

Spin foam models and the Duflo map

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Abstract

We give a general definition of spin foam models, and then of models of 4d quantum gravity based on constraining BF theory. We highlight the construction and quantization ambiguities entering model building, among which the choice of quantization map applied to the B variables carrying metric information after imposing simplicity constraints, and the different strategies for imposing the latter constraints. We then construct a new spin foam model for 4d quantum gravity, using the flux representation of states and amplitudes, based on the Duflo quantization map and the associated non-commutative Fourier transform for Lie groups. The advantages of the new model are the geometrically transparent way in which constraints are imposed, and the underlying mathematical properties of the Duflo map itself. Finally, we study some properties of the resulting spin foam amplitudes, in particular the relation with other models and a few limiting cases.

Contents

1	Introduction	2
2	Spin foam models	3
2.1	Abstract formulation of spin foam models.	3
2.2	The GFT formulation of spin foam models.	9
3	Spin foam models for constrained BF theory	11
3.1	Gravity as a constrained BF theory	11
3.2	The Hilbert space of boundary states	13
3.3	Definition and imposition of the geometricity constraints.	17
3.4	Spin foam amplitudes from imposition of geometricity operators	21
4	Flux variables and the Duflo map. A new model for Holst-Plebanski quantum gravity	26
4.1	Simplicity constraints and non-commutative tetrahedra.	26
4.2	The amplitudes of the new model	30
5	Conclusions.	31
	Acknowledgements	32
A	Overview of non-commutative harmonic analysis.	33
A.1	Non-commutative Fourier transform. Generalities.	33
A.2	The Duflo quantization map.	35
B	Basics of SU(2) recoupling theory.	36
C	The new model's fusion coefficients. Evaluation and limiting cases.	37
D	The new model's vertex and gluing kernels.	41

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1 Introduction

Spin foam models are a covariant definition of the quantum dynamics of spin network-type structures [1]. They are thus a covariant counterpart of canonical Loop Quantum Gravity [2–5], a reformulation of lattice gravity path integrals [6, 7], a natural language for state sum models of topological quantum field theories [8], the perturbative dynamics of group field theories [9–12]. They have by now a long history, dating back almost 25 years, with the ingredients of the first spin foam model actually introduced by Ponzano and Regge in 1968 [13], and the first spin foam models themselves having been proposed in 1992 [14], also as a generalization of tensor models via the group field theory formalism [15, 16]. The corresponding literature is accordingly very large. In particular, in addition to the many results obtained in the context of the mentioned related formalisms (e.g., group field theory and tensor models [17–19]) also directly impacting on spin foam models per se, a lot of work has concentrated on spin foam model building [20–30], and on the semi-classical analysis of the resulting quantum gravity models (for fixed underlying lattice) [31–35]. More recently, the issues of spin foam renormalization and continuum limit (from both lattice [36–46] and GFT [47–56] perspectives) have become central. Progresses have also been made in generalizing and adapting the notions of entanglement entropy and holography, so far mainly in the case of $3d$ quantum gravity models [57–60]. Important steps have also been taken in direction of extracting effective continuum physics out of quantum gravity models, in particular in a cosmological context (both from a canonical perspective [61] and using the GFT reformulation of the same models [62–66]). In parallel, we have deepened our understanding of the formal structure of spin foam models, and explored it at a more mathematical level also illuminating the various choices underlying model building.

This paper tackles the more formal aspects of spin foam construction, as a stepping stone for investigating the more physical ones. First of all, we provide a very general definition of spin foam models and of their construction from their defining building blocks, detailing both the combinatorial aspects and their associated quantum states and amplitudes. We trust this work to have pedagogical value, first of all, for those entering the field but to be also useful for scholars who could appreciate the key features of the formalism beyond specific models or even specific perspectives on the quantum gravity program. Moreover, the general definition we provide will be a convenient starting point for more model building or the analysis of physical consequences, not relying on any specific representation of quantum states or on model building strategy.

Next, we specialize the general construction to spin foam models for $4d$ quantum gravity, in the Riemannian setting, inspired by the formulation of gravity as a constrained BF theory. We introduce the basic ideas of this construction strategy and provide the general definition of the corresponding models, without focusing exclusively on any one of them, but, on the contrary, highlighting their shared features. We do so in different representations for the quantum states and for the amplitudes. We put special emphasis on the construction choices and ambiguities faced in this strategy, in order to make it easier to resolve them or to parametrize them when trying to extract physical aspects of the same models. Also this second part of our work, we believe, has an important pedagogical value. Beyond that, we think this it is very useful to stress and then control the mentioned construction ambiguities, shared by all models, as well as to clearly identify which aspect is specific to a given model and which one is instead a general feature of the formalism, for better comparison. Among the latter, we mention the fact that all spin foam models in this class take the form of non-commutative simplicial gravity path integrals when expressed in the flux (Lie algebra) representation (including the EPRL model which is usually not seen from this perspective). This can greatly facilitate their semi-classical analysis.

Last, we use the previous analysis to construct a *new spin foam model* in the same constrained BF class. The model is constructed focusing on the flux/metric representation of spin foam states and amplitudes, and thus relying on the associated tools from non-commutative geometry, notably the non-commutative Fourier transform for Lie groups [67, 68]. In this respect, it parallels the construction of the model presented in [27]. However, it employs a different quantization map for the Lie algebra variables of discrete BF theory (and thus the metric variables of the gravitational theory): the Duflo map. This is an important improvement. On the one hand, the Duflo map has a number of nice mathematical properties (outlined in the Appendix), making it in many ways the natural quantization map for quantum systems based on group-theoretic structures [69]. On the other hand, and more importantly perhaps, contrary to the quantization map employed in [27] (and in some other related quantum gravity literature [70]) the Duflo map applies to any semi-simple, locally compact Lie group; thus the model we introduce can be straightforwardly generalised to other dimensions, other model-building strategies and, more immediately relevant, the Lorentzian signature. Implicitly, the construction of this new model illustrate the role that quantization ambiguities play in the formulation of spin foam models, a point often downplayed. We give the explicit expression of the new model’s amplitudes in flux, group and spin variables. The availability of all of them in closed form is another useful asset, for concrete computations.

2 Spin foam models

In this section we outline the general definition and construction of spin foam models for Riemannian quantum gravity. Spin foam models associate quantum amplitudes to discrete structures, i.e. lattices, usually in the form of a product of "local" amplitudes, i.e. amplitudes associated to the lower-dimensional cells of the same lattices. We begin by listing and defining the basic building blocks of the combinatorial structures on which spin foam amplitudes are supported. For simplicity, and because there are the most interesting cases for quantum gravity model building, we restrict ourselves to cellular complexes of simplicial type (we often loosely refer to them as 'triangulations').

Then we introduce the Hilbert space of (boundary) states and the set of amplitudes associated to the subsimplices of the chosen complex, obtained as traces of suitably defined operators. Equivalently, the same amplitudes can be thought of as associated to the topological dual complex to the simplicial one. The definitions of these general ingredients are adapted from [71], where we refer for more details, and from which we also take the terminology (a similar set of definitions, with a different nomenclature, was introduced in [72]). The construction we outline is also close to the *operator spin foams* formalism presented in [73].

Finally, we outline the Group field theory (GFT) formulation and completion of spin foam models, based on the same building blocks, removing the dependence of any specific cellular complex, and thus implicitly defining their continuum limit and suggesting an alternative perspective on them and a number of useful field-theoretic tools and approximation schemes.

2.1 Abstract formulation of spin foam models

The discrete building blocks on which spin foam models have support exhibit a 'molecular' structure, which we introduce in steps.

Definition 2.1 (Boundary graph). A boundary graph $\mathfrak{c} \in \mathfrak{C}$ is an ordered pair $\mathfrak{c} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$ comprising a set of vertices or nodes $\bar{\mathcal{V}}$ and an edge (or link) (multi)set $\bar{\mathcal{E}}$ (consisting of unordered two-element subsets of $\bar{\mathcal{V}}$) subjected to the condition that the graph is connected. The set of all boundary graphs is denoted by \mathfrak{C} .

Definition 2.2 (Bisected boundary graph). A bisected boundary graph $\mathfrak{b} \in \mathfrak{C}$ is an ordered pair $\mathfrak{b} = (\mathcal{V}_b, \mathcal{E}_b)$ forming a bipartite graph with vertex partition $\mathcal{V}_b = \bar{\mathcal{V}} \cup \hat{\mathcal{V}}$ such that the vertices $\hat{v} \in \hat{\mathcal{V}}$ are bivalent. The set of all connected bisected boundary graphs is denoted by \mathfrak{B} . It is easy to prove (and intuitive) that exists a bijection between the two sets \mathfrak{C} and \mathfrak{B} [71].

A practical example of these two objects is presented in Figure 1.

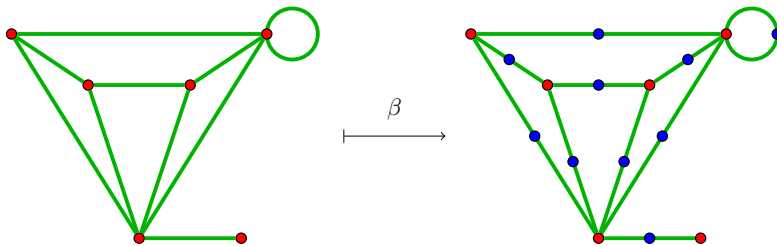


Figure 1: A boundary graph and its bisected counterpart.

Definition 2.3 (Spin foam atom). A spin foam atom $a \in \mathfrak{A}$ is a triple $\mathfrak{a} = (\mathcal{V}_a, \mathcal{E}_a, \mathcal{F}_a)$, constructed in correspondence with a bisected boundary graph \mathfrak{b} , with:

$$\mathcal{V}_a = \{v\} \cup \mathcal{V}_b \quad \mathcal{E}_a = \tilde{\mathcal{E}} \cup \mathcal{E}_b \quad \tilde{\mathcal{E}} = \bigcup_u \{(uv) : u \in \mathcal{V}_b\} \quad \mathcal{F}_a = \bigcup_{\tilde{v} \in \hat{\mathcal{V}}} \{f = (v\tilde{v}\hat{v}) : (\tilde{v}\hat{v}) \in \mathcal{E}_b\} \quad , \quad (2.1)$$

where $\mathcal{V} = \{v\}$ is a one-element vertex set, containing the *bulk* vertex. The set \mathcal{E} reads: $\mathcal{E} = \bigcup_{u \in \mathcal{V}_b} \{(vu) : v \in \mathcal{V}\}$. Note that \mathcal{E} contains precisely one edge for each vertex in \mathcal{V}_b , joining it to the bulk vertex v . Thus, u takes values in $\bar{\mathcal{V}}$ and $\hat{\mathcal{V}}$. Last $(v\tilde{v}\hat{v})$ is the face bounded by the edges connecting the three vertices v on its boundary. The set of spin foam atoms is denoted by \mathfrak{A} .

It is also possible to define two bijective maps: the bulk map $\alpha : \mathfrak{B} \rightarrow \mathfrak{A}$ and its inverse, the boundary map $\delta : \mathfrak{A} \rightarrow \mathfrak{B}$, allowing the set \mathfrak{A} of spin foam atoms to be catalogued by the set \mathfrak{B} of bisected boundary graphs [71]. This completes the definition of the fundamental building blocks. Let us now describe the procedure for obtaining composite structures out of them. For this purpose the notions of *boundary patch* and *gluing map* are required.

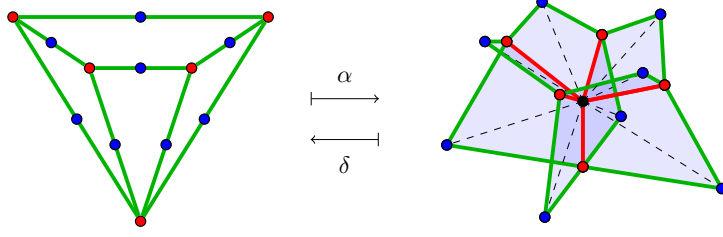


Figure 2: A spin foam atom and its bisected boundary graph.

Definition 2.4 (Boundary patch). A boundary patch \mathfrak{p} is a pair $\mathfrak{p}_{\bar{v}} = (\mathcal{V}_{\mathfrak{p}}, \mathcal{E}_{\mathfrak{p}})$ such that:

$$\mathcal{V}_{\mathfrak{p}} = \{\bar{v}\} \cup \tilde{\mathcal{V}}_{\mathfrak{p}} \quad \tilde{\mathcal{V}}_{\mathfrak{p}} \neq \emptyset \quad \mathcal{E}_{\mathfrak{p}} = \{(\bar{v}\tilde{v}) : \tilde{v} \in \tilde{\mathcal{V}}_{\mathfrak{p}}\} \quad (2.2)$$

where $\mathcal{E}_{\mathfrak{p}}$ is a multiset of edges where each $(\bar{v}\tilde{v})$ occurs at least once and at most twice. A boundary patch can be understood as the closure of the star of a vertex \bar{v} within a bisected boundary graph \mathfrak{b} , in which case

$$\mathcal{V}_{\mathfrak{p}} = \mathcal{V}_{\bar{v}} = \{\bar{v}\} \cup \{\tilde{v} \in \mathcal{V}_{\mathfrak{b}} : (\bar{v}\tilde{v}) \in \mathcal{E}_{\mathfrak{b}}\} \quad \mathcal{E}_{\mathfrak{p}} = \mathcal{E}_{\bar{v}} = \{e = (\bar{v}\tilde{v}) \in \mathcal{E}_{\mathfrak{b}}\} \quad (2.3)$$

Thus a boundary patch is a graph made by the node \bar{v} , all boundary (half)-edges containing it and their end points. The set of boundary patches will be denoted by \mathfrak{P} .

Two patches $\mathfrak{p}_{\bar{v}_1}$ and $\mathfrak{p}_{\bar{v}_2}$ (regardless whether they are distinct or not) are named *bondable* if $|\mathcal{V}_{\mathfrak{p}_1}| = |\mathcal{V}_{\mathfrak{p}_2}|$ and $|\mathcal{E}_{\mathfrak{p}_1}| = |\mathcal{E}_{\mathfrak{p}_2}|$. Hence they also have the same number of loops.

Definition 2.5 (Gluing map). A gluing map $\gamma : \mathfrak{p}_{\bar{v}_1} \rightarrow \mathfrak{p}_{\bar{v}_2}$ is a map identifying elementwise two bondable patches such that:

$$\bar{v}_1 \longrightarrow \bar{v}_2 \quad \mathcal{V}_{\bar{v}_1} - \{\bar{v}_1\} \longrightarrow \mathcal{V}_{\bar{v}_2} - \{\bar{v}_2\} \quad \mathcal{E}_{\bar{v}_1} \longrightarrow \mathcal{E}_{\bar{v}_2} \quad (2.4)$$

where $e_1 = (\bar{v}_1, \tilde{v}_1) \in \mathcal{E}_{\bar{v}_1} \longrightarrow e_2 = (\bar{v}_2, \tilde{v}_2) \in \mathcal{E}_{\bar{v}_2}$ for any identification of vertices $\tilde{v}_1 \in \mathcal{V}_{\bar{v}_1} \longrightarrow \tilde{v}_2 \in \mathcal{V}_{\bar{v}_2}$.

A simple example is given in Figure 3.

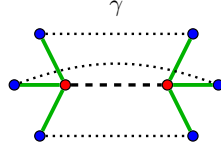


Figure 3: A gluing map identifying two bondable patches.

Definition 2.6 (Spin foam molecule). A spin foam molecule \mathfrak{m} is a triple of vertices, edges and faces $\mathfrak{m} = (\mathcal{V}_{\mathfrak{m}}, \mathcal{E}_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}})$ constructed from a set of spin foam atoms quotiented by a set of gluing maps.

That is, the gluing maps enforce the bonding relations between the spin foam atoms forming a spin foam molecule. To clarify the definition, let us provide an example. Let us consider two spin foam atoms \mathfrak{a}_1 and \mathfrak{a}_2 with their respective (bisected) boundary graphs \mathfrak{b}_1 and \mathfrak{b}_2 and boundary patches $\mathfrak{p}_{\bar{v}_1}$ and $\mathfrak{p}_{\bar{v}_2}$. By taking the quotient of the two spin foam atoms with the gluing map γ we identify the relevant structures and associated sets thus obtaining a spin foam molecule \mathfrak{m} where:

$$\mathcal{V}_{\mathfrak{m}} = \frac{\{\mathcal{V}_{\mathfrak{a}_1}, \mathcal{V}_{\mathfrak{a}_2}\}}{\gamma} \quad \mathcal{E}_{\mathfrak{m}} = \frac{\{\mathcal{E}_{\mathfrak{a}_1}, \mathcal{E}_{\mathfrak{a}_2}\}}{\gamma} \quad \mathcal{F}_{\mathfrak{m}} = \frac{\{\mathcal{F}_{\mathfrak{a}_1}, \mathcal{F}_{\mathfrak{a}_2}\}}{\gamma} \quad (2.5)$$

where the quotient by γ denotes the union of the relevant sets after identifying the corresponding elements.

The boundary map mentioned above can be extended to spin foam molecules and used to identify their boundary graphs, resulting from their individual boundary graphs after the application of the gluing maps.

So far we have been general concerning the allowed types of spin foam atoms and molecule. The next step is to introduce further restrictions on their combinatorial properties, focusing on the case of more direct relevance for the construction of 4d gravity models based on the discretization of Holst-Plebanski gravity. We focus on loopless, n -regular and simplicial structure.

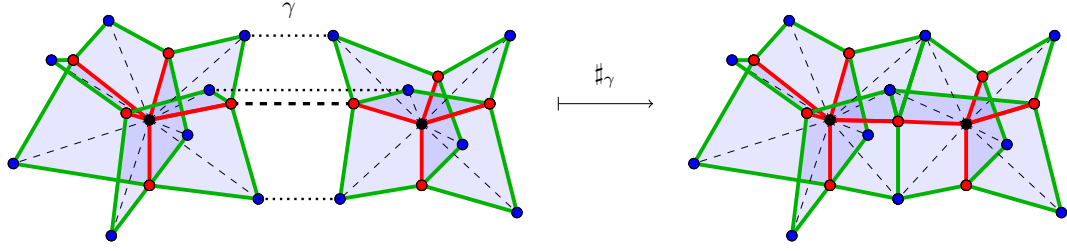


Figure 4: The gluing of two atoms along a shared boundary patch to form a molecule.

Definition 2.7 (Loopless and n -regular structures). A loopless graph is an element $\mathfrak{b} \in \mathfrak{B}$ such that:

$$\forall \bar{v} \in \bar{\mathcal{V}} \text{ s.t. } (\bar{v}\bar{v}) \notin \bar{\mathcal{E}} \quad (2.6)$$

We denote by \mathfrak{B}_L the set of all loopless bisected boundary graphs. They naturally give rise to loopless atoms \mathfrak{A}_L . By applying the gluing map γ , we can construct loopless spin foam molecules \mathfrak{M}_L . A n -regular bisected boundary graph is a graph $\mathfrak{b} \in \mathfrak{B}$ whose nodes $\bar{v} \in \bar{\mathcal{V}}$ are always n -valent. The set of n -regular bisected graphs is denoted by \mathfrak{B}_N . As before we can construct the sets of n -regular atoms \mathfrak{A}_N and molecules \mathfrak{M}_N .

Combining the two above requirements leads us to a much smaller set of graphs $\mathfrak{B}_{L,N}$, atoms $\mathfrak{A}_{L,N}$ and molecules $\mathfrak{M}_{L,N}$. The simplest case is that of simplicial structures.

Definition 2.8 (n -simplicial structures). The set of n -simplicial molecules $\mathfrak{M}_{N,S}$ consists of the set of all molecules obtained as gluings of a single (simplicial) spin foam atom $\mathfrak{a}_{N,S}$ labelled by the complete graph with $n+1$ vertices K_{n+1} .

$$\mathfrak{a}_{N,S} = (\alpha \circ \beta)(\mathfrak{c}_{N,S}) = (\alpha \circ \beta)(K_{n+1}) \quad (2.7)$$

An illustrative example is presented below.

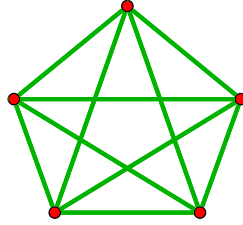


Figure 5: The complete graph with five vertices, labelling the 4-simplex (atom).

Notice that we call ‘simplicial’ the above-defined spin foam molecules because each spin foam atom in itself can be canonically understood as the dual 2-skeleton of an n -simplex. However, this can be done only locally; it has been proven [48] that not every simplicial spin foam molecule can be associated uniquely to a well-defined simplicial complex, as its dual 2-skeleton.

While the restriction to simplicial structures is motivated (in addition to simplicity) by the greater geometric understanding of the corresponding models with respect to those based on non-simplicial complexes, we stress that they remain a special case of a more general formalism and that:

- The use of more general cellular complexes is suggested by canonical Loop Quantum Gravity [72];
- Arbitrary cellular complexes can also be accommodated in the Group Field Theory formulation of spin foam models [71], using techniques from dually weighted tensor models [17].

Having provided the combinatorial foundations let us now review the construction of spin foam models. Again, we provide a general description, without referring (yet) to specific models.

Definition 2.9 (Spin foam model). A spin foam model is a quantum theory prescribed by the assignment of a quadruple $(\mathcal{H}, \mathfrak{M}, W, \mathcal{A})$ and defined by a partition function of the following form:

$$\mathcal{Z}_{\text{SF}} = \sum_{\substack{\mathfrak{m} \in \mathfrak{M} \\ \partial \mathfrak{m} = \emptyset}} W(\mathfrak{m}) \mathcal{A}(\mathfrak{m}) \quad (2.8)$$

- \mathcal{H} is the Hilbert space of states associated to each boundary patch of the spin foam atoms forming the spin foam molecule \mathbf{m} . We will give its
- \mathfrak{M} is the set of spin foam molecules or one of its subsets .
- $\mathcal{A}(\mathbf{m})$ is the *spin foam amplitude* associated by each given spin foam model to the spin foam molecule \mathbf{m} and that is obtained by suitable traces in the Hilbert space \mathcal{H} , as we will detail below.
- $W(\mathbf{m})$ is a further measure factor that weights \mathbf{m} in the sum over all molecules, depending only on the combinatorial structure of the same molecule.

The distinction between W and \mathcal{A} is quite arbitrary at this stage, and one could absorb W in \mathcal{A} . However, we distinguish them here in order to highlight the fact that, while \mathcal{A} can be motivated, with the ambiguities we will emphasize in the following, purely by considering the discretization and quantization of some continuum (gravitational) theory (e.g. for the models we will focus on, the Holst-Plebanski gravity) the prescription for $W(\mathbf{m})$ should come from a different line of reasoning. For example, the Group Field Theory formulation of spin foam models provides a complete (field-theoretic) prescription for both $\mathcal{A}(\mathbf{m})$ and $W(\mathbf{m})$.

Spin foam molecules have been taken above to be without boundary; this corresponds indeed to the definition of the partition function of a spin foam model. The latter can then be used to assign quantum amplitudes to observables and states, as in any sum-over-histories formulation of a quantum mechanical system. In this case, one deals with spin foam molecules with boundary, and the formula given above has to be suitably adapted. We will now detail the general definition of \mathcal{H} , of the Hilbert spaces built from it and useful for the spin foam construction, and of \mathcal{A} .

2.1.1 Hilbert spaces of spin foam states

The quantum states for which spin foam models define probability amplitudes are associated to the boundary graphs of spin foam molecules. The elementary building blocks of such quantum states are associated to boundary patches, and the Hilbert space associated to a single patch is labeled $\mathcal{H}_{\mathbf{p}}$. Given that a boundary patch is uniquely identified by the (boundary) vertex \bar{v} , we can equivalently label the same Hilbert space as $\mathcal{H}_{\bar{v}}$. This is a primary ingredient of the formalism, not derived from any other, and its choice is part of the definition of specific spin foam models.

As far as the general spin foam formalism goes, the Hilbert space $\mathcal{H}_{\bar{v}}$ could be anything, with the only combinatorial proviso that it should assign data to each of the edges $(\bar{v}, \tilde{v}) \in \mathcal{E}_{\mathbf{p}}$ of the boundary patch it corresponds to. In the simplicial case, there are n such edges, if the boundary patch is to be dual to a $(n-1)$ -simplex, and the spin foam atom (having $n+1$ such simplices on its boundary) to a n -simplex. This is necessary in order to correctly implement the bonding relations at the quantum level and to keep track of the combinatorial and topological structure of the resulting spin foam molecule \mathbf{m} . The latter requirement also asks for keeping track of the combinatorial structure of the spin foam atom, i.e. of the relations between the edges of the boundary patches in its bisected boundary graph. This can be done at the level of the Hilbert spaces associated to each spin foam atom or directly in the definition and construction of the spin foam amplitudes $\mathcal{A}(\mathbf{m})$. For simplicity, we will choose the second route. In any case, one way to ensure that the Hilbert space $\mathcal{H}_{\bar{v}}$ for a boundary patch encodes such information is to employ a patch Hilbert space which is the tensor product of Hilbert spaces associated to its edges:

$$\mathcal{H}_{\bar{v}} = \bigotimes_{(\bar{v}\tilde{v}) \in \mathcal{E}} \mathcal{H}_{(\bar{v}\tilde{v})} \quad . \quad (2.9)$$

Given a Hilbert space for each boundary patch, each spin foam atom can be associated straightforwardly a Hilbert space $\mathcal{H}_{\mathbf{a}}$ defined as the simple tensor product of the Hilbert spaces associated to its boundary patches:

$$\mathcal{H}_{\mathbf{a}} = \bigotimes_{\mathbf{p} \in \partial \mathbf{a}} \mathcal{H}_{\mathbf{p}} \quad . \quad (2.10)$$

This is the ambient Hilbert space for the boundary states associated to each spin foam atom. As mentioned above concerning their dynamics, it may be useful to think of the same states as associated to the bisected boundary graphs and to encode their combinatorial structure more explicitly. The specific way to do so depends of course on the nature of the patch Hilbert space and on the spin foam model considered. The general recipe is to introduce operators $O_{\bar{v}}$ associated to the bivalent vertices of the bisected boundary graph, acting on the Hilbert space associated to the (half-)edges incident to the same bivalent vertices and mapping them to a single Hilbert space, with the same constituting data but associated then to the whole edge of the boundary graph:

$$O_{\bar{v}} : \mathcal{H}_{(\bar{v}_1\tilde{v})} \otimes \mathcal{H}_{(\bar{v}_2\tilde{v})} \longrightarrow \mathcal{H}_{(\bar{v}_1\tilde{v}_2)} \quad \tilde{v} \in \mathbf{p}_1, \mathbf{p}_2, \quad (\bar{v}_1\tilde{v}) \in \mathbf{p}_1, \quad (\bar{v}_2\tilde{v}) \in \mathbf{p}_2 \quad (2.11)$$

Obviously, these operators require a tensor product structure for the patch Hilbert spaces, as in $\tilde{\text{refH}}$ -patch-tensor. Using such operators for all the bivalent vertices of a bisected boundary graph one obtains a projection map from the spin foam atom Hilbert space defined above into a Hilbert space $\mathcal{H}_{\mathfrak{b}} \subset \mathcal{H}_{\mathfrak{a}}$ that carries the information about the combinatorics of the boundary graph \mathfrak{b} for the same spin foam atom (with the naturally induced inner product):

$$\mathcal{O}_{\mathfrak{b}} \equiv \times_{\tilde{v} \in \mathfrak{b}} \mathcal{O}_{\tilde{v}} : \mathcal{H}_{\mathfrak{a}(\mathfrak{b})} \longrightarrow \mathcal{H}_{\mathfrak{b}} \quad . \quad (2.12)$$

For explicit examples of such operators and projection maps, for states relevant to loop quantum gravity, see [74]. The same projection can be imposed at the level of the quantum dynamics, rather than the boundary states, i.e. at the level of the spin foam amplitudes acting on them. This is the case we focus on, working thus with the simpler Hilbert spaces 2.10.

The gluing maps identify boundary patches between different spin foam atoms, when bonding them to form spin foam molecules, and can be associated to gluing operators which identify their quantum data as well. The glued patches are not part of the resulting boundary of the spin foam molecule, to which a Hilbert space is associated. This means that the gluing operators amount to tracing away the quantum data of the identified boundary patches, thus removing any dependence on the associated Hilbert spaces. The Hilbert space of the spin foam molecule is then the simple tensor product of the remaining boundary patches:

$$\mathcal{H}_{\mathfrak{m}} = \bigotimes_{\mathfrak{p} \in \partial \mathfrak{m}} \mathcal{H}_{\mathfrak{p}} \quad . \quad (2.13)$$

This concludes the list of relevant Hilbert spaces for any given spin foam model, since for any chosen boundary one can choose the relevant boundary Hilbert space to which a (set of) spin foam amplitudes can then be associated. This is all is needed to compute with the (boundary extension of the) formula 2.8. Before we move to the definition of the spin foam amplitudes themselves, we digress briefly to discuss the issue of how to define a single Hilbert space for a spin foam model, that would include any possible choice of boundary (thus of boundary graphs) in its definition. This issue is relevant for any operator/states formulation of spin foam models that aims at incorporating arbitrary numbers of quantum degrees of freedom (as likely necessary for a continuum limit connecting to relativistic field theories) and for removing any reliance on fixed discrete structures. It is also crucial for relating the formalism to canonical quantum gravity, in particular loop quantum gravity.

