \text{ for all } x$$

$$(iii) \quad \text{med}(SUV^* - SA) \leq (1 + \varepsilon^3)\|UV^* - A\|_1$$

$$(iv) \quad q_{1-\varepsilon/2}(S(UV^* - A)) \leq O\left(\frac{1}{\varepsilon}\right)\|UV^* - A\|_1$$

*Then  $\text{med}(SUV - SA) \geq (1 - O(\varepsilon))\|UV - A\|_1$  for arbitrary  $V$ .*

*Proof.* We say a column index  $i$  is *good* if

$$\text{med}(S([U \ A_{:,i}]y)) \geq (1 - \varepsilon^3)\|[U \ A_{:,i}]y\|_1$$

for all  $y \in \mathbb{R}^{k+1}$ , and *bad* otherwise. We say a bad column index is *large* if

$$\varepsilon\|(UV - A)_{:,i}\|_1 \geq \frac{1}{1 - \varepsilon}q_{1-\varepsilon/2}(S(UV^* - A)_{:,i}) + \|(UV^* - A)_{:,i}\|_1$$

and *small* otherwise.

By (ii), we know that  $\mathbb{E}[\sum_{\text{bad } i} \|(UV^* - A)_{:,i}\|_1] \leq \varepsilon^3\|UV^* - A\|_1$ . By Markov's inequality, we know that with probability  $1 - 1/\Omega(1)$ ,

$$\sum_{\text{bad } i} \|(UV^* - A)_{:,i}\|_1 \leq O(\varepsilon^3\|UV^* - A\|_1). \quad (6)$$

Since we were only using the probability that a column  $i$  was bad in the bound above, then by a similar Markov's inequality argument we know that with probability  $1 - 1/\Omega(1)$ ,

$$\sum_{\text{bad } i} q_{1-\varepsilon/2}(S(UV^* - A)_{:,i}) \leq O(\varepsilon^3)q_{1-\varepsilon/2}(S(UV^* - A)). \quad (7)$$

By (iii)

$$\begin{aligned} (1 + \varepsilon^3)\|UV^* - A\|_1 &\geq \text{med}(S(UV^* - A)) \\ &\geq (1 - \varepsilon^3) \sum_{\text{good } i} \|(UV^* - A)_{:,i}\|_1 + \sum_{\text{bad } i} \text{med}(S(UV^* - A)_{:,i}) \\ &\geq (1 - \varepsilon^3)(1 - \Theta(\varepsilon^3))\|UV^* - A\|_1 + \sum_{\text{bad } i} \text{med}(S(UV^* - A)_{:,i}), \end{aligned}$$

where the second inequality comes from the definition of good, and the third inequality comes from (6).

Thus

$$\sum_{\text{bad } i} \text{med}(S(UV^* - A)_{:,i}) \leq O(\varepsilon^3)\|UV^* - A\|_1 \quad (8)$$

We also have

$$\begin{aligned} \sum_{\text{small } i} \|(UV - A)_{:,i}\|_1 &\leq \frac{1}{\varepsilon(1 - \varepsilon)} \sum_{\text{small } i} q_{1-\varepsilon/2}(S(UV^* - A)_{:,i}) + \frac{1}{\varepsilon} \sum_{\text{small } i} \|(UV^* - A)_{:,i}\|_1 \\ &\leq \frac{1}{\varepsilon(1 - \varepsilon)} \sum_{\text{bad } i} q_{1-\varepsilon/2}(S(UV^* - A)_{:,i}) + O(\varepsilon^2)\|UV^* - A\|_1 \\ &\leq O(\varepsilon^2)q_{1-\varepsilon/2}(S(UV^* - A)) + O(\varepsilon^2)\|UV^* - A\|_1 \\ &\leq O(\varepsilon)\|UV^* - A\|_1 + O(\varepsilon^2)\|UV^* - A\|_1 \\ &\leq O(\varepsilon)\|UV^* - A\|_1 \end{aligned} \quad (9)$$

where the first inequality comes from the definition of small, the second inequality comes from (6) and the fact that small columns are bad columns, the third inequality comes from (7), and the fourth inequality comes from (iv).

**Claim 2.**

$$\sum_{\text{large } i} \text{med}(S(UV - A)_{:,i}) \geq (1 - O(\varepsilon)) \sum_{\text{large } i} \|(UV - A)_{:,i}\|_1$$

*Proof.* Let  $i$  be large. We can write  $S(UV - A)_{:,i} = SU(V - V^*)_{:,i} + S(UV^* - A)_{:,i}$ .

By (i), we know at least  $\frac{1}{2} + \varepsilon$  entries of  $SU(V - V^*)_{:,i}$  are larger than  $(1 - O(\varepsilon))\|U(V - V^*)_{:,i}\|_1$  which is at least

$$(1 - O(\varepsilon))(\|(UV - A)_{:,i}\|_1 - \|(UV^* - A)_{:,i}\|_1)$$

by the triangle inequality. By the definition of large, this is at least

$$(1 - O(\varepsilon))((1 - \varepsilon)\|(UV - A)_{:,i}\|_1 + \left(\frac{1}{1 - \varepsilon}\right)q_{1-\varepsilon/2}(S(UV^* - A)_{:,i}))$$

or

$$(1 - O(\varepsilon))^2\|(UV - A)_{:,i}\|_1 + q_{1-\varepsilon/2}(S(UV^* - A)_{:,i}).$$

By definition, less than an  $\varepsilon/2$  fraction of the entries of  $S(UV^* - A)_{:,i}$  are greater than  $q_{1-\varepsilon/2}(S(UV^* - A)_{:,i})$  so at least half of the entries of  $S(UV - A)_{:,i}$  are greater than  $(1 - O(\varepsilon))^2\|(UV - A)_{:,i}\|_1$ . The result follows.  $\square$

Finally

$$\begin{aligned}
\text{med}(S(UV - A)) &\geq \sum_{\text{good } i} \text{med}(S(UV - A)_{:,i}) + \sum_{\text{large } i} \text{med}(S(UV - A)_{:,i}) \\
&\geq (1 - \varepsilon^3) \sum_{\text{good } i} \|(UV - A)_{:,i}\|_1 + (1 - O(\varepsilon)) \sum_{\text{large } i} \|(UV - A)_{:,i}\|_1 \\
&\geq (1 - O(\varepsilon))\|UV - A\|_1 - (1 - O(\varepsilon)) \sum_{\text{small } i} \|(UV - A)_{:,i}\|_1 \\
&\geq (1 - O(\varepsilon))\|UV - A\|_1 - (1 - O(\varepsilon))O(\varepsilon)\|UV^* - A\|_1 \\
&\geq (1 - O(\varepsilon))\|UV - A\|_1
\end{aligned}$$

where the first inequality occurs because large  $i$  are bad  $i$ , the second inequality comes from the definition of good and Claim 2, the third inequality comes from the definition of small, the fourth inequality comes from (9), and the last inequality holds because  $V^*$  is a minimizer.  $\square$

**Corollary 2.** *Let  $A$  be an  $n \times d$  matrix with integer entries bounded by  $\text{poly}(n)$  and let  $k$  be a constant. There is a PTAS for finding the closest rank  $k$  matrix to  $A$  in the entrywise  $\ell_1$  norm.*

*Proof.* Let  $U^* \in \mathbb{R}^{n \times k}$ ,  $V^* \in \mathbb{R}^{k \times d}$  be minimizers for  $\|U^*V^* - A\|_1$ . It suffices to prove that an  $m \times n$  ( $m = \Theta(\text{poly}(\frac{k}{\varepsilon}))$ ) matrix  $S$  with i.i.d. standard Cauchy entries satisfies the conditions of Theorem 11 with  $U = U^*$ , then use Theorem 10.

Indeed,  $S$  satisfies (i) through Lemma 5 and (ii) with probability  $1 - 1/\Omega(1)$  through Corollary 1.  $S$  satisfies (iii) with probability  $1 - 1/\Omega(1)$  via Lemma 6 and (iv) with probability  $1 - 1/\Omega(1)$  via Lemma 7.  $\square$

### 3.2 $1 < p < 2$

We can extend these  $\ell_1$  results to  $\ell_p$  for  $1 < p < 2$  by using  $p$ -stable variables (with scale 1) instead of Cauchy variables (or 1-stable variables). These have the property that if  $x \in \mathbb{R}^n$  and  $Z, Z_i$  are i.i.d  $p$ -stable variables (for  $i = 1, \dots, n$ ) then  $\sum_{i=1}^n x_i Z_i \sim \|x\|_p Z$ .

**Definition 6.** *We let  $\text{med}_p$  denote the median of the absolute value of a  $p$ -stable variable.*

There is no convenient closed form expression for  $\text{med}_p$  unless  $p = 1$ , in which case  $\text{med}_1 = 1$ . However, in Appendix A.2 of [35] it is shown that a  $1 \pm \varepsilon$  approximation of  $\text{med}_p$  can be computed efficiently. Since we are only interested in  $\varepsilon$  approximations, then this will suffice for our purposes. Our main sketch will be  $\text{med}\left(\frac{(Sx)}{\text{med}_p}\right)$  ( $S$  has i.i.d  $p$ -stable entries with scale 1) which will concentrate around  $(1 \pm \varepsilon)\|x\|_p$ .

We can cite similar concentration / tail bounds for  $p$ -stable variables like the ones we used for Cauchy variables. We can also state a series of claims analagous to the ones we used in the  $\ell_1$  case.

**Fact 2.** *If  $Z$  is a  $p$ -stable variable with scale  $\gamma$ , then*

1. For  $\tau > 1$ ,  $\Pr[|Z| > \tau\gamma\text{med}_p] \leq \Theta(\frac{1}{\tau^p})$
2. For small  $\varepsilon > 0$ ,  $\Pr[|Z| > (1 + \varepsilon)\gamma\text{med}_p] < \frac{1}{2} - \Theta(\varepsilon)$
3. For small  $\varepsilon > 0$ ,  $\Pr[|Z| < (1 - \varepsilon)\gamma\text{med}_p] < \frac{1}{2} - \Theta(\varepsilon)$

**Lemma 10.** *Let  $S \in \mathbb{R}^{m \times n}$  have entries that are i.i.d.  $p$ -stable variables with scale 1 and let  $x \in \mathbb{R}^n$ . Then*

1.  $\Pr[q_{\frac{1}{2} - \Theta(\varepsilon)}(Sx) < (1 - \varepsilon)\|x\|_p \text{med}_p] < \exp(-\Theta(\varepsilon^2)m)$
2.  $\Pr[q_{\frac{1}{2} + \Theta(\varepsilon)}(Sx) > (1 + \varepsilon)\|x\|_p \text{med}_p] < \exp(-\Theta(\varepsilon^2)m)$
3. For  $M > 3$ ,  $\Pr[q_{1 - \frac{1}{M}}(Sx) > \frac{M}{\varepsilon}\|x\|_p \text{med}_p] < \exp(-\Theta(\varepsilon)Mm)$
4. For  $M > 3$ ,  $\Pr[\text{med}(Sx) > M\|x\|_p \text{med}_p] < \exp(-\Theta(m)M)$

*Proof.* The proof follows the same structure as the proof for Lemma 4. We use Fact 2 in combination with Chernoff bounds.  $\square$

Since  $1 < p$ , then we can take advantage of Minkowski's inequality and use the triangle inequality with  $\|\cdot\|_p$ .

**Lemma 11.** *Let  $X \subset \mathbb{R}^n$  be a  $k$ -dimensional space and  $\varepsilon, \delta > 0$ . Let  $S$  have  $O(\frac{1}{\varepsilon^2}k \log \frac{k}{\varepsilon\delta})$  rows,  $n$  columns, and i.i.d.  $p$ -stable entries with scale 1. Then with probability at least  $1 - \Theta(\delta)$ , for all  $x \in X$ ,*

$$(1 - \Theta(\varepsilon))\|x\|_p \leq q_{\frac{1}{2}-\varepsilon}(Sx \text{ med}_p) \leq q_{\frac{1}{2}+\varepsilon}(Sx \text{ med}_p) \leq (1 + O(\varepsilon))\|x\|_p$$

*Proof.* The proof follows the same structure as the proof for Lemma 5 except we use Fact 2 and  $p$ -well conditioned bases ([22]) to bound  $\|(Sz)/\text{med}_p\|_\infty$  for any  $z \in X$ . We also use the  $\ell_p$  ball (which is still convex) instead of the  $\ell_1$  ball.  $\square$

This automatically gives us the following corollary.

**Corollary 3.** *Let  $X \subset \mathbb{R}^n$  be a  $k$ -dimensional space and  $\varepsilon, \delta > 0$ . Let  $S$  have  $O(\frac{1}{\varepsilon^2}k \log \frac{k}{\varepsilon\delta})$  rows,  $n$  columns, and i.i.d.  $p$ -stable entries with scale 1. With probability at least  $1 - \Theta(\delta)$ , for all  $x \in X$ ,*

$$(1 - \varepsilon)\|x\|_p \leq \text{med}(Sx \text{ med}_p) \leq (1 + \varepsilon)\|x\|_p$$

We also have analogous versions of our bounds on fixed matrices. The proof structures are the same as those of Lemmas 6 and 7.

**Lemma 12.** *Let  $S$  be an  $m \times n$  matrix ( $m = \Theta(1/\text{poly}(\varepsilon))$ ) with i.i.d. standard  $p$ -stable entries and let  $M$  be an  $n \times d$  matrix. For  $\varepsilon > 0$ , with probability  $1 - 1/\Omega(1)$ ,*

$$(1 - \varepsilon) \sum_i \|M\|_p \leq \left( \sum_i \text{med}(SM_{:,i})^p \right)^{1/p} / \text{med}_p \leq (1 + \varepsilon) \sum_i \|M\|_p$$

**Lemma 13.** *When  $S$  is an  $m \times n$  matrix with i.i.d. standard  $p$ -stable entries,  $m = \Theta(1/\text{poly}(\varepsilon))$ , and  $M$  is  $n \times d$ , then with probability  $1 - 1/\Omega(1)$ ,*

$$\left( \sum_i q_{1-\varepsilon/2}(SM_{:,i})^p \right)^{1/p} / \text{med}_p \leq O\left(\frac{1}{\varepsilon}\right) \sum_i \|M\|_p$$

Finally, we have an  $\ell_p$  form of Theorem 11 and it is proved analogously.

**Theorem 12.** *Let  $U \in \mathbb{R}^{n \times k}$ ,  $A \in \mathbb{R}^{n \times d}$ . Let  $V^*$  be chosen to minimize  $\|UV^* - A\|_p$ . Suppose  $S$  is an  $m \times n$  matrix satisfying*

1.  $q_{\frac{1}{2}-\varepsilon}(SU \text{ med}_p) \geq (1 - \Theta(\varepsilon))\|Ux\|_p$
2. For each  $i$  with probability at least  $1 - \varepsilon^3$ ,  $\text{med}(S[U A_{:,i}] \text{ med}_p) \geq (1 - \varepsilon^3)\|[U A_{:,i}]x\|_p$  for all  $x$
3.  $(\sum_i \text{med}(SUV_{:,i}^* - SA_{:,i})^p)^{1/p} / \text{med}_p \leq (1 + \varepsilon^3)\|UV^* - A\|_p$
4.  $(\sum_i q_{1-\varepsilon/2}(S(UV^* - A)_{:,i})^p)^{1/p} / \text{med}_p \leq O\left(\frac{1}{\varepsilon}\right)\|UV^* - A\|_p$

*Then  $(\sum_i \text{med}(SUV_{:,i} - SA_{:,i})^p)^{1/p} / \text{med}_p \geq (1 - O(\varepsilon)) \sum_i \|UV - A\|_p$  for arbitrary  $V$ .*

It follows that we have a PTAS for rank  $k \ell_p$  low rank approximation.

**Corollary 4.** *Let  $A$  be an  $n \times d$  matrix with entries bounded by  $\text{poly}(n)$  and let  $k$  be a constant. There is a PTAS for finding the closest rank  $k$  matrix to  $A$  in entrywise  $\ell_p$  norm for  $1 < p < 2$ .*

*Proof.* The algorithm is analogous to Algorithm 1. Correctness follows from the fact that there exist  $\ell_p$  well-conditioned bases and that  $\ell_p$  regression is a convex optimization problem.

Indeed, if  $p > 1$  then  $\|UV_{:,i} - A_{:,i}\|_p$  is convex over vectors  $V_{:,i}$  and we can calculate minima in polynomial time.  $\square$

### 3.3 $0 < p < 1$

For  $v \in \mathbb{R}^n$  we will denote  $v^p$  to mean we raise each entry of  $v$  to the  $p$ th power, i.e.  $(v^p)_i = v_i^p$ .

We can extend these results to  $\ell_p$  for  $0 < p < 1$  as well, but more care needs to be taken for this range of  $p$  because among other issues,  $\|\cdot\|_p$  is no longer a norm. However,  $\|\cdot\|_p^p$  satisfies the triangle inequality which will be enough for our purposes. We will prove that  $\text{med}\left(\frac{(Sx)^p}{\text{med}_p^p}\right)$  ( $S$  has i.i.d  $p$ -stable entries) will concentrate around  $(1 \pm \varepsilon)\|x\|_p^p$ .

**Lemma 14.** *Let  $S \in \mathbb{R}^{m \times n}$  have entries that are i.i.d.  $p$ -stable variables with scale 1 and let  $x \in \mathbb{R}^n$ . Then*

1.  $\Pr[q_{\frac{1}{2}-\Theta(\varepsilon)}(Sx)^p < (1 - \varepsilon)\|x\|_p^p \text{med}_p^p] < \exp(-\Theta(\varepsilon^2)m)$
2.  $\Pr[q_{\frac{1}{2}+\Theta(\varepsilon)}(Sx)^p > (1 + \varepsilon)\|x\|_p^p \text{med}_p^p] < \exp(-\Theta(\varepsilon^2)m)$
3. For  $M > 3$ ,  $\Pr[q_{1-\frac{1}{M}}(Sx)^p > \frac{M}{\varepsilon}\|x\|_p^p \text{med}_p^p] < \exp(-\Theta(\varepsilon)Mm)$
4. For  $M > 3$ ,  $\Pr[\text{med}(Sx)^p > M\|x\|_p^p \text{med}_p^p] < \exp(-\Theta(m)M)$

*Proof.* These results follow from Lemma 10 and the fact that for  $0 < p < 1$ , we have  $(1 - \varepsilon)^p > 1 - \varepsilon$  and  $(1 + \varepsilon)^p < 1 + \varepsilon$ .  $\square$

Using the above quantile results we can prove an embedding result similar to Lemma 11 by using the fact that  $\|\cdot\|_p^p$  satisfies the triangle inequality.

**Lemma 15.** *Let  $X \subset \mathbb{R}^n$  be a  $k$ -dimensional space and  $\varepsilon, \delta > 0$ . Let  $S$  have  $O(\frac{1}{\varepsilon^2}k \log \frac{k}{\varepsilon\delta})$  rows,  $n$  columns, and i.i.d.  $p$ -stable entries with scale 1. Then with probability at least  $1 - \Theta(\delta)$ , for all  $x \in X$ ,*

$$(1 - \Theta(\varepsilon))\|x\|_p^p \leq q_{\frac{1}{2}-\varepsilon}((Sx)^p / \text{med}_p^p) \leq q_{\frac{1}{2}+\varepsilon}((Sx)^p / \text{med}_p^p) \leq (1 + O(\varepsilon))\|x\|_p^p$$

This automatically gives us the following corollary.

**Corollary 5.** *Let  $X \subset \mathbb{R}^n$  be a  $k$ -dimensional space and  $\varepsilon, \delta > 0$ . Let  $S$  have  $O(\frac{1}{\varepsilon^2}k \log \frac{k}{\varepsilon\delta})$  rows,  $n$  columns, and i.i.d.  $p$ -stable entries with scale 1. With probability at least  $1 - \Theta(\delta)$ , for all  $x \in X$ ,*

$$(1 - \varepsilon)\|x\|_p^p \leq \text{med}((Sx)^p / \text{med}_p^p) \leq (1 + \varepsilon)\|x\|_p^p$$

We also have analogous versions of our bounds on fixed matrices. Again, the proof structures are the same as those of Lemmas 6 and 7.

**Lemma 16.** *Let  $S$  be an  $m \times n$  matrix ( $m = \Theta(1/\text{poly}(\varepsilon))$ ) with i.i.d. standard  $p$ -stable entries and let  $M$  be an  $n \times d$  matrix. For  $\varepsilon > 0$ , with probability  $1 - O(1)$ ,*

$$(1 - \varepsilon)\|M\|_p^p \leq \sum_i \text{med}((SM_{:,i})^p / \text{med}_p^p) \leq (1 + \varepsilon)\|M\|_p^p$$

**Lemma 17.** *When  $S$  is an  $m \times n$  matrix with i.i.d standard  $p$ -stable entries,  $m = \Theta(1/\text{poly}(\varepsilon))$ , and  $M$  is  $n \times d$ , then with probability  $1 - O(1)$ ,*

$$\sum_i q_{1-\varepsilon/2}((SM_{:,i})^p / \text{med}_p^p) \leq O\left(\frac{1}{\varepsilon}\right) \sum_i \|M_{:,i}\|_p^p$$

As expected, we have an  $\ell_p$  form of Theorem 11 and it is proved analogously.

**Theorem 13.** *Let  $U \in \mathbb{R}^{n \times k}$ ,  $A \in \mathbb{R}^{n \times d}$ . Let  $V^*$  be chosen to minimize  $\|UV^* - A\|_p^p$ . Suppose  $S$  is an  $m \times n$  matrix satisfying*

1.  $q_{\frac{1}{2}-\varepsilon}((SUx)^p / \text{med}_p^p) \geq (1 - \Theta(\varepsilon))\|Ux\|_p^p$
2. For each  $i$  with probability at least  $1 - \varepsilon^3$ ,  $\text{med}((S[U A_{:,i}]x)^p / \text{med}_p^p) \geq (1 - \varepsilon^3)\|[U A_{:,i}]x\|_p^p$  for all  $x$
3.  $\sum_i \text{med}((SUV^* - SA)_{:,i}^p / \text{med}_p^p) \leq (1 + \varepsilon^3) \sum_i \|(UV^* - A)_{:,i}\|_p^p$

$$4. \sum_i q_{1-\varepsilon/2}(S(UV^* - A)_i^p / \text{med}_p^p) \leq O\left(\frac{1}{\varepsilon}\right) \sum_i \|(UV^* - A)_i\|_p^p$$

Then  $\sum_i \text{med}((SUV - SA)_{:,i}^p / \text{med}_p^p) \geq (1 - O(\varepsilon)) \|(UV - A)\|_p^p$  for arbitrary  $V$ .

The results above can give us the desired PTAS.

**Corollary 6.** *Let  $A$  be an  $n \times d$  matrix with entries bounded by  $\text{poly}(n)$  and let  $k$  be a constant. There is a PTAS for finding the closest rank  $k$  matrix to  $A$  in entrywise  $\ell_p$  norm when  $0 < p < 1$ .*

*Proof.* The algorithm is slightly different from Algorithm 1, because  $\ell_p$  regression is no longer a convex optimization problem when  $0 < p < 1$ . Thus after sketching to find a minimizing  $V$ , we need a different approach to find a minimizing  $U$ . We accomplish this by sketching  $UV - A$  again, but from the right and guessing the sketched  $V$ . We use the guessed  $V$  to solve for  $U$ .

Besides the above modification, we rely on the fact that  $\|\cdot\|_p^p$  satisfies the triangle inequality. We also note that for  $0 < p < 1$ , we may not have a well-conditioned basis. However, we know that an  $\ell_1$  well-conditioned basis exists so there exist  $qr = \text{poly}(k)$  such that  $\frac{\|x\|_1}{q} \leq \|x^T V^*\|_1 \leq r\|x\|_1$ . By Holder's inequality, we know  $\|x^T V^*\|_p^p \leq d^{1-p} \|x^T V^*\|_1^p \leq d^{1-p} r^p \|x\|_p^p$  and  $\|x^T V^*\|_p^p \geq \|x^T V^*\|_1^p \geq \|x\|_1^p / q^p \geq d^{p-1} \|x\|_p^p / q^p$  so we can get a similar well-conditioned basis result saying there exist  $\tilde{q} \tilde{r} = \text{poly}(d)$  such that  $\frac{\|x\|_p}{\tilde{q}} \leq \|x^T V^*\|_p \leq \tilde{r} \|x\|_p$  which will suffice for our proof.  $\square$

### 3.4 $p > 2$

There are no  $p$ -stable random variables when  $p > 2$  so any  $\ell_p$ -approximation algorithms in this setting will need to rely on a different technique. Our sketch will be lifted from [22]. Rather than a matrix of  $p$ -stable random variables, we use a sampling matrix that samples  $m$  rows of  $A$  with each row  $i$  having some probability  $p_i$  of being sampled. Furthermore, each sampled row is reweighted by  $1/p_i$ . The following claim (adapted from Theorem 5 of [22]) says we can get a subspace embedding from the right sampling matrix.

**Claim 3.** *Suppose  $U$  is an  $n \times k$  matrix. Then there exists a  $m \times n$  sampling matrix  $S$  with  $m = \text{poly}(k/\varepsilon)$  such that  $\|SUX\|_p = (1 \pm \varepsilon)\|UX\|_p$  for all  $x$ .*

**Theorem 14.** *If  $A$  is an  $n \times d$  matrix with entries bounded by  $\text{poly}(n)$ , then there is a  $(3 + \varepsilon)$ -approximation algorithm running in time  $n^{\text{poly}(k/\varepsilon)}$  for finding the closest rank  $k$  matrix to  $A$  in the entrywise  $\ell_p$  norm for  $p > 2$ .*

*Proof.* Let  $S$  be the sampling matrix of the above claim. Let  $\hat{V}$  be a minimizer for the expression  $\|SU^*\hat{V} - SA\|_p$ . Again, by a similar argument as that of the proof of Theorem 10, we can guess  $SU^*$  using  $\text{poly}(n)$  tries. We can round the sampling probabilities and the entries of  $U^*$  to the nearest  $1/\text{poly}(n)$  value.

