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Marco Finocchiaro and Daniele Oriti

1 Max Planck Institute for Gravitational Physics (Albert Einstein Institute), Am Muehlenberg 1, D-14476 Potsdam-Golm, EU, Germany
2 Arnold-Sommerfeld-Center for Theoretical Physics, Ludwig-Maximilians-Universität, Theresienstrasse 37, D-80333 München, EU, Germany
3 Institute for Physics, Humboldt-Universität zu Berlin, Newtonstraße 15, 12489 Berlin, EU, Germany

E-mail: marco.finocchiaro@aei.mpg.de and daniele.oriti@aei.mpg.de

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Abstract
We give a general definition of spin foam models, and then of models of 4d quantum gravity based on constraining BF theory. We highlight the construction and quantization ambiguities entering model building, among which the choice of quantization map applied to the $B$ variables carrying metric information after imposing simplicity constraints, and the different strategies for imposing the latter constraints. We then construct a new spin foam model for 4d quantum gravity, using the flux representation of states and amplitudes, based on the Duflo quantization map and the associated non-commutative Fourier transform for Lie groups. The advantages of the new model are the geometrically transparent way in which constraints are imposed, and the underlying mathematical properties of the Duflo map itself. Finally, the presence of a closed analytical formula for the model’s amplitudes is another valuable asset for future applications.

Keywords: spin foam models, loop quantum gravity, noncommutative geometry, Duflo map

(Some figures may appear in colour only in the online journal)
1. Introduction

Spin foam models are a covariant definition of the quantum dynamics of spin network-type structures [1]. They are thus a covariant counterpart of canonical loop quantum gravity [2, 3], a reformulation of lattice gravity path integrals [4, 5], a natural language for state sum models of topological quantum field theories [6–8] and the perturbative dynamics of group field theories (GFT) [9]. In addition to the many results obtained in the context of the mentioned related formalisms (e.g. group field theory and tensor models [10, 11]) also directly impacting on spin foam models per se, a lot of work has concentrated on spin foam model building [12–19], and on the semi-classical analysis of the resulting quantum gravity models (for fixed underlying lattice) [20]. More recently, the issues of spin foam renormalization and continuum limit (from both lattice [21–26] and GFT [27–31] perspectives) have become central. Progresses have also been made in generalizing and adapting the notions of entanglement entropy and holography, so far mainly in the case of 3d quantum gravity models [32–34]. Important steps have also been taken in direction of extracting effective continuum physics out of quantum gravity models, in particular in a cosmological context (both from a canonical perspective [35] and using the GFT reformulation of the same models [36–38]). In parallel, we have deepened our understanding of the formal structure of spin foam models, and explored it at a more mathematical level also illuminating the various choices underlying model building.

This paper tackles the more formal aspects of spin foam construction, as a stepping stone for investigating the more physical ones. First of all, we provide a very general definition of spin foam models and of their construction from their defining building blocks, detailing both the combinatorial aspects and their associated quantum states and amplitudes. The general definition we provide will be a convenient starting point for more model building or the analysis of physical consequences, not relying on any specific representation of quantum states or on model building strategy. Next, we specialize the general construction to spin foam models for 4d quantum gravity, in the Riemannian setting, inspired by the formulation of gravity as a constrained BF theory. We introduce the basic ideas of this construction strategy and provide the general definition of the corresponding models, without focusing exclusively on any one of them, but, on the contrary, highlighting their shared features. We do so in different representations for the quantum states and for the amplitudes. Beside its pedagogical value, the main result of this part is to identify the relevant construction ambiguities and choices characterizing each model as well as the general properties shared by all spin foam models.

A good control over both specific and general features will be a useful asset when trying to extract physics from them. Among the shared features, we mention the fact that all spin foam models in this class take the form of non-commutative simplicial gravity path integrals when expressed in the flux (Lie algebra) representation (including the EPRL model which is usually not seen from this perspective). This can greatly facilitate their semi-classical analysis [39].

Last, we use the previous analysis to construct a new spin foam model in the same constrained BF class. The model is constructed focusing on the flux/metric representation of spin foam states and amplitudes, and thus relying on the associated tools from non-commutative geometry, notably the non-commutative Fourier transform for Lie groups [40, 41]. A direct advantage of the flux construction is the presence of a closed integral formula for the model’s amplitudes in group variables enabling the use of heat Kernel methods in the renormalizability analysis [28]. In this respect, our model parallels the formulation of the one presented in [18]. However it employs a different quantization map for the Lie algebra variables of discrete BF theory: the Duflo map. This is another important improvement. On the one hand, the Duflo map has a number of nice mathematical properties, making it in many ways the natural quantization map for quantum systems based on group-theoretic structures [42]. On the other hand,
contrary to the quantization map employed in [18] (and in some other related quantum gravity literature [43]) the Duflo map applies to any semi-simple, locally compact Lie group; thus the model we introduce can be straightforwardly generalised to other dimensions, other model-building strategies and, more immediately relevant, the Lorentzian signature [44]. Last, the Duflo map simplifies the expression of the kernel enforcing the simplicity constraints which, in contrast to [18], allowed us to derive an explicit formula for fusion coefficients (4.13) and (3.42). This is clearly a valuable asset in trying to extract quantitative results about the model and its amplitudes in different regimes by using numerical methods [45, 46].

2. Abstract formulation of Spin foam models

In this section we outline the general construction of spin foam models for Riemannian quantum gravity. Spin foam models associate quantum amplitudes to discrete structures, usually in the form of product of amplitudes associated to the lower-dimensional cells of the same lattice. We begin by listing and defining the combinatorial building blocks on which spin foam amplitudes are supported. Then we introduce the Hilbert spaces of boundary states and the set of corresponding amplitudes, as done in [47] to which we refer for more details (see also [48, 49]). This formulation is also in direct parallel with the GFT completion of spin foam models which implicitly defines their continuum limit suggesting a number of useful field-theoretic tools. Given the formal nature of this section we provide below a synoptic table of all used symbols.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
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<tbody>
<tr>
<td>( \mathcal{B} )</td>
<td>Set of bisected boundary graphs.</td>
</tr>
<tr>
<td>( \mathcal{V}_b, \mathcal{E}_b )</td>
<td>Full vertex and edge sets of a bisected boundary graph ( b \in \mathcal{B} ).</td>
</tr>
<tr>
<td>( \check{\mathcal{V}}_b )</td>
<td>Disjoint vertex sets forming a partition of ( \mathcal{V}_b ).</td>
</tr>
<tr>
<td>( \mathcal{P} )</td>
<td>Set of all boundary patches of the boundary graphs in ( \mathcal{B} ).</td>
</tr>
<tr>
<td>( \mathcal{V}_p, \mathcal{E}_p )</td>
<td>Full vertex and edge sets of a boundary patch ( p \in \mathcal{P} ).</td>
</tr>
<tr>
<td>( \mathcal{A} )</td>
<td>Set of all spin foam atoms, i.e. the set of the dual polytopes of all ( d )-dimensional cells.</td>
</tr>
<tr>
<td>( \mathcal{V}_a, \mathcal{E}_a, \mathcal{F}_a )</td>
<td>Vertex, edge and face sets of a spin foam atom ( a \in \mathcal{A} ).</td>
</tr>
<tr>
<td>( \mathcal{M} )</td>
<td>Set of all spin foam molecule or equivalently of all ( d )-dimensional cellular complexes.</td>
</tr>
<tr>
<td>( \mathcal{V}_m, \mathcal{E}_m, \mathcal{F}_m )</td>
<td>Vertex, edge and face sets of a spin foam molecule ( m \in \mathcal{M} ).</td>
</tr>
<tr>
<td>( \mathcal{H}_p \equiv \mathcal{H}_e )</td>
<td>Patch Hilbert space (e.g. tetrahedron Hilbert space in the 4d simplicial case).</td>
</tr>
<tr>
<td>( \mathcal{A}(m) )</td>
<td>Spin foam amplitude associated to a spin foam molecule ( m ).</td>
</tr>
<tr>
<td>( \mathcal{K}_g, \mathcal{K}_v )</td>
<td>Gluing and Vertex kernels of a given spin foam model.</td>
</tr>
</tbody>
</table>

The discrete building blocks of spin foam models exhibit a `molecular structure, here introduced in steps.

**Definition 2.1 (Bisected boundary graph).** A bisected boundary graph \( b \in \mathcal{B} \) is an ordered pair \( b = (\mathcal{V}_b, \mathcal{E}_b) \) forming a bipartite graph with vertex partition \( \mathcal{V}_b = \check{\mathcal{V}}_b \cup \tilde{\mathcal{V}}_b \) such that the vertices \( \check{v} \in \check{\mathcal{V}}_b \) are bivalent.

**Definition 2.2 (Boundary patch).** A boundary patch \( p \in \mathcal{P} \) is a pair \( p = (\mathcal{V}_p, \mathcal{E}_p) \) such that:

\[
\mathcal{V}_p = \{ \check{v} \} \cup \tilde{\mathcal{V}}_p \quad \check{\mathcal{V}}_p \neq \emptyset \quad \mathcal{E}_p = \{ (\check{v}, \check{v}) : \check{v} \in \check{\mathcal{V}}_p \}.
\] (2.1)
Thus a boundary patch is a graph made by the node $\bar{v}$, all boundary (half)-links containing it and their end points. For example, the graph in figure 1 has five boundary patches. Each one of them consists of one red node $\bar{v}$ (e.g. the one in the top left corner) together with its incident green half-links and their blue end points $\tilde{v}$.

Two patches $p_{\bar{v}_1}$ and $p_{\bar{v}_2}$ are named bondable if $|V_{p_{\bar{v}_1}}| = |V_{p_{\bar{v}_2}}|$ and $|E_{p_{\bar{v}_1}}| = |E_{p_{\bar{v}_2}}|$. Bondable patches can be elementwise identified via a gluing map $\gamma : p_{\bar{v}_1} \to p_{\bar{v}_2}$.

**Definition 2.3 (Spin foam atom).** A spin foam atom $a \in \mathcal{A}$ is a triple $a = (V_a, E_a, F_a)$, constructed in correspondence with a bisected boundary graph $b$, with:

$$V_a = \{v\} \cup V_b \quad E_a = \mathcal{E} \cup \mathcal{E}_b \quad \mathcal{E} = \bigcup_u \{(uv) : u \in V_b\} \quad F_a = \bigcup_{\tilde{v} \in \tilde{V}} \{f = (v\bar{v}\tilde{v}) : (\bar{v}\tilde{v}) \in \mathcal{E}_b\}.$$  

The set $\mathcal{E}$ contains one edge for each node in $V_b$, joining it to the bulk vertex $v$. Last $(v\bar{v}\tilde{v})$ is the boundary face bounded by the edges connecting the three vertices. The set of spin foam atoms is denoted by $\mathcal{A}$.

An example of spin foam atom is illustrated in figure 1. With respect to the picture the set $V_a$ is made by the bulk black vertex $v$ together with all the nodes (in red and $\bar{v}$ in blue) of the atom’s boundary graph belonging to $V_b$. The set $E$ contains the red edges connecting the five red nodes $\bar{v}$ to the black vertex $v$. Thus the full atom edge set $E_a$ consists of the five red edges together with all the green half-links of the atom’s boundary graph (which are elements of $E_b$). With respect to the picture (figure 1) all the spin foam atom’s faces are cycles made by four distinct vertices $(v\bar{v}\tilde{v}\bar{v}')$ and by the edges connecting them. For example one face is identified by the black vertex, the two red nodes on the bottom and on the right, the blue node on the bottom right corner and the half-edges joining them (note however that not all faces have been drawn in the picture).

Summarizing a spin foam atoms is the 2-skeleton of the dual polytope of a $d$-dimensional fundamental cell, e.g. the dual 2-skeleton of a 4-simplex in the four-dimensional simplicial case. Moreover, as argued in [47], the set $\mathcal{A}$ of atoms to be catalogued by the set $\mathcal{B}$ of bisected boundary graphs.

**Definition 2.4 (Spin foam molecule).** A spin foam molecule $m \in \mathcal{M}$ is a triple of vertices, edges and faces $m = (V_m, E_m, F_m) = (\cup_a V_a, \cup_a E_a, \cup_a F_a)$ (constructed from a set of spin foam atoms quotiented by a set of gluing maps enforcing the bonding relations between the atoms forming the molecule. An example of spin foam molecule is illustrated in figure 2.