The simplest extension of the Hilbert space for individual spin foam molecules that accommodate any spin foam boundary into a single Hilbert space is the direct sum of tensor products of arbitrary numbers of boundary patches, or, *choosing* a bosonic statistics, the Fock space:

$$\mathcal{H}_{SF}^1 = \bigoplus_{N=0}^{\infty} \text{Sym} \left(\bigotimes_{\mathfrak{p}_i / i=1, \dots, N} \mathcal{H}_{\mathfrak{p}_i} \right) \quad . \quad (2.14)$$

This is the Hilbert space of (bosonic) group field theories (working with the unsymmetric pre-Fock space is also a possibility, but it does not lead to a nice QFT framework) and of the formulation of spin foam models defined via their perturbative expansion. Its is a natural choice from the perspective of standard QFT and from the one that sees quantum spacetime as a peculiar quantum many-body system and gravity as emergent [75]. Coming from the perspective of canonical loop quantum gravity, the focus is instead on the Hilbert spaces associated to (boundary) graphs $\mathcal{H}_{\mathfrak{b}}$. This, as we have seen, can be embedded into the tensor product Hilbert spaces associated to boundary patches, and thus in the above Fock space.

Another possibility, emphasizing such graph Hilbert spaces is to define a Hilbert space from the union of all possible graph Hilbert spaces $\bigcup_{\mathfrak{b}} \mathcal{H}_{\mathfrak{b}}$. The simplest way to do so is to turn the union into a direct sum:

$$\mathcal{H}_{SF}^2 = \bigoplus_{\mathfrak{b} \in \mathfrak{B}} \mathcal{H}_{\mathfrak{b}} \quad . \quad (2.15)$$

This is indeed a choice that can be found in the spin foam literature. Closer to the canonical loop quantum gravity construction is the alternative one, based on imposing appropriate equivalence relations among graph-based Hilbert spaces, which also imply an embedding of any such Hilbert space into one defined for a finer (more vertices, more edges) graph. The imposition of such *cylindrical equivalence* relations defines the Hilbert space:

$$\mathcal{H}_{SF}^3 = \underbrace{\bigcup_{\mathfrak{b} \in \mathfrak{B}} \mathcal{H}_{\mathfrak{b}}}_{\sim} \quad . \quad (2.16)$$

and leads indeed, for appropriate choice of $\mathcal{H}_{\mathbf{p}}$ and of equivalence relations \sim , to the same Hilbert space used in loop quantum gravity. At the spin foam level the task becomes however to show that the spin foam amplitudes are compatible with the same equivalence relations. We will not be concerned any further with these larger Hilbert spaces, however, and turn instead to the construction of spin foam amplitudes themselves (for more details on this issue, see [74]).

2.1.2 Spin foam amplitudes

The abstract definition of the spin foam amplitude $\mathcal{A}_{\mathbf{m}}$ for the spin foam molecule \mathbf{m} can now be detailed as follows. It uses the fact that the spin foam molecule itself is *oriented*, with a consistent orientation of the edges and faces, and of boundary graphs. Whether or not this orientation is reflected in the associated amplitudes is one of the choices entering model building.

In our operator spin foam formalism, a spin foam model is specified by a set of operators that define maps between the Hilbert spaces associated to the boundary patches. The basic one is the vertex operator associated to each spin foam atom:

$$O_v \equiv V_{\mathbf{a}} : \bigotimes_{\mathbf{p}_{in} \in \partial \mathbf{a}} \mathcal{H}_{\mathbf{p}_{in}} \longrightarrow \bigotimes_{\mathbf{p}_{fin} \in \partial \mathbf{a}} \mathcal{H}_{\mathbf{p}_{fin}} , \quad (2.17)$$

where we have split the Hilbert space associated to the spin foam atom according to a split of its boundary patches into ‘initial’ and ‘final’ ones, reflecting the orientation chosen for the atom itself. Assuming the isomorphism $\mathcal{H}_{\mathbf{p}} \simeq \mathcal{H}_{\mathbf{p}}^*$ (where the duality also reflects the change in orientation), the same operator defines the function:

$$\mathcal{V}_{\mathbf{a}} : \bigotimes_{\mathbf{p} \in \partial \mathbf{a}} \mathcal{H}_{\mathbf{p}} \longrightarrow \mathbb{C} \quad (2.18)$$

which we call *vertex kernel* and which gives, when applied to any basis in the Hilbert spaces $\mathcal{H}_{\mathbf{p}}$, the generalised ‘matrix elements’ of the operator $V_{\mathbf{a}}$. It should be clear from the definition that the set of such functions is in one-to-one to the (bisected) boundary graphs \mathfrak{B} , since they account for how patches are combined in the boundary of spin foam atoms, i.e. how patch edges are glued to form such graphs.

Similarly, one can associate a gluing operator to each gluing map $\gamma : \mathbf{p}_{\bar{v}_1} \rightarrow \mathbf{p}_{\bar{v}_2}$ between spin foam atoms, as a map between the corresponding patch Hilbert spaces:

$$O_{\gamma} \equiv O_e : \mathcal{H}_{\mathbf{p}_{\bar{v}_1}} \longrightarrow \mathcal{H}_{\mathbf{p}_{\bar{v}_2}} \quad (2.19)$$

and a corresponding *gluing kernel* which coincides with its matrix elements when evaluated in any given basis:

$$\mathcal{K}_{\gamma} \equiv \mathcal{K}_e : \mathcal{H}_{\mathbf{p}_{\bar{v}_1}} \otimes \mathcal{H}_{\mathbf{p}_{\bar{v}_2}} \longrightarrow \mathbb{C} \quad e \equiv (\bar{v}_1, \bar{v}_2) \quad (2.20)$$

where again we have identified the patch Hilbert space with its dual.

The gluing operators can be composed with the vertex operators, on the basis of their action on shared patch Hilbert spaces, and this in turn allows to compose vertex operators for different spin foam atoms, to obtain operators associated to spin foam molecules. Consider for example two spin foam atoms \mathbf{a}_1 , with boundary patches $\mathbf{p}_1, \dots, \mathbf{p}_n$, and \mathbf{a}_2 , with boundary patches $\mathbf{p}_{n+1}, \dots, \mathbf{p}_{n+m}$ (one can take $n = m$, for the regular case), glued along the two boundary patches \mathbf{p}_n and \mathbf{p}_{n+1} by the map $\gamma_{n,n+1}$. The composite operator $V_{\mathbf{a}_1} O_{n,n+1} V_{\mathbf{a}_2}$ is the operator associated to the resulting spin foam molecule, defining a map between the boundary Hilbert spaces, built out of the ones of the two individual spin foam atoms, with the patches \mathbf{p}_n and \mathbf{p}_{n+1} having been removed (since they are now in the interior of the cellular complex). Equivalently, one can associate to the spin foam molecule the composite function $\mathcal{V}_{\mathbf{a}_1} \circ \mathcal{K}_{n,n+1} \circ \mathcal{V}_{\mathbf{a}_2} : \bigotimes_{\mathbf{p}_i \in \partial \mathbf{m} | i=1, \dots, n-1, n+2, \dots, n+m} \mathcal{H}_{\mathbf{p}_i} \longrightarrow \mathbb{C}$, where \circ indicates convolution on the shared patch Hilbert spaces, thus a summation over a basis of states in the same Hilbert spaces, defining a resolution of the identity.

One can generalise from the above construction to associate a spin foam operator to any spin foam molecule, defining a map from an ‘initial’ multi-patch Hilbert space to a ‘final’ one, whose matrix elements can be interpreted as a spin foam transition amplitude between the corresponding quantum states. It should be clear that, in the case of spin foam molecules without boundaries, thus corresponding to closed cellular complexes, the same prescription produces a trace of the full spin foam operator over the last patch Hilbert space identified by the gluing map. One thus associates a (complex) number to the full closed spin foam molecule, rather than an operator, consistently with the fact that now all patches lie in the interior of the dual cellular complex. This number can be understood as a spin foam partition function associated to an individual molecule, the key ingredient of

the full spin foam model, incorporating a sum over all such molecules as in 2.8.

The general formula for the spin foam amplitude associated to a generic molecule, and depending on its combinatorial structure, i.e. the connectivity pattern between spin foam atoms and their subcells, is best given in terms of the vertex and gluing kernels:

$$\mathcal{A}(\mathbf{m}) = \text{Tr}_{\mathfrak{p} \in \mathbf{m}} \left(\prod_{\gamma | \mathbf{m}} \mathcal{K}_{\gamma} \prod_{\alpha \in \mathbf{m}} \mathcal{V}_{\alpha} \right) \quad (2.21)$$

which are bound together and evaluated by a tracing operation over a complete basis in each of the shared patch Hilbert spaces (producing the convolution of the corresponding functions evaluated in the same basis). An equivalent way of expressing the same amplitude is by labelling the vertex kernels by the vertices defining each spin foam atom (they are then called ‘spin foam vertex amplitudes’) and the gluing kernels by the edges resulting from the gluing of spin foam atoms (they are then called ‘edge amplitudes’). When the factorized form 2.9 for the patch Hilbert space is used, the tracing operation takes place in each factor associated to each boundary edge of the patch. Following the gluing pattern effected by the gluing maps, one identifies a closed cycle and thus a spin foam face associated to the same patch (for internal patches), and one can then single out a contribution to the spin foam amplitude associated to such internal face. The final spin foam amplitude can then also be written in terms of individual contributions associated to the faces, edges and vertices of the spin foam molecule. This remains just an equivalent rewriting of the general formula 2.21.

2.2 The GFT formulation of spin foam models

The construction of spin foam amplitudes presented in the previous subsection is completely general but, as we have stressed, it does not deal with all the ingredients defining a spin foam model. In particular, it says nothing about how to specify the operators defining the same amplitudes (or the corresponding kernels), and it does not give a general rule for defining the additional combinatorial weights $W(\mathbf{m})$ entering the sum over spin foam molecules completing the definition of the spin foam model, once the amplitudes have been chosen. This second issue is important also because the sum over spin foam molecules is, as we have mentioned, a way to define implicitly the continuum limit of the given spin foam model. We will discuss the first issue at length in the remainder of this paper, presenting the guidelines, ambiguities and strategies of model building for the class of spin foam models inspired by the formulation of gravity as a constrained BF theory (in the Riemannian setting), and introducing as well as new spin foam model in the same class. We will not deal much, instead, with the second issue. We limit ourselves to outlining the general formulation of spin foam models in the language of Group Field Theory (GFT), which is one solution to it, and provides a complete definition of such models (as well as new technical and conceptual tools for dealing with them). For more details, we refer again to [71].

In a sentence, spin foam models including the sum over spin foam molecules arise as the perturbative Feynman expansion of a combinatorially non-local quantum field theory, with the spin foam amplitudes being the Feynman amplitudes obtained, together with any additional combinatorial weight, by standard Feynman rules and the combinatorial structures derived by the non-local pairing of field arguments in the interactions. Restricting our attention, for simplicity, to the n -regular case, the correspondence between the ingredients of spin foam models as outlined in the previous subsection and GFT ones works as follows.

The Hilbert space of the GFT theory, assuming once more bosonic statistics (other choices are possible), is the Fock space 2.14; thus, the quanta of the field theory are associated to the patches \mathfrak{p} and have individual Hilbert space $\mathcal{H}_{\mathfrak{p}}$, which may have in turn the tensorial structure 2.9. Assuming that this patch Hilbert space is an L^2 space over a domain $\mathcal{D} \simeq \times_i \mathcal{D}_i$, itself a direct product of sub-domains to match the tensorial nature of the Hilbert space, the GFT field is also a complex L^2 function $\phi : \mathcal{D} \rightarrow \mathbb{C}$. In the following, we focus on real fields for simplicity of presentation. The subdomain \mathcal{D}_i is usually taken to be a (Lie) group manifold, hence the name of the formalism. GFT observables are then indexed by (bisected) boundary graphs \mathfrak{B} , being defined as convolutions of GFT fields corresponding to a number of patches convoluted with appropriate kernels encoding how the same patches are glued along their edges to form such graphs:

$$\mathcal{O}_{\mathfrak{b}}[\phi] = \int [dg] \mathcal{O}_{\mathfrak{b}}(\{g_{\bar{v}}\}_{\bar{\mathfrak{V}}}) \prod_{\bar{v} \in \bar{\mathfrak{V}}} \phi_{\mathfrak{p}_{\bar{v}}}(g_{\bar{v}}) \quad (2.22)$$

and the possible GFT interactions are taken from the set of such observables. In particular, they can be put in correspondence with the vertex operators for spin foam atoms, defining spin foam amplitudes, by identifying

vertex kernels with GFT interaction kernels:

$$\mathcal{O}_a[\phi] = \lambda_b \int [dg] \mathcal{V}_b(\{g_{\bar{v}}\}) \prod_{\bar{v} \in \bar{\mathcal{V}}} \phi_{\mathfrak{p}_{\bar{v}}}(g_{\bar{v}}). \quad (2.23)$$

Finally, the gluing operators can be also encoded in the GFT dynamics by identifying their kernels with the GFT propagators, thus the GFT kinetic kernels with their inverse:

$$\frac{1}{2} \int [dg] \phi_{\mathfrak{p}}(g_{\bar{v}_1}) \mathcal{K}_{\mathfrak{p}}(g_{\bar{v}_1}, g_{\bar{v}_2}) \phi_{\mathfrak{p}}(g_{\bar{v}_2}), \quad \text{where} \quad \mathcal{K}_{\mathfrak{p}}(g_{\bar{v}_1}, g_{\bar{v}_2}) = \mathcal{K}(\{g_{\bar{v}_1 \bar{v}}, g_{\bar{v}_2 \bar{v}}\}) = \mathcal{G}^{-1}(g_{\bar{v}_1}, g_{\bar{v}_2}) \quad (2.24)$$

is a function of group elements for each $(\bar{v}_i \hat{v}) \in \mathcal{E}_{\mathfrak{p}}$. In this way, the GFT model defined by the action

$$S[\phi] = \frac{1}{2} \int [dg] \phi_{\mathfrak{p}}(g_{\bar{v}_1}) \mathcal{K}_{\mathfrak{p}}(g_{\bar{v}_1}, g_{\bar{v}_2}) \phi_{\mathfrak{p}}(g_{\bar{v}_2}) + \sum_{b \in \mathfrak{B}} \lambda_b \int [dg] \mathcal{V}_b(\{g_{\bar{v}}\}) \prod_{\bar{v} \in \bar{\mathcal{V}}} \phi_{\mathfrak{p}_{\bar{v}}}(g_{\bar{v}}) \quad (2.25)$$

and partition function

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-S[\phi]} \quad (2.26)$$

corresponds to the spin foam models defined by the same operator kernels \mathcal{K} and \mathcal{V}_b in the sense that observables can be estimated perturbatively, generating Feynman series of the type:

$$\langle \mathcal{O}_{b_1} \cdots \mathcal{O}_{b_l} \rangle_{\text{GFT}} = \frac{1}{\mathcal{Z}_{\text{GFT}}} \int \mathcal{D}\phi \mathcal{O}_{b_1}[\phi] \cdots \mathcal{O}_{b_l}[\phi] e^{-S[\phi]} = \sum_{\substack{\mathfrak{m} \in \mathfrak{M} \\ \delta(\mathfrak{m}) = \sqcup_{i=1}^l b_i}} \frac{1}{C(\mathfrak{m})} \mathcal{A}(\mathfrak{m}, \{\lambda_b\}_{\mathfrak{B}}) \quad (2.27)$$

with the Feynman diagrams characterized as spin foam molecules, the combinatorial weight in front of the spin foam amplitude depending on the automorphism group of each diagram, and the convolution of gluing (propagator) kernels with vertex kernels for spin foam atoms arising naturally from Wick contractions. The simplest case corresponds, of course, to the partition function itself.

3 Spin foam models for constrained BF theory

Having given the general definition and structure of spin foam models, we now focus on the class of gravitational (or geometrical) spin foam models arising from the Holst-Plebanski formulation of GR as a constrained BF theory in $4d$. We restrict ourselves to the Riemannian setting and to simplicial structures. Extension to the Lorentzian context and to arbitrary polyhedral complexes can be found in the literature [20, 23, 24, 29, 72]. In particular, we discuss the issues concerning the imposition of the geometricity constraints on simplicial structures (the quantum and discrete counterpart of the Holst-Plebanski constraints), which give rise to different spin foam models, and we write down the resulting spin foam amplitudes in different representations. We also emphasize a number of construction ambiguities that can be seen to give rise to variations of the same spin foam models, with different analytic properties. Regardless of the choice of variables, we will emphasize how these share the same universal structure, differing only by the analytical expression of the model-dependent coefficients encoding the geometricity constraints (usually referred to as ‘simplicity’ constraints).

3.1 Gravity as a constrained BF theory

Let us briefly review the Holst-Plebanski formulation of GR as a constrained BF theory (in the Lagrangian formalism), both at the classical continuum and discrete level, focusing, for shortness, on the small subset of results directly relevant for quantum gravity model building in the Spinfoam (and GFT) context [76–83].

Consider first the Palatini-Holst Lagrangian in natural units without cosmological constant.

$$S_{\text{HP}}[e, \omega, \lambda] = \int_{\mathcal{M}} *(e^I \wedge e^J) \wedge F^{IJ}[\omega] + \frac{1}{\gamma} e^I \wedge e^J \wedge F_{IJ}[\omega] \quad (3.1)$$

where ω is a Spin(4)-valued connection one-form field, $F[\omega]$ is the curvature two-form and e is a $\mathfrak{spin}(4)$ -valued tetrad one-form field representing an orthonormal frame. When there are no fermions the term involving the Barbero-Immirzi parameter γ , known as the Holst term, is irrelevant at the classical level since it vanishes on shell. It is, however, of fundamental importance in Loop quantum gravity (LQG) as well as in the formulation of Spinfoam models for quantum gravity aiming to provide, broadly speaking, a covariant counterpart to the LQG canonical framework [84]. The Palatini-Holst action (3.1) can be recovered, on shell, from a topological BF theory action with additional polynomial constraints $C_\alpha[B]$ called simplicity constraints.

$$S[B, \omega, \lambda] = \int B^{IJ} \wedge F_{IJ}[\omega] + \lambda^\alpha C_\alpha[B] \quad (3.2)$$

Indeed, upon taking the variation with respect to the Lagrange multiplier λ^α , they enforce the field B to be of the form (3.2), thus turning the BF action to the Holst-Palatini action for $4D$ gravity in the first-order formalism. The variation with respect to the connection field, instead, yields us the Gauß constraint. As we said BF theory is purely topological. Having non local degrees of freedom the quantization of such theories is rather simple and well understood, thus providing a convenient starting-point for the construction of $4d$ Spinfoam models for quantum gravity. The issue is then the correct implementation of the appropriate set of constraints leading back to (3.2), either at the level of the quantum states or in a path integral formulation. Indeed the bulk of the work in the spinfoam approach (and in GFT as well), in recent years, has been devoted to this task [1]. Let us first consider the following quadratic simplicity constraints.

$$\epsilon_{IJKL}(B - \gamma^* B)_{ab}^{IJ}(B - \gamma^* B)_{cd}^{KL} = V \epsilon_{abcd} \quad V \equiv \frac{1}{4!} \epsilon^{abcd} \epsilon_{IJKL}(B - \gamma^* B)_{ab}^{IJ}(B - \gamma^* B)_{cd}^{KL} \quad (3.3)$$

The traditional way of imposing such restrictions is to add them to the BF action which therefore reads:

$$S_{\text{HP}}[B, \omega, \lambda] = \int \left(B^{IJ} \wedge F_{IJ}[\omega] + \frac{\gamma^2}{(1 - \gamma^2)^2} \lambda_{IJKL} (B - \gamma^* B)^{IJ} \wedge (B - \gamma^* B)^{KL} \right) \quad (3.4)$$

Under the assumption $V \neq 0$, the equation (3.3) admits the following four classes of solutions:

$$B^{IJ} = \pm *(e^I \wedge e^J) + \frac{1}{\gamma} e^I \wedge e^J \quad B^{IJ} = \pm e^I \wedge e^J + \frac{1}{\gamma} *(e^I \wedge e^J) \quad (3.5)$$

In principle one would like to select only the first class of solutions, which upon substitution, yields the Holst-Palatini action (3.1). Interestingly General relativity can also be formulated as a constrained BF theory with

linear simplicity constraints. Among other features, this formulation has the advantage of picking up the right sector of solutions. The constraints read:

$$k_{Icd}(B - \gamma^* B)_{ab}^{IJ} = 0 \quad \{a, b\} \subset \{c, d, l\} \quad k_{Ic[dl}(B - \gamma^* B)_{a]b}^{IJ} = 0 \quad \{a, b\} \cap \{c, d, l\} = \emptyset \quad (3.6)$$

where, for non degenerate tetrads, $k_{Icd} \equiv k_{I[cd]} \equiv \epsilon_{IPKL} e_c^P e_d^K e_l^L$ can be interpreted as a $3d$ volume form for the submanifold parametrized by the coordinates (x^c, x^d, x^l) embedded in the $4d$ spacetime \mathcal{M} , whose internal index I gives the normal to this $3d$ submanifold. The new variables k_I provide a basis of three-forms at each spacetime point and thus can be used as an alternative choice to the tetrad field. The first condition in (3.6) is the linear simplicity constraint while the second one is the linear volume constraint. As shown in [78] these two sets of constraints are indeed sufficient to ensure the simplicity of the two form B^{IJ} which is therefore given by the first expressions in (3.5). The resulting formulation of the Holst-Plebanski action with linear constraint is given by:

$$S_{\text{HP}}[B, \omega, \Xi] = \frac{\gamma}{1 - \gamma^2} \int \left(\frac{1}{4} (B - \gamma^* B)^{IJ} \wedge F_{IJ}[\omega] + \frac{\gamma}{8} \epsilon_{IJKL} (B - \gamma^* B)^{IJ} \wedge F^{KL} + \Xi^{abcd} k_{Icd}(B - \gamma^* B)_{ab}^{IJ} \right) \\ \sum_{acd} \delta_{bc} \epsilon_{adl} \Xi_I^{abcd} = 0 \quad \Xi_I^{abcd} = \Xi_I^{[ab][cd]} \quad (3.7)$$

It is easy to show that, upon substituting back the solution of the constraint Eqs. (3.6), we recover GR.

As we saw earlier, Spinfoam models are usually defined in a piecewise flat context. In this paper we restrict ourselves to discrete structures (i.e. cellular complexes) of simplicial type. Once a simplicial decomposition (triangulation) $\mathfrak{m} \in \mathfrak{M}_S$ of the spacetime has been chosen, the neatest and most direct approach to the quantization of the Holst-Plebanski action would be to encode the simplicity constraints into the definition of the measure in the simplicial path integral reformulation of the corresponding model's lattice amplitude $\mathcal{A}(\mathfrak{m})$. Nevertheless the standard model building strategy, leading to the currently known models, consists of two steps:

1. First discretize and quantize the topological part of the classical (constrained) BF Action (3.2).
2. Then implement, directly at the quantum level, a suitable version of the required geometricity constraints.

Focusing on the first item, we illustrate the procedure in the simplest case of a single (simplicial) Spinfoam atom $\mathfrak{a} \in \mathfrak{A}_S$. The generalization to more refined triangulations is straightforward. The $\text{Spin}(4)$ -valued spin connection ω appearing in the classical BF action is a one-form field. Thus in the discrete setting it can naturally be replaced by its holonomy (strictly speaking by its parallel transport) $H_e \equiv \in \text{Spin}(4)$ along the edges $e \equiv (\bar{v}v)$ of \mathfrak{a} . The field B , instead is a $\mathfrak{spin}(4)$ -valued two-form field. Its discrete counterpart is the lie algebra element $B_t \equiv B_{\bar{v}\bar{v}} \in \mathfrak{spin}(4) \simeq \Lambda^2 \mathbb{R}^4$, namely the bivector or flux variable obtained by smearing the original continuum field on the triangles of the 4-simplex. Summarizing the discrete BF action reads:

$$\omega \mapsto H_{\bar{v}v} \equiv \mathcal{P} e^{\int_e \omega_a dx^a} \quad B \mapsto X_{\bar{v}\bar{v}} \equiv \int_t B_{ab}^{IJ} dx^a \wedge dx^b \quad t \simeq (\bar{v}\tilde{v}) \in \mathfrak{p}_{\bar{v}} \quad e \equiv (\bar{v}v) \in \mathcal{E}_{\mathfrak{a}} \quad (3.8)$$

$$S_{\text{BF}}[H_{\bar{v}v}, X_{\bar{v}\bar{v}}] = \text{Tr} \left(\vec{\zeta}(H_f) X_{\bar{v}\bar{v}} \right) = \vec{\zeta}(H_f) \cdot X_{\bar{v}\bar{v}} \quad H_f \equiv \prod_{\bar{v}, v, \bar{v}' \in f} H_{\bar{v}v} H_{\bar{v}'v}^{-1} \quad f \equiv (\bar{v}v\bar{v}') \in \mathcal{F}_{\mathfrak{a}} \quad (3.9)$$

The coordinate on the group $\vec{\zeta}(H)$, selected by the choice of the quantization map, dictates the prescription for discretizing the curvature two-form. In case of the Duflo map, for instance, the expression of the discrete curvature involves the logarithm of the holonomy. The expression of the BF action for a simplicial molecule, follows from the previous one, valid for a single atom, after tracing out the variables in each of the shared patch Hilbert spaces (with respect to the same basis) according to Eq (2.21). When the patch Hilbert space takes the factorized form (2.9) the trace is evaluated in each factor associated to each boundary half-link of the patch. Following the molecule's combinatorial pattern one can then identify a closed spinfoam face (cycle) associated to the same patch and its corresponding face amplitude. Such amplitude depends on one bivector X_f (the one left after doing all the identifications) and on the individual holonomies along the half-edges binding the face.

After discussing the discretization of the BF action we now give the discrete version of the constraints (3.3).

$$\epsilon_{IJKL} (X_{\bar{v}\bar{v}} - \gamma^* X_{\bar{v}\bar{v}})^{IJ} (X_{\bar{v}\bar{v}'} - \gamma^* X_{\bar{v}\bar{v}'})^{KL} = 0 \quad \forall \tilde{v}, \tilde{v}' \in \mathfrak{p}_{\bar{v}} \quad (3.10)$$

$$\epsilon_{IJKL} (X_{\bar{v}\bar{v}} - \gamma^* X_{\bar{v}\bar{v}})^{IJ} (X_{\bar{v}'\bar{v}'} - \gamma^* X_{\bar{v}'\bar{v}'})^{KL} = \pm 12V(\mathfrak{a}) \quad \forall \tilde{v} \in \mathfrak{p}_{\bar{v}}, \quad \tilde{v}' \in \mathfrak{p}_{\bar{v}'}, \quad \bar{v}, \bar{v}' \in \bar{\mathcal{V}} \subset \mathcal{V}_{\mathfrak{a}} \quad (3.11)$$

where, in the last equation, V is interpreted, on the solution of the constraints, as the volume of the 4-simplex. The previous conditions, namely the discrete quadratic and volume simplicity constraints, form a set of second

class constraints for arbitrary values of the Immirzi parameter. In particular as shown in [85] the Volume constraint (3.11) is automatically satisfied if the other conditions have been consistently enforced in all tetrahedra of the 4-simplex. This is a useful fact since the closure constraint, being linear in the X 's and local in each tetrahedron, is clearly easier to impose at the discrete level and in the quantum theory. This concludes our short overview on the Holst-Plebanski formulation of GR with quadratic simplicity constraints both in the continuum and discrete framework.

In recent years the Spinfoam approach to quantum gravity has witnessed many interesting developments, ranging from the introduction of a whole new class of model with nice semiclassical properties and a tighter connection to canonical LQG to a preliminary investigation of their phase diagram, RG flow and continuum limit based on lattice and Tensor network techniques [32, 41, 42]. One of the main features of the new models is the replacement of the discrete quadratic simplicity constraints (3.10) by discrete linear constraints of the form:

$$(X_{\bar{v}\bar{v}}^{IJ} - \gamma^* X_{\bar{v}\bar{v}}^{IJ})k_{\bar{v}J} = 0 \quad \forall \bar{v} \in \mathfrak{p}_{\bar{v}}, \quad \bar{v} \in \mathcal{V}_{\mathfrak{a}} \quad (3.12)$$

where the variable $k_{\bar{v}}$ can be interpreted as the normal to the tetrahedron dual to the patch node \bar{v} . Concretely, using the canonical decomposition of bivectors into selfdual and anti-self dual $\mathfrak{su}(2)$ -components, we find:

$$k_{\bar{v}}x_{\bar{v}\bar{v}}^-k_{\bar{v}}^{-1} + \beta x_{\bar{v}\bar{v}}^+ = 0 \quad k_{\bar{v}}x_{\bar{v}\bar{v}}^-k_{\bar{v}}^{-1} = \epsilon_{jk}^i X_{\bar{v}\bar{v}}^{jk} - X_{\bar{v}\bar{v}}^{i0} \quad x_{\bar{v}\bar{v}}^+ = \epsilon_{jk}^i X_{\bar{v}\bar{v}}^{jk} + X_{\bar{v}\bar{v}}^{i0} \quad \beta = \frac{\gamma - 1}{\gamma + 1} \quad (3.13)$$

One could also write down a discrete version of the linear volume constraint, though this will not be needed here. Indeed, as we said earlier in the quadratic case, the volume constraints are automatically satisfied if the closure and the simplicity constraints (3.10) are correctly imposed¹. Since, for non-degenerate two-forms (bivectors), the quadratic constraints follow from the linear ones (3.13), as can be easily checked, we can safely replace them and neglect the redundant (quadratic) volume constraint according to the same argument. Thus we are left with a smaller set of independent conditions to be implemented in each 4-simplex, collectively denoted in the future as geometricity constraints.

