

Tullio Regge's legacy: Regge calculus and discrete gravity

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The review paper “Discrete Structures in Physics”,¹ written in 2000, describes how Regge’s discretization of Einstein’s theory has been applied in classical relativity and quantum gravity. Here, developments since 2000 are reviewed briefly, with particular emphasis on progress in quantum gravity through spin foam models and group field theories.

1. Introduction

Regge’s seminal 1961 paper, “General Relativity without Coordinates”,² is a brief and rigorous description of how to approximate curved spaces and space-times by simplicial complexes, with the curvature distributed on simplicial nets of co-dimension 2. This was motivated by the desire to model complicated topologies and to obtain a deeper geometrical

insight. The action for a simplicial space was written down and the analogue of the Einstein equations derived, making use of the Schläfli identity. In the last sentence, Regge mentions that the approach could be useful in numerical work. The paper shows clearly Regge's profound geometrical intuition; he admitted later that, having seen how to triangulate space-time manifolds, he was really interested in triangulating group manifolds. His approach became known as Regge calculus.

The paper opened up whole new areas of research. At first it was used for calculations of the evolution of model universes in classical general relativity, and later in efforts to formulate a theory of simplicial quantum gravity. Regge himself never worked extensively in these fields, but maintained an interest and in fact made a connection which led to one of the most important approaches to discrete quantum gravity.

In the classical theory, in an unpublished paper with Lund, "Simplicial Approximation to some Homogeneous Cosmologies",³ Regge contributed to the continuous time three-plus-one formulation of Regge calculus by writing down the form of the action for homogeneous and isotropic spaces, giving an explicit form for the Hamiltonian constraint for such spaces. (The momentum constraints are identically satisfied in this case, but finding a simplicial form for them is more challenging.)

The paper which proved so influential in simplicial quantum gravity was written with Ponzano in 1968, and entitled "Semiclassical limit of Racah coefficients",⁴ these coefficients being a very useful tool in the calculation of matrix elements in atomic physics. Almost in passing, the authors point out the relationship between a sum involving the asymptotic values of $6j$ -symbols associated with a triangulated three-manifold, and the path integral for three-dimensional simplicial gravity with the Regge action. This work was largely neglected until the 1990s when mathematicians⁵ were writing down invariants of three-manifolds, hoping that they would help in the classification, and it was realised that there was a very close connection with the work of Ponzano and Regge. In fact the the Turaev-Viro model appeared to be a regularised version of the Ponzano-Regge model.

There were many attempts to generalise this work to four dimensions, in the general language of spin foam models, the first being the Barrett-Crane model, which will be discussed in a later section. This, in its turn, renewed interest in group field theories, first introduced for the Ponzano-Regge-Turaev-Viro models, which are described in the penultimate section.

Regge was clearly still thinking about discretization, and sometime in the 1970s produced a draft paper entitled "Discrete Yang-Mills theories". (He gave his address as 'The New Jersey Mental Hospital for Retired Physicists and Compulsory Psychopaths' - in mirror writing on one copy; he was at the Institute for Advanced Study in Princeton at the time!) A final version was published in the Festschrift for Yuval Ne'eman.⁶

For the millenium edition of the Journal of Mathematical Physics, Regge co-authored a review paper¹ on Discrete Structures in Physics. This describes in detail forty years of development of Regge calculus and formulations of quantum gravity which have grown out of it. (When the review was being written, Regge was fascinated by computer-generated art work, and produced a picture he called "Johann Sebastian Beach", with piano keys forming the sea wall!) This article will not attempt to cover the material in that review, but rather to describe some of the developments since it was written, under the headings Classical Formalism, Numerical Relativity, Simplicial Quantum Gravity, the Barrett-Crane

Model and Group Field Theories.