**Definition 2.5 ($n$-simplicial structures).** The set of $n$-simplicial molecules $\mathcal{M}_S$ consists of all molecules obtained as gluings of a single (simplicial) atom $a_S$ labelled by the complete
Notice that we call simplicial, the above-defined spin foam molecules because each spin foam atom in itself can be canonically understood as the dual 2-skeleton of an \( n \)-simplex. However, this can be done only locally; it has been proven that not every simplicial spin foam molecule can be associated uniquely to a well-defined simplicial complex, as its dual 2-skeleton \[50\]. While the restriction to simplicial structures is motivated (in addition to simplicity) by the greater geometric understanding of the corresponding models with respect to those based on non-simplicial complexes, we stress that they remain a special case of a more general formalism. The use of arbitrary cellular complexes is suggested by canonical LQG \[48\] and can also be accommodated in the GFT formulation of spin foam models \[47\], using techniques from dually weighted tensor models.

**Definition 2.6 (Spin foam model).** A spin foam model is a quantum theory prescribed by the assignment of a quadruple \((\mathcal{H}_p, \mathcal{M}_S, W, \mathcal{A})\) and defined by a partition function of the following form:

\[
Z_{\text{SF}} = \sum_{m \in \mathcal{M}_S} W(m) \mathcal{A}(m).
\]  

(2.3)

Here \( \mathcal{H}_p \) is the Hilbert space associated to each boundary patch of the atoms forming the molecule, \( \mathcal{A}(m) \) is the spin foam amplitude assigned to \( m \) by each given model and \( W(m) \) is a further weight factor in the sum over all molecules. While \( \mathcal{A} \) can be motivated, purely by considering the discretization and quantization of some continuum (gravitational) theory the prescription for \( W(m) \) should come from a different line of reasoning. For example, the GFT approach to spin foam models provides a field-theoretic prescription for both of them.

The quantum states for which spin foam models define probability amplitudes are associated to the boundary graphs of spin foam molecules. The primary ingredient is the patch Hilbert space, denoted by \( \mathcal{H}_p \equiv \mathcal{H}_\gamma \). One can then associate an Hilbert space to each spin foam atom \( \mathcal{H}_a \) and to each spin foam molecule \( \mathcal{H}_m \).

\[
\mathcal{H}_a = \bigotimes_{p \in \partial a} \mathcal{H}_p \quad \mathcal{H}_m = \bigotimes_{p \in \partial m} \mathcal{H}_p.
\]  

(2.4)

---

4 In combinatorics an finite abstract \( k \)-simplicial complex \( \mathcal{C} \) (e.g. an abstract \( k \)-simplex) is a collection of subsets \( \sigma \) of a set of vertices \( \mathcal{C}_0 = \{v_1, \ldots, v_n\} \) such that the following two properties are satisfied:

1. For all \( \sigma, \sigma' \in \mathcal{C} \) and \( \sigma' \subset \sigma \) then \( \sigma' \in \mathcal{C} \).
2. If \( \sigma, \sigma' \in \mathcal{C} \) then \( \sigma \cup \sigma' \in \mathcal{C} \).

All subsets of cardinality \( p + 1 \) are called \( p \)-simplices \( \sigma_p \in \mathcal{C}_p \). The dimension \( k \) of \( \mathcal{C} \) is defined as the maximal cardinality of simplices in \( \mathcal{C} \). From a topological point of view a \( k \)-simplex is the convex hull of a set of \( k + 1 \) affinely independent points in \( \mathbb{R}^k \).
One might also want to define a single Hilbert space for a spin foam model, that would accommodate any possible choice of boundary. This is indeed a crucial issue to tackle the continuum limit and relate the formalism to canonical quantum gravity. From this point of view the simplest proposal is that of a (bosonic) Fock space. This is a natural choice from a QFT/emergent-gravity perspective that sees quantum spacetime as a peculiar quantum many-body system. Another possibility is to define a Hilbert space as the direct sum of all possible graph Hilbert spaces. A third alternative is the one inspired by the canonical LQG construction based on the imposition of cylindrical equivalence relations. A comprehensive discussion of these issues can be found in [51]. Let us now turn instead to the construction of the spin foam amplitudes themselves.

In order to specify the spinfoam amplitudes $A_m$ we need a set of operators defining maps between the various boundary patches’ Hilbert spaces. The basic ones are the vertex and gluing operators.

$$V_a : \bigotimes_{p \in \partial a} \mathcal{H}_p \rightarrow \mathcal{H}_a$$
$$V_a : \bigotimes_{p \in \partial a} \mathcal{H}_p \rightarrow \mathbb{C} \quad (2.5)$$

$$K_e : \mathcal{H}_{p_1} \rightarrow \mathcal{H}_{p_2}$$
$$\mathcal{K}_e : \mathcal{H}_{p_1} \otimes \mathcal{H}_{p_2} \rightarrow \mathbb{C} \quad (2.6)$$

The associated functions $V_a$ and $K_e$, called the vertex and gluing kernels, give, when applied to any basis in the Hilbert spaces $\mathcal{H}_p$, the generalised ‘matrix elements’ of the corresponding operators. The general formula of the spin foam amplitude for a generic molecule, depending on its combinatorial structure, i.e. the connectivity pattern between spin foam atoms and their subcells, is given by:

$$A(m) = \text{Tr}_{p \in m} \left( \prod_{e \in m} \mathcal{K}_e \prod_{a \in m} V_a \right) \quad (2.7)$$

The trace is evaluated over a complete basis in each of the shared patch Hilbert spaces (producing the convolution of the corresponding functions). Following the gluing pattern effected by the gluing maps, one identifies a closed cycle and thus a spin foam face associated to the same patch (for internal patches). Thus the final spin foam amplitude can also be written in terms of individual contributions associated to the faces, edges and vertices of the spin foam molecule. Last these amplitudes, together with an additional combinatorial factor, can be recovered as the perturbative Feynman amplitudes of a Group field theory whose propagator and the interaction kernels are the same gluing and vertex kernels of the corresponding (dual) spin foam model [9, 47].

3. Spin foam models for constrained BF theory

Having given the general definitions, let us now focus on the class of gravitational or geometrical Riemannian spin foam models arising from the Holst–Plebanski formulation of general relativity in 4d [52]. From now on we restrict ourselves to simplicial structures. Extensions to the Lorentzian context and to arbitrary cellular complexes can be found in the literature [1, 47, 48]. In this section, we emphasize and illustrate two points: the construction ambiguities and the universal structure of the resulting amplitudes.

3.1. Gravity as a constrained BF theory

Our starting point is the Holst–Palatini action without cosmological constant and matter fields, given below:
\[ S_{\text{HP}}[\epsilon, \omega, \lambda] = \int_{\mathcal{M}}^* (\epsilon^I \wedge \epsilon^J) \wedge F^{I\lambda}[\omega] + \frac{1}{\gamma} \epsilon^I \wedge \epsilon^J \wedge F_{IJ}[\omega]. \] (3.1)

Here \( \omega \) is a Spin(4)-valued connection one-form field, \( F[\omega] \) is the curvature two-form and \( \epsilon \) is a spin(4)-valued tetrad one-form field representing an orthonormal frame. The topological term proportional to the Barbero–Immirzi parameter, though irrelevant at the classical level, is vital in the formulation of both loop quantum gravity (LQG) and Spin foam models. The Holst–Palatini action can be recovered, on the solution of the constraint equations, from a topological BF theory action with additional polynomial constraints \( C_{\alpha}[B] \).

\[ S[B, \omega, \lambda] = \int BIJ \wedge FIJ[\omega] + \lambda^\alpha C_{\alpha}[B]. \] (3.2)

Several formulation can be given [53, 54]. In particular the constraints can be taken to be the linear ones:

\[ k_{\text{cd}}(B - \gamma^* B)^{IJ}_{ab} = 0 \quad k_{\text{cd}} = \epsilon_{IPKL} e^P e^r e^K e^L \] (3.3)

where \( k_{\text{cd}} \) can be interpreted as a 3d volume form for the submanifold parametrized by the coordinates \( (x^I, x^J, x^L) \) embedded in the 4d spacetime \( \mathcal{M} \), whose internal index \( I \) gives the normal to the 3d submanifold. Ideally one would want to constrain the classical variables first and then quantize the resulting geometric structures. Nevertheless the standard approach is to first discretize quantize the topological discrete BF action and then to impose, directly at the quantum level, a suitable version of the required geometricity constraints hoping to recover the correct gravitational degrees of freedom. On a simplicial lattice \( m \in \mathcal{M}_s \), the spin connection \( \omega \) can be naturally replaced by its holonomy \( H_e \) along the molecule’s half-edges. Seemingly the bivector \( B \) is replaced by its own flux \( X_f \) across the triangle dual to a face of the molecule.

\[ \omega \mapsto H_e \equiv \mathcal{P} e^{I} e^L e^J e^x \wedge dx^x \quad B \mapsto X_f \equiv \int_{\ell} BIJ dx^a \wedge dx^b \quad H_e \in \text{Spin}(4), \quad X_f \in \text{spin}(4). \quad (3.4) \]

Therefore the discretized BF theory’s action take the following form:

\[ S_{\text{BF}}[H_f, X_f] = \text{Tr} \left( \zeta (H_f) X_f \right) \quad \zeta : \text{Spin}(4) \rightarrow \mathbb{R} \] (3.5)

where \( H_f \) is the holonomy of the connection around the face (defined as the ordered product of the holonomies associated to the half-edges bounding that face) and the trace is evaluated in the fundamental representation. The coordinate on the group \( \zeta \) dictates the prescription for discretizing the curvature 2-form [40]. We can also write down a discrete version of the linear simplicity constraints (3.3). Indeed by using the canonical decomposition of bivectors into selfdual and anti-selfdual \( \mathfrak{su}(2) \) components \( X = (x^-, x^+) \), the discrete simplicity constraints read:

\[ (X_{IJ}^- - \gamma^* X_{IJ}^+) k_{\alpha \beta} = 0 \quad k_x x^- k_{\alpha \beta}^{-1} + \beta x^+ = 0 \quad \beta = \frac{\gamma - 1}{\gamma + 1} \] (3.6)

where the variable \( k_x \) can be interpreted as the normal to the tetrahedron dual to the patch node \( \bar{\nu} \).

Let us conclude this section with few important remarks. As show in [53] the linear simplicity constraints (3.6) are not sufficient to ensure the simpliciality of the bivectors \( X_{IJ}^\lambda \). The missing conditions are the linear volume constraints. These indeed follow from the closure and the linear simplicity constraints provided one also enforces a four-dimensional closure.
constraint on the normals to the tetrahedra in each 4-simplex. Such restriction however would highly complicate the structure of the resulting spin foam models. For this reason it has never been implemented in the literature as confirmed by numerous studies on the asymptotic analysis of partition function for various models, showing the presence of non-geometrical configurations in the semiclassical limit \([20, 55]\). In the next section we discuss the quantization of the phase space of discrete BF theory and related geometricity constraints (i.e. the closure and linear simplicity constraint).

3.2. The Hilbert space of boundary states

The building block of spinfoam boundary states is the patch Hilbert space \(\mathcal{H}_v\). It admits different realizations as an \(L^2\) space depending on the choice of variables. Here we consider three distinct choices defining three equivalent formulations of spin foam models, denoted as the Flux, Holonomy and Spin representations.

\[
\mathcal{H}_v \equiv L^2[\text{spin}(4)^\times 4] \otimes L^2[S^3] \quad \mathcal{H}_v \equiv L^2[\text{Spin}(4)^\times 4] \otimes L^2[S^3] \quad \mathcal{H}_v^{(L)} \equiv \otimes_{i=1}^4 \mathcal{H}^I \otimes L^2[S^3].
\]

These Hilbert spaces, related by a change of basis, naturally arise in the quantization of phase space of BF theory which is given by one copy of the cotangent bundle of structure group \(T^*\text{Spin}(4)\). The quantization of the cotangent bundle of a compact Lie group \(G\) has been widely studied in the literature. Here we shortly review this topic following the presentation of \([40]\) to which we refer the reader for further details.

We begin by outlining the general strategy. The first step in the quantization procedure is to promote the Poisson algebra \(P_G\) to an abstract operator algebra \(X\) by acting on it with a quantization map \(Q\) preserving the commutation relations. The choice of the quantization map (thus the operator ordering) is indeed the first ambiguity entering the spin foam model building. Given an abstract quantum algebra of observables \(X\), the next task is to construct explicit representations of it as a concrete operator algebra on suitable Hilbert spaces. Last one has to find suitable (unitary) relations between these representations.

\[
\mathcal{P}_G \xrightarrow{Q} X \xrightarrow{\pi} \text{Aut}(\mathcal{H}).
\]

Following the above quantization strategy, we now introduce the notions of quantization map, star-product and non-commutative Fourier transform. We provide first the mathematical definition before discussing few examples. These tools play a crucial role in the flux construction/formulation of spin foam models and will be used throughout the paper, especially in section 4.