Summarizing we reviewed the Holst-Plebanski formulation of GR as a constrained BF theory both with quadratic and linear simplicity constraints. We started with the continuum theory and then we moved to the discrete (simplicial) setting giving the corresponding expressions of the smeared BF action and constraint equations. The linear constraints (3.13) are slightly stronger than their quadratic counterparts and thus they can be used to replace them leading, together with the closure constraint, to a smaller set of geometricity conditions. The implementation of such constraints, as we mentioned earlier, will play a crucial role in the construction of Spinfoam models for quantum gravity. So far we have only addressed the first step of our model building programme. The next one, namely the quantization of the discrete BF theory's classical phase space and related geometricity constraints, will be discussed in the coming section.

3.2 The Hilbert space of boundary states

As shown in the previous section, the building block of spin foam quantum states (and amplitudes) is a choice of patch Hilbert space. In the simplicial setting, this corresponds to a Hilbert space for quantized tetrahedra. The association with tetrahedra is purely topological at first, since the states will encode the simplicial geometry of quantum tetrahedra only if appropriate conditions are imposed. So we start from states labelled by the variables of discrete BF theory, and move to the imposition of geometricity constraints next.

The patch Hilbert space $\mathcal{H}_{\mathfrak{p}} = \mathcal{H}_{\bar{v}}$ admits different realizations as an L^2 space, depending on the choice of variables. These different realizations are related by (generalised) Fourier transforms for Lie groups, which amount in the end to basis changes in the same Hilbert space [67, 68]. We consider three distinct choices of variables: Lie algebra elements, holonomies and spins. They define three equivalent formulations of spin foam models denoted as the *Flux*, *Holonomy* and *Spin representations*². In formulas we have:

$$\mathcal{H}_{\bar{v}} \equiv L_{\star}^2[\mathfrak{spin}(4)^{\times 4}] \otimes L^2[S^3] \quad \text{Flux representation.} \quad (3.14)$$

$$\mathcal{H}_{\bar{v}} \equiv L^2[\text{Spin}(4)^{\times 4}] \otimes L^2[S^3] \quad \text{Group representation.} \quad (3.15)$$

$$\mathcal{H}_{\bar{v}}^{\{J_i\}} \equiv \otimes_{i=1}^4 \mathcal{H}^{J_i} \quad \text{Spin representation.} \quad (3.16)$$

¹This is not true in the linear case. The linear volume constraint does not follow from the closure and the linear simplicity constraint, unless an additional "4d closure condition" involving the normals of the five tetrahedra is imposed.

²Strictly speaking, the label 'spin' is appropriate only when the Lie group is $SU(2)$, but we take it here to indicate more generally a formulation in terms of group representations for the given Lie group.

The space L_*^2 is the space of square integrable functions on the lie algebra $\mathfrak{g} = \mathfrak{spin}(4)$ endowed with a non-commutative \star -product deforming the usual pointwise multiplication. We have also added to the domain of the wavefunctions an extra variable $k \in S^3 \simeq \text{Spin}(4)/\text{SU}(2)$, which plays an important role in the imposition of the geometricity constraints, while being immaterial in the simpler BF case. The variable J_i denotes a unitary irreducible representation of the group $\text{Spin}(4)$, labeled by pair of half-integers $J_i = (j_i^-, j_i^+)$.

These Hilbert spaces naturally arise when seeking to construct explicit representations of the quantum algebra of observables of simplicial BF theory, as a concrete algebra of operators on suitable Hilbert spaces (in our cases L^2 spaces). The quantum algebra of observables stems from the quantization of the theory's classical Poisson algebra and requires the choice of a quantization map. Therefore to explain the origin of the representations (3.14)-(3.16), we need to review the quantization procedure. In order to make the discussion as pedagogical and comprehensible as possible, we proceed as follows.

- We recall the classical Poisson algebra of discrete BF theory that we wish to quantize.
- We review in full generality the quantization of physical systems whose phase space is given by the cotangent bundle of a Lie group $T^*G \simeq G \times \mathfrak{g}^*$.
- We explicitly perform the quantization of simplicial BF theory by applying the abstract procedure introduced earlier. The representations of the resulting quantum algebra yield the patch Hilbert spaces (3.14)-(3.16).
- We conclude by briefly discussing the quantization of the discrete geometricity constraints.

Before moving forward let us make few important remarks.

Remark 3.0.1 (The quantization strategy). According to our strategy we start with the quantization of topological BF theory and proceed by imposing the geometricity constraints at the quantum level. Ideally one would want to constraint the discrete data at the classical level and then quantizing the resulting geometric structures as one would do in a simplicial path integral approach. Thus in principle one should first compute the full constraint hypersurface and perform a symplectic reduction. One then quantize the resulting reduced phase space with its associated symplectic structure. However for non-linear systems, like GR, this has proven to be a remarkably difficult task up to now, partly because of the discrete setting where the construction takes place. In order to circumvent this technical obstruction people have proposed to implement the constraints not before but after the quantization hoping to obtain the correct result despite the various ambiguity entering the construction process. Such ambiguities include the choice of the quantization map (operator ordering) and the different available methods to define and implement the geometricity constraints (the simplicity constraints in particular) as suitable restrictions on the states and amplitudes quantum variables.

According to the expression of the discretized BF action (3.9) the phase space of simplicial BF theory is given by one copy of the cotangent bundle of the structure group $\text{Spin}(4)$ for each dual half-link in a boundary patch,

$$\mathcal{P}_{\bar{v}\tilde{v}} \equiv T^*\text{Spin}(4) \simeq \text{Spin}(4) \times \mathfrak{spin}^*(4) \quad \bar{v} \in \bar{\mathcal{V}}, \quad \tilde{v} \in \tilde{\mathcal{V}}, \quad (\bar{v}\tilde{v}) \in \mathfrak{p}_{\bar{v}} \quad (3.17)$$

$$\mathcal{P}_{\bar{v}\tilde{v}} \equiv T^*\text{Spin}(4) \simeq T^*\text{SU}(2) \times T^*\text{SU}(2) \simeq [\text{SU}(2) \times \mathfrak{su}^*(2)]^{\times 2} \quad (3.18)$$

which is locally equivalent to the cartesian product of the group times the dual of its Lie algebra. On a given (simplicial) molecule the individual half-link phase spaces $\mathcal{P}_{\bar{v}\tilde{v}}$ can be organized according to the connectivity pattern of the molecule's boundary graph. Thus we have:

$$\mathcal{P}_{\bar{v}} \equiv \mathcal{P}_{\mathfrak{p}_{\bar{v}}} \equiv \times_{\tilde{v} \in \mathfrak{p}_{\bar{v}}} \mathcal{P}_{\bar{v}\tilde{v}} \quad \mathcal{P}_{\mathfrak{a}(\mathfrak{b})} \equiv \times_{\bar{v} \in \mathcal{V}_{\mathfrak{a}}} \mathcal{P}_{\bar{v}} \equiv \times_{\mathfrak{p}_{\bar{v}} \in \mathfrak{P}_{\mathfrak{b}}} \mathcal{P}_{\mathfrak{p}_{\bar{v}}} \quad (3.19)$$

Since they exhibit, by construction, a global direct product structure we can restrict ourselves, without loss of generality, to the quantization of a single copy of the phase space $\mathcal{P}_{\bar{v}\tilde{v}}$. Its decomposition formula, in the case of Euclidean BF theory, further simplifies the problem to the quantization of one copy of the cotangent bundle of $\text{SU}(2)$. Such topic has been largely investigated in the literature for arbitrary Lie groups. Here we only summarize the main steps of the quantization strategy, providing further details in Appendix. A. For the sake of clarity we begin with a synoptic table of the main symbols and their meaning.

Symbol.	Meaning.
\mathcal{L}_i	Lie derivative with respect to a basis of right-invariant vector fields.
\mathcal{P}_G	Poisson algebra indexed by the symplectic structure on the group manifold G .
\mathcal{A}	Maximal subalgebra of the Poisson algebra \mathcal{P}_G .
$\mathcal{A}_G, \mathcal{A}_{\mathfrak{g}^*} \subset \mathcal{A}$	Subalgebras of the function in \mathcal{A} constant, in order, on \mathfrak{g}^* and on G .
$C_c^\infty(G), C_c^\infty(\mathfrak{g}^*)$	Spaces of smooth compactly supported functions on G and on \mathfrak{g}^* .
\mathfrak{X}	Quantum algebra of observables obtained from \mathcal{P}_G upon quantization.
π	Representation of \mathfrak{X} , i.e a linear \star -homomorphism preserving commutators.

The phase space of a physical system is a symplectic manifold. Its canonical symplectic structure together with the pointwise multiplication uniquely determines the Poisson algebra and therefore the Poisson brackets.

Definition 3.1 (Poisson brackets). Let $\mathcal{P}_G = (C^\infty(T^*G), \{\bullet, \bullet\}, \cdot)$ the Poisson algebra on the group manifold induced by the symplectic structure on $\mathcal{P} = T^*G$. Then the Poisson brackets can be defined as follows:

$$\forall f, g \in C^\infty(T^*G) \quad \{f, g\} \equiv \frac{\partial f}{\partial X_i} \mathcal{L}_i g - \mathcal{L}_i f \frac{\partial g}{\partial X_i} + c_{ij}^k \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_j} X_k \quad (3.20)$$

where X_i are euclidean coordinates on the Lie algebra \mathfrak{g} and c_{ij}^k are the structure constants.

Seeking to quantize a maximal subalgebra $\mathcal{A} \subset \mathcal{P}_G$ as an operator *-algebra \mathfrak{X} , we define a quantization map \mathcal{Q} .

Definition 3.2 (Quantization map). A quantization map \mathcal{Q} is a linear map between algebras defined as:

$$\mathcal{Q}: \mathcal{A} \rightarrow \mathfrak{X} \quad \text{s.t.} \quad \forall f \in \mathcal{A}_G \subset \mathcal{A} \subset C^\infty(G \times \mathfrak{g}^*) \quad f \equiv \mathcal{Q}(f) \quad X_i \equiv \mathcal{Q}(X_i) \quad (3.21)$$

$$[f, g] = 0 \quad [X_j, f] = \widehat{i\mathcal{L}_j f} \in \mathfrak{X}_G \quad [X_i, X_j] = ic_{ij}^k X_k \quad \forall f, g \in \mathfrak{X}_G \quad \mathfrak{X}_G = \mathcal{Q}(\mathcal{A}_G) \quad (3.22)$$

Given the quantum algebra of observables \mathfrak{X} , the next task is to construct explicit representations π of it as a concrete operator algebra on suitable Hilbert spaces, where $\pi: \mathfrak{X} \rightarrow \text{Aut}(\mathcal{H})$ preserves the commutators.

Definition 3.3 (Group representation π_G). The group representation π_G on $\mathcal{H} = L^2(G)$ is defined as the one diagonalizing all the operators $f \in \mathfrak{X}_G$. Let $O \in \mathfrak{X}_G$ be a generic operator (e.g. $O = f, X$). Then we have,

$$\forall f, \psi \in C_c^\infty(G) \subset \mathcal{A}_G \quad \left(\pi_G(f)(\psi) \right)(g) \equiv f(g)\psi(g) \quad \left(\pi_G(X_i)(\psi) \right)(g) \equiv i\mathcal{L}_i \psi(g) \quad (3.23)$$

where we have purposefully restricted the domain of $\pi_G(f)$ and $\pi_G(X_i)$ so that $f\psi \in C_c^\infty(G)$, $\forall \psi \in C_c^\infty(G)$.

We now move on to define a representation in terms of functions on the classical dual space \mathfrak{g}^* . Since the flux operators $X_i \in \mathfrak{X}_{\mathfrak{g}^*} = \mathcal{Q}(\mathcal{A}_{\mathfrak{g}^*})$ do not commute we *deform* the action of $\pi_{\mathfrak{g}^*}(X_i)$ by introducing a star product operation such that the commutation relations are satisfied.

Definition 3.4 (Star product). A star product, denoted by \star , is an operation such that:

$$\left(\pi_{\mathfrak{g}^*}(f(X))(\psi) \right)(g) \equiv f_\star(X) \star \psi(X) \quad \forall f_\star \in \mathcal{A}_{\mathfrak{g}^*} \subset C^\infty(\mathfrak{g}^*) \quad \text{s.t.} \quad f(X) = \mathcal{Q}(f_\star) \in \mathfrak{X}_{\mathfrak{g}^*} \quad (3.24)$$

$$\left(\pi_{\mathfrak{g}^*}(\mathcal{Q}(f_\star)\mathcal{Q}(\tilde{f}_\star))(\psi) \right)(X) = \left(\pi_{\mathfrak{g}^*}(\mathcal{Q}(f_\star \star \tilde{f}_\star))(\psi) \right)(X) \quad \forall f_\star, \tilde{f}_\star \in \mathcal{A}_{\mathfrak{g}^*} \quad (3.25)$$

This ensures that f_\star can be interpreted as the function that upon quantization gives $f(X_i)$, thus establishing a connection between the classical phase space structure and the quantum operators. Furthermore the choice of the quantization map uniquely dictates the \star -product according to following relation:

$$f_\star \star \tilde{f}_\star = \mathcal{Q}^{-1}(\mathcal{Q}(f_\star)\mathcal{Q}(\tilde{f}_\star)) \quad (3.26)$$

Given a star product we define the flux representation π to be:

Definition 3.5 (Flux representation $\pi_{\mathfrak{g}^*}$). Let us consider the operators $\zeta^i, X_i \in \mathfrak{X}_{\mathfrak{g}^*}$. Their action reads:

$$\forall \psi \in C_c^\infty(\mathfrak{g}^*) \subset \mathcal{A}_{\mathfrak{g}^*} \subset L_\star^2(\mathfrak{g}^*) \quad \left(\pi_{\mathfrak{g}^*}(X_i)(\psi) \right)(X) \equiv X_i \star \psi(X) \quad \left(\pi_{\mathfrak{g}^*}(\zeta^i)(\psi) \right)(X) \equiv -i \frac{\partial}{\partial X_i} \psi(X) \quad (3.27)$$

It can be shown that the space $C_c^\infty(\mathfrak{g}^*)$ is closed under the action of the above operators.

Definition 3.6 (Spin unitary irreducible representations π_J). A unitary irreducible representation π_J of a (compact) Lie group G is a homomorphism $\pi: G \rightarrow \mathcal{B}(\mathcal{H}^J)$, $g \mapsto \pi(g)$ where $\mathcal{B}(\mathcal{H}^J)$ denotes the bounded linear operators on some Hilbert space \mathcal{H}^J , called the representation space, such that:

$$\pi(g_1 g_2) = \pi(g_1) \pi(g_2) \quad [\pi(g)]^\dagger = [\pi(g)]^{-1} \quad \forall g, g_1, g_2 \in G \quad (3.28)$$

and the (multi)index J , called spin, takes values in a countable set.

The next task is to find the relation between the two representations π_G and $\pi_{\mathfrak{g}^*}$ and between the holonomy and the spin representation. The non-commutative Fourier transform and the Peter-Weyl decomposition, introduced below, provides such correspondence.

Definition 3.7 (Non-commutative Fourier transform). We assume the existence of a unitary map between the two representations $\mathcal{F} : L^2(G) \rightarrow L^2_{\star}(g^*)$ which can be expressed as an integral transform,

$$\tilde{\psi}(X) \equiv \mathcal{F}(\psi)(X) \equiv \int_G dg E_g(X) \psi(g) \quad \tilde{\psi} \in L^2_{\star}(g^*) \quad (3.29)$$

$$\psi(g) = \mathcal{F}^{-1}(\tilde{\psi})(g) \equiv \int_{g^*} \frac{d^D X}{(2\pi)^D} \overline{E_g(X)} \star \tilde{\psi}(X) \quad \psi(g) \in L^2(G) \quad (3.30)$$

where $E_g(X)$ denotes the integral kernel of the transform, i.e the non-commutative plane wave. Its actual existence has to be verified once an explicit choice of the quantization map has been made. Further details on the non-commutative Fourier transform can be found in Appendix. A.

Theorem 3.1 (Peter-Weyl). Choose once and for all a representative π_J from each equivalence class of finite dimensional unitary irreducible representations of a compact Lie group G on representation spaces \mathcal{H}^J . Let d_J be the dimension of the representation π_J and

$$D_{MN}^J(g) \equiv d_J [\pi_J(g)]_{MN} \quad M, N = 1, \dots, d_J. \quad (3.31)$$

the Wigner functions. Consider the Hilbert space $\mathcal{H} = L^2(G, d\mu_{\text{Haar}})$ where μ_{Haar} is the unique Haar measure on G . Then the system of function $D_{MN}^J(g)$ is a complete orthonormal basis for \mathcal{H} .

So far we kept the discussion general, without referring to a specific Lie group or to a particular choice of the quantization map. In the case of quantum gravity spin foam models inspired by the Holst-Plebanski formulation of General Relativity the structure group G is chosen to be the local gauge group of gravity in 4d and in the appropriate space-time signature, thus $G = \text{Spin}(4), \text{SL}(2, \mathbb{C})$. We consider the Riemannian case only, here, thus $G = \text{Spin}(4) \simeq SU(2) \times SU(2)$, $\mathfrak{g} = \mathfrak{spin}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, and the (unitary) irreducible representations are labeled by two half-integers (the irreps of the two $SU(2)$ factors) $J = (j_+, j_-)$. In this case by applying the quantization procedure outlined before together with the definitions (3.19), one immediately recovers The topological interpretation is the following. Each Hilbert space $\mathcal{H}_{\bar{v}}$ provides the space of states of a single "quantum of space", a quantum gravity "atom". It can be pictured as a fundamental spin-network vertex, represented by a node $\bar{v} \in \bar{\mathcal{V}}$ with $d = 4$ outgoing half-links and their one-valent end points (i.e. a boundary patch $\mathfrak{p}_{\bar{v}} \in \mathfrak{P}_{\bar{v}}$) labelled, depending on the choice of polarization, by Lie algebra elements, group elements or group representations. Equivalently it can also be depicted as a tetrahedron, whose $d = 4$ faces (triangles) are labelled by the same variables. The Lie algebra elements can be understood as the discretization (covariant smearing) of the B 2-form fields along the same triangles (i.e. fluxes of the B field), the group elements as discretized parallel transports of the BF 1-form connection along the dual links, and the group representations are quantum numbers labelling eigenstates of the modulus of the B field. Moreover, our quantum (spin network) states are supplemented with an additional variable $k \in S^3 \simeq \text{Spin}(4)/SU(2)$ interpreted as the unit normal to the tetrahedron in its local \mathbb{R}^4 embedding. The presence of such extra variable, and the way it will enter the imposition of the geometricity constraints, implies that our boundary states correspond, more precisely, to *projected spin networks* [86,87]. This is the case for all current spin foam models based on constrained BF theory.

So far we reviewed the quantization of the phase space of simplicial BF theory without addressing the quantization of the constraints. This is easily achieved upon promoting the classical bivectors to quantum operators by acting on them with the quantization map. Using the spitting into selfdual and anti-selfdual components they read:

$$\mathbf{X} \equiv \mathcal{Q}(X) = (\mathcal{Q}(x^-), \mathcal{Q}(x^+)) \equiv (\mathbf{X}^-, \mathbf{X}^+) \quad \mathbf{X}^{\pm} |x^{\pm}\rangle = x^{\pm} \star |x^{\pm}\rangle \quad x^-, x^+ \in \mathfrak{su}(2) \quad (3.32)$$

$$(\mathbf{X}^{\pm})^2 |j^{\pm}, m^{\pm}, n^{\pm}\rangle = (j^{\pm}(j^{\pm} + 1) + c) |j^{\pm}, m^{\pm}, n^{\pm}\rangle \quad \mathbf{X}_z^{\pm} |j^{\pm}, m^{\pm}, n^{\pm}\rangle = m^{\pm} |j^{\pm}, m^{\pm}, n^{\pm}\rangle \quad (3.33)$$

As one may easily check, the operators $\mathbf{X}^-, \mathbf{X}^+$ can be naturally identified with the angular momentum generators (associated to left-invariant vector fields) of the corresponding $SU(2)^{\pm}$ subgroups of $\text{Spin}(4)^3$. The ambiguity constant c entering the spectrum of the Casimir operators is dictated by the choice of the quantization map. Thus the quantum linear simplicity constraint read.

$$S^{\beta} = k \mathbf{X}^- k^{-1} + \beta \mathbf{X}^+ \quad \mathbf{X}^-, \mathbf{X}^+ \in \mathfrak{X}_{\mathfrak{g}^*} \quad k \in S^3 \quad (3.34)$$

where $\mathfrak{X}_{\mathfrak{g}^*}$ with $\mathfrak{g} = \mathfrak{su}(2)$ is the image via \mathcal{Q} of the classical subalgebra $\mathcal{A}_{\mathfrak{g}^*}$. As we already said, this relation cannot be enforced strongly as an operatorial equation. Thus in order to proceed one first prescribes a constraint imposition method and solves the resulting equations, obtaining the explicit form of constraints as restrictions

³A complete orthonormal set in $L^2[SU(2)]$ is given by the eigenstates $|j, m, n\rangle$ of maximal commuting set of operators J^2, J_z, L, J_z, R .

on the chosen set of variables. Then one implements those restrictions by acting on the quantum amplitude with a purposefully defined operator. This still leaves us the freedom to decide where (a power of) the constraint operator should be inserted, i.e. on which ones of the patch Hilbert spaces entering the definition of the spinfoam amplitudes it should act. We discuss these issues in sections 3.3 and 3.4.

One final note: each $\text{Spin}(4)$ representation can be decomposed into irreps of the diagonal $\text{SO}_k(3)$ subgroup (which in the case $k = \mathbb{I}$ are those used in usual spin networks in the context of LQG), i.e. the stabilizer subgroup of the normal vector k . Indeed in formulas we have:

$$\begin{aligned} \mathcal{H}^J &\equiv \mathcal{H}^{j^-} \otimes \mathcal{H}^{j^+} = \bigoplus_{j=|j^- - j^+|}^{j^- + j^+} \mathcal{H}^j \quad |J, M, N\rangle \equiv |j^-, m^-, n^-\rangle \otimes |j^+, m^+, n^+\rangle \in \mathcal{H}^{j^-} \otimes \mathcal{H}^{j^+} \\ |j^-, m^-, n^-\rangle \otimes |j^+, m^+, n^+\rangle &= \sum_{jmn} \overline{C_{m^- m^+ m}^{j^- j^+ j}(k)} C_{n^- n^+ n}^{j^- j^+ j}(k) |(j^-, j^+), j, m, n\rangle \end{aligned} \quad (3.35)$$

The notation goes as follows. \mathcal{H}^J and \mathcal{H}^j denote, in order, a $\text{Spin}(4)$ and an $\text{SO}(3) \simeq \text{SU}(2)$ representation space. As we can see we can always factorize an element of the basis of \mathcal{H}^J into the tensor product of the base vectors in each of its $\text{SU}(2)$ components and further expand them in terms of a complete orthonormal set in \mathcal{H}^j . By linearity the above argument can be extended to any state $|\psi\rangle \in \mathcal{H}^J$. The coefficients $C^{j^- j^+ j}$ are rotated Clebsch-Gordan coefficients. Their definition and properties are listed in Appendix. B. We will make extensive use of such decomposition in the definition of the spin foam amplitudes for quantum gravity in 4d.

This decomposition can also be used to re-write every $\text{Spin}(4)$ structure (boundary states, spin foam amplitudes) in terms of the analogous structures using only $\text{SU}(2)$ data (group elements, fluxes, irreps) with respects to the same normal vector. In fact, the geometricity constraints coming from the Holst-Plebanski formulation can be implemented, for any spin foam models, as a modification of this map (and its inverse) between $\text{Spin}(4)$ and $\text{SU}(2)_k$ data, with respect to the general one. The definition and properties of this modified map for the EPRL model have been investigated in [87, 88], while the general case will be the subject of forthcoming work [89]. Even if we will be using extensively the decomposition into $\text{SU}(2)$ representations, we will base our presentation of spin foam models on $\text{Spin}(4)$ data.

3.3 Definition and imposition of the geometricity constraints.

We now focus on the geometricity constraints for *simplicial* spin foam models for Riemannian quantum gravity in $d = 4$ spacetime dimensions, turning, at least in principle, simplicial topological BF theory into simplicial quantum gravity. Their implementation procedure is not unique and there exist different model-building prescriptions. In the following subsections we will present the general form of the constraint operators, highlight the construction ambiguities, and indicate briefly the key elements of specific models.

3.3.1 The closure constraint.

As we anticipated, the relevant constraints to be imposed are the closure and the (linear) simplicity constraints. The closure constraint is a first class constraint, from a canonical point of view. This means that we can expect to be imposed by a projector at the level of quantum states. In the simplicial setting, it requires that the sum of the bivectors X_i associated to the four boundary triangles of the same tetrahedron vanishes; equivalently, looking at the dual cellular complex, the bivectors labelling the four faces sharing a given edge sum to zero:

$$\sum_{\bar{v} \in \bar{v}} X_{\bar{v}\bar{v}} \equiv \sum_{i=1}^4 X_i = 0 \quad X_{\bar{v}\bar{v}} \in \text{Spin}(4) \quad X_{\bar{v}\bar{v}} = (x_{\bar{v}\bar{v}}^-, x_{\bar{v}\bar{v}}^+) \quad (3.36)$$

At the quantum level, it can be implemented via an orthogonal projector P_{cl} acting separately on each node of a spin network state, or on each tetrahedron state, taking into account its normal vector $k \in S^3$.

Definition 3.8 (Closure constraint operator). Let $\Psi_{\text{p}\bar{v}} \equiv \Psi_{\bar{v}} \equiv \Psi_k \in \mathcal{H}_{\bar{v}}$ be a single tetrahedron state. The closure constraint projector is defined as follows:

$$P_{\text{cl}} : \mathcal{H}_{\bar{v}} \rightarrow \mathcal{H}_{\bar{v}} \quad P_{\text{cl}}^2 = P_{\text{cl}} \quad P_{\text{cl}}^\dagger = P_{\text{cl}} \quad (3.37)$$

In the flux representation the operators P_{cl} reads:

$$P_{\text{cl}}(X_i, Y_i, \tilde{k}, k) = \int [dH] \delta[\tilde{k}^{-1}(H \triangleright k)] \prod_{i=1}^4 \left(\delta_{X_i} \star E_H \right) (Y_i)$$

$$\tilde{\Psi}_{H \triangleright k}(X_i) = (\text{P}_{\text{cl}} \Psi_{\tilde{k}})(X_i) = \int [dH] \left(\prod_{i=1}^4 E_H \star \Psi_{H \triangleright k} \right) (X_i) \quad (3.38)$$

Seemingly, upon non-commutative Fourier transform, in the group representation we find:

$$\begin{aligned} \text{P}_{\text{cl}}(G_i, \tilde{G}_i, \tilde{k}, k) &= \int [dH] \delta[\tilde{k}^{-1}(H \triangleright k)] \prod_{i=1}^4 \delta(G_i^{-1} H \tilde{G}_i) \\ \tilde{\Psi}_{H \triangleright k}(G_i) &= (\text{P}_{\text{cl}} \Psi_{\tilde{k}})(G_i) = \int [dH] \Psi_{H \triangleright k}(H G_i) \end{aligned} \quad (3.39)$$

where $H \triangleright k = h^+ k (h^-)^{-1}$ denotes the action of a Spin(4) rotation H on the normal k .

In the above formula the *star*-product acts on repeated Lie algebra variables. Upon integration over the normal k , which is automatically enforced for every bulk tetrahedron in the simplicial spin foam amplitudes, the (extended) closure constraint so defined reduces to the usual gauge invariance of the Ooguri spin foam model [27, 28], i.e. to the invariance under diagonal group action on the four group elements on which each tetrahedron state depends. In absence of simplicity constraints, one would then get the usual Ooguri spin foam model for 4d Spin(4) BF theory also when using this extended notion of gauge invariance.

Remark 3.8.1 (Invariance under the stabilizer group). The Spin(4) covariance of the spin network state Ψ_k induces an invariance under the stabilizer group $\text{SO}_k(3) \subset \text{Spin}(4)$ of the normal k .

$$\text{SO}_k(3) = \{H \in \text{Spin}(4) \mid h^+ k (h^-)^{-1} = k\} \Leftrightarrow H = (k^{-1} h k, h) \quad \forall H \in \text{SO}_k(3), h \in \text{SU}(2), k \in S^3 \quad (3.40)$$

Upon Peter-Weyl decomposition the $\text{SO}_k(3)$ gauge invariant state Ψ_k reads:

$$\Psi(G_i, k) = \sum_{\text{All indices}} \Psi_{N_1, \dots, N_4}^{J_1, \dots, J_4, j_1, \dots, j_4, l}(k) \left[\prod_{i=1}^4 C_{p_i^- p_i^+ p_i}^{j_i^- j_i^+ j_i}(k) D_{P_i N_i}^{J_i}(G_i) \right] (\mathcal{I}_{P_1 P_2 P_3 P_4}^{j_1 j_2 j_3 j_4 l}) \quad (3.41)$$

The field modes are labelled by four irreducible Spin(4) representations $J_i = (j_i^-, j_i^+)$, one for each (half) link of the patch, which can be further decomposed into $\text{SO}_k(3)$ representations j_i . The $D^J(G)$ are Spin(4) Wigner functions and $(\mathcal{I})^{j_i, l}$ denote a basis of four-valent $\text{SO}(3)$ intertwiners with virtual spin l . The coefficients $C^{j^- j^+ j}(k)$ are SU(2) rotated Clebsch-Gordan coefficients intertwining the action of the stabilizer group $\text{SO}_k(3)$ in the $j_i^- \otimes j_i^+$ representation and the action of $\text{SO}(3)$ in the j_i representation.