2. Classical Formalism

Many of the developments in classical Regge calculus since 2000 have come from Warner Miller and his collaborators. Gentle and Miller⁷ developed an algorithm which produced spacelike simplicial hypersurfaces with constant mean curvature. The equations resulting from a variational approach to this problem were solved together with the Regge equations, and a formulation was proposed which was compatible with Sorkin evolutions.⁸ McDonald and Miller⁹ have emphasised the important role that the dual lattice plays in Regge calculus. They derived a vertex-based scalar curvature using a new lattice obtained both from the simplicial lattice and its dual; this was a vertex-based weighted average of deficit angles per weighted average of dual areas. Mean curvature on a simplicial lattice was also discussed by Conbove, Miller and Ray.¹⁰

Turning their attention to non-vacuum space-times, Gentle, Khefets, McDonald and Miller¹¹ derived a conservation law in Regge calculus by equating the discrete Bianchi identity to a sum of components of the stress-energy tensor projected along the edges of the simplicial lattice. In principle, this extends Regge calculus in a natural way to non-vacuum space-times, provided an appropriate form of the stress-energy tensor can be written down. McDonald and Miller¹² considered the coupling of non-gravitational fields to simplicial space-times, and constructed the lattice action for scalar fields, the Maxwell field tensor and Dirac particles, using discrete differential forms.

An exact form for the Bianchi identity on a simplicial lattice has been given by Hamber and Kegel.¹³ It is valid for arbitrarily curved manifolds, but is not linear in the curvatures in general.

Arjwahjoedi and Zen¹⁴ have focussed on (2+1)-Regge calculus and obtained expressions for discrete curvatures, the Bianchi identity and the Gauss-Codazzi equation, in an attempt to give geometrical clarification to earlier work. It was shown that the standard formulae for these can be obtained in the continuum limit. The main result is that the Gauss-Codazzi equation is very closely related to the dihedral angle formula, which relates an n -dimensional angle between two $(n-1)$ -dimensional simplices, to $(n-1)$ -dimensional angles between two $(n-2)$ -dimensional simplices. This work should now be extended to 3+1 dimensions.

In a contrasting approach, Höhn¹⁵ has studied canonical Regge calculus expanded to linear order about a flat background. He showed how to use the Pachner moves in the evolution of a spacelike hypersurface and identified the gauge and ‘graviton’ degrees of freedom. The constraints generating the vertex displacement symmetry are consistent with the dynamics and are preserved by the Pachner moves. However it seems that the gauge symmetries will be broken in higher order approximations.

Area Regge calculus is an alternative approach to Regge’s scheme, first suggested by Rovelli¹⁶ and motivated by loop quantum gravity and spinfoam models. In this, the triangle areas rather than the edge lengths are the fundamental variables. The variational principle leads to vanishing deficit angles, and metric discontinuities. Wainwright and Williams¹⁷ showed how a simple class of geometries with a discontinuity across a hyperplane leads to refractive wave geometries, which are generalised solutions of general relativity¹⁸.

Neiman¹⁹ posed the question of whether area Regge calculus is a valid discretisation of general relativity. It was found that in cases of interest (Euclidean, or Lorentzian with spacelike tetrahedra) the distributional scalar curvature is non-zero and has the same sign round all triangles, so cannot average to zero, which would be required in general relativity. A non-zero cosmological constant does not solve the problem. If all tetrahedra were null, the argument would not hold, but it is combinatorically impossible to triangulate space-time with null tetrahedra. A carefully-constructed space-time with both timelike and spacelike tetrahedra might solve the problem but that seems unlikely. Neiman's result has implications for the Barrett-Crane model.

In a series of papers, Bahr and Dittrich²⁰ constructed a further interesting variation on conventional Regge calculus. Motivated by the fact that diffeomorphism symmetry is broken if the simplicial space-time is not flat, which leads to pseudo-constraints in the canonical theory, they considered the replacement of flat simplices by ones with constant sectional curvature (consistent with the presence of a cosmological constant). The lengths were replaced by dihedral angles as the basic variables and there were no constraints in the first order formalism.

Christiansen²¹ studied the convergence of linearised three-dimensional Regge calculus, and showed that the first non-trivial terms in the Regge action (the second variations) agree with the Einstein-Hilbert action.