We know that

$$\begin{aligned} \|U^*\hat{V} - A\|_p &\leq \|U^*(\hat{V} - V^*)\|_p + \|U^*V^* - A\|_p \\ &\leq (1 + O(\varepsilon))\|SU^*(\hat{V} - V^*)\|_p + \|U^*V^* - A\|_p \\ &\leq (1 + O(\varepsilon))\|SU^*\hat{V} - SA\|_p + (1 + O(\varepsilon))\|SU^*V^* - SA\|_p + \|U^*V^* - A\|_p \\ &\leq 2(1 + O(\varepsilon))\|SU^*V^* - SA\|_p + \|U^*V^* - A\|_p \\ &\leq (3 + \varepsilon)\|U^*V^* - A\|_p \end{aligned}$$

where the second inequality follows from the embedding property of  $S$  and the fourth inequality comes from the definition of  $\hat{V}$  as a minimizer.

The final inequality comes from a Markov bound on  $S$ . More specifically, since  $S$  is a sampling matrix, then for an arbitrary matrix  $M$ ,  $\mathbb{E}[SM] = \|M\|_p$ . Thus Markov's Inequality says that with probability  $1 - O(1)$ , we have  $SM \leq O(1)\|M\|_p$ . This concludes the proof.  $\square$



### 3.5 Finite Fields

We can also study low rank approximation over finite fields. The  $\ell_p$  metrics are not defined over finite fields for  $p \gg 0$ , but we can look at low rank approximation over the entrywise  $\ell_0$  metric (where  $\|M\|_0 = |\{(i,j) : M_{i,j} \neq 0\}|$ ). For the rest of this section we will work over a finite field  $\mathbb{F}_q$ , for some prime power  $q$ .

The structure of the algorithm will be similar to that of the case  $0 < p < 2$  but our sketch will be based on hashing rather than  $p$ -stable random variables. Furthermore, we will be able to sketch in the dimension  $d$  row space rather than the dimension  $n$  column space and get a running time better than that of the  $0 < p < 2$  algorithms. We now describe a  $(1 + \varepsilon)$ -approximation sketch for the  $\ell_0$  metric, where  $\varepsilon$  will be sufficiently small. This sketch is inspired by the  $L_0$  streaming algorithm in [35]. Throughout this section, we will refer to constants  $C$  and  $C'$  that are sufficiently large.

Let  $S_i$  denote a  $n \times n$  matrix where column  $i$  is the standard basis column  $e_i$  with probability  $p_i = \min(\frac{1}{2^i}, 1)$  or the all zeroes column otherwise. In other words,  $S_i$  is a sampling matrix that takes  $x$  and preserves each coordinate with probability  $p_i$  and otherwise maps the coordinate to 0. Note that  $p_0 = 1$ . We can generate our matrices  $S_i$  by uniformly sampling  $n$  integers between 0 to  $n$  and sampling column  $j$  in  $S_i$  if the leading 1 in the  $j$ th integer (written in binary, with indexing starting from 1) is before the  $i$ th position. Observe that under this procedure, our subsampling is nested so that if  $S_i$  does not sample entry  $j$ , then neither will  $S_{i'}$  for any  $i' > i$ .

Note that by this nestedness property, we have  $\|x\|_0 = \|S_0 x\|_0 \geq \|S_1 x\|_0 \geq \|S_2 x\|_0 \geq \dots \geq \|S_{\log n - 1} x\|_0$ . Let  $S$  denote the  $n \log n \times n$  block matrix

$$\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ \vdots \\ S_{\log n - 1} \end{bmatrix}.$$

Let  $h$  be a pairwise independent hashing function from  $[n]$  to  $[\frac{C'}{\varepsilon^8}]$  and let  $H_0$  denote a  $\frac{C'}{\varepsilon^8} \times n$  hashing matrix where each column equals  $e_{h(i)}$ . Let  $H$  denote the  $\frac{C'}{\varepsilon^8} \log n \times n$  block matrix

$$\begin{bmatrix} H_0 S_0 \\ H_0 S_1 \\ H_0 S_2 \\ \vdots \\ H_0 S_{\log n - 1} \end{bmatrix} \quad \text{with } H^{(i)} = H_0 S_i.$$

Suppose that  $x = \begin{bmatrix} x^{(0)} \\ x^{(1)} \\ \vdots \\ x^{(\log n - 1)} \end{bmatrix}$  is a block vector. Then we let  $\widetilde{\text{nnz}}(x)$  denote  $\begin{bmatrix} \|x^{(0)}\|_0 \\ \|x^{(1)}\|_0 \\ \vdots \\ \|x^{(\log n - 1)}\|_0 \end{bmatrix}$ .

We will abuse notation and let  $\mathcal{C}^S(x) = \widetilde{\text{nnz}}(Sx)$  and  $\mathcal{C}(x) = \widetilde{\text{nnz}}(HSx)$  with the understanding that  $HSx$  and  $Sx$  are of different dimensions but have the same number of blocks.

The main idea of the sketch is that if  $\|x\|_0$  is less than a small constant and the coordinates of  $x$  are hashed into a number of buckets that is a large constant, then with high probability it will be a perfect hash. Thus the number of non-zero buckets will equal  $\|x\|_0$ . If  $x$  is subsampled with a low enough probability, then the subsampled vector will have an  $\ell_0$  value that is sufficiently small and it can be hashed as we described.

We should note that the hash is needed for dimensionality reduction, not for the sketch to be an accurate estimator. For certain proofs we will analyze properties of the sketch without the hashing step (as in  $\mathcal{C}^S(x)$ ).

So  $S$  will sample  $x$  with different subsampling probabilities and we will expect that one will be small enough. We can then hash that subsampled vector, count the number of non-zero entries, and rescale by the sampling probability to approximate  $\|x\|_0$ . It then suffices to identify a suitably subsampled vector.

To do so, we will let  $\tau := \frac{C}{\varepsilon^4}$  and define estimation functions  $\text{est}_j : \mathbb{R}^{\log n} \rightarrow \mathbb{R}$ , where  $\text{est}_j(v) = \frac{v_j}{p_j}$ . If  $j^*$  denotes the maximum index such that  $v_{j^*} > \gamma$  then  $\text{est}(v, \gamma) = \text{est}_{j^*}(v)$ . If such an index does not exist, then  $\text{est}(v, \gamma) = \text{est}_0(v)$ . We let  $\mathcal{E}(x, \gamma) = \text{est}(\mathcal{C}(x), \gamma)$  and  $\mathcal{E}_j(x, \gamma) = \text{est}_j(\mathcal{C}(x), \gamma)$ . We will also let  $\mathcal{E}^S(x, \gamma) = \text{est}(\mathcal{C}^S(x), \gamma)$  and  $\mathcal{E}_j^S(x, \gamma) = \text{est}_j(\mathcal{C}^S(x), \gamma)$ .

Note that we can replace all instances of  $n$  in the above definitions with  $d$  and our algorithm

will just sketch the row space rather than the column space. We use  $n$  in our discussion just to keep the exposition similar to the case of  $0 < p < 2$  and to emphasize the similarities in technique.

For ease of notation in our proofs, we will omit the parameter  $\gamma$  in  $\mathcal{E}(x), \mathcal{E}^S(x), \mathcal{E}_i(x), \mathcal{E}_i^S(x)$  if it is clear that  $\gamma = \tau$ .

The idea is that past  $j^*$  we can be confident that we are subsampling  $x$  with too small of a probability and if all subsampled values are too small, then we can be confident that  $\|x\|_0$  itself was small.

To sketch a vector it is enough to show that at the index  $j^*$ , a  $p_{j^*}$  fraction of  $x$  is sampled up to a relative error of  $\varepsilon$ . For the purposes of our low rank approximation algorithm, we will want a slightly stronger condition that the indices around  $j^*$  will be sampled "as expected" and that the value of  $j^*$  will be approximately  $\log(\|x\|_0/\gamma)$  as expected.

**Definition 7.** Given a threshold  $\gamma$ , let  $j = \max(0, \lfloor \log_2(\|x\|_0/\gamma) \rfloor)$ , so  $\gamma \leq \frac{\|x\|_0}{2^j} < 2\gamma$ . Let  $j^*$  be the maximum index such that  $\mathcal{C}(x)_{j^*} \geq \gamma$ , or 0 if none exists.

We say that  $\mathcal{E}(x_\gamma)$  is a well-behaved sampling if

1.  $j^* = j - 1, j$  or  $j + 1$
2. If  $\|x\|_0 \geq \gamma$ , then  $\mathcal{E}_i(x_\gamma) = (1 \pm \Theta(\varepsilon))\|x\|_0$  for  $i = j - 1, j, j + 1$ , and  $j + 2$
3. If  $\|x\|_0 < \gamma$ , then  $L_1 < 3\gamma/4$

To prove the correctness of our sketch, it will suffice to prove that with high probability our samplings are well-behaved samplings. We will need a folklore fact about pairwise independent hashing (the proof is included for completeness).

**Fact 3.** If  $h : [n] \rightarrow [m]$  is a pairwise independent hash function and  $m \geq \Omega(n^2/\varepsilon)$ , then with probability at least  $1 - \Theta(\varepsilon)$ ,  $h$  will perfectly hash  $[n]$ .

*Proof.* For  $i \neq j \in [n]$  let  $I_{i,j}$  be an indicator variable for the event  $h(i) = h(j)$ . Then  $I = \sum_{i \neq j} I_{i,j}$  is the total number of collisions. We have  $\mathbb{E}[I] = \sum_{i \neq j} \mathbb{E}[I_{i,j}] = \sum_{i \neq j} \frac{1}{m} \leq \frac{n^2}{m} \leq O(\varepsilon)$ . By Markov's Inequality,  $\Pr[I \geq 1] \leq O(\varepsilon)$  and the result follows.  $\square$

To make use of this fact we will set  $C' \gg C^2$ . These hash sizes are chosen such that they are at least  $\Omega(\gamma^2)$ . Thus any subsampling past level  $j^*$  will likely result in a perfect hashing.

Throughout this section, we will let  $L_j$  denote  $\|S_j x\|_0$  so

$$\mathbb{E}[L_j] = p_j \|x\|_0 \text{ and } \text{Var}[L_j] = p_j(1 - p_j)\|x\|_0 \leq \mathbb{E}[L_j].$$

**Lemma 18.** If  $O(1/\varepsilon^3) > \gamma > \Omega(1/\varepsilon^2)$ , then with probability at least  $1 - \Theta(\varepsilon)$  over the randomness of  $S$  and  $H$ ,  $\mathcal{E}(x_\gamma)$  is a well-behaved sampling.

In particular, this holds when  $\gamma = \tau$  or  $\gamma = \varepsilon\tau$ .

*Proof.* We let  $j^*$  and  $v$  be as given in the definition of well-behaved. First we consider the case when  $\|x\|_0 \geq \gamma$ .

Note that for  $i = j - 1, j, j + 1$ , or  $j + 2$ , we have  $\mathbb{E}[L_i] \geq \|x\|_0/2^{j+2} \geq \gamma/4$ . Since  $\text{Var}[L_i] \leq \mathbb{E}[L_i]$ , then by Chebyshev's Inequality, we know

$$\Pr[L_i \notin (1 \pm \varepsilon)\mathbb{E}[L_i]] \leq \left( \frac{\sqrt{\text{Var}[L_i]}}{\varepsilon \mathbb{E}[L_i]} \right)^2 \leq \frac{1}{\varepsilon^2 \mathbb{E}[L_i]} \leq \frac{4}{\varepsilon^2 \gamma} \leq O(\varepsilon).$$

For the given values of  $i$ , we have  $\mathbb{E}[L_i] \leq \|x\|_0/2^{j-1} \leq 4\gamma$ . Since  $H_0$  hashes to a range of size  $C'/\varepsilon^6 > (4\gamma)^2$ , then by Fact 3,  $H_0$  will perfectly hash the non-zero entries of  $S_i x$  for the given values of  $i$  with probability at least  $1 - \Theta(\varepsilon)$ .

By a union bound,  $\mathcal{C}(x)_{:,i} = (1 \pm \varepsilon)\mathbb{E}[L_i]$  for  $i = j - 1, j, j + 1$ , or  $j + 2$  with probability at least  $1 - \Theta(\varepsilon)$ . Thus

$$\Pr[\mathcal{E}_i(x) = (1 \pm \varepsilon)\|x\|_0] = \Pr[\mathcal{C}(x)_{:,i} = (1 \pm \varepsilon)L_i] \geq 1 - \Theta(\varepsilon)$$

which satisfies (ii).

As we argued above, with probability at least  $1 - \Theta(\varepsilon)$  both  $\mathcal{C}(x)_{j-1} \geq (1 - \varepsilon)\mathbb{E}[L_{j-1}] \geq 3\gamma/2$  and  $\mathcal{C}(x)_{j+2} \leq (1 + \varepsilon)\mathbb{E}[L_{j+2}] \leq 3\gamma/4$  hold. By the nestedness of our sampling procedure, for any  $i > j + 2$  we have  $\mathcal{C}(x)_{:,i} \leq 3\gamma/4$ . Thus  $j^* = j - 1, j$ , or  $j + 1$  which satisfies (i).

Now suppose  $\|x\|_0 < \gamma$ . This implies  $j = 0$  and  $j^* = 0$  by definition which satisfies (i). If  $\|x\|_0 \geq \gamma/2$ , then by our reasoning above,  $L_1 < 3\gamma/4$  with probability at least  $1 - \Theta(\varepsilon)$ . If  $\|x\|_0 < \gamma/2$ , then  $L_1 < \gamma/2$  by the nestedness property of our sampling procedure. Therefore (iii) is satisfied.  $\square$

It follows that for a given  $x$ , with probability at least  $1 - \Theta(\varepsilon)$ ,  $\mathcal{E}(x) = (1 \pm \varepsilon)\|x\|_0$ .

We can also get tail bounds for  $\mathcal{E}_j(x)$  ( $j = 1, \dots, \log n$ ) and  $\mathcal{E}(x)$ .

**Lemma 19.** *Let  $M > 1$ . Then  $\Pr[\mathcal{E}_j(x) > M\|x\|_0] \leq \frac{1}{M}$  and  $\Pr[\mathcal{E}(x) > M\|x\|_0] \leq \frac{1}{M}$ . Furthermore,  $\Pr[\mathcal{E}_j^S(x) > M\|x\|_0] \leq \frac{1}{M}$  and  $\Pr[\mathcal{E}^S(x) > M\|x\|_0] \leq \frac{1}{M}$ .*

*Proof.* Let  $j^*$  be chosen so that  $\mathcal{E}(x) = \frac{\mathcal{C}_{j^*}(x)}{p_{j^*}}$ .

By Markov's Inequality, we have

$$\begin{aligned} \Pr[\mathcal{E}_j(x) > M\|x\|_0] &= \Pr[\mathcal{C}_j(x)/p_j > M\|x\|_0] \\ &\leq \Pr[L_j/p_j > M\|x\|_0] \\ &\leq \frac{\mathbb{E}[L_j]}{p_j M\|x\|_0} \\ &= \frac{1}{M} \end{aligned}$$

as desired. This implies that

$$\begin{aligned} \Pr[\mathcal{E}(x) > M\|x\|_0] &= \sum_i \Pr[\mathcal{C}_i(x)/p_i > M\|x\|_0] \Pr[j^* = i] \\ &\leq \frac{1}{M} \sum_i \Pr[j^* = i] \\ &= \frac{1}{M} \end{aligned}$$

and the result follows.  $\square$

Let  $K = \text{poly}(k, 1/\delta, 1/\varepsilon)$  for some  $\delta > 0$  and  $\mathcal{E}^{(1)}, \dots, \mathcal{E}^{(K)}$  be independent instances of the sketching procedure  $\mathcal{E}$ . Let  $\mathcal{A}(x) = \begin{bmatrix} \mathcal{E}^{(1)}(x) \\ \vdots \\ \mathcal{E}^{(K)}(x) \end{bmatrix}$ .

For a matrix  $M$ , we let  $\mathcal{A}(M)$  denote the matrix whose  $i$ th column is  $\mathcal{A}(M_{:,i})$ .

We can also define  $\mathcal{A}^S(M)$  the natural way.

We can study medians and quantiles of  $\mathcal{A}(M)$  like we did the medians and quantiles of our sketches based on  $p$ -stable variables.

**Lemma 20.** 1.  $\Pr[q_{\frac{1}{2} - \Theta(\varepsilon)}(\mathcal{A}(x)) < (1 - \varepsilon)\|x\|_0 < \exp(-\Theta(\varepsilon^2)K)$

2.  $\Pr[q_{\frac{1}{2} + O(\varepsilon)}(\mathcal{A}(x)) > (1 + \varepsilon)\|x\|_0 < \exp(-\Theta(\varepsilon^2)K)$

3. For  $T > 2$ ,  $\Pr[q_{1 - \frac{\varepsilon}{T}}(\mathcal{A}(x)) > \frac{T}{\varepsilon}\|x\|_0 < \exp(-\Theta(\varepsilon)TK)$

4. For  $T > 2$ ,  $\Pr[\text{med}(\mathcal{A}(x)) > T\|x\|_0 < \exp(-\Theta(T)K)$

The analogous bounds for  $\mathcal{A}^S(x)$  also hold.

*Proof.* We can use Chernoff bounds, Lemma 18, and Lemma 19 to prove this in a similar way to how the proof of Lemma 4 used Chernoff bounds and the tail bounds on Cauchy sketches.  $\square$

We can now deduce a finite field subspace embedding result.

**Corollary 7.** Let  $X \subset \mathbb{F}_q^n$  be a  $k$ -dimensional space. With probability at least  $1 - \Theta(\delta)$ , for all  $x \in X$ ,

$$(1 - \varepsilon)\|x\|_0 \leq q_{\frac{1}{2} - \Theta(\varepsilon)}(\mathcal{A}(x)) \leq q_{\frac{1}{2} + O(\varepsilon)}(\mathcal{A}(x)) \leq (1 + \varepsilon)\|x\|_0$$

and

$$(1 - \varepsilon)\|x\|_0 \leq q_{\frac{1}{2} - \Theta(\varepsilon)}(\mathcal{A}^S(x)) \leq q_{\frac{1}{2} + O(\varepsilon)}(\mathcal{A}^S(x)) \leq (1 + \varepsilon)\|x\|_0$$

*Proof.* We can use Lemma 20 and the fact that  $|X| = q^k$  to deduce the result with a union bound.  $\square$

We can also bound the median of an  $\ell_0$  sketch of a fixed matrix.

**Lemma 21.** Let  $M$  be an  $n \times d$  matrix. For  $\varepsilon > 0$ , with probability  $1 - 1/\Omega(1)$ ,

$$(1 - \varepsilon)\|M\|_0 \leq \text{med}(\mathcal{A}(M)) \leq (1 + \varepsilon)\|M\|_0$$

and

$$(1 - \varepsilon)\|M\|_0 \leq \text{med}(\mathcal{A}^S(M)) \leq (1 + \varepsilon)\|M\|_0$$

*Proof.* The proof follows the same structure as the proof of Lemma 6 where we bound the expected sum of  $\text{med}(\mathcal{A}(M_{:,i}))$  over values of  $i$  where  $\text{med}(\mathcal{A}(M_{:,i}))$  is large and conclude with Markov's Inequality. Instead of Fact 1 and Lemma 4, we use Lemma 18 and Lemma 20.  $\square$

We can also bound the  $(1 - \varepsilon/2)$ -quantile of an  $\ell_0$  sketch and we can bound the  $(1 - \varepsilon/2)$ -quantile of a fixed index  $\ell_0$  sketch of a fixed matrix.

**Lemma 22.** Let  $M$  be an  $n \times d$  matrix. Let  $J$  be a set of indices with  $|J| = d$ . With probability  $1 - 1/\Omega(1)$ ,

$$q_{1-\varepsilon/2}(\mathcal{A}(M)) \leq O\left(\frac{1}{\varepsilon}\right)\|M\|_0$$

and

$$q_{1-\varepsilon/2}(\mathcal{A}^S(M)) \leq O\left(\frac{1}{\varepsilon}\right)\|M\|_0$$

*Proof.* These inequalities can be proved using the same argument that was used to prove Lemma 7, but using Lemma 20 instead of Lemma 4.  $\square$

**Theorem 15.** Let  $U \in \mathbb{F}_q^{n \times k}$ ,  $A \in \mathbb{F}_q^{n \times d}$ . With probability  $1 - 1/\Omega(1)$ ,

$$\text{med}(\mathcal{A}(UV - A)) \geq (1 - O(\varepsilon))\|UV - A\|_0$$

for arbitrary  $V$ .

*Proof.* Let  $V^*$  be chosen to minimize  $\|UV^* - A\|_0$ .

For column indices  $i$ , let  $J_i = \max(0, \log(\|(UV^* - A)_{:,i}\|_0/\gamma))$

By Lemmas 18, 21, and 22, the following statements hold with probability  $1 - 1/\Omega(1)$ :

1.  $\mathcal{E}$  is well-behaved on  $Ux$  for all  $x$
2. For each  $i$  with probability at least  $1 - \varepsilon^3$ ,  $\text{med}(\mathcal{A}([U A_{:,i}]x)) \geq (1 - \varepsilon^3)\|[U A_{:,i}]x\|_0$  for all  $x$
3.  $\text{med}(\mathcal{A}(UV^* - A)) \leq (1 + \varepsilon^3)\|UV^* - A\|_0$
4.  $q_{1-\varepsilon/2}(\mathcal{A}^S(UV^* - A, \varepsilon\tau)) \leq O\left(\frac{1}{\varepsilon}\right)\|UV^* - A\|_0$

We say a column index  $i$  is *good* if

$$\text{med}(\mathcal{A}([U A_{:,i}]y)) \geq (1 - \varepsilon^3)\|[U A_{:,i}]y\|_0$$

for all  $y \in \mathbb{R}^{k+1}$ , and *bad* otherwise. Let  $Q_i = q_{1-\varepsilon/2}(\mathcal{A}^S(UV^* - A, \varepsilon\gamma)_{:,i})$ . We say a bad column index is *large* if

$$\varepsilon\|(UV - A)_{:,i}\|_0 \geq \frac{2}{1 - \varepsilon}Q_i + \|(UV^* - A)_{:,i}\|_0.$$

By (ii), we know that  $\mathbb{E}[\sum_{\text{bad } i} \|(UV^* - A)_{:,i}\|_0] \leq \varepsilon^3 \|UV^* - A\|_0$ . By Markov's inequality, we know that with probability  $1 - 1/\Omega(1)$ ,

$$\sum_{\text{bad } i} \|(UV^* - A)_{:,i}\|_0 \leq O(\varepsilon^3) \|UV^* - A\|_0 \quad (10)$$

By (iii)

$$\begin{aligned} (1 + \varepsilon^3) \|UV^* - A\|_0 &\geq \text{med}(\mathcal{A}(UV^* - A)) \\ &\geq (1 - \varepsilon^3) \sum_{\text{good } i} \|(UV^* - A)_{:,i}\|_0 + \sum_{\text{bad } i} \text{med}(\mathcal{A}(UV^* - A)_{:,i}) \\ &\geq (1 - \varepsilon^3)(1 - \Theta(\varepsilon^3)) \|UV^* - A\|_0 + \sum_{\text{bad } i} \text{med}(\mathcal{A}(UV^* - A)_{:,i}), \end{aligned}$$

where the second inequality comes from the definition of good, and the third inequality comes from (10).

Thus

$$\sum_{\text{bad } i} \text{med}(\mathcal{A}(UV^* - A)_{:,i}) \leq O(\varepsilon^3) \|UV^* - A\|_0 \quad (11)$$

We also have

$$\begin{aligned} \sum_{\text{small } i} \|(UV - A)_{:,i}\|_0 &\leq \frac{2}{\varepsilon(1 - \varepsilon)} \sum_{\text{small } i} Q_i + \frac{1}{\varepsilon} \sum_{\text{small } i} \|(UV^* - A)_{:,i}\|_0 \\ &\leq O\left(\frac{1}{\varepsilon^2(1 - \varepsilon)}\right) \left( \sum_{\text{small } i} \|(UV^* - A)_{:,i}\|_0 \right) + O(\varepsilon^2) \|UV^* - A\|_0 \\ &\leq O(\varepsilon) \|UV^* - A\|_0 \end{aligned} \quad (12)$$

where the first inequality comes from the definition of small, the second inequality comes from (iv) and (10) and the third inequality comes from (11).

**Claim 4.**

$$\sum_{\text{large } i} \text{med}(\mathcal{A}(UV - A)_{:,i}) \geq (1 - O(\varepsilon)) \sum_{\text{large } i} \|(UV - A)_{:,i}\|_0$$

*Proof.* Let column  $i$  be large. We have  $H(UV - A)_{:,i} = HU(V - V^*)_{:,i} + H(UV^* - A)_{:,i}$ .

By the triangle inequality, we have

$$\begin{aligned} &(1 - \Theta(\varepsilon)) \|U(V - V^*)_{:,i}\|_0 \\ &\geq (1 - \Theta(\varepsilon)) (\|(UV - A)_{:,i}\|_0 - \|(UV^* - A)_{:,i}\|_0) \\ &\geq (1 - \Theta(\varepsilon)) ((1 - \varepsilon) \|(UV - A)_{:,i}\|_0 + \frac{2}{1 - \varepsilon} Q_i) \\ &\geq (1 - \Theta(\varepsilon)) \|(UV - A)_{:,i}\|_0 + Q_i \\ &\geq (1 - \Theta(\varepsilon)) \|(UV - A)_{:,i}\|_0 \end{aligned}$$

where the second inequality follows from the definition of large.

Since  $\|U(V - V^*)_{:,i}\|_0 \geq (1 - \Theta(\varepsilon)) \|(UV - A)_{:,i}\|_0$  and  $\varepsilon \|(UV - A)_{:,i}\|_0 \geq Q_i$ , then  $Q_i/\varepsilon \leq \|U(V - V^*)_{:,i}\|_0$ .

If we run  $K$  independent instances of  $H$ , then by (i), we know that at least  $\frac{1}{2} + \varepsilon$  of those instances will have estimations  $\mathcal{E}(U(V - V^*)_{:,i})$  that are well-behaved and satisfy  $\mathcal{E}(U(V - V^*)_{:,i}) \geq (1 - \Theta(\varepsilon)) \|U(V - V^*)_{:,i}\|_0$ .