The Fourier expansion of the constrained state Ψ_k corresponds, as we anticipated, to the structure of so-called projected spin networks of covariant LQG. Thus they label a basis for polynomial gauge invariant observables of all known SF models based on quantum simplicial BF theory.

3.3.2 The linear simplicity constraint.

Beside the closure constraint, the simplicity constraints are the other ingredient reducing BF theory to gravity in the Plebanski-Holst formulation. The issue is how to impose such conditions in terms of operators acting on quantum states. The general form of such operator, encompassing all known models, will be given first.

Definition 3.9 (Simplicity constraint operator). Let $\Psi_{p_{\bar{v}}} \equiv \Psi_{\bar{v}} \equiv \Psi_k \in \mathcal{H}_{\bar{v}}$ be a single tetrahedron state. The simplicity constraint operator is defined as follows:

$$S^\beta : \mathcal{H}_{\bar{v}} \rightarrow \mathcal{H}_{\bar{v}} \quad S^\beta : \Psi_k \rightarrow \Psi_k^\beta = S^\beta[\Psi_k] \quad (3.42)$$

Its expression and action in the metric (or flux) representation are given by:

$$\begin{aligned} S_k^\beta(X_i, Y_i) &= \prod_{i=1}^4 \left(\delta_{-X_i} \star S_k^\beta \right) (Y_i) \quad \Psi_k^\beta(X_i) = \int \left[\frac{d^6 Y_i}{(2\pi)^6} \right] S_k^\beta(X_i, Y_i) \star \Psi_k(Y_i) \\ S_k^\beta(X) &= \sum_{J j} d_J d_j w(J, j, \beta) \Theta^J(X) \star \chi^j(k x^- k^{-1} + x^+) \end{aligned} \quad (3.43)$$

The $\mathfrak{su}(2)$ Lie algebra character $\chi^j(x)$ can be found by taking the non-commutative Fourier transform of the ordinary SU(2) character $\chi^j(g)$. Such definition trivially extends to the group Spin(4), as shown below.

$$\chi^j(x) = \int dg E_g(x) \chi^j(g) \quad \Theta^J(X) = \int dG E_G(X) \Theta^J(G) \quad x \in \mathfrak{su}(2) \quad X \in \mathfrak{spin}(4) \quad (3.44)$$

Further details are provided in Appendix. A. In the spin formulation the simplicity constraint operator read:

$$\begin{aligned} S_{M_1, \dots, M_4 N_1, \dots, N_4}^{J_1, \dots, J_4}(k, \beta) &= \sum_{j_i m_i} \prod_{i=1}^4 \overline{C_{m_i^- m_i^+ m_i}^{j_i^- j_i^+ j_i}(k) C_{n_i^- n_i^+ m_i}^{j_i^- j_i^+ j_i}(k)} w(J_i, j_i, \beta) \\ \Psi_k^\beta(G_i) &= \sum_{\text{All}} \Psi_{M_1, \dots, M_4 N_1, \dots, N_4}^{J_1, \dots, J_4}(k) S_{N_1, \dots, N_4 P_1, \dots, P_4}^{J_1, \dots, J_4}(k, \beta) \prod_{i=1}^4 D_{P_i M_i}^{J_i}(G_i) \end{aligned} \quad (3.45)$$

The function $w(J, j, \beta)$, encoding the simplicity constraints as restrictions on the spins, depends on the specific way in which they are imposed on quantum states. We will say more about it in the following.

Let us comment on the general structure and properties of such operator.

Remark 3.9.1 (General structure and main properties of S^β).

1. Because of the factorised form of the tetrahedron Hilbert space (in absence of additional conditions), the simplicity constraint operator, modulo the dependence on the normal k , factorizes as well into the product of operators associated to its four triangles.
2. Different constraint implementation methods yield different definitions of the function w , but they always result in the general structure for the simplicity operator given above.
3. This universal structure of S^β is a direct consequence of its invariance under the action of the stabilizer group $\text{SO}_k(3)$, and of the required covariance under the action of $\text{Spin}(4)$, which is indeed a property of the classical simplicity constraints $kx^-k^{-1} + \beta x^+ = 0$ as well.

$$S^\beta(HX_iH^{-1}, Y_i, k) = S^\beta(X_i, HY_iH^{-1}, k) = S^\beta(X_i, Y_i, k) \quad H \in \text{SO}_k(3) \quad (3.46)$$

In turn such property implies the presence of the characters (class functions) or of the Clebsch-Gordan coefficients (invariant tensors) encoding this underlying rotational symmetry. This symmetry also guarantees the correct interplay between the closure constraint (which encodes the action of the local gauge symmetry) and the simplicity constraints, ensuring a consistent imposition of them.

4. In accordance with the previous statements, the operator S^β commutes with the closure projector P_{cl} up to an overall rotation of the normal k :

$$\left(P_{\text{cl}} \circ S_{H^{-1} \triangleright k}^\beta \right) [\Phi_{H^{-1} \triangleright k}] = \left(S_k^\beta \circ P_{\text{cl}} \right) [\Phi_{H^{-1} \triangleright k}] \quad . \quad (3.47)$$

Geometrically this expresses the fact that rotating a bivector which is simple with respect to a given normal k gives a bivector which is simple with respect to the rotated normal $H \triangleright k$, and thus the correct covariant implementation of the constraints⁴. The closure and simplicity operators can thus be combined into a single (well defined, i.e unambiguous) operator $G_k^\beta = P_{\text{cl}} \circ S_k^\beta$ enforcing all required geometricity constraints.

5. The operator S^β is not an orthogonal projector, in general⁵, for arbitrary values of the Barbero-Immirzi parameter (it becomes a projector when $\gamma \rightarrow \infty$). This is to be expected, since it corresponds to a second class constraint from the canonical perspective. The same parameter, moreover, in some cases must be restricted to rational values (as in the EPRL-FK models) for consistency of the chosen imposition procedure.
6. The simplicity constraint operator S^β does not admit a closed formula for any choice of basis, in general. In order to implement the simplicity constraint we have to choose a constraint imposition method. Such criteria usually come with a preferred (if not mandatory) choice of variables. For instance the Master constraint criterion is naturally formulated in the spin basis. The Holomorphic construction, instead, relies on the spinorial representation [30, 90–92]. The lack of a closed formula simply follows from the fact that, depending on the model, it is not always possible to analytically perform the integrals (or the summations) introduced by a change of basis. Remarkably, both the Barrett-Crane model [22, 27] and the *new model*, based on the Duflo map, do admit a closed formula for S^β (and their amplitudes) in the flux, the holonomy and the spin representations.

For later convenience let us define the following single-link fusion coefficients. These functions are often used in the literature to characterize the spin foam amplitudes in presence of simplicity constraints.

⁴Imposing the simplicity constraint with respect to a gauge fixed normal may fail to correctly implement such covariance.

⁵It is, however, for some versions of the EPRL model, depending on the specific choice of the function w .

Definition 3.10 (Single-link fusion coefficients). The single-link fusion coefficients are defined as follows:

$$F^{j^- j^+ j}(k, \beta) : [-1, 1] \times S^3 \times \mathcal{H}^{j^-} \times \mathcal{H}^{j^+} \times \mathcal{H}^j \rightarrow \mathbb{C} \quad F_{m^- m^+ m}^{j^- j^+ j}(k, \beta) = C_{m^- m^+ m}^{j^- j^+ j} w(j^-, j^+, j, \beta) \quad (3.48)$$

Like the Clebsch-Gordan coefficients the fusion coefficients F intertwine the action of the stabilizer group $\text{SO}_k(3)$ in the representation $j^- \otimes j^+$ with the action of $\text{SO}(3)$ in the representation j .

$$F_{m^- m^+ m}^{j^- j^+ j}(k, \beta) D_{m^- n^-}^{j^-}(k^{-1} h k) D_{m^+ n^+}^{j^+}(h) = F_{n^- n^+ n}^{j^- j^+ j}(k, \beta) D_{nm}^j(h) \quad \forall H \in \text{SO}_k(3). \quad (3.49)$$

where for repeated lower indices the Einstein convention is assumed. Therefore they can be seen as the matrix elements of a map from $\text{SO}(3) \simeq \text{SU}(2)$ irreducible representations to $\text{Spin}(4)$ irreducible representations.

So far we have presented the general properties of the constraints operators P_{cl} and S^β as well as their action on the single tetrahedron wave function Ψ_k . The main question now is how to implement the constraints at the quantum level directly on the spin foam amplitudes. We discuss this issue in the next subsection. Before doing so, we list briefly some of the current proposals in the Riemannian context, resulting from different implementation procedures, as found in the literature, to which we refer for more details [1, 21, 23, 24, 27, 28, 30, 93, 94].

Classes of Models	$w(J, j, \beta)$	Method
Barrett-Crane, $\beta = 1$.	$\delta_{j^- j^+} \delta_{j0}$.	Strong imposition.
EPRL, Alexandrov, $\beta < 0$.	$\delta_{j^- \beta j^+} \delta_{j(1+ \beta)j^+}$.	Master constraint criterion.
FK, $\beta < 0$.	$\delta_{j^- \beta j^+} \delta_{j, j^- + j^+} C_{j^- j^+ j}^{j^- j^+ j} = w_{\text{EPRL}}^{\beta < 0}$	Perelomov coherent states.
EPRL, $\beta > 0$.	$\delta_{j^- \beta j^+} \delta_{j(1- \beta)j^+}$.	Master constraint criterion.
Alexandrov, $\beta > 0$.	$\delta_{j^- \beta (j^+ + 1)} \delta_{j, j^+ (1- \beta) - \beta }$.	Master constraint criterion.
FK, $\beta > 0$.	$\delta_{j^- \beta j^+} \delta_{j, j^+ - j^-} C_{-j^- j^+ j}^{j^- j^+ j} $	Perelomov coherent states.
Baratin-Oriti.	$\frac{(-1)^{j^- + j^+ + j}}{\pi \sqrt{d_{j^-} d_{j^+}}} \sum_{a=0}^{\lambda} \mathfrak{S}^a(\beta) \left\{ \begin{matrix} a & j^- & j^- \\ j & j^+ & j^+ \end{matrix} \right\} \mathcal{T}_a^{j^- j^+}(\beta)$	Flux variables.

In the last row, $\mathfrak{S}^a(\beta)$ is the *sign* function, $\lambda = 2\text{Min}(j^-, j^+)$ and the coefficients $\mathcal{T}_a^{j^- j^+}(\beta)$ is given in terms of the $\text{SU}(2)$ generalized characters (B.9) by the following formula

$$\mathcal{T}_a^{j^- j^+}(\beta) = (-1)^a (2a + 1) \int_0^{2\pi} d\psi \sin^2\left(\frac{\psi}{2}\right) \chi_a^{j^-}(\psi) \chi_a^{j^+}(\psi_\beta) \quad \sin \psi_\beta = |\beta| \sin \psi \quad (3.50)$$

Let us also say something more about the function w .

Remark 3.10.1 (Comments on the coefficient w).

1. The function w captures the explicit form of the simplicity constraints phrased as restrictions on the spin quantum numbers. In the trivial case, namely when no constraint is imposed, the coefficient w is equal to one, $w(J, j, \beta) = 1$. This reduce the simplicity constraint operator to the identity. Indeed we have:

$$S_{M_1, \dots, M_4 N_1, \dots, N_4}^{J_1, \dots, J_4}(k, \beta) = \sum_{j_i m_i} \prod_{i=1}^4 \overline{C_{m_i^- m_i^+ m_i}^{j_i^- j_i^+ j_i}}(k) C_{n_i^- n_i^+ m_i}^{j_i^- j_i^+ j_i}(k) = \prod_{i=1}^4 \delta_{M_i N_i} \quad (3.51)$$

2. It encodes the choice of the quantization map and thus operator ordering conventions. The latter enter, for example, the expression of the spectrum of Casimir operators (used in the EPRL procedure for implementing the constraints as operator equations, solved in expectation value [24]) or the explicit form of the plane waves and star product (used in the imposition of the same constraints in the flux representation [27], adopted also in the present paper to construct a new spin foam model based on the Duflo map).
3. The coefficient w can be rescaled by an arbitrary function of the representation $\Delta_{j^- j^+}^j$ compatible with the definition of the operator S^β as a map between L^2 spaces. The choice of this function is not dictated by the quantization map or by the constraint imposition strategy and thus must be prescribed by hand. If we

are mainly interested in the amplitudes' large- j behaviour, such ambiguity, common to all models, can be conveniently parametrized by taking the function to be of the form $\Delta_{j-j+}^j = d_{j-}^a d_{j+}^b d_j^c$, $a, b, c \in \mathbb{R}$, namely a product of representations' dimension factors usually called face weights. Although such weights are in principle arbitrary (modulo the above mentioned limitations), several requirements have been proposed to restrict the allowed choices, like the invariance of the corresponding Spinfoam amplitudes $\mathcal{A}(\mathbf{m})$ under edge or face subdivision [36]. Additional restrictions on the parameter ranges can be obtained by enforcing further conditions on the resulting models perturbative UV behaviour as well as on their (non-perturbative) RG flow, as recently done in the literature [46, 95].

4. Obviously, different choices of this function w , including the above mentioned face weights ambiguities, determine the exact functional form of the spin foam amplitudes, as we will discuss in the next subsection, and in particular, as we emphasized before, they will affect their large- j (semi-classical) behaviour as well as their renormalization flow (and divergences).

3.4 Spin foam amplitudes from imposition of geometricity operators

Once a given prescription to define the quantum geometricity constraints has been selected (yielding a specific expression of the single-link fusion coefficients) we still have to choose how to impose them on the spin foam amplitudes. The main strategies differ according to *where*, in the simplicial complex, the geometricity operator (combining closure constraint operator and simplicity constraints operator) is chosen to act; more precisely, one has to choose on which ones of the patch Hilbert spaces entering the definition of the spin foam amplitudes the operator is acting. Were the geometricity constraint operator an orthogonal projector (consistently with the intuition about constraint imposition, which however applies strictly speaking only to first class or gauge invariance constraints), all these strategies would lead to the same spin foam amplitudes. As we have mentioned already, however, this is not always the case for the simplicity constraints for generic values of the Barbero-Immirzi parameter, even if it remains a possibility for some models in the literature.

Once the above has been decided, there is one more choice to make, in principle. Instead of acting with the geometricity operator itself, one could choose to act with any power of the same; once more, if the geometricity constraint is a projector, this further ambiguity is immaterial. In the following, we keep the treatment general and include a parameter indicating such power (which affects the simplicity constraint operator only). Accordingly, one can identify five (in the end, four) main imposition strategies:

Vertex Hilbert spaces. Impose the geometricity constraints by acting (p times) with G^β on the states in the patch Hilbert spaces for each spin foam vertex (atom), before contracting spin foam atoms with one another.

Edge Hilbert spaces. Enforce the geometricity constraints by acting (q' times) with G^β on the states entering the gluing map, before using it to connect spin foam atoms.

Gluing map. Impose the geometricity constraints by inserting G^β (with power q) in the definition of the gluing map operator; it is clear that this case ends up producing the same spin foam amplitudes of the previous strategy, with $q = 2q'$, so we do not treat it separately

Gluing and Vertex maps. Implement the geometricity constraints by adopting both the above prescriptions, inserting the geometricity constraint both in the vertex and gluing operators.

GFT covariance. The above focus exclusively on the spin foam amplitudes; coming from the GFT perspective on them, it may be natural to focus instead on the GFT covariance determining the spin foam gluing kernel, and inserting the geometricity constraint operator there instead.

The above choices lead a priori to different models, whose vertex and gluing kernels we report below; still, the amplitudes have the same general structure, which we will also present, in flux, connection and spin bases. In order to do so, we first outline the definition of the constrained spin foam vertex operator, used in the 1st and in the 4th strategies listed above, and of the spin foam gluing operator, used in the 2nd (and 3rd) and 4th strategy, and then give an explicit representation of the corresponding kernels in different bases. From them, one can also deduce the GFT kinetic kernel, used in the 5th strategy.

Definition 3.11 (Vertex operator). The (constrained BF) vertex operator takes five single-tetrahedron states, enforces the geometricity constraints on them by acting (p times) with the geometricity operator formed by S^β and by the closure constraint operator, and then connects them together by identification of variables, according to the combinatorial pattern of the 4-simplex boundary graph \mathfrak{b}_5 . Setting $\mathcal{H}_{\mathfrak{p}_v} \equiv \mathcal{H}_{\bar{v}}$, schematically we have:

$$\mathcal{O}_v : \mathcal{H}_{\bar{v}}^{\times 5} \rightarrow \mathcal{H}_{\mathfrak{a}(\mathfrak{b}_5)} \equiv \mathcal{H}_v \quad \mathfrak{a} \in \mathcal{A}_S$$

$$\mathcal{O}_v \equiv \prod_{\bar{v} \in \mathfrak{b}_5} S_{\bar{v}}^\beta \circ P_{\bar{v}}^{\text{cl}} \equiv \prod_{\bar{v} \in \mathfrak{b}_5} G_{\bar{v}}^{\beta,p} \equiv G_{\bar{v}_1}^{\beta,p} \circ_{\mathfrak{b}_5} G_{\bar{v}_2}^{\beta,p} \circ_{\mathfrak{b}_5} G_{\bar{v}_3}^{\beta,p} \circ_{\mathfrak{b}_5} G_{\bar{v}_4}^{\beta,p} \circ_{\mathfrak{b}_5} G_{\bar{v}_5}^{\beta,p}, \quad (3.52)$$

where the convolution symbols denote the contractions of variables with the combinatorics of the 4-simplex boundary graph. Notice that it is these contractions acting on the states resulting from the action of the geometricity operators that produce a (complex) number out of the vertex operator. It defines the function $\mathcal{V}_\alpha : \mathcal{H}_{\bar{v}}^{\times 5} \rightarrow \mathbb{C}$ which we call *Vertex kernel*, giving the generalized "matrix elements" of the operator \mathcal{O}_v .

Definition 3.12 (Gluing operator). The gluing operator for constrained BF does the same operation, acting on the two single-tetrahedron states corresponding to the same tetrahedron/patch, shared as part of the boundary of the two spin foam atoms (4-simplices) to be glued along it. Again, schematically:

$$\mathcal{O}_e : \mathcal{H}_{\bar{v}} \times \mathcal{H}_{\bar{v}'} \rightarrow \mathcal{H}_{(\bar{v}\bar{v}')} \equiv \mathcal{H}_{\mathfrak{a}(\mathfrak{b}_2)} \quad \mathcal{O}_e \equiv G_{\bar{v}}^{\beta,q} \circ G_{\bar{v}'}^{\beta,q} = P_{\text{cl}} \circ S^{\beta,2q} \circ P_{\text{cl}} \quad (3.53)$$

where now the contraction is direct identification of variables according to the one enforced by the corresponding gluing map or *gluing kernel* $\mathcal{K}_e : \mathcal{H}_{\bar{v}} \times \mathcal{H}_{\bar{v}'} \rightarrow \mathbb{C}$.

In both cases, the BF case is recovered by enforcing simple identification of the variables and the closure constraint (gauge invariance) only, without any insertion of the simplicity constraint operator.

Before giving the expressions of the Vertex and Gluing kernels let us briefly recall the notation. A simplicial atom (4-simplex) $\mathfrak{a} \in \mathfrak{A}_S$ consists of a triple $(\mathcal{V}_\alpha, \mathcal{E}_\alpha, \mathcal{F}_\alpha)$ with oriented boundary-bulk half-edges $e = (\bar{v}v) \in \mathcal{E} \subset \mathcal{E}_\alpha$ connecting the centre of each boundary tetrahedron to the centre of the 4-simplex (or viceversa depending on the orientation). Thus v is a *bulk* vertex and $\bar{v} \in \bar{\mathcal{V}} \subset \mathcal{V}_\alpha$ is a *boundary* node of the graph $\mathfrak{b}_5 = \partial\mathfrak{a} \in \mathfrak{B}$. Each of the ten faces of \mathfrak{a} consists of four half-edges such that: $f \in \mathcal{F}_\alpha$, $f = \{e, \bar{e}, \bar{e}', e'\}$ where $e, e' \in \mathcal{E}$ are (oriented) half-edges and $\{\bar{e}, \bar{e}'\} = \{(\bar{v}\bar{v}), (\bar{v}\bar{v}')\} \in \bar{\mathcal{E}} \subset \mathcal{E}_\alpha$, with $\bar{v} \in \bar{\mathcal{V}} \subset \mathcal{V}_\alpha$, are the (oriented) half-links of the (bisected) *boundary* graph \mathfrak{b}_5 . Thus a face of a simplicial Spinfoam atom (4-simplex) can be unambiguously identified either by a triple of vertices $f \equiv (\bar{v}\bar{v}\bar{v}') \sim (\bar{v}\bar{v}\bar{v}')$ or, equivalently, by a pair of half-edges or half-links $f \equiv (ee') \sim (\bar{e}\bar{e}')$. Every half-edge can inherit different orientations from the four faces sharing it. We now provide the concrete expressions for the above operator kernels (which, we stress again, can be used in combination with the BF ones or together, depending on which of the outlined imposition strategies one adopts), focusing only on the flux and spin basis. The notation goes as follows: uppercase letters denotes Spin(4) variables (i.e. fluxes, holonomies and spin) while lowercase symbols will be used instead for SU(2) data.

The Vertex kernel.

$$\begin{aligned} \mathcal{V}_\alpha^{\beta,p}(X_{\bar{v}\bar{v}}, Y_{\bar{v}'\bar{v}'}, k_{\bar{v}}) &= \sum_{J_f, j_{\bar{v}\bar{v}}} \int \left[\prod_{\bar{v} \in \bar{\mathcal{V}}} dH_{\bar{v}\bar{v}} \right] \prod_{f \in \mathcal{F}_\alpha} d_{J_f} d_{j_{\bar{v}\bar{v}}} d_{j_{\bar{v}'\bar{v}'}} w^p(J_f, j_{\bar{v}\bar{v}}, \beta) w^p(J_f, j_{\bar{v}'\bar{v}'}, \beta) \left(\delta_{-X_{\bar{v}\bar{v}}} \star \Theta^{J_f} \star \chi_{k_{\bar{v}}}^{j_{\bar{v}\bar{v}}} \star \chi_{H_{\bar{v}\bar{v}'} \triangleright k_{\bar{v}'}}^{j_{\bar{v}'\bar{v}'}} \star E_{H_{\bar{v}\bar{v}} H_{\bar{v}'\bar{v}'}}^{-1} \right) (Y_{\bar{v}'\bar{v}'}) \\ \mathcal{V}_\alpha^{\beta,p}(J_f, M_{\bar{v}\bar{v}}, k_{\bar{v}}) &= \sum_{j_{\bar{v}\bar{v}} j_{\bar{v}'\bar{v}'}} \sum_{I_{\bar{v}\bar{v}} I_{\bar{v}'\bar{v}'}} \left[\prod_{\bar{v} \in \bar{\mathcal{V}}} dI_{\bar{v}\bar{v}} dI_{\bar{v}'\bar{v}'} \right] \left[\prod_{f \in \mathcal{F}_\alpha} d_{J_f} d_{j_{\bar{v}\bar{v}}} d_{j_{\bar{v}'\bar{v}'}} \right] \left[\prod_{\bar{v} \in \bar{\mathcal{V}}, \bar{v} \in \mathfrak{p}_\alpha} f_{I_{\bar{v}\bar{v}}}^{i_{\bar{v}\bar{v}}, p}(J_f, j_{\bar{v}\bar{v}}, \beta) \mathcal{B}_{M_{\bar{v}\bar{v}}}^{J_f}(i_{\bar{v}\bar{v}}, j_{\bar{v}\bar{v}}, k_{\bar{v}}) \right] \{15J_f\}_v \end{aligned} \quad (3.54)$$

In the first formula we commuted all characters with the non-commutative plane waves according to the identities (A.20) and (A.21). The group element $H_{\bar{v}\bar{v}'}^{-1} \equiv H_{\bar{v}\bar{v}} H_{\bar{v}'\bar{v}'}^{-1}$ is the parallel transport from the node \bar{v} to the node \bar{v}' along the bulk vertex v . In the second one we put in evidence the normalization factors of the Clebsch-Gordan coefficients and of the 4-valent intertwiners. The function $f_{I_{\bar{v}\bar{v}}}^{i_{\bar{v}\bar{v}}, p}$ denotes the so called fusion coefficient. It can be seen as the kernel of a map between the singlet space in the tensor product of four Spin(4) representations and the singlet space in the tensor product of four SU(2) representations. It is labelled by a node $\bar{v} \in \bar{\mathcal{V}}$ (or by the corresponding half-edge e) carrying the intertwiner degrees of freedom and by the four half-links in the same patch to which the spins are associated.

$$\begin{aligned} f : \text{Inv}_{\text{Spin}(4)} \left[\bigotimes_{i=1}^4 \mathcal{H}^{j_i^-} \otimes \mathcal{H}^{j_i^+} \right] &\rightarrow \text{Inv}_{\text{SU}(2)} \left[\bigotimes_{i=1}^4 \mathcal{H}^{j_i} \right] \\ f_{I_{\bar{v}\bar{v}}}^{i_{\bar{v}\bar{v}}, p}(J_f, j_{\bar{v}\bar{v}}, k_{\bar{v}}, \beta) &= \sum_{Q_{\bar{v}\bar{v}} q_{\bar{v}\bar{v}}} (\mathcal{I}_{Q_{\bar{v}\bar{v}}}^{J_f I_{\bar{v}\bar{v}}}) (\mathcal{I}_{q_{\bar{v}\bar{v}}}^{j_{\bar{v}\bar{v}} i_{\bar{v}\bar{v}}}) \prod_{\bar{v} \in \bar{\mathcal{V}}, \bar{e} = (\bar{v}\bar{v})} F_{q_{\bar{v}\bar{v}}^+ q_{\bar{v}\bar{v}}^-}^{j_{\bar{v}\bar{v}}^- j_{\bar{v}\bar{v}}^+ j_{\bar{v}\bar{v}}, p}(k_{\bar{v}}, \beta) \prod_{\bar{v} \in \bar{\mathcal{V}}, \bar{e} = (\bar{v}\bar{v})} \bar{F}_{q_{\bar{v}\bar{v}}^- q_{\bar{v}\bar{v}}^+}^{j_{\bar{v}\bar{v}}^- j_{\bar{v}\bar{v}}^+ j_{\bar{v}\bar{v}}, p}(k_{\bar{v}}, \beta) \end{aligned} \quad (3.55)$$

The above formula takes into account the orientation of the half-links \bar{e} attached to the boundary node \bar{v} , induced by the orientation of the faces $f \in \mathcal{F}_\alpha$ to which those half-links belong. This issue becomes irrelevant whenever we are allowed to gauge fix the normal k to the identity, since the single-link Fusion coefficients F and their complex conjugate coincide in this case. Upon using the appropriate recoupling identities, the previous definition can be rewritten in terms of Nine- J symbols as reported in Appendix. B. The tensor $\mathcal{B}_{M_{\bar{v}\bar{v}}}^{J_f}$, given below, encodes the boundary data labelling the nodes and half-links of the boundary graph \mathfrak{b}_5 .

$$\mathcal{B}_{M_{\bar{v}\bar{v}}}^{J_f}(i_{\bar{v}\bar{v}}, j_{\bar{v}\bar{v}}, k_{\bar{v}}) = \sum_{m_{\bar{v}\bar{v}}} (\mathcal{I}_{m_{\bar{v}\bar{v}}}^{j_{\bar{v}\bar{v}} i_{\bar{v}\bar{v}}}) \prod_{\bar{v} \in \bar{\mathcal{V}}, \bar{e} = (\bar{v}\bar{v})} C_{m_{\bar{v}\bar{v}}^- m_{\bar{v}\bar{v}}^+ m_{\bar{v}\bar{v}}}^{j_{\bar{v}\bar{v}}^- j_{\bar{v}\bar{v}}^+ j_{\bar{v}\bar{v}}} (k_{\bar{v}}) \prod_{\bar{v} \in \bar{\mathcal{V}}, \bar{e} = (\bar{v}\bar{v})} \bar{C}_{m_{\bar{v}\bar{v}}^- m_{\bar{v}\bar{v}}^+ m_{\bar{v}\bar{v}}}^{j_{\bar{v}\bar{v}}^- j_{\bar{v}\bar{v}}^+ j_{\bar{v}\bar{v}}} (k_{\bar{v}}) \quad (3.56)$$

We now give the explicit formulas of the (constrained) gluing kernel.