Regge calculus as originally formulated is a torsion-free theory. Schmidt and Kohler²² have generalised it to include dislocations in the simplicial lattice, corresponding to torsion singularities. Similarly to curvature, torsion is distributed on the simplices of co-dimension two, and in four dimensions the contribution to the action (a discrete version of the Einstein-Cartan action) involves the square of $b_{(i)}$, the part of the Burgers vector parallel to the triangle. The variables of the theory are taken to be not only the edge lengths but also the $b_{(i)}$, and the field equations show that the Burgers vector couples algebraically to the matter term. Thus the torsion vanishes in the absence of matter. See also work by Drummond²³ on torsion in Regge calculus and work by Xue²⁴ on the Einstein-Cartan theory in the quantum Regge calculus context.

Teleparallel gravity is a version of gravity where the curvature vanishes but the torsion is non-zero and acts as a force. Pereira and Vargas²⁵ formulated the teleparallel version of Regge calculus, with the action again proportional to the square of the Burgers vector parallel to the triangular hinge. The variables are taken to be just the edge lengths and, in the variation, the Burgers vector is treated as constant. The result is that in this theory, the Burgers vector does not necessarily vanish in the absence of matter and torsion is a propagating field.

3. Numerical Relativity

Many of the applications of Regge calculus in numerical relativity have been relatively small-scale calculations, where the space-time has considerable symmetry and where, in many cases, the continuum solution is known. Examples of such work will be described first, then there will be brief more general comments about Regge calculus as a tool in numerical relativity.

A suitable cosmology for testing Regge calculus is the Kasner model, which was first examined in this way by Lewis,²⁶ who was unable to obtain the full set of Kasner-Einstein equations in the continuum limit. Gentle²⁷ re-examined the problem and discovered that Lewis had neglected one type of curvature. With this included, the full set of equations was derived in the continuum limit and accurate numerical solutions obtained. Brewin and Gentle,²⁸ again in the context of the Kasner cosmology, investigated how the simplicial solutions were second-order accurate approximations to the continuum solution, but the residual of the Regge equations evaluated on the continuum solution did not converge, because of a wave-like disturbance with high frequency but low amplitude in the simplicial solution. The amplitude of this wave converged to zero as the discretisation was refined, and it did not affect the overall second-order accuracy of the simplicial solution. Brewin²⁹ also looked at evolution of the Kasner cosmology using both Regge calculus and his smooth lattice method; both produced convergent approximations to the exact solution, but Regge calculus was two orders of magnitude slower.

Closed FLRW universes with positive cosmological constant were considered in three dimensions by Tsuda and Fujiwara.³⁰ The Cauchy surfaces were taken to be the surfaces of the regular polyhedral solids in three dimensions, with prisms connecting the surfaces at subsequent times. The numerical solution was found to deviate from the continuum solution at large times. The triangles on the Cauchy surface were then subdivided and the new vertices projected onto a sphere to give a geodesic dome. It was found that the numerical solution agreed with the continuum one in the limit of infinite subdivision. This work should now be extended to four dimensions, and matter included.

In a series of papers, Liu and Williams discussed various aspects of the time evolution of simple model universes. In the context of a closed empty universe with positive cosmological constant, they investigated whether to apply the variational principle to the action then impose symmetries (local variation) or to impose symmetries then vary (global variation). It emerged that local variation does not generally lead to a viable set of Regge equations.³¹ They then considered lattice universes with point masses distributed on a regular lattice on the Cauchy surface. Constraints were obtained on the distribution of the masses for the model to be stable. The evolution resembled that of a closed FLRW dust-filled universe. When one mass was perturbed, the model's evolution was well-behaved, with the expansion increasing in magnitude as the perturbation increased.³²

Progress with the classical evolution of a spacelike hypersurface was made in the early 1990s when it was realised that, in general, the Regge equations decouple into a collection of much smaller groups, leading to the so-called 'Sorkin evolution',⁸ already mentioned. De Felice and Fabri³³ showed that the vertices in the 600-tetrahedron tessellation of the three-sphere fall into *five* distinct classes, not four as in the original paper on the Parallelisable Implicit Evolution Scheme.⁸ They also investigated the so-called 'stopping point' which has plagued many numerical calculations of the evolution of model universes using Regge calculus; this point is where the evolution stops well before the spatial volume becomes zero. Brewin³⁴ had suggested that this occurs when the 'vertical' edges become spacelike as the evolution is so fast. De Felice and Fabri showed that it is a causality-breaking singularity in the effective metric in Regge calculus in such calculations. The issue of causality in this approach was also investigated by Khavari,³⁵ who generalised the triangle