At least  $1 - \varepsilon/2$  of those instances will satisfy  $Q_i > \text{est}^S((UV^* - A)_{:,i}, \varepsilon\tau)$ . In each of these instances, there is some index  $t$  which is the maximum index where  $\mathcal{C}^S((UV^* - A)_{:,i}) > \varepsilon\tau$ . This index  $t$  satisfies  $\text{est}((UV^* - A)_{:,i}, \varepsilon\tau) \geq 2^t \varepsilon\tau$  which implies that

$$t \leq \log_2 \left( \frac{\text{est}^S((UV^* - A)_{:,i}, \varepsilon\tau)}{\varepsilon\tau} \right) < \log_2 \left( \frac{Q_i}{\varepsilon\tau} \right) \leq \log_2 \left( \frac{\|U(V - V^*)_{:,i}\|_0}{\tau} \right) \leq J_i$$

and by the nestedness property of  $S$ , for every index  $l \geq J_i - 1$  we have  $\mathcal{C}^S((UV^* - A)_{:,i})_l < \varepsilon\tau$ . Furthermore,  $\mathcal{C}((UV^* - A)_{:,i})_l < \varepsilon\tau$  because  $\|H_0 y\|_0 \leq \|y\|_0$  for all  $y$ .

Thus, for at least  $\frac{1}{2} + \varepsilon/2$  instances of  $H$ , it is true that  $\mathcal{E}(U(V - V^*)_{:,i})$  is well-behaved and for every index  $l \geq J_i - 1$  we have  $\mathcal{C}((UV^* - A)_{:,i})_l < \varepsilon\tau$ . We first consider the case that  $\|U(V - V^*)_{:,i}\|_0 > \tau$ .

We know that for  $l = J_i - 1, J_i$ , or  $J_i + 1$ , one of those values will be the maximum value such that the  $l$ th block of  $HU(V - V^*)_{:,i}$  has at least  $\tau$  non-zero entries, and all the later blocks will have at most  $3\tau/4$  non-zero entries. Each block of  $H(UV^* - A)_{:,i}$  after the  $J_i - 1$ th one will have fewer than  $\varepsilon\tau$  non-zero entries. By well-behavedness, it follows that  $\mathcal{E}(UV - A)_{:,i} = \mathcal{E}(U(V - V^*)_{:,i} + (UV^* - A)_{:,i}) \geq (1 - \Theta(\varepsilon))\|U(V - V^*)_{:,i}\|_0$  because the salient blocks of  $H(UV - A)_{:,i}$  will have a number of non-zero entries differing from those blocks of  $HU(V - V^*)_{:,i}$  by an additive  $\Theta(\varepsilon)$  error.

If  $\|U(V - V^*)_{:,i}\|_0 \leq \tau$ , then by well-behavedness we know that block 1 of  $HU(V - V^*)_{:,i}$  will have fewer than  $3\tau/4$  non-zero entries. In this case all blocks of  $H(UV^* - A)_{:,i}$  will have fewer than  $\varepsilon\tau$  non-zero entries so all blocks of  $H(UV - A)_{:,i}$  besides the zeroth block will have fewer than  $\tau$  non-zero entries. Thus,  $\mathcal{E}(UV - A)_{:,i} \geq (1 - \Theta(\varepsilon))\|U(V - V^*)_{:,i}\|_0$ .

Therefore in a majority of the instances of  $H$ , we have  $\mathcal{E}(UV - A)_{:,i} \geq (1 - \Theta(\varepsilon))\|U(V - V^*)_{:,i}\|_0 \geq (1 - \Theta(\varepsilon))\|(UV - A)_{:,i}\|_0$  and the result follows.  $\square$

Finally

$$\begin{aligned} \text{med}(\mathcal{A}(UV - A)) &\geq \sum_{\text{good } i} \text{med}(\mathcal{A}(UV - A)_{:,i}) + \sum_{\text{large } i} \text{med}(\mathcal{A}(UV - A)_{:,i}) \\ &\geq (1 - \varepsilon^3) \sum_{\text{good } i} \|(UV - A)_{:,i}\|_0 + (1 - O(\varepsilon)) \sum_{\text{large } i} \|(UV - A)_{:,i}\|_0 \\ &\geq (1 - O(\varepsilon))\|UV - A\|_0 - (1 - O(\varepsilon)) \sum_{\text{small } i} \|(UV - A)_{:,i}\|_0 \\ &\geq (1 - O(\varepsilon))\|UV - A\|_0 - (1 - O(\varepsilon))O(\varepsilon)\|UV^* - A\|_0 \\ &\geq (1 - O(\varepsilon))\|UV - A\|_0 \end{aligned}$$

where the first inequality occurs because large  $i$  are bad  $i$ , the second inequality comes from the definition of good and Claim 4, the third inequality comes from the definition of small, the fourth inequality comes from (12), and the last inequality holds because  $V^*$  is a minimizer.  $\square$

**Theorem 3** (Alternate  $\mathbb{F}_q$  PTAS for  $p = 0$ ). *For  $\varepsilon \in (0, 1)$  there is a  $(1 + \varepsilon)$ -approximation algorithm to Entrywise  $\ell_0$ -Rank- $k$  Approximation over  $\mathbb{F}_q$  running in  $n \cdot d^{\text{poly}(k/\varepsilon)}$  time.*

*Proof.* Suppose  $U^*$  and  $V^*$  ( $n \times k$  and  $k \times d$  respectively) are minimizers for  $\|UV - A\|_0$ . By Theorem 15,  $\text{med}(\mathcal{A}(U^*V^* - A)) = (1 \pm \varepsilon)\|U^*V^* - A\|_0$ . Since  $H$  is a  $\frac{C'}{\varepsilon^4} \log n \times n$  block matrix, then  $HU^*$  has  $\frac{C'}{\varepsilon^4} k \cdot \log n$  entries and we need  $K$  instances of  $HU^*$  for a total of  $(\log n) \cdot \text{poly}(k\varepsilon)$  entries each having  $q$  possible values. Thus we can exhaustively guess all possible values of  $HU^*$  in  $n^{\text{poly}(k/\varepsilon)}$  time.

For each guess of  $HU^*$  and each column  $i$ , we can try all  $q^k$  possible vectors  $V_i$  and choose the minimizer. Once a  $V$  has been identified, we can solve for its optimal  $U$  and throughout this whole process keep the best  $U$  and  $V$  that minimize  $\|UV - A\|_0$ . Since there are  $d$  rows, the algorithm will have a total runtime of  $d \cdot n^{\text{poly}(k/\varepsilon)}$ .

As we stated in the opening exposition of this section, we could have sketched over the dimension  $d$  row space instead. In this case we would be guessing values for  $H(V^*)^T$ , a  $\frac{C'}{\varepsilon^4} \log d \times k$  matrix, which would take  $d^{\text{poly}(k/\varepsilon)}$  time. We would then minimize over each of the  $n$  rows of  $U$  for a total runtime of  $n \cdot d^{\text{poly}(k/\varepsilon)}$ .  $\square$

## 4 Generalized Binary Approximation

Given a matrix  $A \in \{0, 1\}^{n \times d}$  with  $n \geq d$ , an integer  $k$ , and an inner product function  $\langle \cdot, \cdot \rangle: \{0, 1\}^k \times \{0, 1\}^k \rightarrow \mathbb{R}$ , the Generalized Binary  $\ell_0$ -Rank- $k$  problem asks to find matrices  $U \in \{0, 1\}^{n \times k}$  and  $V \in \{0, 1\}^{k \times d}$  minimizing  $\|A - U \cdot V\|_0$ , where the product  $U \cdot V$  is the  $n \times d$  matrix  $B$  with  $B_{i,j} = \langle U_{i,\cdot}, V_{\cdot,j} \rangle$ . An algorithm for the Generalized Binary  $\ell_0$ -Rank- $k$  problem is an  $\alpha$ -approximation, if it outputs matrices  $U \in \{0, 1\}^{n \times k}$  and  $V \in \{0, 1\}^{k \times d}$  satisfying

$$\|A - U \cdot V\|_0 \leq \alpha \cdot \min_{U' \in \{0,1\}^{n \times k}, V' \in \{0,1\}^{k \times d}} \|A - U' \cdot V'\|_0.$$

Choosing an appropriate inner product function  $\langle \cdot, \cdot \rangle$  which also runs in time  $O(k)$ , we obtain the Binary  $\ell_0$ -Rank- $k$  problem over the reals,  $\mathbb{F}_2$ , and the Boolean semiring. We assume that the function  $\langle \cdot, \cdot \rangle$  can be evaluated in time  $2^{O(k)}$ , in order to simplify our running time bounds.

In this section, we prove Theorem 2, restated here for convenience.

**Theorem 2** (PTAS for  $p = 0$ ). *For any  $\varepsilon \in (0, \frac{1}{2})$ , there is a  $(1 + \varepsilon)$ -approximation algorithm for the Generalized Binary  $\ell_0$ -Rank- $k$  problem running in time  $(2/\varepsilon)^{2^{O(k)}/\varepsilon^2} \cdot \mathbf{d}^{1+o(1)}$  and succeeds with constant probability<sup>4</sup>, where  $o(1)$  hides a factor  $(\log \log d)^{1.1} / \log d$ .*

Our algorithm achieves a substantial generalization of the standard clustering approach and applies to the situation with constrained centers. This yields the first randomized almost-linear time approximation scheme (PTAS) for the Generalized Binary  $\ell_0$ -Rank- $k$  problem. The time complexity of the algorithm in Theorem 2 is close to optimal, in the sense that the running time of any PTAS for the Generalized Binary  $\ell_0$ -Rank- $k$  problem must depend exponentially on  $1/\varepsilon$  and doubly exponentially on  $k$ , assuming the Exponential Time Hypothesis. For reader's convenience, we restate our result.

**Theorem 5** (Hardness for Generalized Binary  $\ell_0$ -Rank- $k$ ). *Assuming the Exponential Time Hypothesis, Generalized Binary  $\ell_0$ -Rank- $k$  has no  $(1 + \varepsilon)$ -approximation algorithm in time  $2^{1/\varepsilon^{o(1)}} \cdot 2^{n^{o(1)}}$ . Further, for any  $\varepsilon \geq 0$ , Generalized Binary  $\ell_0$ -Rank- $k$  has no  $(1 + \varepsilon)$ -approximation algorithm in time  $2^{2^{o(k)}} \cdot 2^{n^{o(1)}}$ .*

Regarding the dependence on  $\varepsilon$ , assume w.l.o.g. that  $n \geq d$ , and thus the input size is  $O(n)$ . Even for  $k = 1$  the problem is known to be NP-hard [21, 27]. Under ETH, no NP-hard problem has a  $2^{n^{o(1)}}$ -time algorithm<sup>5</sup>. We can restrict to  $\varepsilon \geq 1/n^2$ , since any better approximation already yields an optimal solution. It follows for  $k = 1$  that the Generalized Binary  $\ell_0$ -Rank- $k$  problem has no  $(1 + \varepsilon)$ -approximation algorithm in time  $2^{1/\varepsilon^{o(1)}} \cdot 2^{n^{o(1)}}$ . In other words, in order to improve our exponential dependence on  $1/\varepsilon$  to subexponential, we would need to pay an exponential factor in  $n$ .

Regarding the dependence on  $k$ , note that for any  $\varepsilon$  a  $(1 + \varepsilon)$ -approximation algorithm for our problem decides whether the answer is 0 or larger. In particular, over the Boolean semiring it solves the problem whether a given bipartite graph can be covered with  $k$  bicliques. For this problem, Chandran et al. [16] proved that even for  $k = O(\log n)$  there is no  $2^{2^{o(k)}}$ -time algorithm, unless ETH fails. It follows that for any  $\varepsilon \geq 0$ , Generalized Binary  $\ell_0$ -Rank- $k$  has no  $(1 + \varepsilon)$ -approximation algorithm in time  $2^{2^{o(k)}} \cdot 2^{o(n)}$ . In other words, in order to improve our doubly exponential dependence on  $k$ , we would need to pay an exponential factor in  $n$ .

**Organization** In Subsection 4.1, we state our core sampling result. In Subsection 4.2, we give a simple but inefficient deterministic PTAS for the Generalized Binary  $\ell_0$ -Rank- $k$  problem, which serves as a blueprint for our efficient randomized PTAS. We present first the deterministic PTAS as it is conceptually simple and exhibits the main algorithmic challenge, namely, to design an efficient sampling procedure. In Subsection 4.3, we prove our core sampling result by extending the analysis of Alon and Sudakov [2] to clustering problems with constrained centers, and by further strengthening an additive  $\pm \varepsilon m$  approximation guarantee to a multiplicative

<sup>4</sup> The success probability can be further amplified to  $1 - \delta$  for any  $\delta > 0$  by running  $O(\log(1/\delta))$  independent trials of the preceding algorithm.

<sup>5</sup> ETH postulates that 3-SAT is not in time  $2^{o(n)}$ . Here we only need the weaker hypothesis that 3-SAT is not in time  $2^{n^{o(1)}}$ .

factor  $(1 + \varepsilon)$ -approximation. In Subsection 4.4, we design an efficient sampling procedure, and this yields our efficient randomized PTAS. Our approach uses ideas from clustering algorithms pioneered by Kumar et al. [43] and refined in [1, 44].

#### 4.1 Setup - A Sampling Theorem

We denote the optimal value of Generalized Binary  $\ell_0$ -Rank- $k$  by

$$\text{OPT} = \text{OPT}_k \stackrel{\text{def}}{=} \min_{U \in \{0,1\}^{n \times k}, V \in \{0,1\}^{k \times d}} \|A - U \cdot V\|_0.$$

Further, for a fixed matrix  $V \in \{0,1\}^{k \times d}$  we let

$$\text{OPT}_k^V \stackrel{\text{def}}{=} \min_{U \in \{0,1\}^{n \times k}} \|A - U \cdot V\|_0,$$

and we say that a matrix  $U \in \{0,1\}^{n \times k}$  is a *best response* to  $V$ , if  $\|A - U \cdot V\|_0 = \text{OPT}_k^V$ .

Given a matrix  $A \in \{0,1\}^{n \times d}$ , a positive integer  $k$ , and an inner product function  $\langle \cdot, \cdot \rangle: \{0,1\}^k \times \{0,1\}^k \rightarrow \mathbb{R}$ , let  $V \in \{0,1\}^{k \times d}$  be arbitrary and  $U \in \{0,1\}^{n \times k}$  be a *best response* to  $V$ . Partition the columns of  $V$  (equivalently the columns of  $A$ ) into

$$C_y^V := \{j \mid V_{:,j} = y\},$$

for  $y \in \{0,1\}^k$ . For any row  $i$ , vector  $y \in \{0,1\}^k$ , and  $c \in \{0,1\}$  we consider

$$Z_{i,y,c} := |\{j \in C_y^V \mid A_{ij} = c\}| \quad \text{and} \quad Z_{i,y,\neq c} := |\{j \in C_y^V \mid A_{ij} \neq c\}|.$$

We define the *exact cost* of a row  $i$  for any vector  $x \in \{0,1\}^k$  as

$$E_{i,x} := \|A_{i,:} - x^T \cdot V\|_0 = \sum_{y \in \{0,1\}^k} Z_{i,y,\neq \langle x,y \rangle}. \quad (13)$$

Observe that  $U_{i,:} \in \{0,1\}^k$  is a vector  $x$  minimizing  $E_{i,x}$  (this follows from  $U$  being a best response to  $V$ ), and let  $E_i := E_{i,U_{i,:}}$ .

We do not know the partitioning  $C_y^V$ , however, as we will see later we can assume that (1) we can sample elements from each  $C_y^V$  and (2) we know approximations of the sizes  $|C_y^V|$ .

For (1), to set up notation let  $\tilde{C} = (\tilde{C}_y)_{y \in \{0,1\}^k}$  be a family, where  $\tilde{C}_y$  is a random multiset with elements from  $C_y^V$ . Specifically, we will work with the following *distribution*  $\mathcal{D}_{V,t}$  for some  $t \in \mathbb{N}$ : For any  $y \in \{0,1\}^k$ , if  $|C_y^V| < t$  let  $\tilde{C}_y = C_y^V$ , otherwise sample  $t$  elements from  $C_y^V$  with replacement and let the resulting multiset be  $\tilde{C}_y$ .

For (2), we say that a sequence  $\alpha = (\alpha_y)_{y \in \{0,1\}^k}$  is a sequence of  $\delta$ -*approximate cluster sizes* if for any  $y \in \{0,1\}^k$  with  $|C_y^V| < t$  we have  $\alpha_y = |C_y^V|$ , and for the remaining  $y \in \{0,1\}^k$  we have

$$|C_y^V| \leq \alpha_y \leq (1 + \delta)|C_y^V|.$$

Then corresponding to  $Z_{i,y,c}$  and  $Z_{i,y,\neq c}$  we have random variables

$$\tilde{Z}_{i,y,c} := |\{j \in \tilde{C}_y \mid A_{ij} = c\}| \quad \text{and} \quad \tilde{Z}_{i,y,\neq c} := |\{j \in \tilde{C}_y \mid A_{ij} \neq c\}|.$$

Given  $\tilde{C}$  and  $\alpha$ , we define the *estimated cost* of row  $i$  and vector  $x \in \{0,1\}^k$  as

$$\tilde{E}_{i,x} := \sum_{y \in \{0,1\}^k} \frac{\alpha_y}{|\tilde{C}_y|} \tilde{Z}_{i,y,\neq \langle x,y \rangle}. \quad (14)$$

If  $C_y^V = \emptyset$  for some  $y \in \{0,1\}^k$ , then  $\tilde{Z}_{i,y,\neq \langle x,y \rangle} = 0$  and we define the corresponding summand in (14) to be 0. Observe that if the approximation  $\alpha_y$  is exact, i.e.,  $\alpha_y = |C_y^V|$ , then  $\tilde{E}_{i,x}$  is an unbiased estimator for the exact cost  $E_{i,x}$ .

We now simplify the problem to optimizing the estimated cost instead of the exact cost. Specifically, we construct a matrix  $\tilde{U} \in \{0,1\}^{n \times k}$  by picking for each row  $i$  any

$$\tilde{U}_{i,:} \in \text{argmin}\{\tilde{E}_{i,x} \mid x \in \{0,1\}^k\}.$$



Note that matrix  $\tilde{U}$  depends on the input  $(A, k \langle \cdot, \cdot \rangle)$ , on the sequence  $\alpha$ , and on the sampled multisets  $\tilde{C} = (\tilde{C}_y)_{y \in \{0,1\}^k}$ . When it is clear from the context, we suppress the dependence on  $A, k \langle \cdot, \cdot \rangle$ , and write  $\tilde{U} = \tilde{U}(\tilde{C}, \alpha)$ . We show that this matrix yields a good approximation to the optimal cost.

**Theorem 16.** *For any matrix  $V \in \{0,1\}^{k \times d}$ , let  $\alpha$  be a sequence of  $\frac{\varepsilon}{6}$ -approximate cluster sizes and draw  $\tilde{C}$  according to distribution  $\mathcal{D}_{V,t}$  for  $t = t(k\varepsilon) := 2^{4k+14}/\varepsilon^2$ . Then we have*

$$\mathbb{E}_{\tilde{C}}[\|A - \tilde{U}(\tilde{C}, \alpha) \cdot V\|_0] \leq (1 + \varepsilon)\text{OPT}_k^V.$$

We defer the proof of Theorem 16 to Section 4.3, and first show how it yields a simple but inefficient deterministic PTAS for the Generalized Binary  $\ell_0$ -Rank- $k$  problem running in time  $n \cdot d^{\text{poly}(2^k/\varepsilon)}$ , see Section 4.2. Then, in Section 4.4, we design a sampling procedure that improves the running time to  $(2/\varepsilon)^{2^{O(k)}/\varepsilon^2} \cdot m^{1+o(1)}$ , where  $o(1)$  hides a factor  $(\log \log d)^{1.1} / \log d$ .

## 4.2 Simple PTAS

In this subsection, we show how Theorem 16 leads to a simple but inefficient deterministic PTAS, see Algorithm 2, for the Generalized Binary  $\ell_0$ -Rank- $k$  problem.

A basic, but crucial property used in our analysis is that given a matrix  $A \in \{0,1\}^{n \times d}$ , an integer  $k$  and a matrix  $V$ , we can compute a *best response* matrix  $U$  minimizing  $\|A - U \cdot V\|_0$  in time  $2^{O(k)} \mathbf{d}$ . Indeed, we can split  $\|A - U \cdot V\|_0 = \sum_{i=1}^d \|A_{i,:} - U_{i,:} \cdot V\|_0$  and brute-force the optimal solution  $U_{i,:} \in \{0,1\}^k$  minimizing the  $i$ -th summand  $\|A_{i,:} - U_{i,:} \cdot V\|_0$ . Symmetrically, given  $U$  we can compute a best response  $V$  in time  $2^{O(k)} \mathbf{d}$ . In particular, if  $(U, V)$  is an optimal solution then  $U$  is a best response for  $V$ , and  $V$  is a best response for  $U$ .

We now present the pseudocode of Algorithm 2.

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**Algorithm 2** (PTAS for Generalized Binary  $\ell_0$ -Rank- $k$ )

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**Input:** A matrix  $A \in \{0,1\}^{n \times d}$ , an integer  $k$ , an inner product  $\langle \cdot, \cdot \rangle$ , and  $\varepsilon \in (0, 1)$ . **Output:** Matrices  $\tilde{U} \in \{0,1\}^{n \times k}$ ,  $\tilde{V} \in \{0,1\}^{k \times d}$  such that  $\|A - \tilde{U} \cdot \tilde{V}\|_0 \leq (1 + \varepsilon)\text{OPT}_k$ .

1. (*Guess column set sizes*) Let  $U, V$  be an optimal solution. Exhaustively guess all sizes  $|C_y^V| =: \alpha_y$  for  $y \in \{0,1\}^k$ . There are  $d^{2^k}$  possibilities.
2. (*Guess column multisets*) Theorem 16 implies existence of a family  $\tilde{C} = (\tilde{C}_y)_{y \in \{0,1\}^k}$  such that  $\|A - \tilde{U}(\tilde{C}, \alpha) \cdot V\|_0 \leq (1 + \varepsilon)\text{OPT}_k$ , where each  $\tilde{C}_y$  is a multiset consisting of at most  $t$  indices in  $\{1, \dots, d\}$ . Exhaustively guess such a family  $\tilde{C}$ . There are  $d^{O(t \cdot 2^k)}$  possibilities.
3. (*Compute  $\tilde{U}$* ) Now we know  $A, k \langle \cdot, \cdot \rangle, |C_y^V|$  for all  $y \in \{0,1\}^k$ , and  $\tilde{C}$ , thus we can compute the matrix  $\tilde{U} = \tilde{U}(\tilde{C}, \alpha)$ , where row  $\tilde{U}_{i,:}$  is any vector  $x$  minimizing the estimated cost  $\tilde{E}_{i,x}$ . Since each row  $\tilde{U}_{i,:} \in \{0,1\}^k$  can be optimized independently, this takes time  $2^{O(k)} \mathbf{d}$ . If we guessed correctly, we have  $\|A - \tilde{U} \cdot V\|_0 \leq (1 + \varepsilon)\text{OPT}_k$ .
4. (*Compute  $\tilde{V}$* ) Compute  $\tilde{V}$  as a best response to  $\tilde{U}$ . This takes time  $2^{O(k)} \mathbf{d}$ . If we guessed correctly, by best-response and Step 3, we have

$$\|A - \tilde{U} \cdot \tilde{V}\|_0 \leq \|A - \tilde{U} \cdot V\|_0 \leq (1 + \varepsilon)\text{OPT}_k.$$

5. **Return** the pair  $(\tilde{U}, \tilde{V})$  minimizing  $\|A - \tilde{U} \cdot \tilde{V}\|_0$  over all exhaustive guesses.
- 

The correctness of Algorithm 2 immediately follows from Theorem 16. The running time is dominated by the exhaustive guessing in Step 2, so we obtain time  $m \cdot n^{\text{poly}(2^k/\varepsilon)}$ .

## 4.3 Proof of the Sampling Theorem 16

We follow the notation in Section 4.1, in particular  $V \in \{0,1\}^{k \times d}$  is an arbitrary matrix and  $U \in \{0,1\}^{n \times k}$  is a best response to  $V$ . We define  $D_{i,x}$  as the difference of the cost of row  $i$  w.r.t.

a vector  $x$  and the cost of row  $i$  w.r.t. the optimal vector  $U_{i,:}$ , i.e.,

$$\begin{aligned} D_{i,x} &:= E_{i,x} - E_i = \|A_{i,:} - x^T \cdot V\|_0 - \|A_{i,:} - U_{i,:} \cdot V\|_0 \\ &= \sum_{y \in \{0,1\}^k} Z_{i,y,\neq\langle x,y \rangle} - Z_{i,y,\neq\langle U_{i,:},y \rangle}. \end{aligned} \quad (15)$$

Note that a vector  $x$  is suboptimal for a row  $i$  if and only if  $D_{i,x} > 0$ . By a straightforward splitting of the expectation, we obtain the following.