The Gluing kernel.

$$\begin{aligned} \mathcal{K}^{\beta,2q}(X_{\bar{v}\bar{v}}, Y_{\bar{v}'\bar{v}}, k_{\bar{v}}, k_{\bar{v}'}) &= \delta(k_{\bar{v}}k_{\bar{v}'}^{-1}) \sum_{J_{\bar{v}}j_{\bar{v}}} \int [dH_{\bar{v}}dH_{\bar{v}'}] \prod_{\bar{v} \in \mathfrak{p}_{\bar{v}}, \mathfrak{p}_{\bar{v}'}} d_{J_{\bar{v}}}d_{j_{\bar{v}}}w^{2q}(J_{\bar{v}}, j_{\bar{v}}, \beta) \left(\delta_{-X_{\bar{v}\bar{v}}} \star E_{H_{\bar{v}}^{-1}} \star \Theta^{J_{\bar{v}}} \star \chi_{k_{\bar{v}}}^{j_{\bar{v}}} \star E_{H_{\bar{v}'}} \right) (Y_{\bar{v}'\bar{v}}) \\ \mathcal{K}^{\beta,2q}(J_{\bar{v}}, M_{\bar{v}\bar{v}}, M_{\bar{v}'\bar{v}}, k_{\bar{v}}, k_{\bar{v}'}) &= \delta(k_{\bar{v}}k_{\bar{v}'}^{-1}) \sum_{I_{\bar{v}}I_{\bar{v}'}} \sum_{i_{\bar{v}}j_{\bar{v}}} d_{I_{\bar{v}}}d_{I_{\bar{v}'}}d_{i_{\bar{v}}} \left[\prod_{\bar{v} \in \mathfrak{p}_{\bar{v}}, \mathfrak{p}_{\bar{v}'}} d_{J_{\bar{v}}}d_{j_{\bar{v}}} \right] (\mathcal{I}_{M_{\bar{v}\bar{v}}}^{J_{\bar{v}}I_{\bar{v}}}(\mathcal{I}_{M_{\bar{v}'\bar{v}}}^{J_{\bar{v}'}}f_{I_{\bar{v}}}^{i_{\bar{v}},q}(J_{\bar{v}}, j_{\bar{v}}, \beta) f_{I_{\bar{v}'}}^{i_{\bar{v}',q}}(J_{\bar{v}'}, j_{\bar{v}'}, \beta) \end{aligned} \quad (3.57)$$

where, as before we have extracted and collected all the normalization factors of the Clebsch-Gordan coefficients and 4-valent intertwiners. The gluing kernel enforces the identification of the variables labelling the two boundary patches associated to the same node $\bar{v} \sim \bar{v}'$ as seen from the two spinfoam atoms sharing it.

Having laid down the relevant definitions, let us now comment on their properties.

Remark 3.12.1 (Structure of the vertex and edge kernels).

1. The vertex kernel depends on twenty lie algebra variables, $(X_{\bar{v}\bar{v}}, Y_{\bar{v}'\bar{v}})$, associated to the ten pairs of half-links $(\bar{e}, \bar{e}') \equiv (\bar{v}\bar{v}, \bar{v}'\bar{v}) \in \mathcal{E}_{\mathfrak{b}_5}$, $\bar{v}, \bar{v}' \in \bar{\mathcal{V}}_{\mathfrak{b}_5}$ of the bisected boundary graph \mathfrak{b}_5 . Equivalently, in terms of simplicial geometry, they are associated to the ten triangles of a geometric 4-simplex (as seen from the reference frames of the two tetrahedra sharing each of those triangles).
2. The vertex kernel encodes the identification of the boundary data associated to the same triangle in different tetrahedral frames up to parallel transport between these frames, given by the group elements $H_{\bar{v}\bar{v}}H_{\bar{v}'\bar{v}}^{-1}$; the sign flip reflects the fact that in an oriented 4-simplex a triangle inherit opposite orientation from the two tetrahedra sharing it.
3. The integrations over the group elements $H_{\bar{v}\bar{v}}$ enforce the closure constraint. Upon Peter-Weyl decomposition they give raise to the four-valent $\text{Spin}(4)$ intertwiners $I_{\bar{v}}$.
4. Notice that we have inserted an harmless integration over an $\text{SO}_k(3)$ holonomy for every edge in the simplicial atom. This can be safely reabsorbed by exploiting the invariance of the Haar measure $dH_{\bar{v}\bar{v}}$, $H_{\bar{v}\bar{v}} \in \text{Spin}(4)$. Using the properties of the single-link fusion coefficients F , this amounts to the insertion of a $\text{SO}(3)$ orthogonal projector P_{h_e} in every edge. The appearance, after performing the integration, of a pair of $\text{SO}(3)$ intertwiners $i_{\bar{v}}$ allows us to recast the Vertex and Gluing kernel's expressions in terms of fusion coefficients $f_{I_{\bar{v}}}^{i_{\bar{v}}}$.
5. The normal vectors $k_{\bar{v}}$ are correlated only indirectly in each 4-simplex through their relation with the bivector boundary variables, in turn dictated by the generalized gauge covariance condition and by the simplicity constraints S^β . The edge kernel \mathcal{K}^β , on the other hand, strictly identifies the normals associated to the same tetrahedron in the two neighbouring 4-simplexes sharing it up to a parallel transport from one to the other.

We now provide the formal expressions of the spin foam amplitudes for an arbitrary (connected) simplicial complexes $\mathfrak{m} \in \mathfrak{M}_S$, for different choices of variables, emphasizing both the general structure they are endowed with and the construction ambiguities they depend on.

3.4.1 Simplicial path integral representation of the spin foam amplitudes in flux variables

Let us consider a generic (connected) simplicial complex and the corresponding spin foam molecule $\mathfrak{m} \in \mathfrak{M}_S$, without boundary. The spin foam amplitude $\mathcal{A}_{\mathfrak{m}}$ in flux variables is obtained by adopting the corresponding representation of the patch Hilbert spaces and contracting the vertex and gluing kernels, following the combinatorics of the molecule \mathfrak{m} , in accordance with the general formula 2.21. A spinfoam molecule, as we said earlier, is a triple $\mathfrak{m} = (\mathcal{V}_{\mathfrak{m}}, \mathcal{E}_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}})$ consisting of a vertex, an edge and a face set built out of the individual atoms corresponding sets after enforcing the bonding relations among them (i.e after gluing the single building blocks to form the molecule). In the following we will omit the pedex \mathfrak{m} , when there is no room for confusion. Upon choosing a reference tetrahedron and an orientation for each face f (together with a consistent enumeration of all vertices and edges belonging to it) the amplitude reads:

$$\mathcal{A}^\beta(\mathfrak{m}) = \int \left[\prod_{v \in \mathcal{V}} \prod_{e \ni v} dH_{ve} \right] \left[\prod_{e \in \mathcal{E}_{\text{int}}} dk_e \right] \prod_{f \in \mathcal{F}} \mathcal{A}_f^\beta(H_{ve}, k_e) \quad (3.58)$$

$$\mathcal{A}_f^\beta(H_{ve}, k_e) = \int \left[\frac{d^6 X_f}{(2\pi)^6} \right]_{v, e \in f} \star \left(E_{H_{ve}^{-1}} \star S_{k_e}^{\beta \star n} \star E_{H_{v'e}} \right) (X_f) \quad (3.59)$$

where $n = 2p + 2q$ and $S^{\beta \star n}$ denotes the (star-product) n th power of the simplicity function S^β . We denoted by H_{ve} the parallel transport along an incoming half-edge e and by X_f the flux associated to the face f . In deriving

this formula we have considered the 4th strategy listed above, which includes the others as sub-cases, whose corresponding amplitudes can thus be read out of the same formula; we have also relied on the commutation properties of simplicity and closure constraint operators, and on the projector nature of the latter. Notice that the amplitude \mathcal{A}_κ is invariant under the following gauge transformations:

$$H_{ve} \rightarrow \xi_e H_{ve} \xi_v^{-1} \quad k_e \rightarrow \xi_e^+ k_e (\xi_e^-)^{-1} \quad X_f \rightarrow \xi_e X_f \xi_e^{-1} \quad \forall H, \xi \in \text{SO}(4), \quad k \in S^3, \quad X_f \in \mathfrak{so}(4) \quad (3.60)$$

Geometrically they can be interpreted as rotations of all local frames, those located at the (center of the) 4-simplices, rotated by the group elements ξ_v , and those located at the (center of the) tetrahedra, rotated by the group elements ξ_e . This gauge invariance also allows to drop all the bulk normals from the amplitude, for example choosing the time gauge $\omega_e \equiv (\omega_e^-, \omega_e^+) = (k_e, \mathbb{I})$. Of course in the case of simplicial complexes with boundary, the amplitude still has an explicit dependence on the normals labelling the boundary tetrahedra.

Importantly, this general expression for the spin foam amplitudes in flux variables can be understood as a simplicial path integral for constrained BF theory. In order to see this more clearly, we can commute all simplicity functions S_k^β with the non-commutative plane waves. This operation also encodes a change of frame of the flux variables by means of the parallel transports of the discrete connection H_{ve} , so that all the fluxes refer to the same reference frame (which can be arbitrarily chosen). Thus the amplitude $\mathcal{A}(\mathbf{m})$ reads:

$$\mathcal{A}^\beta(\mathbf{m}) = \int \left[\prod_{f \in \mathcal{F}} \frac{d^6 X_f}{(2\pi)^6} \right] \left[\prod_{v \in \mathcal{V}} \prod_{e \ni v} dH_{ve} \right] \left[\prod_{e \in \mathcal{E}_{\text{int}}} dk_e \right] \prod_{f \in \mathcal{F}} \left[\star_{e, e' \in f} S_{H_{ee'} \triangleright k_{e'}}^{\beta \star n} \star E_{H_f} \right] (X_f) \quad (3.61)$$

where e denotes the reference frame edge while e' any other edge in the same face. The star product runs on all possible couple of edge indices (e, e') in a given face with the reference edge kept fixed. The simplicity function $S_{H \triangleright k}(\beta)$ imposes on X_f the simplicity condition with respect to the rotated normal $H_{ee'} \triangleright k_{e'}$, namely the pull back of the normal $k_{e'}$ to the chosen reference frame. Upon collecting the simplicity functions all together, the amplitude $\mathcal{A}(\mathbf{m})$ can be further rewritten as follows:

$$\mathcal{A}^\beta(\mathbf{m}) = \int \left[\prod_{f \in \mathcal{F}} \frac{d^6 X_f}{(2\pi)^6} \right] \left[\prod_{e \in \mathcal{E}_{\text{int}}} dk_e \right] \mathcal{D}_\beta^{H_{ve}, k_e}(X_f) \star \prod_{f \in \mathcal{F}} E_{H_f}(X_f) \quad (3.62)$$

where the measure term is detailed as follows:

$$\mathcal{D}_\beta^{H_{ve}, k_e}(X_f) = \prod_{v \in \mathcal{V}} \prod_{e \ni v} dH_{ve} \prod_{f \in \mathcal{F}} \left(\star_{e, e' \in f} S_{H_{ee'} \triangleright k_{e'}}^{\beta \star n} \right) (X_f) \quad (3.63)$$

$$\mathcal{D}_\beta^{H_{ve}, k_e}(X_f) = \prod_{v \in \mathcal{V}} \prod_{e \ni v} dH_{ve} \prod_{f \in \mathcal{F}} \sum_{J_f j_{ef}} d_{J_f} \left[\prod_{e \in f} d_{j_{ef}} w^n(J_f, j_{ef}, \beta) \right] \Theta^{J_f}(X_f) \star \star_{e \in f} \chi_{H_{ee'} \triangleright k_{e'}}^{j_{ef}}(x_f^-, x_f^+) \quad (3.64)$$

Let us interpret the various ingredients of this formula, to confirm our anticipated interpretation as a (non-commutative) simplicial gravity path integral. First, the non-commutative plane waves, once collected together in a single exponential for all faces of the dual complex, give us the exponential of the discrete BF action. In particular the quantization map dictates the prescription for discretizing the curvature two-form in terms of the holonomy of the discrete connection associated to the dual face, via the choice of coordinates $\rho_{\mathcal{Q}}(H_f)$ on the group manifold. For example the FLM map (used in [27]) discretizes it as the holonomy itself, producing the discrete BF action: $\sum_f \text{Tr}(X_f H_f(H_{ve \in f}))$, while the Duflo map, see the new model presented in the following section, expresses it as the logarithm of the same holonomy (see Appendix. A, Eqs. A.25-A.26 for more details). The measure term contains two types of factors.

$$\mathcal{D}_\beta^{H_{ve}, k_e}(X_f) = \prod_{v \in \mathcal{V}} \prod_{e \ni v} dH_{ve} \prod_{f \in \mathcal{F}} \left[S_{k_e}^{\beta \star n} \star \left(\star_{e, e' \in f, e' \neq e} S_{H_{ee'} \triangleright k_{e'}}^{\beta \star n} \right) \right] (X_f) \quad (3.65)$$

One, depending only on the fluxes associated to the tetrahedron chosen as reference, impose the simplicity constraints (with respect to the normal vector of that tetrahedron) on them, by the chosen prescription characterizing the spin foam model. The remaining part can be viewed as (flux-dependent) constraints on the holonomies modifying the Haar measures dH_{ve} on discrete connection parallel transports, used in discrete BF theory. These modifications enforce the requirement that the same discrete connection transports correctly the simplicity constraints across different simplicial frames, i.e. it enforces their covariance with respect to the same discrete connection. The measure also absorbs within it the other construction ambiguities we have mentioned earlier (so it depends on the parameter n). Its origin can be traced back to the use of the extended states, depending explicitly on the tetrahedral normals and to the consequent generalization of the closure constraint

to account for the transformation of the normals upon a change of frame (which had been advocated for a while in the spin foam and loop quantum gravity literature [25, 86, 96, 97]). Both are required in order to ensure a consistent covariant imposition of the geometricity constraints.

Thus the vacuum amplitudes of any spin foam model (of the class considered) in flux variables take always the form of non-commutative first order simplicial path integrals for a constrained BF theory of the Holst-Plebanski type with a discretization of the BF action depending on the chosen quantization map for the fluxes, and a covariant (flux-dependent) measure on discrete connection $\mathcal{D}_\beta^{k,H,X}$ encoding the geometricity constraints, their covariance, and other model-dependent features. We stress once more that this result, first obtained for a specific model in [27, 28] (and for BF theory in [98]) is *general*; it is a feature of the flux representation of all models in this constrained BF class, and not on specific choices of constraint imposition.

3.4.2 Holonomy representation of the amplitudes.

The same amplitudes can be rewritten in the group representation, exploiting the duality between the metric and the holonomy formulation of spin foam models, at the root of the non-commutative Fourier transform.

$$\mathcal{A}^\beta(\mathbf{m}) = \int \left[\prod_{v \in \mathcal{V}} \prod_{e \ni v} dH_{ve} \right] \left[\prod_{e \in \mathcal{E}_{\text{int}}} dk_e \right] \prod_{f \in \mathcal{F}_{\text{cl}}} \mathcal{A}_f^\beta(H_{ve}, k_e) \quad (3.66)$$

$$\mathcal{A}_f^\beta(H_{ve}, k_e) = \sum_{J_f j_{ef}} d_{J_f} \int \left[\prod_{e \in f} du_{ef} \right] \Theta^{J_f} \left[\prod_{e \in f} H_{ve} U_{ef} (H_{v'e})^{-1} \right] \left[\prod_{e \in f} d_{j_{ef}} w^n(J_f, j_{ef}, \beta) \chi^{j_{ef}}(u_{ef}) \right] \quad (3.67)$$

The expression naturally generalizes to simplicial complexes with boundary, with appropriate additional boundary data (depending on the boundary conditions chosen, here the boundary discrete connection elements).

$$\mathcal{A}_m^\beta(G_f, \tilde{G}_f, k_e) = \int \left[\prod_{v \in \mathcal{G}} \prod_{e \ni v} dH_{ve} \right] \left[\prod_{e \in \mathcal{E}_{\text{int}}} dk_e \right] \prod_{f \in \mathcal{F}_{\text{cl}}} \mathcal{A}_f^\beta(H_{ve}, k_e) \prod_{f \in \mathcal{F}_{\text{op}}} \mathcal{A}_f^\beta(H_{ve}, k_e, G_f, \tilde{G}_f) \quad (3.68)$$

Each open face has two boundary edges, namely two edges, say $e_{\text{in}}, e_{\text{out}} \in \mathcal{E}_{\text{ext}}$, with one of their end points being a boundary node. Since we are enforcing the constraints on both kernels \mathcal{K} and \mathcal{V}^β , bulk and boundary edges will pick up different powers of the simplicity function $w(J_f, j_{ef}, \beta)$. Hence, keeping this in mind, the amplitude \mathcal{A}_f associated to an open face $f \in \mathcal{F}_{\text{op}}$ is given by:

$$\mathcal{A}_f^\beta(H_{ve}, k_e, G_f, \tilde{G}_f) = \sum_{J_f j_{ef}} d_{J_f} \int \left[\prod_{e \in f} du_{ef} \right] \left[\prod_{s=\text{in}, \text{out}} d_{j_{esf}} w^{p+q}(J_f, j_{esf}, \beta) \chi^{j_{esf}}(u_{esf}) \right] \quad (3.69)$$

$$\times \Theta^{J_f} \left[G_f H_{\bar{v}e_{\text{in}}} U_{e_{\text{in}}f} \prod_{e \in \mathcal{E}_{\text{int}}} H_{ve} U_{ef} H_{v'e}^{-1} U_{e_{\text{out}}f} H_{\bar{v}'e_{\text{out}}}^{-1} \tilde{G}_f^{-1} \right] \left[\prod_{e \in \mathcal{E}_{\text{int}}} d_{j_{ef}} w^n(J_f, j_{ef}, \beta) \chi^{j_{ef}}(u_{ef}) \right] \quad (3.70)$$

Thus in the holonomy formulation, the spinfoam amplitude takes the form of a lattice gauge theory amplitude. This is a *general* result, common to all spinfoam models for constrained BF theory.

3.4.3 Spin representation of the amplitudes

The spin representation of the amplitudes can be obtained, for example, by Peter-Weyl decomposition of the above lattice gauge theory amplitudes. The expression for them amplitudes is then:

$$\mathcal{A}^\beta(\mathbf{m}) = \sum_{J_f j_{ef}} \sum_{I_{ve} i_e} \prod_{f \in \mathcal{F}} d_{J_f} \prod_{e \in f} d_{j_{ef}} \prod_{v \in \mathcal{V}} \{15J_f\}_v \prod_{(\bar{v}v) \equiv e \in \mathcal{E}} d_{I_{ve}} \sqrt{d_{i_e}} f_{I_{ve}}^{i_e, r}(J_f, j_{ef}, k_e, \beta) \quad (3.71)$$

where $r = p + n$. In the case of molecules/triangulations with boundary the amplitude reads:

$$\mathcal{A}_m^\beta(J_f, M_{ef}, k_e) = \sum_{J_f, f \in \mathcal{F}_{\text{cl}}} \sum_{I_{ve}, i_e, j_{ef}} \prod_{f \in \mathcal{F}} d_{J_f} \prod_{e \in f} d_{j_{ef}} \prod_{v \in \mathcal{V}} \{15J_f\}_v \times \prod_{(\bar{v}v) \equiv e \in \mathcal{E}_{\text{int}}} d_{I_{ve}} \sqrt{d_{i_e}} f_{I_{ve}}^{i_e, r}(J_f, j_{ef}, k_e, \beta) \prod_{(\bar{v}v) \equiv e \in \mathcal{E}_{\text{ext}}} d_{I_{ve}} d_{i_e} f_{I_{ve}}^{i_e, p}(J_f, j_{ef}, k_e, \beta) \mathcal{B}_{M_{ef}}^{J_f}(i_e, j_{ef}, k_e) \quad (3.72)$$

The notation goes as follows. As before, the pedices v, e, f denote the vertices, the edges and the faces of \mathbf{m} . Furthermore we have a Spin(4) representation J_f labelling each face, a pair of Spin(4) four-valent intertwiners $I_{ve}, I_{v'e}$ for every edge and an SO(3) spin j_{ef} for each edge in a given face. Once more, the above formulae are general for this class of constrained BF models, and the specificities of the model lie into the exact form of the fusion coefficients and in the parameter r .

4 Flux variables and the Duflo map. A new spin foam model for Holst-Plebanski quantum gravity

We now present a new spin foam model, constructed focusing on flux/metric variables, which allow for a geometrically transparent imposition of the simplicity constraints, and quantizing them using the Duflo quantization map (more details on it in the Appendix. A.). The flux-based procedure that we use has been introduced in [27, 28] (see also [99–101] for its use in LQG), so the main novelty of our model is only the latter. The use of the Duflo map is however a potentially important aspect, since the Duflo map has a number of attractive mathematical properties. A notable one is that it represents faithfully the subalgebra of invariant functions of the Lie algebra it is applied to, which already makes it the ‘most natural’ quantization map for systems in which the gauge invariance is a key aspect. A second, and possibly even more important in our context, is that it is defined for any semi-simple (locally finite) group, thus in particular it allows an immediate generalization of the construction to the Lorentzian setting, based on the group $SL(2, \mathbb{C})$, which we leave for future work. This was not the case, for example, for the FLM map used to derive the model of [27].

4.1 Simplicity constraints and non-commutative tetrahedra.

As we have seen, the discrete linear simplicity constraint, in the Riemannian setting, amounts to a relation between the selfdual and anti-selfdual part of the bivectors associated to the four triangles of any tetrahedron in the triangulation dual to the spin foam molecule, where the split is defined with respect to the normal vector to the same tetrahedron. The flux/metric formulation of spin foam models, based on the non-commutative Lie algebra representation of the same where the basic variables are the bivectors themselves, allows us to implement this simplicity condition in a geometrical transparent way: the simplicity condition is imposed directly on flux variables by star-multiplication of states (and then, amplitudes) with a non-commutative delta function. We have already presented the general construction of arbitrary spin foam models in this constrained BF theory class, and in particular how the simplicity constraint operator is defined. Therefore, in order to define the new model, we only need to introduce the key elements of the non-commutative flux formulation based on the Duflo quantization map, the corresponding plane wave, and then the associated non-commutative delta function imposing the constraints. Below, then, we report the basic definitions, according to the results obtained for $SU(2)$ in [67], which can be extended to $Spin(4)$ using its selfdual/anti-selfdual split.

$$\mathcal{D} = \mathcal{S} \circ \mathcal{J}^{\frac{1}{2}}(\partial) \quad \mathcal{J}(x) = \det \left(\frac{\sinh \frac{1}{2} \text{ad}_x}{\frac{1}{2} \text{ad}_x} \right) = \left(\frac{\sinh |x|}{|x|} \right)^2 \quad x \in \mathfrak{su}(2) \quad (4.1)$$

where \mathcal{S} denotes the symmetric quantization map. The Duflo map is thus an extension of the symmetric map lifting it an algebra isomorphism between the subalgebra of invariant polynomials $\text{Sym}(\mathfrak{g})^{\mathfrak{g}}$ and the center of the universal enveloping algebra $U(\mathfrak{g})$, denoted by $U(\mathfrak{g})^{\mathfrak{g}}$. For the Duflo map the associated star-product, the non-commutative plane waves and the integral expansion of the delta function take the following form:

$$x_i \star_{\mathcal{D}} x_l = \mathcal{D}^{-1}(\mathcal{D}(x_i)\mathcal{D}(x_l)) = x_i x_l + i \epsilon_{il}^k x_k - \frac{1}{3} \delta_{il} \quad x_i, x_l \in \mathfrak{su}(2) \quad (4.2)$$

$$E_{\star_{\mathcal{D}}}(g, x) = \frac{|\vec{k}|}{\sin |\vec{k}|} e^{i\vec{k} \cdot \vec{x}} \quad \delta_{\star_{\mathcal{D}}}(x - y) = \int dg E_g(x) E_{g^{-1}}(y) = \int \frac{d^3 \vec{k}}{(2\pi)^3} e^{i\vec{k} \cdot \vec{x}} e^{-i\vec{k} \cdot \vec{y}} \quad |\vec{k}| \in [0, \pi[\quad (4.3)$$

where the vector $\vec{k}(g) \in \mathbb{R}^3$ parametrizes the group element and $\vec{x} \equiv \{x_i\}$ denotes the vector of components of the flux variable $x = x_i \sigma^i \in \mathfrak{su}(2)$. Further details can be found in Appendix. A. As anticipated the simplicity constraint imposition amounts to the insertion of a non-commutative delta-function in the BF theory’s amplitudes, thus corresponding to a simplicity constraint operator defined as follows:

Definition 4.1 (Simplicity constraint op. for the new model). Let S_{new}^{β} be an operator such that:

$$\forall \Psi_k \in \mathcal{H}_{\vec{v}} = L_{\vec{x}}^2[\mathfrak{spin}(4)^{\times 4} \times S^3] \quad \tilde{\Psi}_k^{\beta}(X_i) = (S_{\text{new}}^{\beta} \Psi_k)(X_i) = \int \left[\prod_{i=1}^4 \frac{d^6 Y_i}{(2\pi)^6} \right] S_{\text{new}}^{\beta}(X_i, Y_i, k) \star \Psi_k(Y_i) \quad (4.4)$$

$$S_{\text{new}}^{\beta}(X_i, Y_i, k) = \prod_{i=1}^4 \left(\delta_{-X_i} \star S_k^{\beta} \right) (Y_i) \quad S_k^{\beta}(X) = \delta_{-kx-k^{-1}}(\beta x^+) = \int du E_{k^{-1}uk}(x^-) E_u(\beta x^+) \quad (4.5)$$

For convenience of notation we drop the pedex ”new” from now on whenever there is no room for ambiguity.

Before turning to the resulting spin foam amplitudes, let us show that the action of this operator is well defined. To be able to take the star product with the state $\Psi_{\vec{v}}$ we need the simplicity functions S_k^{β} to be in the image

of the non-commutative Fourier transform. Therefore we have to rewrite the expression (4.5) in such a way that the plane waves are functions of the same lie algebra variable x^-, x^+ labelling the single tetrahedron wave function. The above requirement is satisfied if the following equality holds:

$$E_u(\beta x) = \Omega(\psi_u, \beta) E_{u^\beta}(x) \quad (4.6)$$

namely if and only if it is possible to encode the Duflo plane wave's dependence on the Immirzi parameter into a suitable defined $SU(2)$ group element u^β up to a function of β and of the class angle ψ_u . If the above equation admits a unique non-trivial solution then the action of the simplicity operator S_k^β is well defined. For the case of the FLM quantization map this was proven in [27]. In our case (Duflo map) it is shown below.

Proposition 4.1 (Modified plane wave). Let $E_u(x) : \mathfrak{su}(2) \times SU(2) \rightarrow \mathbb{C}$ be a Duflo plane wave. Then:

$$\exists! u^\beta \in SU(2), \quad \Omega : [0, 2\pi[\times [-1, 1] \rightarrow \mathbb{R}, \quad \text{s.t.} \quad E_u(\beta x) = \Omega(\psi_u, \beta) E_{u^\beta}(x) \quad \beta \in [-1, 1] \quad \Rightarrow \quad (4.7)$$

$$S_k^\beta(X) = \int du \Omega(\beta, \psi_u) E_{k^{-1}uk}(x^-) E_{u^\beta}(x^+) = \int \mathcal{D}^\beta \mathcal{U} E_{\mathcal{U}}(X) \quad \mathcal{U} = (k^{-1}uk, u^\beta) \quad \mathcal{D}^\beta \mathcal{U} = \Omega(\psi_u, \beta) du$$

Proof. The proof is straightforward. Upon using the explicit formulas of the Duflo plane waves we find:

$$u_\beta = e^{i \frac{\psi_\beta}{2} \hat{n}_\beta \cdot \vec{\sigma}} \quad \psi_\beta = |\beta| \psi \quad \hat{n}_\beta = \text{sign}(\beta) \hat{n} \quad \Omega(\beta, \psi) = \frac{\sin \frac{|\beta| \psi}{2}}{|\beta| \sin \frac{\psi}{2}} \quad (4.8)$$

where the function Ω is needed to reconstruct the correct prefactor of the Duflo plane wave. Moreover the values' ranges of the angles sparametrizing the group manifold $S^3 \simeq SU(2)$, namely $(\psi, \theta, \phi) \in [0, 2\pi[\times [0, \pi[\times [0, 2\pi[$, ensure the uniqueness of the above stated solution. It is worth to stress once again that the star-product with the modified delta function S_k^β is only defined via its integral expression in term of non-commutative plane waves.

Before giving the expressions for the same operator in group and spin basis, let us stress few of its properties.

Remark 4.1.1 (Properties of S_{new}^β).

1. The simplicity constraint operator S^β is not an orthogonal projector for generic values of β . It is a projector only for the special cases $\beta = 0, 1$ (where the simplicity constraints become first class in the canonical theory).
2. It commutes with the closure projector P_{cl} up to a rotation of the normal k , due to the properties of the non-commutative plane waves (see Appendix. A). This is due to the fact that S_k^β transforms covariantly under a $\text{Spin}(4)$ transformation of both bivectors and normals (it is invariant under the action of the stabilizer group).

$$S_{H \triangleright k}^\beta(HX_i H^{-1}, Y_i) = S_k^\beta(X_i, Y_i) \quad \forall H \in \text{Spin}(4), \quad k \in S^3, \quad X_i, Y_i \in \mathfrak{spin}(4) \quad (4.9)$$

Geometrically this relation expresses the fact that rotating a bivector which is simple with respect to a given normal k gives a bivector which is simple with respect to the rotated normal $H \triangleright k$.

3. It does not require any restriction, e.g. any rationality condition, on the (Immirzi) parameter β .
4. It can be rewritten according to the general formulas (3.43). This statement rests on the following identity

$$S_k^\beta(X) = \delta_{kx^- k^{-1}}(\beta x^+) = \sum_{Jj} d_J d_j w_{\text{new}}(J, j, \beta) \Theta^J(X) \star \chi^j(kx^- k^{-1} + x^+) \quad (4.10)$$

proved in Appendix. C. The coefficient w_{new} , given below, encodes the new model's simplicity constraints.