inequalities in Euclidean space to inequalities on edges of triangles in Minkowski space. When causality is included in this way in the Parallelisable Implicit Evolution Scheme, the ‘stopping point’ problem is resolved, which is significant progress. The revised algorithm was applied successfully to the FLRW universe.³⁶ Khavari also set up a Regge calculus version of the Raychaudhuri equation.³⁵ In both 2+1 and 3+1 dimensions, analogues for the average expansion and the shear scalar were found.

More generally, in a brief review, Gentle and Miller³⁷ set out a programme for large-scale numerical work using Regge calculus and progress was discussed by Gentle.³⁸ In the spherically symmetric case, the static Schwarzschild solution was modelled to high accuracy. Axisymmetric initial data was constructed for Brill waves, and for a black hole with Brill wave perturbations. Regge calculus methods produced a very accurate approximation to Misner’s analytic solution for initial data for the head-on collision of two equal mass non-rotating black holes. The generic axisymmetric code developed by Gentle, Miller and collaborators was used to obtain the time evolution of the Brill wave initial data, the first successful time evolution of gravitational radiation on a lattice. These successes are very encouraging but there is still much work to be done, particularly on the inclusion of matter and the relation of lattice approaches to standard finite element methods for solving differential equations. Because of the investment of time and computer power needed to develop Regge lattice methods further, it seems that current work in numerical relativity relies more on the very successful standard finite difference techniques to perform calculations in astrophysics, including the recent focus on gravitational waves.

4. Simplicial quantum gravity

This section reports on some of the recent work in quantum Regge calculus, but excludes the Barrett-Crane model and group field theories, which are discussed separately.

In a new approach to quantum Regge calculus, Xue²⁴ translated the Einstein-Cartan theory to a Euclidean lattice, with the tetrad field, the gauge field and the spin connection assigned to the edges. The partition function and effective action are constructed, and the vacuum expectation values of diffeomorphism and local gauge-invariant quantities can in principle be calculated. Some calculations in two dimensions are presented.

In a paper dedicated to Rafael Sorkin, one of the pioneers of Regge calculus, Gambini and Pullin³⁹ apply their ‘consistent discretization’ approach to Regge calculus. They set up a canonical formalism which is free of constraints, with the time evolution equivalent to a canonical transformation. Some of the edge lengths are effectively Lagrange multipliers and can be eliminated using their equations of motion. This seems to avoid the common problem of ‘spikes’ (thin, elongated simplices) in the Lorentzian case. Quantization is achieved by writing down a path integral with the ‘Lagrange multipliers’ eliminated; the measure is determined uniquely by the unitary transformation that implements the dynamics. It is straightforward to include topology change in this framework.

Although it is perhaps more relevant to the section on the Barrett-Crane model, it is worth mentioning the relation between Regge calculus and BF theory obtained by Kisielowski.⁴⁰ A smooth manifold is obtained by removing the hinges of a simplicial complex. The Regge geometry is then encoded in a BF theory on the boundary of this manifold.

The process amounts to replacing the degrees of freedom of Regge calculus with discrete degrees of freedom of topological BF theory. Bonzom⁴¹ has also explored the relation between BF theory and Regge calculus, showing that together, the gluing relations and the simplicity constraints which turn BF theory into simplicial gravity, contain the constraints of area-angle Regge calculus.²⁰ The action includes the contribution from the Immirzi parameter.

The possibility that the strength of gravitational interactions might increase with distance has been explored in a series of papers by Hamber, Toriumi and Williams. A set of effective field equations is formulated incorporating the gravitational, vacuum-polarization induced running of Newton's constant, G . This results in an accelerated power law expansion for the Robertson-Walker universe, at times of the order of the inverse of the Hubble constant.⁴² The implications for cosmological density perturbations are also considered.⁴³ The requirement of general covariance for the effective field equations restricts the value of the gravitational scaling exponent to be an integer greater than one.⁴⁴ The running of the cosmological constant, on the other hand, is shown in a number of approaches to be inconsistent with general covariance.⁴⁵ To make the connection with experiment, Hamber⁴⁶ argues that the lattice results suggest that the growth of Newton's constant, G , with distance, should become observable only on very large distance scales, comparable to the observed scaled cosmological constant. The hope is that future high precision satellite experiments will be able to detect this small quantum correction.