**Definition 3.1 (Poisson brackets).** Let \(\mathcal{P}_G = (C^\infty(T^*G), \{\cdot, \cdot\}, \cdot)\) the Poisson algebra on the group manifold induced by the canonical simplectic structure on \(T^*G\). The Poisson brackets can be defined as follows:

\[
\forall f, g \in C^\infty(T^*G) \quad \{f, g\} \equiv \frac{\partial f}{\partial X_i} \mathcal{L}_g - \mathcal{L}_f \frac{\partial g}{\partial X_i} + c^k_{ij} \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_j} X_k
\]

where \(\mathcal{L}_i\) are the Lie derivative with respect to a basis of right-invariant vector fields, \(X_i\) are euclidean coordinates on the Lie algebra \(g\), \(c^k_{ij}\) are the structure constants and repeated lower indeces are summed over.

We now seek to quantize a maximal subalgebra \(\mathcal{A}\) of this Poisson algebra as an abstract operator \(^*\)-algebra \(\mathfrak{X}\).
Definition 3.2 (Quantization map). A quantization map \( Q \) is a linear map between algebras defined as:

\[
Q : \mathcal{A} \rightarrow \mathcal{X} \quad \forall f \in \mathcal{A} \subset \mathcal{A} \subset C^\infty(G \times \mathfrak{g}^*) \quad f \equiv Q(f) \quad X_i \equiv Q(X_i)
\]  

(3.10)

\[
[f, g] = 0 \quad [X_j, f] = i\hbar L^j f \in \mathcal{X}_G \quad [X_i, X_j] = i\hbar \delta_{ij} X_k \quad \forall f, g \in \mathcal{X}_G = Q(\mathcal{A}) \quad X \in \mathcal{X}_G \quad X \equiv Q(\mathcal{A})
\]

(3.11)

where \( \mathcal{A}_G \) is the subalgebra of functions in \( \mathcal{A} \) constant on the Lie algebra. Namely \( \mathcal{A}_G \subset \mathcal{A} \) is the subalgebra of function constant on the group. In general we cannot introduce differentiable coordinates \( \zeta \) on \( G \). However they are approximated arbitrarily well by functions in \( C^\infty(G) \), which allows us to define the operators \( \zeta \).

Given an abstract algebra of observables \( \mathcal{X} \) we construct explicit representations \( \pi : \mathcal{X} \rightarrow \text{Aut}(\mathcal{H}) \) of it as a concrete operator algebra on suitable Hilbert spaces. The holonomy representation \( \pi_G \) is defined as the one diagonalizing all operators \( f \equiv Q(f) \). The flux representation \( \pi_{\mathfrak{g}^*} \) is the one diagonalizing all flux operators \( X_i \equiv Q(X_i) \). However since the fluxes do not commute we have to introduce a suitable star-product operation which, by deforming the ordinary pointwise multiplication, allows us to satisfy the commutators [40].

Definition 3.3 (Star product). A star product, denoted by \( \star \), is an operation such that:

\[
\forall f, \tilde{f} \in \mathcal{A}_G \quad Q(f \star \tilde{f}) = Q(f)Q(\tilde{f}).
\]  

(3.12)

This ensures that \( f \) can be interpreted as the function that upon quantization gives \( f(X_i) = Q(f) \), hence establishing a connection between the classical phase space structure and the quantum operators. Therefore the choice of the quantization map determines uniquely the corresponding \( \star \)-product.

Definition 3.4 (Holonomy and flux representations). Let \( \pi_G \) and \( \pi_{\mathfrak{g}^*} \) denote the holonomy and flux representations on the Hilbert spaces \( \mathcal{H} = L^2(G) \) and \( \mathcal{H} = L^2(\mathfrak{g}^*) \). The action of the operators \( \zeta \) and \( X_i \) read:

\[
\forall \psi \in C^\infty(G) \subset L^2(G) \quad \left( \pi_G(\zeta)(\psi) \right)(g) \equiv \zeta(g) \psi(g) \quad \left( \pi_G(X_i)\psi \right)(g) \equiv i\hbar \frac{\partial}{\partial X_i} \psi(g)
\]  

(3.13)

\[
\forall \psi \in C^\infty(\mathfrak{g}^*) \subset L^2(\mathfrak{g}^*) \quad \left( \pi_{\mathfrak{g}^*}(\zeta)(\psi) \right)(x) \equiv -i \frac{\partial}{\partial \zeta_i} \psi(x) \quad \left( \pi_{\mathfrak{g}^*}(X_i)(\psi) \right)(x) \equiv x_i \psi(x).
\]  

(3.14)

The spaces of smooth compactly supported functions, dense on the corresponding \( L^2 \) spaces, are closed under the action of the above operators. Moreover the commutators are satisfied upon quantization.

Definition 3.5 (Spin representation). The spin representation \( \pi_J \) is defined as \( \pi_J : \mathcal{X} \rightarrow \mathcal{B}(\mathcal{H}^J) \), where \( \mathcal{B}(\mathcal{H}^J) \) is the set of bounded linear operators on the group’s unitary irreducible representations space \( \mathcal{H}^J \).

The holonomy and the spin representations are related by Peter–Weyl decomposition. The non-commutative Fourier transform, instead, provides a map between the flux and the group (or holonomy) representations.
Definition 3.6 (Non-commutative Fourier transform). We assume the existence of a unitary isometric isomorphism $\mathcal{F}: L^2(G) \to L^2(g^*)$ which can be expressed as an integral transform,
\begin{equation}
\hat{\psi}(x) \equiv \mathcal{F}(\psi)(x) \equiv \int_G dg E_g(x)\psi(g) \quad \psi(g) = \mathcal{F}^{-1}(\hat{\psi})(g) \equiv \int_{g^*} \frac{d^4g}{(2\pi)^n} E_g(x) \ast \hat{\psi}(x)
\end{equation}
where the functions $E : G \times g^* \to \mathbb{C}$ are the non-commutative plane wave. Their existence has to be verified once an explicit choice of the quantization map has been made.

Below we list the main properties of the non-commutative plane waves
\begin{equation}
E_g(x+y) = E_g(x)E_g(y) \quad E_{gh}(x) = (E_g \ast E_h)(x)
\end{equation}
\begin{equation}
\overline{E_g(x)} = E_g(-x) = E_{g^{-1}}(-x) \quad E_g(x) \ast E_h(x) = E_{gh}(x) \ast E_{g^{-1}h}(x)
\end{equation}
as well as the integral expansion of the Dirac-delta function on the group and on the lie algebra
\begin{equation}
\delta(x, y) = \delta_s(x - y) = \int dg E_g(x)E_{g^{-1}}(y) \quad \delta(gh) = \int dx E_{gh}(x).
\end{equation}
These identities are completely general. The proofs can be found in [40].

So far we reviewed the quantization of the cotangent bundle of a (compact) Lie group $G$, discussing the mathematical tools required to define the flux representation. In the case of Riemannian spin foam models we chose the structure group $G$ to be the local gauge group of gravity (i.e. $G = \text{Spin}(4)$). By applying the previous quantization procedure to the phase space of BF theory (in four copies) we immediately recover the single-patch Hilbert spaces (3.7). From a combinatorial point of view each Hilbert space $H_P$ provides the space of one-particle states associated to a single ‘atom of space’, i.e. a quantum tetrahedron in the simplicial setting [51, 56]. It can be pictured as a fundamental spin-network vertex, represented by a node $\bar{v} \in \bar{V}$ with $d = 4$ outgoing half-links and their one-valent end points (i.e. a boundary patch $P_e \in \mathcal{P}_b$) labelled, depending on the choice of polarization, by Lie algebra elements, group elements or group representations. The Lie algebra elements can be understood as the covariant smearing of the $B$ 2-form fields (i.e. fluxes of the B field) across the triangles of the 4-simplex, the group elements as discretized parallel transports of the BF 1-form connection along the dual links, and the group representations are quantum numbers labelling eigenstates of the modulus of the $B$ field. Moreover, our quantum (spin network) states are supplemented with an additional variable $k \in S^3 \simeq \text{Spin}(4)/SU(2)$ interpreted as the unit normal to the tetrahedron in its local $\mathbb{R}^3$ embedding. The presence of such extra variable, and the way it will enter the imposition of the geometricity constraints, implies that our boundary states correspond, more precisely, to projected spin networks [57].

3.3. Definition and imposition of the geometricity constraints

The geometricity constraints play a crucial role in the formulation of spin foam models for quantum gravity. The closure constraint is a first class constraint, from the canonical point of view. It requires that the sum of the bivectors associated to the four boundary triangles of the same tetrahedron vanishes. This geometrical interpretation is explicit in the flux representation [17]. From the group perspective it corresponds to a requirement of covariance under the action of the local gauge group. At the quantum level, it can be implemented by an orthogonal projector $\mathcal{P}_d$ acting separately on each node of a spin network state, or on each tetrahedron state, taking into account its normal vector $k \in S^3$. 

10
Definition 3.7 (Closure constraint operator). Let $\Psi_k \equiv \Psi_k \in \mathcal{H}_v$ be a single tetrahedron state and $P_{cl} : \mathcal{H}_v \to \mathcal{H}_v$ the closure constraint projector. Its expression in the group representation reads:

$$
P_{cl}(G_i, \tilde{G}_i, k, k') = \int [dH] \delta([k^{-1}k']) \prod_{i=1}^{4} \delta(G_i^{-1}H\tilde{G}_i) \quad \Psi_k(G_i) = (P_{cl} \Psi_k)(G_i) = \int [dH] \Psi_k(HG_i)
$$

(3.19)

where $k' \equiv H \triangleright k = h^+k(h)^{-1}$ denotes the action of a Spin(4) rotation $H$ on the normal $k$. Upon integration over the normal the (extended) closure constraint reduces to the usual gauge invariance of the Ooguri model [8]. The Spin(4) covariance of $\Psi_k$ induces its invariance under the action of the stabilizer group $SO(3)$.

The simplicity constraints (3.6) are second class and cannot be enforced strongly as operatorial equations. Still, in each representation, they correspond to restrictions on the given variables, that are imposed on the spin foam amplitudes via suitable operators. Different strategies have been employed so far leading to the currently known spin foam models. In the following we provide the general form of the constraint operators, highlight the construction ambiguities have been employed so far leading to the currently known spin foam models. In the following we provide the general form of the constraint operators, highlight the construction ambiguities and indicate briefly the key elements of specific models.

Definition 3.8 (Simplicity constraint operator). Given a state $\Psi_0 \equiv \Psi_k \in \mathcal{H}_v$, the simplicity constraint operator is defined as a map $S^\beta : \mathcal{H}_v \to \mathcal{H}_v$. Its matrix elements in the group and spin basis are given by:

$$
S^\beta(G_i, \tilde{G}_i, k, \beta) = \sum_{j_i} \int [d J] [d \iota] \prod_{i=1}^{4} d_j d_{j_i} w(J, j_i, \beta) \Theta^\beta(G_iU_i \tilde{G}_i^{-1}) \chi^\beta(j_i) \quad U = (k^{-1}uk, u)
$$

(3.20)

$$
S_{M_1 \cdots M_4 N_1 \cdots N_4}^{j_1 \cdots j_4}(k, \beta) = \sum_{j, m_i} \prod_{i=1}^{4} C^{--}_{m_i - m_i'}(k) C^{++}_{m_i' m_i}(k) w(J, j_i, \beta) \quad J = (j_-, j_+).
$$

(3.21)

The expression of the simplicity constraint operator in the spin basis (3.21) follows from the (3.20) upon Peter–Weyl decomposition. The action of $S^\beta$ on a single tetrahedron state reads:

$$
\Psi_k^{\beta}(G_1, G_2, G_3, G_4) = \sum_{\text{All}} \Psi_{M_1 \cdots M_4 N_1 \cdots N_4}^{j_1 \cdots j_4} \prod_{i=1}^{4} D_{M_i}^{j_i}(G_i) \prod_{i=1}^{4} S_{N_i \cdots N_4}^{j_1 \cdots j_4}(k, \beta) D_{N_k}^{\beta}(C).
$$

(3.22)

The notation goes as follows. Uppercase letters denote Spin(4) group elements and representations while lowercase letters label SU(2) data. Moreover $\Theta^\beta$ are Spin(4) characters and $C^{j_+ j_+}(k)$ are rotated Clebsch–Gordan coefficients. The function $w(J, j_i, \beta)$, encoding the constraints as restrictions on the spins, depends on the chosen imposition criterion. This is the second main ambiguity in the construction of spin foam models.