In the group and spin polarizations, the simplicity operator is given as follows (note $U = (k^{-1}uk, u)$):

$$\begin{aligned} S^\beta(G_i, \tilde{G}_i, k) &= \int \left[\prod_{i=1}^4 \mathcal{D}^\beta u_i \right] \prod_{i=1}^4 \delta[G_i \mathcal{U}_i \tilde{G}_i^{-1}] = \sum_{J_i j_i} \int \left[\prod_{i=1}^4 du_i \right] \prod_{i=1}^4 d_{J_i} d_{j_i} w(J_i, j_i, \beta) \Theta^{J_i} [G_i U_i \tilde{G}_i^{-1}] \chi^{j_i}(u_i) \\ S_{M_1, \dots, M_4 N_1, \dots, N_4}^{J_1, \dots, J_4}(k, \beta) &= \prod_{i=1}^4 S_{m_i^- m_i^+ n_i^- n_i^+}^{J_i^- j_i^+}(k, \beta) = \int \left[\prod_{i=1}^4 \mathcal{D}^\beta u_i \right] \prod_{i=1}^4 D_{m_i^- n_i^-}^{J_i^-}(k^{-1} u_i k) D_{m_i^+ n_i^+}^{j_i^+}(u_i^\beta) \end{aligned} \quad (4.11)$$

where for shortness $U = (k^{-1}uk, u) \in \text{SO}_k(3)$. Like in the flux representation we have two equivalent expressions of the simplicity constraint operator in the holonomy picture as well. The presence of a closed integral formula both in flux and group variables is a peculiar feature of our construction which in turn leads to an alternative formulation of the vertex and gluing kernels and thus of the resulting spinfoam amplitudes. We can exhibit a few

more expressions for the same functions, which turn out to be useful for explicit computations. Further details are provided in Appendix. C. By using the formula (B.8) for the Wigner functions we find:

$$S_{m^-m^+n^-n^+}^{j^-j^+}(\beta, \mathbb{I}) = \frac{1}{\pi(2j^-+1)(2j^++1)} \sum_{a=0}^{\lambda} \sum_{\mu=-a}^a (\text{Sign}(\beta))^a (-1)^{-\mu} C_{m^-\mu n^-}^{j^-aj^-} C_{m^+-\mu n^+}^{j^+aj^+} \mathcal{T}_a^{j^-j^+}(|\beta|) \quad (4.12)$$

$$\mathcal{T}_a^{j^-j^+}(|\beta|) = (-1)^a (2a+1) \int_0^{2\pi} d\psi \Omega(\beta, \psi) \sin^2 \frac{\psi}{2} \chi_a^{j^-}(\psi) \chi_a^{j^+}(\psi_\beta) \quad (4.13)$$

where $\lambda = 2 \text{Min}(j^-, j^+)$ and χ_a^j denotes the generalized character of the SU(2) representations (see Eq. B.10). Using the previous formulas, the matrix elements of the simplicity operator $S_{MN}^J(\beta)$ takes the following form:

$$S_{m^-m^+n^-n^+}^{j^-j^+}(\beta, \mathbb{I}) = \sum_{jm} C_{m^-m^+m}^{j^-j^+j} C_{n^-n^+m}^{j^-j^+j} w_{\text{new}}(j^-, j^+, j, \beta) \quad (4.14)$$

$$w_{\text{new}}(j^-, j^+, j, \beta) = \frac{(-1)^{j^-+j^++j}}{\pi \sqrt{(2j^-+1)(2j^++1)}} \sum_{a=0}^{\lambda} (\text{Sign}(\beta))^a \left\{ \begin{matrix} a & j^- & j^- \\ j & j^+ & j^+ \end{matrix} \right\} \mathcal{T}_a^{j^-j^+}(|\beta|) \quad (4.15)$$

The form factor $w(J, j, \beta)$, characterizing the simplicity constraint imposition in our model, behaves like a modulating weight factor peaking on different configurations, including the ones selected for example by the EPRL model. A detailed numerical investigation of its behaviour and properties in different kinematical regimes will appear in a forthcoming paper [102]. The presence of an explicit closed formula for w_{new} is a key asset in trying to extract quantitative consequences of the model's amplitudes, for example the scaling of the leading order radiative corrections of the (connected) n -point functions $\mathcal{W}^{(n)}$ in the large- j regime. This is the object of a larger project whose results will be published in [103, 104]. The presence of an explicit closed formula of its amplitudes in the group representation is another very useful property for calculations in the context of spin foam and GFT renormalization, enabling the use of heat kernel methods in the power counting analysis. This was true also for the flux-based model using the FLM map [27]. However the linear relation between ψ_β and ψ induced by the Dufo map, contrasted to the non-linear one dictated by the FLM map (namely $\frac{\psi_\beta}{2} = \arcsin(|\beta| \sin \frac{\psi}{2})$) simplifies the evaluation of the integrals.

Let us now discuss three interesting limiting cases of these coefficients, thus of the new model. We restrict our attention to the weight factor w_{new} , which characterizes it. A more detailed derivation of the results hereafter summarized can be found in Appendix. C.

Remark 4.1.2 (Limiting cases). For $\beta = 1$, corresponding to the Plebanski case in which the Holst term disappears, we can immediately evaluate the coefficient w_{new} , to find:

$$w(j^-, j^+, j, 1) = \delta_{j-j^+} \delta_{j0} \quad (4.16)$$

The simplicity constraint operator also becomes a projector. The result coincides with the restriction on representations defining the Barrett-Crane model in the euclidean signature modulo the choice of the face weights. These factors are not determined by the quantization map and thus must be prescribed by hand. In our case, in particular, we recover the (revised) version of the euclidean Barrett-Crane model already presented in [28].

For $\beta = 0$, corresponding to $\gamma = 1$, one expects the theory to corresponds to selfdual gravity. However, some of the change of variables used in the definition and discretization of the simplicity constraints may become singular, thus the limit has to be taken with care in the classical theory. Still, the constraint operator \mathcal{S}^β and the resulting spin foam model are well defined in this limit. In particular the function w_{new} is continuous at $\beta = 0$, where it reads⁶

$$w(j^-, j^-, j, 0) = \frac{2(-1)^{2j^-}}{(2j^-+1)^2} \quad (4.17)$$

The simplicity constraint operator for $\beta = 0$ acts on the single-tetrahedron state $\Psi_{\bar{v}}$ by projecting its bivectors onto the selfdual part of $\mathfrak{spin}(4)$, and the constrained model reduces to the Ooguri spin foam model for topological SU(2) BF theory [16].

The case $\beta = -1$, i.e. $\gamma = 0$, corresponds classically to the so-called Holst sector of the Plebanski gravity. This denomination come from the fact that the Holst term of the classical Plebanski-Holst action dominates in this limit. This term vanishes on shell due to the torsion freeness requirement of the spin connection, thus

⁶The simplicity function $S_{MN}^J(\beta)$ and the coefficient $w_{\text{new}}(J, j, \beta)$ are both continuous functions of β in the full interval $\beta \in [-1, 1]$ up to an harmless redefinition in $\beta = 0$ where they have a removable singularity (see Appendix. C for an elementary proof).

one would be tempted to assume that the corresponding theory is topological. However, the limit in the formal quantum theory for the Plebanski-Holst action in the continuum is subtle, and it has been argued that the resulting quantum theory would correspond to a spin foam quantization of 2nd order metric gravity with no torsion [7, 105]. In this limit the coefficient w_{new} is given by:

$$w(j^-, j^+, j, -1) = \frac{(-1)^{2j^-+j}}{2j^-+1} \delta_{j-j^+} \{j^-, j^+, j\} \quad (4.18)$$

Therefore for $\beta = -1$ the simplicity constraint operator projects onto simple Spin(4) representation of the type $J = (j^-, j^+) = (l, l)$ without imposing any restriction (apart from the triangular inequalities) on the SU(2) representation j labelling the direct sum expansion of the tensor product of simple representations $\mathcal{H}^l \otimes \mathcal{H}^l = \bigoplus_{j=0}^{2l} \mathcal{H}^j$. Moreover it multiplies each pair (J, j) by an extra phase factor $(-1)^{2j^-+j}$. Summarizing the resulting spin foam model can be obtained from the ordinary Spin(4) Ooguri model by restricting its representations to simple ones as dictated by the function $w_{\text{new}}(\beta = -1)$.

Let us also point out that the simplicity function (4.5) itself (and not just its Fourier modes $S_{MN}^J(\beta)$) is well behaved in the limits $\beta = \pm 1$ and $\beta = 0$. In the first case this is manifestly true. Thus no proof is required. In the last one for the Duflo (and the FLM) quantization maps a quick calculation yields:

$$\lim_{\beta \rightarrow 0} S_{\text{new}}^\beta(x^-, x^+) = \int du E_u(x^-) \lim_{\beta \rightarrow 0} \Omega(\psi, \beta) E_{u^\beta}(x^+) = \int du \Omega(\psi) E_u(x^-) \quad \Omega(\psi) = \frac{\frac{\psi}{2}}{\sin \frac{\psi}{2}} \quad (4.19)$$

$$\lim_{\beta \rightarrow 0} S_{\text{FLM}}^\beta(x^-, x^+) = \int du E_u(x^-) \lim_{\beta \rightarrow 0} E_{u^\beta}(x^+) = \int du E_u(x^-) \quad (4.20)$$

since in the above limit $u^\beta \rightarrow \mathbb{I}$ and $E_{\mathbb{I}}(x) = 1$ regardless of the choice of the quantization map.

Remark 4.1.3 (The role of the quantization map). In the limit $|\beta| \rightarrow 1$ the simplicity function $S_k^\beta(X)$ reduces, by construction, to an ordinary non-commutative delta function both for the Duflo and for the FLM quantization maps⁷. Thus one could ask whether, in this limit, the choice of the quantization map becomes irrelevant. This turns out not to be the case since the non-commutative delta depends, by definition, on the *star*-product, as one can easily check by writing down the corresponding integral expression in terms of non-commutative plane waves. Gauge fixing the S^3 normal for convenience, this shows that the two delta distributions, $\delta_{\star\text{FLM}}$ and $\delta_{\star\varnothing}$, seen as functions on the lie algebra $\mathfrak{su}^*(2)$, do not coincide.

$$S_{\text{new}}^{|\beta|=1}(x^-, x^+) = \delta_{\star\varnothing}(x^- \pm x^+) = \frac{1}{(2\pi)^2} \int dS_2 \int d|\vec{k}| |\vec{k}|^2 e^{i\vec{k} \cdot \vec{x}^-} e^{\pm i\vec{k} \cdot \vec{x}^+} \quad (4.22)$$

$$S_{\text{FLM}}^{|\beta|=1}(x^-, x^+) = \delta_{\star\text{FLM}}(x^- \pm x^+) = \frac{1}{(2\pi)^2} \int dS_2 \int d|\vec{k}| \sin^2 |\vec{k}| e^{i \frac{\sin |\vec{k}|}{|\vec{k}|} \vec{k} \cdot \vec{x}^-} e^{\pm i \frac{\sin |\vec{k}|}{|\vec{k}|} \vec{k} \cdot \vec{x}^+} \quad (4.23)$$

However they do behave as regular Dirac delta functions under integration upon \star -multiplication (with respect to the associated \star -product) by an arbitrary function $f \in L^2(\mathfrak{g}^*)$. Therefore, in the above limit, the Fourier components of the simplicity function (obtained upon non-commutative Fourier transform) and more in general the spinfoam amplitude for molecules without boundaries will always coincide regardless of the choice of the quantization map. Indeed using (one copy of) the Eq. (4.11) one can explicitly check that:

$$S_{MN}^J(|\beta| = 1, \mathbb{I})_{\text{new}} = S_{MN}^J(|\beta| = 1, \mathbb{I})_{\text{FLM}} \quad w_{\text{new}}(J, j, \beta = \pm 1) = w_{\text{FLM}}(J, j, \beta = \pm 1) \quad (4.24)$$

The fact that the vacuum Spinfoam amplitudes coincide is not a puzzling issue. Indeed the physical content of a quantum theory is encoded by the observables rather than by the vacuum amplitudes. As emphasized in [67] (see also Appendix. A, Eq. A.5), there is a precise relation between a quantum operator O_f and the classical function f_\star which upon quantization gives O_f . Such relation, establishing a connection between the classical phase space and the quantum operators, is controlled by the star product and thus by the choice of the quantization map. Thus, according to general argument, the quantum observables have to depend on the

⁷The FLM quantization map is only defined for SU(2) and can be trivially extended to Spin(4) by "double copy" [68]. The expression of the simplicity function with the FLM map (BO model [27]) is formally identical to our own one (based on Duflo). The key differences are encoded by the deformed group element u^β , the prefactor $\Omega(\beta, \psi)$ and the non-commutative plane wave. Namely:

$$u_\beta = e^{i \frac{\psi_\beta}{2} \hat{k}_\beta \cdot \vec{\sigma}} \quad |\vec{k}_\beta| \equiv \frac{\psi_\beta}{2} = \arcsin(|\beta| \sin \frac{\psi}{2}) \quad \hat{k}_\beta = \text{Sign}(\beta) \hat{k} \quad |\vec{k}| \equiv \frac{\psi}{2} \in [0, \pi[$$

$$E_{\star\text{FLM}}(g, x) = e^{i \text{Tr}[xg]} = e^{i \sin |\vec{k}| \vec{k} \cdot \vec{x}} \quad \Omega_{\text{FLM}}(\beta, \psi) = 1 \quad (4.21)$$

The previous formulas can be used to compute the plane wave expansion of the constraint delta $S_{\text{FLM}}^\beta(X, \mathbb{I}) \equiv \delta_{\star\text{FLM}}(x^- + \beta x^+)$

operator ordering conventions even in the limit $|\beta| \rightarrow 1$ (though the model's vacuum amplitudes do not). Unlike the previous case, for arbitrary values of $\beta \neq \pm 1$ the model's spinfoam amplitudes do explicitly depend on the operator ordering even for molecules without boundaries. In the holonomy picture, for instance, the expressions of the simplicity functions and of the vacuum Spinfoam amplitudes for the Duflo and the FLM map differ by the parametric deformation of the group element u^β and by the prefactor $\Omega(\psi_u, \beta)$.

For arbitrary values of β our model encompasses several distinct models, interpolating between them. It would be interesting to further investigate and characterize its critical properties by studying its RG flow.

4.2 The amplitudes of the new model

We now present the spin foam amplitudes of the new model. They can be derived straightforwardly from the general definitions introduced in Section. 3.4. The formulas for the new models's kernels \mathcal{K}^β and \mathcal{V}^β in the flux and group representations are provided in Appendix. D.

4.2.1 Flux representation.

Let us introduce the following short-hand notation. Since we are going to insert the simplicity constraint operator both in the edge and vertex kernels (in multiple copies according to the Eqs. 3.52,3.53) it is useful to define a group element denoting the product of the "lagrange multipliers" enforcing these constraints. Thus we have:

$$\mathcal{Q}^{(n)} = \left(k^{-1} \prod_{a=1}^n u_a k, \prod_{a=1}^n u_a^\beta \right) \quad \mathcal{D}^\beta \mathcal{Q}^{(n)} = \prod_{a=1}^n du_a \Omega(\beta, \psi_{u_a}) \quad u \in \text{SU}(2) \quad (4.25)$$

where the integer n counts the number of insertion of the simplicity constraint operator. The amplitude \mathcal{A}_m can be computed exactly as in the general case by taking the appropriate convolution of vertex and gluing kernels. For a closed simplicial complex dual to a spin foam molecule \mathbf{m} without boundary $\partial \mathbf{m} = \emptyset$, we find:

$$\mathcal{A}_{\text{new}}^\beta(\mathbf{m}) = \int \left[\prod_{v \in \mathcal{V}} \prod_{e \ni v} dH_{ve} \right] \left[\prod_{e \in \mathcal{E}_{\text{int}}} dk_e \right] \prod_{f \in \mathcal{F}_{\text{cl}}} \mathcal{A}_f^\beta(H_{ve}, k_e) \quad (4.26)$$

$$\mathcal{A}_f^\beta(H_{ve}, k_e) = \int \left[\frac{d^6 X_f}{(2\pi)^6} \right] \left(\star_{v, e \in f} E_{H_{ve}^{-1}} \star S_{k_e}^{\beta \star n} \star E_{H_{v'e}} \right) (X_f) \quad (4.27)$$

where $n = 2p + 2q$ and $S_k^{\beta \star n}$ denotes the star-product n th power of the simplicity function S_k^β . Like in the general case we have adopted the 4th strategy among those listed in Section 3 which encompasses all other cases. Upon expanding the simplicity function S^β into plane waves, the faces amplitudes \mathcal{A}_f take the following form:

$$\mathcal{A}_f^\beta(H_{ve}, k_e) = \int \left[\frac{d^6 X_f}{(2\pi)^6} \right] \left[\prod_{e \in f} \mathcal{D}^\beta \mathcal{Q}_e^{(n)} \right] E_{\left(\prod_{e \in f} H_{ve}^{-1} \mathcal{Q}_e^{(n)} H_{v'e} \right)} (X_f) \quad (4.28)$$

The amplitude $\mathcal{A}(\mathbf{m})$ is invariant under the simultaneous rotation of all local frames. Therefore it can be evaluated in the time gauge by setting $k_e = \mathbb{I}$ for all the internal edges. As we have discussed for the general case, in order to recast the amplitudes of this model in the form of a simplicial path integral for constrained BF theory all simplicity functions S_k^β have to be commuted with the plane waves. The following property holds:

$$\forall l \in \mathbb{N}, l \leq N_{ef} \quad E_{H_{e_0 e_l}} \star S_{k_{e_l}}^\beta = S_{H_{e_0 e_l} \triangleright k_{e_l}}^\beta \star E_{H_{e_0 e_l}} \quad (4.29)$$

where the group element $H_{e_0 e_l}$ is the parallel transport from the frame "l" to the chosen reference frame. Samely H_f denotes the holonomy of the connection around a given face $f \in \mathcal{F}_{\text{cl}}$. Therefore, the amplitude $\mathcal{A}(\mathbf{m})$ reads:

$$\mathcal{A}_{\text{new}}^\beta(\mathbf{m}) = \int \left[\prod_{f \in \mathcal{F}_{\text{cl}}} \frac{d^6 X_f}{(2\pi)^6} \right] \left[\prod_{e \in \mathcal{E}_{\text{int}}} dk_e \right] \mathcal{D}_{\text{new}}^\beta(H_{ve}, k_e, X_f) \star \prod_f E_{H_f}(X_f) \quad (4.30)$$

$$\mathcal{D}_{\text{new}}^\beta(H_{ve}, k_e, X_f) = \left[\prod_{v \in \mathcal{V}} \prod_{e \ni v} dH_{ve} \right] \prod_{f \in \mathcal{F}_{\text{cl}}} \left[\star_{e, e' \in f} \delta_{(H_{ee'} \triangleright k_{e'})_{X_f} (H_{ee'} \triangleright k_{e'})^{-1}}^{\beta \star n} (\beta x_f^+) \right] \quad (4.31)$$

where e denotes the reference frame edge while e' any other edge in the same face. The star-product runs on all possible pairs of edge labels (e, e') in a given face with the reference edge kept fixed. Moreover the non-commutative plane wave gives us the exponential of the discretized BF action

$$E_{H_f}(X_f) = \eta(H_f) e^{i \text{Tr}[\log(H_f) X_f]} = \eta(H_f) e^{i \tilde{\zeta}(H_f) \cdot X_f} = \eta(H_f) e^{i S_f^{\text{BF}}[H_f, X_f]} \quad (4.32)$$

and the effective covariant measure satisfies the general properties discussed in Section. 3.4.

We see again the explicit expression for a non-commutative simplicial path integrals for a constrained BF theory of Holst-Plebansky type, with a discretized BF action depending on the chosen quantization map for the fluxes, and with a measure $\mathcal{D}_\beta^{k,H}$ capturing the construction choices (and quantization map) characterizing the model. The appearance of an effective measure on the space of discrete connection is a direct consequence of the use of extended states and of the relaxation of the closure constraint (as advocated in the literature [25, 94, 96, 97]) both required to ensure a consistent imposition of the simplicity constraints.

4.2.2 Group and Spin representation.

The pure lattice gauge formulation of the model can be obtained either by non-commutative Fourier, exploiting the duality between the metric and the holonomy representations or by using the corresponding expressions of the vertex and gluing kernels in group variables. In the holonomy picture, the spinfoam amplitude \mathcal{A}_m of an arbitrary simplicial molecule without boundary takes the following form

$$\mathcal{A}_{\text{new}}^\beta(\mathbf{m}) = \int \left[\prod_{v \in \mathcal{V}} \prod_{e \ni v} dH_{ve} \right] \left[\prod_{e \in \mathcal{E}_{\text{int}}} dk_e \right] \prod_{f \in \mathcal{F}_{\text{cl}}} \mathcal{A}_f^\beta(H_{ve}, k_e) \quad (4.33)$$

where the face amplitude \mathcal{A}_f associated to the closed faces $f \in \mathcal{F}_{\text{cl}}$ of \mathbf{m} is given by:

$$\mathcal{A}_f^\beta(H_{ve}, k_e) = \int \left[\prod_{e \in f} \mathcal{D}^\beta \mathcal{Q}_{ef}^{(n)} \right] \delta \left[\prod_{e \in f} H_{ve} \mathcal{Q}_{ef}^{(n)} H_{v'e}^{-1} \right] \quad (4.34)$$

As expected, this is the expression a lattice gauge theory amplitude for constrained BF theory. The notation goes as follows: the group elements $H_{ve} \in \text{Spin}(4)$ are the parallel transports of the connection along an incoming bulk half-edge e which, upon integration, implement the closure constraint while the variables U_{ef} are the Lagrange multipliers enforcing the linear simplicity constraints. The amplitude is invariant under simultaneous rotation of all local frames and therefore can be evaluated in the time gauge. .

The above amplitude can be easily rewritten in the spin representation. First, one can pass to the equivalent form upon Peter-Weyl decomposition:

$$\mathcal{A}_f^\beta(H_{ve}, k_e) = \sum_{J_f j_{ef}} d_{J_f} \int \left[\prod_{e \in f} du_{ef} \right] \Theta^{J_f} \left[\prod_{e \in f} H_{ve} U_{ef} (H_{v'e})^{-1} \right] \left[\prod_{e \in f} d_{j_{ef}} w_{\text{new}}^q(J_f, j_{ef}, \beta) \chi^{j_{ef}}(u_{ef}) \right] \quad (4.35)$$

and then in the complete quantum number basis by carrying out all the group integrals:

$$\mathcal{A}^\beta(\mathbf{m}) = \sum_{J_f j_{ef}} \sum_{I_{ve} i_e} \prod_{f \in \mathcal{F}} d_{J_f} \prod_{e \in f} d_{j_{ef}} \prod_{v \in \mathcal{V}} \{15J_f\}_v \prod_{(\bar{v}v) \equiv e \in \mathcal{E}} d_{I_{ve}} \sqrt{d_{i_e}} f_{\text{new}}^{i_e, p+q}(J_f, j_{ef}, I_{ve}, k_e, \beta) \quad (4.36)$$

As before, the pedices v, e, f denote the vertices, the edges and the faces of the molecule \mathbf{m} . Uppercase letters denotes Spin(4) representations and intertwiners while lowercase letters label SU(2) quantum numbers. Furthermore we have a representation J_f for each face, a pair of four-valent intertwiners $I_{ve}, I_{v'e}$ for every edge and a spin j_{ef} for each edge in a given face. The amplitude is written in terms of the Wigner Fifteen-J Symbol (B.19) and fusion coefficients. These coefficients contain the single-link weight factor w_{new} and can be interpreted as a map between the spaces of four-valent Spin(4) and SU(2) intertwiners according to the Eqs. (C.36) and (C.37).

5 Conclusions.

We have discussed in detail the general structure of spin foam models for quantum gravity, for what concerns both their combinatorial aspects, their quantum states and their quantum amplitudes. We have then specialized to the case of (Riemannian) quantum gravity models in 4d based on the formulation of gravity as a constrained BF theory, which is the class to which most current spin foam models belong. In the latter case, we have emphasized quantization and construction choices that enter model building and differentiate different models, but also the features and mathematical structures that are common to all of them. We believe that this analysis, beyond its pedagogical usefulness, can be important in providing a clearer base for further model building, as well as for extracting physics out of these models, allowing to better parametrize these ambiguities and to focus instead on shared properties. Next, we have constructed a new spin foam model, in the same constrained BF class, focusing on the flux/metric representation of spin foam states and amplitudes, and thus relying on the

associated tools from non-commutative geometry. It is based on the Duflo map, for quantizing the Lie algebra variables of discrete BF theory (and thus the metric variables of the gravitational theory). This is important because the Duflo map has a number of nice mathematical properties that make it the natural quantization map for quantum systems based on group-theoretic structures, and because, thanks to them, the model we introduce can be straightforwardly generalise, in particular to the Lorentzian signature. We give the explicit expression of the amplitudes of the new model in flux, group and spin representation, and the availability of all of them in closed form is another useful asset, for concrete computations.

These concrete computation are the next developments we envisage, based on the new model. First, the model lends itself nicely to numerical evaluations, which allow to compare it straightforwardly with other models in the literature, in particular in the semi-classical (i.e. large- j) regime. A first detailed study will be presented in [102]. Second, the scaling of the amplitudes, divergences structure and radiative corrections of the new model can be investigated, as a first step towards a more complete renormalization group flow analysis of it. This is also forthcoming work [103, 104]. The mentioned Lorentzian extension of the new model is another important development that can be already targeted, since we have all the elements for its completion (the mathematical basis for it, i.e. the Lorentzian non-commutative Fourier transform stemming from the Duflo map being worked out in [106]). Having all these more formal results at hand, we can expect a further strong impulse to the ongoing efforts to extract physics out of spin foam models (also within their GFT reformulation). This is indeed our main goal.

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A Overview of non-commutative harmonic analysis.

A.1 Non-commutative Fourier transform. Generalities.

The quantum geometry underlying both spin foam models, LQG and GFTs in manifest in the lie algebra (or flux) representation of their quantum states and amplitudes. It relies on new mathematical tools from non-commutative geometry. In this section a short review of them is provided following the presentation of [67] to which we refer for further details. As we saw the single-atom Hilbert space of spin foam models' boundary states is given, in the holonomy representation, by the space of square-integrable function on d copies of the Lie group G . Hence the classical phase space we are interested in is given by (one copy of) the cotangent bundle $T^*G \cong G \times \mathfrak{g}^*$ for a lie group G . The extension to several copies is straightforward. Its canonical symplectic structure together with the pointwise multiplication uniquely determines the Poisson algebra and thus the Poisson brackets.

Definition A.1 (Poisson brackets). Let $\mathcal{P}_G = (C^\infty(T^*G), \{\bullet, \bullet\}, \cdot)$ the induced Poisson algebra on the group manifold. The Poisson brackets can be defined as follows:

$$\forall f, g \in C^\infty(T^*G) \quad \{f, g\} \equiv \frac{\partial f}{\partial X_i} \mathcal{L}_i g - \mathcal{L}_i f \frac{\partial g}{\partial X_i} + c_{ij}^k \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_j} X_k \quad (\text{A.1})$$

where \mathcal{L}_i are the Lie derivative with respect to a basis of right-invariant vector fields, X_i are euclidean coordinates on the Lie algebra \mathfrak{g} , c_{ij}^k are the structure constants and repeated lower indices are summed over.

We now seek to quantize a maximal subalgebra \mathcal{A} of this Poisson algebra as an abstract operator $*$ -algebra \mathfrak{X} . To this aim we define a quantization map \mathcal{Q} .

Definition A.2 (Quantization map). A quantization map \mathcal{Q} is a linear map between algebras defined as:

$$\mathcal{Q} : \mathcal{A} \rightarrow \mathfrak{X} \quad \text{s.t.} \quad \forall f \in \mathcal{A}_G \subset \mathcal{A} \subset C^\infty(G \times \mathfrak{g}^*) \quad f \equiv \mathcal{Q}(f) \quad X_i \equiv \mathcal{Q}(X_i) \quad (\text{A.2})$$

$$[f, g] = 0 \quad [X_j, f] = \widehat{i\mathcal{L}_j f} \in \mathfrak{X}_G \quad [X_i, X_j] = ic_{ij}^k X_k \quad \forall f, g \in \mathfrak{X}_G \quad \mathfrak{X}_G = \mathcal{Q}(\mathcal{A}_G) \quad (\text{A.3})$$

where \mathcal{A}_G is the subalgebra of functions in \mathcal{A} constant in the second argument.

In general we cannot introduce differentiable coordinates $\zeta^i \in C^\infty(G)$ on G . However they can be approximated arbitrarily well by elements in $C^\infty(G)$. This allows us to define coordinate operators $\hat{\zeta}^i$ not necessarily in \mathfrak{X}_G corresponding to a set of coordinates $\zeta^i : G \rightarrow \mathbb{R}$. We refer the reader to [67] for further details. Given the quantum algebra of observables \mathfrak{X} , the next task is to construct explicit representations π of it as a concrete operator algebra on suitable Hilbert spaces, where $\pi : \mathfrak{X} \rightarrow \text{Aut}(\mathcal{H})$ is a linear $*$ -homomorphism preserving commutators.

Definition A.3 (Group representation π_G). The group representation π_G on $\mathcal{H} = L^2(G)$ is defined as the one diagonalizing all the operators $f \in \mathfrak{X}_G$. Let $\mathcal{O} \in \mathfrak{X}_G$ be a generic operator (e.g. $\mathcal{O} = f, X$). Then we have,

$$\begin{aligned} \pi_G(\mathcal{O}) : C_c^\infty(G) &\rightarrow C_c^\infty(G) & \left(\pi_G(f)(\psi) \right)(g) &\equiv f(g)\psi(g) \\ \left(\pi_G(X_i)(\psi) \right)(g) &\equiv i\mathcal{L}_i \psi(g) & \forall f, \psi \in C_c^\infty(G) \subset \mathcal{A}_G & \end{aligned} \quad (\text{A.4})$$

where we have purposefully restricted the domain of $\pi_G(f)$ to the space of smooth compactly supported functions on G , dense in $L^2(G)$, so that $f\psi \in C_c^\infty(G)$, $\forall \psi \in C_c^\infty(G)$. Similar remarks hold about the domain of $\pi_G(X_i)$.