Quantum gravity in the limit of a large number of space-time dimensions has been considered by Hamber and Williams.⁴⁷ For a simplicial lattice dual to a hypercube, a critical point was found, separating a weak coupling from a strong coupling phase, where dominant contributions to the curvature correlation functions were described by large closed random polygonal surfaces.

Hamber and Williams⁴⁸ have used the gravitational Wilson loop to obtain information about the large-scale curvature properties of the geometry. By comparing the resulting quantum averages to expected semi-classical forms valid for macroscopic observers, it is possible to identify the gravitation correlation length in the Wilson loop with the observed large-scale curvature. The results imply a positive effective cosmological constant at large distances.

Most of the quantum gravity work of Hamber and collaborators has used the Euclidean lattice path integral approach, with numerical simulations using Monte Carlo methods. To complement this, Hamber and Williams⁴⁹ have obtained a discrete form of the Wheeler-De Witt equation for the quantum wave functional of the lattice. In the strong coupling limit, the wave functional depends only on geometric quantities such as areas and volumes. Explicit solutions in 2+1 dimensions are found;⁵⁰ a finite correlation length emerges, that cuts off any infra-red divergences, and there seems to be no weak-coupling perturbative phase. By contrast, in 3+1 dimensions,⁵¹ the critical point in G has a non-zero value, but the weak-coupling perturbative ground state appears to be non-perturbatively unstable. The results obtained seem to suggest that the Lorentzian and Euclidean formulations of lattice quantum gravity belong to the same field-theoretic universality class.

In work related to that of Khavari,³⁵ using the causal structure in Lorentzian space-time and the fact that certain simplices are not constrained by the triangle inequalities, Tate and

Visser⁵² set up Lorentzian signature models of quantum Regge calculus, showing that these are not related to Euclidean models by a simple Wick rotation. The lack of the triangle inequality constraints means that it is easier to do analytical calculations and that numerical simulations are more computationally efficient. They set up the path integral for a model in 1+1 dimensions, obtaining scaling relations for the path integral and showing that spikes are absent, then discuss the model in higher dimensions.

Causal dynamical triangulations are based on a Lorentzian simplicial lattice and use the Regge action, but rather than integrating over the edge lengths in the path integral, sums are performed over triangulations using the Pachner moves. A sophisticated and detailed analysis of the phase structure in 3+1 dimensions has been performed and we refer the reader to reviews by Ambjorn, Loll and collaborators.⁵³

5. The Barrett-Crane model

The original development of the Barrett-Crane model is explained in detail in the earlier review.¹ The two versions of the model are based on representations of the group $SO(4)$ in the Euclidean case or $SO(3,1)$ in the Lorentzian case. After the initial flurry of excitement of the model as a potential four-dimensional quantum gravity theory, the hard work of establishing its properties continued at a slower pace. Different proposals for the normalisation factors on the lower-dimensional faces were examined, arriving at a reasonable scheme after numerical simulations of several proposals.⁵⁴ The large-spin asymptotics of the amplitude of a single 4-simplex was established in a precise way by analytical means^{55,56} and confirmed by numerical means.⁵⁷ This showed that although the desired four-dimensional ‘Regge calculus’ geometries are present and contribute the Regge action to the amplitude, some degenerate configurations typically dominate. This presents a physical picture that is not so clear overall.

The issue of understanding the geometry of several 4-simplexes glued together proves to be a hard problem. The main issue is that the data that is matched on a pair of 4-simplexes is the areas of the four common triangles, but this information is not enough to guarantee a continuous metric spanning the two. A geometrical interpretation of this gluing is lacking, except perhaps across a null surface in the Lorentzian case.¹⁸

The Pachner moves for the gluing have been analysed recently⁵⁸ and it is argued there that the action of ‘area Regge calculus’ (see section 2) provides a plausible semiclassical model for the phase of the Barrett-Crane amplitude. There is evidence that when more than two simplexes are considered, the possible metric discontinuity is rather more restricted; for example along planes the discontinuity vanishes except in the null case.¹⁷ There is much about the geometry of the Barrett-Crane model that is still to be understood.