Remark 1 (Expression in the flux representation). The simplicity constraint operator can also be written in the flux basis. By taking the non-commutative Fourier transform of equation (3.20) we find:

$$
S_k^{\beta}(X, Y) = \prod_{i=1}^{4} \left( \delta_{-\iota_i} \ast S_i^{\beta}(Y) \right)(Y_i) \quad S_k^{\beta}(X) = \sum_{j} d_j d_{j_1} w(J, j_1, \beta) \Theta^\beta(X) \ast \chi^\beta(kx^{-1}k + x_+^{-1})
$$

(3.23)
$$\chi^j(x) = \int dg \chi^j(g) E_g(x)$$

$$\Theta^j(X) \equiv \chi^j_-(x^-) \chi^j_+(x^+) \quad x \in \mathfrak{su}(2), \ g \in \text{SU}(2)$$

(3.24)

where $\chi^j$ and $\Theta^j$ are the non-commutative Fourier transform of the group characters. The star-multiplication between the two characters in the above expression is well defined thanks to the following identity:

$$\Theta^j(X) \star \chi^j(kx^- k^{-1} + x^+) \equiv \int dg \chi^j(g) \left( \chi^j_-(x^-) \star E_{k^{-1} g} \right) \chi^j_+(g) \left( \chi^j_+(x^+) \star E_g \right).$$

(3.25)

Next we comment on the general structure and main properties of the simplicity constraint operator.

**Remark 2 (General structure and main properties of $S^\beta$).**

The simplicity constraint operator $S^\beta$ must satisfy two important properties which, in turn, dictate its structure.

- In order to be well defined its image must lay in $H_{\ell}$.

$$\forall \Psi_k \in L^2[\text{Spin}(4) \times 4] \quad \| S^\beta \Psi_k \|_{L^2} < \infty.$$  \hfill (3.26)

- It must commute with closure projector (up to an overall rotation of the normal $k$) in order to ensure a consistent imposition of the geometricity constraint [17, 18].

$$S^\beta(HG_i, \tilde{G}_i, k) = S^\beta(G_i, \tilde{G}_i, H, k) \quad H = (k^{-1} h k, \ h) \in \text{SO}_q(3).$$  \hfill (3.27)

According to the second property the constraint operator transforms covariantly under the action of Spin(4) and therefore it is invariant with respect to the stabilizer group $\text{SO}_q(3)$ of the normal $k$. The expression (3.21) of the simplicity constraint operator satisfies the above criteria (the first one only for suitable choices of the coefficient $w$) and is sufficiently general to encompass all known (Euclidean) spin foam models in the constrained BF theory class (although it is not the most general one). It becomes the unique possible solution compatible with all criteria under the additional requirement that for each link the constraint operator intertwines SU(2) and Spin(4) representations via an invariant tensor. This property completely fixes the structure of the operator’s matrix elements to be given, for each link, by a pair of Clasch–Gordan coefficients up to a function $w$ of the spins. The simplicity constraint operator $S^\beta$ is not an orthogonal projector for arbitrary values of the Immirzi parameter. Moreover it does not always admit a closed formula for any choice of basis.

The expression of the single-link fusion coefficients $w$ can be found by demanding that the operator $S^\beta$ enforces the simplicity constraints with respect to a given criterion. In other words for each constraint imposition method we have to choose the coefficient $w$ such that the resulting constrained state (3.22) satisfies the simplicity constraints with respect to the same method. In table 1 we list the expressions of the fusion coefficients for the most studied models, as found in the literature, to which we refer for more details [1, 13–15].

**Example 1 (The EPRL model’s fusion coefficients).**

The EPRL model imposes the simplicity constraint by applying the Master constraint criterion. More precisely it selects a
Table 1. Expressions of the single-link fusion coefficients for some of the most studied spin foam models.

<table>
<thead>
<tr>
<th>Classes of Models</th>
<th>$w(J, j, \beta)$</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Barrett–Crane, $\beta = 1$.</td>
<td>$\delta_{J^{-}, J^{+}}$.</td>
<td>Strong imposition.</td>
</tr>
<tr>
<td>EPRL, Alexandrov, $\beta &lt; 0$.</td>
<td>$\delta_{J^{-},</td>
<td>\beta</td>
</tr>
<tr>
<td>EPRL, $\beta &gt; 0$.</td>
<td>$\delta_{J^{-},</td>
<td>\beta</td>
</tr>
<tr>
<td>Alexandrov, $\beta &gt; 0$.</td>
<td>$\delta_{J^{-},</td>
<td>\beta</td>
</tr>
<tr>
<td>FK, $\beta &lt; 0$.</td>
<td>$\delta_{J^{-},</td>
<td>\beta</td>
</tr>
<tr>
<td>FK, $\beta &gt; 0$.</td>
<td>$\delta_{J^{-},</td>
<td>\beta</td>
</tr>
</tbody>
</table>

particular class of configurations which minimize (and annihilate in the semiclassical limit) the eigenvalue of the Master constraint operator.

$$M = (kJ^{-}k^{-1} + |\beta|J^{+})^2 = k(J^{-})^2k^{-1} + |\beta|^2(J^{+})^2 + \beta \left[(J^{-})^2 - k(J^{-})^2k^{-1} - (J^{+})^2\right]$$  \hspace{1cm} (3.28)

$$J = kJ^{-}k^{-1} + J^{+} \quad M|J, M, N\rangle = m(j^{-}, j^{+}, j, \beta)|J, M, N\rangle$$  \hspace{1cm} (3.29)

EPRL solutions: $j^{-} = |\beta|j^{+} \quad j = (1 + |\beta|)j^{+} \quad \beta < 0, \quad j = (1 - |\beta|)j^{+} \quad \beta \geq 0$  \hspace{1cm} (3.30)

where $J^{-}, J^{+}$ are the angular momentum generators. Notice that the condition $j^{-} = |\beta|j^{+}$ forces the Immirzi parameter to be a rational number. The operator $S^{0}$ will implement the simplicity constraints according to the Master constraint criterion if the general constrained state (3.22) satisfies the same criterion on average.

Therefore we have the following equation:

$$\forall \Psi_k \in \mathcal{H}_{\beta} \quad \Psi_k^{\beta} = S^{0} \Psi_k \quad \langle \Psi_k^{\beta}|M|\Psi_k^{\beta}\rangle \approx 0$$  \hspace{1cm} (3.31)

$$\langle \Psi_k^{\beta}|M|\Psi_k^{\beta}\rangle = \sum_{\text{All}} \Psi_{k_{\beta}, \beta_{\pi}, p}^{\beta} \prod_{i=1}^{4} C_{i^{	ext{in}}, i^{	ext{out}}_{\beta}, i_{\pi}}^{\text{in}, \text{out}, \beta} C_{i^{	ext{in}}, i_{\pi}}^{\text{out}, \text{out}, \beta} w(J^{-}, j^{-}, j, \beta) m(j^{-}, j^{+}, j, \beta).$$  \hspace{1cm} (3.32)

The above expression is minimized when the fusion coefficient $w$ encodes the restrictions corresponding to a solution of the Master constraint (e.g. the EPRL solution). In particular we can choose $w$ to be of the form:

$$w^{2}(J^{-}, j^{+}, j, \beta) = \delta_{J^{-}, |\beta|} \delta_{J^{+}, (1-|\beta|)} \quad \beta < 0 \quad w(J^{-}, j^{+}, j, \beta) = \delta_{J^{-}, |\beta|} \delta_{J^{+}, (1-|\beta|)} \quad \beta \geq 0.$$  \hspace{1cm} (3.33)

By construction the resulting constraint operator will then implement the EPRL simplicity conditions. Clearly the above procedure can be applied to any constraint imposition methods in any basis.

Summarizing the universal structure of the simplicity constraint operator follows from the general requirements we imposed. Different constraint implementation methods yield different definitions of the single-link fusion coefficient $w$ (and thus different models) encoding the simplicity constraints as restrictions on the spins. This coefficient also encodes the choice of the quantization map and thus the operator ordering conventions. Furthermore it can be
rescaled by an arbitrary function of the representations $\Delta_j^j$, compatible with the definition of $S^\beta$ as a map between $L^2$ spaces. This is a third ambiguity in the construction of spin foam models for 4d gravity. The choice of this function is not dictated by the quantization map or by the constraint imposition strategy and thus must be prescribed by hand. Several requirements have been proposed to restrict the allowed choices [21]. Last when no simplicity constraints are imposed, we have $w(J,j,\beta) = 1$ and one recovers the spin foam model for BF theory.

3.4. Spin foam amplitudes

Once a given prescription to define the quantum geometricity constraints has been selected (yielding a specific expression of the single-link fusion coefficients) we still have to choose how to impose them on the spin foam amplitudes. The main strategies differ according to where, i.e. on which ones of the patch Hilbert spaces in the amplitude’s definition, the geometricity operator is chosen to act. Once the above decision has been made, we also have to specify whether we intend to act with the geometricity operator itself or with any arbitrary power of the same. This is the fourth and last ambiguity in the spin foam construction. When the geometricity operator is an orthogonal projector the above ambiguities are irrelevant. In the following we keep the treatment general and include a parameter indicating such power (which affects the simplicity constraint operator only). Accordingly we can identify three main imposition strategies.

**Vertex Hilbert spaces.** Impose the geometricity constraints by acting ($p$ times) with $G^\beta$ on the states in the patch Hilbert spaces for each spin foam vertex (atom), before contracting spin foam atoms with one another.

**Edge Hilbert spaces.** Enforce the geometricity constraints by acting ($q$ times) with $G^\beta$ on the states entering the gluing map, before using it to connect spin foam atoms.

**Gluing and Vertex maps.** Implement the geometricity constraints by adopting both the above prescriptions, inserting the geometricity constraint both in the vertex and gluing operators.

The above choices lead apriori to different models. We adopt the third strategy, providing the expressions of the vertex and gluing kernels in the flux representation.

\[
\mathcal{V}^\beta\rho(X_{\bar{v}}, Y_{\bar{v}}, k_\bar{v}) = \int \left[ \prod_{v \in V_a} dH_{\bar{v}} \right] \prod_{f \in F_a} \left( \delta_{-X_{\bar{v}} \delta S_{k_\bar{v}}^{\beta \rho} \times E_{H_{\bar{v}};H_{\bar{v}'}^{-1}} \times S_{k_\bar{v}'}^{\beta \rho} \right)(Y_{\bar{v}}) \tag{3.34}
\]

\[
\mathcal{K}^\beta\rho(X_{\bar{v}}, Y_{\bar{v}}, k_\bar{v}, k_{\bar{v}'}) = \delta(k_{\bar{v}} k_{\bar{v}'}) \int [dH_{\bar{v}}dH_{\bar{v}'}] \prod_{v \in P; P'} \left( \delta_{-X_{\bar{v}} \delta S_{k_\bar{v}}^{\beta \rho} \times E_{H_{\bar{v}}} \times H_{\bar{v}'}^{-1} \right)(Y_{\bar{v}'}) \tag{3.35}
\]

Here $S^{\beta \rho}$ denotes the star-product $\rho$th power of the function $S^\beta$. The vertex kernel depends on twenty lie algebra variables associated to the ten triangles of a geometric 4-simplex as seen from the reference frames of the two tetrahedra sharing them. The group element $H_{\bar{v}}$ is the parallel transport along the edge $e = v\bar{v}$ from the centre of the 4-simplex to one of its boundary tetrahedra. The gluing kernel enforces the identification of the variables labelling the two boundary patches associated to the same node/tetrahedron as seen from the two 4-simplexes sharing it.

Despite the above ambiguities the spin foam amplitudes share the same general structure. Their expressions are given below in the flux and spin basis.
3.4.1. Simplicial path integral representation of the amplitudes in flux variables. Let us consider a generic (connected) simplicial complex without boundary and the corresponding spin foam molecule $m \in \mathcal{M}_S$. The spin foam amplitude $A_m$ in flux variables is obtained by adopting the corresponding representation of the patch Hilbert spaces and contracting the vertex and gluing kernels (3.34) and (3.35) following the combinatorics of the molecule $m$. Upon choosing an orientation for each face $f$ the amplitude reads:

$$A^3(m) = \int \left[ \prod_{v \in V_m} \prod_{e \in E_v} dH_{ve} \right] \left[ \prod_{e \in E_m} dk_e \right] \prod_{f \in \mathcal{F}_m} A^3_f (H_{ve}, k_e)$$

(3.36)

$$A^3_f (H_{ve}, k_e) = \int \left[ \frac{d^6 X_f}{(2\pi)^6} \right] \bigg[ \star \left( E_{H_{ve}^{-1} \star S_{k_e} \star E_{H_{ve}}} \right) (X_f) \bigg]$$

(3.37)

where $n = 2p + 2q$. We denoted by $H_{ve}$ the parallel transport along an incoming half-edge $e$ and by $X_f$ the flux associated to the face $f$. Notice that the amplitude $A_m$ is invariant under the following gauge transformations

$$H_{ve} \rightarrow \xi_e H_{ve} \xi_e^{-1}, \quad k_e \rightarrow \xi_e^+ k_e \xi_e^- \quad X_f \rightarrow \xi F \xi_e^{-1} \quad \forall H, \xi \in SO(4), \ k \in S^1, \ X_f \in \mathfrak{so}(4)$$

(3.38)

geometrically interpreted as rotations of all local frames. Such symmetry allows us to drop all the bulk normals from the amplitude, for example by choosing the time gauge $\xi_e = (k_e, I)$.

