

ALGEBRAIC STRUCTURES IN THE COUPLING OF GRAVITY TO GAUGE THEORIES

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Abstract

This article is an extension of the authors second master thesis [1]. It aims to introduce the theory of perturbatively quantized General Relativity coupled to Spinor Electrodynamics, provide the results thereof and set the notation to serve as a starting point for further research in this direction. It includes the differential geometric and Hopf algebraic background, as well as the corresponding Lagrange density and some renormalization theory. Then, a particular problem in the renormalization of Quantum General Relativity coupled to Quantum Electrodynamics is addressed and solved by a generalization of Furry's Theorem. Next, the restricted combinatorial Green's functions for all two-loop propagator and all one-loop divergent subgraphs thereof are presented. Finally, relations between these one-loop restricted combinatorial Green's functions necessary for multiplicative renormalization are discussed. One of those relations suggests that it is unphysical to consider the coupling to Spinor Electrodynamics alone and instead consider the coupling to the whole Electroweak Sector.

1 Introduction

The theory of General Relativity (GR) and Quantum Field Theory (QFT) are the two great achievements of physics in the 20th century. GR, on the one side, describes nature on very large scales when huge masses are involved. QFT, on the other side, describes nature on very small scales when tiny masses are involved. Being very successful in their regimes, there are situations when both conditions appear at the same time, i.e. when huge masses are compressed into small scales. For example, this situation occurs in models of the big bang and in models of black holes. For these situations a theory of Quantum Gravity (QG) is needed to understand nature. In particular, a theory of QG should be able to clarify how the universe emerged, i.e. through a big bang or otherwise. Therefore, it was soon tried to apply the usual techniques of perturbative QFT to the dynamical part of the metric in spacetimes of GR [2]. These works, in [2] called "The covariant line of research", were started by M. Fierz, W. Pauli and L. Rosenfeld in the 1930s. Then, R. Feynman [3] and B. DeWitt [4, 5, 6, 7] calculated Feynman rules of GR in the 1960s. Next, D. Boulware, S. Deser, P. van Nieuwenhuizen [8] and G. 't Hooft [9] and M. Veltman [10] found evidence of the non-renormalizability of Quantum General Relativity (QGR) in the 1970s. We stress, that by QGR we mean a quantization of GR using QFT methods, whereas by QG we mean any theory of quantized gravitation, such as e.g. Loop Quantum Gravity, String Theory or Supergravity. In this article we continue the work on perturbative QGR as started by D. Kreimer in the 2000s [11, 12]. D. Kreimer used the modern techniques of Hopf algebraic renormalization developed by A. Connes and himself in the 1990s and 2000s [13, 14, 15]. Similar

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situations were studied in the so-called core Hopf algebra by D. Kreimer and W.D. van Suijlekom in the 2000s [16, 17].

We start this article in Section 2 with the differential geometric background needed to understand the Lagrange density of Quantum General Relativity coupled to Quantum Electrodynamics (QGR-QED). Then, we introduce the Lagrange density of QGR-QED,

$$\mathcal{L}_{\text{QGR-QED}} = \left(-\frac{1}{2\lambda^2}R + \frac{1}{4e^2}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} + \bar{\Psi} \left(i\nabla^{U(1)\times\rho\Sigma M} - m \right) \Psi \right) dV_g + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{Ghost}}, \quad (1)$$

which consists of the usual Einstein-Hilbert Lagrange density, the canonical generalization of the Maxwell-Dirac Lagrange density to curved spacetimes and the gauge fixing and ghost Lagrange densities. Finally, the Lagrange density of QGR-QED is discussed in detail. Then, in Section 3 we introduce Hopf algebras in general and the Connes-Kreimer renormalization Hopf algebra in particular. Next, we discuss a problem which can occur when associating the Connes-Kreimer renormalization Hopf algebra to a given local QFT. As we remark, this problem occurs already in QED and in particular in QGR-QED. More concretely, there can exist divergent Feynman graphs whose residue is not in the residue set of the given local QFT. Thus, there are no vertex residues in the theory present which are able to absorb the corresponding divergences. Then, we present three different solutions to this problem in Solutions 3.35, 3.36 and 3.37 and discuss their physical interpretation. We proceed by analyzing the structure of Hopf ideals in the renormalization Hopf algebra which represent the symmetries compatible with renormalization. Then, we define the Hopf algebra of QGR-QED. Therefore, we formulate and prove a generalization of Furry's Theorem in Theorem 3.53 which holds also for amplitudes with an arbitrary number of external gravitons. This is in particular useful, since at least for the calculations done in the realm of this article, these are the only Feynman graphs which need to be set to zero when constructing the renormalization Hopf algebra of QGR-QED, besides from pure self-loop Feynman graphs, which vanish for kinematic renormalization schemes. Finally, in Section 4 we present all one- and two-loop propagator graphs and all one-loop three-point graphs. Then, we present their coproduct structure, for which the coproduct of 155 Feynman graphs has been computed. Using these relations, we study the obstructions to multiplicative renormalizability of QGR-QED which results in a generalization of Ward-Takahashi and 't Hooft-Slavnov-Taylor identities [18, 19, 20, 21, 22]. In particular, we arrive at the conclusion, that we need to consider the whole Electroweak Sector instead of Spinor Electrodynamics alone to be able to absorb the divergences in a physically sensible way.

2 Differential geometric background and the Lagrange density of QGR-QED

We start with the differential geometric background and the Lagrange density of QGR-QED. In this work, we use the Einstein summation convention if not stated otherwise. Furthermore, we require sections to be smooth, i.e. $\Gamma(U, \cdot) := \Gamma^\infty(U, \cdot)$ for any open U . Moreover, we underline the coupling constants for the electric charge e and for the gravitational charge λ in order to avoid confusion with Euler's number, vielbeins and inverse vielbeins.

2.1 Differential geometric background

Definition 2.1 (Spacetime). Let (M, g) be a Lorentzian manifold. (M, g) is called a spacetime if it is smooth, connected, 4-dimensional and time-orientable. Furthermore, we choose the West

coast (mostly minus) signature signature for g . Moreover, we denote the constant Minkowski metric by η and the unit matrix by δ , i.e.

$$\eta := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \delta := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2)$$

Remark 2.2. We assume spacetimes to be 4-dimensional as e.g. the gravity-matter Feynman rules depend directly on the spacetime dimension, c.f. Remark 2.25. However, the corresponding calculations could also be carried out in dimensions different than 4.

Definition 2.3 (Matter-compatible spacetime). Let (M, g) be a spacetime. We call (M, g) a matter-compatible spacetime if it is diffeomorphic to the Minkowski spacetime (\mathbb{M}, η) .

Remark 2.4. The motivation of Definition 2.3 comes from the definition of the graviton field $h_{\mu\nu}$ relative to the Minkowski metric $\eta_{\mu\nu}$, c.f. Definition 2.21.¹ Then, Definition 2.3 could be weakened to: *Let (M, g) be a spacetime. We call (M, g) a matter-compatible spacetime if it is globally hyperbolic, parallelizable, oriented, time-oriented and such that its second de Rham cohomology vanishes.* This is then motivated by the following facts: We want to consider spacetimes with well-defined Cauchy problems, which require (M, g) to be globally hyperbolic. Furthermore, we want (M, g) to admit a spin-structure in order to define matter via a spinor bundle, which was shown by Geroch to be equivalent to (M, g) being parallelizable, oriented and time-oriented [23]. Finally, we want to consider spacetimes which allow the exclusion of magnetic monopoles, which is ensured if the second de Rham cohomology vanishes, as this allows for a global definition of the local connection form ieA_μ on the base-manifold M .

Definition 2.5 (Spacetime-matter bundle). Let (M, g) be a matter-compatible spacetime. Then we define the spacetime-matter bundle to be the globally trivial bundle

$$\mathcal{S} := M \times TM \times E \times (U(1) \times_\rho \Sigma M) \times \bigwedge T^*M \times \bigwedge \mathfrak{u}(1)^*. \quad (3)$$

TM is the tangent bundle and E a real 4-dimensional vector bundle used for the definition of vielbeins and inverse vielbeins (then considered as the tensor product bundle $TM \otimes_{\mathbb{R}} E$ to stress that vielbeins and inverse vielbeins are multilinear maps), c.f. Definition 2.8. $U(1)$ is the principle bundle modeling electrodynamics and acting via the representation ρ on the spinor bundle ΣM , which is a 4-dimensional complex vector bundle modeling fermions, c.f. Definition 2.21. $U(1) \times_\rho \Sigma M$ denotes the corresponding fiber product bundle, to which we refer to as twisted spinor bundle. Finally, $\bigwedge T^*M$ and $\bigwedge \mathfrak{u}(1)^*$ denote the sheaves of Grassmann algebras modeling the graviton ghost and the photon ghost, respectively. We equip the spacetime-matter bundle with metrics in Definition 2.7 which, in turn, naturally includes the corresponding dual bundles.² Additionally, we also equip it with connections in Definition 2.10 such that we have a notion of curvature, c.f. Definition 2.17.

Remark 2.6. The global triviality of the spacetime-matter bundle in Definition 2.5 is motivated by the following facts: The tangent bundle TM and the spinor bundle ΣM are globally trivial since matter-compatible spacetimes are defined to be diffeomorphic to the Minkowski spacetime

¹It is in principle also possible to choose a different background metric, which might require a different topology of M . This is for example necessary, when a non-vanishing cosmological constant is included which, in the quantum theory, behaves like a mass term for the graviton propagator.

²We denote the dual bundles via the asterisk, $*$, except for the spinor bundle, twisted spinor bundle and the sheaves of Grassmann algebras for which we use the overline, $\bar{}$.

and are thus in particular parallelizable, c.f. Definition 2.3. Furthermore, the vector bundle E is chosen to be trivial for the definition of vielbeins and inverse vielbeins, c.f. Definition 2.8. Moreover, the $U(1)$ principle bundle is globally trivial since matter-compatible spacetimes are defined to be diffeomorphic to the Minkowski spacetime and thus in particular have vanishing second de Rham cohomology which implies the first Chern class of the associated line bundle to vanish, c.f. Definition 2.3. Finally, the sheaves of Grassmann algebras $\bigwedge T^*M$ and $\bigwedge \mathfrak{u}(1)^*$ are globally trivial because of the global triviality of the bundles TM and $U(1)$.

Definition 2.7 (Metrics on the spacetime-matter bundle). We consider the following metrics on the spacetime-matter bundle \mathcal{S} : On the tangent bundle TM we consider the Lorentzian metric g with West coast (mostly minus) signature, mapping vector fields $X_1, X_2 \in \Gamma(M, TM)$ to the real number

$$\langle X_1, X_2 \rangle_{TM} := g_{\mu\nu} X_1^\mu X_2^\nu \in \mathbb{R}. \quad (4)$$

Furthermore, on the real vector bundle E we consider the constant Minkowski metric η with West coast (mostly minus) signature, mapping vector fields $Y_1, Y_2 \in \Gamma(M, E)$ to the real number

$$\langle Y_1, Y_2 \rangle_E := \eta_{mn} Y_1^m Y_2^n \in \mathbb{R}. \quad (5)$$

Moreover, on the $U(1)$ principle bundle we use the Hermitian metric, mapping sections $s_1, s_2 \in \Gamma(M, U(1))$ to the complex number³

$$\langle s_1, s_2 \rangle_{U(1)} := s_1^* s_2 \in \mathbb{C}. \quad (6)$$

Additionally, on the spinor bundle ΣM we use the Hermitian metric together with the Clifford multiplication by a timelike vector field, mapping spinor fields $\psi_1, \psi_2 \in \Gamma(M, \Sigma M)$ to the complex number

$$\langle \psi_1, \psi_2 \rangle_{\Sigma M} := \overline{\psi_1} \psi_2 \in \mathbb{C}, \quad (7)$$

where we have set $\overline{\psi} := e_0^m \psi^\dagger \gamma_m$,⁴ with the timelike components of a vielbein e_0^m ,⁵ being introduced in Definition 2.8, and the Clifford multiplication γ_m , being introduced in Definition 2.12. Thus, on the twisted spinor bundle $U(1) \times_\rho \Sigma M$ we consider the induced fiber product metric, mapping twisted spinor fields $\Psi_1, \Psi_2 \in \Gamma(M, U(1) \times_\rho \Sigma M)$ to the complex number

$$\langle \Psi_1, \Psi_2 \rangle_{U(1) \times_\rho \Sigma M} := \overline{\Psi_1} \Psi_2 \in \mathbb{C}. \quad (8)$$

where, again, we have set $\overline{\Psi} := e_0^m \Psi^\dagger \gamma_m$. Finally, on the sheaves of Grassmann algebras $\bigwedge T^*M$ and $\bigwedge \mathfrak{u}(1)^*$ we consider the metrics induced by the metrics on TM and $U(1)$.

Definition 2.8 (Vielbeins and inverse vielbeins). Let \mathcal{S} be the spacetime-matter bundle. Then we can define global vector bundle isomorphisms $e \in \Gamma(M, T^*M \otimes_{\mathbb{R}} E)$, called vielbeins, such that

$$g_{\mu\nu} = \eta_{mn} e_\mu^m e_\nu^n. \quad (9)$$

Furthermore, we can define global inverse vector bundle isomorphisms $e^* \in \Gamma(M, TM \otimes_{\mathbb{R}} E^*)$, called inverse vielbeins, such that⁶

$$\eta_{mn} = g_{\mu\nu} e_m^\mu e_n^\nu. \quad (10)$$

³Where we denote complex conjugation via the asterisk, $*$.

⁴Where we denote Hermitian conjugation via the dagger, \dagger .

⁵We remark that since we consider matter-compatible spacetimes to be diffeomorphic to the Minkowski spacetime and thus in particular globally hyperbolic, it is also possible to consider charts in which $e_0^m \equiv \delta_0^m$ such that in particular $e_0^m \gamma_m \equiv \gamma_0$, and some references use this implicitly, e.g. [24].

⁶We omit the asterisk, $*$, for inverse vielbeins e^* when the abstract index notation is used because of Equation (11).

Greek indices, here μ and ν , belong to the tangent bundle TM and are referred to as curved indices. Furthermore, they are raised and lowered using the usual metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. Latin indices, here m and n , belong to the vector bundle E and are referred to as flat indices. Furthermore, they are raised and lowered using the Minkowski metric η_{mn} and its inverse η^{mn} . Therefore, inverse vielbeins are related to vielbeins via

$$e_m^\mu = g^{\mu\nu} \eta_{mn} e_\nu^n. \quad (11)$$

Moreover, notice that Equation (10) is equivalent to

$$g_{\mu\nu} e_n^\nu = \eta_{mn} e_\mu^m = e_{\mu n}, \quad (12)$$

which states that, when suppressing the Einstein summation convention on flat indices and viewing them as numbers, inverse vielbeins $\{e_m^\mu\}_{m \in \{1,2,3,4\}}$ are a set of 4 eigenvector fields of the metric $g_{\mu\nu}$ with eigenvalues $\eta_{mm} \in \{\pm 1\}$, which are normalized to unit length

$$\|e_m^\mu\|_2 = \sqrt{|g_{\mu\nu} e_m^\mu e_m^\nu|} = \sqrt{|\eta_{mm}|} = 1. \quad (13)$$

Finally, we remark that the global definition of vielbeins and inverse vielbeins is only possible for parallelizable manifolds, such as the matter-compatible spacetimes of Definition 2.3. This is because the tangent frame bundle FM allows for a global section if and only if the manifold is parallelizable, i.e. it is then possible to choose a global coordinate system on TM that consists of eigenvector fields of the metric g .

Remark 2.9. Given the situation of Definition 2.8, notice that vielbeins and inverse vielbeins are not unique, since for any local Lorentz transformation acting on E^* , i.e. a section of the corresponding orthogonal frame bundle $\Lambda \in \Gamma(U, F_O E^*)$ acting via the standard representation on E^* , we have:

$$\begin{aligned} g_{\mu\nu} e_m^\mu e_n^\nu &= \eta_{mn} \\ &= \eta_{rs} \Lambda^r_m \Lambda^s_n \\ &= g_{\mu\nu} e_r^\mu e_s^\nu \Lambda^r_m \Lambda^s_n \\ &= g_{\mu\nu} \tilde{e}_m^\mu \tilde{e}_n^\nu \end{aligned} \quad (14)$$

Here, we denoted the transformed inverse vielbeins as $\tilde{e}_m^\mu := e_r^\mu \Lambda^r_m$. Obviously, the same calculations also holds for vielbeins instead of inverse vielbeins with local Lorentz transformations acting on E . This ambiguity will lead to the first term in the spin connection, $e_\nu^n (\partial_\mu e_\tau^\nu)$, c.f. Equation (18). In fact, the first term in the spin connection can be viewed as the gauge field associated to local Lorentz transformations.

Definition 2.10 (Connections on the spacetime-matter bundle). We use the following connections on the spacetime-matter bundle \mathcal{S} : For the tangent bundle TM of the manifold M we use the Levi-Civita connection ∇_μ^{TM} , acting on a vector field $X \in \Gamma(M, TM)$ via

$$\nabla_\mu^{TM} X^\nu := \partial_\mu X^\nu + \Gamma_{\mu\lambda}^\nu X^\lambda \quad (15a)$$

and on covector fields via

$$\nabla_\mu^{TM} X_\nu := \partial_\mu X_\nu - \Gamma_{\mu\nu}^\lambda X_\lambda, \quad (15b)$$

with the Christoffel symbol $\Gamma_{\mu\lambda}^\nu$, given in the case of the Levi-Civita connection via

$$\Gamma_{\mu\lambda}^\nu = \frac{1}{2} g^{\nu\tau} (\partial_\mu g_{\lambda\tau} + \partial_\lambda g_{\tau\mu} - \partial_\tau g_{\mu\lambda}). \quad (16)$$

Furthermore, for the vector bundle E we use the covariant derivative ∇_μ^E , induced via the connection on the tangent bundle using vielbeins and inverse vielbeins and acting on a vector field $Y \in \Gamma(M, E)$ via

$$\nabla_\mu^E Y^n := \partial_\mu Y^n + \omega_{\mu l}^n Y^l \quad (17a)$$

and on covector fields via

$$\nabla_\mu^E Y_n := \partial_\mu Y_n - \omega_{\mu n}^l Y_l, \quad (17b)$$

with the spin connection

$$\begin{aligned} \omega_{\mu l}^n &:= e_\nu^n \left(\nabla_\mu^{TM} e_l^\nu \right) \\ &= e_\nu^n \left(\partial_\mu e_l^\nu \right) + e_\nu^n \Gamma_{\mu\lambda}^\nu e_l^\lambda. \end{aligned} \quad (18)$$

We remark, that the first term in the spin connection, $e_\nu^n (\partial_\mu e_l^\nu)$, is due to the ambiguity in the definition of vielbeins and inverse vielbeins, as was discussed in Remark 2.9. Moreover, for the $U(1)$ -principle bundle, we use the covariant derivative $\nabla_\mu^{U(1)}$, acting on a section $s \in \Gamma(M, U(1))$ via⁷

$$\nabla_\mu^{U(1)} s := \partial_\mu s + ieA_\mu s \quad (19a)$$

and on complex conjugated sections via

$$\nabla_\mu^{U(1)} s^* := \partial_\mu s^* - ieA_\mu s^*, \quad (19b)$$

with the $\mathfrak{u}(1) \cong i\mathbb{R}$ -valued connection form ieA_μ .⁸ Additionally, for the spinor bundle we use the covariant derivative $\nabla_\mu^{\Sigma M}$, acting on a spinor field $\psi \in \Gamma(M, \Sigma M)$ via

$$\nabla_\mu^{\Sigma M} \psi := \partial_\mu \psi + \varpi_\mu \psi \quad (20a)$$

and on cospinor fields via

$$\nabla_\mu^{\Sigma M} \bar{\psi} := \partial_\mu \bar{\psi} - \bar{\psi} \varpi_\mu, \quad (20b)$$

with the spinor bundle spin connection

$$\varpi_\mu := -\frac{i}{4} \omega_\mu^{rs} \sigma_{rs}, \quad (21)$$

where $\sigma_{rs} := \frac{i}{2} [\gamma_r, \gamma_s]$ is a representation of $\mathfrak{spin}(1, 3)$ on ΣM . Thus, on the twisted spinor bundle $U(1) \times_\rho \Sigma M$ we use the covariant derivative $\nabla_\mu^{U(1) \times_\rho \Sigma M}$, acting on twisted spinor fields $\Psi \in \Gamma(M, U(1) \times_\rho \Sigma M)$ via⁹

$$\nabla_\mu^{U(1) \times_\rho \Sigma M} \Psi = \partial_\mu \Psi + \varpi_\mu \Psi + ieA_\mu \Psi \quad (22a)$$

and on twisted cospinor fields via

$$\nabla_\mu^{U(1) \times_\rho \Sigma M} \bar{\Psi} = \partial_\mu \bar{\Psi} - \bar{\Psi} \varpi_\mu - ieA_\mu \bar{\Psi}. \quad (22b)$$

Finally, on the sheaves of Grassmann algebras $\bigwedge T^*M$ and $\bigwedge \mathfrak{u}(1)^*$ we use the connections induced by the connections on TM and $U(1)$, respectively. In particular, we don't consider odd superderivations.