Discontent with the gluing conditions led to some substantial extensions of the Barrett-Crane model. These new models were formulated by Freidel-Krasnov⁵⁹ and by Engle-Levine-Pereira-Rovelli,⁶⁰ and in the Lorentzian case by FK and Pereira.^{59,61} The idea is to allow a vector space of intertwiners on each tetrahedron, so that the gluing between two 4-simplexes involves an inner product in this intertwiner space.

Representations of $SO(4)$ are determined by a pair of half-integers (j, j') which label the total spin of each factor in the spin covering $\text{Spin}(4) \cong \text{SU}(2) \times \text{SU}(2)$. Choosing to

sum over all representations (j, j') on each triangle and the full intertwiner space on each tetrahedron leads back to the Ooguri model (or Crane-Yetter model in the q -deformed case), whereas limiting to the simple representations (j, j) and the one-dimensional intertwiner space determined by the ‘canonical vertex’ gives the Barrett-Crane model again. The idea of the new models is to take an intermediate case where $j' = cj$ for a global constant c , and on each tetrahedron a vector space of intertwiners for the group $SU(2)$ (not $SO(4)$). These intertwiners are supposed to give a more complete description of the quantum geometry of a tetrahedron. This means that these degrees of freedom can propagate into the neighbouring 4-simplex in a manifold, giving a more geometric gluing.

The constant c is determined by an Immirzi parameter, as introduced in loop quantum gravity, giving an asymmetry in the action between the self-dual and anti-self-dual parts of the curvature. For some values of c , the ELPR and FK models are identical, and for other values they are different but similar in construction.

As before, the first semiclassical analysis of the amplitudes for the new models was for the single 4-simplex, in the Euclidean case⁶² and in the Lorentzian case.⁶³ These papers also provide a more precise definition of the models themselves. The boundary data is now more complicated because, beside the areas of the triangles, there is also data for the geometry of each tetrahedron. The amplitude for most boundary data is exponentially damped because the integral representation for the amplitude contains no stationary points. The interesting cases are where there are stationary points so that the amplitudes are not damped. The analysis shows that there is a class of such boundary data called ‘Regge-like’ where all of the data is consistent with a flat metric in the 4-simplex. In terms of this metric, the phase of the 4-simplex amplitude is again the Regge action for a 4-simplex. There are however some other important configurations, in the same way as in the original Barrett-Crane model, but with different details. These other configurations are three-dimensional ‘vector geometries’ or even lower-dimensional geometries, and do not have a Regge calculus interpretation.

Probably the simplest explanation of this asymptotics is given by considering the 4-simplex amplitudes as squares of $15j$ symbols and analysing the rather simpler asymptotics of the $15j$ symbols.⁶⁴ Amazingly, the $15j$ symbols can also have Regge-like boundary data with asymptotics determined by the Regge calculus action, even though these symbols belong to the Ooguri model, which is not normally regarded as a gravity model at all.

The ELPR and FK models still have difficulties with gluing. Considering a spatial slice of spacetime (i.e., a 3-manifold), each tetrahedron now has a quantum state space that can distinguish between the flat geometries with the same area for each triangle. But unfortunately, this data does not always glue together continuously from one tetrahedron to another across a common triangle. In essence, it is like the problem with the original Barrett-Crane model but in one lower dimension. It is also similar to the gluing problem for the geometries associated to states in loop quantum gravity.⁶⁵

Several works have analysed the ELPR/FK models when a number of 4-simplexes are glued together to form a manifold. Replacing each 4-simplex amplitude with its asymptotic expression for large spins results in an amplitude for a triangulated manifold whose phase part is the Regge calculus action, but with additional independent variables.^{66,67} This procedure is only a heuristic one as the spin variables on interior triangles are summed over all values, so that it is not clear whether the asymptotic formula is a useful approximation.