**Claim 5.** *For every  $V \in \{0,1\}^{k \times d}$ , we have*

$$\mathbb{E}_{\tilde{C}}[\|A - \tilde{U} \cdot V\|_0] = \text{OPT}_k^V + \sum_{i=1}^n \sum_{\substack{x \in \{0,1\}^k \\ D_{i,x} > 0}} \Pr[\tilde{U}_{i,:} = x] \cdot D_{i,x}.$$

*Proof.* We split  $\|A - \tilde{U} \cdot V\|_0 = \sum_{i=1}^n \|A_{i,:} - \tilde{U}_{i,:} \cdot V\|_0$ . This yields

$$\mathbb{E}_{\tilde{C}}[\|A - \tilde{U} \cdot V\|_0] = \sum_{i=1}^n \mathbb{E}_{\tilde{C}}[\|A_{i,:} - \tilde{U}_{i,:} \cdot V\|_0] = \sum_{i=1}^n \sum_{x \in \{0,1\}^k} \Pr[\tilde{U}_{i,:} = x] \cdot \|A_{i,:} - x^T \cdot V\|_0.$$

By definition of  $D_{i,x}$ , we have

$$\begin{aligned} \mathbb{E}_{\tilde{C}}[\|A - \tilde{U} \cdot V\|_0] &= \sum_{i=1}^n \sum_{x \in \{0,1\}^k} \Pr[\tilde{U}_{i,:} = x] \cdot (\|A_{i,:} - U_{i,:} \cdot V\|_0 + D_{i,x}) \\ &= \sum_{i=1}^n \left( \|A_{i,:} - U_{i,:} \cdot V\|_0 + \sum_{x \in \{0,1\}^k} \Pr[\tilde{U}_{i,:} = x] \cdot D_{i,x} \right) \\ &= \text{OPT}_k^V + \sum_{i=1}^n \sum_{\substack{x \in \{0,1\}^k \\ D_{i,x} > 0}} \Pr[\tilde{U}_{i,:} = x] \cdot D_{i,x}. \quad \square \end{aligned}$$

Similarly to  $D_{i,x}$ , we define an estimator

$$\tilde{D}_{i,x} := \tilde{E}_{i,x} - \tilde{E}_{i,U_{i,:}} = \sum_{y \in \{0,1\}^k} \frac{\alpha_y}{|\tilde{C}_y|} \cdot \left( \tilde{Z}_{i,y,\neq\langle x,y \rangle} - \tilde{Z}_{i,y,\neq\langle U_{i,:},y \rangle} \right). \quad (16)$$

Note that  $\tilde{U}_{i,:}$  is chosen among the vectors  $x \in \{0,1\}^k$  minimizing  $\tilde{D}_{i,x}$ . Hence, our goal is to show that significantly suboptimal vectors (with  $D_{i,x} > \frac{\varepsilon}{3} \cdot E_i$ ) satisfy  $\tilde{D}_{i,x} > 0$  with good probability, and thus these vectors are not picked in  $\tilde{U}$ .

To this end, we split the rows  $i$  and suboptimal vectors  $x$  into:

$$\begin{aligned} L_0 &:= \{(i,x) \mid 0 < D_{i,x} \leq \frac{\varepsilon}{3} \cdot E_i\}, \\ L_1 &:= \{(i,x) \mid \frac{\varepsilon}{3} \cdot E_i < D_{i,x} \leq E_i\}, \\ L_2 &:= \{(i,x) \mid D_{i,x} > E_i\}. \end{aligned}$$

Observe that  $\sum_{(i,x) \in L_0} \Pr[\tilde{U}_{i,:} = x] \cdot D_{i,x} \leq \frac{\varepsilon}{3} \cdot \text{OPT}_k^V$ . By Claim 5, we can ignore all tuples  $(i,x) \in L_0$ , since

$$\mathbb{E}_{\tilde{C}}[\|A - \tilde{U} \cdot V\|_0] \leq (1 + \frac{\varepsilon}{3}) \text{OPT}_k^V + \sum_{(i,x) \in L_1 \cup L_2} \Pr[\tilde{U}_{i,:} = x] \cdot D_{i,x}. \quad (17)$$

Hence, our goal is to upper bound the summation  $\sum_{(i,x) \in L_1 \cup L_2} \Pr[\tilde{U}_{i,:} = x] \cdot D_{i,x}$ .

We next establish a sufficient condition for  $\tilde{U}_{i,:} \neq x$ , for any suboptimal vector  $x$ . Note that by definition of  $D_{i,x}$  we have

$$D_{i,x} = \sum_{y \in \{0,1\}^k} Z_{i,y,\neq\langle x,y \rangle} - Z_{i,y,\neq\langle U_{i,:},y \rangle} = \sum_{y \in \hat{Y}_{i,x}} Z_{i,y,\neq\langle x,y \rangle} - Z_{i,y,\neq\langle U_{i,:},y \rangle}, \quad (18)$$

where  $\hat{Y}_{i,x} := \{y \in \{0,1\}^k \mid \langle xy \rangle \neq \langle U_{i,\cdot}, y \rangle\}$ . Similarly, for the estimator we have

$$\tilde{D}_{i,x} = \sum_{y \in \{0,1\}^k} \frac{\alpha_y}{|\tilde{C}_y|} \cdot \left( \tilde{Z}_{i,y,\neq(x,y)} - \tilde{Z}_{i,y,\neq(U_{i,\cdot},y)} \right) = \sum_{y \in \hat{Y}_{i,x}} \frac{\alpha_y}{|\tilde{C}_y|} \cdot \left( \tilde{Z}_{i,y,\neq(x,y)} - \tilde{Z}_{i,y,\neq(U_{i,\cdot},y)} \right). \quad (19)$$

Let  $\mathcal{W}_{i,x}$  be the event that for every  $y \in Y_{i,x} := \{y \in \hat{Y}_{i,x} \mid |\tilde{C}_y| = t\}$  and every  $c \in \{0,1\}$ , we have

$$\left| \tilde{Z}_{i,y,c} - \frac{|\tilde{C}_y|}{|C_y^V|} \cdot Z_{i,y,c} \right| \leq \Delta_y, \quad \text{where} \quad \Delta_y := \frac{t \cdot D_{i,x}}{2^{k+2} \cdot \alpha_y}.$$

We now show that conditioned on the event  $\mathcal{W}_{i,x}$ , we have  $\tilde{D}_{i,x} > 0$  for any  $(i,x) \in L_1 \cup L_2$ , and thus  $\tilde{U}_{i,\cdot} \neq x$ .

**Lemma 23.** *For any vector  $x \in \{0,1\}^k$  and row  $i \in [m]$ , if event  $\mathcal{W}_{i,x}$  occurs then we have  $\tilde{D}_{i,x} \geq \frac{1}{2} \cdot D_{i,x} - \frac{\varepsilon}{6} \cdot E_i$ . In particular, if additionally  $D_{i,x} > \frac{\varepsilon}{3} \cdot E_i$  then  $\tilde{D}_{i,x} > 0$ .*

*Proof.* Observe that  $\tilde{Z}_{i,y,\neq c} \in \{\tilde{Z}_{i,y,0}, \tilde{Z}_{i,y,1}, \tilde{Z}_{i,y,0} + \tilde{Z}_{i,y,1}\}$  for any  $iy,c$ . Since  $\mathbb{E}[\tilde{Z}_{i,y,0} + \tilde{Z}_{i,y,1}] = |\tilde{C}_y| = \tilde{Z}_{i,y,0} + \tilde{Z}_{i,y,1}$ , conditioned on the event  $\mathcal{W}_{i,x}$  for any  $y \in Y_{i,x}$  all three random variables  $\tilde{Z}_{i,y,0}$ ,  $\tilde{Z}_{i,y,1}$ ,  $\tilde{Z}_{i,y,0} + \tilde{Z}_{i,y,1}$  differ from their expectation by at most  $\Delta_y$ . Hence, we have

$$\left| \frac{\alpha_y}{|\tilde{C}_y|} \tilde{Z}_{i,y,\neq(x,y)} - \frac{\alpha_y}{|C_y^V|} Z_{i,y,\neq(x,y)} \right| \leq \frac{D_{i,x}}{2^{k+2}}.$$

The same inequality also holds for  $y \in \hat{Y}_{i,x} \setminus Y_{i,x}$ , since then  $\tilde{Z}_{i,y,\neq(x,y)} = Z_{i,y,\neq(x,y)}$  and  $|\tilde{C}_y| = |C_y^V|$  (by definition of the distribution  $\mathcal{D}_{V,t}$ ). In combination with (19) we obtain

$$\tilde{D}_{i,x} \geq -\frac{D_{i,x}}{2} + \sum_{y \in \hat{Y}_{i,x}} \frac{\alpha_y}{|C_y^V|} \cdot \left( Z_{i,y,\neq(x,y)} - Z_{i,y,\neq(U_{i,\cdot},y)} \right). \quad (20)$$

Let  $\alpha_y = (1 + \gamma_y)|C_y^V|$  with  $0 \leq \gamma_y \leq \frac{\varepsilon}{6}$  for any  $y \in \{0,1\}^k$ . By (18), and since  $\alpha_y = |C_y^V| = |\tilde{C}_y|$  for every  $y \in \hat{Y}_{i,x} \setminus Y_{i,x}$  (by definition of distribution  $\mathcal{D}_{V,t}$ ), we have

$$\begin{aligned} \sum_{y \in \hat{Y}_{i,x}} \frac{\alpha_y}{|C_y^V|} (Z_{i,y,\neq(x,y)} - Z_{i,y,\neq(U_{i,\cdot},y)}) &= D_{i,x} + \sum_{y \in Y_{i,x}} \gamma_y (Z_{i,y,\neq(x,y)} - Z_{i,y,\neq(U_{i,\cdot},y)}) \\ &\geq D_{i,x} - \sum_{y \in Y_{i,x}} \gamma_y Z_{i,y,\neq(U_{i,\cdot},y)} \\ &\geq D_{i,x} - \frac{\varepsilon}{6} \sum_{y \in \{0,1\}^k} Z_{i,y,\neq(U_{i,\cdot},y)} = D_{i,x} - \frac{\varepsilon}{6} E_i. \end{aligned}$$

Together with (20), we have  $\tilde{D}_{i,x} \geq \frac{1}{2} D_{i,x} - \frac{\varepsilon}{6} E_i$ .  $\square$

We next upper bound the probability of picking a suboptimal vector  $x$ .

**Claim 6.** *For any  $x \in \{0,1\}^k$  with  $D_{i,x} > \frac{\varepsilon}{3} \cdot E_i$ , we have*

$$\Pr[\tilde{U}_{i,\cdot} = x] \leq \sum_{y \in Y_{i,x}} \min_{c \in \{0,1\}} \Pr \left[ |\tilde{Z}_{i,y,c} - \mathbb{E}[\tilde{Z}_{i,y,c}]| > \Delta_y \right].$$

*Proof.* For any  $y \in \{0,1\}^k$ , we have  $\tilde{Z}_{i,y,0} + \tilde{Z}_{i,y,1} = |\tilde{C}_y| = \mathbb{E}[\tilde{Z}_{i,y,0}] + \mathbb{E}[\tilde{Z}_{i,y,1}]$ . Further, it holds that  $|\tilde{Z}_{i,y,0} - \mathbb{E}[\tilde{Z}_{i,y,0}]| = |\tilde{Z}_{i,y,1} - \mathbb{E}[\tilde{Z}_{i,y,1}]|$ , and thus  $\Pr[|\tilde{Z}_{i,y,0} - \mathbb{E}[\tilde{Z}_{i,y,0}]| \leq \Delta_y] = \Pr[|\tilde{Z}_{i,y,1} - \mathbb{E}[\tilde{Z}_{i,y,1}]| \leq \Delta_y] = \Pr[|\tilde{Z}_{i,y,0} - \mathbb{E}[\tilde{Z}_{i,y,0}]| \leq \Delta_y \text{ and } |\tilde{Z}_{i,y,1} - \mathbb{E}[\tilde{Z}_{i,y,1}]| \leq \Delta_y]$ . Since  $\tilde{U}_{i,\cdot} = x$  can only hold if  $\tilde{D}_{i,x} \leq 0$ , the claim follows by Lemma 23 and a union bound over  $y \in Y_{i,x}$ .  $\square$

In the following subsections, we bound the summation in (17) over the sets  $L_1$  and  $L_2$ .

### 4.3.1 Case 1: Small Difference

We show first that  $|L_1|$  is small (see Claim 7). Then, we use a simple bound for  $\Pr[\tilde{U}_{i,:} = x]$  which is based on Lemma 9 (see Claim 8).

**Claim 7.** We have  $\sum_{(i,x) \in L_1} \sum_{y \in Y_{i,x}} |C_y^V| \leq 2^{k+2} \cdot \text{OPT}_k^V$ .

*Proof.* Fix  $(i,x) \in L_1$  and let  $y \in Y_{i,x}$ . Note that since  $\langle xy \rangle \neq \langle U_{i,:}, y \rangle$  we have

$$\{j \in C_y^V \mid A_{i,j} \neq \langle xy \rangle\} \cup \{j \in C_y^V \mid A_{i,j} \neq \langle U_{i,:}, y \rangle\} = C_y^V.$$

Note that this union is not necessarily disjoint, e.g., if  $\langle xy \rangle \notin \{0, 1\}$ . Since  $E_{i,x} = D_{i,x} + E_i$  (by (15)) and  $D_{i,x} \leq E_i$  (by definition of  $L_1$ ), we have

$$\sum_{y \in Y_{i,x}} |C_y^V| \leq \sum_{y \in Y_{i,x}} Z_{i,y,\neq\langle xy \rangle} + Z_{i,y,\neq\langle U_{i,:}, y \rangle} \leq E_{i,x} + E_i \leq 3E_i. \quad (21)$$

Fixing  $x$  and summing over all  $i$  with  $(i,x) \in L_1$ , the term  $E_i$  sums to at most  $\text{OPT}_k^V$ . Also summing over all  $x \in \{0, 1\}^k$  yields another factor  $2^k$ . Therefore, the claim follows.  $\square$

**Claim 8.** We have  $\sum_{(i,x) \in L_1} \Pr[\tilde{U}_{i,:} = x] \cdot D_{i,x} \leq \frac{\varepsilon}{3} \cdot \text{OPT}_k^V$ .

*Proof.* Note that for any row  $i$ , vector  $y \in Y_{i,x}$ , and  $c \in \{0, 1\}$ , the random variable  $\tilde{Z}_{i,y,c}$  is a sum of independent Bernoulli random variables, since the  $t$  samples from  $C_y^V$  forming  $\tilde{C}_y$  are independent, and each sample contributes either 0 or 1 to  $\tilde{Z}_{i,y,c}$ . Hence, our instantiations of Chebyshev's inequality, Lemmas 8 and 9, are applicable. We use Lemma 9 to bound  $\Pr[|\tilde{Z}_{i,y,c} - \mathbb{E}[\tilde{Z}_{i,y,c}]| > \Delta_y] \leq \sqrt{t} \Delta_y$ . Since  $\Delta_y = t \cdot D_{i,x} / (2^{k+2} \cdot \alpha_y)$  and  $\alpha_y \leq (1 + \frac{\varepsilon}{6}) |C_y^V| < 2|C_y^V|$ , we have  $\Pr[|\tilde{Z}_{i,y,c} - \mathbb{E}[\tilde{Z}_{i,y,c}]| > \Delta_y] \leq 2^k 2^3 |C_y^V| / (\sqrt{t} \cdot D_{i,x})$ , and thus by Claim 6, we obtain  $\Pr[\tilde{U}_{i,:} = x] \leq 2^{k+3} / (\sqrt{D_{i,x}}) \cdot \sum_{y \in Y_{i,x}} |C_y^V|$ . Claim 7 now yields

$$\sum_{(i,x) \in L_1} \Pr[\tilde{U}_{i,:} = x] \cdot D_{i,x} \leq \frac{2^{k+3}}{\sqrt{t}} \sum_{(i,x) \in L_1} \sum_{y \in Y_{i,x}} |C_y^V| \leq \frac{2^{2k+5}}{\sqrt{t}} \text{OPT}_k^V.$$

Since we chose  $t \geq 2^{4k+14} / \varepsilon^2$ , see Theorem 16, we obtain the upper bound  $\frac{\varepsilon}{3} \text{OPT}_k^V$ .  $\square$

### 4.3.2 Case 2: Large Difference

We use here the stronger instantiation of Chebyshev's inequality, Lemma 8, and charge  $\mu = \mathbb{E}[\tilde{Z}_{i,y,c}]$  against  $\text{OPT}_k^V$ .

**Claim 9.** We have  $\sum_{(i,x) \in L_2} \Pr[\tilde{U}_{i,:} = x] \cdot D_{i,x} \leq \frac{\varepsilon}{3} \cdot \text{OPT}_k^V$ .

*Proof.* Fix  $(i,x) \in L_2$  and let  $y \in Y_{i,x}$ . As in the proof of Claim 8, we see that our instantiation of Chebyshev's inequality, Lemma 8, is applicable to  $\tilde{Z}_{i,y,c}$  for any  $c \in \{0, 1\}$ . We obtain  $\Pr[|\tilde{Z}_{i,y,c} - \mathbb{E}[\tilde{Z}_{i,y,c}]| > \Delta_y] \leq \mathbb{E}[\tilde{Z}_{i,y,c}] / \Delta_y^2$ . Note that  $\mathbb{E}[\tilde{Z}_{i,y,c}] = Z_{i,y,c} \cdot \# |C_y^V|$ , since  $|\tilde{C}_y| = t$ . Using  $\min_{c \in \{0,1\}} Z_{i,y,c} \leq Z_{i,y,\neq\langle U_{i,:}, y \rangle}$ , we have

$$\min_{c \in \{0,1\}} \Pr[|\tilde{Z}_{i,y,c} - \mathbb{E}[\tilde{Z}_{i,y,c}]| > \Delta_y] \leq \frac{t}{|C_y^V| \Delta_y^2} \cdot Z_{i,y,\neq\langle U_{i,:}, y \rangle}.$$

Since  $\Delta_y = t \cdot D_{i,x} / (2^{k+2} \cdot \alpha_y)$  and  $\alpha_y \leq (1 + \varepsilon/6) |C_y^V| < 2|C_y^V|$ , we have

$$\min_{c \in \{0,1\}} \Pr[|\tilde{Z}_{i,y,c} - \mathbb{E}[\tilde{Z}_{i,y,c}]| > \Delta_y] \leq \frac{2^{2k+6} \cdot |C_y^V|}{t \cdot (D_{i,x})^2} \cdot Z_{i,y,\neq\langle U_{i,:}, y \rangle}.$$

Summing over all  $y \in Y_{i,x}$ , Claim 6 yields

$$\Pr[\tilde{U}_{i,:} = x] \leq \sum_{y \in Y_{i,x}} \frac{2^{2k+6} \cdot |C_y^V|}{t \cdot (D_{i,x})^2} \cdot Z_{i,y,\neq\langle U_{i,:}, y \rangle}. \quad (22)$$

We again use inequality (21), i.e.,  $\sum_{y \in Y_{i,x}} |C_y^V| \leq E_{i,x} + E_i$ . Since  $E_{i,x} = D_{i,x} + E_i$  (by (15)) and  $E_i < D_{i,x}$  (by definition of  $L_2$ ), we have  $|C_y^V| \leq 3D_{i,x}$  for any  $y \in Y_{i,x}$ . Together with (22), and then using the definition of  $E_i$ , we have

$$\Pr[\tilde{U}_{i,:} = x] \cdot D_{i,x} \leq \frac{2^{2k+8}}{t} \sum_{y \in Y_{i,x}} Z_{i,y, \neq \langle U_{i,:}, y \rangle} \leq \frac{2^{2k+8}}{t} E_i,$$

Fixing  $x$  and summing over all  $i$  with  $(i, x) \in L_2$ , the term  $E_i$  sums to at most  $\text{OPT}_k^V$ . Also summing over all  $x \in \{0, 1\}^k$  yields another factor  $2^k$ . Thus, it follows that

$$\sum_{(i,x) \in L_2} \Pr[\tilde{U}_{i,:} = x] \cdot D_{i,x} \leq \frac{2^{3k+8}}{t} \text{OPT}_k^V.$$

Since we chose  $t \geq 2^{3k+10}/\varepsilon$ , see Theorem 16, we obtain the upper bound  $\frac{\varepsilon}{3} \cdot \text{OPT}_k^V$ .  $\square$

### 4.3.3 Finishing the Proof

Taken together, Claims 5, 8, and 9 prove Theorem 16.

*Proof of Theorem 16.* Using Claim 5, splitting into  $L_0$ ,  $L_1$  and  $L_2$ , and using Claims 8 and 9, we obtain for any  $\varepsilon \in (0, 1)$  and  $t = 2^{4k+12}/\varepsilon^2$  that

$$\begin{aligned} \mathbb{E}_{\tilde{V}}[\|A - \tilde{U} \cdot V\|_0] &\leq (1 + \frac{\varepsilon}{3}) \text{OPT}_k^V + \sum_{(i,x) \in L_1} \Pr[\tilde{U}_{i,:} = x] \cdot D_{i,x} + \sum_{(i,x) \in L_2} \Pr[\tilde{U}_{i,:} = x] \cdot D_{i,x} \\ &\leq (1 + \varepsilon) \text{OPT}_k^V. \end{aligned}$$

This completes the proof.  $\square$

## 4.4 Efficient Sampling Algorithm

The conceptually simple PTAS in Section 4.2 has two running time bottlenecks, due to the exhaustive enumeration in Step 1 and Step 2. Namely, Step 1 guesses exactly the sizes  $|C_y^V|$  for each  $y \in \{0, 1\}^k$ , and there are  $d^{O(2^k)}$  possibilities; and Step 2 guesses among all columns of matrix  $A$  the multiset family  $\tilde{C}$ , guaranteed to exist by Theorem 16 and there are  $d^{O(t \cdot 2^k)}$  possibilities.

Since Theorem 16 needs only approximate cluster sizes, it suffices in Step 1 to guess numbers  $\alpha_y$  with  $|C_y^V| \leq \alpha_y \leq (1 + \frac{\varepsilon}{6})|C_y^V|$  if  $|C_y^V| \geq t$ , and  $\alpha_y = |C_y^V|$  otherwise, where  $t = 2^{4k+12}/\varepsilon^2$ . Hence, the runtime overhead for Step 1 can be easily improved to  $(t + \varepsilon^{-1} \log d)^{2^k}$ .

To reduce the exhaustive enumeration in Step 2, we design an efficient sampling procedure, see Algorithm 3, that uses ideas from clustering algorithms pioneered by Kumar et al. [43] and refined in [1, 44]. Our algorithm reduces the total exhaustive enumeration in Step 2 and the guessing overhead for the approximate cluster sizes in Step 1 to  $(2^k/\varepsilon)^{2^{O(k)}} \cdot (\log d)^{(\log \log d)^{0.1}}$  possibilities.

This section is structured as follows. We first replace an optimal solution  $(U, V)$  by a “well-clusterable” solution  $(U, W)$ , which will help in our correctness proof. In Subsection 4.4.2 we present pseudocode for our sampling algorithm. We then prove its correctness in Subsection 4.4.3 and analyze its running time in Subsection 4.4.4. Finally, we show how to use the sampling algorithm designed in Subsection 4.4.2 together with the ideas of the simple PTAS from Subsection 4.2 to prove Theorem 2, see Subsection 4.4.5.

### 4.4.1 Existence of a $(U, V, \varepsilon)$ -Clusterable Solution

For a matrix  $B \in \{0, 1\}^{n \times d}$  we denote by  $\text{ColSupp}(B)$  the set of unique columns of  $B$ . Note that if the columns of  $U$  are linearly independent then  $U \cdot \text{ColSupp}(V)$  denotes the set of distinct columns of  $U \cdot V$ . In the clustering formulation of the Generalized Binary  $\ell_0$ -Rank- $k$  problem as discussed in the introduction, the set  $U \cdot \text{ColSupp}(V)$  corresponds to the set of cluster centers.

Given matrices  $U, V$ , we will first replace  $V$  by a related matrix  $W$  in a way that makes all elements of  $U \cdot \text{ColSupp}(W)$  sufficiently different without increasing the cost too much, as formalized in the following.

**Lemma 24.** For any  $U \in \{0, 1\}^{n \times k}$ ,  $V \in \{0, 1\}^{k \times d}$  and  $\varepsilon \in (0, 1)$ , there exists a matrix  $W \in \{0, 1\}^{k \times n}$  such that  $\|A - U \cdot W\|_0 \leq (1 + \varepsilon)\|A - U \cdot V\|_0$  and for any distinct  $yz \in \text{ColSupp}(W)$  we have

- (i)  $\|Uy - Uz\|_0 > \varepsilon \cdot 2^{-k} \cdot \|A - U \cdot V\|_0 / \min\{|C_y^W|, |C_z^W|\}$ , and
- (ii)  $\|A_{:,j} - Uy\|_0 \leq \|A_{:,j} - Uz\|_0$  for every  $j \in C_y^W$ .

We say that such a matrix  $W$  is  $(U, V, \varepsilon)$ -clusterable.

*Proof.* The proof is by construction of  $W$ . We initialize  $W := V$  and then iteratively resolve violations of (i) and (ii). In each step, resulting in a matrix  $W'$ , we ensure that  $\text{ColSupp}(W') \subseteq \text{ColSupp}(W)$ . We call this *support-monotonicity*.

We can resolve all violations of (ii) at once by iterating over all columns  $j \in [d]$  and replacing  $W_{:,j}$  by the vector  $z \in \text{ColSupp}(W)$  minimizing  $\|A_{:,j} - Uz\|_0$ . This does not increase the cost  $\|A - U \cdot W\|_0$  and results in a matrix  $W'$  without any violations of (ii).

So assume that there is a violation of (i). That is, for distinct  $yz \in \text{ColSupp}(W)$ , where we can assume without loss of generality that  $|C_y^W| \leq |C_z^W|$ , we have

$$\|Uy - Uz\|_0 \leq \varepsilon \cdot 2^{-k} \cdot \|A - U \cdot V\|_0 / |C_y^W|.$$

We change the matrix  $W$  by replacing for every  $j \in C_y^W$  the column  $W_{:,j} = y$  by  $z$ . Call the resulting matrix  $W'$ . Note that the cost of any replaced column  $j$  changes to

$$\begin{aligned} \|A_{:,j} - U \cdot W'_{:,j}\|_0 &= \|A_{:,j} - Uz\|_0 \leq \|A_{:,j} - Uy\|_0 + \|Uy - Uz\|_0 \\ &\leq \|A_{:,j} - U \cdot W_{:,j}\|_0 + \varepsilon \cdot 2^{-k} \cdot \|A - U \cdot V\|_0 / |C_y^W|. \end{aligned}$$

Since the number of replaced columns is  $|C_y^W|$ , it follows that the overall cost increases by at most  $\varepsilon \cdot 2^{-k} \cdot \|A - U \cdot V\|_0$ . Note that after this step the size of  $\text{ColSupp}(W)$  is reduced by 1, since we removed any occurrence of column  $y$ . By support-monotonicity, the number of such steps is bounded by  $2^k$ . Since resolving violations of (ii) does not increase the cost, the final cost is bounded by  $(1 + \varepsilon)\|A - U \cdot V\|_0$ .