⁷The existence of global sections in the $U(1)$ principle bundle is ensured by its global triviality, c.f. Definition 2.5, contrary to vector bundles, such as TM , E or ΣM , which always allow for the global zero section.

⁸The imaginary unit i is included into the definition, such that A_μ is real-valued. Moreover, the coupling constant e is included to introduce the right proportionality in the interaction terms in the Lagrange density \mathcal{L}_{MD} when taking covariant derivatives of fermion fields, c.f. Definition 2.21.

⁹Being pedantic, Equations (22) should actually read $\nabla_\mu^{U(1) \times_\rho \Sigma M} \Psi = \partial_\mu \Psi + (\text{Id} \times_\rho \varpi_\mu) \Psi + (ieA_\mu \times_\rho \text{Id}) \Psi$ and $\nabla_\mu^{U(1) \times_\rho \Sigma M} \bar{\Psi} = \partial_\mu \bar{\Psi} - \bar{\Psi} (\text{Id} \times_\rho \varpi_\mu) - \bar{\Psi} (ieA_\mu \times_\rho \text{Id})$.

Remark 2.11 (Tetrad postulate). Given the situation of Definition 2.8, then the tetrad postulate states that vielbeins and inverse vielbeins are parallel sections in $\Gamma(U, T^*M \otimes_{\mathbb{R}} E)$ and $\Gamma(U, TM \otimes_{\mathbb{R}} E^*)$, respectively, with respect to the corresponding tensor product space connections, c.f. Definition 2.10, i.e. we have

$$\nabla_{\mu}^{TM \otimes_{\mathbb{R}} E} e_{\nu}^n = \partial_{\mu} e_{\nu}^n - \Gamma_{\mu\nu}^{\lambda} e_{\lambda}^n + \sigma_{\mu l}^n e_{\nu}^l \equiv 0 \quad (23a)$$

and

$$\nabla_{\mu}^{TM \otimes_{\mathbb{R}} E} e_n^{\nu} = \partial_{\mu} e_n^{\nu} + \Gamma_{\mu\lambda}^{\nu} e_n^{\lambda} - \sigma_{\mu n}^l e_l^{\nu} \equiv 0. \quad (23b)$$

In particular, this implies that it is irrelevant whether vielbeins and inverse vielbeins are placed before or after the covariant derivative $\nabla_{\mu}^{TM \otimes_{\mathbb{R}} E}$ on the product bundle $TM \otimes_{\mathbb{R}} E$. Finally, we remark that despite its name the tetrad postulate is not a postulate but always true as the connection on E is defined using the connection on TM via vielbeins and inverse vielbeins.

Definition 2.12 (Clifford multiplication). Given the spacetime-matter bundle \mathcal{S} , we define the Clifford multiplication $\gamma \in \Gamma(M, E^* \otimes_{\mathbb{R}} \text{End}(\Sigma M))$ for vector fields $Y \in \Gamma(M, E)$ and spinor fields $\psi \in \Gamma(M, \Sigma M)$ via

$$Y \cdot \psi := Y^m \gamma_m \psi. \quad (24)$$

Furthermore, using vielbeins $e \in \Gamma(M, T^*M \otimes_{\mathbb{R}} E)$, we can define the Clifford multiplication $\gamma \circ e \in \Gamma(M, T^*M \otimes_{\mathbb{R}} \text{End}(\Sigma M))$ for vector fields $X \in \Gamma(M, TM)$ via

$$X \cdot \psi := X^{\mu} e_{\mu}^m \gamma_m \psi. \quad (25)$$

Moreover, we extend this definition to the twisted spinor bundle $U(1) \times_{\rho} \Sigma M$ via its action on ΣM and denote it for simplicity via the same symbol $\gamma \in \Gamma(M, E^* \otimes_{\mathbb{R}} \text{End}(U(1) \times_{\rho} \Sigma M))$. Then, we obtain for vector fields $Y \in \Gamma(M, E)$ and twisted spinor fields $\Psi \in \Gamma(M, U(1) \times_{\rho} \Sigma M)$

$$Y \cdot \Psi = Y^m \gamma_m \Psi. \quad (26)$$

Additionally, using vielbeins $e \in \Gamma(M, T^*M \otimes_{\mathbb{R}} E)$, we can extend this definition to obtain $\gamma \circ e \in \Gamma(M, T^*M \otimes_{\mathbb{R}} \text{End}(U(1) \times_{\rho} \Sigma M))$ for vector fields $X \in \Gamma(M, TM)$

$$X \cdot \Psi = X^{\mu} e_{\mu}^m \gamma_m \Psi. \quad (27)$$

Remark 2.13. We remark, that the Clifford multiplication γ or $\gamma \circ e$ turns the spaces of spinor fields $\Gamma(M, \Sigma M)$ and twisted spinor fields $\Gamma(M, U(1) \times_{\rho} \Sigma M)$ into modules over the space of vector fields $\Gamma(M, E)$ or $\Gamma(M, TM)$, respectively.¹⁰ Furthermore, it induces an automorphism if and only if the corresponding vector field $Y \in \Gamma(M, E)$ or $X \in \Gamma(M, TM)$ has nowhere vanishing seminorm with respect to the metric η or g , respectively, i.e. $\|Y\|_{\eta} := \sqrt{\eta_{mn} Y^m Y^n} \neq 0$ or $\|X\|_g := \sqrt{g_{\mu\nu} X^{\mu} X^{\nu}} \neq 0$.¹¹

¹⁰This is similar to the fact, that all spaces of super vector fields are modules over spaces of functions, i.e. $\Gamma(M, TM)$, $\Gamma(M, E)$, $\Gamma(M, \wedge T^*M)$ and $\Gamma(M, \wedge \mathfrak{u}(1)^*)$ are modules over the space $\Gamma(M, \mathbb{R})$ and $\Gamma(M, \Sigma M)$ and $\Gamma(M, U(1) \times_{\rho} \Sigma M)$ are modules over the space $\Gamma(M, \mathbb{C})$.

¹¹Again, this is similar to the fact, that the multiplication of a super vector field with a function induces an automorphism if and only if the corresponding function $f \in \Gamma(M, \mathbb{K})$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ has nowhere vanishing absolute value, i.e. $|f| \neq 0$. The only difference is, that the Clifford multiplication depends also on the metric, which only induces a norm if the metric is positive or negative definite.

Definition 2.14 (Clifford relation). We set the Clifford relation for the vector bundle E as

$$\{\gamma_m, \gamma_n\} = 2\eta_{mn} \text{Id}, \quad (28)$$

or equivalently, using vielbeins $e \in \Gamma(M, T^*M \otimes_{\mathbb{R}} E)$, for the tangent bundle TM as

$$e_\mu^m e_\nu^n \{\gamma_m, \gamma_n\} = 2g_{\mu\nu} \text{Id}, \quad (29)$$

where the identity automorphism Id is either considered on the spinor bundle ΣM or extended to the twisted spinor bundle $U(1) \times_\rho \Sigma M$, c.f. Definition 2.12.

Remark 2.15. We remark, that the West coast (“mostly minus”) signature for the metrics g and η together with the “plus signed” Clifford relation induces a quaternionic representation for the Clifford algebra $\text{Cliff}(1, 3)$ as the matrix algebra $\text{Mat}(2, \mathbb{H})$. Choosing the Pauli matrices as a representation for the quaternions, we obtain the usual complex Dirac representation as the matrix algebra $\text{Mat}(4, \mathbb{C})$, whose generators are Hermitian.

Definition 2.16 (Twisted Dirac operator). Let \mathcal{S} be the spacetime-matter bundle from Definition 2.5. Then we define the twisted Dirac operator $\nabla^{U(1) \times_\rho \Sigma M}$ on twisted spinors $\Psi \in \Gamma(M, U(1) \times_\rho \Sigma M)$ such that the following diagram commutes:

$$\begin{array}{ccc} \Gamma(M, U(1) \times_\rho \Sigma M) & \xrightarrow{\nabla^{U(1) \times_\rho \Sigma M}} & \Gamma(M, U(1) \times_\rho \Sigma M) \\ \nabla^{U(1) \times_\rho \Sigma M} \downarrow & & \uparrow \gamma \circ e \\ \Gamma(M, T^*M \otimes_{\mathbb{R}} (U(1) \times_\rho \Sigma M)) & \xrightarrow{g^{-1} \otimes_{\mathbb{R}} \text{Id}} & \Gamma(M, TM \otimes_{\mathbb{R}} (U(1) \times_\rho \Sigma M)) \end{array} \quad (30)$$

Here, $\nabla_\mu^{U(1) \times_\rho \Sigma M} = \partial_\mu + \varpi_\mu + ieA_\mu$ is the covariant derivative on the bundle $U(1) \times_\rho \Sigma M$, and $e_\nu^n \gamma_n$ is the local representation of the Clifford-multiplication, with γ_n being the usual Minkowski spacetime Dirac matrices. Thus, the local description of the Dirac-operator on the bundle $U(1) \times_\rho \Sigma M$ is given via¹²

$$\begin{aligned} \nabla^{U(1) \times_\rho \Sigma M} &:= e^{\mu m} \gamma_m \nabla_\mu^{U(1) \times_\rho \Sigma M} \\ &= e^{\mu m} \gamma_m (\partial_\mu + \varpi_\mu + ieA_\mu). \end{aligned} \quad (31)$$

Definition 2.17 (Curvatures of the spacetime-matter bundle). Using the connections from Definition 2.10 on the spacetime-matter bundle \mathcal{S} , we can construct curvature tensors as commutators of the corresponding covariant derivatives. We start with the Riemann tensor of the tangent bundle, acting on a vector field $X \in \Gamma(M, TM)$ as

$$R^\rho_{\sigma\mu\nu} X^\sigma := \left[\nabla_\mu^{TM}, \nabla_\nu^{TM} \right] X^\rho, \quad (32a)$$

which reads

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}. \quad (32b)$$

Starting from the Riemann tensor, we can define the Ricci tensor as the following contraction

$$R_{\mu\nu} := R^\rho_{\mu\rho\nu}, \quad (33)$$

¹²Notice, that by the tetrad postulate given in the previous Remark 2.11 it does not matter whether we place the inverse vielbeins $e^{\mu m}$ before or after the covariant derivative $\nabla_\mu^{U(1) \times_\rho \Sigma M}$ (if we place it after, however, we need to consider the covariant derivative $\nabla_\mu^{TM \otimes_{\mathbb{R}} E \times U(1) \times_\rho \Sigma M}$).

and finally the Ricci scalar as

$$R := g^{\mu\nu} R_{\mu\nu} . \quad (34)$$

Furthermore, we can construct the curvature two-form on the $U(1)$ principle bundle, acting on a section $s \in \Gamma(M, TM)$ as¹³

$$F_{\mu\nu} s := \left[\nabla_{\mu}^{U(1)}, \nabla_{\nu}^{U(1)} \right] s , \quad (35a)$$

which reads

$$F_{\mu\nu} = ie (\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}) . \quad (35b)$$

Finally, we remark that being forms, the connection form ieA_{μ} and the curvature form $F_{\mu\nu}$, both have naturally lower indices.

Proposition 2.18 (Ricci scalar for the Levi-Civita connection). *Using the Levi-Civita connection, the Ricci scalar can be expressed in terms of partial derivatives of the metric and the inverse metric:*

$$\begin{aligned} R &= g^{\mu\rho} g^{\nu\sigma} (\partial_{\mu} \partial_{\nu} g_{\sigma\rho} - \partial_{\mu} \partial_{\rho} g_{\nu\sigma}) \\ &\quad + g^{\mu\rho} g^{\nu\sigma} g^{\kappa\lambda} \left\{ (\partial_{\mu} g_{\kappa\lambda}) \left[(\partial_{\nu} g_{\sigma\rho}) - \frac{1}{4} (\partial_{\rho} g_{\nu\sigma}) \right] + (\partial_{\nu} g_{\rho\kappa}) \left[\frac{3}{4} (\partial_{\sigma} g_{\mu\lambda}) - \frac{1}{2} (\partial_{\mu} g_{\sigma\lambda}) \right] \right. \\ &\quad \left. - (\partial_{\mu} g_{\rho\kappa}) (\partial_{\nu} g_{\sigma\lambda}) \right\} , \end{aligned} \quad (36)$$

Proof. The claim is verified by the calculation

$$\begin{aligned} R &= g^{\nu\sigma} R^{\mu}_{\sigma\mu\nu} \\ &= g^{\nu\sigma} \left(\partial_{\mu} \Gamma^{\mu}_{\nu\sigma} - \partial_{\nu} \Gamma^{\mu}_{\mu\sigma} + \Gamma^{\mu}_{\mu\kappa} \Gamma^{\kappa}_{\nu\sigma} - \Gamma^{\mu}_{\nu\kappa} \Gamma^{\kappa}_{\mu\sigma} \right) \\ &= g^{\nu\sigma} \left((\partial_{\mu} g^{\mu\rho}) \left(\partial_{\nu} g_{\sigma\rho} - \frac{1}{2} \partial_{\rho} g_{\nu\sigma} \right) - \frac{1}{2} (\partial_{\nu} g^{\mu\rho}) (\partial_{\sigma} g_{\mu\rho}) + g^{\mu\rho} (\partial_{\mu} \partial_{\nu} g_{\sigma\rho} - \partial_{\mu} \partial_{\rho} g_{\nu\sigma}) \right) \\ &\quad + g^{\mu\rho} g^{\nu\sigma} g^{\kappa\lambda} \left\{ (\partial_{\mu} g_{\kappa\lambda}) \left[\frac{1}{2} (\partial_{\nu} g_{\sigma\rho}) - \frac{1}{4} (\partial_{\rho} g_{\nu\sigma}) \right] + (\partial_{\nu} g_{\rho\kappa}) \left[\frac{1}{4} (\partial_{\sigma} g_{\mu\lambda}) - \frac{1}{2} (\partial_{\mu} g_{\sigma\lambda}) \right] \right\} \quad (37) \\ &= g^{\mu\rho} g^{\nu\sigma} (\partial_{\mu} \partial_{\nu} g_{\sigma\rho} - \partial_{\mu} \partial_{\rho} g_{\nu\sigma}) \\ &\quad + g^{\mu\rho} g^{\nu\sigma} g^{\kappa\lambda} \left\{ (\partial_{\mu} g_{\kappa\lambda}) \left[(\partial_{\nu} g_{\sigma\rho}) - \frac{1}{4} (\partial_{\rho} g_{\nu\sigma}) \right] + (\partial_{\nu} g_{\rho\kappa}) \left[\frac{3}{4} (\partial_{\sigma} g_{\mu\lambda}) - \frac{1}{2} (\partial_{\mu} g_{\sigma\lambda}) \right] \right. \\ &\quad \left. - (\partial_{\mu} g_{\rho\kappa}) (\partial_{\nu} g_{\sigma\lambda}) \right\} , \end{aligned}$$

where we have used $(\partial_{\rho} g^{\nu\sigma}) g_{\mu\sigma} = -g^{\nu\sigma} (\partial_{\rho} g_{\mu\sigma})$, which results from

$$\begin{aligned} 0 &= \partial_{\rho} \delta_{\mu}^{\nu} \\ &= \partial_{\rho} (g_{\mu\sigma} g^{\nu\sigma}) \\ &= (\partial_{\rho} g_{\mu\sigma}) g^{\nu\sigma} + g_{\mu\sigma} (\partial_{\rho} g^{\nu\sigma}) . \end{aligned} \quad (38)$$

¹³We remark that the definition of $F_{\mu\nu}$ has nothing to do with the connection on the tangent bundle, i.e. for a general connection with torsion we have $F_{\mu\nu} \neq ie (\nabla_{\mu}^{TM} A_{\nu} - \nabla_{\nu}^{TM} A_{\mu})$. However, in the absence of torsion the expressions are similar since the Christoffel symbols are then symmetric in their lower two indices, i.e. $\Gamma_{\mu\nu}^{\rho} \equiv \Gamma_{\nu\mu}^{\rho}$. Rather, $F_{\mu\nu}$ is defined as the covariant exterior derivative of the connection form ieA_{μ} , which for abelian gauge groups is just given as the ordinary exterior derivative, as stated in Equation (35b). ■

Remark 2.19. Assuming the Levi-Civita connection, the Riemann tensor $R^\rho{}_{\sigma\mu\nu}$ and the Ricci tensor $R_{\mu\nu}$ are not sensitive to the choice of the signature of the metric $g_{\mu\nu}$, whereas the Riemann tensor $R_{\rho\sigma\mu\nu} := g_{\rho\lambda}R^\lambda{}_{\sigma\mu\nu}$ and the Ricci scalar R are. Therefore, the Einstein-Hilbert Lagrange density is sensitive to this choice as well, and this is the reason for the minus sign in our West coast (mostly minus) signature convention, c.f. Definition 2.1.