Importantly the general expression of the spinfoam amplitudes in flux variables can be understood as a simplicial path integral for constrained BF theory. In order to see this more clearly we must first commute all simplicity functions $S^2_k$ with the non-commutative plane waves by using the second one of the identities (3.17). Hence the amplitude can be rewritten as follows:

$$A^3(m) = \int \left[ \prod_{f \in \mathcal{F}_m} \frac{d^6 X_f}{(2\pi)^6} \right] \left[ \prod_{e \in E_m} dk_e \right] \mathcal{D}^{H_{ve} \cdot k_e} (X_f) \star \prod_{f \in \mathcal{F}_m} E_{H_f} (X_f)$$

(3.39)

$$\mathcal{D}^{H_{ve} \cdot k_e} (X_f) = \prod_{v \in V_m} \prod_{e \in E_v} dH_{ve} \prod_{f \in \mathcal{F}_m} \left[ S^2_{k_e} \star \left( \left[ \Sigma_{e', e \neq e'} S^2_{H_{ve} \cdot k_e} \right] (X_f) \right) \right]$$

(3.40)

Here $e$ denotes the reference frame edge while $e'$ any other edge in the same face. The star product runs on all possible couple of edge indeces $(e, e')$ in a given face with the reference edge kept fixed. The simplicity function $\mathcal{D}_{H_{ve} \cdot k_e} (\beta)$ imposes on $X_f$ the simplicity condition with respect to the rotated normal $H_{ve} \cdot k_e$, namely the pull back of the normal $k_e$ to the chosen reference frame. The non-commutative plane waves, collected together for each face of the molecule, gives us the exponential of the discretized BF action. In particular the quantization map dictates the prescription for discretizing the curvature two-from in terms of the holonomy on the dual face, via the choice of coordinates $\{ H_f \}$ on the group manifold. For example the Freidel–Livine–Majid (FLM) map (used in [18, 39, 43]) discretizes it as the holonomy itself, producing the discrete BF action $\sum_{f} Tr (X_f H_f)$, while the Duflo map [40, 42] expresses it as the logarithm of the same holonomy. Last the measure term contains two types of factors. One, depending only on the fluxes associated to the tetrahedron chosen as reference, impose the simplicity constraints on them, by the chosen prescription characterizing the spin foam model. The remaining part can be viewed as a (flux-dependent) constraint on the holonomies modifying the Haar measures $dH_{ve}$ on discrete connection parallel transports, used in discrete BF theory. These modifications enforce the requirement that the same discrete connection transports correctly the simplicity constraints across different simplicial frames. The measure
also absorbs within it the other construction ambiguities we have mentioned earlier. Its origin can be traced back to the use of the extended states, depending explicitly on the tetrahedral normals and to the consequent generalization of the closure constraint to account for the transformation of the normals (which had been advocated in the spin foam and LQG literature [58, 59]). Both are required in order to ensure a consistent covariant imposition of the geometricity constraints.

Thus the vacuum amplitudes of any spin foam model (of the class considered) in flux variables take always the form of non-commutative first order simplicial path integrals for a constrained BF theory of the Holst–Plebanski type with a discretization of the BF action depending on the chosen quantization map for the fluxes, and a covariant measure on discrete connection $\mathcal{D}_\beta^{k,H,X}$ encoding the geometricity constraints, their covariance, and other model-dependent features. We stress once more that this result, first obtained for a specific model in [17, 18] (and for BF theory in [39]) is general; it is a feature of the flux representation of all models in this constrained BF class, and not the outcome of specific choices in the constraints imposition.

### 3.4.2. Spin representation of the amplitudes.

The expressions of the amplitudes in the spin basis can be found by taking the non-commutative Fourier transform and the Peter–Weyl decomposition of the above formulas in flux variables. Thus we have:

$$\mathcal{A}^3(m) = \sum_{J_f} \sum_{I_{ef}} \prod_{j \in \mathcal{F}_m} d_{J_f} \prod_{v \in \mathcal{V}_m} \{15J_f\} \cdot \prod_{e \in \mathcal{E}_m} d_{I_{ef}} \sqrt{\mathcal{J}}_{J_f} \mathcal{J}_{I_{ef}} \mathcal{J}_{I_{ef}} \mathcal{J}_{I_{ef}}(J_f, j_f, k_e, \beta).$$

(3.41)

The fusion coefficients $f^l_i$ are the matrix elements of a map between Spin(4) and SU(2) intertwiners’ spaces.

$$f : \text{Inv}_{\text{Spin}(4)} \left[ \bigotimes_{i=1}^{4} \mathcal{H}^{I_i} \otimes \mathcal{H}^{I_i} \right] \rightarrow \text{Inv}_{\text{SU}(2)} \left[ \bigotimes_{i=1}^{4} \mathcal{H}^{I_i} \right]$$

$$f^l_i(I_i, j_i, k, \beta) = \sum_{M, m_i} (I_i)^{j_i,j_i,j_i,j_i}_{M_i M_i M_i M_i} \prod_{l=1}^{4} C_{m_i m_i m_i m_i}^{j_i j_i j_i j_i} (k) \omega^p (J_f, j_f, k_e, \beta) \langle I_i \rangle_{I_i I_i I_i I_i}.$$ $$f^l_i(j_i, j_i, k, \beta) = \sum_{M, m_i} (I_i)^{j_i,j_i,j_i,j_i}_{M_i M_i M_i M_i} \prod_{l=1}^{4} C_{m_i m_i m_i m_i}^{j_i j_i j_i j_i} (k) \omega^p (J_f, j_f, k_e, \beta) \langle I_i \rangle_{I_i I_i I_i I_i}.$$ (3.42)

Upon using the appropriate recoupling identities the previous definition can be rewritten in terms of NineJ symbols [60]. The notation goes as follows. We have a Spin(4) representation $J_f$ labelling each face, a pair of Spin(4) four-valent intertwiners $I_{ef}$, $I_{ef}$ for every edge and an SO(3) spin $j_f$ for each edge in a given face. Once more, the above formulas are general for this class of constrained BF models, and the specificities of the model lie into the exact form of the single-link fusion coefficients.

So far we provided a short overview on the formulation of Spin foam models for constrained BF theory. We started by listing the combinatorial building blocks on which spin foam models are supported. Then we restricted ourselves to the simplicial setting and discussed the quantization of the BF theory’s phase space (with an emphasis on the flux representation and related mathematical tools) as well as the definition and imposition of the geometricity constraints giving the general expressions of the resulting spin foam amplitudes in different basis. At each stage we highlighted the various construction choices entering the model building. In particular we identified four ambiguities in the spin foam construction: the choice of the quantization map, the prescriptions defining the simplicity constraint, the choice of the face weights and the strategies to implement the constraints on the spin foam amplitudes. Different choices lead apriori to different models. The resulting kernels and amplitudes, however, share
the same general structure and differ only by the explicit form of the model-dependent single-link fusion coefficient encoding all the construction choices. This is an important feature since it allows us to parametrize the possible expressions of the simplicial interaction kernel in terms of the underlying model building prescriptions (i.e. in terms of the coefficient \( w \) and its integer power \( n \) counting the number of constraint insertions). To achieve a full characterization of the theory space of spin foam models (and GFTs) one must also constrain the set of allowed interactions terms, i.e. the combinatorial structure of the corresponding spin foam atoms, and analyze the stability of a given ansatz under renormalization group flow [23, 24, 26, 30, 31]. In the next section we introduce a new spin foam model for constrained BF theory.

4. Flux variables and the Duflo map: a new spin foam model for quantum gravity

We now present a new spin foam model for constrained BF theory, based on the flux/metric procedure introduced in [17, 18] (see also [40]) and on the use of the Duflo quantization map [42]. The choice of the Duflo map is a potentially important aspect since it has a number of attracting mathematical properties. A notable one is that it represents faithfully the subalgebra of invariant functions of the Lie algebra it is applied to, which already makes it the most natural quantization map for systems in which the gauge invariance is a key aspect. A second one, possibly even more important in our context, is that it is defined for any semi-simple (locally finite) Lie group, thus in particular it allows an immediate generalization of the construction to the Lorentzian setting, based on the group \( SL(2, \mathbb{C}) \), which can be found in [44].

4.1. Simplicity constraints and non-commutative tetrahedra

The flux formulation of spin foam models allow us to implement the simplicity condition in a geometrical transparent way: the simplicity constraints are imposed directly on flux variables by star-multiplication of states (and the amplitudes) with a non-commutative delta function. In order to construct the model we need to define the Duflo map and provide the corresponding expressions of the non-commutative plane waves.

Definition 4.1 (Symmetric quantization map). Let \( G \) be a semisimple Lie group and \( g^\ast \) the dual of its Lie algebra. The Symmetric quantization map can be defined as follows:

\[
S : \text{Sym}(g) \rightarrow U(g) \quad S(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_k} X_{x_{\sigma_1}} \cdots X_{x_{\sigma_n}}
\]

(4.1)

where \( \mathfrak{S}_k \) is the symmetric group of order \( k \). Here \( \text{Sym}(g) \) is the symmetric algebra of \( g \) and \( U(g) \) is its Universal enveloping algebra. The symmetric map is not an algebra isomorphism unless \( g \) is abelian.

The Duflo map [40, 42] provides an algebra isomorphism between the subalgebra of invariant polynomials under the adjoint group action, denoted by \( \text{Sym}(g)^\delta \), and the center of the universal enveloping algebra \( U(g)^\delta \).

Definition 4.2 (Duflo quantization map). The Duflo map is defined as follows:

\[
\mathcal{D} : \text{Sym}(g) \rightarrow U(g) \quad \mathcal{D} = S \circ \mathcal{J}^\dagger (\partial) \quad \mathcal{J}(x) = \det \left( \frac{\sinh \frac{1}{2}\text{ad}_x}{\frac{1}{2}\text{ad}_x} \right) = \left( \frac{\sinh |x|}{|x|} \right)^2
\]

(4.2)
where the last expression for $\mathcal{J}$ holds for $x \in \mathfrak{su}(2)$. When applied to exponentials the Duflo map gives:

$$E(g, x) = e^{i\tilde{\xi}} = \eta(g)e^{i\tilde{\zeta}(g)} \quad \tilde{\zeta}(g) = -i\ln(g) = \tilde{k}(g) \quad \eta(g) = \frac{|\tilde{k}(g)|}{\sin |\tilde{k}(g)|} \quad |\tilde{k}| \in [0, \pi].$$

(4.3)

This is the well known expression of the Duflo non-commutative planes wave for SU(2) in the $k$-parametrization, first obtained in [40], which can be trivially extended to Spin(4) by double copy.

We can now define the simplicity constraint operator for the new model.

**Definition 4.3 (Simplicity constraint operator).** In the flux basis the operator $S^\beta : \mathcal{H}_\beta \to \mathcal{H}_\beta$ reads:

$$S^\beta(X_i, Y_i, k) = \prod_{i=1}^4 \left( \delta_{-X_i \cdot S^\beta_k} (Y_i) \right) \quad S^\beta_k(X) = \delta_{-k \cdot \beta \cdot k^{-1}} (\beta x^+ ) = \int du E_{k^{-1}uk}(x^-)E_u(\beta x^+)$$

(4.4)

$$\Phi^\beta_k(X) = (S^\beta \Psi_k)(X) = \int \prod_{i=1}^4 \left( \frac{d^4 Y_i}{(2\pi)^6} \right) S^\beta(Y_i, k) \Psi_k(Y_i) \quad X, Y \in \text{spin}(4).$$

(4.5)

Let us now show that the action of this operator is well defined.