After at most  $2^k$  times resolving a violation of (i) and then all violations of (ii), we end up with a matrix  $W$  without violations and the claimed cost bound.  $\square$

#### 4.4.2 The Algorithm Sample

Given  $A \in \{0, 1\}^{n \times d}$ ,  $k \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ , and  $t \in \mathbb{N}$ , fix any optimal solution  $U, V$ , that is  $\|A - U \cdot V\|_0 = \text{OPT}_k$ . Our proof will use the additional structure provided by well clusterable solutions. Therefore, fix any  $(U, V, \varepsilon)$ -clusterable matrix  $W$  as in Lemma 24. Since  $\|A - U \cdot W\|_0 \leq (1 + \varepsilon)\|A - U \cdot V\|_0$ , we can restrict to matrix  $W$ . Specifically, we fix the optimal partitioning  $C^W$  of  $[d]$  for the purpose of the analysis and for the guessing steps of the algorithm. Our goal is to sample from the distribution  $\mathcal{D}_{W,t}$ .

Pseudocode of our sampling algorithm **Sample** $_{A,k,\varepsilon,t}(M, \mathcal{N}, \tilde{R}, \tilde{C}, \alpha)$  is given below. The arguments of this procedure are as follows. Matrix  $M$  is the current submatrix of  $A$  (initialized as the full matrix  $A$ ). Set  $\mathcal{N} \subseteq \{0, 1\}^k$  is the set of clusters that we did not yet sample from (initialized to  $\{0, 1\}^k$ ). The sequence  $\tilde{R}$  stores “representatives” of the clusters that we already sampled from (initialized to undefined entries  $(\perp, \dots, \perp)$ ). The sequence  $\tilde{C}$  contains our samples, so in the end we want  $\tilde{C}$  to be drawn according to  $\mathcal{D}_{W,t}$  ( $\tilde{C}$  is initialized such that  $\tilde{C}_y = \emptyset$  for all  $y \in \{0, 1\}^k$ ). Finally,  $\alpha$  contains guesses for the sizes of the clusters that we already sampled from, so in the end we want it to be a sequence of  $\frac{\varepsilon}{6}$ -approximate cluster sizes ( $\alpha$  is initialized such that  $\alpha_y = 0$  for all  $y \in \{0, 1\}^k$ ). This algorithm is closely related to algorithm “Irred- $k$ -means” by Kumar et al. [43], see the introduction for a discussion.

In this algorithm, at the base case we call **EstimateBestResponse** $_{A,k}(\tilde{C}, \alpha)$ , which computes matrix  $\tilde{U} = \tilde{U}(\tilde{C}, \alpha)$  and a best response  $\tilde{V}$  to  $\tilde{U}$ . Apart from the base case, there are three phases of algorithm **Sample**. In the *sampling* phase, we first guess some  $y \in \mathcal{N}$  and an approximation  $\alpha_y$  of  $|C_y^W|$ . Then, from the current matrix  $M$  of dimension  $n \times d_M$ , we sample  $\min\{t \alpha_y\}$  columns to form a multiset  $\tilde{C}_y$ , and we sample one column from  $M$  to form  $\tilde{R}_y$ . We make a recursive call with  $y$  removed from  $\mathcal{N}$  and updated  $\tilde{R}, \tilde{C}, \alpha$  by the values  $\tilde{R}_y, \tilde{C}_y, \alpha_y$ .

As an intermediate solution, we let  $U^{(1)}, V^{(1)}$  be the best solution returned by the recursive calls over all exhaustive guesses. In the *pruning* phase, we delete the  $d_M/2$  columns of  $M$  that are closest to  $\tilde{R}$ , and we make a recursive call with the resulting matrix  $M'$ , not changing the remaining arguments. Denote the returned solution by  $U^{(2)}, V^{(2)}$ . Finally, in the *decision* phase we return the better solution between  $U^{(1)}, V^{(1)}$  and  $U^{(2)}, V^{(2)}$ .

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**Algorithm 3** Estimating Best Response

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**Sample** $_{A,k,\varepsilon,t}(M, \mathcal{N}, \tilde{R}, \tilde{C}, \alpha)$

let  $d_M$  be the number of columns of  $M$   
set  $\nu := (\varepsilon/2^{k+4})^{2^k+2-|\mathcal{N}|}$

1. if  $\mathcal{N} = \emptyset$  or  $d_M = 0$ : **return**  $(\tilde{U}, \tilde{V}) = \mathbf{EstimateBestResponse}_{A,k}(\tilde{C}, \alpha)$
  - \* **Sampling phase** \*
  2. **guess**  $y \in \mathcal{N}$
  3. **guess** whether  $|C_y^W| < t$ :
  4.   **if**  $|C_y^W| < t$ : **guess**  $\alpha_y := |C_y^W|$  exactly, i.e.  $\alpha_y \in \{0, 1, \dots, t-1\}$
  5.   **otherwise**: **guess**  $\nu \cdot d_M \leq \alpha_y \leq d_M$  such that  $|C_y^W| \leq \alpha_y \leq (1 + \frac{\varepsilon}{6})|C_y^W|$
  6. **if**  $\alpha_y = 0$ :  $(U^{(y,\alpha_y)}, V^{(y,\alpha_y)}) = \mathbf{EstimateBestResponse}_{A,k}(\tilde{C}, \alpha)$
  7. **else**
  8.   sample u.a.r.  $\min\{t\alpha_y\}$  columns from  $M$ ; let  $\tilde{C}_y$  be the resulting multiset<sup>6</sup>
  9.   sample u.a.r. one column from  $M$ ; call it  $\tilde{R}_y$
  10.    $(U^{(y,\alpha_y)}, V^{(y,\alpha_y)}) = \mathbf{Sample}_{A,k,\varepsilon,t}(M, \mathcal{N} \setminus \{y\}, \tilde{R} \cup \{\tilde{R}_y\}, \tilde{C} \cup \{\tilde{C}_y\}, \alpha \cup \{\alpha_y\})$
  11. let  $(U^{(1)}, V^{(1)})$  be the pair minimizing  $\|A - U^{(y,\alpha_y)}V^{(y,\alpha_y)}\|_0$  over all guesses  $y$  and  $\alpha_y$
  - \* **Pruning phase** \*
  12. let  $M'$  be matrix  $M$  after the deleting  $d_M/2$  closest columns to  $\tilde{R}$ ,  
i.e., the  $d_M/2$  columns  $M_{:,j}$  with smallest values  $\min_{y \in \{0,1\}^k \setminus \mathcal{N}} \|M_{:,j} - \tilde{R}_y\|_0$
  13.  $(U^{(2)}, V^{(2)}) = \mathbf{Sample}_{A,k,\varepsilon,t}(M', \mathcal{N}, \tilde{R}, \tilde{C}, \alpha)$
  - \* **Decision** \*
  14. **return**  $(U^{(\ell)}, V^{(\ell)})$  with the minimal value  $\|A - U^{(\ell)}V^{(\ell)}\|_0$  over  $\ell \in \{1, 2\}$ .
- 

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**Algorithm 4** Estimating Best Response

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**EstimateBestResponse** $_{A,k}(\tilde{C}, \alpha)$

1. (*Compute*  $\tilde{U}$ ) Compute a matrix  $\tilde{U} = \tilde{U}(\tilde{C}, \alpha)$ , where row  $\tilde{U}_{i,:}$  is any vector  $x$  minimizing the estimated cost  $\tilde{E}_{i,x}$ . Note that each row  $\tilde{U}_{i,:} \in \{0, 1\}^k$  can be optimized independently.
  2. (*Compute*  $\tilde{V}$ ) Compute  $\tilde{V}$  as a best response to  $\tilde{U}$ .
  3. **Return**  $(\tilde{U}, \tilde{V})$
- 

#### 4.4.3 Correctness of Algorithm Sample

With notation as above, we now prove correctness of algorithm **Sample**.

**Theorem 17.** *Algorithm **Sample** $_{A,k,\varepsilon,t}$  generates a recursion tree which with probability at least  $(\frac{\varepsilon}{2t})^{2^{O(k)} \cdot t}$  has a leaf calling  $\mathbf{EstimateBestResponse}_{A,k}(\tilde{C}, \alpha)$  such that*

- (i)  $\alpha$  is a sequence of  $\frac{\varepsilon}{6}$ -approximate cluster sizes (w.r.t. the fixed matrix  $W$ ), and
- (ii)  $\tilde{C}$  is drawn according to distribution  $D_{W,t}$ .

The rest of this section is devoted to proving Theorem 17. Similarly as in the algorithm, we define parameters

$$\gamma := \varepsilon/2^{k+4} \quad \text{and} \quad \nu_i := \gamma^{2^k+2-i}.$$

---

<sup>6</sup> Given a submatrix  $M$  of  $A$ , and  $t$  columns sampled u.a.r. from  $M$ , we denote by  $\tilde{C}_y$  the resulting multiset of column indices with respect to the original matrix  $A$ .

Sort  $\{0, 1\}^k = \{y_1, \dots, y_{2^k}\}$  such that  $|C_{y_1}^W| \leq \dots \leq |C_{y_{2^k}}^W|$ . We construct the leaf guaranteed by the theorem inductively. In each depth  $\tau = 0, 1, \dots$  we consider one recursive call

$$\text{Sample}_{A,k,\varepsilon,t}(M^{(\tau)}, \mathcal{N}^{(\tau)}, \tilde{R}^{(\tau)}, \tilde{C}^{(\tau)}, \alpha^{(\tau)}).$$

We consider the partitioning  $P^{(\tau)} := \{P_y^{(\tau)}\}_{y \in \{0,1\}^k}$  induced by the partitioning  $C^W$  on  $M^{(\tau)}$ , i.e.,  $P_y^{(\tau)}$  is the set  $C_y^W$  restricted to the columns of  $A$  that appear in the submatrix  $M^{(\tau)}$ . We claim that we can find a root-to-leaf path such that the following inductive invariants hold with probability at least  $(\nu_0/t)^{(2^k - |\mathcal{N}^{(\tau)}|)(t+1)}$ :

- I1.  $P_y^{(\tau)} = C_y^W$  for all  $y \in \mathcal{N}^{(\tau)}$ , i.e., no column of an unsampled cluster has been removed,
- I2.  $\mathcal{N}^{(\tau)} = \{y_1, \dots, y_{|\mathcal{N}^{(\tau)}|}\}$ , i.e., the remaining clusters are the  $|\mathcal{N}^{(\tau)}|$  smallest clusters,
- I3. For any  $y \in \{0, 1\}^k \setminus \mathcal{N}^{(\tau)}$  the value  $\alpha_y^{(\tau)}$  is an  $\frac{\varepsilon}{6}$ -approximate cluster size, i.e., if  $|C_y^W| < t$  we have  $\alpha_y^{(\tau)} = |C_y^W|$ , and otherwise  $|C_y^W| \leq \alpha_y^{(\tau)} \leq (1 + \frac{\varepsilon}{6})|C_y^W|$ ,
- I4. For any  $y \in \{0, 1\}^k \setminus \mathcal{N}^{(\tau)}$  the multiset  $\tilde{C}_y^{(\tau)}$  is sampled according to distribution  $\mathcal{D}_{W,t}$ , i.e., if  $|C_y^W| < t$  then  $\tilde{C}_y^{(\tau)} = C_y^W$  and otherwise  $\tilde{C}_y^{(\tau)}$  consists of  $t$  uniformly random samples from  $C_y^W$  with replacement, and
- I5. For any  $y \in \{0, 1\}^k \setminus \mathcal{N}^{(\tau)}$  the vector  $\tilde{R}_y^{(\tau)}$  satisfies  $\|\tilde{R}_y^{(\tau)} - Uy\|_0 \leq 2\|A - UW\|_0/|C_y^W|$ .

For shorthand, we set  $d^{(\tau)} := d_{M^{(\tau)}}$  and  $\nu^{(\tau)} := \nu_{|\mathcal{N}^{(\tau)}|}$ .

**Base Case:** Note that the recursion may stop in Step 1 with  $\mathcal{N}^{(\tau)} = \emptyset$  or  $d^{(\tau)} = 0$ , or in Step 6 with  $\alpha_y^{(\tau)} = 0$  for some guessed  $y \in \mathcal{N}$ . Since we only want to show existence of a leaf of the recursion tree, in the latter case we can assume that we guessed  $y = y_{|\mathcal{N}^{(\tau)}|}$  and  $\alpha_y^{(\tau)} = |C_y^W|$ , and thus we have  $|C_y^W| = 0$ . Hence, in all three cases we have  $|C_y^W| = 0$  for all  $y \in \mathcal{N}^{(\tau)}$ , by invariant I2 and sortedness of  $y_1, \dots, y_{2^k}$ . Since we initialize  $\tilde{C}_y^{(0)} = \emptyset$  and  $\alpha_y^{(0)} = 0$ , we are done for all  $y \in \mathcal{N}^{(\tau)}$ . By invariants I3 and I4, we are also done for all  $y \in \{0, 1\}^k \setminus \mathcal{N}^{(\tau)}$ . The total success probability is at least

$$\left(\frac{\nu_0}{t}\right)^{2^k(t+1)} = \left(\frac{\varepsilon}{2^{k+4}t}\right)^{2^k(2^k+2)(t+1)} = \left(\frac{\varepsilon}{2t}\right)^{2^{O(k)} \cdot t}.$$

The proof of the inductive step proceeds by case distinction.

**Case 1 (Sampling):** Suppose  $|P_y^{(\tau)}| \geq \nu^{(\tau)} \cdot d^{(\tau)}$  for some  $y \in \mathcal{N}^{(\tau)}$ . Since  $P_y^{(\tau)} = C_y^W$  (by invariant I1) and sortedness, we have  $|C_y^W| \geq \nu^{(\tau)} d^{(\tau)}$  for  $y := y_{|\mathcal{N}^{(\tau)}|}$ . We may assume that we guess  $y = y_{|\mathcal{N}^{(\tau)}|}$  in Step 2, since we only want to prove existence of a leaf of the recursion tree. Note that there is a number

$$\nu^{(\tau)} d^{(\tau)} \leq \alpha_y \leq d^{(\tau)} \quad \text{with} \quad |C_y^W| \leq \alpha_y \leq (1 + \frac{\varepsilon}{6})|C_y^W|$$

(in particular  $\alpha_y = |C_y^W|$  would work), so we can guess such a number in Step 5. Together with Steps 3 and 4, we can assume that  $\alpha^{(\tau+1)}$  satisfies invariant I3.

In Step 8 we sample a multiset  $\tilde{C}_y$  of  $\min\{t, \alpha_y\}$  columns from  $M$ . If  $|C_y^W| \geq t$ , we condition on the event that all these columns lie in  $C_y^W$ . Then  $\tilde{C}_y$  forms a uniform sample from  $C_y^W$  of size  $t$ . Since  $|C_y^W| \geq \nu^{(\tau)} d^{(\tau)}$ , this event has probability at least  $(\nu^{(\tau)})^t$ . Otherwise, if  $|C_y^W| = \alpha_y < t$ , we condition on the event that all  $\alpha_y$  samples lie in  $C_y^W$  and are distinct. Then  $\tilde{C}_y = C_y^W$ . The probability of this event is at least

$$(1/d^{(\tau)})^{\alpha_y} \geq (\nu^{(\tau)}/\alpha_y)^{\alpha_y} \geq (\nu^{(\tau)}/t)^t.$$

In total,  $\tilde{C}^{(\tau+1)}$  satisfies invariant I4 with probability at least  $(\nu^{(\tau)}/t)^t$ .



In Step 9 we sample one column  $\tilde{R}_y$  uniformly at random from  $M$ . With probability at least  $\nu^{(\tau)}$ ,  $\tilde{R}_y$  belongs to  $C_y^W$ , and conditioned on this event  $\mathcal{E}_y$  we have

$$\mathbb{E}_{\tilde{R}_y} \left[ \|\tilde{R}_y - Uy\|_0 \mid \mathcal{E}_y \right] = \frac{1}{|C_y^W|} \sum_{j \in C_y^W} \|A_{:,j} - Uy\|_0 \leq \frac{\|A - UW\|_0}{|C_y^W|}.$$

By Markov's inequality, with probability at least  $\nu^{(\tau)}/2$  we have  $\|\tilde{R}_y - Uy\|_0 \leq 2\|A - UW\|_0/|C_y^W|$ , and thus invariant I5 holds for  $\tilde{R}^{(\tau+1)}$ .

Finally, since we did not change  $M^{(\tau)}$ , invariant I1 is maintained. We conditioned on events that hold with combined probability at least

$$(\nu^{(\tau)}/t)^t \cdot \nu^{(\tau)}/2 \geq (\nu_0/t)^{t+1}.$$

Since we decrement  $|\mathcal{N}^{(\tau)}|$  by removing  $y = y_{|\mathcal{N}^{(\tau)}|}$  from  $\mathcal{N}^{(\tau)}$ , we maintain invariant I2, and we obtain total probability at least

$$(\nu_0/t)^{(2^k - |\mathcal{N}^{(\tau+1)}|)(t+1)}.$$

**Case 2 (Pruning):** Suppose  $|P_y^{(\tau)}| < \nu^{(\tau)} \cdot d^{(\tau)}$  for every  $y \in \mathcal{N}^{(\tau)}$ . (Note that cases 1 and 2 are complete.) In this case, we remove the  $d^{(\tau)}/2$  columns of  $M^{(\tau)}$  that are closest to  $\tilde{R}^{(\tau)}$ , resulting in a matrix  $M^{(\tau+1)}$ , and then start a recursive call on  $M^{(\tau+1)}$ . Since we do not change  $\mathcal{N}^{(\tau)}$ ,  $\tilde{R}^{(\tau)}$ ,  $\tilde{C}^{(\tau)}$ , and  $\alpha^{(\tau)}$ , invariants I2-I5 are maintained.

Invariant I1 is much more difficult to verify, as we need to check that the  $d^{(\tau)}/2$  deleted columns do not contain any column from an unsampled cluster. We first show that *some* column of a cluster we already sampled from *survives* to depth  $\tau + 1$  and has *small* distance to  $\tilde{R}^{(\tau)}$  (see Claim 10). Then we show that *every* column of a cluster that we did not yet sample from has *large* distance to  $\tilde{R}^{(\tau)}$  (see Claim 12). Since we delete the  $d^{(\tau)}/2$  closest columns to  $\tilde{R}^{(\tau)}$ , it follows that every column of a cluster that we did not yet sample from *survives*.

**Claim 10.** *There exists  $x \in \{0, 1\}^k \setminus \mathcal{N}^{(\tau)}$  and column  $j \in P_x^{(\tau+1)}$  with*

$$\|A_{:,j} - \tilde{R}_x^{(\tau)}\|_0 \leq 2^{k+4}\|A - UW\|_0/d^{(\tau)}.$$

*Proof.* By Case 2, we have  $|P_y^{(\tau)}| < \nu^{(\tau)} \cdot d^{(\tau)}$  for every  $y \in \mathcal{N}^{(\tau)}$ , and since  $\nu^{(\tau)} \leq \nu_{2^k} \leq 2^{-k-2}$  it follows that

$$\sum_{y \in \mathcal{N}^{(\tau)}} |P_y^{(\tau)}| < 2^k \nu^{(\tau)} d^{(\tau)} \leq d^{(\tau)}/4.$$

Combining  $|P_y^{(\tau)}| \geq |P_y^{(\tau+1)}|$  and  $\sum_{y \in \{0,1\}^k} |P_y^{(\tau+1)}| = d^{(\tau)}/2$ , yields

$$\sum_{y \in \{0,1\}^k \setminus \mathcal{N}^{(\tau)}} |P_y^{(\tau+1)}| \geq d^{(\tau)}/4.$$

Hence, there is  $x \in \{0, 1\}^k \setminus \mathcal{N}^{(\tau)}$  such that

$$|P_x^{(\tau+1)}| \geq 2^{-k-2}d^{(\tau)}. \quad (23)$$

By the minimum-arithmetic-mean inequality, some  $j \in P_x^{(\tau+1)}$  satisfies

$$\begin{aligned} \|A_{:,j} - \tilde{R}_x^{(\tau)}\|_0 &\leq \frac{1}{|P_x^{(\tau+1)}|} \cdot \sum_{j' \in P_x^{(\tau+1)}} \|A_{:,j'} - \tilde{R}_x^{(\tau)}\|_0 \\ &\leq \|\tilde{R}_x^{(\tau)} - Ux\|_0 + \frac{1}{|P_x^{(\tau+1)}|} \cdot \sum_{j' \in P_x^{(\tau+1)}} \|A_{:,j'} - Ux\|_0, \end{aligned}$$

where the last step uses the triangle inequality. For the first summand we use invariant I5, and for the second we use that  $P_x^{(\tau+1)}$  is by definition part of an induced partitioning of  $C^W$  on a smaller matrix, and thus the summation is bounded by  $\|A - UW\|_0$ . This yields

$$\|A_{:,j} - \tilde{R}_x^{(\tau)}\|_0 \leq \left( \frac{2}{|C_x^W|} + \frac{1}{|P_x^{(\tau+1)}|} \right) \cdot \|A - UW\|_0.$$

By  $P_x^{(\tau+1)} \subseteq C_x^W$  and by (23), we obtain the claimed bound.  $\square$

**Claim 11.** For any  $y \in \{0, 1\}^k \setminus \mathcal{N}^{(\tau)}$  we have  $|C_y^W| \geq \nu^{(\tau)} d^{(\tau)} / \gamma$ .

*Proof.* Since  $y \notin \mathcal{N}$ , we sampled from this cluster in some depth  $\tau' < \tau$ . In the call corresponding to  $\tau'$ , we had  $\mathcal{N}^{(\tau')} \supseteq \mathcal{N}^{(\tau)} \cup \{y\}$  and thus  $|\mathcal{N}^{(\tau')}| \geq |\mathcal{N}^{(\tau)}| + 1$ , and we had  $d^{(\tau')} \geq d^{(\tau)}$ . Since we sampled from  $C_y^W$  in depth  $\tau'$ , Case 1 was applicable, it follows that

$$\begin{aligned} |C_y^W| &\stackrel{(I1)}{=} |P_y^{(\tau')}| \geq \nu^{(\tau')} \cdot d^{(\tau')} = \nu_{|\mathcal{N}^{(\tau')}|} \cdot d^{(\tau')} \\ &\geq \nu_{|\mathcal{N}^{(\tau)}|+1} \cdot d^{(\tau)} = \nu_{|\mathcal{N}^{(\tau)}|} \cdot d^{(\tau)} / \gamma = \nu^{(\tau)} \cdot d^{(\tau)} / \gamma. \end{aligned} \quad \square$$

**Claim 12.** For any  $y \in \{0, 1\}^k \setminus \mathcal{N}^{(\tau)}$ ,  $z \in \mathcal{N}^{(\tau)}$ , and  $j \in P_z^{(\tau)}$  we have

$$\|A_{:,j} - \tilde{R}_y^{(\tau)}\|_0 > 2^{k+4} \|A - UW\|_0 / d^{(\tau)}.$$

*Proof.* By triangle inequality, we have

$$\|Uy - Uz\|_0 \leq \|A_{:,j} - Uy\|_0 + \|A_{:,j} - Uz\|_0.$$

Since  $j \in P_z^{(\tau)} = C_z^W$  and by property (ii) of  $(U, V, \varepsilon)$ -clustered (see Lemma 24), the first summand is at least as large as the second, and we obtain

$$\|Uy - Uz\|_0 \leq 2\|A_{:,j} - Uy\|_0.$$

We use this and the triangle inequality to obtain

$$\begin{aligned} \|A_{:,j} - \tilde{R}_y^{(\tau)}\|_0 &\geq \|A_{:,j} - Uy\|_0 - \|\tilde{R}_y^{(\tau)} - Uy\|_0 \\ &\geq \frac{1}{2} \|Uy - Uz\|_0 - \|\tilde{R}_y^{(\tau)} - Uy\|_0. \end{aligned}$$

For the first summand we use property (i) of  $(U, V, \varepsilon)$ -clustered (see Lemma 24), for the second we use invariant I5. This yields

$$\|A_{:,j} - \tilde{R}_y^{(\tau)}\|_0 > \frac{\varepsilon}{2^{k+1}} \cdot \frac{\|A - UV\|_0}{|C_z^W|} - \frac{2\|A - UW\|_0}{|C_y^W|}.$$

Since  $y \in \{0, 1\}^k \setminus \mathcal{N}^{(\tau)}$ , Claim 11 yields  $|C_y^W| \geq \nu^{(\tau)} d^{(\tau)} / \gamma$ . Since  $z \in \mathcal{N}^{(\tau)}$ , by invariant I1, and since we are in Case 2, we have

$$|C_z^W| = |P_z^{(\tau)}| < \nu^{(\tau)} \cdot d^{(\tau)}.$$

Moreover, by the properties of  $(U, V, \varepsilon)$ -clustered (see Lemma 24), it follows that  $\|A - UW\|_0 \leq (1 + \varepsilon)\|A - UV\|_0$  and thus  $\|A - UV\|_0 \geq \frac{1}{2}\|A - UW\|_0$ . Together, this yields

$$\begin{aligned} \|A_{:,j} - \tilde{R}_y^{(\tau)}\|_0 &> \left( \frac{\varepsilon}{2^{k+2}} - 2\gamma \right) \cdot \frac{\|A - UW\|_0}{\nu^{(\tau)} d^{(\tau)}} \\ &= \frac{\varepsilon}{2^{k+3\nu^{(\tau)}}} \cdot \frac{\|A - UW\|_0}{d^{(\tau)}} \\ &\geq 2^{k+4} \cdot \frac{\|A - UW\|_0}{d^{(\tau)}}, \end{aligned}$$

since  $\gamma = \varepsilon/2^{k+4}$  and

$$\nu^{(\tau)} \leq \nu_{2^k} = \gamma^2 \leq \varepsilon/2^{2k+7}.$$

□

Together, Claims 10 and 12 prove that no column  $j \in P_y^{(\tau)}$  with  $y \in \mathcal{N}^{(\tau)}$  is removed. Indeed, we remove the columns with smallest distance to  $\tilde{R}^{(\tau)}$ , some of the columns in distance  $2^{k+4}\|A - UW\|_0/d^{(\tau)}$  survives, and any column  $j \in P_y^{(\tau)}$  with  $y \in \mathcal{N}$  has larger distance to  $\tilde{R}^{(\tau)}$ . It follows that invariant I1 is maintained, completing our proof of correctness.