Definition 2.20 (Riemannian volume form). We define the Riemannian volume form for the spacetime (M, g) as

$$dV_g := \sqrt{-\text{Det}(g)} dt \wedge dx \wedge dy \wedge dz. \quad (39)$$

In particular, the Riemannian volume form for the Minkowski spacetime (\mathbb{M}, η) takes the form

$$dV_\eta = dt \wedge dx \wedge dy \wedge dz. \quad (40)$$

Definition 2.21 (Fermion, photon and graviton field). The mathematical objects correspond in the following way to physical particles: The deviation of the metric $g_{\mu\nu}$ from the Minkowski metric $\eta_{\mu\nu}$ of flat space is defined to be proportional to the graviton field $h_{\mu\nu}$, with the proportionality factor given by the gravitational coupling constant $\lambda := \sqrt{8\pi G}$,¹⁴ i.e.

$$h_{\mu\nu} := \frac{1}{\lambda} (g_{\mu\nu} - \eta_{\mu\nu}) \iff g_{\mu\nu} = \eta_{\mu\nu} + \lambda h_{\mu\nu}. \quad (41)$$

The graviton field can be thought of as a $(0, 2)$ -tensor field living on the flat background Minkowski spacetime (\mathbb{M}, η) . Moreover, the connection form $i e A_\mu$ is defined to be proportional to the photon field A_μ , with the proportionality factor given by the imaginary unit i and the electromagnetic coupling constant e , such that the photon field is real and induces the right interaction couplings with fermions. It induces the Faraday or electromagnetic field strength tensor $F_{\mu\nu}$, c.f. Definition 2.17. Moreover, a section in the spinor bundle $\psi \in \Gamma(M, \Sigma M)$, corresponds to a fermion field. In particular, in our case it is a linear combination of the electron and the positron field with charge $\pm e$, respectively. In the following, we are interested in the coupling of the fermion field ψ to the photon field A_μ . Mathematically, this is obtained by viewing fermions as sections in the twisted spinor bundle $\Psi \in \Gamma(M, U(1) \times_\rho \Sigma M)$. Therefore, in the following we use twisted spinor fields, i.e. sections in the twisted bundle $U(1) \times_\rho \Sigma M$, if not stated otherwise, which are denoted by Ψ , compared to ψ which denote sections in ΣM . Spinors and twisted spinors induce the four-current j^μ , via

$$j^\mu := e^{\mu m} \bar{\psi} \gamma_m \psi \equiv e^{\mu m} \bar{\Psi} \gamma_m \Psi. \quad (42)$$

Remark 2.22 (Physical interpretation of the connections on the spacetime-matter bundle). In this remark we stress, that even though the connections on the spacetime-matter bundle are mathematically similarly defined, c.f. Definition 2.10, their physical interpretation is rather different, c.f. Definition 2.21: The Christoffel symbols $\Gamma_{\mu\lambda}^\nu$ and the spin connection $\omega_{\mu\lambda}^n$ are proportional to a series in the graviton field, i.e. to a sum of arbitrary many particles. Contrary, the connection form $i e A_\mu$ on the $U(1)$ principle bundle corresponds directly to the photon field, i.e. to a single particle.

Remark 2.23 (Gauge transformations). The Lagrange density of QGR-QED, Equation (61), is invariant under two different symmetry transformations, in the following called gauge transformations: The first are diffeomorphisms of spacetime (M, g) homotopic to the identity $\phi \in \text{Diff}^0(M, g)$ and the second are $U(1)$ principle bundle automorphisms $\varphi_f \in \text{Aut}(M \times U(1))$. While the first affects the whole spacetime-matter bundle \mathcal{S} , the second affects only the $U(1)$

¹⁴Where G is Newtons constant and we have $\lambda = \sqrt{\kappa}$, where $\kappa := 8\pi G$ is Einsteins constant.

principle bundle together with the spinor bundle ΣM , due to the action ρ . We remark, that we only consider diffeomorphisms $\phi \in \text{Diff}^0(M, g)$ which are flows of vector fields $X \in \Gamma(M, TM)$ for some parameter $\tau \in \mathcal{I} \subseteq \mathbb{R}$ open, denoted ϕ_X^τ , as they generate $\text{Diff}^0(M, g)$.¹⁵ Furthermore, given $\phi_X^\tau \in \text{Diff}^0(M, g)$, all affected objects transform via the induced pullback $(\phi_X^\tau)^*$ or pushforward $(\phi_X^\tau)_*$.¹⁶ Given an arbitrary tensor field of type (r, s) , $\mathcal{T} \in \Gamma(M, T_s^r M)$, we can consider the Taylor expansion of its pullback $(\phi_X^\tau)^* \mathcal{T}$ in the parameter τ around $\tau = 0$, which converges for some $\tau \in \mathcal{J} \subseteq \mathcal{I}$ open,¹⁷

$$\begin{aligned} (\phi_X^\tau)^* \mathcal{T} &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\partial_\sigma^k ((\phi_X^\sigma)^* \mathcal{T}) \right) \Big|_{\sigma=0} \tau^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\mathcal{L}_X^k \mathcal{T} \right) \tau^k, \end{aligned} \quad (43)$$

where we have set

$$\left(\mathcal{L}_X^k \mathcal{T} \right)_p := \left(\partial_\sigma^k ((\phi_X^\sigma)^* \mathcal{T}_p) \right) \Big|_{\sigma=0} \quad (44)$$

for all $p \in M$. Observe, that $\mathcal{L}_X^0 \mathcal{T} = \mathcal{T}$ is the identity and $\mathcal{L}_X^1 \mathcal{T} = \mathcal{L}_X \mathcal{T}$ is the Lie derivative.

Remark 2.24 (Transformation properties). Given the situation of Remark 2.23, the graviton field $h_{\mu\nu}$ and the connection form $\text{ie}A_\mu$ transform in the following way under the flow $\phi_X^\tau \in \text{Diff}^0(M, g)$ in the first order approximation of Equation (43)

$$(\phi_X^\tau)_* h_{\mu\nu} = h_{\mu\nu} + \left(X^\rho (\partial_\rho h_{\mu\nu}) + \partial_\mu X_\nu + \partial_\nu X_\mu \right) \tau + \mathcal{O}(\tau^2) \quad (45a)$$

and

$$(\phi_X^\tau)_* A_\mu = A_\mu + \left(X^\nu \partial_\nu A_\mu + A_\nu \partial_\mu X^\nu \right) \tau + \mathcal{O}(\tau^2). \quad (45b)$$

Furthermore, the connection form $\text{ie}A_\mu$ transforms in the following way under the principle bundle automorphism $\varphi_f \in \text{Aut}(M \times U(1))$

$$(\varphi_f)^* A_\mu = A_\mu + \partial_\mu f, \quad (46)$$

where $f \in \Gamma(M, \mathbb{R})$ is the real-valued function on the spacetime (M, g) associated to the bundle automorphism φ_f .¹⁸ Finally, we remark that only the linearized Riemann tensor is invariant under the gauge transformation of Equation (45a). This is similar to non-abelian gauge theories where the field strength tensor is also not invariant under gauge transformations. However, the Ricci scalar is invariant, as is the Yang-Mills Lagrange density due to the ad-invariance of the Killing form.

¹⁵However, not every diffeomorphism homotopic to the identity can be written as the flow of a vector field, c.f. e.g. [25, p. 456, 43.2. Example.].

¹⁶In the following we will not make a distinction between pullbacks and pushforwards if they are applied to mixed tensor fields, as for diffeomorphisms f the pushforward is the pullback of the inverse, i.e. $f_* = (f^{-1})^*$, and vice versa. Nevertheless, when regarding flows we need to take signs of the parameter τ into account, as $(\phi_X^\tau)^{-1} \equiv \phi_X^{-\tau}$, i.e. contravariant indices transform via $(\phi_X^\tau)^*$, whereas covariant indices transform via $(\phi_X^\tau)_* \equiv (\phi_X^{-\tau})^*$.

¹⁷Be aware, that here $\partial_\sigma \equiv \frac{\partial}{\partial \sigma}$ denotes the ordinary differentiation w.r.t. the parameter σ , i.e. is not to be confused with $\frac{\partial}{\partial x^\sigma}$. Furthermore, we denote the k -fold derivative via $\partial_\sigma^k := \frac{\partial^k}{\partial \sigma^k}$.

¹⁸Given that $\mathfrak{u}(1) \cong \text{ie}\mathbb{R}$, the convention is such that the connection form $\text{ie}A_\mu \in \Gamma(M, T^*M \otimes_{\mathbb{R}} \mathfrak{u}(1))$ is purely imaginary and thus $A_\mu \in \Gamma(M, T^*M \otimes_{\mathbb{R}} \mathbb{R}) \cong \Gamma(M, T^*M)$ is real (and in particular Hermitian, as is the case when this definition is used for non-abelian gauge theories). Thus, in particular $\text{ie}f \in \Gamma(M, \mathfrak{u}(1)) \cong \Gamma(M, \text{ie}\mathbb{R})$.

Remark 2.25 (Feynman rules). To calculate the corresponding gravity Feynman rules, we need to express the inverse metric, vielbeins and inverse vielbeins and the prefactor of the Riemannian volume form in terms of the graviton field. Since we need only general properties of the Feynman rules in this work, we just motivate their derivation and postpone their detailed treatment to [26]. The interested reader can find some of the Feynman rules for linearized GR in [24, 27].¹⁹ Furthermore, we remark that all following series converge if and only if $|\lambda| \|h_{\mu\nu}\|_\infty < 1$. The inverse metric in terms of the graviton field is given by the corresponding Neumann series, i.e.

$$g^{\mu\nu} = \sum_{k=0}^{\infty} (-\lambda)^k (h^k)^{\mu\nu}, \quad (47)$$

where

$$h^{\mu\nu} := \eta^{\mu\rho} \eta^{\nu\sigma} h_{\rho\sigma}, \quad (48a)$$

$$(h^0)^{\mu\nu} := \eta^{\mu\nu} \quad (48b)$$

and

$$(h^k)^{\mu\nu} := \underbrace{h_{\kappa_1}^\mu h_{\kappa_2}^{\kappa_1} \dots h^{\kappa_{k-1}\nu}}_{k\text{-times}}, \quad k \in \mathbb{N}. \quad (48c)$$

Furthermore, the expressions for vielbeins and inverse vielbeins read

$$e_\mu^m = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} (h^k)_\mu^m, \quad (49a)$$

with $h_\mu^m := \eta^{m\nu} h_{\mu\nu}$, and

$$e_m^\mu = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} (h^k)_m^\mu, \quad (49b)$$

with $h_m^\mu := \eta^{\mu\nu} \delta_m^\rho h_{\nu\rho}$. Moreover, the prefactor of the Riemannian volume form,

$$\begin{aligned} \sqrt{-\text{Det}(g)} &= \sqrt{-\text{Det}(\eta + \lambda h)} \\ &= \sqrt{-\text{Det}(\eta) \text{Det}(\delta + \lambda \eta^{-1} h)} \\ &= \sqrt{\text{Det}(\delta + \lambda \eta h)}, \end{aligned} \quad (50)$$

can be obtained by first expressing the determinant in terms of traces and then plug the result into the Taylor series expansion of the square-root. More precisely, the determinant of a 4×4 matrix²⁰ \mathfrak{M} can be expressed via Newton's identities in terms of its trace as

$$\text{Det}(\mathfrak{M}) = \frac{1}{4!} \text{Det} \begin{pmatrix} \text{Tr}(\mathfrak{M}) & 1 & 0 & 0 \\ \text{Tr}(\mathfrak{M}^2) & \text{Tr}(\mathfrak{M}) & 2 & 0 \\ \text{Tr}(\mathfrak{M}^3) & \text{Tr}(\mathfrak{M}^2) & \text{Tr}(\mathfrak{M}) & 3 \\ \text{Tr}(\mathfrak{M}^4) & \text{Tr}(\mathfrak{M}^3) & \text{Tr}(\mathfrak{M}^2) & \text{Tr}(\mathfrak{M}) \end{pmatrix}. \quad (51)$$

Now, we set

$$\mathfrak{M} := \delta + \lambda \eta h. \quad (52)$$

¹⁹We refer again to Footnote 5 for a comment concerning the fermion Feynman rules in [24].

²⁰This works for any $d \times d$ matrix with the obvious generalizations of the following equations.

Since the trace is linear, we have

$$\begin{aligned}\mathrm{Tr}(\delta + \lambda\eta h) &= \mathrm{Tr}(\delta) + \lambda \mathrm{Tr}(\eta h) \\ &= 4 + \lambda\eta^{\mu\nu}h_{\mu\nu},\end{aligned}\tag{53}$$

and similar expressions for the higher powers $\mathrm{Tr}\left((\delta + \lambda\eta h)^k\right)$ for $k \in \{2, 3, 4\}$. Thus, the determinant is a polynomial in traces of powers of $\lambda\eta h$, i.e.

$$\mathrm{Det}(\delta + \lambda\eta h) \in \mathbb{R} \left[\lambda \mathrm{Tr}(\eta h), \lambda^2 \mathrm{Tr}\left((\eta h)^2\right), \lambda^3 \mathrm{Tr}\left((\eta h)^3\right), \lambda^4 \mathrm{Tr}\left((\eta h)^4\right) \right].\tag{54}$$

We separate the constant term, which is 1, and write the non-constant term in the polynomial as $\mathfrak{P}(\lambda\eta h)$, i.e.

$$\mathfrak{P}(\lambda\eta h) := \mathrm{Det}(\delta + \lambda\eta h) - 1 \iff \mathrm{Det}(\delta + \lambda\eta h) = \mathfrak{P}(\lambda\eta h) + 1.\tag{55}$$

Then, we plug this expression into the Taylor series expansion of the square root around 1, i.e.

$$\sqrt{x+1} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} x^k,\tag{56}$$

which converges for $|x| < 1$. Finally, we to obtain

$$\begin{aligned}\sqrt{-\mathrm{Det}(g)} &= \sqrt{\mathrm{Det}(\delta + \lambda\eta h)} \\ &= \sqrt{\mathfrak{P}(\lambda\eta h) + 1} \\ &= \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} (\mathfrak{P}(\lambda\eta h))^k.\end{aligned}\tag{57}$$

However, for the realm of this article it suffices to know that the inverse metric, vielbeins and inverse vielbeins and the prefactor of the Riemannian volume form are all power series in the graviton field $h_{\mu\nu}$.

Remark 2.26 (Feynman rules 2). In this article we consider two-loop propagator Feynman graphs and one-loop propagator and three-point Feynman graphs. Therefore, it is actually sufficient to consider the Lagrange density $\mathcal{L}_{\mathrm{QGR-QED}}$ only up to order λ^2 , since the higher valent graviton vertices would only contribute to graphs with self-loops. Thus, the formulas from Remark 2.25 read as follows ($\mathcal{O}(\cdot)$ denotes the Landau symbol): The inverse metric is given by

$$g^{\mu\nu} = \eta^{\mu\nu} - \lambda\eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma} + \lambda^2\eta^{\mu\rho}\eta^{\sigma\kappa}\eta^{\tau\nu}h_{\rho\sigma}h_{\kappa\tau} + \mathcal{O}\left(\lambda^3\right).\tag{58}$$

Furthermore, vielbeins and inverse vielbeins, as defined in Definition 2.8, are given by

$$e_{\mu}^m = \delta_{\mu}^m + \frac{1}{2}\lambda\eta^{\nu m}h_{\mu\nu} - \frac{1}{8}\lambda^2\eta^{\rho\sigma}\eta^{\kappa m}h_{\mu\rho}h_{\sigma\kappa} + \mathcal{O}\left(\lambda^3\right)\tag{59a}$$

and

$$e_m^{\mu} = \delta_m^{\mu} - \frac{1}{2}\lambda\eta^{\mu\nu}h_{\nu m} + \frac{3}{8}\lambda^2\eta^{\mu\rho}\eta^{\sigma\kappa}h_{\rho\sigma}h_{\kappa m} + \mathcal{O}\left(\lambda^3\right).\tag{59b}$$

And finally, the prefactor of the Riemannian volume form, defined in Definition 2.20, is given by

$$\sqrt{-\mathrm{Det}(g)} = 1 + \frac{1}{2}\lambda\eta^{\mu\nu}h_{\mu\nu} + \frac{1}{8}\lambda^2\left(\eta^{\mu\nu}\eta^{\rho\sigma}h_{\mu\nu}h_{\rho\sigma} - 2\eta^{\mu\sigma}\eta^{\nu\rho}h_{\mu\nu}h_{\rho\sigma}\right) + \mathcal{O}\left(\lambda^3\right).\tag{60}$$

2.2 The Lagrange density of QGR-QED

Having introduced the necessary differential geometric background in Subsection 2.1, we can now turn our attention to physics: In this article we consider QGR-QED, described via the following Lagrange density as a functional on the spacetime-matter bundle \mathcal{S} :

$$\mathcal{L}_{\text{QGR-QED}} = \mathcal{L}_{\text{GR-ED}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{Ghost}}, \quad (61a)$$

where $\mathcal{L}_{\text{GR-ED}}$ is the classical Lagrange density of General Relativity (GR) with Spinor Electrodynamics (ED)

$$\mathcal{L}_{\text{GR-ED}} = \left(-\frac{1}{2\lambda^2} R + \frac{1}{4e^2} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + \bar{\Psi} \left(i\mathcal{D}^{U(1)\times\rho\Sigma M} - m \right) \Psi \right) dV_g, \quad (61b)$$

\mathcal{L}_{GF} are the gauge fixing terms

$$\mathcal{L}_{\text{GF}} = \left(\frac{1}{4\lambda^2\zeta} g_{\mu\nu} dD^\mu dD^\nu + \frac{1}{2e^2\xi} L^2 \right) dV_g, \quad (61c)$$

with $dD^\mu := g^{\rho\sigma} \Gamma_{\rho\sigma}^\mu$ and $L := g^{\mu\nu} \nabla_\mu^{TM} i e A_\nu$, and $\mathcal{L}_{\text{Ghost}}$ are the ghost terms

$$\begin{aligned} \mathcal{L}_{\text{Ghost}} = & \left(g^{\mu\nu} g^{\rho\sigma} \partial_\mu \bar{\chi}_\rho \partial_\nu \chi_\sigma + g^{\mu\nu} \partial_\mu \bar{\theta} \partial_\nu \theta \right. \\ & \left. + \lambda g^{\mu\nu} g^{\rho\sigma} \bar{\theta} \left((\partial_\mu A_\rho) (\partial_\sigma \chi_\nu) + (\partial_\mu \partial_\rho A_\nu) \chi_\sigma + A_\mu (\partial_\rho \partial_\sigma \chi_\nu) + (\partial_\mu A_\rho) (\partial_\nu \chi_\sigma) \right) \right) dV_g. \end{aligned} \quad (61d)$$

Now, we discuss the parts of the QGR-QED Lagrange density $\mathcal{L}_{\text{QGR-QED}}$ in detail:

2.2.1 The Lagrange density of GR-ED

The classical Lagrange density of General Relativity with Spinor Electrodynamics is the sum of the Einstein-Hilbert, Maxwell and Dirac Lagrange densities:

$$\mathcal{L}_{\text{GR-ED}} = \left(-\frac{1}{2\lambda^2} R + \frac{1}{4e^2} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + \bar{\Psi} \left(i\mathcal{D}^{U(1)\times\rho\Sigma M} - m \right) \Psi \right) dV_g, \quad (62)$$

where we have rescaled the Einstein-Hilbert term by $1/\lambda^2$ and the Maxwell term by $1/e^2$ such that the graviton and photon propagators are of order $\mathcal{O}(\lambda^0)$ and $\mathcal{O}(e^0)$, respectively. Here, $R = g^{\nu\sigma} R^\mu_{\sigma\nu}$ is the Ricci scalar of the tangent bundle, c.f. Definition 2.17, $F_{\mu\nu}$ is the curvature two-form on the $U(1)$ principle bundle, c.f. Definition 2.17, $\mathcal{D}^{U(1)\times\rho\Sigma M}$ the twisted Dirac operator, c.f. Definition 2.16 and dV_g is the Riemannian volume form, c.f. Definition 2.20. The classical Lagrange density $\mathcal{L}_{\text{GR-ED}}$ leads to the Einstein field equations:

$$G_{\mu\nu} = \lambda^2 T_{\mu\nu}, \quad (63a)$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \quad (63b)$$

is the Einstein tensor and

$$\begin{aligned} T_{\mu\nu} = & \frac{1}{e^2} \left(g^{\rho\sigma} F_{\mu\rho} F_{\sigma\nu} - \frac{1}{4} g_{\mu\nu} g^{\rho\kappa} g^{\sigma\tau} F_{\rho\sigma} F_{\kappa\tau} \right) \\ & + \frac{i}{4} e_\mu^m \left(\bar{\Psi} \gamma_m \left(\nabla_\nu^{U(1)\times\rho\Sigma M} \Psi \right) - \left(\nabla_\nu^{U(1)\times\rho\Sigma M} \bar{\Psi} \right) \gamma_m \Psi \right) \\ & + \frac{i}{4} e_\nu^n \left(\bar{\Psi} \gamma_n \left(\nabla_\mu^{U(1)\times\rho\Sigma M} \Psi \right) - \left(\nabla_\mu^{U(1)\times\rho\Sigma M} \bar{\Psi} \right) \gamma_n \Psi \right) \end{aligned} \quad (63c)$$

is the corresponding (generalized) Hilbert energy-momentum tensor for Spinor Electrodynamics.²¹ We remark, that considering linearized GR is equivalent to the vanishing of the first bracket in the equation derived in Proposition 2.18, i.e. considering locally geometries such that

$$g^{\mu\rho}g^{\nu\sigma}(\partial_\mu\partial_\nu g_{\sigma\rho} - \partial_\mu\partial_\rho g_{\nu\sigma}) \equiv 0, \quad (64)$$

which could otherwise be interpreted as a source term for the graviton field $h_{\mu\nu}$. Thus, when considering Feynman rules it would correspond to a graviton half-edge — however, the corresponding Feynman rule vanishes if momentum conservation is considered. Nevertheless, it produces additional contributions to the higher valent pure graviton vertex Feynman rules. Observe, that the coupling of the graviton field to the matter fields is given via inverse metrics, inverse vielbeins, the spin-connection in the Dirac-operator and the prefactor of the Riemannian volume form.