One can easily prove that the commutation relations are correctly reproduced upon quantization. We now move on to define a representation in terms of functions on the classical dual space \mathfrak{g}^* . Since the flux operators $X_i \in \mathfrak{X}_{\mathfrak{g}^*}$ do not commute the previous prescription can no longer be used. The strategy we adopt is then to *deform* the action of $\pi_{\mathfrak{g}^*}(X_i)$ by introducing a star product operation giving it the needed freedom to satisfy the commutation relations.

Definition A.4 (Star product). A star product, denoted by \star , is an operation such that:

$$\left(\pi_{\mathfrak{g}^*}(f(X))(\psi) \right)(g) \equiv f_\star(x) \star \psi(x) \quad \forall f_\star \in \mathcal{A}_{\mathfrak{g}^*} \subset C^\infty(\mathfrak{g}^*) \quad \text{s.t.} \quad f(X) = \mathcal{Q}(f_\star) \in \mathfrak{X}_{\mathfrak{g}^*} \quad (\text{A.5})$$

$$\left(\pi_{\mathfrak{g}^*}(\mathcal{Q}(f_\star)\mathcal{Q}(\tilde{f}_\star))(\psi) \right)(x) = \left(\pi_{\mathfrak{g}^*}(\mathcal{Q}(f_\star \star \tilde{f}_\star))(\psi) \right)(x) \quad \forall f_\star, \tilde{f}_\star \in \mathcal{A}_{\mathfrak{g}^*} \quad (\text{A.6})$$

This ensures that f_\star can be interpreted as the function that upon quantization gives $f(X_i)$, thus establishing a connection between the classical phase space structure and the quantum operators.

Moreover, according to the previous definition, the star product and quantization map \mathcal{Q} are related by:

$$f_\star \star \tilde{f}_\star = \mathcal{Q}^{-1}(\mathcal{Q}(f_\star)\mathcal{Q}(\tilde{f}_\star)) \quad (\text{A.7})$$

Therefore the choice of the quantization map determines uniquely the \star -product to be used in representing the quantum algebra in terms of function on \mathfrak{g}^* . Given a star product we define the flux representation of the operators ζ and X_i acting on the space of smooth compactly supported functions on \mathfrak{g}^* to be:

Definition A.5 (Flux representation $\pi_{\mathfrak{g}^*}$). Let us consider the operators $\zeta^i, X_i \in \mathfrak{X}_{\mathfrak{g}^*}$. Their action reads:

$$\forall \psi \in C_c^\infty(\mathfrak{g}^*) \subset \mathcal{A}_{\mathfrak{g}^*} \subset L_\star^2(\mathfrak{g}^*) \quad \left(\pi_{\mathfrak{g}^*}(X_i)(\psi)\right)(x) \equiv x_i \star \psi(X) \quad \left(\pi_{\mathfrak{g}^*}(\zeta^i)(\psi)\right)(x) \equiv -i \frac{\partial}{\partial x_i} \psi(x) \quad (\text{A.8})$$

It can be shown that the space $C_c^\infty(\mathfrak{g}^*)$ is closed under the action of the above operators.

In order to prove that we have a faithful representation of the fundamental quantum algebra \mathfrak{X} we must check that the commutator $[X_i, \zeta]$ is preserved. This amounts to show the compatibility between the star product and the coproduct of the observable algebra. Such property provides then a conditions on the quantization map (and thus on the star product) to allow for the existence of the flux representation as it was built here.

The next task is to find the relation between the two representations π_G and $\pi_{\mathfrak{g}^*}$ of \mathfrak{X} . The non-commutative Fourier transform, defined below, provides such correspondence.

Definition A.6 (Non-commutative Fourier transform). We assume the existence of a unitary map between the two representations $\mathcal{F} : L^2(G) \rightarrow L_\star^2(\mathfrak{g}^*)$ which can be expressed as an integral transform,

$$\tilde{\psi}(x) \equiv \mathcal{F}(\psi)(x) \equiv \int_G dg E_g(x) \psi(g) \quad \tilde{\psi} \in L_\star^2(\mathfrak{g}^*) \quad (\text{A.9})$$

where $\psi \in L^2(G)$ and $E_g(x)$ denotes the integral kernel of the transform, i.e the non-commutative plane wave.

Then the goal is to identify the defining equations for the kernel $E_g(x)$, using the fact that the intertwined functional spaces define a representation of the same quantum algebra and applying the action of \mathfrak{X} in different representation. Its actual existence has to be verified once an explicit choice for the quantization map has been made. The intertwining property of \mathcal{F} can be written generally as $\mathcal{F} \circ \pi_G(\mathbb{T}) = \pi_{\mathfrak{g}^*}(\mathbb{T}) \circ \mathcal{F}$ where $\mathbb{T} \in \mathfrak{X}$. By applying it to the operators ζ^i and X_i we find that the non-commutative plane wave is given by:

$$E_g(x) = e_{\star}^{i\vec{k}(g)\cdot\vec{x}} = \mathcal{Q}^{-1}(e^{i\vec{k}(g)\cdot\mathbb{X}}) = \sum_{n=0}^{\infty} \frac{i^n}{n!} k(g)^{i_1} \dots k(g)^{i_n} x_{i_1} \star \dots \star x_{i_n} \quad E_g(x) = \eta(g) e^{i\vec{\zeta}(g)\cdot\vec{x}} \quad (\text{A.10})$$

with $k(g) = -i \log(g)$ taken from any given branch of the logarithm. The prefactor $\eta(g) = E_g(0)$ depends on the \star -product or, equivalently, on the quantization map \mathcal{Q} chosen. The above expressions are solutions to two differential equations and therefore, for consistency, they must define the same function. For a given \star -product, determining coordinates for which both forms are satisfied might be difficult and, in general, there is no guarantee that such coordinates exist. It is a requirement for the existence of the non-commutative Fourier transform.

Let us now summarize some important properties of the non-commutative Fourier transform and plane waves. For shortness we skip the proofs, referring the reader to [67] for further details.

Remark A.6.1 (Properties of the non-commutative Fourier transform).

- \mathcal{F} is an isomorphism between the space $L^2(G)$ and $L_\star^2(\mathfrak{g}^*)$. Its inverse is defined as follows:

$$\psi(g) = \mathcal{F}^{-1}(\tilde{\psi})(g) \equiv \int_{\mathfrak{g}^*} \frac{d^D x}{(2\pi)^D} \overline{E_g(x)} \star \tilde{\psi}(x) \quad (\text{A.11})$$

- \mathcal{F} is an isometry from $L^2(G)$ to $L_\star^2(\mathfrak{g}^*)$, i.e. it preserves the inner product $\langle \tilde{\psi}, \tilde{\psi}' \rangle_{\mathfrak{g}^*} = \langle \psi, \psi' \rangle_G$.

Remark A.6.2 (Properties of the non-commutative plane waves). Some properties of $E_g(x)$ are:

$$\overline{E_g(x)} = E_g(-x) = E_{g^{-1}}(-x) \quad E_g(h^{-1}xh) = E_{hg h^{-1}}(x) \quad (\text{A.12})$$

$$E_g(x+y) = E_g(x)E_g(y) \quad E_{gh}(x) = (E_g \star E_h)(x) \quad (\text{A.13})$$

$$E_g(x) \star E_h(x) = E_h(x) \star E_{h^{-1}gh}(x) \quad \psi(x) \star E_h(x) = E_h(x) \star \psi(hxh^{-1}) \quad (\text{A.14})$$

$$\delta_x y = \delta_\star(x-y) = \delta_\star(y-x) = \int du E_u(x) E_{u^{-1}}(y) \quad \delta(g) = \int \frac{d^3 x}{(2\pi)^3} E_g(x) \quad (\text{A.15})$$

To conclude this section we introduce the Lie algebra (or flux) counterpart of the SU(2) Wigner matrices and characters, providing few useful identities.

Definition A.7 (Lie algebra Wigner functions and characters). Let $G = \text{SU}(2)$, $\mathfrak{g}^* = \mathfrak{su}^*(2)$. The Lie algebra Wigner functions and characters are defined as follows:

$$D_{mn}^j(x) = \int dg D_{mn}^j(g) E_g(x) \quad \chi^j(x) = \int dg \chi^j(g) E_g(x) \quad x \in \mathfrak{su}^*(2), g \in \text{SU}(2) \quad (\text{A.16})$$

The previous definitions trivially extends to the case of $G = \text{Spin}(4)$. In particular for the characters we have:

$$\Theta^J(X) \equiv \chi^{j^-}(x^-) \chi^{j^+}(x^+) \quad \chi_k^j(X) \equiv \chi^j(kx^- k^{-1} + x^+) \quad X \equiv (x^-, x^+) \in \mathfrak{spin}(4), k \in \text{SU}(2). \quad (\text{A.17})$$

where we recalled the definition of the character $\chi_k^j(X)$, used in the paper as a convenient short-hand notation.

Remark A.7.1 (Identities involving the Lie algebra Wigner functions and characters). The lie algebra Wigner functions and characters satisfy the following identities:

$$\sum_n D_{mn}^j(x) \star D_{np}^j(x) = D_{mp}^j(x) \quad \int \frac{d^3x}{(2\pi)^3} D_{mn}^j(x) \star D_{m'n'}^{j'}(x) = \frac{1}{d_j} \delta_{jj'} \delta_{mn'} \delta_{nm'} \quad (\text{A.18})$$

$$\sum_{jj'} \chi^j(x) \star \chi^{j'}(x) = \sum_j \frac{1}{d_j} \chi^j(x) \quad \chi^j(x) = \chi^j(h^{-1}xh) = \chi^j(hxh^{-1}) \quad (\text{A.19})$$

$$\chi^j(x) \star E_h(x) = E_h(x) \star \chi^j(x) \quad E_h(x) E_{\tilde{h}}(y) \star \chi^j(x+y) = \chi^j(h^{-1}xh + \tilde{h}^{-1}y\tilde{h}) \quad (\text{A.20})$$

$$E_h(x) E_{h'}(y) \star \chi_k^j(x+y) = \chi_{H \triangleright k}^j(x+y) \star E_h(x) E_{h'}(y) \quad H = (h, h') \quad H \triangleright k = h'k(h)^{-1} \quad (\text{A.21})$$

As we saw a key ingredient in defining the non-commutative Fourier transform is the non-commutative plane wave. This can be computed as soon as a quantization map and thus a \star -product is chosen. In the next section we discuss one explicit example.

A.2 The Duflo quantization map.

Let us begin by defining the Symmetric quantization map.

Definition A.8 (Symmetric quantization map). Let us consider a (weakly) semisimple Lie Group G and the dual of its Lie algebra \mathfrak{g}^* . Formally the Symmetric quantization map can be defined as follows:

$$\mathcal{S} : \text{Sym}(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \quad \mathcal{S}(x_{i_1}, \dots, x_{i_n}) = \frac{1}{n!} \sum_{\sigma \in \sigma_k} X_{i_{\sigma_1}} \cdots X_{i_{\sigma_n}} \quad (\text{A.22})$$

where σ_k is the symmetric group of order k . In the above formula $\text{Sym}(\mathfrak{g})$ is the Symmetric algebra of \mathfrak{g} and $U(\mathfrak{g})$ is the Universal enveloping algebra of \mathfrak{g} .

In other words given a set of \mathfrak{g} coordinates x_{i_1}, \dots, x_{i_n} the map \mathcal{S} takes the symmetric ordering of the corresponding coordinate operators. The idea of \mathcal{S} is to map as surjectively as possible a commutative algebra to a non-commutative algebra and therefore it is obviously *not* an algebra isomorphism unless \mathfrak{g} is abelian (though it is a *linear* isomorphism). Invariant polynomial, i.e. elements of $\text{Sym}(\mathfrak{g})$ invariant under the adjoint action of G , denoted by $\text{Sym}(\mathfrak{g})^{\mathfrak{g}}$ are particular important since they map to Casimirs under any quantization scheme. In fact there exists an algebra isomorphism between the subalgebras $\text{Sym}(\mathfrak{g})^{\mathfrak{g}}$ and $U(\mathfrak{g})^{\mathfrak{g}}$, the latter being an alternative definition for the center of $U(\mathfrak{g})$. The map providing such isomorphism is the Duflo quantization map.

Definition A.9 (Duflo quantization map). The Duflo map is explicitly given by:

$$\mathcal{D} = \mathcal{S} \circ \mathcal{J}^{\frac{1}{2}}(\partial) \quad (\text{A.23})$$

where \mathcal{S} is the Symmetric quantization map and \mathcal{J} is the following function on \mathfrak{g} ,

$$\mathcal{J}(x) = \det \left(\frac{\sinh \frac{1}{2} \text{ad}_x}{\frac{1}{2} \text{ad}_x} \right) = \left(\frac{\sinh |x|}{|x|} \right)^2 \quad (\text{A.24})$$

where the last expression holds for $x \in \mathfrak{su}(2)$.

When applied to exponential it gives:

$$E_{\star_{\mathcal{D}}}(g, x) \equiv e_{\star_{\mathcal{D}}}^{i\vec{k}\cdot\vec{x}} \equiv \mathcal{D}^{-1}(e^{i\vec{k}\cdot X}) = \frac{|\vec{k}|}{\sin|\vec{k}|} e^{i\text{Tr}[x \log g]} = \frac{|\vec{k}|}{\sin|\vec{k}|} e^{i\vec{k}\cdot\vec{x}} \quad \vec{k} \in \mathbb{R}^3 \sim \mathfrak{su}(2) \quad (\text{A.25})$$

$$E_{\star_{\mathcal{D}}}(g, x) \equiv e_{\star_{\mathcal{D}}}^{i\vec{k}\cdot\vec{x}} = \eta(g) e^{i\vec{\zeta}(g)\cdot\vec{x}} \quad \vec{\zeta}(g) = -i \ln(g) = \vec{k}(g) \quad \eta(g) = \frac{|\vec{k}(g)|}{\sin|\vec{k}(g)|} \quad |\vec{k}| \in [0, \pi[\quad (\text{A.26})$$

that is the expression of the plane wave under \mathcal{D} with the k -parametrization. Samely on monomials we find:

$$x_i \star_{\mathcal{D}} x_j = x_i x_j + i\epsilon_{ij}^k x_k - \frac{1}{3} \delta_{ij} \quad (\text{A.27})$$

Last the non-commutative plane waves together with the associated \star -product satisfy the following identity

$$\frac{|\vec{k}_1|}{\sin|\vec{k}_1|} e^{i\vec{k}_1\cdot\vec{x}} \star_{\mathcal{D}} \frac{|\vec{k}_2|}{\sin|\vec{k}_2|} e^{i\vec{k}_2\cdot\vec{x}} = \frac{|\mathcal{B}(\vec{k}_1, \vec{k}_2)|}{\sin|\mathcal{B}(\vec{k}_1, \vec{k}_2)|} e^{i\mathcal{B}(\vec{k}_1, \vec{k}_2)\cdot\vec{x}} \quad (\text{A.28})$$

where $\mathcal{B}(\vec{k}_1, \vec{k}_2)$ is the Lie algebra element resulting from the BCH formula. This concludes our overview.

B Basics of SU(2) recoupling theory.

In this appendix we collect definitions and identities involving Wigner- D functions, invariant tensors and recoupling coefficients used so far throughout the paper and in Appendix. C. For further can be found in [107].

Gegenbauer polynomials and Spherical Harmonics. The Gegenbauer or ultraspherical polynomials are a complete orthonormal set (c.o.s.) in $L_{\omega}^2[-1, 1]$ with weight $\omega = (1 - x^2)^{a-\frac{1}{2}}$.

$$C_m^a(t) = \frac{\Gamma(m+2a)}{\Gamma(2a)\Gamma(m+1)} {}_2F_1\left(-m, m+2a; a + \frac{1}{2}; \frac{1-t}{2}\right) \quad (\text{B.1})$$

$$\int_{-1}^1 dt (1-t^2)^{a-\frac{1}{2}} C_m^a(t) C_n^a(t) = \frac{\pi 2^{1-2a} \Gamma(n+2a)}{n! [\Gamma(a)]^2 (a+n)} \delta_{mn} \quad (\text{B.2})$$

The Spherical Harmonics are a c.o.s. in $L^2(S^2)$. Some useful identities are reported below.

$$Y_{lm}^*(\theta, \phi) = Y_{lm}(\theta, -\phi) = (-1)^m Y_{l-m}(\theta, \phi) \quad Y_{lm}(\pi - \theta, \pi + \phi) = (-1)^l Y_{lm}(\theta, \phi) \quad (\text{B.3})$$

$$\int_0^{\pi} \int_0^{2\pi} d\theta d\phi \sin\theta Y_{l'm'}^*(\theta, \phi) Y_{lm}(\theta, \phi) = \delta_{l'l} \delta_{m'm'} \quad \int_0^{\pi} \int_0^{2\pi} d\theta d\phi \sin\theta Y_{lm}(\theta, \phi) = \sqrt{4\pi} \delta_{l0} \delta_{m0} \quad (\text{B.4})$$

Wigner matrices. The normalized left (and right) invariant Haar measure on the group SU(2) reads:

$$d\mu_{\text{Haar}} = \frac{1}{4\pi^2} \sin^2 \frac{\psi}{2} \sin\theta d\psi d\theta d\phi \quad \theta \in [0, \pi[\quad \psi, \phi \in [0, 2\pi[\quad (\text{B.5})$$

$$d\mu_{\text{Haar}} = \frac{1}{4\pi^2} \left(\frac{\sin|\vec{k}|}{|\vec{k}|} \right)^2 d^3\vec{k} = \frac{1}{4\pi^2} \frac{d^3\vec{p}}{\sqrt{1-|\vec{p}|^2}} \quad |\vec{k}| = \frac{\psi}{2} \in [0, \pi[\quad |\vec{p}| \in [0, 1[\quad (\text{B.6})$$

Wigner matrices are a complete orthonormal set in $L^2[\text{SU}(2)]$. Their explicit formula is given below.

$$\int dg D_{mn}^j(g) \overline{D_{pq}^l(g)} = \frac{\delta_{jl} \delta_{mp} \delta_{nq}}{(2j+1)} \quad \overline{D_{mn}^j(g)} = (-1)^{m-n} D_{-m-n}^j(g) \quad (\text{B.7})$$

$$D_{mn}^j(\psi, \theta, \phi) = \sum_{a=0}^{2j} \sum_{\mu=-a}^a (-i)^a \left(\frac{4\pi}{2a+1} \right)^{\frac{1}{2}} \left(\frac{2a+1}{2j+1} \right) C_{m\mu n}^{jaj} \chi_a^j(\psi) Y_{a\mu}(\theta, \phi) \quad D_{mn}^j(0, 0, \phi) = \delta_{mn} \quad (\text{B.8})$$

where $\chi_a^j(\psi)$ denote the generalized character of SU(2) representations introduced next.

Generalized characters. The SU(2) generalized characters, denoted by χ_a^j , are given by:

$$\chi_a^j(\psi) = 2^a a! \left[\frac{(2j+1)(2j-a)!}{(2j+a+1)} \right]^{\frac{1}{2}} \left[\sin \frac{\psi}{2} \right]^a C_{2j-a}^{a+1} \left[\cos \frac{\psi}{2} \right] \quad \chi_a^j(\psi) = i^a \sum_{p=-j}^j e^{-ip\psi} C_{p0p}^{jaj} \quad (\text{B.9})$$

$$\int_0^{2\pi} d\psi \sin^2 \frac{\psi}{2} \chi_a^j(\psi) \chi_a^l(\psi) = \pi \delta_{jl} \quad \chi_a^j(0) = \delta_{a0} \chi^j(0) \quad \chi_0^j(\psi) = \chi^j(\psi) \quad (\text{B.10})$$

We move forward by introducing the (rotated) Clebsch-Gordan coefficients and 4-valent Intertwiners.

Invariant tensors. Clebsch-Gordan coefficients. Orthogonality of rows and columns. Summation rules.

$$\sum_{j_2 m_2} d_{j_2} C_{m_1 m_2 m_3}^{j_1 j_2 j_3} C_{n_1 m_2 n_3}^{j_1 j_2 j_3} = \delta_{m_1 n_1} \delta_{m_3 n_3} \quad (\text{B.11})$$

$$\sum_{m_1 m_2 m_4} (-1)^{j_2 - m_2} C_{m_1 m_2 m_3}^{j_1 j_2 j_3} C_{m_4 - m_2 m_5}^{j_4 j_2 j_5} C_{m_1 m_4 m_6}^{j_1 j_4 j_6} = (-1)^{j_1 + j_2 + j_5 + j_6} \sqrt{d_{j_3} d_{j_5}} C_{m_3 m_5 m_6}^{j_3 j_5 j_6} \begin{Bmatrix} j_2 & j_1 & j_3 \\ j_6 & j_5 & j_4 \end{Bmatrix} \quad (\text{B.12})$$

Column permutations, sign flip and degenerate cases. Definition of rotated CG coefficients.

$$C_{m_1 m_2 0}^{j_1 j_2 0} = \frac{(-1)^{j_1 - m_1}}{\sqrt{2j_1 + 1}} \delta_{j_1 j_2} \delta_{m_1 - m_2} \quad C_{m_1 0 m_3}^{j_1 0 j_3} = \delta_{j_1 j_3} \delta_{m_1 m_3} \quad (\text{B.13})$$

$$C_{m_1 m_2 m_3}^{j_1 j_2 j_3} = (-1)^{j_2 + m_2} \left(\frac{2j_3 + 1}{2j_1 + 1} \right)^{\frac{1}{2}} C_{-m_3 m_2 - m_1}^{j_3 j_2 j_1} \quad C_{m_1 m_2 m_3}^{j_1 j_2 j_3} = (-1)^{j_1 + j_2 + j_3} C_{-m_1 - m_2 - m_3}^{j_1 j_2 j_3} \quad (\text{B.14})$$

The Clebsch series and other useful identities (rotated case). The usual case is recovered for $k = \mathbb{I}$.

$$\sum_{m_1 m_2} C_{m_1 m_2 m_3}^{j_1 j_2 j_3}(k) D_{m_1 n_1}^{j_1}(k^{-1} h k) D_{m_2 n_2}^{j_2}(h) = \sum_{n_3} C_{n_1 n_2 n_3}^{j_1 j_2 j_3}(k) D_{m_3 n_3}^{j_3}(h) \quad (\text{B.15})$$

$$\overline{C_{m_1 m_2 m_3}^{j_1 j_2 j_3}(k)} C_{n_1 n_2 n_3}^{j_1 j_2 j_3}(k) = (2j_3 + 1) \int dh D_{m_1 n_1}^{j_1}(k^{-1} h k) D_{m_2 n_2}^{j_2}(h) \overline{D_{m_3 n_3}^{j_3}(h)} \quad (\text{B.16})$$

Let us now define the four-valent SU(2) intertwiners.

$$\int dh \prod_{i=1}^4 D_{m_i n_i}^{j_i}(h) = \sum_{i=i_{\min}}^{i_{\max}} (\mathcal{I}_{m_i}^{j_i i}) (\mathcal{I}_{n_i}^{j_i i}) \quad (\mathcal{I}_{m_i}^{j_i i}) = \sum_m (-1)^{i-m} \sqrt{d_i} \begin{pmatrix} j_1 & j_2 & i \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} i & j_4 & j_3 \\ -m & m_4 & m_3 \end{pmatrix} \quad (\text{B.17})$$

Wigner SixJ Symbol. Sum rules The Wigner SixJ Symbol, defined in [107], obeys the following identities:

$$\sum_{j_3} d_{j_3} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_1 & j_2 & j_6 \end{Bmatrix} = (-1)^{2j_6} \{j_1, j_2, j_6\} \quad \sum_{j_6} (-1)^{j_1 + j_2 + j_6} d_{j_6} \begin{Bmatrix} j_1 & j_1 & j_3 \\ j_2 & j_2 & j_6 \end{Bmatrix} = \delta_{j_3 0} \sqrt{d_{j_1} d_{j_2}} \quad (\text{B.18})$$

Wigner FifteenJ Symbol. The FifteenJ symbol (first type) can be written as follows:

$$\begin{Bmatrix} j_{12} & j_{13} & j_{14} & j_{15} & j_{23} \\ j_{24} & j_{25} & j_{34} & j_{35} & j_{45} \\ i_1 & i_2 & i_3 & i_4 & i_5 \end{Bmatrix} = \sum_{\{m_{ij}\}} \left[\prod_{i=1, l>i}^5 (-1)^{j_{il} - m_{il}} \right] (\mathcal{I}_{m_{12} m_{13} m_{14} m_{15}}^{j_{12} j_{13} j_{14} j_{15} i_1}) \\ \times (\mathcal{I}_{-m_{12} m_{23} m_{24} m_{25}}^{j_{12} j_{23} j_{24} j_{25} i_2}) (\mathcal{I}_{-m_{13} - m_{23} m_{34} m_{35}}^{j_{13} j_{23} j_{34} j_{35} i_3}) (\mathcal{I}_{-m_{14} - m_{24} - m_{34} m_{45}}^{j_{14} j_{24} j_{34} j_{45} i_4}) (\mathcal{I}_{-m_{15} - m_{25} - m_{35} - m_{45}}^{j_{15} j_{25} j_{35} j_{45} i_5}) \quad (\text{B.19})$$

There are five type of FifteenJ symbols. The number of inequivalent Wigner $3nj$ -symbols ($n \geq 3$) quickly increases with the order n . An excellent overview can be found in [107, 108].

C The new model's fusion coefficients. Evaluation and limiting cases.

In this appendix we provide the explicit derivation of the main results concerning the *new model* simplicity functions and fusion coefficients, previously stated in this paper. We set $k = \mathbb{I}$ for convenience.

Definition C.1 (Simplicity functions). Setting $k = \mathbb{I}$ for convenience, the simplicity functions, i.e. the matrix elements of the kernel of the *new model* simplicity constraint operator are defined as follows:

$$S_{m^- m^+ n^- n^+}^{j^- j^+}(\beta) = \int du \Omega(\beta, \psi_u) D_{m^- n^-}^{j^-}(u) D_{m^+ n^+}^{j^+}(u_\beta) \quad \beta \in [-1, 1] \quad \Omega(\beta, \psi) = \frac{\sin \frac{|\beta| \psi}{2}}{|\beta| \sin \frac{\psi}{2}} \quad (\text{C.1})$$

where the SU(2) group elements u and u_β are given by:

$$u = e^{i \frac{\psi}{2} \hat{n} \cdot \vec{\sigma}} \quad \hat{n} = \hat{n}(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad \psi, \phi \in [0, 2\pi] \quad \theta \in [0, \pi[\\ u_\beta = e^{i \frac{\psi_\beta}{2} \hat{n}_\beta \cdot \vec{\sigma}} \quad \psi_\beta = |\beta| \psi \quad \hat{n}_\beta = \mathfrak{S}(\beta) \hat{n} \Rightarrow \theta_\beta = \mathcal{S}(\beta) \pi + \mathfrak{S}(\beta) \theta \quad \phi_\beta = \mathcal{S}(\beta) \pi + \phi \quad (\text{C.2})$$

Here $\mathcal{S}(\beta)$ is the piecewise (step) function while $\mathfrak{S}(\beta)$ is the sign function.

The next step is to calculate the integral. We find the following result.

Proposition C.1 (Simplicity functions explicit formula). The function $S_{MN}^J(\beta)$ can be rewritten as follows:

$$\mathcal{S}_{m^-m^+n^-n^+}^{j^-j^+}(\beta) = \frac{1}{\pi d_{j^-}d_{j^+}} \sum_{a=0}^{\lambda} \sum_{\mu=-a}^a \mathfrak{S}^a(\beta)(-1)^{-\mu} C_{m^- \mu n^-}^{j^- a j^-} C_{m^+ - \mu n^+}^{j^+ a j^+} \mathcal{T}_a^{j^- j^+}(|\beta|) \quad (\text{C.3})$$

$$\mathcal{T}_a^{j^- j^+}(|\beta|) = (-1)^a (2a+1) \int_0^{2\pi} d\psi \frac{1}{|\beta|} \sin \frac{\psi}{2} \sin \frac{|\beta|\psi}{2} \chi_a^{j^-}(\psi) \chi_a^{j^+}(|\beta|\psi) \quad \lambda = 2 \text{Min}(j^-, j^+) \quad (\text{C.4})$$

Proof. We substitute the formulas (B.8) and perform the integrals by using the Eqs. (B.3) and (B.4).

$$\begin{aligned} \mathcal{S}_{m^-m^+n^-n^+}^{j^-j^+}(\beta) &= \frac{1}{4\pi^2} \int_0^{2\pi} d\psi \Omega(\beta, \psi) \sin^2 \frac{\psi}{2} \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi D_{m^- n^-}^{j^-}(\psi, \theta, \phi) D_{m^+ n^+}^{j^+}(\psi, \theta, \phi) \\ &= \frac{1}{\pi d_{j^-}d_{j^+}} \sum_{a=0}^{2j^-} \sum_{\mu=-a}^a \sum_{a'=0}^{2j^+} \sum_{\mu'=-a'}^{a'} (-i)^{a+a'} \sqrt{d_a d_{a'}} C_{m^- \mu n^-}^{j^- a j^-} C_{m^+ \mu' n^+}^{j^+ a' j^+} \int_0^{2\pi} d\psi \Omega(\beta, \psi) \sin^2 \frac{\psi}{2} \chi_a^{j^-}(\psi) \chi_{a'}^{j^+}(\psi) \\ &\times \int_0^\pi \int_0^{2\pi} d\theta d\phi \sin \theta Y_{a\mu}[\theta, \phi] Y_{a'\mu'}[\mathcal{S}(\beta)\pi + \mathfrak{S}(\beta)\theta, \mathcal{S}(\beta)\pi + \phi] \\ &= \frac{1}{\pi d_{j^-}d_{j^+}} \sum_{a=0}^{2j^-} \sum_{\mu=-a}^a \sum_{a'=0}^{2j^+} \sum_{\mu'=-a'}^{a'} \delta_{aa'} \delta_{\mu-\mu'} \mathfrak{S}^{a'}(\beta) (-i)^{a+a'} (-1)^{\mu'} \sqrt{d_a d_{a'}} C_{m^- \mu n^-}^{j^- a j^-} C_{m^+ \mu' n^+}^{j^+ a' j^+} \\ &\times \int_0^{2\pi} d\psi \frac{1}{|\beta|} \sin \frac{\psi}{2} \sin \frac{|\beta|\psi}{2} \chi_a^{j^-}(\psi) \chi_{a'}^{j^+}(|\beta|\psi) = \frac{\pi^{-1}}{d_{j^-}d_{j^+}} \sum_{a=0}^{\lambda} \sum_{\mu} \mathfrak{S}^a(\beta)(-1)^{-\mu} C_{m^- \mu n^-}^{j^- a j^-} C_{m^+ - \mu n^+}^{j^+ a j^+} \mathcal{T}_a^{j^- j^+}(|\beta|) \end{aligned}$$

which are precisely the results stated above. This concludes the proof.