#### 4.4.4 Running Time Analysis of Algorithm Sample

We now analyze the running time of Algorithm 4.

**Lemma 25.** *Algorithm EstimateBestResponse runs in time  $2^{O(k)} \mathbf{d}$ .*

*Proof.* Note that if  $\tilde{C}$  is drawn according to distribution  $\mathcal{D}_{W,t}$ , then its total size  $\sum_{y \in \{0,1\}^k} |\tilde{C}_y|$  is at most  $n$ . Hence, we can ignore all calls violating this inequality. We can thus evaluate the estimated cost  $\tilde{E}_{i,x}$  in time  $2^{O(k)} d$ . Optimizing over all  $x \in \{0,1\}^k$  costs another factor  $2^k$ , and iterating over all rows  $i$  adds a factor  $n$ . Thus, Step 1 runs in time  $2^{O(k)} \mathbf{d}$ . Further, Step 2 finds a best response matrix, which can be computed in the same running time.  $\square$

We proceed by analyzing the time complexity of Algorithm 3.

**Lemma 26.** *For any  $t = \text{poly}(2^k/\varepsilon)$ , Algorithm Sample $_{A,k,\varepsilon,t}$  runs in time  $(2/\varepsilon)^{2^{O(k)}} \cdot \mathbf{d}^{1+o(1)}$ , where  $o(1)$  hides a factor  $(\log \log d)^{1.1} / \log d$ .*

*Proof.* Consider any recursive call **Sample** $_{A,k,\varepsilon,t}(M, \mathcal{N}, \tilde{R}, \tilde{C}, \alpha)$ . We express its running time as  $T(a, b)$  where  $a := |\mathcal{N}|$  and  $b := \log(d_M)$ . For notational convenience, we let  $\log(0) =: -1$  and assume that  $d_M$  is a power of 2.

If we make a call to algorithm **EstimateBestResponse** then this takes time  $2^{O(k)} \mathbf{d}$  by the preceding lemma. Note that here we indeed have the size  $d$  of the original matrix and not the size  $d_M$  of the current submatrix, since we need to determine the cost with respect to the original matrix.

In the sampling phase, in Step 2 we guess  $y$ , with  $|\mathcal{N}| \leq 2^k$  possibilities. Moreover, in Steps 3,4,5 we guess either  $\alpha_y \in \{0, 1, \dots, t-1\}$  or  $\nu_{|\mathcal{N}|} \cdot d_M \leq \alpha_y \leq d_M$  such that  $|C_y^W| \leq \alpha_y \leq (1 + \frac{\varepsilon}{6}) |C_y^W|$ . Note that there are  $O(\log(1/\nu_{|\mathcal{N}|}) / \log(1 + \varepsilon/6)) = \text{poly}(2^k/\varepsilon)$  possibilities for the latter, and thus  $\text{poly}(2^k/\varepsilon)$  possibilities in total. For each such guess we make one recursive call with a decremented  $a$  and we evaluate the cost of the returned solution in time  $2^{O(k)} \mathbf{d}$ .

In the pruning phase, we delete the  $d_M/2$  columns that are closest to  $\tilde{R}$ , which can be performed in time  $2^{O(k)} \mathbf{d}_M$  (using median-finding in linear time). We then make one recursive call with a decremented  $b$ .

Together, we obtain the recursion

$$T(a, b) \leq \text{poly}(2^k/\varepsilon) \mathbf{d} + \text{poly}(2^k/\varepsilon) \cdot T(a-1, b) + T(a, b-1),$$

with base cases  $T(0, b) = T(a, -1) = 2^{O(k)} \mathbf{d}$ . The goal is to upper bound  $T(2^k, \log d)$ .

Let  $Y = \text{poly}(2^k/\varepsilon)$  and  $X = Y \cdot \mathbf{d}$  such that

$$T(a, b) \leq X + Y \cdot T(a-1, b) + T(a, b-1),$$

and  $T(0, b), T(a, -1) \leq X$ . We prove by induction that  $T(a, b) \leq X \cdot (2Y(b+2))^a$ . This works in the base cases where  $a = 0$  or  $b = -1$ . Inductively, for  $a > 0$  and  $b \geq 0$  we bound <sup>7</sup>

$$\begin{aligned} T(a, b) &\leq X + Y \cdot X \cdot (2Y(b+2))^{a-1} + X \cdot (2Y(b+1))^a \\ &= X \cdot (2Y(b+2))^a \cdot \left( \frac{1}{(2Y(b+2))^a} + \frac{1}{2(b+2)} + \left( \frac{b+1}{b+2} \right)^a \right) \\ &\leq X \cdot (2Y(b+2))^a \cdot \left( \frac{1}{2(b+2)} + \frac{1}{2(b+2)} + \frac{b+1}{b+2} \right) \\ &= X \cdot (2Y(b+2))^a. \end{aligned} \tag{24}$$

Let  $C$  be a constant to be determined soon. Using (24), the total running time is bounded by

$$T(2^k, \log d) \leq X \cdot (2Y(\log(d) + 2))^{2^k} \leq (2/\varepsilon)^{2^{(C+1) \cdot k}} \cdot \mathbf{d} \cdot \log^{2^k} d \leq (2/\varepsilon)^{2^{(C+1) \cdot k}} \cdot \mathbf{d}^{1+o(1)},$$

where the last inequality follows by noting that  $\log^{2^k} d > (2/\varepsilon)^{2^{(C+1) \cdot k}}$  iff  $k < \log \left( \frac{\log \log d}{\log(2/\varepsilon)} \right)^{1/C}$  and in this case

$$(\log d)^{2^k} \leq (\log d)^{\left( \frac{\log \log d}{\log(2/\varepsilon)} \right)^{1/C}} \leq n^{o(1)},$$

where  $o(1)$  hides a factor  $(\log \log d)^{1+1/C} / \log d$ . The statement follows for any  $C \geq 10$ .  $\square$

<sup>7</sup> Using similar arguments, for any  $\alpha \in [0, 1]$  the recurrence  $T(a, b) \leq (1 + \alpha)^b \cdot X + Y \cdot T(a-1, b) + T(a, b-1)$  is upper bounded by  $X \cdot (2Y)^a \cdot (b^{1-\alpha} + 2)^a \cdot (1 + \alpha)^b$ . In particular, we obtain the following upper bound  $T(a, b) \leq X \cdot (2Y)^a \cdot \min \{ (b+2)^a, 2^{a+b} \}$  and thus  $T(2^k, \log d) \leq (2/\varepsilon)^{2^{O(k)}} \cdot nd \cdot \min \{ (\log d)^{2^k}, d \}$ .

#### 4.4.5 The Complete PTAS

Finally, we use Algorithm **Sample** to obtain an efficient PTAS for the Generalized Binary  $\ell_0$ -Rank- $k$  problem. Given  $A, k, \varepsilon$ , we call **Sample** $_{A,k,\varepsilon/4,t}$  with

$$t = t(k, \frac{\varepsilon}{4}) := 2^{4k+16}/\varepsilon^2.$$

(This means that we replace all occurrences of  $\varepsilon$  by  $\varepsilon/4$ , in particular we also assume that  $W$  is  $(U, V, \frac{\varepsilon}{4})$ -clusterable.) By Theorem 17, with probability at least

$$\left(\frac{\varepsilon}{2t}\right)^{2^{O(k)} \cdot t} = (\varepsilon/2)^{2^{O(k)}/\varepsilon^2}$$

at least one leaf of the recursion tree calls **EstimateBestResponse** $_{A,k}(\tilde{C}, \alpha)$  with proper  $\tilde{C}$  and  $\alpha$  such that the Sampling Theorem 16 is applicable. By choice of  $t = t(k, \frac{\varepsilon}{4})$ , this yields

$$\mathbb{E}[\|A - \tilde{U}(\tilde{C}, \alpha) \cdot W\|_0] \leq (1 + \frac{\varepsilon}{4})\text{OPT}_k^W \leq (1 + \frac{\varepsilon}{4})^2\text{OPT}_k,$$

where we used that  $W$  is  $(U, V, \frac{\varepsilon}{4})$ -clusterable in the second step (see Lemma 24). The algorithm **EstimateBestResponse** computes the matrix  $\tilde{U} = \tilde{U}(\tilde{C}, \alpha)$  and a best response  $\tilde{V}$  to  $\tilde{U}$ . This yields

$$\mathbb{E}[\|A - \tilde{U} \cdot \tilde{V}\|_0] \leq \mathbb{E}[\|A - \tilde{U} \cdot W\|_0] \leq (1 + \frac{\varepsilon}{4})^2\text{OPT}_k.$$

By Markov's inequality, with probability at least  $1 - 1/(1 + \varepsilon/4) \geq \frac{\varepsilon}{5}$  we have

$$\|A - \tilde{U} \cdot \tilde{V}\|_0 \leq (1 + \frac{\varepsilon}{4}) \cdot \mathbb{E}[\|A - \tilde{U} \cdot \tilde{V}\|_0] \leq (1 + \frac{\varepsilon}{4})^3\text{OPT}_k \leq (1 + \varepsilon)\text{OPT}_k.$$

Hence, with probability at least  $p = (\varepsilon/2)^{2^{O(k)}/\varepsilon^2}$  at least one solution  $\tilde{U}, \tilde{V}$  generated by our algorithm is a  $(1 + \varepsilon)$ -approximation. Since we return the best of the generated solutions, we obtain a PTAS, but its success probability  $p$  is very low.

The success probability can be boosted to a constant by running  $O(1/p) = (2/\varepsilon)^{2^{O(k)}/\varepsilon^2}$  independent trials of Algorithm **Sample**. By Lemma 26, each call runs in time  $(2/\varepsilon)^{2^{O(k)}} \cdot \mathbf{d}^{1+o(1)}$ , where  $o(1)$  hides a factor  $(\log \log d)^{1.1} / \log d$ , yielding a total running time of  $(2/\varepsilon)^{2^{O(k)}/\varepsilon^2} \cdot \mathbf{d}^{1+o(1)}$ . This finishes the proof of Theorem 2. The success probability can be further amplified to  $1 - \delta$  for any  $\delta > 0$ , by running  $O(\log(1/\delta))$  independent trials of the preceding algorithm.

## 5 Hardness

In this section, we prove hardness of approximately computing the best rank  $k$ -approximation of a given  $n \times d$  matrix  $A$ , where  $n \geq d$ . Indeed all hardness results in this section hold when  $k = d - 1$ , indicating that reducing the rank by 1 is indeed hard to even approximate. This complements our efficient approximation schemes when  $k = O(1)$ .

Our results for  $p \in (1, 2)$  assume the Small Set Expansion Hypothesis. Originally conjectured by Raghavendra and Stuerer [59], it is still the only assumption that implies strong hardness results for various graph problems such as Uniform Sparsest Cut [61] and Bipartite Clique [47]. Assuming this hypothesis, we prove even stronger results than above that rules out *any* constant factor approximation in  $\text{poly}(nk)$ . The following theorem immediately implies Theorem 4 in the introduction.

**Theorem 18.** *Fix  $p \in (1, 2)$  and  $r > 1$ . Assuming the Small Set Expansion Hypothesis, there is no  $r$ -approximation algorithm for rank  $k$  approximation of a matrix  $A \in \mathbb{R}^{n \times d}$  with  $n \geq d$  and  $k = d - 1$  in the entrywise  $\ell_p$  norm that runs in time  $\mathfrak{h}(n)$ .*

*Consequently, additionally assuming the Exponential Time Hypothesis, there exists  $\delta := \delta(p, r) > 0$  such that there is no  $r$ -approximation algorithm for rank  $k$  approximation of a matrix  $A \in \mathbb{R}^{n \times d}$  with  $n \geq d$  and  $k = d - 1$  in the entrywise  $\ell_p$  norm that runs in time  $2^{n^\delta}$ .*

For  $p \in (2, \infty)$ , we do not rely on the Small Set Expansion Hypothesis though the hardness factor is bounded by a constant. Recall that  $\gamma_p := \mathbb{E}_g[|g|^p]^{1/p}$  where  $g$  is a standard Gaussian, which is strictly greater than 1 for  $p > 2$ .

**Theorem 19.** *Fix  $p \in (2, \infty)$  and  $\varepsilon > 0$ . Assuming  $P \neq NP$ , there is no  $(\gamma_p^p - \varepsilon)$ -approximation algorithm for rank  $k$  approximation of a matrix  $A \in \mathbb{R}^{n \times d}$  with  $n \geq d$  and  $k = d - 1$  in the entrywise  $\ell_p$  norm that runs in time  $\mathfrak{h}(n)$ .*

*Consequently, assuming the Exponential Time Hypothesis, there exists  $\delta := \delta(p, \varepsilon) > 0$  such that there is no  $(\gamma_p^p - \varepsilon)$ -approximation algorithm for rank  $k$  approximation of a matrix  $A \in \mathbb{R}^{n \times d}$  with  $n \geq d$  and  $k = d - 1$  in the entrywise  $\ell_p$  norm that runs in time  $2^{n^\delta}$ .*

We also prove similar hardness results for  $\ell_0$ -low rank approximation in finite fields. The following theorem immediately implies Theorem 6 in the introduction.

**Theorem 20.** *Fix a finite field  $\mathbb{F}$  and  $r > 1$ . Assuming  $P \neq NP$ , there is no  $r$ -approximation algorithm for rank  $k$  approximation of a matrix  $A \in \mathbb{F}^{n \times d}$  with  $n \geq d$  and  $k = d - 1$  in the entrywise  $\ell_0$  metric that runs in time  $\mathfrak{h}(n)$ .*

*Consequently, assuming the Exponential Time Hypothesis, there exists  $\delta := \delta(r) > 0$  such that there is no  $r$ -approximation algorithm for rank  $k$  approximation of a matrix  $A \in \mathbb{F}^{n \times d}$  with  $n \geq d$  and  $k = d - 1$  in the entrywise  $\ell_0$  metric that runs in time  $2^{n^\delta}$ .*

Section 5.1 proves Lemma 1, showing that computing  $\min_{p^* \rightarrow p}(A)$  is equivalent to finding the best rank  $k$  approximation of  $A \in \mathbb{R}^{n \times d}$  when  $n \geq d$  and  $k = d - 1$ . Section 5.2 proves Lemma 2, reducing  $\|\cdot\|_{2 \rightarrow p^*}$  to  $\min_{p^* \rightarrow p}(\cdot)$ . Section 5.3 presents the Barak et al. [7]'s proof of hardness of  $\|\cdot\|_{2 \rightarrow p^*}$  with modifications for all  $q > 2$ , finishing the proof of Theorem 18 for  $p \in (1, 2)$ . Section 5.4 proves the hardness of  $\min_{p^* \rightarrow p}(\cdot)$  for  $p > 2$ , using the result of [29], and finishes the proof of Theorem 19. Finally, Theorem 20 is proved in Section 5.5.

**Numerical issues.** In the proofs of Theorem 18 and Theorem 19, we consider our matrices as having real entries for simplicity, but our results will hold even when all entries are rescaled to polynomially bounded integers. The instance in Theorem 19 is explicitly constructed and it can be easily checked that all entries are polynomially bounded integers. For Theorem 18, our hard instance  $B$  for  $\|\cdot\|_{p \rightarrow p^*}$  is simply a projection matrix and the final instance  $A$  is obtained by  $(\varepsilon I + B)^{-1}$ , so by ensuring that  $\varepsilon \geq 1/\text{poly}(n)$ , we can ensure that eigenvalues of  $A$  are within  $[1, \text{poly}(n)]$ .

## 5.1 $\ell_p$ -Low Rank Approximation and $\min_{p^* \rightarrow p}(A)$

In this subsection, we prove the following lemma showing that computing  $\min_{p^* \rightarrow p}(A)$  is equivalent to finding the best rank  $k$  approximation of  $A \in \mathbb{R}^{n \times d}$  when  $n \geq d$  and  $k = d - 1$ .

**Lemma 27** (Restatement of Lemma 1). *Let  $p \in (1, \infty)$ . Let  $A \in \mathbb{R}^{n \times d}$  with  $n \geq d$  and  $k = d - 1$ . Then*

$$\min_{U \in \mathbb{R}^{n \times k}, V \in \mathbb{R}^{k \times d}} \|UV - A\|_p = \min_{x \in \mathbb{R}^d, \|x\|_{p^*} = 1} \|Ax\|_p.$$

*Proof.* Assume that the rank of  $A$  is  $d$ ; otherwise the lemma becomes trivial. We first prove  $(\geq)$ . Given  $V^* \in \mathbb{R}^{k \times d}$  that achieves the best rank  $k$  approximation, assume without loss of generality that the rank of  $V^*$  is  $k = d - 1$ . Let  $x \in \mathbb{R}^d$  be the unique vector (up to sign) that is orthogonal to the rowspace of  $V^*$  and  $\|x\|_{p^*} = 1$ . Let  $a_1, \dots, a_n$  be the rows of  $A$ . For fixed  $V^*$ , for  $i \in [n]$ , the  $i$ th row  $u_i^* \in \mathbb{R}^k$  of  $U^*$  must be obtained by computing

$$\min_{u_i^* \in \mathbb{R}^k} \|u_i^* V^* - a_i\|_p = \min_{y \in \text{rowspace}(V^*)} \|y - a_i\|_p = \min_{z \in \mathbb{R}^d: \langle x, z \rangle = -\langle x, a_i \rangle} \|z\|_p.$$

Note that by Hölder's inequality, the last quantity is at least  $|\langle xz \rangle| / \|x\|_{p^*} = |\langle x a_i \rangle| / \|x\|_{p^*} = |\langle x a_i \rangle|$ . Indeed, taking  $z \in \mathbb{R}^d$  with  $z_j := (-\langle x a_i \rangle) \cdot (\text{sgn}(x_j) |x_j|^{p^*/p})$  for each  $j \in [d]$  implies

$$\langle xz \rangle = -\langle x a_i \rangle \cdot \sum_{j \in [d]} \text{sgn}(x_j) |x_j|^{p^*/p} \cdot x_j = -\langle x a_i \rangle \cdot \|x\|_{p^*}^{p^*} = -\langle x a_i \rangle,$$

and

$$\|z\|_p = |\langle x a_i \rangle| \cdot \left( \sum_{j \in [d]} |x_j|^{p^*} \right)^{1/p} = |\langle x a_i \rangle| \cdot \|x\|_{p^*}^{p^*/p} = |\langle x a_i \rangle|,$$

so we can conclude  $\|u_i^* V^* - a_i\|_p = \min_{z \in \mathbb{R}^d: \langle x, z \rangle = -\langle x, a_i \rangle} \|z\|_p = |\langle x a_i \rangle|$ . Summing over  $i \in [n]$ ,

$$\|U^* V^* - A\|_p = \left( \sum_{i \in [n]} \|u_i^* V^* - a_i\|_p^p \right)^{1/p} = \left( \sum_{i \in [n]} |\langle x a_i \rangle|^p \right)^{1/p} = \|Ax\|_p.$$

This proves that  $\min_{U \in \mathbb{R}^{n \times k}, V \in \mathbb{R}^{k \times d}} \|UV - A\|_p \geq \min_{x \in \mathbb{R}^d, \|x\|_{p^*} = 1} \|Ax\|_p$ . For the other direction, given  $x \in \mathbb{R}^d$  with  $\|x\|_{p^*} = 1$ , let  $V^* \in \mathbb{R}^{k \times d}$  be a matrix whose rowspace is a  $k$ -dimensional subspace orthogonal to  $x$ , and compute  $U^*$  as above. The above analysis shows that  $\|U^* V^* - A\|_p = \|Ax\|_p$ , which completes the proof.  $\square$

## 5.2 Reducing $\|\cdot\|_{2 \rightarrow p^*}$ to $\min_{p^* \rightarrow p}(\cdot)$

In this subsection, we show that computing  $\min_{p^* \rightarrow p}(\cdot)$  is as hard as computing  $\|\cdot\|_{2 \rightarrow p^*}$ , proving the following lemma.

**Lemma 28** (Restatement of Lemma 2). *For any  $\varepsilon > 0, p \in (1, \infty)$ , there is an algorithm that runs in  $\text{poly}(n \log(1/\varepsilon))$  and on a non-zero input matrix  $A$ , computes a matrix  $B$  satisfying*

$$(1 - \varepsilon) \|A\|_{2 \rightarrow p^*}^{-2} \leq \min_{p^* \rightarrow p}(B) \leq (1 + \varepsilon) \|A\|_{2 \rightarrow p^*}^{-2}.$$

The lemma is proved in the following two steps.

**Reducing  $\|\cdot\|_{2 \rightarrow p^*}$  to  $\|\cdot\|_{p \rightarrow p^*}$ .** We first prove the following claim. This follows from standard tools from Banach space theory that factor an operator from  $\ell_p$  to  $\ell_p^*$  via  $\ell_2$ .

**Claim 13.**  $\|AA^T\|_{p \rightarrow p^*} = \|A\|_{2 \rightarrow p^*}^2$ .

*Proof.* By the definitions of  $p \rightarrow q$  norms,

$$\|AA^T\|_{p \rightarrow p^*} = \sup_x \frac{\|AA^T x\|_{p^*}}{\|x\|_p} \leq \sup_x \frac{\|A\|_{2 \rightarrow p^*} \|A^T x\|_2}{\|x\|_p} \leq \|A\|_{2 \rightarrow p^*} \|A^T\|_{p \rightarrow 2} = \|A\|_{2 \rightarrow p^*}^2,$$

where the last line follows from the fact that

$$\|A\|_{2 \rightarrow p^*} = \sup_{\|y\|_p = 1} \sup_{\|x\|_2 = 1} \langle yAx \rangle = \sup_{\|x\|_2 = 1} \sup_{\|y\|_p = 1} \langle A^T yx \rangle = \|A^T\|_{p \rightarrow 2}.$$

For the other direction,

$$\begin{aligned}\|AA^T\|_{p \rightarrow p^*} &= \sup_{\|x\|_p=1} \sup_{\|y\|_{p^*}=1} \langle yAA^T x \rangle = \sup_{\|x\|_p=1} \sup_{\|y\|_p=1} \langle A^T y A^T x \rangle \\ &\geq \sup_{\|x\|_p=1} \|A^T x\|_2^2 = \|A^T\|_{p \rightarrow 2}^2 = \|A\|_{2 \rightarrow p^*}^2,\end{aligned}$$

which completes the proof.  $\square$

**Reducing  $\|\cdot\|_{p \rightarrow p^*}$  to  $\min_{p^* \rightarrow p}(\cdot)$ .** We now relate two quantities  $\|A\|_{p \rightarrow p^*}$  and  $\min_{p^* \rightarrow p}(B)$  for two related matrices  $A$  and  $B$ . If  $A$  is invertible, this can be seen easily.

**Fact 4.** *If  $A$  is an invertible matrix, then  $\min_{p^* \rightarrow p}(A^{-1}) = (\|A\|_{q \rightarrow p})^{-1}$*

*Proof.* First observe that the condition  $A^{-1}x \neq 0$  is equivalent to the condition  $x \neq 0$  since  $A$  is invertible. Then we have,

$$\inf_{x \neq 0} \frac{\|A^{-1}x\|_q}{\|x\|_p} = \inf_{Ax \neq 0} \frac{\|A^{-1}x\|_q}{\|x\|_p} = \left( \sup_{A^{-1}x \neq 0} \frac{\|x\|_p}{\|A^{-1}x\|_q} \right)^{-1} = \left( \sup_{y \neq 0} \frac{\|A^{-1}y\|_p}{\|y\|_q} \right)^{-1}.$$

The leftmost quantity is  $\min_{p^* \rightarrow p}(A^{-1})$  and the rightmost quantity is  $(\|A\|_{q \rightarrow p})^{-1}$ .  $\square$

Even if  $A$  is not invertible, there is an invertible matrix  $B$  whose  $p \rightarrow q$  norm is close to that of  $A$  for any  $p$  and  $q$ .

**Claim 14.** *Let  $A$  be a non-zero  $n \times d$  matrix. For any  $p, q \in (1, \infty)$  and any  $\varepsilon > 0$ , there is an invertible and polynomial time computable  $\max(n, d) \times \max(n, d)$  matrix  $B$  such that  $(1 - \varepsilon)\|A\|_{p \rightarrow q} \leq \|B\|_{p \rightarrow q} \leq (1 + \varepsilon)\|A\|_{p \rightarrow q}$ .*

*Proof.* Let  $\oplus$  denote vector concatenation. We start by exhibiting a square matrix with the same norm. If  $d \geq n$ , we pad 0's to the bottom of  $A$  to obtain an  $d \times d$  matrix  $A'$ . Now for any  $x \in \mathbb{R}^d$ ,  $\|A'x\|_q = \|Ax \oplus 0^{d-n}\|_q = \|Ax\|_q$ . So  $\|A\|_{p \rightarrow q} = \|A'\|_{p \rightarrow q}$ .

If  $d \leq n$ , we pad 0's to the right of  $A$  to obtain an  $n \times n$  matrix  $A'$ . Consider any  $y \in \mathbb{R}^n$  and let  $x \in \mathbb{R}^d$ ,  $z \in \mathbb{R}^{n-d}$  be such that  $y = x \oplus z$ . Then we have  $\|A'y\|_q = \|Ax\|_q$ . Now since  $\|y\|_p \geq \|x\|_p$ , we have  $\|A\|_{p \rightarrow q} \geq \|A'\|_{p \rightarrow q}$ . On the other hand,  $\|A\|_{p \rightarrow q} \leq \|A'\|_{p \rightarrow q}$  since  $\|A'(x \oplus 0^{n-d})\|_q = \|Ax\|_q$  and  $\|x \oplus 0^{n-d}\|_p = \|x\|_p$ .