2.2.2 Gauge fixing Lagrange density

In QGR-QED there are two gauge symmetries present: One coming from General Relativity and affecting the whole spacetime-matter-bundle \mathcal{S} and the other one coming from electrodynamics and affecting the $U(1)$ principle bundle only. These gauge transformations are described in Remark 2.23 and lead to the transformations of the graviton and photon fields as given in Remark 2.24. Thus, the gauge fixing part of the Lagrange density consists also of two parts: One for the gravitational part, where we choose the de Donder gauge

$$g^{\rho\sigma}\Gamma_{\rho\sigma}^\mu \stackrel{!}{=} 0 \iff g^{\rho\sigma}(\partial_\rho g_{\sigma\mu}) \stackrel{!}{=} \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\rho\sigma}), \quad (65)$$

and one for the electrodynamic part, where we choose the Lorenz gauge

$$g^{\mu\nu}\nabla_\mu^{TM}A_\nu \stackrel{!}{=} 0. \quad (66)$$

The de Donder gauge is motivated by the fact, that in this gauge the divergence of a vector or covector field reduces to the following simpler expression involving only a partial derivative

$$\begin{aligned} \nabla_\mu^{TM}X^\mu &= \partial_\mu X^\mu + \Gamma_{\mu\rho}^\mu X^\rho \\ &= g^{\mu\nu}(\partial_\mu X_\nu - \Gamma_{\mu\nu}^\sigma X_\sigma) \\ &= g^{\mu\nu}\partial_\mu X_\nu \\ &= \partial_\mu X^\mu. \end{aligned} \quad (67)$$

Thus, in particular the Beltrami-Laplace operator takes on the simple form

$$\Delta_{\text{Beltrami}}^{TM} = g^{\mu\nu}\partial_\mu\partial_\nu. \quad (68)$$

Furthermore, we remark that there does in general not exist a gauge such that the evolution of the graviton field is governed by a wave equation, because the graviton field is in general non-linear. This comes from the first term in Equation (37). If, however, the linearized Einstein-Hilbert Lagrange density is considered together with the de Donder gauge, then the evolution of the graviton field is given via a wave equation — a fact which is used in gravitational wave analysis, c.f. [28]. On the other hand, the Lorenz-Gauge is motivated by the fact, that then the photon

²¹Where we have already used the equations of motion for the fermionic energy-momentum contribution.

field A_ρ satisfies a wave equation with a source term, given by the dual four-current $g_{\rho\sigma}j^\sigma$ and the Ricci curvature tensor applied to it $R_{\nu\rho}^\kappa A_\kappa$,²²

$$\begin{aligned}
\Delta_{\text{Bochner}}^{TM} A_\rho &= g^{\mu\nu} \nabla_\mu^{TM} \nabla_\nu^{TM} A_\rho \\
&= g^{\mu\nu} \nabla_\mu^{TM} \left(\nabla_\nu^{TM} A_\rho - \nabla_\rho^{TM} A_\nu + \nabla_\rho^{TM} A_\nu \right) \\
&= g^{\mu\nu} \nabla_\mu^{TM} \left(F_{\nu\rho} + \nabla_\rho^{TM} A_\nu \right) \\
&= g_{\rho\sigma} j^\sigma + g^{\mu\nu} \left(\left[\nabla_\mu^{TM}, \nabla_\rho^{TM} \right] + \nabla_\rho^{TM} \nabla_\mu^{TM} \right) A_\nu \\
&= g_{\rho\sigma} j^\sigma + g^{\mu\nu} R_{\nu\rho\mu}^\kappa A_\kappa \\
&= g_{\rho\sigma} j^\sigma + R_\rho^\kappa A_\kappa,
\end{aligned} \tag{69}$$

where $\Delta_{\text{Bochner}}^{TM}$ is the Bochner-Laplace operator. Observe, that when we apply the de Donder gauge of Equation (65), the expression in the Lorenz gauge of Equation (66) simplifies to

$$\begin{aligned}
g^{\mu\nu} \nabla_\mu^{TM} A_\nu &= g^{\mu\nu} \left(\partial_\mu A_\nu - \Gamma_{\mu\nu}^\rho A_\rho \right) \\
&= g^{\mu\nu} \partial_\mu A_\nu,
\end{aligned} \tag{70}$$

as was shown in general in Equation (67). For the following, we write the de Donder gauge as

$$dD^\mu := g^{\rho\sigma} \Gamma_{\rho\sigma}^\mu \stackrel{!}{=} 0 \tag{71}$$

and the Lorenz gauge as

$$L := g^{\mu\nu} \nabla_\mu^{TM} A_\nu \stackrel{!}{=} 0. \tag{72}$$

We implement the de Donder gauge and the Lorenz gauge in the Lagrange density by using the Lagrange multipliers $1/\xi$ and $1/\zeta$, i.e. by adding the following Lagrange density:

$$\mathcal{L}_{\text{GF}} = \left(\frac{1}{4\lambda^2 \zeta} g_{\mu\nu} dD^\mu dD^\nu + \frac{1}{2e^2 \xi} L^2 \right) dV_g, \tag{73}$$

where dV_g is the Riemannian volume form, c.f. Definition 2.20. If $1/\xi$ and $1/\zeta$ are interpreted as parameters rather than Lagrange multipliers, then $\zeta = 1$ corresponds to the de Donder gauge and $\xi = 1$ to the Feynman gauge.

2.2.3 Ghost Lagrange density

We denote the graviton ghost and graviton anti-ghost by χ_μ and $\bar{\chi}_\mu$, respectively, and the photon ghost and photon anti-ghost by θ and $\bar{\theta}$, respectively. The ghost Lagrange density is obtained as the variation of the gauge fixing condition via a gauge transformation and then replacing the transformation fields via the corresponding ghost fields:

$$\begin{aligned}
\mathcal{L}_{\text{Ghost}} &= \left(g^{\mu\nu} \bar{\chi}_\mu \left((\phi_X^\tau)_* dD_\nu \right) \Big|_{\substack{\mathcal{O}(\tau)=1 \\ \tau=1 \\ X_{\rho^\dagger} \mapsto \chi_\rho}} + \bar{\theta} \left((\phi_X^\tau)_* L \right) \Big|_{\substack{\mathcal{O}(\tau)=1 \\ \tau=1 \\ X_{\rho^\dagger} \mapsto \chi_\rho}} + \bar{\theta} \left((\phi_f)_* L \right) \Big|_{f \mapsto \theta} \right) dV_g \\
&= \left(g^{\mu\nu} g^{\rho\sigma} \partial_\mu \bar{\chi}_\rho \partial_\nu \chi_\sigma + g^{\mu\nu} \partial_\mu \bar{\theta} \partial_\nu \theta \right. \\
&\quad \left. + \lambda g^{\mu\nu} g^{\rho\sigma} \bar{\theta} \left((\partial_\mu A_\rho) (\partial_\sigma \chi_\nu) + (\partial_\mu \partial_\rho A_\nu) \chi_\sigma + A_\mu (\partial_\rho \partial_\sigma \chi_\nu) + (\partial_\mu A_\rho) (\partial_\nu \chi_\sigma) \right) \right) dV_g,
\end{aligned} \tag{74}$$

²²Where we have used that the Levi-Civita connection is torsion-free, i.e. $\Gamma_{\mu\nu}^\rho \equiv \Gamma_{\nu\mu}^\rho$, to match the definition of $F_{\mu\nu}$, c.f. Definition 2.17 and Footnote 13.

where $\left. \left((\phi_X^\tau)_* dD_\nu \right) \right|_{\mathcal{O}(\tau)=1}$, $\left. \left((\phi_X^\tau)_* L \right) \right|_{\mathcal{O}(\tau)=1}$ and $(\varphi_f)_* L$ denote the linearized gauge transformations, given in Equations (45a), (45b) and (46) applied to the gauge conditions given in Equations (65) and (66), and dV_g is the Riemannian volume form, c.f. Definition 2.20. Finally, we remark that the photon ghost is a scalar particle, whereas the graviton ghost is a spin-one particle.

3 Hopf algebras, the renormalization Hopf algebra and QGR-QED

In this section, we introduce Hopf algebras in general and the Connes-Kreimer renormalization Hopf algebra in particular. The intention is to review the basic notions and give the relevant definitions. We refer the reader who wishes a more detailed treatment on Hopf algebras in general and its connection to affine group schemes to [29, 30] and the reader who wishes a more detailed treatment on the construction of the Connes-Kreimer renormalization Hopf algebra to [13, 14, 15, 31]. In this work, we set k to be a commutative ring with one and then later consider the case $k := \mathbb{Q}$ for the Connes-Kreimer renormalization Hopf algebra.²³ Furthermore, we set the tensor product over the corresponding ring if not stated otherwise, i.e. $\otimes := \otimes_k$. Finally, we study Hopf ideals in the renormalization Hopf algebra in order to understand which symmetries are compatible with renormalization.

3.1 Hopf algebras

We start by defining Hopf algebras in general. In this article, we set k to be a commutative ring with one. Furthermore, by algebra we mean associative algebra with identity and by coalgebra we mean coassociative coalgebra with coidentity. Moreover, all algebras, coalgebras, bialgebras and Hopf algebras are all considered over the ring k .

Definition 3.1 (Algebra). The triple (A, μ, \mathbb{I}) is an algebra, where A is a k -module, $\mu : A \otimes A \rightarrow A$ an associative multiplication map, i.e. the following diagram commutes

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes \text{Id}} & A \otimes A \\ \text{Id} \otimes \mu \downarrow & & \downarrow \mu \\ A \otimes A & \xrightarrow{\mu} & A \end{array}, \quad (75)$$

and $\mathbb{I} : k \rightarrow A$ the identity with respect to μ ,²⁴ i.e. the following diagram commutes

$$\begin{array}{ccccc} k \otimes A & \xrightarrow{\mathbb{I} \otimes \text{Id}} & A \otimes A & \xleftarrow{\text{Id} \otimes \mathbb{I}} & A \otimes k \\ & \searrow \cong & \downarrow \mu & \swarrow \cong & \\ & & A & & \end{array}. \quad (76)$$

²³Actually, the physical needs require k only to be a field with characteristic 0. Since \mathbb{Q} is the smallest such field, it is the canonical choice.

²⁴We denote both the neutral element in the algebra A with respect to the multiplication map μ and the inclusion function $k \rightarrow A$ by the symbol \mathbb{I} .

Definition 3.2 ((Connected) graded algebra). An algebra A is called graded, if the k -module A can be written as a direct sum

$$A = \bigoplus_{i=0}^{\infty} A_i, \quad (77)$$

which is respected by the multiplication μ , i.e.

$$\mu(A_i \otimes A_j) \subseteq A_{i+j}, \quad \forall i, j \in \mathbb{N}_0, \quad (78)$$

and the identity \mathbb{I} , i.e.

$$\mathbb{I} : A_0 \hookrightarrow A, \quad a_0 \mapsto a_0 \quad (79)$$

Furthermore, a graded algebra A is called connected, if the grade zero component is isomorphic to the base ring, i.e.

$$A_0 \cong k. \quad (80)$$

Definition 3.3 (Coalgebra). The triple $(C, \Delta, \hat{\mathbb{I}})$ is a coalgebra, where C is a k -module, $\Delta : C \rightarrow C \otimes C$ a coassociative comultiplication map, i.e. the following diagram commutes

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \Delta \downarrow & & \downarrow \text{Id} \otimes \Delta \\ C \otimes C & \xrightarrow{\Delta \otimes \text{Id}} & C \otimes C \otimes C \end{array}, \quad (81)$$

and $\hat{\mathbb{I}} : C \rightarrow k$ the coidentity with respect to Δ ,²⁵ i.e. the following diagram commutes

$$\begin{array}{ccccc} & & C & & \\ & \cong \swarrow & \downarrow \Delta & \searrow \cong & \\ k \otimes C & \xleftarrow{\hat{\mathbb{I}} \otimes \text{Id}} & C \otimes C & \xrightarrow{\text{Id} \otimes \hat{\mathbb{I}}} & C \otimes k \end{array}. \quad (82)$$

Definition 3.4 ((Connected) graded coalgebra). A coalgebra C is called graded, if the k -module C can be written as a direct sum

$$C = \bigoplus_{i=0}^{\infty} C_i, \quad (83)$$

which is respected by the comultiplication Δ , i.e.

$$\Delta(C_i) \subseteq \sum_{j=0}^i C_j \otimes C_{i-j}, \quad \forall i \in \mathbb{N}_0, \quad (84)$$

and the coidentity $\hat{\mathbb{I}}$, i.e.

$$\hat{\mathbb{I}} : C \rightarrow C_0, \quad c_i \mapsto \delta_{0i} c_i, \quad (85)$$

where δ_{0i} is the Kronecker delta and $c_i \in C_i$ are homogeneous elements of degree i . Furthermore, a graded coalgebra C is called connected, if the grade zero component is isomorphic to the base ring, i.e.

$$C_0 \cong k. \quad (86)$$

²⁵We denote both the neutral element in the coalgebra C with respect to the comultiplication map Δ and the projection function $C \rightarrow k$ by the symbol $\hat{\mathbb{I}}$.

Definition 3.5 (Homomorphism of (graded) algebras). Let A_1 and A_2 be two algebras. Then a map $f : A_1 \rightarrow A_2$ is called a homomorphism of algebras, if f is compatible with the products μ_1 on A_1 and μ_2 on A_2 , i.e. the following diagram commutes

$$\begin{array}{ccc} A_1 \otimes A_1 & \xrightarrow{\mu_1} & A_1 \\ f \otimes f \downarrow & & \downarrow f \\ A_2 \otimes A_2 & \xrightarrow{\mu_2} & A_2 \end{array}, \quad (87)$$

and f maps the identity \mathbb{I}_1 on A_1 to the identity \mathbb{I}_2 on A_2 , i.e. the following diagram commutes

$$\begin{array}{ccc} & k & \\ \mathbb{I}_1 \swarrow & & \searrow \mathbb{I}_2 \\ A_1 & \xrightarrow{f} & A_2 \end{array}. \quad (88)$$

If the algebras A_1 and A_2 are both graded, then f additionally has to respect this structure to be a homomorphism of graded algebras, i.e.

$$f((A_1)_i) \subseteq (A_2)_i, \quad \forall i \in \mathbb{N}_0. \quad (89)$$

Definition 3.6 (Homomorphism of (graded) coalgebras). Let C_1 and C_2 be two coalgebras. Then a map $g : C_1 \rightarrow C_2$ is called a homomorphism of coalgebras, if g is compatible with the two coproducts Δ_1 on C_1 and Δ_2 on C_2 , i.e. the following diagram commutes

$$\begin{array}{ccc} C_1 & \xrightarrow{\Delta_1} & C_1 \otimes C_1 \\ g \downarrow & & \downarrow g \otimes g \\ C_2 & \xrightarrow{\Delta_2} & C_2 \otimes C_2 \end{array}, \quad (90)$$

and g is compatible with the two coidentities $\hat{\mathbb{I}}_1$ on C_1 and $\hat{\mathbb{I}}_2$ on C_2 , i.e. the following diagram commutes

$$\begin{array}{ccc} C_1 & \xrightarrow{g} & C_2 \\ \hat{\mathbb{I}}_1 \searrow & & \swarrow \hat{\mathbb{I}}_2 \\ & k & \end{array}. \quad (91)$$

If the coalgebras C_1 and C_2 are both graded, then g additionally has to respect this structure to be a homomorphism of graded coalgebras, i.e.

$$g((C_1)_i) \subseteq (C_2)_i, \quad \forall i \in \mathbb{N}_0. \quad (92)$$

Remark 3.7 (Relation between algebra and coalgebra). Finite dimensional ((connected) graded) algebras A are related to ((connected) graded) coalgebras C via dualization, i.e. applying the functor $\text{Hom}_{k\text{-Alg}}(\cdot, k)$.

Definition 3.8 ((Connected graded) bialgebra). The quintuple $(B, \mu, \mathbb{I}, \Delta, \hat{\mathbb{I}})$ is a bialgebra, where the triple (B, μ, \mathbb{I}) is an algebra and the triple $(B, \Delta, \hat{\mathbb{I}})$ is a coalgebra. Furthermore, the coproduct Δ and the coidentity $\hat{\mathbb{I}}$ are homomorphisms of the graded algebra (B, μ, \mathbb{I}) , or, equivalently, the multiplication μ and the identity \mathbb{I} are homomorphisms of the graded coalgebra $(B, \Delta, \hat{\mathbb{I}})$. Furthermore, a bialgebra B is called graded, if it is graded as an algebra and as a coalgebra. Moreover, a bialgebra B is called connected, if it is connected as an algebra, or, equivalently, as a coalgebra.²⁶

Definition 3.9 ((Connected graded) Hopf algebra). The sextuple $(H, \mu, \mathbb{I}, \Delta, \hat{\mathbb{I}}, S)$ is a Hopf algebra, where the quintuple $(H, \mu, \mathbb{I}, \Delta, \hat{\mathbb{I}})$ is a bialgebra and $S : H \rightarrow H$ is an anti-automorphism,²⁷ called the antipode, and is defined such that the following diagram commutes:

$$\begin{array}{ccccc}
 & H \otimes H & \xrightarrow{S \otimes \text{Id}} & H \otimes H & \\
 & \nearrow \Delta & & \searrow \mu & \\
 H & \xrightarrow{\hat{\mathbb{I}}} & k & \xrightarrow{\mathbb{I}} & H \\
 & \searrow \Delta & & \nearrow \mu & \\
 & H \otimes H & \xrightarrow{\text{Id} \otimes S} & H \otimes H &
 \end{array} \tag{93}$$

Furthermore, a Hopf algebra H is called graded, if it is graded as a bialgebra.²⁸ Moreover, a Hopf algebra H is called connected, if it is connected as a bialgebra.

Definition 3.10 (Hopf ideals and Hopf subalgebras). Let H be a Hopf algebra and \mathfrak{i} an ideal in H . Then \mathfrak{i} is called a Hopf ideal if it satisfies additionally the following three conditions

$$\Delta(\mathfrak{i}) \subseteq \mathfrak{i} \otimes H + H \otimes \mathfrak{i}, \tag{94a}$$

$$\hat{\mathbb{I}}(\mathfrak{i}) = 0 \tag{94b}$$

and

$$S(\mathfrak{i}) \subseteq \mathfrak{i}. \tag{94c}$$

Then, the quotient $h := H/\mathfrak{i}$ is a Hopf algebra as well, called a Hopf subalgebra of H . If H is graded, then h inherits a grading from H via

$$h = H \cap h = \left(\bigoplus_{i=0}^{\infty} H_i \right) \cap h = \bigoplus_{i=0}^{\infty} (H_i \cap h) = \bigoplus_{i=0}^{\infty} h_i, \tag{95}$$

i.e. we define the grade i subspace of h as $h_i := (H_i \cap h)$ for all $i \in \mathbb{N}_0$. Furthermore, if H is connected, then h is also connected, since $\mathbb{I} \neq 0$ in h and thus $h_0 = k\mathbb{I} \cong k$.