In order to be able to take the star product with the state $\Psi_k$ we need the simplicity functions $S^\beta_k$ to be in the image of the non-commutative Fourier transform. Therefore we have to rewrite the (4.4) so that the plane waves are evaluated on the same variables $(x^-, x^+)$ labelling the state $\Psi_k$. This requirement implies:

$$E_u(\beta x) = \Omega(\psi_u, \beta)E_{\beta u^\beta}(x) \quad u^\beta \in \text{SU}(2) \quad \beta \in [-1, 1].$$

(4.6)

If the previous equality is satisfied for a suitable group element $u^\beta$ then the simplicity operator is well defined. The above equation admits a unique non-trivial solution given below

$$u_\beta = e^{i\frac{\theta}{2} \beta \cdot \beta} \quad \psi_\beta = |\beta|\psi \quad \tilde{n}_\beta = \text{sign}(\beta)\tilde{n} \quad \Omega(\beta, \psi) = \frac{\sin |\beta|\psi}{|\beta|\sin \frac{\psi}{2}},$$

(4.7)

where the function $\Omega$ is needed to reconstruct the correct prefactor of the Duflo plane wave. The values of the angles parametrizing the group manifold $S^\beta \simeq \text{SU}(2)$, namely $(\psi, \theta, \phi) \in [0, 2\pi] \times [0, \pi] \times [0, 2\pi]$, ensure the uniqueness of the solution. Thus the simplicity constraint operator becomes:

$$S^\beta_k(X) = \int du \Omega(\beta, \psi_u)E_{k^{-1}uk}(x^-)E_{\beta u^\beta}(x^+) = \int \mathcal{D}^\beta \mathcal{W} E_{\mathcal{W}}(X) \quad \mathcal{D}^\beta \mathcal{W} = \Omega(\psi_u, \beta)du$$

(4.8)

where, for convenience, we have introduced the following short hand notation $\mathcal{W} = (k^{-1}uk, u^\beta)$.

**Remark 3 (Properties of $S^\beta$).** The simplicity constraint operator $S^\beta$ is not an orthogonal projector for generic values of $\beta$, except for the special cases $\beta = 0, 1$ (where the simplicity constraints become first class). Due to the properties of the non-commutative plane waves (3.17) it commutes with the closure projector $P_{cl}$, up to a rotation of the normal $k$. Last it does not require any restriction, e.g. any rationality condition, on the Immirzi parameter $\beta$.

In the holonomy representation the simplicity constraint operator takes the following form:

$$S^\beta(G_i, \tilde{G}_i, k) = \int \left[ \prod_{i=1}^4 \mathcal{D}^\beta_i \mathcal{W}_i \right] \prod_{i=1}^4 \delta \left[ G_i \mathcal{W}_i \tilde{G}_i^{-1} \right].$$

(4.9)
The above expression can be found by taking the non-commutative Fourier transform of (4.4) where $S^\beta$ is given by (4.8) as shown in appendix B. The presence of a closed formula for the constraint operator (and thus for the amplitudes) both in flux and group variables is a peculiar feature of our construction.

Upon Peter–Weyl decomposition the kernel of the simplicity constraint operator reads:

$$S_{M_1, \ldots, M_n}^{I_1, \ldots, I_n} (I, \beta) = \prod_{i=1}^{4} \prod_{j=1}^{4} D_{m_i n_i}^{L_i R_i} (u_i) D_{m_i n_i}^{L_i R_i} (u_i)$$

where, for shortness, we setted $k = \parallel$. In order to compute the matrix elements of $S^\beta$ and obtain the expression of the single-link fusion coefficients for the new model we need to evaluate the above integral. Here we only present and discuss the main results, derived in details in appendix B. The matrix elements of the simplicity constraint operator in the spin basis are given by:

$$S_{m^- n^+ a^- a^+}^{j^- j^+} (\beta) = \frac{1}{2 \pi d_{j^-} d_{j^+}} \sum_{a=0}^{\lambda} \sum_{\mu=-a}^{a} \bar{S}^{\alpha}(\beta) (-1)^{\mu} C_{m^- \mu}^{j^- a^+} C_{m^+ \mu}^{a^- j^+} \mathcal{T}_{a}^{j^- j^+} (\beta)$$

where $\chi_a^{j}$ are the generalized characters of SU(2) representation (A.7). The function $\mathcal{T}_{a}^{j^- j^+} (\beta)$ can be computed analytically and is given by in appendix B. The single-link fusion coefficient are given below.

$$w(j-, j^+, j, \beta) = \frac{-1)^{j^- j^+ + j}}{\sqrt{2j^- + 1}} \sum_{\mu=-a}^{a} \bar{S}^{\alpha}(\beta) (-1)^{\mu} C_{m^- \mu}^{j^- a^+} C_{m^+ \mu}^{a^- j^+} \mathcal{T}_{a}^{j^- j^+} (\beta)$$

The form factor $w(J, j, \beta)$, characterizing the simplicity constraint imposition in our model, behaves like a modulating weight factor peaking on different configurations, including the ones selected for example by the EPRL model. A detailed numerical investigation of its behaviour and properties in different kinematical regimes will appear in a forthcoming paper [45]. It would be interesting to investigate whether this feature leads to a solution of the flatness problem.

Before discussing the model’s limiting cases let us summarize the main features of our construction based on the flux representation and on the Duflo map. A very interesting property of the flux construction (compared to other approaches) is the presence of an explicit closed formula for the constraint operator and thus for the model’s amplitudes in the group formulation enabling the use of heat kernel methods in the renormalizability analysis of both spin foam models and GFTs. The use of Duflo map is another important improvement with respect to the existing models based on the same flux/metric procedure [18]. Indeed beside its good mathematical properties, the linear relation between $\psi_\beta$ and $\psi$ induced by the Duflo map, contrasted to the non-linear one dictated by the FLM map [18] (namely $\frac{\delta\psi}{\delta\beta} = \text{arcsin}(|\beta| \sin \frac{\psi}{2})$) simplifies the integrals in the expressions of the constraint operators (e.g. (4.10)) allowing us to derive an explicit formula for the single-link fusion coefficients. The presence of a closed expression for $w$ is a key asset in trying to extract quantitative consequences of the model’s amplitudes by using numerical methods, for example the scaling of the leading order radiative corrections of the $n$-point functions $W^{(n)}$ in the large-$j$ regime. This is the object of a larger
project whose results will be published in [46]. Let us now discuss three interesting limiting cases of our model.

**Barrett–Crane limit.** For $\beta = 1$, corresponding to the Plebanski case in which the Holst term disappears, we can immediately evaluate the coefficient $w$, to find:

$$w(j^-, j^+, j, 1) = \delta_{j^+}^{j^-} \delta_{j^+}^1.$$

(4.14)

The simplicity constraint operator also becomes a projector. The result coincides with the restriction on representations defining the Barrett–Crane model in the euclidean signature modulo the choice of the face weights. These factors are not determined by the quantization map and thus must be prescribed by hand. In our case, in particular, we recover the version of the euclidean Barrett–Crane model already presented in [17].

**Topological BF theory limit.** For $\beta = 0$, corresponding to $\gamma = 1$, one expects the theory to correspond to selfdual gravity. The constraint operator $S^\beta$ and the resulting spin foam model are still well defined in this limit despite the fact that the classical constraints have no clear geometrical interpretation. The function $w_{\text{new}}$ is continuous (strictly speaking it has a removable singularity in $\beta = 0$) and reads:

$$w(j^-, j^-, j, 0) = \frac{2(-1)^{j^-} j^2}{(2j^- + 1)^2}.$$

(4.15)

The simplicity operator acts on the single-tetrahedron state $\Psi_k$ by projecting its bivectors onto the selfdual part of spin $\mathbf{4}$, and the constrained model reduces to the Ooguri spin foam model for $\text{SU}(2)$ BF theory [8].

**Holst limit.** The case $\beta = -1$, i.e. $\gamma = 0$, corresponds classically to the so-called Holst sector of the Plebanski gravity. This denomination come from the fact that the Holst term of the classical Plebanski–Holst action dominates in this limit. This term vanishes on shell due to the torsion freeness requirement of the spin connection, thus one would be tempted to assume that the corresponding theory is topological. However, the limit in the formal quantum theory for the Plebanski–Holst action in the continuum is subtle, and it has been argued that the resulting quantum theory would correspond to a spin foam quantization of 2nd order metric gravity with no torsion [1, 14, 18]. In this limit the coefficient $w$ is given by:

$$w(j^-, j^+, j, -1) = \frac{(-1)^{2j^-} j}{2j^- + 1} \delta_{j^+}^{j^-} \{j^-, j^+, j\}.$$

(4.16)

Therefore for $\beta = -1$ the simplicity constraint operator projects onto simple Spin($4$) representation of the type $J = (j^-, j^+) = (l, l)$ without imposing any restriction (apart from the triangular inequalities) on the representation $j$ labelling the decomposition of the tensor product of simple representations $H^l \otimes H^l = \bigoplus_{j^+ = 0}^{2l} H^j$. Moreover it multiplies each pair $(J, j)$ by an extra phase factor $(-1)^{2j^-} j$. Summarizing the resulting spin foam model can be obtained from the ordinary Spin($4$) Ooguri model by restricting its representations to simple ones as dictated by the function $w(\beta = -1)$.

**Remark 4 (The role of the quantization map).** In the limit $|\beta| \to 1$ the simplicity function $S^\beta(X)$ reduces, by construction, to an ordinary non-commutative delta function both for the Duflo and for the FLM quantization maps. However the two delta distributions $\delta_{\text{Duflo}}$ and $\delta_{\text{FLM}}$, seen as functions on the lie algebra, do not coincide since they still depend by definition on the star-product, as one can easily check by writing down the corresponding integral.
expressions in terms of non-commutative plane waves.

\[ S_{\text{new}}^{f}(x^{-}, x^{+}) = \delta_{x,0}(x^{-} \pm x^{+}) = \frac{1}{(2\pi)^2} \int d\vec{k} |\vec{k}|^2 e^{i\vec{k} \cdot \vec{x}^-} e^{\pm i\vec{k} \cdot \vec{x}^+} \]

(4.17)

\[ S_{\text{FLM}}^{f}(x^{-}, x^{+}) = \delta_{x,0}(x^{-} \pm x^{+}) = \frac{1}{(2\pi)^2} \int d|\vec{k}| |\vec{k}|^2 e^{i\vec{k} \cdot \vec{x}^-} e^{\pm i\vec{k} \cdot \vec{x}^+} . \]

(4.18)

Nevertheless they behave as regular Dirac delta functions under integration upon \( \ast \)-multiplication (with respect to the associated \( \ast \)-product) by an arbitrary function \( f \in L^1_s(\text{su}(2)) \). Therefore, in the above limits \( \beta \rightarrow \pm 1 \), their Fourier components (4.9) and (4.10) for the Duflo map, [18] for the FLM and thus the spin foam amplitudes for molecules without boundary \textit{will always coincide} regardless of the choice of the quantization map. The fact that the vacuum spin foam amplitudes coincide is not a puzzling issue. Indeed the physical content of a quantum theory is encoded by the observables rather than by the vacuum amplitudes. As emphasized in [40], there is a precise relation between a quantum operator \( O_f \) and the classical function \( f \), which upon quantization gives \( O_f \). Such relation, establishing a connection between the classical phase space and the quantum operators, is controlled by the star-product and thus by the choice of the quantization map. Thus the quantum observables have to depend on the operator ordering conventions even in the limit \( \beta \rightarrow 1 \) (though the model’s vacuum amplitudes do not). For arbitrary values of \( \beta \neq \pm 1 \) the model’s spin foam amplitudes do explicitly depend on the operator ordering even for molecules without boundaries. In the flux picture, for instance, the expressions of the simplicity functions and of the vacuum amplitudes for the Duflo map (e.g. the equations (4.8) and (4.21)) and for the FLM map [18] differ by the parametric deformation of the group element \( w^3 \) and by the prefactor \( \Omega(\psi_u, \beta) \).

4.2. The amplitudes of the new model

In this section we present the spin foam amplitudes of the new model. They can be derived straightforwardly from the general definitions introduced in section 3.4. Since we are going to insert the simplicity constraint operator both in the edge and vertex kernels we introduce, for convenience, the following shorthand notation.

\[ \Psi^{(n)} = \left( k^{-1} \prod_{a=1}^{n} u_{\alpha}, \prod_{a=1}^{n} u_{\alpha}^{\mathrm{c}} \right), \quad D^\beta \Psi^{(n)} = \prod_{a=1}^{n} du \Omega(\beta, \psi_u) \quad u \in \text{SU}(2) \]

(4.19)

where the integer \( n \) counts the number of insertion of the simplicity constraint operator.