Next to obtain an invertible matrix, we set  $B := A' + \varepsilon' \cdot I$  where  $\varepsilon' := \varepsilon \cdot M / \|I\|_{p \rightarrow q}$  and  $M$  is the max magnitude of an entry of  $A$  which must be non-zero since  $A$  is non-zero. First we observe that  $\|A\|_{p \rightarrow q} \geq M$  since one can substitute  $x = e_i$  where  $i$  is the index of the column containing the max magnitude entry. Lastly, applying triangle inequality (since  $\|\cdot\|_{p \rightarrow q}$  is a norm) implies the claim.  $\square$

### 5.3 Hardness of $2 \rightarrow q$ norm for all $q \in (2, \infty)$

In this subsection, we prove Theorem 7 for hardness of  $\|\cdot\|_{2 \rightarrow q}$  for  $q \in (2, \infty)$ . Barak et al. [7] proved that under the Small Set Expansion Hypothesis, for any  $r > 1$  and an even integer  $q \geq 4$ , it is NP-hard to approximate the  $2 \rightarrow q$  norm problem within a factor  $r$ . The same proof essentially works for all  $q \in (2, \infty)$  with slight modifications. For completeness, we present their proof here, with additional remarks when we generalize an even integer  $q \geq 4$  to all  $q \in (2, \infty)$ .

**Preliminaries for Small Set Expansion.** For a vector  $x \in \mathbb{R}^d$ , every  $p$ -norm in this subsection denotes the expectation norm defined as  $\|x\|_{L_p} := (\mathbb{E}_{i \in [d]} [|x_i|^p])^{1/p}$ . For a regular graph  $G = (V, E)$  and a subset  $S \subseteq V$ , we define the measure of  $S$  to be  $\mu(S) = |S|/|V|$  and we define  $G(S)$  to be the distribution obtained by picking a random  $x \in S$  and then outputting a random neighbor  $y$  of  $x$ . We define the expansion of  $S$  to be

$$\Phi_G(S) = \Pr_{y \in G(S)} [y \in S].$$

For  $\delta \in (0, 1)$ , we define  $\Phi_G(\delta) = \min_{S \subseteq V: \mu(S) \leq \delta} \Phi_G(S)$ . We identify  $G$  with its normalized adjacency matrix. For every  $\lambda \in [-1, 1]$ , we denote by  $V_{\geq \lambda}(G)$  the subspace spanned by the

eigenvectors of  $G$  with eigenvalue at least  $\lambda$ . The projector into this subspace is denoted  $P_{\geq \lambda}(G)$ . For a distribution  $D$ , we let  $\text{cp}(D)$  denote the collision probability of  $D$  (the probability that two independent samples from  $D$  are identical). The Small Set Expansion Hypothesis, posed by Raghavendra and Steurer [59] states the following.

**Hypothesis 8.** *For any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that it is NP-hard to decide whether  $\Phi_G(\delta) \leq \varepsilon$  or  $\Phi_G(\delta) \geq 1 - \varepsilon$ .*

This implies strong hardness results for various graph problems such as Uniform Sparsest Cut [61] and Bipartite Clique [47]. The main theorem of this subsection is the following, which corresponds to Theorem 2.4 of [7].

**Theorem 21.** *For every regular graph  $G$ ,  $\lambda \in (0, 1)$ , and  $q \in (2, \infty)$ ,*

1. *For all  $\delta > 0, \varepsilon > 0$ ,  $\|P_{\geq \lambda}(G)\|_{L_2 \rightarrow L_q} \leq \varepsilon/\delta^{(q-2)/2q}$  implies that  $\Phi_G(\delta) \geq 1 - \lambda - \varepsilon^2$ .*
2. *There is a constant  $a = a(q)$  such that for all  $\delta > 0$ ,  $\Phi_G(\delta) > 1 - a\lambda^{2q}$  implies*

$$\|P_{\geq \lambda}(G)\|_{L_2 \rightarrow L_q} \leq 2/\sqrt{\delta}.$$

Given this theorem, the hardness of  $2 \rightarrow q$  norm can be proved as follows. This corresponds to Corollary 8.1 of [7].

*Proof of Theorem 7.* Using [60], the Small Set Expansion Hypothesis implies that for any sufficiently small numbers  $0 < \delta \leq \delta'$ , there is no polynomial time algorithm that can distinguish between the following cases for a given graph  $G$ :

- Yes case:  $\Phi_G(\delta) < 0.1$ .
- No case:  $\Phi_G(\delta') > 1 - 2^{-a' \log(1/\delta')}$ . ( $a'$  is a fixed universal constant.)

In particular, for all  $\eta > 0$ , if we let  $\delta' = \delta^{(q-2)/8q}$  and make  $\delta$  small enough, then in the No case  $\Phi_G(\delta^{(q-2)/8q}) > 1 - \eta$ . (Since  $q > 2$ ,  $\delta' \rightarrow 0$  as  $\delta \rightarrow 0$ .)

Using Theorem 21, in the Yes case we know  $\|P_{\geq 1/2}\|_{L_2 \rightarrow L_q} \geq 1/(10\delta^{(q-2)/2q})$ , while in the No case, if we choose  $\delta$  sufficiently small so that  $\eta$  is smaller than  $a(1/2)^{2q}$ , then we know that  $\|P_{\geq 1/2}\|_{L_2 \rightarrow L_q} \leq 2/\sqrt{\delta'} = 2/\delta^{(q-2)/4q}$ . The gap between the Yes case and the No case is at least  $\delta^{-(q-2)/4q}/20$ , which goes to  $\infty$  as  $\delta$  decreases.  $\square$

We now prove Theorem 21. The first part that proves small set expansion of  $G$  given a  $2 \rightarrow q$  norm bound indeed follows from older work (e.g., [40]).

**Lemma 29** (Lemma B.1 of [7]). *For all  $\delta > 0, \varepsilon > 0$ ,  $\|P_{\geq \lambda}(G)\|_{L_2 \rightarrow L_q} \leq \varepsilon/\delta^{(q-2)/2q}$  implies that  $\Phi_G(\delta) \geq 1 - \lambda - \varepsilon^2$*

*Proof.* Let  $q^* = q/(q-1)$  be the Hölder conjugate of  $q$  such that  $1/q + 1/q^* = 1$ . Since  $P_{\geq \lambda}$  is a projector,

$$\|P_{\geq \lambda}(G)\|_{L_{q^*} \rightarrow L_2} = \|P_{\geq \lambda}(G)^T\|_{L_{q^*} \rightarrow L_2} = \|P_{\geq \lambda}(G)\|_{L_2 \rightarrow L_q}.$$

Given  $S \subseteq V$  with  $\mu(S) = \mu \leq \delta$ , let  $f = 1_S/\sqrt{\mu}$  be the normalized indicator vector of  $S$  so that  $\|f\|_{L_2} = 1$ . Let  $f = f' + f''$  where  $f'$  is its projection to the eigenvalues at least  $\lambda$  (i.e.,  $f' = P_{\geq \lambda}f$ ) and  $f''$  is its projection to the eigenvalues strictly less than  $\lambda$ . Since  $\|1_S\|_{L_{q^*}} = \mu^{1/q^*} = \mu^{(q-1)/q}$ , we have  $\|f\|_{L_{q^*}} = \mu^{((q-1)/q)-1/2} \leq \delta^{((q-1)/q)-1/2}$  (since  $q > 2$  and  $\delta \geq \mu$ ), and

$$\|f'\|_{L_2} \leq \|f\|_{L_{q^*}} \cdot \|P_{\geq \lambda}(G)\|_{L_{q^*} \rightarrow L_2} \leq \delta^{((q-1)/q)-1/2} \cdot (\varepsilon/\delta^{(q-2)/2q}) = \varepsilon.$$

Then

$$\langle fGf \rangle = \langle f', Gf' \rangle + \langle f'', Gf'' \rangle \leq \|f'\|_{L_2}^2 + \lambda \|f''\|_{L_2}^2 \leq \varepsilon^2 + \lambda.$$

Since  $\Phi_G(S) = 1 - \langle fGf \rangle$ , the lemma follows.  $\square$

The second part of Theorem 21 requires more technical proofs.

**Lemma 30** (Lemma 8.2 of [7]). *There is a constant  $a = a(q)$  such that for all  $\delta > 0$ ,  $\Phi_G(\delta) > 1 - a\lambda^{2q}$  implies  $\|P_{\geq \lambda}(G)\|_{L_2 \rightarrow L_q} \geq 2/\sqrt{\delta}$ .*



*Proof.* Let  $f$  be a function in  $V_{\geq \lambda}$  with  $\|f\|_{L_2} = 1$  that maximizes  $\|f\|_{L_q}$ . We write  $f = \sum_{i=1}^m \alpha_i \chi_i$  where  $\chi_1, \dots, \chi_m$  denote the eigenfunctions of  $G$  with values  $\lambda_1, \dots, \lambda_m$  that are at least  $\lambda$ . Assume towards contradiction that  $\|f\|_{L_q} < 2/\sqrt{\delta}$ . We will prove that  $g = \sum_{i=1}^m (\alpha_i/\lambda_i) \chi_i$  satisfies  $\|g\|_{L_q} \geq 5\|f\|_{L_q}/\lambda$ . Note that  $g$  is defined such that  $f = Gg$ . This is a contradiction since (using  $\lambda_i \in [\lambda, 1]$ )  $\|g\|_{L_2} \leq \|f\|_{L_2}/\lambda$ , and we assumed  $f$  is a function in  $V_{\geq \lambda}$  with a maximal ratio  $\|f\|_{L_q}/\|f\|_{L_2}$ .

Let  $U \subseteq V$  be the set of vertices such that  $|f(x)| \geq 1/\sqrt{\delta}$  for all  $x \in U$ . Using the Markov inequality and the fact that  $\mathbb{E}_{x \in V}[f(x)^2] = 1$ , we know that  $\mu(U) = |U|/|V| \leq \delta$ . On the other hand, because  $\|f\|_{L_q}^q \geq 2^q/\delta^{q/2}$ , we know that  $U$  contributes at least half of the term  $\|f\|_{L_q}^q = \mathbb{E}_{x \in V}[|f(x)|^q]$ . That is, if we define  $\alpha$  to be  $\mu(U)\mathbb{E}_{x \in U}[|f(x)|^q]$  then  $\alpha \geq \|f\|_{L_q}^q/2$ . We will prove the lemma by showing that  $\|g\|_{L_q}^q \geq (10\lambda^{-1})^q \alpha$ .

Let  $c = c(q)$  and  $d = d(c, q)$  be sufficiently large constants that will be determined later, and  $e = d \cdot \lambda^{-q}$ . By the variant local Cheeger bound obtained in Theorem 2.1 of [69], there exists  $a = a(d, q)$  such that  $\Phi_G(\delta) > 1 - a\lambda^{2q}$  implies that  $\text{cp}(G(S)) \leq 1/(e|S|)$  for all  $S$  with  $\mu(S) \leq \delta$ .

We define  $U_i$  to be the set  $\{x \in U : f(x) \in [c^i/\sqrt{\delta}, c^{i+1}/\sqrt{\delta}]\}$ , and let  $I$  be the maximal  $i$  such that  $U_i$  is non-empty. Thus, the sets  $U_0, \dots, U_I$  form a partition of  $U$  (where some of these sets may be empty). We let  $\alpha_i$  be the contribution of  $U_i$  to  $\alpha$ . That is,  $\alpha_i = \mu_i \mathbb{E}_{x \in U_i}[|f(x)|^q]$ , where  $\mu = \mu(U_i)$ . Note that  $\alpha = \alpha_0 + \dots + \alpha_I$ . We will show that there are some indices  $i_1, \dots, i_J$  such that

1.  $\alpha_{i_1} + \dots + \alpha_{i_J} \geq \alpha/(2c^q)$ .
2. For all  $j \in [J]$ , there is a non-negative function  $g_j : V \rightarrow \mathbb{R}$  such that  $\mathbb{E}_{x \in V}[|g_j(x)|^q] \geq \alpha_{i_j}/(10c^2)^{q/2}$ .
3. For every  $x \in V$ ,  $g_1(x) + \dots + g_J(x) \leq |g(x)|$ .

Showing these will complete the proof, since it is easy to see that for non-negative functions  $g', g''$  and  $q \in [1, \infty)$

$$\mathbb{E}[(g'(x) + g''(x))^q] \geq \mathbb{E}[g'(x)^q] + \mathbb{E}[g''(x)^q],$$

and hence 2. and 3. imply that

$$\|g\|_{L_q}^q = \mathbb{E}[|g(x)|^q] \geq (e/(10c^2)^{q/2}) \sum_j \alpha_{i_j}. \quad (25)$$

Using 1., we conclude that for  $e \geq 2c^q \cdot (10c^2)^{q/2} \cdot (10/\lambda)^q$ , the right-hand side of (25) will be larger than  $(10/\lambda)^q \alpha$ . In particular, we set  $d = d(c, q) = 2c^q \cdot (10c^2)^{q/2} \cdot 10^q$ .

We find the indices  $i_1, \dots, i_J$  iteratively. We let  $\mathcal{I}$  be initially the set  $\{0, \dots, I\}$  of all indices. For  $j = 1, 2, \dots$ , we do the following as long as  $\mathcal{I}$  is not empty:

- Let  $i_j$  be the largest index in  $\mathcal{I}$ .
- Remove from  $\mathcal{I}$  every index  $i$  such that  $\alpha_i \leq c^q \alpha_{i_j}/2^{i-i_j}$ .

We let  $J$  denote the step we stop. Note that our indices  $i_1, \dots, i_J$  are sorted in descending order. For every step  $j$ , the total of the  $\alpha_i$ 's for all indices we removed is less than  $c^q \alpha_{i_j}$  and hence we satisfy 1. We use the following claim, whose proof is omitted here since it does not involve  $q$  at all. This follows from the fact that  $\text{cp}(G(S)) \leq 1/(e|S|)$  for all  $S$  with  $\mu(S) \leq \delta$ .

**Claim 15** (Claim 8.3 of [7]). *Let  $S \subseteq V$  and  $\beta > 0$  such that  $\mu(S) \leq \delta$  and  $|f(x)| \geq \beta$  for all  $x \in S$ . Then there is a set of size at least  $e|S|$  such that  $\mathbb{E}_{x \in T}[g(x)^2] \geq \beta^2/4$ .*

We will construct the functions  $g_1, \dots, g_J$  by applying iteratively Claim 15. We do the following for  $j = 1, \dots, J$ :

1. let  $T_j$  be the set of size  $e|U_{i_j}|$  that is obtained by applying Claim 15 to the function  $f$  and the set  $U_{i_j}$ . Note that  $\mathbb{E}_{x \in T_j}[g(X)^2] \geq \beta_{i_j}^2/4$ , where we let  $\beta_i = c^i/\sqrt{\delta}$  (and hence for every  $x \in U_i$ ,  $\beta_i \leq |f(x)| \leq c\beta_i$ ).
2. Let  $g'_j$  be the function on input  $x$  that outputs  $\gamma \cdot |g(x)|$  if  $x \in T_j$  and 0 otherwise, where  $\gamma \leq 1$  is a scaling factor that ensures that  $\mathbb{E}_{x \in T_j}[g'(x)^2]$  equals exactly  $\beta_{i_j}^2/4$ .

3. We define  $g_j(x) = \max(0, g'_j(x) - \sum_{k < j} g_k(x))$ .

Note that the second step ensures  $g'_j(x) \leq |g(x)|$ , while the third step ensures that  $g_1(x) + \dots + g_j(x) \leq g'_j(x)$  for all  $j$ , and in particular  $g_1(x) + \dots + g_J(x) \leq |g(x)|$ . Hence the only thing left to prove is the following.

**Claim 16** (Claim 8.5 of [7]).  $\mathbb{E}_{x \in V}[|g_j(x)|^q] \geq \alpha_{i_j}/(10c^2)^{q/2}$ .

*Proof.* Recall that for every  $i$ ,  $\alpha_i = \mu_i \mathbb{E}_{x \in U_i}[|f(x)|^q]$ , and hence (using  $f(x) \in [\beta_i, c\beta_i]$  for  $x \in U_i$ ):

$$\mu_i \beta_i^q \leq \alpha_i \leq \mu_i c^q \beta_i^q. \quad (26)$$

Now fix  $T = T_j$ . Since  $\mathbb{E}_{x \in V}[|g_j(x)|^q] = \mu(T) \cdot \mathbb{E}_{x \in T}[|g_j(x)|^q]$  and  $\mu(T) = \mu(U_{i_j})$ , we can use (26) and  $\mathbb{E}_{x \in T}[|g_j(x)|^q] \geq (\mathbb{E}_{x \in T}[g_j(x)^2])^{q/2}$  (since  $q > 2$ ), to reduce proving the claim to showing the following:

$$\mathbb{E}_{x \in T}[g_j(x)^2] \geq (c\beta_{i_j})^2/(10c^2) = \beta_{i_j}^2/10. \quad (27)$$

We know that  $\mathbb{E}_{x \in T}[g'_j(x)^2] = \beta_{i_j}^2/4$ . We claim that (27) will follow by showing that for every  $k < j$ ,

$$\mathbb{E}_{x \in T}[g'_k(x)^2] \leq 100^{-i'} \cdot \beta_{i_j}^2/4, \quad (28)$$

where  $i' = i_k - i_j$ . (Note that  $i' > 0$  since in our construction the indices  $i_1, \dots, i_J$  are sorted in descending order.)

Indeed, (28) means that if we let momentarily  $\|g_j\|_{L_2}$  denote  $\sqrt{\mathbb{E}_{x \in T}[g_j(x)^2]}$  then

$$\|g_j\|_{L_2} \geq \|g'_j\|_{L_2} - \left\| \sum_{k < j} g_k \right\|_{L_2} \geq \|g'_j\|_{L_2} - \sum_{k < j} \|g_k\|_{L_2} \geq \|g'_j\|_{L_2} \left(1 - \sum_{i'=1}^{\infty} 10^{-i'}\right) \geq 0.8 \|g'_j\|_{L_2}. \quad (29)$$

The first inequality holds we can write  $g_j$  as  $g'_j - h_j$ , where  $h_j = \min(g'_j, \sum_{k < j} g_k)$ . Then, on the other hand,  $\|g_j\|_{L_2} \geq \|g'_j\|_{L_2} - \|h_j\|_{L_2}$ , and on the other hand,  $\|h_j\|_{L_2} \leq \left\| \sum_{k < j} g_k \right\|_{L_2}$  since  $g'_j \geq 0$ . The second inequality holds because  $\|g_k\|_{L_2} \leq \|g'_k\|_{L_2}$ . By squaring (29) and plugging in the value of  $\|g'_j\|_{L_2}^2$  we get (27).

*Proof of (28).* By our construction, it must hold that

$$c^q \alpha_{i_k}/2^{i'} \leq \alpha_{i_j}, \quad (30)$$

since otherwise the index  $i_j$  would have been removed from the  $\mathcal{I}$  at the  $k$ th step. Since  $\beta_{i_k} = \beta_{i_j} c^{i'}$ , we can plug (26) in (30) to get

$$\mu_{i_k} c^{q+qi'}/2^{i'} \leq c^q \mu_{i_j}$$

or

$$\mu_{i_k} \leq \mu_{i_j} \cdot 2^{i'} \cdot c^{-qi'}.$$

Since  $|T_i| = e|U_i|$  for all  $i$ , it follows that  $|T_k|/|T| \leq 2^{i'} \cdot c^{-qi'}$ . On the other hand, we know that  $\mathbb{E}_{x \in T_k}[g'_k(x)^2] = \beta_{i_k}^2/4 = c^{2i'} \beta_{i_j}^2/4$ . Thus,

$$\mathbb{E}_{x \in T}[g'_k(x)^2] \leq 2^{i'} c^{2i'-qi'} \beta_{i_j}^2/4 = (2/c^{q-2})^{i'} \beta_{i_j}^2/4,$$

and we now just choose  $c$  sufficiently large so that  $2/c^{q-2} > 100$ . □

□

□

## 5.4 Hardness of $\min_{p^* \rightarrow p}(\cdot)$

In this subsection, we prove Theorem 8 that for any  $\varepsilon > 0$  and  $p \in (2, \infty)$ , it is NP-hard to approximate  $\min_{p^* \rightarrow p}(\cdot)$  within a factor  $(\gamma_p - \varepsilon)$ , where  $\gamma_p = (\mathbb{E}_{g \sim \mathcal{N}(0,1)}[|g|^p])^{1/p} > 1$  is the absolute  $p$ th moment of the standard Gaussian.

Our result is obtained by using the result of Guruswami et al. [29] that proved the same hardness of  $\min_{2 \rightarrow p}(\cdot)$ . When  $\|\cdot\|_{L_p}$  denotes expectation  $p$ -norm defined by  $\|x\|_{L_p} := \mathbb{E}_i[|x_i|^p]^{1/p}$ , since  $p^* < 2$ , any  $x$  satisfies  $\|x\|_{L_{p^*}} \leq \|x\|_{L_2}$ . This implies that for any matrix  $A$ , the optimal value of  $\min_{p^* \rightarrow p}(A)$  is at least the optimal value of  $\min_{2 \rightarrow p}(A)$ . We modify the reduction of [29] slightly such that in the Yes case,  $x$  that minimizes  $\min_{2 \rightarrow p}(A)$  has either  $+1$  or  $-1$  in each coordinate. This implies  $\|x\|_{L_2} = \|x\|_{L_{p^*}}$ , and certifies that  $\min_{p^* \rightarrow p}(A) = \min_{2 \rightarrow p}(A)$ . In the No case,  $\min_{p^* \rightarrow p}(A)$  is always at least  $\min_{2 \rightarrow p}(A)$ , so the gap between the Yes case and the No case for  $\min_{p^* \rightarrow p}(\cdot)$  is at least as large as the gap for  $\min_{2 \rightarrow p}(\cdot)$ .

Our presentation closely follows the recent work by Bhattiprolu et al. [10]. To present the reduction, we introduce standard backgrounds on Fourier analysis and Label Cover problems.

**Fourier Analysis.** We introduce some basic facts about Fourier analysis of Boolean functions. Let  $R \in \mathbb{N}$  be a positive integer, and consider a function  $f : \{\pm 1\}^R \rightarrow \mathbb{R}$ . For any subset  $S \subseteq [R]$  let  $\chi_S := \prod_{i \in S} x_i$ . Then we can represent  $f$  as

$$f(x_1, \dots, x_R) = \sum_{S \subseteq [R]} \widehat{f}(S) \cdot \chi_S(x_1, \dots, x_R), \quad (31)$$

where

$$\widehat{f}(S) = \mathbb{E}_{x \in \{\pm 1\}^R} [f(x) \cdot \chi_S(x)] \text{ for all } S \subseteq [R]. \quad (32)$$

The *Fourier transform* refers to a linear operator  $F$  that maps  $f$  to  $\widehat{f}$  as defined as (32). We interpret  $\widehat{f}$  as a  $2^R$ -dimensional vector whose coordinates are indexed by  $S \subseteq [R]$ . In this subsection, we let  $\|\cdot\|_{\ell_p}$  to denote the counting  $p$ -norm and  $\|\cdot\|_{L_p}$  to denote the expectation  $p$ -norm. Endow the expectation norm and the expectation norm to  $f$  and  $\widehat{f}$  respectively; i.e.,

$$\|f\|_{L_p} := (\mathbb{E}_{x \in \{\pm 1\}^R} |f(x)|^p)^{1/p} \quad \text{and} \quad \|\widehat{f}\|_{\ell_p} := \left( \sum_{S \subseteq [R]} |\widehat{f}(S)|^p \right)^{1/p}.$$

as well as the corresponding inner products  $\langle fg \rangle$  and  $\langle \widehat{f} \widehat{g} \rangle$  consistent with their 2-norms. We also define the *inverse Fourier transform*  $F^T$  to be a linear operator that maps a given  $\widehat{f} : 2^R \rightarrow \mathbb{R}$  to  $f : \{\pm 1\}^R \rightarrow \mathbb{R}$  defined as in (31). We state the following well-known facts from Fourier analysis.

**Observation 9** (Parseval's Theorem). *For any  $f : \{\pm 1\}^R \rightarrow \mathbb{R}$ ,  $\|f\|_{L_2} = \|Ff\|_{\ell_2}$ .*

**Observation 10.**  *$F$  and  $F^T$  form an adjoint pair; i.e., for any  $f : \{\pm 1\}^R \rightarrow \mathbb{R}$  and  $\widehat{g} : 2^R \rightarrow \mathbb{R}$ ,*

$$\langle \widehat{g} Ff \rangle = \langle F^T \widehat{g} f \rangle.$$

**Observation 11.**  *$F^T F$  is the identity operator.*

**Smooth Label Cover.** An instance of Label Cover is given by a quadruple  $\mathcal{L} = (G, [R], [L], \Sigma)$  that consists of a regular connected graph  $G = (V, E)$ , a label set  $[R]$  for some positive integer  $n$ , and a collection  $\Sigma = ((\pi_{e,v}, \pi_{e,w}) : e = (vw) \in E)$  of pairs of maps both from  $[R]$  to  $[L]$  associated with the endpoints of the edges in  $E$ . Given a *labeling*  $\ell : V \rightarrow [R]$ , we say that an edge  $e = (vw) \in E$  is *satisfied* if  $\pi_{e,v}(\ell(v)) = \pi_{e,w}(\ell(w))$ . Let  $\text{OPT}(\mathcal{L})$  be the maximum fraction of satisfied edges by any labeling.

The following hardness result for Label Cover, given in [29], is a slight variant of the original construction due to [39]. The theorem also describes the various structural properties, including smoothness, that are identified by the hard instances.

**Theorem 22.** *For any  $\xi > 0$  and  $J \in \mathbb{N}$ , there exist positive integers  $R = R(\xi, J)$ ,  $L = L(\xi, J)$  and  $D = D(\xi)$ , and a Label Cover instance  $(G, [R], [L], \Sigma)$  as above such that*

- (Hardness): It is NP-hard to distinguish between the following two cases:
  - Yes case:  $\text{OPT}(\mathcal{L}) = 1$ .
  - No case:  $\text{OPT}(\mathcal{L}) \leq \xi$ .
- (Structural Properties):
  - (J-Smoothness): For every vertex  $v \in V$  and distinct  $i, j \in [R]$ , we have

$$\Pr_{e \ni v} \left[ \pi_{e,v}(i) = \pi_{e,v}(j) \right] \leq 1/J.$$

- (D-to-1): For every vertex  $v \in V$ , edge  $e \in E$  incident on  $v$ , and  $i \in [L]$ , we have  $|\pi_{e,v}^{-1}(i)| \leq D$ ; that is at most  $D$  elements in  $[R]$  are mapped to the same element in  $[L]$ .
- (Weak Expansion): For any  $\delta > 0$  and vertex set  $V' \subseteq V$  such that  $|V'| = \delta \cdot |V|$ , the number of edges among the vertices in  $|V'|$  is at least  $(\delta^2/2)|E|$ .