Definition 3.11 (Augmentation ideal). Given a bi- or a Hopf algebra B , then the kernel of the coidentity $\hat{\mathbb{I}}$ is an ideal, called the augmentation ideal.

²⁶It can be shown, that a connected graded bialgebra possesses an antipode and thus is a Hopf algebra, c.f. Definition 3.9.

²⁷Meaning that in the case of non-commutative Hopf algebras the antipode is an order reversing endomorphism, i.e. given $x, y \in H$ then we have $S(xy) = S(y)S(x)$.

²⁸Observe, that if a Hopf algebra H is graded as a bialgebra, then the antipode S is automatically an anti-automorphism of graded algebras and an anti-automorphism of graded coalgebras.

Definition 3.12 (Convolution product). Let A be an algebra and C a coalgebra. Then using the multiplication μ_A on A and the comultiplication Δ_C on C , we can turn the k -module $\text{Hom}_{k\text{-Mod}}(C, A)$ of k -linear maps from C to A into a k -algebra by defining the convolution product \star for given $f, g \in \text{Hom}_{k\text{-Mod}}(C, A)$ via

$$f \star g := \mu_A \circ (f \otimes g) \circ \Delta_C . \quad (96)$$

Obviously, this definition extends trivially if A or C possesses additionally a bi- or Hopf algebra structure, since it only requires a coalgebra structure in the source algebra and an algebra structure in the target algebra.

3.2 The Connes-Kreimer renormalization Hopf algebra

From now on, we consider the Connes-Kreimer renormalization Hopf algebra, which is a Hopf algebra over $k = \mathbb{Q}$.²⁹ Furthermore, let in this subsection \mathcal{Q} be a local QFT such, that the residue for each divergent Feynman graph is in the residue set of \mathcal{Q} — the general case will be discussed in Subsection 3.3, c.f. Problem 3.31.

Definition 3.13 (Weighted residue set of a local QFT). Let \mathcal{Q} be a local QFT. Then \mathcal{Q} is either given via a Lagrange density $\mathcal{L}_{\mathcal{Q}}$ or via a set of residues $\mathcal{R}_{\mathcal{Q}}$ together with a weight function $\omega : \mathcal{R}_{\mathcal{Q}} \rightarrow \mathbb{N}_0$. The set of residues is a disjoint union of all vertex-types $\mathcal{R}_{\mathcal{Q}}^{[0]}$ and all edge-types $\mathcal{R}_{\mathcal{Q}}^{[1]}$ of \mathcal{Q} , i.e. $\mathcal{R}_{\mathcal{Q}} = \mathcal{R}_{\mathcal{Q}}^{[0]} \amalg \mathcal{R}_{\mathcal{Q}}^{[1]}$. If \mathcal{Q} is given via a Lagrange density, then the set of residues $\mathcal{R}_{\mathcal{Q}}$ and the weight $\omega(R)$ of each residue $R \in \mathcal{R}_{\mathcal{Q}}$ is given as follows: Each field in the Lagrange density $\mathcal{L}_{\mathcal{Q}}$ corresponds to a particle type of \mathcal{Q} . Therefore, every monomial in $\mathcal{L}_{\mathcal{Q}}$ consisting of one field corresponds to a source-term, i.e. a vertex-residue in $\mathcal{R}_{\mathcal{Q}}^{[0]}$ consisting of a half-edge of that particle type. Every monomial in $\mathcal{L}_{\mathcal{Q}}$ consisting of two equivalent fields corresponds to a propagation term, i.e. an edge-residue in $\mathcal{R}_{\mathcal{Q}}^{[1]}$ consisting of an edge of that particle type. And every monomial in $\mathcal{L}_{\mathcal{Q}}$ consisting of different fields corresponds to an interaction term, i.e. a vertex-residue in $\mathcal{R}_{\mathcal{Q}}^{[0]}$ consisting of half-edges of that particle types. Finally, the weight $\omega(R) \in \mathbb{N}_0$ of a residue $R \in \mathcal{R}_{\mathcal{Q}}$ is set to be the number of derivative operators involved in the corresponding field monomial in $\mathcal{L}_{\mathcal{Q}}$.

Definition 3.14 (Feynman graphs generated by residue sets). Let \mathcal{Q} be a local QFT with residue set $\mathcal{R}_{\mathcal{Q}}$. Then we denote by $\mathcal{G}_{\mathcal{Q}}$ the set of all one-particle irreducible (1PI) Feynman graphs³⁰ that can be generated by the residue set $\mathcal{R}_{\mathcal{Q}}$ of \mathcal{Q} .

Definition 3.15 (Residue of a Feynman graph). Let \mathcal{Q} be a local QFT with residue set $\mathcal{R}_{\mathcal{Q}}$ and Feynman graph set $\mathcal{G}_{\mathcal{Q}}$. Then the residue of a Feynman graph $\Gamma \in \mathcal{G}_{\mathcal{Q}}$, denoted by $\text{Res}(\Gamma)$, is the vertex residue or edge residue $\text{Res}(\Gamma)$, not necessary in the residue set $\mathcal{R}_{\mathcal{Q}}$, obtained by shrinking all internal edges of Γ to a single vertex.

Definition 3.16 (First Betti number of a Feynman graph, [32]). Let \mathcal{Q} be a local QFT and $\mathcal{G}_{\mathcal{Q}}$ the set of its Feynman graphs. Let furthermore $\Gamma \in \mathcal{G}_{\mathcal{Q}}$ be a Feynman graph. Then we define the first Betti number of Γ as

$$b_1(\Gamma) := \#H_1(\Gamma) , \quad (97)$$

where $\#H_1(\Gamma)$ is the rank of the first singular homology group of Γ .

²⁹Again, the physical needs require k only to be a field with characteristic 0. Since \mathbb{Q} is the smallest such field, it is the canonical choice.

³⁰The use of 1PI Feynman graphs, rather than connected Feynman graphs, is justified by Theorem 3.53, as is discussed in Remark 3.54.

Definition 3.17 (Superficial degree of divergence). Let \mathcal{Q} be a local QFT with weighted residue set $\mathcal{R}_{\mathcal{Q}}$ and Feynman graph set $\mathcal{G}_{\mathcal{Q}}$. We turn $\mathcal{G}_{\mathcal{Q}}$ into a weighted set as well by declaring the function

$$\omega : \mathcal{G}_{\mathcal{Q}} \rightarrow \mathbb{Z}, \quad \Gamma \mapsto db_1(\Gamma) + \sum_{v \in \Gamma^{[0]}} \omega(v) - \sum_{e \in \Gamma^{[1]}} \omega(e), \quad (98)$$

where d is the dimension of spacetime of \mathcal{Q} and $b_1(\Gamma)$ the first Betti number of the Feynman graph $\Gamma \in \mathcal{G}_{\mathcal{Q}}$. Then, the weight $\omega(\Gamma)$ of a Feynman graph $\Gamma \in \mathcal{G}_{\mathcal{Q}}$ is called the superficial degree of divergence of Γ . A Feynman graph $\Gamma \in \mathcal{G}_{\mathcal{Q}}$ is called superficially divergent if $\omega(\Gamma) \geq 0$, otherwise it is called superficially convergent if $\omega(\Gamma) < 0$.

Remark 3.18. The definition of the superficial degree of divergence of a Feynman graph, Definition 3.17, is motivated by the fact, that the Feynman integral corresponding to a given Feynman graph via the Feynman rules converges, if the Feynman graph itself and all its subgraphs are superficially convergent [33].

Definition 3.19 (Set of superficially divergent subgraphs of a Feynman graph). Let \mathcal{Q} be a local QFT and $\Gamma \in \mathcal{G}_{\mathcal{Q}}$ a Feynman graph of \mathcal{Q} . Then we denote by $\mathcal{D}(\Gamma)$ the set of superficially divergent subgraphs of Γ , i.e.

$$\mathcal{D}(\Gamma) := \left\{ \gamma \subseteq \Gamma \mid \gamma = \prod_{m=1}^M \gamma_m, M \in \mathbb{N} : \omega(\gamma_m) \geq 0, \forall m \right\}. \quad (99a)$$

Furthermore, we define the set $\mathcal{D}'(\Gamma)$ of superficially divergent proper subgraphs of Γ , i.e.

$$\mathcal{D}'(\Gamma) := \{ \gamma \in \mathcal{D}(\Gamma) \mid \emptyset \subsetneq \gamma \subsetneq \Gamma \}. \quad (99b)$$

Definition 3.20 (Renormalization Hopf algebra of a local QFT). Let \mathcal{Q} be a local QFT, $\mathcal{R}_{\mathcal{Q}}$ the set of its weighted residues and $\mathcal{G}_{\mathcal{Q}}$ the set of its weighted Feynman graphs. We assume \mathcal{Q} to be such, that the residues of all superficially divergent Feynman graphs are in the residue set $\mathcal{R}_{\mathcal{Q}}$, i.e. $\{ \Gamma \mid \Gamma \in \mathcal{G}_{\mathcal{Q}} : \omega(\Gamma) \geq 0 : \text{Res}(\Gamma) \notin \mathcal{R}_{\mathcal{Q}} \} = \emptyset$. The general and more involving case is discussed in Subsection 3.3. Then, the connected graded, c.f. Definition 3.24, renormalization Hopf algebra $(\mathcal{H}_{\mathcal{Q}}, \mu, \mathbb{I}, \Delta, \hat{\mathbb{I}}, S)$ is defined as follows: We set $\mathcal{H}_{\mathcal{Q}}$ to be the vector space over \mathbb{Q} generated by the set $\mathcal{G}_{\mathcal{Q}}$. The associative multiplication $\mu : \mathcal{H}_{\mathcal{Q}} \otimes \mathcal{H}_{\mathcal{Q}} \rightarrow \mathcal{H}_{\mathcal{Q}}$ is defined as the disjoint union of Feynman graphs, i.e.

$$\mu : \mathcal{H}_{\mathcal{Q}} \otimes \mathcal{H}_{\mathcal{Q}} \rightarrow \mathcal{H}_{\mathcal{Q}}, \quad \gamma \otimes \Gamma \mapsto \gamma \Gamma. \quad (100)$$

Then, the identity $\mathbb{I} : \mathbb{Q} \rightarrow \mathcal{H}_{\mathcal{Q}}$ is set to be the empty graph, i.e.

$$\mathbb{I} := \emptyset. \quad (101)$$

Moreover, we define the coproduct of a Feynman graph Γ such that it maps to the following sum over all possible combinations of divergent subgraphs of Γ : The left-hand side of the tensor product of each summand is given by a superficially divergent subgraph of the Feynman graph Γ , while the right-hand-side of the tensor product is given by returning the Feynman graph Γ with the corresponding subgraph shrunken to zero length, i.e.

$$\Delta : \mathcal{H}_{\mathcal{Q}} \rightarrow \mathcal{H}_{\mathcal{Q}} \otimes \mathcal{H}_{\mathcal{Q}}, \quad \Gamma \mapsto \sum_{\gamma \in \mathcal{D}(\Gamma)} \gamma \otimes \Gamma/\gamma, \quad (102)$$

where the quotient Γ/γ is defined as follows: If γ is a proper subgraph of Γ , then Γ/γ is defined by shrinking all internal edges of γ in Γ to a single vertex for each connected component of γ .

Otherwise, if $\gamma = \Gamma$ we define the quotient to be the identity, i.e. $\Gamma/\Gamma := \mathbb{I}$. The coidentity $\hat{\mathbb{I}} : \mathcal{H}_{\mathcal{Q}} \rightarrow \mathbb{Q}$ is set such that its kernel is the Hopf algebra without the subalgebra generated by \mathbb{I} , i.e.

$$\hat{\mathbb{I}} : \mathcal{H}_{\mathcal{Q}} \rightarrow \mathbb{Q}, \quad \Gamma \mapsto \begin{cases} q & \text{if } \Gamma = q\mathbb{I} \text{ with } q \in \mathbb{Q} \\ 0 & \text{else} \end{cases}. \quad (103)$$

Finally, we define the antipode recursively via

$$S : \mathcal{H}_{\mathcal{Q}} \rightarrow \mathcal{H}_{\mathcal{Q}}, \quad \Gamma \mapsto \begin{cases} q\mathbb{I} & \text{if } \Gamma = q\mathbb{I} \text{ with } q \in \mathbb{Q} \\ -\sum_{\gamma \in \mathcal{D}(\Gamma)} S(\gamma) \Gamma/\gamma & \text{else} \end{cases}, \quad (104)$$

where the quotient Γ/γ is defined as in the definition of the coproduct after Equation (102).

Definition 3.21 (Reduced coproduct). Let \mathcal{Q} be a local QFT as in Definition 3.20, $\mathcal{R}_{\mathcal{Q}}$ the set of its weighted residues and $(\mathcal{H}_{\mathcal{Q}}, \mu, \mathbb{I}, \Delta, \hat{\mathbb{I}}, S)$ its renormalization Hopf algebra. Then we define the reduced coproduct as the non-trivial part of the coproduct, i.e.

$$\Delta' : \mathcal{H}_{\mathcal{Q}} \rightarrow \mathcal{H}_{\mathcal{Q}} \otimes \mathcal{H}_{\mathcal{Q}}, \quad \Gamma \mapsto \sum_{\gamma \in \mathcal{D}'(\Gamma)} \gamma \otimes \Gamma/\gamma. \quad (105)$$

In particular, the coproduct and the reduced coproduct are related via

$$\Delta(\Gamma) = \Delta'(\Gamma) + \mathbb{I} \otimes \Gamma + \Gamma \otimes \mathbb{I}. \quad (106)$$

Definition 3.22 (Product of coupling constants of a Feynman graph). Let \mathcal{Q} be a local QFT and $\mathcal{G}_{\mathcal{Q}}$ the set of its Feynman graphs. Let furthermore $\Gamma = \prod_{m=1}^M \Gamma_m$ be a product of $M \in \mathbb{N}$ connected Feynman graphs $\Gamma_m \in \mathcal{G}_{\mathcal{Q}}$, $1 \leq m \leq M$. Then we define the product of coupling constants of Γ as

$$\text{Cpl}(\Gamma) := \prod_{m=1}^M \left(\frac{1}{\text{Cpl}(\text{Res}(\Gamma_m))} \prod_{v \in \Gamma_m^{[0]}} \text{Cpl}(v) \right), \quad (107)$$

with

$$\text{Cpl}(\text{Res}(\Gamma_m)) := \begin{cases} \text{coupling constant of the vertex } \text{Res}(\Gamma_m) & \text{if } \text{Res}(\Gamma_m) \in \mathcal{R}_{\mathcal{Q}}^{[0]} \\ 1 & \text{else} \end{cases}. \quad (108)$$

Definition 3.23 (Multi-index corresponding to the product of coupling constants of a Feynman graph). Let \mathcal{Q} be a local QFT and $\mathcal{G}_{\mathcal{Q}}$ the set of its Feynman graphs. Let furthermore $\Gamma = \prod_{m=1}^M \Gamma_m$ be a product of $M \in \mathbb{N}$ connected Feynman graphs $\Gamma_m \in \mathcal{G}_{\mathcal{Q}}$, $1 \leq m \leq M$ and $\text{Cpl}(\Gamma)$ the product of its coupling constants. Then we define the multi-index $\mathbf{C} \in \mathbb{Z}^{\#\mathcal{R}_{\mathcal{Q}}^{[0]}}$ corresponding to $\text{Cpl}(\Gamma)$ as the vector counting the multiplicities of the several coupling constants in $\text{Cpl}(\Gamma)$. Sums and direct sums over multi-indices are understood componentwise, e.g. let $N := \#\mathcal{R}_{\mathcal{Q}}^{[0]}$ and $\mathbf{C}^+ = (C_1^+, \dots, C_N^+) \in \mathbb{Z}^N$ and $\mathbf{C}^- = (C_1^-, \dots, C_N^-) \in \mathbb{Z}^N$, then we set

$$\sum_{\mathbf{c}=\mathbf{C}^-}^{\mathbf{C}^+} := \sum_{c_1=C_1^-}^{C_1^+} \cdots \sum_{c_N=C_N^-}^{C_N^+} \quad \text{and} \quad \bigoplus_{\mathbf{c}=\mathbf{C}^-}^{\mathbf{C}^+} := \bigoplus_{c_1=C_1^-}^{C_1^+} \cdots \bigoplus_{c_N=C_N^-}^{C_N^+}. \quad (109)$$

Furthermore, we set $\mathbf{0} := (0_1, \dots, 0_N)$, $-\infty := (-\infty_1, \dots, -\infty_N)$ and $\infty := (\infty_1, \dots, \infty_N)$.

Definition 3.24 (Connectedness and gradings of the renormalization Hopf algebra). Let \mathcal{Q} be a local QFT as in Definition 3.20, $\mathcal{R}_{\mathcal{Q}}$ the set of its residues and $\mathcal{H}_{\mathcal{Q}}$ its renormalization Hopf algebra. Then we consider the following three gradings of $\mathcal{H}_{\mathcal{Q}}$ as a Hopf algebra which are further refinements of each other: The first grading comes from the first Betti number, i.e.

$$\mathcal{H}_{\mathcal{Q}} = \bigoplus_{L=0}^{\infty} (\mathcal{H}_{\mathcal{Q}})_L . \quad (110)$$

The second grading comes from the multi-index corresponding to the coupling constants of a Feynman graph, i.e.

$$\mathcal{H}_{\mathcal{Q}} = \bigoplus_{\mathbf{C}=-\infty}^{\infty} (\mathcal{H}_{\mathcal{Q}})_{\mathbf{C}} . \quad (111)$$

Finally, the third grading comes from the multi-index corresponding to the vertex-residues of a Feynman graph, i.e.

$$\mathcal{H}_{\mathcal{Q}} = \bigoplus_{\mathbf{R}=-\infty}^{\infty} (\mathcal{H}_{\mathcal{Q}})_{\mathbf{R}} . \quad (112)$$

Clearly, $(\mathcal{H}_{\mathcal{Q}})_{L=0} \cong (\mathcal{H}_{\mathcal{Q}})_{\mathbf{C}=0} \cong (\mathcal{H}_{\mathcal{Q}})_{\mathbf{R}=0} \cong \mathbb{Q}$, and thus $\mathcal{H}_{\mathcal{Q}}$ is connected in all three gradings. In this article we use only the second grading.

Definition 3.25 (Symmetry factor of a Feynman graph). Let \mathcal{Q} be a local QFT and $\mathcal{G}_{\mathcal{Q}}$ the set of its Feynman graphs. Let furthermore $\Gamma \in \mathcal{G}_{\mathcal{Q}}$ be a Feynman graph. Then we define the symmetry factor of Γ as

$$\text{Sym}(\Gamma) := \# \text{Aut}(\Gamma) , \quad (113)$$

where $\# \text{Aut}(\Gamma)$ is the rank of the automorphism group of Γ , leaving its external leg structure fixed and respecting its vertex and edge types $v \in \mathcal{R}_{\mathcal{Q}}$ and $e \in \mathcal{R}_{\mathcal{Q}}$ for all $v \in \Gamma^{[0]}$ and $e \in \Gamma^{[1]}$, respectively.