Nevertheless, continuing with this formula and looking for the stationary points when the spin is varied leads to the naive conclusion that the geometry is flat, at least in an asymptotic limit.⁶⁸ More careful analyses are underway^{69,70} but it is too soon for a definitive conclusion.

6. Group field theories

Group field theories (GFTs)⁷¹ bring Regge's intuition about describing space-time, geometry and gravity in terms of piecewise-flat, simplicial structures to a whole new level. They extend some of the simplicial quantum gravity approaches discussed above, merging them into a single framework. They were first introduced⁷² as an enrichment of tensor models, themselves a generalization of matrix models for two-dimensional gravity, to higher dimensions, and a generating functional for Euclidean dynamical triangulations. This enrichment defined instead a generating functional for sums over triangulations weighted by the state sum models for topological BF theory in three (Ponzano-Regge model) and four dimensions, based on the representation theory of $SU(2)$. Then, group field theories were found⁷³ to provide a generating functional for the Barrett-Crane model and a tentative definition of the dynamics of four-dimensional quantum gravity, in the language of loop quantum gravity and spin foam models. Many developments over the last twenty years form now a massive body of work making group field theories a promising formalism for a quantum gravity theory based on discrete structures. They have also led to a new perspective, in which spacetime and geometry are emergent notions, to be extracted from the collective behaviour of the discrete GFT structures, seen as fundamental entities.⁷⁴

GFTs can be understood as quantum field theories for elementary objects (the 'quanta' of the GFT field) corresponding to quantum tetrahedra. They rely on a description of simplicial geometry in terms of group- and representation-theoretic data,⁷⁵ also important in the context of spin foam models and loop quantum gravity. The relevant phase spaces are cotangent bundles of group manifolds (usually $SU(2)$, $Spin(4)$ or $SL(2, \mathbb{C})$) and the quantization leads to Hilbert spaces which admit a complete orthonormal basis of spin network states. In fact, the fundamental tetrahedra (in four-dimensional models) can be seen as dual to spin network vertices, with the outgoing links corresponding to the faces of the tetrahedra. The variables encoding the geometry of the simplicial structures are then group elements (corresponding to a discretized connection) and group representations, corresponding to eigenvalues of quantized face areas. A non-commutative description of the states can also be given,⁷⁶ in terms of Lie algebra elements corresponding to metric variables associated to the same faces. Generic quantum states are then many-body states formed by several quantum tetrahedra, including those in which the tetrahedra are glued to form extended three-dimensional triangulations. A Fock space description of the GFT Hilbert space has been given by Oriti,⁷⁷ clarifying also the nature of these models as second quantized descriptions of loop quantum gravity-type theories.

On these kinematical states, different dynamical prescriptions can be imposed. They are encoded in a choice of GFT action for a field whose arguments are the same data characterizing the geometry of a tetrahedron. The defining feature is a non-local pairing of field arguments in the interaction terms, reflecting the gluing of the tetrahedra to form the boundary of four-dimensional cells, taken as the fundamental building block of their inter-

action processes. The perturbative expansion of the GFT partition function defines a sum over Feynman diagrams dual to four-dimensional cellular complexes, with the interaction vertices defining the constituting cells and the propagator enforcing their gluing. One route towards model building⁷⁸ is based on the fact that GFT fields can be seen as generalised tensors with indices on the group manifold, transforming under unitary groups acting on the same indices. Thus, a theory space can be defined including all possible unitary invariants as interaction terms. Such tensorial GFTs can then take advantage directly of the many results obtained in the context of tensor models,⁷⁸ concerning the control over the topology of the GFT interaction diagrams, and their statistical analysis and critical behaviour. In fact, when the Lie group defining the domain of GFT fields is replaced by a finite group or simply by some finite set, then GFT models become tensor models, and their Feynman amplitudes depend only on the combinatorics of the underlying diagram; they actually become akin to those of the dynamical triangulations approach. Beside this tensorial setting, there have been few analyses of symmetries in GFT models,⁷⁹ and thus there is little control over possible theory spaces. Model building has been based, then, on the choice of having Feynman diagrams corresponding to triangulations (which suggests interactions with the combinatorics of 4-simplices) and on the desire to have the corresponding Feynman amplitudes coinciding with the most promising spin foam models, proposed in the context of simplicial quantum gravity and loop quantum gravity,^{71,73,80} and inspired by the formulation of (continuum and discrete) GR as a constrained BF theory. The latter requirement translates into specific choices of kinetic and interaction terms in the GFT action, for each desired model. In fact, one can show that the correspondence between GFTs and spin foam models is generic:⁸¹ for any given spin foam model defined on a given cellular complex, there is a GFT model whose Feynman amplitudes coincide with it on the Feynman diagram dual to the same complex (and vice versa, any given GFT model defines a corresponding spin foam model, in its perturbative expansion). Similarly generic is the fact that, when one formulates the same GFT models in terms of Lie algebra variables, the corresponding Feynman amplitudes take the form of (non-commutative) simplicial gravity path integrals in first order variables, on the same complex.⁸⁰