**Reduction.** Let  $\mathcal{L} = (G, [R], [L], \Sigma)$  be an instance of Label Cover with  $G = (V, E)$ . Our reduction will construct a linear operator  $\mathbf{A} : \mathbb{R}^N \rightarrow \mathbb{R}^M$  with  $N = |V| \cdot 2^R$  and  $M = 2|V| \cdot 2^R - |V| + |E| \cdot |L|$ . The space  $\mathbb{R}^N$  will be endowed the expectation norm (and call its elements functions) and  $\mathbb{R}^M$  will be endowed the counting norm (and call its elements vectors). We define  $\mathbf{A}$  by giving a linear transformation from a function  $\mathbf{f} : V \times \{\pm 1\}^R \rightarrow \mathbb{R}$  to a vector  $\mathbf{a} \in \mathbb{R}^M$ . Let  $C := M^3$ . Given  $\mathbf{f}$ , a vertex  $v \in V$  induces  $f_v \in \mathbb{R}^{2^R}$  defined by  $f_v(x) := \mathbf{f}(vx)$  for  $x \in \{\pm 1\}^R$ . Let  $\hat{\mathbf{g}} \in V \times [R]$  be the vectors of linear coefficients;  $\hat{\mathbf{g}}(vi) = \hat{f}_v(i)$  for  $v \in V, i \in [R]$ . Given  $\mathbf{f}$  (that determines  $\{\hat{f}_v\}_v \in V$  and  $\hat{\mathbf{g}}$ ),  $\mathbf{a} = \mathbf{A}\mathbf{f}$  is defined as follows.

- For  $v \in V$  and  $x \in \{\pm 1\}^R$ ,  $\mathbf{a}(vx) = \sum_{i=1}^R \hat{\mathbf{g}}(vi) x_i$ .
- For  $v \in V$  and  $S \subseteq [R]$  with  $|S| \neq 1$ ,  $\mathbf{a}(vS) = C \cdot \hat{f}_v(S)$ .
- For  $e = (vw) \in E$  and  $i \in [L]$ ,  $\mathbf{a}(ei) = C \cdot \left( \sum_{j \in \pi_{e,v}^{-1}(i)} \hat{f}_v(j) - \sum_{j \in \pi_{e,w}^{-1}(i)} \hat{f}_w(j) \right)$ .

Since  $\hat{\mathbf{g}}$  and  $\mathbf{a}$  are all linear in  $\mathbf{f}$ , the matrix  $\mathbf{A}$  that satisfies  $\mathbf{a} = \mathbf{A}\mathbf{f}$  is well-defined, which is our instance of  $\min_{p^* \rightarrow p}(\cdot)$ . Intuitively,  $C$  will be chosen large enough so that every  $\hat{f}_v$  has almost all Fourier mass on its linear coefficients, and their linear coefficients correctly indicate the labels that satisfy all constraints of the Label Cover instance.

**Completeness.** We prove the following lemma for the Yes case.

**Lemma 31** (Completeness). *Let  $\ell : V \rightarrow [R]$  be a labeling that satisfies every edge of  $\mathcal{L}$ . There exists a function  $\mathbf{f} \in \mathbb{R}^{V \times 2^R}$  such that  $\mathbf{f}(vx)$  is either  $+1$  or  $-1$  for all  $v \in V, x \in \{\pm 1\}^R$  and  $\|\mathbf{A}\mathbf{f}\|_{\ell_p} = (|V| \cdot 2^R)^{1/p}$ . In particular,  $\|\mathbf{A}\mathbf{f}\|_{\ell_p} / \|\mathbf{f}\|_{L_{p^*}} = (|V| \cdot 2^R)^{1/p}$ .*

*Proof.* Let  $\mathbf{f}(vx) := x_{\ell(v)}$  for every  $v \in V, x \in \{\pm 1\}^R$ . Consider  $\mathbf{a} = \mathbf{A}\mathbf{f}$ . Since every  $\hat{f}_v$  is linear, for each  $v \in V$  and  $S \subseteq [R]$  with  $|S| \neq 1$ ,  $\mathbf{a}(vS) = 0$ . For each  $v \in V$  and  $i \in [R]$ ,  $\hat{\mathbf{g}}(vi) = 1$  if and only if  $i = \ell(v)$  and 0 otherwise. Since  $\ell$  satisfies every edge of  $\mathcal{L}$ ,  $\mathbf{a}(ei) = 0$  for every  $e \in E$  and  $i \in [L]$ . This implies that for every  $v \in V, x \in \{\pm 1\}^R$ ,  $\mathbf{a}(vx) = x_{\ell(v)} = \mathbf{f}(vx)$ . Therefore,  $\|\mathbf{A}\mathbf{f}\|_{\ell_p} = (|V| \cdot 2^R)^{1/p}$ .  $\square$

**Soundness.** We prove the following lemma for the soundness. Combined with Theorem 22 for hardness of Label Cover and observing that  $\|\mathbf{f}\|_{L_{p^*}} \leq \|\mathbf{f}\|_{L_2}$ , it finishes the proof of Theorem 8.

**Lemma 32.** *For any  $\eta > 0$ , there exists  $\xi > 0$  (that determines  $D = D(\xi)$  as in Theorem 22) and  $J \in \mathbb{N}$  such that if  $\text{OPT}(\mathcal{L}) \leq \xi$ ,  $\mathcal{L}$  is D-to-1 and  $\mathcal{L}$  is J-smooth, for every  $\mathbf{f}$  with  $\|\mathbf{f}\|_{L_2} = 1$ ,  $\|\mathbf{A}\mathbf{f}\|_{\ell_p} \geq (\gamma_p - \eta)(|V| \cdot 2^R)^{1/p}$ .*

*Proof.* We will prove contrapositive; if  $\|\mathbf{A}\mathbf{f}\|_{\ell_p} \leq (\gamma_p - \eta)(|V| \cdot 2^R)^{1/p}$  for some  $\mathbf{f}$  is small then  $\text{OPT}(\mathcal{L}) \geq \xi$  with the choice of the parameters that will be determined later. Fix such an  $\mathbf{f}$  with  $\|\mathbf{f}\|_{L_2} = 1$  that determines  $f_v$  and  $\hat{f}_v$  for each  $v \in V$ . Let  $\mathbf{a} = \mathbf{A}\mathbf{f}$ . Suppose that there is  $v \in V$  and  $S \subseteq [R]$  with  $|S| \neq 1$  such that  $|\hat{f}_v(S)| > 1/M^2$ . It means that  $|\mathbf{a}(vS)| > C/M^2$ . Since  $C = M^3$ , it already implies  $\|\mathbf{a}\|_{\ell_p} \geq M \gg (\gamma_p - \eta)(|V| \cdot 2^R)^{1/p}$ , so suppose that there is no such  $v$  and  $S$ .

Let  $\hat{\mathbf{g}} \in V \times [R]$  be defined as above;  $\hat{\mathbf{g}}(vi) = \hat{f}_v(i)$  for  $v \in V, i \in [R]$ . By Parseval,

$$\sum_{v \in V} \|\hat{f}_v\|_{\ell_2}^2 = \sum_{v \in V} \|f_v\|_{L_2}^2 = |V| \cdot \mathbb{E}_{v \in V} \|f_v\|_{L_2}^2 = |V| \cdot \|\mathbf{f}\|_{L_2}^2 = |V|.$$

and the fact that  $|\hat{f}_v(S)| < 1/M^2$  for every  $v \in V, S \subseteq [R]$  with  $|S| \neq 1$ , we have  $\|\hat{\mathbf{g}}\|_{\ell_2} \in [\sqrt{|V|} - 1/M, \sqrt{|V|}]$ .

Furthermore, suppose that there is  $e = (uv) \in E$  and  $i \in [L]$  such that

$$\left| \sum_{j \in \pi_{e,u}^{-1}(i)} \hat{\mathbf{g}}(uj) - \sum_{j \in \pi_{e,v}^{-1}(i)} \hat{\mathbf{g}}(vj) \right| \geq 1/M^2.$$

This implies that  $|\mathbf{a}(ei)| \geq C/M^2$ . Since  $C = M^3$ , it already implies  $\|\mathbf{a}\|_{\ell_p} \geq M \gg (\gamma_p - \eta)(|V| \cdot 2^R)^{1/p}$ , so we can assume that there is no such  $e$  and  $i$ .

To bound  $\|\mathbf{a}\|_{\ell_p}$ , it only remains to analyze

$$\sum_{v \in V} \sum_{x \in \{\pm 1\}^R} \left| \mathbf{a}(vx) \right|^p = \sum_{v \in V} \sum_{x \in \{\pm 1\}^R} \left| \sum_{i \in [R]} \hat{\mathbf{g}}(vi)x_i \right|^p. \quad (33)$$

The rest of the proof closely follows [29], and we explain high-level intuitions and why their proofs work in our settings. First, let us assume that  $\|\hat{\mathbf{g}}\|_{\ell_2} = \sqrt{|V|}$ . It involves a multiplicative error of  $(1 - 1/M)$ , which is negligible in our proof. To simplify notations, let  $\hat{g}_v \in \mathbb{R}^R$  be such that  $\hat{g}_v(i) := \hat{f}_v(\{i\}) = \hat{\mathbf{g}}(vi)$  for each  $v \in V$  and  $i \in [R]$ . Call a vertex  $v \in V$   $\tau$ -irregular if there exists  $i \in [R]$  such that  $|\hat{\mathbf{g}}(vi)| > \tau \|\hat{g}_v\|_{\ell_2}^2$ . If not,  $v$  is  $\tau$ -regular. Also, call a vertex  $v \in V$  small if  $\|\hat{g}_v\|_{\ell_2} < 1/M$ . Otherwise, call it big.

For each  $v \in V$ , we consider  $\sum_{x \in \{\pm 1\}^R} \left| \sum_{i \in [R]} \hat{g}_v(i)x_i \right|^p$ . By Khintchine inequality, it is at most  $2^R \cdot \gamma_p^p \cdot \|\hat{g}_v\|_{\ell_2}^p$ . The following lemma, based on standard applications of the Berry-Esseen theorem, shows that the converse is almost true when  $v$  is  $\tau$ -regular, implying the contribution from irregular vertices to (33) is large.

**Lemma 33** ([42]). *For sufficiently small  $\tau$  (depending only on  $p$ ), if  $v \in V$  is  $\tau$ -regular, then*

$$\sum_{x \in \{\pm 1\}^R} \left| \sum_{i \in [R]} \hat{g}_v(i)x_i \right|^p \geq 2^R \cdot \gamma_p^p \cdot \|\hat{g}_v\|_{\ell_2}^p (1 - \sqrt{\tau}).$$

Let  $S$  be the set of big  $\tau$ -irregular vertices. Based on the above, the following lemma shows that  $S$  must be a large set. Originally, [29] only argued for  $\tau$ -irregular vertices. (The notion of big and small vertices does not appear there.) However, since the contribution of small vertices to (33) is negligible, the same proof essentially works.

**Lemma 34** (Lemma 4.4 of [29]). *There are  $\tau$  and  $\theta$ , depending only on  $p$  and  $\eta$ , such that  $S$ , the set of big  $\tau$ -irregular vertices, satisfies  $|S| \geq \theta|V|$ .*

By the weak expansion property of  $\mathcal{L}$  guaranteed in Theorem 22,  $S$  induces at least  $\theta^2|E|$  edges of  $\mathcal{L}$ . To finish the proof, [29] showed that we can satisfy a significant fraction of the edges from  $\mathcal{L}$ . The only difference in their setting and our setting is that

- [29]:  $S$  is the set of all  $\tau$ -irregular vertices. For each  $e = (uv)$  and  $i \in [L]$ ,

$$\sum_{j \in \pi_{e,u}^{-1}(i)} \hat{g}_u(j) = \sum_{j \in \pi_{e,v}^{-1}(i)} \hat{g}_v(j). \quad (34)$$

- Here:  $S$  is the set of all big  $\tau$ -irregular vertices. For each  $e = (uv)$  and  $i \in [L]$ ,

$$\left| \sum_{j \in \pi_{e,u}^{-1}(i)} \widehat{g}_u(j) - \sum_{j \in \pi_{e,v}^{-1}(i)} \widehat{g}_v(j) \right| < 1/M^2. \quad (35)$$

These differences do not affect their proof since in the only place (34) was used for  $e = (uv)$  and  $i \in [L]$ , they indeed used the fact the left-hand side of (35) is at most  $0.3\tau \cdot \max(\|\widehat{g}_u\|_{\ell_2}, \|\widehat{g}_v\|_{\ell_2})$ . Since we additionally assumed that  $S$  is big,  $\|\widehat{g}_u\|_{\ell_2} \geq 1/M$  for each  $u \in S$ , so it is always satisfied from (35).

**Lemma 35** ([29]). *Let  $\beta := 10000D^4/\tau^4J$ . Then  $\text{OPT}(\mathcal{L}) \geq (\tau^4/16)(\theta^2 - 2/\beta)$ .*

Since  $\theta$  and  $\tau$  only depend on  $\eta$  and  $p$ , fixing small enough  $\xi$  (that determines  $D$ ) and large enough  $J$  will ensure  $\text{OPT}(\mathcal{L}) \geq (\tau^4/16)(\theta^2 - 2/\beta) \geq \xi$ , finishing the proof of the lemma.  $\square$

## 5.5 Hardness for Finite Fields

In this subsection, we prove Lemma 3, which in turn finishes the proof of Theorem 20 for hardness of  $\ell_0$ -row rank approximation for matrices whose entries are from a finite field  $\mathbb{F}$ .

**Lemma 36** (Restatement of Lemma 3). *Let  $\mathbb{F}$  be a finite field and  $A \in \mathbb{F}^{n \times d}$  with  $n \geq d$  and  $k = d - 1$ . Then*

$$\min_{U \in \mathbb{F}^{n \times k}, V \in \mathbb{F}^{k \times d}} \|UV - A\|_0 = \min_{x \in \mathbb{F}^d, x \neq 0} \|Ax\|_0.$$

*Proof.* Assume that the rank of  $A$  is  $d$ ; otherwise the lemma becomes trivial. We first prove ( $\geq$ ). Given  $V^* \in \mathbb{F}^{k \times d}$  that achieves the best rank  $k$  approximation, assume without loss of generality that the rank of  $V^*$  is  $k = d - 1$ . Let  $x \in \mathbb{F}^d$  be a nonzero vector that is orthogonal to the row space of  $V^*$ ; i.e.,  $\langle vx \rangle = 0$  if and only if  $v \in \text{row space}(V^*)$ . Note that unlike in  $\mathbb{R}$ ,  $x$  can be in  $\text{row space}(V^*)$ , but it does not affect the proof. Let  $a_1, \dots, a_n$  be the rows of  $A$ . For fixed  $V^*$  and  $i \in [n]$ , if  $a_i \in \text{row space}(V^*)$ , then we can compute the  $i$ th row of  $U^*$  (denoted by  $u_i^*$ ) such that  $u_i^* V^* = a_i$ . Otherwise,  $\langle a_i, x \rangle = b$  for some  $b \neq 0$ , since  $x$  is nonzero, there is  $u_i^*$  such that  $\|u_i^* V^* - a_i\|_0 = 1$ . Therefore,  $\|U^* V^* - A\|_0 = \|Ax\|_0$ , which implies that  $\min_{U \in \mathbb{F}^{n \times k}, V \in \mathbb{F}^{k \times d}} \|UV - A\|_0 \geq \min_{x \in \mathbb{F}^d, x \neq 0} \|Ax\|_0$ .

For the other direction, given  $x \in \mathbb{F}^d \setminus \{0\}$ , the set of vectors  $u$  with  $\langle ux \rangle = 0$  forms a  $k$ -dimensional subspace. (Again, this space may contain  $x$  unlike in  $\mathbb{R}$ , but it does not matter.) Let  $V^* \in \mathbb{F}^{k \times d}$  be a matrix whose rows span that space, and compute  $U^*$  as above. The above analysis shows that  $\|U^* V^* - A\|_0 = \|Ax\|_0$ , which completes the proof.  $\square$

## 6 Additional Results

Here we list some additional results on variants of the  $\ell_p$  low rank approximation problem.

### 6.1 Bicriteria Algorithm

In this section we show that we can develop low rank approximations that apply to matrices whose entries are not bounded by  $\text{poly}(n)$  so long as we accept bicriteria algorithms. That is, instead of a target rank  $k$  approximation, the algorithm will output an approximating matrix of rank  $3k$ .

**Theorem 23.** *If  $A$  is an  $n \times d$  matrix, our target rank  $k$  is a constant, and  $1 \leq p < 2$ , then there exists a polynomial time algorithm that outputs a matrix  $M$  of rank at most  $3k$  such that  $\|M - A\|_p \leq (1 + \epsilon)OPT$  where  $OPT$  is the best rank  $k$   $\ell_p$ -low rank approximation value for  $A$  with probability  $1 - O(1)$ .*

*Proof.* Let  $C_l$  denote the best rank  $l$  approximation to a matrix  $C$  in the  $\ell_p$  norm (i.e. the matrix that minimizes  $\|C_l - C\|_p$ ).

Let  $B$  be the best rank  $k$  approximation to  $A$  in the Frobenius norm. Then

$$\|A - B\|_p \leq \text{poly}(n)\|A - B\|_F \leq \text{poly}(n)\|A - A_k\|_F \leq \text{poly}(n)OPT.$$

We can find a rank  $2k$   $(1 + \epsilon)$ -approximation to  $A - B$  using the same techniques as in Theorem 10, where we sample a matrix  $S$  of  $p$ -stable variables, guess values for  $SU^*$ , and then minimize  $\|SU^*V^* - S(A - B)\|_p$ . Now the entries of  $A - B$  are not necessarily bounded by  $\text{poly}(n)$  so we need to justify that it suffices to guess  $\text{poly}(n)$  values for  $SU^*$ .

Indeed, by a well-conditioned basis argument, no entry of  $U^*$  has absolute value greater than  $\text{poly}(n)\|A - B\|_p$ . Furthermore, we can round each entry of  $U^*$  (similar to the proof of Theorem 10) to the nearest multiple of  $\frac{\epsilon\|A - B\|_p}{\text{poly}(n)}$  and incur an additive error of at most  $\epsilon OPT$  because  $\|A - B\|_p \leq \text{poly}(n)OPT$ . This error is small enough for the purposes of our approximation.

Let  $(A - B)_{2k}^* = U^*V^*$  and let  $M = (A - B)_{2k}^* + B$ . We have

$$\begin{aligned} \|M - A\|_p &= \|(A - B)_{2k}^* + B - A\|_p \\ &= \|(A - B)_{2k}^* - (A - B)\|_p \\ &\leq (1 + \epsilon)\|(A - B)_{2k} - (A - B)\|_p + \epsilon OPT \\ &\leq (1 + \epsilon)\|A_k - B - (A - B)\|_p + \epsilon OPT \\ &\leq (1 + \epsilon)OPT \end{aligned}$$

where the first inequality follows from our argument above and the second inequality follows because  $A_k - B$  has rank at most  $k + k = 2k$ .

Since  $M$  has rank at most  $2k + k = 3k$ , then the result follows.  $\square$

### 6.2 Weighted Low Rank Approximation

For  $0 < p < 2$ , we can also design a PTAS for the weighted  $\ell_p$  low rank approximation problem. In this setting we have a matrix  $A$ , a weight matrix  $W$  of rank  $r$ , and we want to output a rank  $k$  matrix  $A'$  such that, for  $\epsilon > 0$ ,

$$\|W \circ (A - A')\|_p^p \leq (1 + \epsilon) \min_{\text{rank } k \ A_k} \|W \circ (A - A_k)\|_p^p.$$

Our main tool will be a multiple regression concentration result based on that of [62].

**Theorem 24.** *Let  $S$  be a  $\text{poly}(1/\epsilon) \times n$  matrix whose entries are i.i.d  $p$ -stable random variables with scale 1. Let  $M^{(1)}, M^{(2)}, \dots, M^{(m)}$  be  $n \times d$  matrices and let  $b^{(1)}, b^{(2)}, \dots, b^{(m)} \in \mathbb{R}^n$ . Let*

$$x^{(i)} = \arg \min_x \|M^{(i)}x - b^{(i)}\|_p^p$$

and

$$y^{(i)} = \arg \min_y \text{med}(SM^{(i)}y - Sb^{(i)})/\text{med}_p.$$

Then w.h.p we have

$$\sum_i \|M^{(i)}y^{(i)} - b^{(i)}\|_p^p \leq (1 + O(\epsilon)) \sum_i \|M^{(i)}x^{(i)} - b^{(i)}\|_p^p$$

*Proof.* By Lemmas 12 and 16, w.h.p.

$$\sum_i \frac{\text{med}(S(M^{(i)}x^{(i)} - b^{(i)}))^p}{\text{med}_p^p} \leq (1 + O(\epsilon)) \sum_i \|M^{(i)}x^{(i)} - b^{(i)}\|_p^p.$$

Let  $T$  be the set of all  $i$  such that

$$\frac{\text{med}(S[M^{(i)} b^{(i)}]y)^p}{\text{med}_p^p} \geq (1 - \Theta(\epsilon)) \| [M^{(i)} b^{(i)}]y \|_p^p$$

for all  $y$ . By Corollary 3, we know that for each  $i$ , the probability that  $i \in T$  is at least  $1 - \Theta(\epsilon)$ .

Thus

$$\mathbb{E} \left[ \sum_{i \notin T} \| [M^{(i)} b^{(i)}]y \|_p^p \right] \leq \Theta(\epsilon) \sum_i \| [M^{(i)} b^{(i)}]y \|_p^p$$

so by Markov's inequality, w.h.p we have

$$\sum_{i \notin T} \| [M^{(i)} b^{(i)}]y \|_p^p \leq \Theta(\epsilon) \sum_i \| [M^{(i)} b^{(i)}]y \|_p^p.$$

Let  $y$  be arbitrary. Since

$$\sum_i \frac{\text{med}(S[M^{(i)} b^{(i)}]y)^p}{\text{med}_p^p} \geq \sum_{i \notin T} \frac{\text{med}(S[M^{(i)} b^{(i)}]y)^p}{\text{med}_p^p} \geq (1 - \Theta(\epsilon)) \sum_{i \notin T} \| [M^{(i)} b^{(i)}]y \|_p^p,$$

it follows that for all  $y$  we have

$$\sum_i \frac{\text{med}(S[M^{(i)} b^{(i)}]y)^p}{\text{med}_p^p} \geq (1 - \Theta(\epsilon)) \sum_i \| [M^{(i)} b^{(i)}]y \|_p^p.$$

Therefore w.h.p we have

$$\begin{aligned} (1 - \Theta(\epsilon)) \sum_i \|M^{(i)}y^{(i)} - b^{(i)}\|_p^p &\leq \sum_i \frac{S(M^{(i)}y^{(i)} - b^{(i)})}{\text{med}_p^p} \\ &\leq \sum_i \frac{S(M^{(i)}x^{(i)} - b^{(i)})}{\text{med}_p^p} \leq (1 + O(\epsilon)) \sum_i \|M^{(i)}x^{(i)} - b^{(i)}\|_p^p \end{aligned}$$

because  $0 < p < 2$ . The result follows.  $\square$

**Theorem 25.** Suppose  $A$  and  $W$  are  $n \times d$  matrices with entries bounded by  $\text{poly}(n)$ , and  $r = \text{rank}(W)$ . There is an algorithm that for any integer  $k$ ,  $p \in (0, 2)$  and  $\epsilon \in (0, 1)$ , outputs in time  $n^{r \cdot \text{poly}(k/\epsilon)}$  a  $n \times k$  matrix  $U^*$  and a  $k \times d$  matrix  $V^*$  such that

$$\|W \circ (A - U^*V^*)\|_p^p \leq (1 + O(\epsilon)) \min_{\text{rank-}k A_k} \|W \circ (A - A_k)\|_p^p.$$

*Proof.* To achieve a relative-error low rank approximation  $W \circ (UV - A)$ , for each column  $i$  we can guess sketches for  $W_{:,i} \circ UV_i$  using a similar argument as in Theorem 10. Indeed, we can apply Theorem 24 with  $M^{(i)} = W_{:,i} \circ U^*V_i^*$  and  $b^{(i)} = W_{:,i} \circ A_{:,i}$ . To do so, we need to be able to guess  $SW_{:,i} \circ U^*$ , a  $\text{poly}(\frac{k}{\epsilon}) \times k$  matrix, in  $\text{poly}(n)$  tries. We will follow the same reasoning as in the proof of Theorem 10. Since the entries of  $W$  and  $A$  are bounded by  $\text{poly}(n)$ , then we can bound the entries of  $U^*$  by  $\text{poly}(n)$  using a well-conditioned basis. Furthermore, we can round each entry of  $U^*$  to the nearest multiple of  $\text{poly}(n^{-1})$  while incurring an error factor of only  $(1 + O(\epsilon))$ . Thus, we need only  $n^{\text{poly}(k/\epsilon)}$  guesses.

Of course, there  $d$  columns so this is not enough to achieve a PTAS. However, we only need to guess sketches for  $r$  values of  $j$  because  $W$  has rank  $r$  so we can express any column of  $W$  as a linear combination of those  $r$  columns. That is, we choose a subset  $S$  of the columns such that  $|S| = r$  and guess the sketches of  $W_{:,i} \circ UV_i$  for each  $i \in S$  as described in the previous paragraph. Therefore, we require  $n^{r \cdot \text{poly}(k/\epsilon)}$  time in total for a  $(1 + O(\epsilon))$  approximation algorithm. Since  $k$  and  $r$  are constants this results in a PTAS.  $\square$



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