Definition 3.26 (Combinatorial Green's functions). Let \mathcal{Q} be a local QFT, $\mathcal{R}_{\mathcal{Q}}$ the set of its residues and $\mathcal{G}_{\mathcal{Q}}$ the set of its Feynman graphs. Given $r \in \mathcal{R}_{\mathcal{Q}}$, we set for notational simplicity in this definition

$$g^r := \sum_{\substack{\Gamma \in \mathcal{G}_{\mathcal{Q}} \\ \text{Res}(\Gamma)=r}} \frac{1}{\text{Sym}(\Gamma)} \Gamma . \quad (114)$$

Then, we define the total combinatorial Green's function with residue r as the following sums:

$$G^r := \begin{cases} \mathbb{I} + g^r & \text{if } r \in \mathcal{R}_{\mathcal{Q}}^{[0]} \\ \mathbb{I} - g^r & \text{if } r \in \mathcal{R}_{\mathcal{Q}}^{[1]} \\ g^r & \text{else, i.e. } r \notin \mathcal{R}_{\mathcal{Q}} \end{cases} \quad (115)$$

Finally, we denote the restriction of G^r to one of the gradings \mathbf{g} from Definition 3.24 via

$$G_{\mathbf{g}}^r := G^r \Big|_{\mathbf{g}} . \quad (116)$$

Remark 3.27. We remark, that combinatorial Green's functions are in the literature often denoted via X^r , however in order to avoid confusion with vector fields X^μ we have chosen G^r . Furthermore, restricted combinatorial Green's functions are in the literature often denoted via $c_{\mathbf{g}}^r$ and differ by a minus sign from our definition. Our convention is such, that they are given as the restriction of the total combinatorial Green's function to the corresponding grading, which requires minus signs for non-empty propagator graphs.

Definition 3.28 (Hopf subalgebras for multiplicative renormalization). Let \mathcal{Q} be a local QFT as in Definition 3.20, $\mathcal{R}_{\mathcal{Q}}$ its weighted residue set, $\mathcal{H}_{\mathcal{Q}}$ its renormalization Hopf algebra and $G_{\mathbf{G}}^r \in \mathcal{H}_{\mathcal{Q}}$ its restricted Green's functions, where \mathbf{G} and \mathbf{g} denotes one of the gradings from Definition 3.24. We are interested in Hopf subalgebras which correspond to multiplicative renormalization, i.e. Hopf subalgebras of $\mathcal{H}_{\mathcal{Q}}$ such that the coproduct factors on the restricted combinatorial Green's functions for all multi-indices \mathbf{G} in the following way:

$$\Delta(G_{\mathbf{G}}^r) = \sum_{\mathbf{g}=0}^{\mathbf{G}} \mathfrak{P}_{\mathbf{g}}(G_{\mathbf{G}}^r) \otimes G_{\mathbf{G}-\mathbf{g}}^r, \quad (117)$$

where $\mathfrak{P}_{\mathbf{g}}(G_{\mathbf{G}}^r) \in \mathcal{H}_{\mathcal{Q}}$ is a polynomial in graphs such that each summand has multi-index \mathbf{g} .³¹

Remark 3.29 (Hopf subalgebras and multiplicative renormalization). We shortly remark the connection between Hopf subalgebras in the sense of Definition 3.28 and multiplicative renormalization: Let \mathcal{Q} be a local QFT as in Definition 3.20 and $\mathcal{H}_{\mathcal{Q}}$ its renormalization Hopf algebra. Then we can define renormalized Feynman rules $\Phi_{\mathcal{R}}(\cdot)$ for a given renormalization scheme \mathcal{R} as the following map to the target algebra of the Feynman rules

$$\Phi_{\mathcal{R}}(\cdot) := \left(S_{\mathcal{R}}^{\Phi} \star \Phi \right) (\cdot), \quad (118)$$

where $\Phi(\cdot)$ are the Feynman rules, \star is the convolution product from Definition 3.12 and $S_{\mathcal{R}}^{\Phi}(\cdot)$ is the counterterm map, recursively given via the normalization $S_{\mathcal{R}}^{\Phi}(\mathbb{I}) := \mathbb{I}$ and

$$S_{\mathcal{R}}^{\Phi}(\cdot) := -\mathcal{R} \circ \left(S_{\mathcal{R}}^{\Phi} \star (\Phi \circ \mathcal{P}) \right) (\cdot), \quad (119)$$

else, with \mathcal{P} being the projector onto the augmentation ideal from Definition 3.11. If the renormalization Hopf algebra $\mathcal{H}_{\mathcal{Q}}$ possesses Hopf subalgebras in the sense of Definition 3.28, we can calculate the Z -factor for a given residue $r \in \mathcal{R}_{\mathcal{Q}}$ via

$$Z_{\mathcal{R}}^r = S_{\mathcal{R}}^{\Phi}(G^r). \quad (120)$$

More details on this result can be found in [35, 36] (where the second reference uses a different notation).

Remark 3.30 (Hopf subalgebras and different gradings). Furthermore, we remark that the existence of the Hopf subalgebras from Definition 3.28 depends crucially on the grading \mathbf{g} . In particular, for the grading induced by the first Betti number these Hopf subalgebras exist if and only if the local QFT has only one vertex, for the coupling-constant grading if and only if the local QFT has for each vertex a different coupling constant and always for the residue grading [35].

3.3 Associating the renormalization Hopf algebra to a local QFT

In this subsection we describe a problem which may occur in the construction of the renormalization Hopf algebra $\mathcal{H}_{\mathcal{Q}}$ to a given local QFT \mathcal{Q} using Definition 3.20. Then, we present three different solutions to still obtain a renormalization Hopf algebra (which are not isomorphic if the problem occurs) and discuss their physical interpretation.

³¹There exist closed expressions for the polynomials $\mathfrak{P}_{\mathbf{g}}(G_{\mathbf{G}}^r)$ via $\mathfrak{P}_{\mathbf{g}}(G_{\mathbf{G}}^r) := \left(\overline{G^r Q^{\mathbf{G}}} \right) \Big|_{\mathbf{g}}$, where the overline denotes the restriction to divergent graphs and Q^v denotes so-called combinatorial charges, c.f. e.g. [34, 35] (combinatorial charges were introduced in the first reference, whereas the second reference uses the notations and conventions of the present work).

Problem 3.31. Given a general local QFT \mathcal{Q} , Definition 3.20 may not yield a well-defined Hopf algebra due to the following reason: Let \mathcal{Q} be such, that there exist divergent Feynman graphs $\gamma \in \mathcal{G}_{\mathcal{Q}}$ whose residue is not in the residue set, i.e. we have $\omega(\gamma) \geq 0$ and $\text{Res}(\gamma) \notin \mathcal{R}_{\mathcal{Q}}$. Then given any Feynman graph $\Gamma \in \mathcal{G}_{\mathcal{Q}}$ with $\gamma \in \mathcal{D}(\Gamma)$, the quotients of the form Γ/γ for $\gamma \subsetneq \Gamma$ are ill-defined, as they generate a new vertex $\text{Res}(\gamma) \notin \mathcal{R}_{\mathcal{Q}}^{[0]}$. As a consequence, the definitions of the coproduct and the antipode are ill-defined as well.

Remark 3.32. In order to remedy Problem 3.31 we need to change some of the definitions. This is explained in the following Solutions 3.35, 3.36 and 3.37. In order to distinguish the different objects, we use script letters for the objects as defined in Definition 3.20 and calligraphic letters for the modified definitions.

Definition 3.33 (Feynman graphs generated by residue sets 2). Let \mathcal{Q} be a local QFT with residue set $\mathcal{R}_{\mathcal{Q}}$. Recall from Definition 3.14 that we denote by $\mathcal{G}_{\mathcal{Q}}$ the set of all one-particle irreducible (1PI) Feynman graphs³² that can be generated by the residue set $\mathcal{R}_{\mathcal{Q}}$ of \mathcal{Q} . Moreover, we define the set $\mathcal{G}_{\mathcal{Q}}$ of all 1PI Feynman graphs of \mathcal{Q} which does not contain superficially divergent subgraphs whose residue is not in the residue set $\mathcal{R}_{\mathcal{Q}}$, i.e.

$$\mathcal{G}_{\mathcal{Q}} := \left\{ \Gamma \in \mathcal{G}_{\mathcal{Q}} \left| \Gamma = \prod_{m=1}^M \Gamma_m, M \in \mathbb{N} : \text{Res}(\Gamma_m) \in \mathcal{R}_{\mathcal{Q}}, \forall m : \omega(\Gamma_m) \geq 0 \right. \right\}. \quad (121)$$

This set will be used in Solution 3.35.

Definition 3.34 (Set of superficially divergent subgraphs of a Feynman graph 2). Let \mathcal{Q} be a local QFT and $\Gamma \in \mathcal{G}_{\mathcal{Q}}$ a Feynman graph of \mathcal{Q} . Recall from Definition 3.19 that we denote by $\mathcal{D}(\Gamma)$ the set of superficially divergent subgraphs of Γ and by $\mathcal{D}'(\Gamma)$ the set of superficially divergent proper subgraphs of Γ . Moreover, we define the two additional sets $\mathcal{D}(\Gamma)$ and $\mathcal{D}'(\Gamma)$, corresponding to $\mathcal{D}(\Gamma)$ and $\mathcal{D}'(\Gamma)$, respectively, which do not contain Feynman graphs with superficially divergent subgraphs whose residue is not in the residue set $\mathcal{R}_{\mathcal{Q}}$, i.e.

$$\mathcal{D}(\Gamma) := \left\{ \gamma \in \mathcal{D}(\Gamma) \left| \gamma = \prod_{m=1}^M \gamma_m, M \in \mathbb{N} : \text{Res}(\gamma_m) \in \mathcal{R}_{\mathcal{Q}}, \forall m \right. \right\} \quad (122a)$$

and

$$\mathcal{D}'(\Gamma) := \{ \gamma \in \mathcal{D}(\Gamma) \mid \gamma \subsetneq \Gamma \}, \quad (122b)$$

These sets will be used in Solution 3.36.

Solution 3.35. The first solution to Problem 3.31 to replace the Feynman graph set $\mathcal{G}_{\mathcal{Q}}$ from Definition 3.14 by $\mathcal{G}_{\mathcal{Q}}$ from Definition 3.33. Then, we can construct the renormalization Hopf algebra as in Definition 3.20 which we now denote by $\mathcal{H}_{\mathcal{Q}}$.

Solution 3.36. The second solution to Problem 3.31 is to simply remove all divergent Feynman graphs whose residue is not in the residue set from the sets of divergent subgraphs, i.e. replace the sets $\mathcal{D}(\cdot)$ and $\mathcal{D}'(\cdot)$ from Definition 3.19 by $\mathcal{D}(\cdot)$ and $\mathcal{D}'(\cdot)$ from Definition 3.34, respectively. Then, we can construct the renormalization Hopf algebra as in Definition 3.20 which we now again denote by $\mathcal{H}_{\mathcal{Q}}$.

³²Again, we remark that the use of 1PI Feynman graphs, rather than connected Feynman graphs, is justified by Theorem 3.53, as is discussed in Remark 3.54.

Solution 3.37. The third solution to Problem 3.31 is to add all missing residues to the residue set and set its weights to the value of a divergent Feynman graph with this particular residue (if there exist two or more such graphs with different superficial degree of divergence we take the highest for uniqueness, although it suffices to be divergent). This enlarges also the set of Feynman graphs. Then, we can define the Hopf algebra using this enlarged set of Feynman graphs as in Definition 3.20.

Remark 3.38. Equivalently, the Hopf algebra from Solution 3.35 could be constructed in two different ways: The first possibility is to define $\mathcal{H}_{\mathcal{Q}}$ as the \mathbb{Q} -algebra generated by the set $\mathcal{G}_{\mathcal{Q}}$ of Feynman graphs with the multiplication and unit as in Definition 3.20. Then, we define the ideal

$$\mathfrak{i}_{\mathcal{Q}} := \left(\Gamma \in (\mathcal{G}_{\mathcal{Q}} \setminus \mathcal{R}_{\mathcal{Q}}) \right), \quad (123)$$

generated by all divergent Feynman graphs whose residue is not in the residue set, and consider the the quotient

$$\mathcal{H}_{\mathcal{Q}} := \mathcal{H}_{\mathcal{Q}} / \mathfrak{i}_{\mathcal{Q}}, \quad (124)$$

on which we can define the additional Hopf algebra structures as in Definition 3.20. The set of Feynman graphs from the quotient Hopf algebra $\mathcal{H}_{\mathcal{Q}}$ is then precisely the set as defined in Definition 3.33. The second possibility is to use the Hopf algebra from Solution 3.37 and consider the quotient by the ideal given as the sum of $\mathfrak{i}_{\mathcal{Q}}$ and the ideal generated by all Feynman graphs with vertices which are not in the residue set $\mathcal{R}_{\mathcal{Q}}$, which is a Hopf ideal inside the Hopf algebra from Solution 3.37 via Corollary 3.49 together with Proposition 3.44.

Remark 3.39 (Physical interpretation). Physically, Problem 3.31 states that divergent Feynman graphs whose residue is not in the residue set contribute in principle to a divergent restricted Green's function which cannot be renormalized if the corresponding vertex is missing in the local QFT. However, there could be still two possibilities that the unrenormalized Feynman rules remedy the problem themselves: The first one is, that the problematic Feynman graphs itself or the corresponding restricted combinatorial Green's functions turn out to be in the kernel of the unrenormalized Feynman rules which corresponds to Solution 3.35. The second one is, that the problematic Feynman graphs itself or the corresponding restricted combinatorial Green's functions turn out to be already finite when applying the unrenormalized Feynman rules which corresponds to Solution 3.36. However, if this is not the case, we need to add the corresponding vertices with suitable Feynman rules in order to absorb the divergences of the corresponding restricted combinatorial Green's functions via multiplicative renormalization corresponding to Solution 3.37. Luckily, for all established physical local QFTs this situation did not appear so far.

Example 3.40 (QED). An illuminating example to Problem 3.31 is QED since we need to apply both, Solution 3.35 and Solution 3.37: Consider QED with its combinatorics as a renormalizable local QFT (i.e. the superficial degree of divergence of a Feynman graph depends only on its external leg structure). Then, the Feynman graphs contributing to the three- and four-point function are divergent — however in contrast to non-abelian quantum gauge theories, there is no three- and four-photon vertex present to absorb the corresponding divergences. Luckily, when summing all Feynman graphs of a given loop order we have the following cancellations after applying the unrenormalized Feynman rules: The Feynman graphs contributing to the three-point function cancel pairwise due to Furry's Theorem, c.f. Theorem 3.53 for a generalization thereof and the divergences of the Feynman graphs contributing to the four-point function cancel pairwise due to gauge invariance [37]. Thus, QED is a renormalizable local QFT after all without the need to add a three- and four-photon vertex to the theory.

Remark 3.41 (The situation of QGR-QED). The situation of QGR-QED is worse than the one for QED since QGR is a non-renormalizable as a local QFT (in particular, the superficial degree of divergence of a pure gravity Feynman graph depends only on its loop number). However, for the two-loop propagator Feynman graphs considered in this article the generalization of Furry's theorem given in Theorem 3.53 suffices, since Feynman graphs with self-loops (in the mathematical literature also known as “roses”) vanish in the renormalization process. In particular, this ensures that the graviton-photon 2-point function vanishes, which would be in principle possible via quantum correction (whose corresponding Feynman graphs are divergent due to the superficial degree of divergence).

Definition 3.42 (Renormalization Hopf algebra associated to a local QFT). Let \mathcal{Q} be a local QFT. Then we denote by $\mathcal{H}_{\mathcal{Q}}$ one of the following Hopf algebras: If Definition 3.20 is well-defined, we denote $\mathcal{H}_{\mathcal{Q}}$ the renormalization Hopf algebra of Definition 3.20. Otherwise, we denote by $\mathcal{H}_{\mathcal{Q}}$ the Hopf algebra obtained after applying Solutions 3.35, 3.36 or 3.37 to Definition 3.20. We call $\mathcal{H}_{\mathcal{Q}}$ “the renormalization Hopf algebra associated to \mathcal{Q} ”.

Remark 3.43. The motivation for Definition 3.42 is to simplify notation, as for the realm of this work it is not necessary to distinguish between Definition 3.20 and Solutions 3.35, 3.36 and 3.37.

3.4 Hopf ideals and the renormalization Hopf algebra

Recall the definition of a Hopf ideal from Definition 3.10. Now, we study general properties of Hopf ideals and then specialize to the renormalization Hopf algebra associated to a local QFT. To this end, we prove general results for Hopf ideals and a condition for Hopf ideals in the renormalization Hopf algebra which yields some particular Hopf ideals as corollaries. This is of physical interest since symmetries generating Hopf ideals are compatible with BPHZ and BPHZL renormalization. In particular we show, that the ideal generated by all Feynman graphs having at least one self-loop (“rose”) is a Hopf ideal. This is useful, as these Feynman integrals vanish for kinematic renormalization schemes and can thus already be set to zero in the renormalization Hopf algebra.