Remark C.1.1 (Simplicity functions explicit formula). The coefficient $\mathcal{T}_a^{j^- j^+}(|\beta|)$ can be computed from the expressions of the generalized characters (B.9). We find two equivalent formulas. The first read:

$$\mathcal{T}_a^{j^- j^+}(|\beta|) = \mathcal{N}_a^{j^- j^+} \int_0^{2\pi} d\psi \frac{1}{|\beta|} \left[\sin \frac{\psi}{2} \sin \frac{|\beta|\psi}{2} \right]^{a+1} C_{2j^- - a}^{a+1} \left[\cos \frac{\psi}{2} \right] C_{2j^+ - a}^{a+1} \left[\cos \frac{|\beta|\psi}{2} \right] \quad (\text{C.5})$$

$$\mathcal{N}_a^{j^- j^+} = 2^{2a} (a!)^2 (-1)^a (2a+1) \left[\frac{d_{j^-}d_{j^+} (2j^- - a)! (2j^+ - a)!}{(d_{j^-} + a)! (d_{j^+} + a)!} \right]^{\frac{1}{2}} \quad (\text{C.6})$$

The second one, instead, is given by:

$$\mathcal{T}_a^{j^- j^+}(|\beta|) = (2a+1) \sum_{p=-j^-}^{j^-} \sum_{q=-j^+}^{j^+} C_{p0p}^{j^- a j^-} C_{q0q}^{j^+ a j^+} \Upsilon_{pq}(|\beta|) \quad (\text{C.7})$$

$$\begin{aligned} \Upsilon_{pq}(|\beta|) &= \int_0^{2\pi} d\psi \frac{1}{|\beta|} \sin \frac{\psi}{2} \sin \frac{|\beta|\psi}{2} e^{-i(p+|\beta|q)\psi} \\ &= \begin{cases} \frac{i - ie^{2i\pi|\beta|} + 2\pi|\beta|(|\beta|-1)}{4\beta^2(|\beta|-1)} & \forall p, q; \quad 2(p+|\beta|q) = 1 - |\beta| \\ \frac{i - ie^{2i\pi|\beta|} - 2\pi|\beta|(|\beta|+1)}{4\beta^2(|\beta|+1)} & \forall p, q; \quad 2(p+|\beta|q) = -1 - |\beta| \\ -\frac{i - ie^{-2i\pi|\beta|} + 2\pi|\beta|(|\beta|+1)}{4\beta^2(|\beta|+1)} & \forall p, q; \quad 2(p+|\beta|q) = 1 + |\beta| \\ -\frac{i - ie^{-2i\pi|\beta|} - 2\pi|\beta|(|\beta|-1)}{4\beta^2(|\beta|-1)} & \forall p, q; \quad 2(p+|\beta|q) = -1 + |\beta| \end{cases} \quad (\text{C.8}) \\ &\quad \frac{8i|\beta|(p+|\beta|q)e^{-2i\pi(p+|\beta|q)} \cos |\beta|\pi - 2(-1+\beta^2+(p+|\beta|q)^2)e^{-2i\pi(p+|\beta|q)} \sin |\beta|\pi + 8i|\beta|(p+|\beta|q)}{|\beta|(2|\beta|q+|\beta|+2p-1)(2|\beta|q+|\beta|+2p+1)(2|\beta|q-|\beta|+2p-1)(2|\beta|q-|\beta|+2p+1)} \quad \text{Otherwise.} \end{aligned}$$

The latter expression is very useful for precision numerical evaluation.

Let us now define the single-strand fusion coefficients F of our model as follows:

Definition C.2 (Single-link fusion coefficients). The single-link fusion coefficients are given by:

$$F_{n^-n^+m}^{j^-j^+j}(\beta) = C_{m^-m^+m}^{j^-j^+j} S_{m^-m^+n^-n^+}^{j^-j^+}(\beta) = \int du \Omega(\beta, \psi_u) C_{m^-m^+m}^{j^-j^+j} D_{m^-n^-}^{j^-}(u) D_{m^+n^+}^{j^+}(u_\beta) \quad (\text{C.9})$$

Proposition C.2 (Single-link fusion coefficient's explicit formula). The single-link fusion coefficients can be rewritten as follows:

$$F_{n^-n^+m}^{j^-j^+j}(\beta) = C_{n^-n^+m}^{j^-j^+j} w(J, j, \beta) \quad (\text{C.10})$$

$$w(J, j, \beta) = \frac{(-1)^{j^-+j^++j}}{\pi \sqrt{(2j^-+1)(2j^++1)}} \sum_a \mathfrak{S}^a(\beta) \left\{ \begin{matrix} a & j^- & j^- \\ j & j^+ & j^+ \end{matrix} \right\} \mathcal{T}_a^{j^-j^+}(|\beta|) \quad (\text{C.11})$$

Proof. Upon using the identity for the sum of three CG coefficients, given by Eq. (B.12) we obtain:

$$\begin{aligned} F_{n^-n^+m}^{j^-j^+j}(\beta) &= \frac{1}{\pi d_{j^-} d_{j^+}} \sum_a \sum_{\mu, m^-, m^+} \mathfrak{S}^a(\beta) (-1)^a (-1)^{a-\mu} C_{m^- \mu n^-}^{j^- a j^-} C_{m^+ - \mu n^+}^{j^+ a j^+} C_{m^- m^+ m}^{j^- j^+ j} \mathcal{T}_a^{j^-j^+}(|\beta|) \\ &= C_{n^-n^+m}^{j^-j^+j} \frac{(-1)^{j^-+j^++j}}{\pi \sqrt{d_{j^-} d_{j^+}}} \sum_a \mathfrak{S}^a(\beta) \left\{ \begin{matrix} a & j^- & j^- \\ j & j^+ & j^+ \end{matrix} \right\} \mathcal{T}_a^{j^-j^+}(|\beta|) = C_{n^-n^+m}^{j^-j^+j} w(J, j, \beta) \end{aligned} \quad (\text{C.12})$$

This concludes the proof.

Remark C.2.1 (Simplicity functions alternative formula). As a corollary of the previous statement we can immediately write down an equivalent formula for the simplicity functions S^β , exploiting their relation to the single-link fusion coefficients. Indeed combining Eqs. (C.9) and (C.10)

$$S_{m^-m^+n^-n^+}^{j^-j^+}(\beta) = \sum_{jm} C_{m^-m^+m}^{j^-j^+j} F_{n^-n^+m}^{j^-j^+j}(\beta) = \sum_{jm} C_{m^-m^+m}^{j^-j^+j} C_{n^-n^+m}^{j^-j^+j} w(J, j, \beta) \quad (\text{C.13})$$

As argued in the paper the presence of the Clebsch-Gordan coefficients in the above identity reflects the underlying of S^β under the action of the stabilizer group $\text{SO}_k(3)$.

Proposition C.3 (Single-link fusion coefficient's sum rule). The new model's coefficient $w(J, j, \beta)$ obeys the following identity.

$$S^\beta(x^-, x^+) = \delta_{-x^-}(\beta x^+) = \sum_{Jj} d_J d_j w(J, j, \beta) \Theta^J(X) \star \chi^j(x^- + x^+) \quad (\text{C.14})$$

Proof. It is sufficient to prove the dual version of the previous formula obtained by Fourier transform.

$$S^\beta(g^-, g^+) = \int \mathcal{D}^\beta \mathcal{U} \delta[\mathcal{U}G] = \sum_{Jj} d_J d_j w(J, j, \beta) \int du \Theta^J[UG] \chi^j[u] \quad U \equiv (u, u) \quad (\text{C.15})$$

Upon Peter-Weyl decomposition by using Eq. (B.16) we find:

$$\int du \Omega(\Psi_u, \beta) D_{m^-n^-}^{j^-}(u) D_{m^+n^+}^{j^+}(u^\beta) = \sum_{jm} C_{m^-m^+m}^{j^-j^+j} C_{n^-n^+m}^{j^-j^+j} w(j^-, j^+, j, \beta) \quad (\text{C.16})$$

We recognize the identity (C.13) which was proven before. This concludes the proof. The previous identity can be suitably generalized to the case of multiple copies of the constraint operator.

Let us now discuss the relevant limiting cases $\beta = \{1, 0, -1\}$. We begin by proving the following two identities.

Proposition C.4 (Limiting cases). For the values of the Barbero-Immirzi parameter $|\beta| = 1$ and $\beta = 0$ the function $\mathcal{T}_a^{j^-j^+}(|\beta|)$ can be analytically computed. A direct evaluation yields the following results:

$$\mathcal{T}_a^{j^-j^+}(\pm 1) = \pi (-1)^a (2a+1) \delta_{j^-j^+} \quad \mathcal{T}_a^{j^-j^+}(0) = \frac{2\pi (-1)^{2j^-} (2j^++1)}{(2j^-+1)} \delta_{a0} \quad (\text{C.17})$$

Proof. The first identity follows immediately from the orthogonality of the generalized characters (B.10).

$$\mathcal{T}_a^{j^-j^+}(-1) = \mathcal{T}_a^{j^-j^+}(1) = (-1)^a (2a+1) \int_0^{2\pi} d\psi \sin^2 \frac{\psi}{2} \chi_a^{j^-}(\psi) \chi_a^{j^+}(\psi) = \pi (-1)^a (2a+1) \delta_{j^-j^+} \quad (\text{C.18})$$

The second one can be easily obtained upon employing the Eq. (B.9).

$$\begin{aligned} \mathcal{T}_a^{j^-j^+}(|\beta|)|_{\beta=0} &\equiv \lim_{\beta \rightarrow 0} \mathcal{T}_a^{j^-j^+}(|\beta|) = \lim_{\beta \rightarrow 0} \int_0^{2\pi} d\psi \frac{1}{|\beta|} \sin \frac{\psi}{2} \sin \frac{|\beta|\psi}{2} \chi_a^{j^-}(\psi) \chi_a^{j^+}(|\beta|\psi) \\ &= (2j^++1) \delta_{a0} \int_0^{2\pi} d\psi \frac{\psi}{2} \sin \frac{\psi}{2} \chi^{j^-}(\psi) = \frac{2\pi (-1)^{2j^-} (2j^++1)}{(2j^-+1)} \delta_{a0} \end{aligned} \quad (\text{C.19})$$

Equipped with the above results we are now ready to study the various limiting cases.

Proposition C.5 (The Barrett-Crane model). For the value $\beta = 1$ corresponding to the limit $\gamma \rightarrow \infty$ we recover the *Barrett-Crane model*.

$$S_{m^-m^+n^-n^+}^{j^-j^+}(1) = \frac{(-1)^{m^- - n^-}}{2j^- + 1} \delta_{j^-j^+} \delta_{m^- - m^+} \delta_{n^- - n^+} \quad w(j^-, j^+, j, 1) = \delta_{j^-j^+} \delta_{j0} \quad (\text{C.20})$$

Proof. The result follows immediately from the definition C.1. The calculation is straightforward.

$$S_{m^-m^+n^-n^+}^{j^-j^+}(1) = \int du D_{m^-n^-}^{j^-}(u) D_{m^+n^+}^{j^+}(u) = \frac{(-1)^{m^- - n^-}}{2j^- + 1} \delta_{j^-j^+} \delta_{m^- - m^+} \delta_{n^- - n^+} \quad (\text{C.21})$$

Here we used that $(-1)^{2j^- - m^- - n^-} = (-1)^{m^- - n^-}$. Alternatively we can use Eq. (C.3). Indeed,

$$\begin{aligned} S_{m^-m^+n^-n^+}^{j^-j^+}(1) &= \frac{1}{\pi(2j^- + 1)(2j^+ + 1)} \sum_{a=0}^{\lambda} \sum_{\mu=-a}^a (-1)^{-\mu} C_{m^- \mu n^-}^{j^- a j^-} C_{m^+ - \mu n^+}^{j^+ a j^+} \mathcal{T}_a^{j^- j^+}(1) \\ &= \frac{\delta_{j^-j^+} (-1)^{m^- - n^-}}{(2j^- + 1)^2} \sum_{a=0}^{\lambda} \sum_{\mu=-a}^a (2a + 1) C_{m^- \mu n^-}^{j^- a j^-} C_{-m^+ \mu - n^+}^{j^- a j^-} = \frac{(-1)^{m^- - n^-}}{2j^- + 1} \delta_{j^-j^+} \delta_{m^- - m^+} \delta_{n^- - n^+} \end{aligned} \quad (\text{C.22})$$

where the identities (C.17), (B.11) and (B.14) were used. The single-link fusion coefficient reads:

$$w(j^-, j^+, j, 1) = \frac{\pi \delta_{j^-j^+} (-1)^{j^- + j^+ + j}}{\pi \sqrt{d_{j^-} d_{j^+}}} \sum_a (-1)^a (2a + 1) \left\{ \begin{matrix} a & j^- & j^- \\ j & j^+ & j^+ \end{matrix} \right\} = \delta_{j^-j^+} \delta_{j0} \quad (\text{C.23})$$

$$F_{m^-m^+m}^{j^-j^+j}(1) = C_{m^-m^+m}^{j^-j^+j} w(j^-, j^+, j, 1) = C_{m^-m^+m}^{j^-j^+j} \delta_{j^-j^+} \delta_{j0} = \frac{(-1)^{j^- - m^-}}{\sqrt{2j^- + 1}} \delta_{j^-j^+} \delta_{m^- - m^+} \delta_{j0} \delta_{m0} \quad (\text{C.24})$$

provided we use the appropriate Six-J symbol's sum rule stated in (B.18).

Proposition C.6 (The topological Holst model). The limit $\beta = 1$ gives the topological Holst model.

$$S_{m^-m^+n^-n^+}^{j^-j^+}(-1) = \frac{1}{2j^- + 1} \delta_{j^-j^+} \delta_{m^-n^+} \delta_{n^-m^+} \quad w(j^-, j^+, j, -1) = \frac{(-1)^{2j^- + j}}{(2j^- + 1)} \delta_{j^-j^+} \{j^-, j^+, j\} \quad (\text{C.25})$$

Proof. We can proceed directly from the definition (C.1) or equivalently from the formula (C.3). Thus we have:

$$S_{m^-m^+n^-n^+}^{j^-j^+}(-1) = \int du D_{m^-n^-}^{j^-}(u) D_{m^+n^+}^{j^+}(u^{-1}) = \frac{1}{2j^- + 1} \delta_{j^-j^+} \delta_{m^-n^+} \delta_{n^-m^+} \quad (\text{C.26})$$

Upon using the explicit formula (C.3), together with the identities (C.17), (B.11) and (B.14), we easily find:

$$\begin{aligned} S_{m^-m^+n^-n^+}^{j^-j^+}(-1) &= \frac{1}{\pi(2j^- + 1)(2j^+ + 1)} \sum_{a=0}^{\lambda} \sum_{\mu=-a}^a (-1)^{a-\mu} C_{m^- \mu n^-}^{j^- a j^-} C_{m^+ - \mu n^+}^{j^+ a j^+} \mathcal{T}_a^{j^- j^+}(|-1|) \\ &= \frac{\delta_{j^-j^+}}{(2j^- + 1)^2} \sum_{a=0}^{\lambda} \sum_{\mu=-a}^a (2a + 1) C_{m^- \mu n^-}^{j^- a j^-} C_{n^+ \mu m^+}^{j^- a j^-} = \frac{1}{2j^- + 1} \delta_{j^-j^+} \delta_{m^-n^+} \delta_{n^-m^+} \end{aligned} \quad (\text{C.27})$$

Thus the previous result is recovered as expected. The fusion coefficients w are given by:

$$w(j^-, j^+, j, -1) = \frac{\pi (-1)^{j^- + j^+ + j} \delta_{j^-j^+}}{\pi(2j^- + 1)} \sum_a (2a + 1) \left\{ \begin{matrix} j^- & j^- & a \\ j^+ & j^+ & j \end{matrix} \right\} = \frac{(-1)^{2j^- + j}}{(2j^- + 1)} \delta_{j^-j^+} \{j^-, j^+, j\} \quad (\text{C.28})$$

$$F_{m^-m^+m}^{j^-j^+j}(-1) = C_{m^-m^+m}^{j^-j^+j} w(j^-, j^+, j, -1) = \frac{(-1)^{2j^- + j}}{(2j^- + 1)} C_{m^-m^+m}^{j^-j^+j} \delta_{j^-j^+} \quad (\text{C.29})$$

where again we used the sum rules (B.18).

Proposition C.7 (The SU(2) Ooguri model). . The case $\beta = 0$ corresponds to the SU(2) Ooguri. The single-link fusion coefficients S and w are continuous function in $\beta = 0$. Their expressions are given by:

$$S_{m^-m^+n^-n^+}^{j^-j^+}(0) = \frac{2(-1)^{2j^-}}{(2j^- + 1)^2} \delta_{m^-n^-} \delta_{m^+n^+} \quad w(J, j, 0) = \frac{2(-1)^{2j^-}}{d_{j^-}^2} \quad (\text{C.30})$$

Proof. Two formulas for the coefficients $S_{MN}^J(\beta)$ have been provided, namely Eqs. (C.1) and (C.3). They were proved to be equivalent in the interval $\beta \in [-1, 1] \setminus \{0\}$ therein defining, for fixed values of the indices, a continuous function. The case $\beta = 0$ is subtle. A priori the two formulas may not agree and the *sign* function in (C.3) is discontinuous at the origin. We show that the left and right limits $\beta \rightarrow 0^\pm$ coincide in both cases. Therefore $S(\beta)$ can be made continuous at $\beta = 0$ by setting:

$$S_{m^-m^+n^-n^+}^{j^-j^+}(0) \equiv \lim_{\beta \rightarrow 0^+} S_{m^-m^+n^-n^+}^{j^-j^+}(\beta) = \lim_{\beta \rightarrow 0^-} S_{m^-m^+n^-n^+}^{j^-j^+}(\beta) \quad (\text{C.31})$$

Starting from the Eq. (C.1) and using the explicit formula of the Wigner matrices we have:

$$\begin{aligned} \lim_{\beta \rightarrow 0^\pm} S_{m^-m^+n^-n^+}^{j^-j^+}(\beta) &= \frac{1}{4\pi^2} \int dS_2 \int d\psi \sin \frac{\psi}{2} D_{m^-n^-}^{j^-}(\psi, \theta, \phi) \lim_{\beta \rightarrow 0^\pm} \left[\frac{\sin \frac{|\beta|\psi}{2}}{|\beta|} D_{m^+n^+}^{j^+}(\psi, \theta, \phi) \right] \\ &= \frac{\delta_{m^+n^+}}{\pi(2j^-+1)} C_{m^-0n^-}^{j^-0j^-} \int_0^{2\pi} d\psi \frac{\psi}{2} \sin \frac{\psi}{2} \chi^{j^-}(\psi) = \frac{2(-1)^{2j^-}}{(2j^-+1)^2} \delta_{m^-n^-} \delta_{m^+n^+} \end{aligned} \quad (\text{C.32})$$

On the other hand using Eq. (C.3) we easily find:

$$\begin{aligned} \lim_{\beta \rightarrow 0^\pm} S_{m^-m^+n^-n^+}^{j^-j^+}(\beta) &= \frac{1}{\pi(2j^-+1)(2j^++1)} \sum_{a=0}^{\lambda} \sum_{\mu=-a}^a (\pm 1)^a (-1)^{-\mu} C_{m^- \mu n^-}^{j^- a j^-} C_{m^+ - \mu n^+}^{j^+ a j^+} \lim_{\beta \rightarrow 0^\pm} \mathcal{T}_a^{j^-j^+}(|\epsilon|) \\ &= \frac{2\pi(-1)^{2j^-}(2j^++1)}{\pi(2j^-+1)^2(2j^++1)} \sum_{a=0}^{\lambda} \sum_{\mu=-a}^a \delta_{a0} (\pm 1)^a (-1)^{-\mu} C_{m^- \mu n^-}^{j^- a j^-} C_{m^+ - \mu n^+}^{j^+ a j^+} = \frac{2(-1)^{2j^-}}{(2j^-+1)^2} \delta_{m^-n^-} \delta_{m^+n^+} \end{aligned} \quad (\text{C.33})$$

The same reasoning can be applied to the single-link fusion coefficients as well. In particular we find:

$$w(J, j, 0) \equiv \lim_{\beta \rightarrow 0^\pm} w(J, j, \beta) = \frac{2(-1)^{2j^-+j^-+j^++j}}{d_{j^-}^{\frac{3}{2}} d_{j^+}^{\frac{1}{2}} d_{j^+}^{-1}} \sum_a \delta_{a0} \begin{Bmatrix} a & j^- & j^+ \\ j & j^+ & j^+ \end{Bmatrix} = \frac{2(-1)^{2j^-}}{d_{j^-}^2} \quad (\text{C.34})$$

$$F_{m^-m^+m}^{j^-j^+j}(\beta)|_{\beta=0} \equiv C_{m^-m^+m}^{j^-j^+j} w(j^-, j^+, j, 0) = C_{m^-m^+m}^{j^-j^+j} \frac{2(-1)^{2j^-}}{(2j^-+1)^2} \quad (\text{C.35})$$

This concludes the discussion.

Definition C.3 (Fusion coefficients f_I^j). In our context the fusion coefficients are defined as the matrix elements (in the spin basis) of a map between the spaces of 4-valent Spin(4) and SU(2) intertwiners.

$$\begin{aligned} f : \text{Inv}_{\text{Spin}(4)} \left[\bigotimes_{i=1}^4 \mathcal{H}^{j_i^-} \otimes \mathcal{H}^{j_i^+} \right] &\rightarrow \text{Inv}_{\text{SU}(2)} \left[\bigotimes_{i=1}^4 \mathcal{H}^{j_i} \right] \\ f_I^{i,p}(J_l, j_l, k, \beta) &= \sum_{M_l m_l} (\mathcal{I})_{M_1 M_2 M_3 M_4}^{J_1 J_2 J_3 J_4 I} \left[\prod_{l=1}^4 C_{m_l^- m_l^- m_l}^{j_l^- j_l^- j_l} (k) w^p(J_l, j_l, \beta) \right] (\mathcal{I})_{m_1 m_2 m_3 m_4}^{j_1 j_2 j_3 j_4 i} \end{aligned} \quad (\text{C.36})$$

Uppercase letters denote Spin(4) unitary irreducible representations $J \equiv (j^-, j^+)$ and intertwiners $I \equiv (i^-, i^+)$ while lowercase letters label SU(2) spins and intertwiners.

Remark C.3.1 (Fusion coefficients: alternative formula). Upon recoupling the above defined fusion coefficients can be rewritten in terms of the Wigner NineJ symbol as shown in [107]. In formulas we have:

$$\begin{aligned} f_I^{i,p}(J_l, j_l, \mathbb{I}, \beta) &= (-1)^\Xi \left[\prod_{l=1}^4 w^p(J_l, j_l, \beta) \right] \begin{Bmatrix} j_1^- & i^- & j_2^- \\ j_1^+ & i^+ & j_2^+ \\ j_1 & i & j_2 \end{Bmatrix} \begin{Bmatrix} j_3^- & i^- & j_4^- \\ j_3^+ & i^+ & j_4^+ \\ j_3 & i & j_4 \end{Bmatrix} \\ \Xi &= 2(j_1^- + j_2^-) + (j_3^- - j_3^+ + j_3) + (j_4^- - j_4^+ + j_4) \end{aligned} \quad (\text{C.37})$$

This concludes the Appendix.

D The new model's vertex and gluing kernels.

The formulas of the new model's vertex and gluing kernels can be found by following the strategy outlined in Section. 3.4. Here we report the kernels' expressions both in flux and holonomy variables, obtained upon using the Dirac-delta representation for the simplicity constraint operator. The notation we use is the one introduced by Eq. (4.25) with $k = \mathbb{I}$. By substituting the Eq. (4.5) into the formal definitions (3.52) and (3.53) we find:

The Vertex Kernel.

$$\begin{aligned}\mathcal{V}_\alpha^{\beta,p}(X_{\bar{v}\bar{v}}, Y_{\bar{v}'\bar{v}}, k_{\bar{v}}) &= \int \left[\prod_{\bar{v} \in \bar{\mathcal{V}}} dH_{\bar{v}v} \right] \left[\prod_{\bar{v} \in \bar{\mathcal{V}}} \prod_{\bar{v} \in \mathfrak{p}_{\bar{v}}} \mathcal{D}^\beta \mathcal{U}_{\bar{v}\bar{v}}^{(p)} \right] \prod_{(\bar{v}\bar{v}') \equiv f \in \mathcal{F}_\alpha} \left(\delta_{-X_{\bar{v}\bar{v}}} \star E_{\mathcal{U}_{\bar{v}\bar{v}}^{(p)} H_{\bar{v}v} H_{\bar{v}'v}^{-1} \mathcal{U}_{\bar{v}'\bar{v}}^{(p)}} \right) (Y_{\bar{v}'\bar{v}}) \\ \mathcal{V}_\alpha^{\beta,p}(G_{\bar{v}\bar{v}}, \tilde{G}_{\bar{v}'\bar{v}}, k_{\bar{v}}) &= \int \left[\prod_{\bar{v} \in \bar{\mathcal{V}}} dH_{\bar{v}v} \right] \left[\prod_{\bar{v} \in \bar{\mathcal{V}}} \prod_{\bar{v} \in \mathfrak{p}_{\bar{v}}} \mathcal{D}^\beta \mathcal{U}_{\bar{v}\bar{v}}^{(p)} \right] \prod_{(\bar{v}\bar{v}') \equiv f \in \mathcal{F}_\alpha} \delta \left[G_{\bar{v}\bar{v}} \mathcal{U}_{\bar{v}\bar{v}}^{(p)} H_{\bar{v}v} H_{\bar{v}'v}^{-1} \mathcal{U}_{\bar{v}'\bar{v}}^{(p)} \tilde{G}_{\bar{v}'\bar{v}}^{-1} \right] \quad (\text{D.1})\end{aligned}$$

Samely for the gluing kernel we get:

The Gluing kernel.

$$\begin{aligned}\mathcal{K}^{\beta,2q}(X_{\bar{v}\bar{v}}, Y_{\bar{v}'\bar{v}}, k_{\bar{v}}, k_{\bar{v}'}) &= \delta(k_{\bar{v}} k_{\bar{v}}^{-1}) \int [dH_{\bar{v}} dH_{\bar{v}'}] \left[\prod_{\bar{v} \in \mathfrak{p}_{\bar{v}}, \mathfrak{p}_{\bar{v}'}} \mathcal{D}^\beta \mathcal{U}_{\bar{v}\bar{v}}^{(2q)} \right] \prod_{\bar{v} \in \mathfrak{p}_{\bar{v}}, \mathfrak{p}_{\bar{v}'}} \left(\delta_{-X_{\bar{v}\bar{v}}} \star E_{H_{\bar{v}\bar{v}}^{-1} \mathcal{U}_{\bar{v}\bar{v}}^{(2q)} H_{\bar{v}'\bar{v}}} \right) (Y_{\bar{v}'\bar{v}}) \\ \mathcal{K}^{\beta,2q}(G_{\bar{v}\bar{v}}, \tilde{G}_{\bar{v}'\bar{v}}, k_{\bar{v}}, k_{\bar{v}'}) &= \delta(k_{\bar{v}} k_{\bar{v}}^{-1}) \int [dH_{\bar{v}} dH_{\bar{v}'}] \left[\prod_{\bar{v} \in \mathfrak{p}_{\bar{v}}, \mathfrak{p}_{\bar{v}'}} \mathcal{D}^\beta \mathcal{U}_{\bar{v}\bar{v}}^{(2q)} \right] \prod_{\bar{v} \in \mathfrak{p}_{\bar{v}}, \mathfrak{p}_{\bar{v}'}} \delta \left[G_{\bar{v}\bar{v}} H_{\bar{v}\bar{v}}^{-1} \mathcal{U}_{\bar{v}\bar{v}}^{(2q)} H_{\bar{v}'\bar{v}} \tilde{G}_{\bar{v}'\bar{v}}^{-1} \right] \quad (\text{D.2})\end{aligned}$$

with the same notation and conventions used throughout the paper. Upon Peter-Weyl decomposition we recover the Eqs. (3.54) and (3.57), already discussed in Section. 3.4. The equivalence between the above formulas (D.1) and (D.2) and the corresponding general expressions (3.54) and (3.57) (with $w(J, j, \beta)$ given by C.11) can be easily checked upon using the identity C.14, which, in plain words, allows us to switch between the two formulas of the new model's simplicity function $S^\beta(X)$.

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