4.2.1. Flux representation. The amplitude \( A_{\mathbf{m}} \) can be computed exactly as in the general case by taking the appropriate convolution of vertex and gluing kernels. For a closed simplicial complex dual to a molecule \( \mathbf{m} \) without boundary we find:

\[ A^{\beta}(\mathbf{m}) = \left[ \prod_{v \in V_m} \int dH_{he} \right] \left[ \prod_{e \in E_m} d\kappa_e \right] \left[ \prod_{f \in F_m} A_{f}^{\beta}(H_{ve}, \kappa_e) \right] \]

(4.20)

\[ A_{f}^{\beta}(H_{ve}, \kappa_e) = \left[ \frac{d^6 \mathbf{x}_f}{(2\pi)^6} \right] \left[ \prod_{e \in \mathbf{f}} D^\beta \Psi^{(n)} \right] E_{e} \left( \prod_{e \in \mathbf{f}} H_{ve}, \psi_u^{(n)} \right) \left( \mathbf{x}_f \right). \]

(4.21)
In deriving the previous formula we relied on the expression (4.8) and on the properties of the non-commutative plane waves. The amplitude can be recasted in the form of a simplicial path integral for constrained BF theory:

\[ A^\beta (m) = \int \left[ \prod_{f \in \mathcal{F}_m} \frac{d^6 X_f}{(2\pi)^6} \right] \prod_{e \in \mathcal{E}_m} dk_e D^\beta (H_{ve}, k_e, X_f) \times \prod_{f \in \mathcal{F}_m} E_{H_f}(X_f) \] (4.22)

\[ D^\beta (H_{ve}, k_e, X_f) = \left[ \prod_{e \in \mathcal{E}_m} \prod_{e' \in \mathcal{E}_m} dH_{ve} \right] \prod_{f \in \mathcal{F}_m} \left[ \star \delta_n \left( (H_{ve}, k_e) \right)^{-1} \left( \beta x_f^e \right) \right] . \] (4.23)

In order to show this we commuted all simplicity functions \( S_{\beta}^e \) with the plane waves by using the identity (3.17). The label \( e \) denotes the reference frame edge while \( e' \) any other edge in the same face. The star-product runs on all possible pairs of edge labels \((e, e')\) in a given face with the reference edge kept fixed. The group element \( H_{ve} \) is the parallel transport from the frame \( e' \) to the chosen reference frame \( e \) and \( H_f \) is the holonomy of the connection along the face. The non-commutative plane wave in (4.22) gives us the exponential of the discretized BF action and the effective measure satisfies the general properties discussed in section 3.4.

Thus we recover again the expression of a non-commutative simplicial path integrals for a constrained BF theory of Holst–Plebansky type, with a discretized BF action depending on the chosen quantization map for the fluxes, and with a measure \( D^\beta_{k, H} \) capturing the construction choices (and quantization map) characterizing the model. The appearance of an effective measure on the space of discrete connection is a direct consequence of the use of extended states and of the relaxation of the closure constraint (as advocated in the literature [4, 57–59]) both required to ensure a consistent imposition of the simplicity constraints. Last the presence of simplicial path integral formula with a simple expression of the measure (in closed form) allows us to perform the asymptotic analysis of the amplitudes directly in flux variables as done in [39] for the Ponzano–Regge model. For models built in the flux representation, this approach turns out to be much simpler compared to the standard strategy based on the stationary phase analysis of the amplitudes written in terms of Perelomov states or spinors [55]. For other types of models based on a different constraint imposition method, like the EPRL model, the previous statement is no longer true. Although the spin foam amplitudes can be written as simplicial path integrals for constrained BF theory (see section 3.4), the complicated structure of the constrained measure \( D^\beta_{k, H} \) makes the semiclassical analysis (in flux variables) quite difficult.

4.2.2. Group and spin representation. The pure lattice gauge formulation of the model can be found either by non-commutative Fourier transform, exploiting the duality between the metric and the group representations. For an arbitrary simplicial molecule without boundary the amplitude reads:

\[ A^\beta (m) = \int \left[ \prod_{e \in \mathcal{E}_m} \prod_{e' \in \mathcal{E}_m} dH_{ve} \right] \prod_{e \in \mathcal{E}_m} dk_e A^\beta_f (H_{ve}, k_e) \] (4.24)

\[ A_f^\beta (H_{ve}, k_e) = \left[ \prod_{e \in \mathcal{E}_m} \prod_{e' \in \mathcal{E}_m} D^\beta (\vartheta_{ef}^{(m)} \vartheta_{ef}^{(n)} H_{ve}^{-1}) \right] \delta \left[ \prod_{e \in \mathcal{E}_m} H_{ve} \vartheta_{ef}^{(n)} H_{ve}^{-1} \right] . \] (4.25)

As expected, this is the expression of a lattice gauge theory amplitude for constrained BF theory. The notation goes as follows: the group elements \( H_{ve} \in \text{Spin}(4) \) are the parallel transports.
of the connection along the half-edge $e$ which, upon integration, implement the closure constraint while the variables $U_{ef}$ are the Lagrange multipliers enforcing the linear simplicity constraints. The amplitude is invariant under simultaneous rotation of all local frames and therefore can be evaluated in the time gauge. The presence of a closed formula in the group formulation, following from the flux construction, enables the use of heat Kernel methods which can be employed, for example, in the computation of the amplitude’s degree of divergence via saddle point methods [21, 28]. Moreover the linear relation between $\psi_\beta$ and $\psi$ (and therefore the simpler form of the group element $u^\beta$), induced by the Duflo map, represent an additional improvement with respect to [18].

The above amplitude can be equivalently rewritten in the quantum number basis by carrying out all the group integrals. Upon Peter–Weyl decomposition into irreducible representations it reads:

$$A^0(m) = \sum_{J_f \in F_m} \sum_{j_{ef} \in \mathbb{R}} \prod_{v \in V_m} d_{J_f} \prod_{e \in E_m} \prod_{v \in V_m} \{15J_f\} \prod_{e \in E_m} d_{j_{ef}} \sqrt{d_{Ive} d_{Ive'} d_{Ive} d_{Ive'}} \psi_{J_f, j_{ef}, k_e, \beta}.$$  (4.26)

As before, the pedices $v, e, f$ denote the vertices, the edges and the faces of the molecule $m$. Uppercase letters denote Spin(4) representations and intertwiners while lowercase letters label SU(2) quantum numbers. Furthermore we have a representation $J_f$ for each face, a pair of four-valent intertwiners $Ive, Iv'e$ for every edge and a spin $j_{ef}$ for each edge in a given face. The amplitude is written in terms of the Wigner 15J symbol [60, 61] and fusion coefficients (3.42) which in turn contain the single-link form factor $w(J, j, \beta)$ in (4.13) and (4.12). Thus they encode the choice of quantization map as well as all the other specificities of our model.

5. Conclusions

We have discussed in detail the general structure of spin foam models for quantum gravity, for what concerns both their combinatorial aspects, their quantum states and their quantum amplitudes. We have then specialized to the case of (Riemannian) quantum gravity models in 4d based on the formulation of gravity as a constrained BF theory, which most of the current models belong to [1]. In the latter case, we have emphasized the construction choices that enter the model building and differentiate different models as well as the mathematical structures common to all of them. This allowed us to characterize and parametrize the possible choices for the simplicial vertex kernel in terms of the underlying model building prescription (e.g. the expression and the number of insertions of the fusion coefficient $w$). We believe that this analysis, beyond its pedagogical usefulness, can be important in providing a clearer base for extracting physics out of these models by focusing both on their specific and on their universal features.

Next, we have constructed a new spin foam model, in the same constrained BF class, focusing on the flux/metric representation of spin foam states and amplitudes, and thus relying on the associated tools from non-commutative geometry [40]. A clear advantage of the flux construction, beside the lack of a rationality restriction on the Immirzi parameter, is the presence of an explicit closed formula for the model’s amplitudes in group variables enabling the use of heat kernel methods in the context of spin foam and GFT renormalization. Moreover our model is based on the Duflo map [42], for quantizing the Lie algebra variables of discrete BF theory (and thus the metric variables of the gravitational theory). This is another important improvement. First the Duflo map has a number of nice mathematical properties that make it the natural quantization map for quantum systems based on group-theoretic structures. Second, unlike the FLM map employed in [18] it applies to any semi-simple, locally
compact Lie group; thus the model we introduced can be straightforwardly generalized to the Lorentzian signature. Last it simplifies the expression of the simplicity constraint operator in the spin basis (compared to [18]) allowing us to derive an explicit formula for the fusion coefficient $w$.

We gave then the explicit expression of the amplitudes of the new model in flux, group and spin representations, and the availability of all of them in closed form is another useful asset, for concrete computations. These concrete computation are the next developments we envisage, based on the new model. First the single-link fusion coefficient can be numerically evaluated and tabulated which allow us to compare it with other models in the literature, in particular in the semi-classical (i.e. large-$j$) regime. A first detailed study will be presented in [45]. Second, the scaling of the amplitudes, divergences structure and radiative corrections of the new model can be investigated, as a first step towards a more complete renormalization group flow analysis of it. This is also forthcoming work [46]. The mentioned Lorentzian extension of the new model is another important development that can be already targeted, since we have all the elements for its completion (the mathematical basis for it, i.e. the Lorentzian non-commutative Fourier transform stemming from the Duflo map has been worked out in [44]). Having all these more formal results at hand, we can expect a further strong impulse to the ongoing efforts to extract physics out of spin foam models (also within their GFT reformulation). This is indeed our main goal.

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Appendix A. Basics of SU(2) harmonic analysis

In this appendix we collect some useful definitions and identities on the Wigner-$D$ functions and the recoupling coefficients used throughout the paper and in appendix B. Further details can be found in [60].

Spherical Harmonics. The spherical harmonics are a basis in $L^2(S^2)$. Some identities are given below.

$$Y_{lm}^*(\theta, \phi) = (-1)^m Y_{l-m}(\theta, -\phi) = (-1)^{l+m} Y_{l+m}(\theta, \phi) = (-1)^{l} Y_{0m}(\theta, \phi) \quad Y_{lm}(\pi - \theta, \pi + \phi) = (-1)^l Y_{lm}(\theta, \phi) \quad (A.1)$$

$$\int_0^\pi \int_0^{2\pi} d\theta d\phi \sin \theta Y_{lm}^*(\theta, \phi) Y_{lm}(\theta, \phi) = \delta_{ll'} \delta_{mm'} \quad \int_0^\pi \int_0^{2\pi} d\theta d\phi \sin \theta Y_{lm}(\theta, \phi) = \sqrt{4\pi} \delta_{l0} \delta_{m0}. \quad (A.2)$$

Wigner matrices. The normalized left (and right) invariant Haar measure on the group SU(2) reads:

$$d\mu_{\text{Haar}} = \frac{1}{4\pi^3} \sin^2 \frac{\psi}{2} \sin \theta d\psi d\theta d\phi \quad \theta \in [0, \pi] \quad \psi, \phi \in [0, 2\pi] \quad (A.3)$$

$$d\mu_{\text{Haar}} = \frac{1}{4\pi} \left( \frac{\sin |\vec{k}|}{|\vec{k}|} \right)^2 d^3\vec{k} = \frac{1}{4\pi} \frac{d^3\vec{p}}{\sqrt{1 - |\vec{p}|^2}} \quad |\vec{k}| = \frac{\psi}{2} \in [0, \pi] \quad |\vec{p}| \in [0, 1]. \quad (A.4)$$
Wigner matrices are a complete orthonormal set in $L^2[SU(2)]$. Their explicit formula is given below.

$$\int dg D_{mn}^j(g) D_{pq}^j(g) = \frac{\delta_{mp}\delta_{nq}}{(2j+1)} \quad \text{(A.5)}$$

$$D_{mn}^j(\psi, \theta, \phi) = \sum_{a=0}^{2j} \sum_{\mu=0}^a (-i)^a \left(\frac{4\pi}{2a+1}\right)^{\frac{j}{2}} \left(\frac{2a+1}{2j+1}\right) C_{mja}^{b} \chi_i^{j} \psi \ Y_{\mu}^{j} (\theta, \phi) \quad \text{(A.6)}$$

where $\chi_i^{j}(\psi)$ denote the generalized character of SU(2) representations introduced next.