Proposition 3.44 (Sums of Hopf ideals are Hopf ideals). *Let H be a Hopf algebra over a field with characteristic zero and $\{\mathfrak{i}_n\}_{n=1}^N$ be a set of N non-empty Hopf ideals, where $N \in \mathbb{N}_{\geq 1} \cup \infty$. Then the sum*

$$\mathfrak{i}_{\Sigma} := \sum_{n=1}^N \mathfrak{i}_n, \quad (125)$$

i.e. the ideal \mathfrak{i}_{Σ} generated by sums of the generators of all Hopf ideals in the set $\{\mathfrak{i}_n\}_{n=1}^N$, is also a Hopf ideal in H , i.e. \mathfrak{i}_{Σ} satisfies:

1. $\Delta(\mathfrak{i}_{\Sigma}) \subseteq H \otimes \mathfrak{i}_{\Sigma} + \mathfrak{i}_{\Sigma} \otimes H$
2. $\hat{\mathbb{I}}(\mathfrak{i}_{\Sigma}) = 0$
3. $S(\mathfrak{i}_{\Sigma}) \subseteq \mathfrak{i}_{\Sigma}$

Proof. This follows directly by the linearity of the involved maps. ■

Proposition 3.45 (Special products of Hopf ideals are Hopf ideals). *Let H be a Hopf algebra over a field with characteristic zero and $\{\mathfrak{i}_n\}_{n=1}^N$ be a set of N non-empty Hopf ideals, where*

$N \in \mathbb{N}_{\geq 1} \cup \infty$ and $k \in \{1, \dots, N\}$. Then the special “product”

$$\mathfrak{i}_{\Pi k} := \prod_{n=1}^N \mathfrak{i}_n + \mathfrak{i}_k, \quad (126)$$

i.e. the ideal $\mathfrak{i}_{\Pi k}$ generated by products of the generators of all Hopf ideals and the sum of a particular Hopf ideal in the set $\{\mathfrak{i}_n\}_{n=1}^N$, is also a Hopf ideal in H , i.e. $\mathfrak{i}_{\Pi k}$ satisfies:

1. $\Delta(\mathfrak{i}_{\Pi k}) \subseteq H \otimes \mathfrak{i}_{\Pi k} + \mathfrak{i}_{\Pi k} \otimes H$
2. $\hat{\mathbb{I}}(\mathfrak{i}_{\Pi k}) = 0$
3. $S(\mathfrak{i}_{\Pi k}) \subseteq \mathfrak{i}_{\Pi k}$

Proof. This follows directly by the linearity and multiplicativity of the involved maps. ■

Proposition 3.46 (Condition for Hopf ideals). *Let \mathcal{Q} be a local QFT with residue set $\mathcal{R}_{\mathcal{Q}}$ and $\mathcal{H}_{\mathcal{Q}}$ its renormalization Hopf algebra.³³ Let furthermore $\emptyset \subsetneq \mathcal{S} \subseteq \mathcal{G}_{\mathcal{Q}}$ be a non-empty set of Feynman graphs and denote via*

$$\mathfrak{i}_{\mathcal{S}} := (\Gamma \in \mathcal{S})_{\mathcal{H}_{\mathcal{Q}}} \quad (127)$$

the ideal generated by the set \mathcal{S} .³⁴ Then, $\mathfrak{i}_{\mathcal{S}}$ is a Hopf ideal, i.e. $\mathfrak{i}_{\mathcal{S}}$ satisfies:

1. $\Delta(\mathfrak{i}_{\mathcal{S}}) \subseteq \mathcal{H}_{\mathcal{Q}} \otimes \mathfrak{i}_{\mathcal{S}} + \mathfrak{i}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{Q}}$
2. $\hat{\mathbb{I}}(\mathfrak{i}_{\mathcal{S}}) = 0$
3. $S(\mathfrak{i}_{\mathcal{S}}) \subseteq \mathfrak{i}_{\mathcal{S}}$

if and only if the set \mathcal{S} is such that for all graphs $\Gamma \in \mathcal{S}$ and for all corresponding graphs $\gamma \in \mathcal{D}(\Gamma)$ we have that either $\gamma \in \mathcal{S}$ or $\Gamma/\gamma \in \mathcal{S}$

Proof. By the multiplicativity of the coproduct and the antipode, i.e.

$$\Delta(\Gamma_1 \Gamma_2) = \Delta(\Gamma_1) \Delta(\Gamma_2) \quad (128a)$$

and

$$S(\Gamma_1 \Gamma_2) = S(\Gamma_1) S(\Gamma_2) \quad (128b)$$

for $\Gamma_1, \Gamma_2 \in \mathcal{H}_{\mathcal{Q}}$, it suffices to check all three conditions on the level of generators, i.e. let $\Gamma \in \mathcal{S}$: Then, the first property follows from the definition of the coproduct

$$\begin{aligned} \Delta(\Gamma) &= \sum_{\gamma \in \mathcal{D}(\Gamma)} \gamma \otimes \Gamma/\gamma \\ &\subseteq \mathcal{H}_{\mathcal{Q}} \otimes \mathfrak{i}_{\mathcal{S}} + \mathfrak{i}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{Q}}. \end{aligned} \quad (129)$$

The second property follows directly from the fact that $\mathcal{S} \neq \emptyset$ and thus $\mathfrak{i}_{\mathcal{S}} \neq \emptyset$. Finally, the third property follows from the normalization $S(\mathbb{I}) = \mathbb{I}$ and the recursive definition of the antipode

$$\begin{aligned} S(\Gamma) &= - \sum_{\gamma \in \mathcal{D}(\Gamma)} S(\gamma) \Gamma/\gamma \\ &\subseteq \mathfrak{i}_{\mathcal{S}}. \end{aligned} \quad (130)$$

■

³³Defined either via Definition 3.20 or, if this definition fails, as described in Problem 3.31 via one of the solutions described in Solutions 3.35, 3.36 and 3.37, c.f. Definition 3.42.

³⁴In general $\mathfrak{i}_{\mathcal{S}}$ will not be finitely generated.

Corollary 3.47 (Ideals generated by self-loops are Hopf ideals). *Given the situation of Proposition 3.46, then we denote by $\emptyset \subsetneq \mathcal{S}_S \subsetneq \mathcal{G}_Q$ the set of all Feynman graphs being or containing at least one self-loop (“rose”), i.e.*

$$\mathcal{S}_S := \left\{ \Gamma \in \mathcal{G}_Q \mid \exists \gamma \subseteq \Gamma : \#\gamma^{[0]} = 1, \#\gamma^{[1]} \geq 1 \right\}. \quad (131)$$

Furthermore, we denote by \mathfrak{i}_S the ideal generated by the set \mathcal{S}_S in the renormalization Hopf algebra \mathcal{H}_Q . Then, \mathfrak{i}_S is a Hopf ideal in \mathcal{H}_Q .

Proof. Let $\Gamma \in \mathcal{S}_S$ be a generator of \mathfrak{i}_S , i.e. there exists at least one subgraph $\gamma_S \subseteq \Gamma$ which is a single self-loop, and let $\gamma_{\mathcal{D}} \in \mathcal{D}(\Gamma)$. Then we have either $\gamma_S \subseteq \gamma_{\mathcal{D}}$ or $\gamma_S \subseteq \Gamma/\gamma_{\mathcal{D}}$ and thus either $\gamma_{\mathcal{D}} \in \mathcal{S}_S$ or $\Gamma/\gamma_{\mathcal{D}} \in \mathcal{S}_S$. Finally, applying Proposition 3.46 finishes the proof. ■

Remark 3.48. We remark, that self-loop graphs are in the kernel of the renormalized Feynman rules for kinematic renormalization schemes, i.e. it makes physically sense to set them already in the renormalization Hopf algebra to zero.

Corollary 3.49 (Ideals generated by residues are Hopf ideals). *Given the situation of Proposition 3.46 and let $r \in \mathcal{R}_Q$ be a residue, then we denote by $\emptyset \subsetneq \mathcal{S}_r \subsetneq \mathcal{G}_Q$ the set of all Feynman graphs in \mathcal{H}_Q having residue r or having a subgraph with residue r , i.e.*

$$\mathcal{S}_r := \left\{ \Gamma \in \mathcal{G}_Q \mid \exists \gamma \subseteq \Gamma : \text{Res}(\gamma) = r \right\}. \quad (132)$$

Furthermore, we denote by \mathfrak{i}_r the ideal generated by the set \mathcal{S}_r in the renormalization Hopf algebra \mathcal{H}_Q . Then, \mathfrak{i}_r is a Hopf ideal in \mathcal{H}_Q .

Proof. Let $\Gamma \in \mathcal{G}_Q$ be a generator of \mathfrak{i}_r , i.e. there exists at least one subgraph $\gamma_r \subseteq \Gamma$ with $\text{Res}(\gamma_r) = r$, and let $\gamma_{\mathcal{D}} \in \mathcal{D}(\Gamma)$. Then the following three situations can occur:

1. $\gamma_r \cap \gamma_{\mathcal{D}} = \emptyset$
2. $\gamma_r \cap \gamma_{\mathcal{D}} \neq \emptyset$ and $\gamma_r \subsetneq \gamma_{\mathcal{D}}$
3. $\gamma_r \subseteq \gamma_{\mathcal{D}}$

Observe, that for each respective situation we have:

1. $\gamma_r \subseteq \Gamma/\gamma_{\mathcal{D}}$ and thus $\Gamma/\gamma_{\mathcal{D}} \in \mathcal{S}_r$
2. $\gamma_r/(\gamma_{\mathcal{D}} \cap \gamma_r) \subseteq \Gamma/\gamma_{\mathcal{D}}$ and $\text{Res}(\gamma_r/(\gamma_{\mathcal{D}} \cap \gamma_r)) = r$ and thus $\Gamma/\gamma_{\mathcal{D}} \in \mathcal{S}_r$
3. $\gamma_r \subseteq \gamma_{\mathcal{D}}$ and thus $\gamma_{\mathcal{D}} \in \mathcal{S}_r$

Finally, applying Proposition 3.46 finishes the proof. ■

Remark 3.50. Corollary 3.49 states, that it is compatible with renormalization Hopf algebra to set all Feynman graphs with a given residue to zero in the corresponding renormalization Hopf algebra.

Remark 3.51. Proposition 3.44 and Proposition 3.45 state that, in the case of the renormalization Hopf algebra associated to a local QFT, special linear combinations of Feynman graph sets $\{\mathcal{S}_n\}_{n=1}^N$, which each individually generate Hopf ideals $\mathfrak{i}_n := (\mathcal{S}_n)_H$, again generate Hopf ideals. These results will be used e.g. in [35] to study the renormalization of general local Quantum Gauge Theories, similar to [38, 39].

$$\omega \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = 2 \qquad \text{Cpl} \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = \lambda \qquad (136g)$$

$$\omega \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = 1 \qquad \text{Cpl} \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = \lambda \qquad (136h)$$

$$\omega \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = 2 \qquad \text{Cpl} \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = \lambda \qquad (136i)$$

$$\omega \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = 2 \qquad \text{Cpl} \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = \lambda \qquad (136j)$$

$$\omega \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = 0 \qquad \text{Cpl} \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = e\lambda \qquad (136k)$$

$$\omega \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = 1 \qquad \text{Cpl} \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = \lambda^2 \qquad (136l)$$

$$\omega \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = 2 \qquad \text{Cpl} \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = \lambda^2 \qquad (136m)$$

$$\omega \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = 2 \qquad \text{Cpl} \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = \lambda^2 \qquad (136n)$$

$$\omega \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = 2 \qquad \text{Cpl} \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = \lambda^2 \qquad (136o)$$

$$\omega \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = 2 \qquad \text{Cpl} \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = \lambda^2 \qquad (136p)$$

$$\omega \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = 2 \qquad \text{Cpl} \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = \lambda^2 \qquad (136q)$$

Theorem 3.53 (Generalized Furry's Theorem). *Consider QGR-QED with the Lagrange density $\mathcal{L}_{\text{QGR-QED}}$ given in Equation (61). Let r be the residue of an amplitude, set $p(r)$ to be the sum of its external photon and photon ghost edges and $f(r)$ to be the number of external fermion edges. Then the corresponding restricted combinatorial Green's functions $G_{\mathbf{c}}^r$ are in the kernel of the unrenormalized Feynman rules $\Phi_{\text{QGR-QED}}(\cdot)$ for each coupling constant grading \mathbf{c} individually if $p(r)$ is odd and $f(r)$ is zero.*

Proof. This proof is the only part of this article where we need to consider explicitly the orientation of fermion edges. We denote by $\mathcal{A}_{\text{QGR-QED}}^{\text{Furry}}$ the set of all amplitudes³⁵ with residues r_i such that $p(r_i)$ is odd and $f(r_i)$ is zero. Furthermore, let

$$\mathcal{C} : \mathcal{G}_{\text{QGR-QED}} \rightarrow \mathcal{G}_{\text{QGR-QED}}, \quad \Gamma \mapsto \Gamma^{\mathcal{C}}, \qquad (137)$$

be the fermion charge-conjugation operator on Feynman graphs, i.e. $\Gamma^{\mathcal{C}}$ is the Feynman graph Γ with the orientation of all fermion lines reversed. Notice that \mathcal{C} is an involution. Moreover,

³⁵The union of the residue set $\mathcal{R}_{\text{QGR-QED}}$ with all possible quantum corrections.

to $G_{\mathbf{c}}^r$, by abuse of notation now considered as a set. To show this claim, we first observe that the photon-fermion-antifermion vertex with an arbitrary number of gravitons attached to it is the only residue which allows to change $p(r)$. Thus, the requirement on $p(r)$ being odd implies that every Feynman graph $\Gamma \in G_{\mathbf{c}}^r$ has at least one fermion loop. Thus, \mathcal{C} applied to any Feynman graph $\Gamma \in G_{\mathbf{c}}^r$ is not the identity and furthermore $\Gamma^{\mathcal{C}} \in G_{\mathbf{c}}^r$, since residues and gradings are preserved by \mathcal{C} . Finally, we claim that for $\Gamma \in G_{\mathbf{c}}^r$ the sum $\Gamma + \Gamma^{\mathcal{C}}$ is in the kernel of the Feynman rules, i.e.

$$\Phi(\Gamma + \Gamma^{\mathcal{C}}) = 0. \quad (144)$$

We conclude this by showing that in this case Γ is build from an odd number of residues belonging to the set $\mathcal{R}_{\text{QGR-QED}}^-$, and thus

$$\Phi(\Gamma^{\mathcal{C}}) = -\Phi(\Gamma) \quad (145)$$

which directly implies Equation (144) by the linearity of the Feynman rules. Indeed, since $\text{Res}(\Gamma) \in \mathcal{A}_{\text{QGR-QED}}^{\text{Furry}}$ we conclude that Γ has to consist of an odd number of photon-fermion-antifermion (modulo an arbitrary number of gravitons) vertices and thus in particular at least one closed fermion loop as noted before. Since the fermion loops are closed (as Γ contains no external fermion edges), the number of fermion propagators equals the number of fermion vertices. Moreover, we remind that photon-fermion-antifermion (modulo an arbitrary number of gravitons) vertices are in the set $\mathcal{R}_{\text{QGR-QED}}^+$, whereas fermion propagators and gravitons-fermion-antifermion vertices are in the set $\mathcal{R}_{\text{QGR-QED}}^-$. But we need to have an odd number of photon-fermion-antifermion (modulo an arbitrary number of gravitons) vertices by the previous argument, Γ needs to have an odd number of residues from the set $\mathcal{R}_{\text{QGR-QED}}^-$ which finishes the proof. \blacksquare

Remark 3.54. Theorem 3.53 also justifies the restriction of connected Feynman graphs to 1PI Feynman graphs in Definition 3.14 and Definition 3.33. More precisely, since the amplitude of the two-point vertex function of a photon and a graviton vanishes, the set of connected Feynman graphs which are not 1PI is a trivial extension of the set of 1PI Feynman graphs.

4 Combinatorial Green's functions, the coproduct structure and obstructions to multiplicative renormalization in QGR-QED

Now, we consider the coproduct structure on the two-loop propagator graphs in QGR-QED. We associate to QGR-QED its renormalization Hopf algebra $\mathcal{H}_{\text{QGR-QED}}$, as was described in Subsection 3.5. Let furthermore $G_{\mathbf{c}}^r$ be the combinatorial Green's function with residue $r \in \mathcal{R}_{\text{QGR-QED}}$ and multi-index $\mathbf{c} \in \mathbb{Z}^2$, as was defined in Definition 3.26. The multi-index \mathbf{c} , introduced in Definition 3.23, is defined such that $\mathbf{c} = (m, n)$ corresponds to a graphs with coupling constants of order $\mathcal{O}(\lambda^m e^n)$, with $m, n \in \mathbb{Z}$. In the following subsections, we present the corresponding Green's functions and their coproducts. We remind, that fermion edges are denoted by --- , photon edges by ~~~~ , graviton edges by ~~~~ , photon-ghost edges by --- and graviton-ghost edges by --- . In this work, we draw oriented Feynman graph edges, i.e. fermion, photon-ghost and graviton-ghost edges, without orientation. This is understood as the sum of all Feynman graphs having all possible orientations.

4.1 Combinatorial Green's functions in QGR-QED

Now, we present the restricted combinatorial Green's functions: Non-symmetric graphs are drawn only once, but with the corresponding multiplicity.³⁶ We remark, that in the realm of two loop propagator graphs only one graph in $G_{(2,2)}^{\bullet}$, Equation (171), contains a problematic subgraph which, however, is captured by Generalized Furry's Theorem, stated in Theorem 3.53, and thus is presented in brackets. Below, we also present the one-loop two- and three-point amplitudes which are set to zero due to Generalized Furry's Theorem.

$$G_{(0,2)}^{\bullet} = - \text{[Diagram: a circle with a wavy line on top and a straight line on bottom]} \quad (146)$$

$$G_{(2,0)}^{\bullet} = - \text{[Diagram: a circle with a wavy line on top and a wavy line on bottom]} \quad (147)$$

$$G_{(0,2)}^{\sim} = -\frac{1}{2} \text{[Diagram: a circle with a wavy line on top and a wavy line on bottom]} \quad (148)$$

$$G_{(2,0)}^{\sim} = - \text{[Diagram: a circle with a wavy line on top and a wavy line on bottom]} - \text{[Diagram: a circle with a dashed line on top and a dashed line on bottom]} \quad (149)$$

$$G_{(0,2)}^{\text{coiled}} = 0 \quad (150)$$

$$G_{(2,0)}^{\text{coiled}} = -\frac{1}{2} \text{[Diagram: a circle with a wavy line on top and a wavy line on bottom]} - \frac{1}{2} \text{[Diagram: a circle with a wavy line on top and a wavy line on bottom]} - \frac{1}{2} \text{[Diagram: a circle with a wavy line on top and a wavy line on bottom]} - \frac{1}{2} \text{[Diagram: a circle with a dashed line on top and a dashed line on bottom]} - \frac{1}{2} \text{[Diagram: a circle with a dashed line on top and a dashed line on bottom]} \quad (151)$$

$$G_{(0,2)}^{\text{dashed}} = 0 \quad (152)$$

$$G_{(2,0)}^{\text{dashed}} = - \text{[Diagram: a circle with a wavy line on top and a dashed line on bottom]} - \text{[Diagram: a circle with a dashed line on top and a wavy line on bottom]} \quad (153)$$

$$G_{(0,2)}^{\text{dotted}} = 0 \quad (154)$$

$$G_{(2,0)}^{\text{dotted}} = - \text{[Diagram: a circle with a dashed line on top and a dashed line on bottom]} - \text{[Diagram: a circle with a dashed line on top and a dashed line on bottom]} \quad (155)$$

$$G_{(0,2)}^{\text{triangle}} = \text{[Diagram: a triangle with a wavy line on the left and a straight line on the right]} \quad (156)$$

$$G_{(2,0)}^{\text{triangle}} = \text{[Diagram: a triangle with a wavy line on the left and a wavy line on the right]} + 2 \text{[Diagram: a triangle with a wavy line on the left and a wavy line on the right]} + 2 \text{[Diagram: a triangle with a wavy line on the left and a wavy line on the right]} + \text{[Diagram: a triangle with a wavy line on the left and a wavy line on the right]} \quad (157)$$

$$G_{(0,2)}^{\text{dotted triangle}} = 0 \quad (158)$$

³⁶The minus signs for propagator graphs are due to Definition 3.26, c.f. Remark 3.27.