From the point of view of simplicial quantum gravity, therefore, group field theories define in their perturbative expansion a quantum dynamics combining a sum over discrete geometric data corresponding to a first order version of quantum Regge calculus, associated to a given lattice, with a sum over lattices, including very refined ones, in the spirit of dynamical triangulations. The problem becomes, of course, to show the consistency of such quantum theory, i.e. to control analytically the combined sum, and to ‘resum it’, i.e. to define the full partition function, thus the continuum limit. This is where the quantum field theory nature of GFTs becomes crucial. Both problems, in fact, become problems in the renormalization of GFT models: to prove their perturbative renormalizability, and to define their full RG flow and continuum phase diagram. This has become a very active research direction over the last ten years,⁸² in parallel with the developments in tensor models. Among the many results, it has led to: a better understanding of the divergences of four-dimensional quantum gravity models for constrained BF theories,⁸³ and a complete characterization of them for topological BF models;⁸⁴ the rigorous proof of renormalizability of several tensorial GFT models,⁸⁵ both abelian and non-abelian, in several dimensions;

constructive analyses of the same tensorial GFT models;⁸⁶ the application of functional renormalization group techniques with a first characterization of their flows and phase diagrams (in simple truncations), confirming their asymptotic freedom or safety, and providing hints of new phases.⁸⁷ A first analysis of inequivalent (coherent) representations of GFT observables algebras, possibly corresponding to different phases of GFT systems, has been done by Kegeles, Oriti and Tomlin.⁸⁸

The extraction of effective physics from GFT models can be pursued using methods from simplicial (quantum) gravity and spin foam models, at the level of GFT perturbation theory. The generating functional for superpositions of discrete structures defined by GFT models, as well as their second quantized formalism, are useful tools to do so. They also offer, however, new avenues to study emergent continuum physics bypassing to some extent the discrete gravity picture, and aiming at coarse-grained information encoding many discrete gravity degrees of freedom. In one such line of research,⁸⁹ effective cosmological physics is extracted from the hydrodynamics of GFT models and more precisely from the mean field description corresponding to simple GFT condensate states. Among the many results, we cite the evidence that (within this hydrodynamic approximation): a homogeneous and isotropic universe satisfies a modified Friedmann equation with the correct classical limit but with a bouncing regime replacing the classical big bang singularity,⁹⁰ similar to what is found in loop quantum cosmology; GFT interactions can modify this dynamics to give an extended period of acceleration following the bounce (a sort of quantum gravity-driven inflation), without the need to introduce additional (inflaton-like) degrees of freedom, and a later recollapse and cyclic evolution;⁹¹ the extension of the formalism to include anisotropies⁹² and inhomogeneities,⁹³ with approximate scale invariance found in the spectrum of cosmological perturbations.

7. Conclusions

Applying the geometric concept of piecewise linear spaces to general relativity, Regge provided the basis for very fruitful research in a number of areas, including calculations of the classical evolution of model universes and formulations of quantum gravity. Since the review paper of 2000, in particular there has been outstanding progress in the development of quantum gravity models, which arise directly from Regge's work. These have the potential to provide a consistent theory of quantum gravity with wide-ranging implications in physics.

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