**Generalized characters.** The SU(2) generalized characters, denoted by $\chi_i^{j}$, are given by:

$$\chi_i^{j}(\psi) = 2^a! \left[\frac{(2j+1)(2j-a)!}{(2j+a+1)!}\right] \sum_{m} \sin \psi \left[\sin \frac{\psi}{2}\right]^a C_{2j-a}^{m} \chi_0^{j} \psi \ Y_{i}^{j}.$$

$$\text{(A.7)}$$

We move forward by introducing the (rotated) Clebsch–Gordan coefficients and 4-valent intertwiners.


$$\sum_{j,m_2} d_{j} \bar{C}_{m_1 m_2}^{j_1 j_2} C_{m_2 m}^{j_1 j_2} = \delta_{m_1 m} \delta_{m_2 m} \quad \text{(A.9)}$$

$$\sum_{m_{1},m_{2},m_{3}} (-1)^{i-m_{1}} C_{m_{1} m_{2}}^{j_{1} j_{2}} C_{m_{3} m_{2}}^{j_{3} j_{4}} C_{m_{4} m_{1}}^{j_{3} j_{4}} = \delta_{j_{1} j_{2} j_{3} j_{4}}$$

$$\quad \sum_{j_{1},m_{1},m_{2}} \sqrt{d_{j_{1}}} \bar{C}_{m_{1} m_{2}}^{j_{1} j_{2}} C_{m_{2} m_{1}}^{j_{1} j_{2}} \psi_{i}^{j_{1}}.$$ 

$$\text{(A.10)}$$

Column permutations sign flip and degenerate cases.

$$C_{m_{1} m_{2} m_{3}}^{j_{1} j_{2} j_{3}} = (-1)^{i+m_{1}} \sqrt{\frac{2j_{1} + 1}{2j_{2} + 1}} \delta_{j_{1} j_{2}} \delta_{m_{1} m_{2}} \delta_{m_{2} m_{3}} \quad \text{(A.11)}$$

Let us now define the SU(2) four-valent intertwiners.

$$\int dh \prod_{i=1}^{4} D_{m_{i}}^{j_{i}}(h) = \sum_{m} (-1)^{i-m} \sqrt{d_{i}} \left(\begin{array}{ccc} j_{1} & j_{2} & i \\ m_{1} & m_{2} & m \end{array}\right) \left(\begin{array}{ccc} j_{3} & j_{4} & m_{3} \\ -m & m_{4} & m \end{array}\right).$$

$$\text{(A.13)}$$

Next we provide two sum rules for the Wigner SixJ symbol.

**Wigner SixJ symbol.** The Wigner SixJ Symbol, satisfies the following identities:
Theorem B.1 (Simplicity constraint operator). In the flux basis the operator $S^\beta : \mathcal{H}_\theta \to \mathcal{H}_\theta$ reads:

$$S^\beta (X, Y) = \prod_{i=1}^{i=3} (\delta_{X \cdot i} \star S^\beta) (Y_i)$$

where the relation between the group elements $u$ and $u^\beta$ is given by:

$$u = e^\frac{i \beta \pi}{4} \hat{n} = \hat{n}(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad \psi, \phi \in [0, 2\pi] \quad \theta \in [0, \pi]$$

$$u^\beta = e^\frac{i \beta \pi}{4} \psi_\beta = |\beta| \psi \quad \hat{\beta} = \mathcal{S}(\beta) \hat{n} \quad \theta_\beta = \mathcal{S}(\beta) \pi + \mathcal{S}(\beta) \phi \quad \phi_\beta = \mathcal{S}(\beta) \pi + \phi.$$

Here $\mathcal{S}(\beta)$ is the piecewise (step) function while $\mathcal{S}(\beta)$ is the sign function. Moreover we setted $\bar{1} = \bar{1}$.

Since the operator factorizes we can perform our analysis by working on a single copy. First we compute the expression of the constraint operator in the holonomy representation.

Proposition B.1 (Simplicity operator. Group representation). In the group representation the simplicity constraint operator reads (recall that we setted $k = 1$):

$$S^\beta (G, \tilde{G}) = \int \left[ D^\beta \Psi \right] \delta \left[ G \Psi \tilde{G}^{-1} \right] \quad \Psi = (u, u^\beta) \quad D^\beta \Psi = du \Omega(\psi, \beta)$$

Proof. By taking the non commutative Fourier transform of (B.1) we find:

$$S^\beta (G, \tilde{G}) = \int \left[ \frac{d^6 X}{(2\pi)^6} \frac{d^6 Y}{(2\pi)^6} \right] E_{\tilde{G}^{-1}} (X) E_{G^{-1}} (Y) \star S^\beta (X, Y) = \int \left[ \frac{d^6 X}{(2\pi)^6} \right] \left[ \frac{d^6 \Psi}{(2\pi)^6} \right] E_{\tilde{G}^{-1}} (X) \quad \Psi = (u, u^\beta)$$

where we used the properties of the plane wave (3.16) and (3.17). To complete the proof we use the (3.18).

Next we compute the expression of $S^\beta$ in the spin basis. From the equation (B.3) we have:

$$S^\beta_{m-m+n-n} (\beta) = \int du \Omega(\beta, \psi_u) D^\beta_{m-n} (u) D^\beta_{m+n} (u) \quad \beta \in [-1, 1].$$
Now we have to calculate the above integral. We find the following result.

**Proposition B.2 (Simplicity operator. Spin representation).** The matrix elements of the simplicity constraint operator in the spin basis can be rewritten as follows:

\[ S_{m^\prime m n^\prime n}^{j^\prime j} (\beta) = \frac{1}{\pi d_j^m d_{j^\prime}^{m^\prime}} \sum_{\mu a} \sum_{\mu^\prime a^\prime} \Theta^\mu(\beta)(-1)^{-\mu} C_{m^\prime m n^\prime n}^{\mu a^\prime \mu^\prime a} T_{a^\prime a}^{j^\prime j} (|\beta|) \]  

(B.6)

\[ T_{a^\prime a}^{j^\prime j} (|\beta|) = (-1)^\mu (2\alpha + 1) \int_0^{2\pi} d\psi \frac{1}{|\beta|} \sin \frac{\beta}{2} \psi^- (\psi) \chi_{a^\prime}^{j^\prime} (|\beta| \psi) \chi_{a}^{j} (\psi) \lambda = 2 \min(j^-, j^+) \]  

(B.7)

**Proof.** We substitute the formulas (A.6) and perform the integrals by using the equations (A.1) and (A.2).

\[ S_{m^\prime m n^\prime n}^{j^\prime j} (\beta) = \frac{1}{\pi d_j^m d_{j^\prime}^{m^\prime}} \sum_{\mu a} \sum_{\mu^\prime a^\prime} \Theta^\mu(\beta)(-1)^{-\mu} C_{m^\prime m n^\prime n}^{\mu a^\prime \mu^\prime a} \int_0^{2\pi} d\psi \int_0^{2\pi} d\theta \sin \theta \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' D_{\mu a}^{\mu^\prime a^\prime}(\psi, \theta, \phi) D_{\mu^\prime a^\prime}^{\mu a}(\psi, \theta, \phi) \]  

\[ \times \int_0^{2\pi} d\theta d\phi d\phi' \sin \theta Y_{\mu\theta}(\phi) Y_{\mu'\theta'}(\phi') \Omega(\beta, \psi^2) \frac{1}{|\beta|} \sin \frac{\beta}{2} \psi^- (\psi) \chi_{a^\prime}^{j^\prime} (|\beta| \psi) \chi_{a}^{j} (\psi) \lambda = 2 \min(j^-, j^+) \]  

which is precisely the result stated above. This concludes the proof.

**Remark B.1 (Explicit formula).** The coefficient \( T_{a^\prime a}^{j^\prime j} (|\beta|) \) can be computed from the expressions of the generalized characters. Using the second one of the identities (A.7) we found the following result:

\[ T_{a^\prime a}^{j^\prime j} (|\beta|) = (2\alpha + 1) \sum_{p = -j}^{j} \sum_{q = 0}^{j} C_{p 0 p}^{j^\prime j} C_{q 0 q}^{j^\prime j} \Psi_{pq} (|\beta|) \]  

(B.8)

\[ \Psi_{pq} (|\beta|) = \int_0^{2\pi} d\psi \frac{1}{|\beta|} \sin \frac{\beta}{2} \psi^- (\psi) e^{-i(p + |\beta| q) \psi} \]  

\[ = \left\{ \begin{array}{ll}
\frac{-i e^{-2\beta |j| + 2\beta |q|} \cos |j|}{4 \beta (|\beta| + 1)} & \forall p, q; 2(p + |\beta| q) = 1 - |j| \\
\frac{i e^{2\beta |j| - 2\beta |q|} \cos |j|}{4 \beta (|\beta| + 1)} & \forall p, q; 2(p + |\beta| q) = 1 - |j| \\
\frac{-i e^{2\beta |j| - 2\beta |q|} \cos |j|}{4 \beta (|\beta| + 1)} & \forall p, q; 2(p + |\beta| q) = 1 - |j| \\
\frac{i e^{2\beta |j| + 2\beta |q|} \cos |j|}{4 \beta (|\beta| + 1)} & \forall p, q; 2(p + |\beta| q) = 1 - |j| \\
\frac{8i |\beta| (p + |\beta| q) e^{-2\beta (p + |\beta| q) \cos |j|}}{8i |\beta| (p + |\beta| q) e^{-2\beta (p + |\beta| q) \cos |j|}} & \text{otherwise}
\end{array} \right. \]  

(B.9)
This expression is very useful for precision numerical computations.

The best way to compute the formula of the single-link fusion coefficients is to prove the following identity.

\[
\sum_{m'\rightarrow m^+} C_{m'\rightarrow m^+ m} S_{m'\rightarrow m^+ m} (\beta) = C_{m'\rightarrow m} \langle \beta \rangle (J, j, \beta).
\]  

(B.10)

**Proposition B.3 (Single-link fusion coefficients).** The single link fusion coefficients are given by:

\[
w(J, j, \beta) = \frac{(-1)^{j^- + j^+ + j}}{\pi \sqrt{(2j^- + 1)(2j^+ + 1)}} \sum_{a} \Theta^a_0 (\beta) \left\{ a \begin{array}{c} j^- \\ j^+ \end{array} \right\} T^J_{a} (|\beta|)
\]

(B.11)

**Proof.** We evaluated the left hand side of the (B.10) by using the equations (B.6)–(A.10) and read off the result.

\[
\sum_{m'\rightarrow m^+} C_{m'\rightarrow m^+ m} S_{m'\rightarrow m^+ m} (\beta) = \frac{(-1)^{j^- + j^+ + j}}{\pi \sqrt{(2j^- + 1)(2j^+ + 1)}} \sum_{a} \Theta^a_0 (\beta) \left\{ a \begin{array}{c} j^- \\ j^+ \end{array} \right\} T^J_{a} (|\beta|) = C_{m'\rightarrow m} \langle \beta \rangle (J, j, \beta).
\]

(B.12)

This concludes the proof. \(\Box\)

**Remark B.2 (Limiting cases).** For the values \(|\beta| = 1\) and \(\beta = 0\) the function \(T^J_{a} (|\beta|)\) reads:

\[
T^J_{a} (\pm 1) = \pi (-1)^a (2a + 1) \delta_{j^- j^+} \quad T^J_{a} (0) = \lim_{\beta \to 0} T^J_{a} (|\beta|) = \frac{2\pi (-1)^{2j^-} (2j^+ + 1)}{(2j^- + 1)} \delta_{j^- j^+}.
\]

(B.13)

These identities follows directly from the properties of the generalized characters (A.8). Our model exhibits three interesting limiting cases corresponding to the values of the Immirzi parameter \(\beta = -1, 0, 1\). The corresponding expression of the constraint operator and fusion coefficients are given below.

In the limit \(\beta = 1\), corresponding to the limit \(\gamma \to \infty\) we recover the Barrett–Crane model:

\[
S_{m'\rightarrow m^+ n^-} (1) = \frac{(-1)^{m' - n^-}}{2j^- + 1} \delta_{j^- j^+} \delta_{m' - m, n^- - n^+} \quad w(j^-, j^+, j, 1) = \delta_{j^- j^+} \delta_{\beta}.
\]

(B.14)

The first result follows immediately from the definition (B.5) or, in alternative, from the equation (B.6) upon using the identities (B.13), (A.9) and (A.12). The latter one can be found by using the equations (B.11) and (A.14).

In the limit \(\beta = -1\), we recover the topological Holst model. In particular we find:

\[
S_{m'\rightarrow m^+ n^-} (-1) = \frac{\delta_{j^- j^+}}{2j^- + 1} \delta_{m' - m, n^- - n^+} \quad w(j^-, j^+, j, -1) = \frac{(-1)^{2j^- + j}}{(2j^- + 1)} \delta_{j^- j^+} \{ j^-, j^+, j \}.
\]

(B.15)
As before the first formula follows from the definition (B.5) or equivalently from the (B.6) upon using the equations (B.13), (A.9) and (A.12). The expression of \( w \) can be found using again the identities (B.11) and (A.14).

In the limit \( \beta = 0 \), corresponding to the SU(2) Ooguri model, the functions \( S(\beta) \) and \( w(J,j,\beta) \) become:

\[
S_{m}^{j+} \equiv \lim_{\beta \to 0^+} S_{m}^{j+} = 2^{(2j-1)^2} \delta_{m+n^+}^{n^-}.
\]

\[
w(J,j,0) = \lim_{\beta \to 0^+} w(J,j,\beta) = 2^{(2j-1)^2} \delta_{m+n^-}^{n^+}.
\]

The limits can be easily computed from the definitions (B.6) and (B.11) together with the identity (B.13).

**ORCID iDs**

Marco Finocchiaro [https://orcid.org/0000-0002-2521-0093](https://orcid.org/0000-0002-2521-0093)

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