$$\begin{aligned}
G_{(2,0)} = & \text{[Diagram 1]} + \text{[Diagram 2]} + \text{[Diagram 3]} + \text{[Diagram 4]} + \text{[Diagram 5]} \\
& + \text{[Diagram 6]} + \text{[Diagram 7]}
\end{aligned}
\tag{159}$$

$$G_{(0,2)} = \text{[Diagram 1]} + \text{[Diagram 2]} + 2 \text{[Diagram 3]}
\tag{160}$$

$$G_{(2,0)} = \text{[Diagram 1]} + \text{[Diagram 2]} + 2 \text{[Diagram 3]} + \frac{1}{2} \text{[Diagram 4]}
\tag{161}$$

$$G_{(0,2)} = \text{[Diagram 1]} + \frac{2}{2} \text{[Diagram 2]}
\tag{162}$$

$$\begin{aligned}
G_{(2,0)} = & \text{[Diagram 1]} + \text{[Diagram 2]} + \text{[Diagram 3]} + \text{[Diagram 4]} + \text{[Diagram 5]} \\
& + \frac{1}{2} \text{[Diagram 6]} + 2 \text{[Diagram 7]}
\end{aligned}
\tag{163}$$

$$G_{(0,2)} = 0
\tag{164}$$

$$\begin{aligned}
G_{(2,0)} = & \text{[triangle with wavy lines]} + \text{[triangle with wavy and zigzag lines]} + \text{[triangle with wavy and curly lines]} + \text{[triangle with wavy and dashed lines]} + \text{[triangle with wavy and dotted lines]} \\
& + \frac{3}{2} \text{[loop with wavy lines]} + \frac{3}{2} \text{[loop with zigzag lines]} + \frac{3}{2} \text{[loop with curly lines]} + \frac{3}{2} \text{[loop with dashed lines]} \\
& + \frac{3}{2} \text{[loop with dotted lines]}
\end{aligned} \tag{165}$$

$$G_{(0,2)} = 0 \tag{166}$$

$$\begin{aligned}
G_{(2,0)} = & \text{[triangle with wavy and dashed lines]} + \text{[triangle with wavy and curly lines]} + \text{[triangle with wavy and dotted lines]} + \text{[triangle with wavy and zigzag lines]} + \frac{1}{2} \text{[loop with wavy lines]} \\
& + 2 \text{[loop with wavy and dashed lines]} + 2 \text{[loop with wavy and curly lines]}
\end{aligned} \tag{167}$$

$$G_{(0,2)} = 0 \tag{168}$$

$$\begin{aligned}
G_{(2,0)} = & \text{[triangle with wavy and dashed lines]} + \text{[triangle with wavy and curly lines]} + \text{[triangle with wavy and dotted lines]} + \text{[triangle with wavy and zigzag lines]} + \frac{1}{2} \text{[loop with wavy lines]} \\
& + 2 \text{[loop with wavy and dashed lines]} + 2 \text{[loop with wavy and curly lines]}
\end{aligned} \tag{169}$$

$$G_{(0,4)} = \text{[loop with wavy lines]} - \frac{1}{2} \text{[loop with wavy lines]} - \text{[loop with wavy lines]} \tag{170}$$

$$\begin{aligned}
G_{(2,2)}^{\bullet} &= -2 \left[\text{Diagram 1} \right] - 2 \left[\text{Diagram 2} \right] - 2 \left[\text{Diagram 3} \right] \\
&\quad \left(- \left[\text{Diagram 4} \right] \right) - \left[\text{Diagram 5} \right] - \left[\text{Diagram 6} \right] \\
&\quad - \left[\text{Diagram 7} \right] - \left[\text{Diagram 8} \right] - 2 \left[\text{Diagram 9} \right]
\end{aligned}
\tag{171}$$

$$\begin{aligned}
G_{(4,0)}^{\bullet} &= - \left[\text{Diagram 1} \right] - \left[\text{Diagram 2} \right] - \frac{1}{2} \left[\text{Diagram 3} \right] \\
&\quad - \frac{1}{2} \left[\text{Diagram 4} \right] - \frac{1}{2} \left[\text{Diagram 5} \right] - \frac{1}{2} \left[\text{Diagram 6} \right] \\
&\quad - \frac{1}{2} \left[\text{Diagram 7} \right] - \left[\text{Diagram 8} \right] - \left[\text{Diagram 9} \right] \\
&\quad - \frac{2}{2} \left[\text{Diagram 10} \right] - 2 \left[\text{Diagram 11} \right]
\end{aligned}
\tag{172}$$

$$G_{(0,4)}^{\bullet} = -\frac{1}{2} \left[\text{Diagram 1} \right] - \left[\text{Diagram 2} \right]
\tag{173}$$

$$\begin{aligned}
G_{(2,2)}^{\bullet} &= -\frac{1}{2} \left[\text{Diagram 1} \right] - 2 \left[\text{Diagram 2} \right] - \left[\text{Diagram 3} \right] \\
&\quad - \frac{1}{2} \left[\text{Diagram 4} \right] - \frac{2}{2} \left[\text{Diagram 5} \right] - 2 \left[\text{Diagram 6} \right] \\
&\quad - \frac{2}{2} \left[\text{Diagram 7} \right]
\end{aligned}
\tag{174}$$

$$\begin{aligned}
G_{(4,0)}^{\bullet} &= -\frac{1}{2} \text{[diagram 1]} - \frac{2}{2} \text{[diagram 2]} - \frac{1}{2} \text{[diagram 3]} \\
&- \frac{2}{2} \text{[diagram 4]} - \frac{2}{2} \text{[diagram 5]} - \frac{2}{2} \text{[diagram 6]} \\
&- \frac{1}{2} \text{[diagram 7]} - \frac{2}{2} \text{[diagram 8]} - \frac{2}{2} \text{[diagram 9]} \\
&- \frac{1}{2} \text{[diagram 10]} - \frac{2}{2} \text{[diagram 11]} - \frac{1}{2} \text{[diagram 12]} \\
&- \text{[diagram 13]} - \text{[diagram 14]} - \text{[diagram 15]} \\
&- \frac{1}{2} \text{[diagram 16]} - \frac{1}{2} \text{[diagram 17]} - \frac{1}{2} \text{[diagram 18]} \\
&- \frac{1}{2} \text{[diagram 19]} - \frac{1}{2} \text{[diagram 20]} - \text{[diagram 21]} \\
&- \text{[diagram 22]} - \text{[diagram 23]} - \text{[diagram 24]} \\
&- \frac{2}{4} \text{[diagram 25]} - \frac{2}{4} \text{[diagram 26]} \\
&- \frac{1}{4} \text{[diagram 27]} - \frac{2}{4} \text{[diagram 28]} \\
&- \frac{2}{4} \text{[diagram 29]} - 2 \text{[diagram 30]} - 2 \text{[diagram 31]} \\
&- 2 \text{[diagram 32]} - \frac{2}{2} \text{[diagram 33]} - \frac{2}{2} \text{[diagram 34]} \\
&- \frac{2}{2} \text{[diagram 35]} - \frac{2}{2} \text{[diagram 36]} - \frac{2}{2} \text{[diagram 37]} \\
&- 2 \text{[diagram 38]} - 2 \text{[diagram 39]} - 2 \text{[diagram 40]} \\
&- 2 \text{[diagram 41]}
\end{aligned}$$

(178)

$$\begin{aligned}
G_{(4,0)} &= \text{[Diagram 1]} - 2 \text{[Diagram 2]} - \text{[Diagram 3]} \\
&- 2 \text{[Diagram 4]} - \text{[Diagram 5]} - \text{[Diagram 6]} \\
&- \text{[Diagram 7]} - \text{[Diagram 8]} - \text{[Diagram 9]} \\
&- \text{[Diagram 10]} - \text{[Diagram 11]} - \text{[Diagram 12]} \\
&- \frac{1}{2} \text{[Diagram 13]} - \frac{1}{2} \text{[Diagram 14]} - \frac{1}{2} \text{[Diagram 15]} \\
&- \frac{1}{2} \text{[Diagram 16]} - \frac{1}{2} \text{[Diagram 17]} - 2 \text{[Diagram 18]} \\
&- \text{[Diagram 19]} - 2 \text{[Diagram 20]} - 2 \text{[Diagram 21]} \\
&- 2 \text{[Diagram 22]} - 2 \text{[Diagram 23]} - \frac{2}{2} \text{[Diagram 24]}
\end{aligned} \tag{184}$$

Additionally, we also present the one-loop two- and three-point amplitudes which are set to zero due to Generalized Furry's Theorem, stated in Theorem 3.53:

$$G_{(1,1)}^{\text{[Diagram 1]}} = - \text{[Diagram 2]} \tag{185}$$

$$G_{(0,2)}^{\text{[Diagram 1]}} = \text{[Diagram 2]} \tag{186}$$

$$G_{(2,1)}^{\text{[Diagram 1]}} = \text{[Diagram 2]} \tag{187}$$

4.2 The coproduct structure in QGR-QED

We obtain the following reduced coproduct structure of the two-loop propagator combinatorial Green's functions:³⁷

³⁷We display only the non-vanishing restricted combinatorial Green's functions, c.f. Subsection 4.1.

$$\Delta' \left(G_{(0,4)} \right) = \left(-G_{(0,2)} - G_{(0,2)}^{\text{wavy}} + 2G_{(0,2)}^{\text{triangle}} \right) \otimes G_{(0,2)} \quad (188)$$

$$\Delta' \left(G_{(2,2)} \right) = \left(-G_{(2,0)} - G_{(2,0)}^{\text{wavy}} + 2G_{(2,0)}^{\text{triangle}} \right) \otimes G_{(0,2)} \quad (189)$$

$$+ \left(-G_{(0,2)} + 2G_{(0,2)}^{\text{triangle}} \right) \otimes G_{(2,0)}$$

$$\Delta' \left(G_{(4,0)} \right) = \left(-G_{(2,0)} - G_{(2,0)}^{\text{double wavy}} + 2G_{(2,0)}^{\text{triangle}} \right) \otimes G_{(2,0)} \quad (190)$$

$$\Delta' \left(G_{(0,4)}^{\text{wavy}} \right) = \left(-2G_{(0,2)} + 2G_{(0,2)}^{\text{triangle}} \right) \otimes G_{(0,2)}^{\text{wavy}} \quad (191)$$

$$\Delta' \left(G_{(2,2)}^{\text{wavy}} \right) = \left(-2G_{(2,0)} + 2G_{(2,0)}^{\text{triangle}} \right) \otimes G_{(0,2)}^{\text{wavy}} \quad (192)$$

$$+ \left(-G_{(0,2)}^{\text{wavy}} + 2G_{(0,2)}^{\text{triangle}} \right) \otimes G_{(2,0)}^{\text{wavy}}$$

$$\begin{aligned}
\Delta' \left(G_{(4,0)}^{\text{wavy} \bullet \text{wavy}} \right) &= \left(-G_{(2,0)}^{\text{wavy} \bullet \text{wavy}} - G_{(2,0)}^{\text{wavy} \bullet \text{wavy}} + 2G_{(2,0)}^{\text{wavy} \bullet \text{wavy}} \right) \otimes \text{wavy circle} \\
&+ \left(-G_{(2,0)}^{\text{dashed} \bullet \text{dashed}} - G_{(2,0)}^{\text{dashed} \bullet \text{dashed}} + 2G_{(2,0)}^{\text{dashed} \bullet \text{dashed}} \right) \otimes \text{dashed circle}
\end{aligned} \tag{193}$$

$$\Delta' \left(G_{(0,4)}^{\text{wavy} \bullet \text{wavy}} \right) = 0 \tag{194}$$

$$\begin{aligned}
\Delta' \left(G_{(2,2)}^{\text{wavy} \bullet \text{wavy}} \right) &= \left(-2G_{(0,2)}^{\text{solid} \bullet \text{solid}} + 2G_{(0,2)}^{\text{wavy} \bullet \text{wavy}} \right) \otimes \frac{1}{2} \text{solid circle} \\
&+ \left(-2G_{(0,2)}^{\text{wavy} \bullet \text{wavy}} + 2G_{(0,2)}^{\text{wavy} \bullet \text{wavy}} \right) \otimes \frac{1}{2} \text{wavy circle}
\end{aligned} \tag{195}$$

$$\begin{aligned}
\Delta' \left(G_{(4,0)}^{\text{wavy}\bullet\text{wavy}} \right) &= \left(-2G_{(2,0)}^{\text{wavy}\bullet\text{wavy}} + 2G_{(2,0)}^{\text{wavy}\bullet\text{wavy}} \right) \otimes \frac{1}{2} \text{wavy}\bullet\text{wavy} \text{ circle} \\
&+ \left(-2G_{(2,0)}^{\text{wavy}\bullet\text{wavy}} + 2G_{(2,0)}^{\text{wavy}\bullet\text{wavy}} \right) \otimes \frac{1}{2} \text{wavy}\bullet\text{wavy} \text{ star} \\
&+ \left(-2G_{(2,0)}^{\text{wavy}\bullet\text{wavy}} + 2G_{(2,0)}^{\text{wavy}\bullet\text{wavy}} \right) \otimes \frac{1}{2} \text{wavy}\bullet\text{wavy} \text{ circle} \\
&+ \left(-2G_{(2,0)}^{\text{wavy}\bullet\text{wavy}} + 2G_{(2,0)}^{\text{wavy}\bullet\text{wavy}} \right) \otimes \frac{1}{2} \text{wavy}\bullet\text{wavy} \text{ dashed circle} \\
&+ \left(-2G_{(2,0)}^{\text{wavy}\bullet\text{wavy}} + 2G_{(2,0)}^{\text{wavy}\bullet\text{wavy}} \right) \otimes \frac{1}{2} \text{wavy}\bullet\text{wavy} \text{ dashed circle}
\end{aligned} \tag{196}$$

$$\Delta' \left(G_{(0,4)}^{\text{dashed}\bullet\text{dashed}} \right) = 0 \tag{197}$$

$$\Delta' \left(G_{(2,2)}^{\text{dashed}\bullet\text{dashed}} \right) = -G_{(0,2)}^{\text{wavy}\bullet\text{wavy}} \otimes \text{dashed}\bullet\text{dashed} \text{ star} \tag{198}$$

$$- 0 \otimes \text{dashed}\bullet\text{dashed} \text{ circle} \tag{199}$$

$$\tag{200}$$

$$\begin{aligned}
\Delta' \left(G_{(4,0)}^{\text{dashed}\bullet\text{dashed}} \right) &= \left(-G_{(2,0)}^{\text{wavy}\bullet\text{wavy}} - G_{(2,0)}^{\text{dashed}\bullet\text{dashed}} + 2G_{(2,0)}^{\text{wavy}\bullet\text{wavy}} \right) \otimes \text{dashed}\bullet\text{dashed} \text{ star} \\
&+ \left(-G_{(2,0)}^{\text{wavy}\bullet\text{wavy}} - G_{(2,0)}^{\text{dashed}\bullet\text{dashed}} + 2G_{(2,0)}^{\text{wavy}\bullet\text{wavy}} \right) \otimes \text{dashed}\bullet\text{dashed} \text{ circle}
\end{aligned} \tag{201}$$

$$\Delta' \left(G_{(0,4)}^{\text{dashed}\bullet\text{dashed}} \right) = 0 \tag{202}$$

$$\Delta' \left(G_{(2,2)}^{\text{dashed}\bullet\text{dashed}} \right) = -G_{(0,2)}^{\text{wavy}\bullet\text{wavy}} \otimes \text{dashed}\bullet\text{dashed} \text{ star} \tag{203}$$

$$-0 \otimes \text{---} \circlearrowleft \text{---} \quad (204)$$

$$(205)$$

$$\Delta' \left(G_{(4,0)} \right) = \left(-G_{(2,0)}^{\text{---} \bullet \text{---}} - G_{(2,0)}^{\text{---} \bullet \text{---}} + 2G_{(2,0)}^{\text{---} \bullet \text{---}} \right) \otimes \text{---} \circlearrowleft \text{---} \quad (206)$$

$$+ \left(-G_{(2,0)}^{\text{---} \bullet \text{---}} - G_{(2,0)}^{\text{---} \bullet \text{---}} + 2G_{(2,0)}^{\text{---} \bullet \text{---}} \right) \otimes \text{---} \circlearrowleft \text{---}$$

4.3 Obstructions to multiplicative renormalization in QGR-QED

Using Definition 3.28, we conclude that multiplicative renormalization is possible if the following generalized Slavnov-Taylor identities hold on the level of Feynman rules, i.e. the divergent contributions from each form factor of the corresponding integral expressions coincide:

$$\left(-G_{(2,0)}^{\text{---} \bullet \text{---}} - G_{(2,0)}^{\text{---} \bullet \text{---}} + 2G_{(2,0)}^{\text{---} \bullet \text{---}} \right) = \left(-G_{(2,0)}^{\text{---} \bullet \text{---}} - G_{(2,0)}^{\text{---} \bullet \text{---}} + 2G_{(2,0)}^{\text{---} \bullet \text{---}} \right) \quad (207)$$

$$\left(-2G_{(0,2)}^{\text{---} \bullet \text{---}} + 2G_{(0,2)}^{\text{---} \bullet \text{---}} \right) = \left(-2G_{(0,2)}^{\text{---} \bullet \text{---}} + 2G_{(0,2)}^{\text{---} \bullet \text{---}} \right) \quad (208)$$

$$\left(-2G_{(2,0)}^{\text{---} \bullet \text{---}} + 2G_{(2,0)}^{\text{---} \bullet \text{---}} \right) = \left(-2G_{(2,0)}^{\text{---} \bullet \text{---}} + 2G_{(2,0)}^{\text{---} \bullet \text{---}} \right) \quad (209)$$

$$= \left(-2G_{(2,0)}^{\text{---} \bullet \text{---}} + 2G_{(2,0)}^{\text{---} \bullet \text{---}} \right) \quad (210)$$

$$= \left(\begin{array}{c} \text{---} \bullet \text{---} \\ -2G_{(2,0)} \\ \text{---} \bullet \text{---} \\ + 2G_{(2,0)} \end{array} \right) \quad (211)$$

$$= \left(\begin{array}{c} \text{---} \bullet \text{---} \\ -2G_{(2,0)} \\ \text{---} \bullet \text{---} \\ + 2G_{(2,0)} \end{array} \right) \quad (212)$$

$$G_{(0,2)}^{\text{wavy}} = 0 \quad (213)$$

$$G_{(2,0)}^{\text{dotted}} = G_{(2,0)}^{\text{dashed}} \quad (214)$$

$$\left(\begin{array}{c} \text{---} \bullet \text{---} \\ -G_{(2,0)}^{\text{wavy}} \\ \text{---} \bullet \text{---} \\ + 2G_{(2,0)} \end{array} \right) = \left(\begin{array}{c} \text{---} \bullet \text{---} \\ -G_{(2,0)}^{\text{dotted}} \\ \text{---} \bullet \text{---} \\ + 2G_{(2,0)} \end{array} \right) \quad (215)$$

The study of these relations on a general level will be done in future work [35]. We only remark here, that Equation (213) requires that the one-loop photon propagator with a fermion loop is convergent. This is not true, and thus shows that QGR-QED is not multiplicative renormalizable using only two coupling constants (one for the electric charge and one for the gravitational coupling). The reason for this problem is, that in QED no ghost-vertices and thus ghost-loops are present. This problem could be solved by either introduce two gravitational couplings — one for the pure gravity part and one for the gravity-matter coupling. Or, and this is the physically more appealing solution, by concluding that it is unphysical to consider the coupling to Spinor Electrodynamics alone and rather consider the coupling to the whole Electroweak Sector. Then, the corresponding gauge boson, ghost, Higgs and Goldstone interactions could resolve this problem.

5 Conclusion

In this article we considered Quantum General Relativity coupled to Quantum Electrodynamics (QGR-QED). First, we introduced the necessary differential geometric background and the Lagrange density of QGR-QED in Section 2. Then, in Section 3 we introduced Hopf algebras in general and the Connes-Kreimer renormalization Hopf algebra in particular. Furthermore, we discussed a problem which can occur when associating the renormalization Hopf algebra to a given local QFT and discuss possible solutions. Moreover, we examine Hopf ideals inside the renormalization Hopf algebra which represent the symmetries compatible with renormalization. Next, the application of these general results to QGR-QED is discussed. In particular, a generalization of Furry's Theorem including external gravitons and graviton ghosts is formulated and proved in Theorem 3.53. This is in particular useful, since the calculations showed that, besides from pure self-loop Feynman graphs which vanish in kinematic renormalization schemes, these are the only graphs which need to be set to zero when constructing the renormalization Hopf algebra of QGR-QED for two-loop propagator graphs. Then, in Section 4 we present all combinatorial Green's functions for the one- and two-loop propagator graphs and the one-loop

three-point functions. Then, their coproduct structure is presented, for which the coproduct of 155 Feynman graphs has been computed. Using this result we present the obstructions to multiplicative renormalization. We conclude in particular, that it is physically not sensible to consider the coupling to Spinor Electrodynamics alone and suggest to rather include the whole Electroweak Sector.

A couple of points were postponed to future work, since they went beyond the scope of this article. This includes a detailed treatment of the Feynman rules of QGR-QED which will be examined in [26], as was mentioned in Remark 2.25. Additionally, the obstructions to multiplicative renormalization, as discussed in Subsection 4.3, come from a generalization of the corresponding Ward-Takahashi and 't Hooft-Slavnov-Taylor identities [18, 19, 20, 21, 22] which will be examined in general in [35]. Finally, it is also interesting if it is possible to define a Corolla polynomial [40, 41, 42, 43] creating the Feynman integrands for QGR-QED.

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