

ON KDV CHARACTERS IN LARGE  $c$  CFTs

---

**Enrico M. Brehm <sup>$\beta$</sup>  and Diptarka Das <sup>$\beta,\mu$</sup>** 

*$\beta$ ) Max Planck Institut für Gravitationsphysik,  
Albert-Einstein-Institut,  
Potsdam-Golm, D-14476, Germany.*

*$\mu$ ) Indian Institute of Technology, Kanpur,  
Department of Physics,  
Kanpur, 208016, India.*

*E-mail:* [brehm@aei.mpg.de](mailto:brehm@aei.mpg.de), [didas@iitk.ac.in](mailto:didas@iitk.ac.in)

ABSTRACT: Two-dimensional conformal field theories with just Virasoro symmetry are endowed with integrable structure. We review how to construct the integrable charges in a two-dimensional conformal field theory and how to relate them to the charges of quantum Sinh-Gordon theory when  $c > 25$ . We then explicitly calculate the single charge characters in the large  $c$  limit for all even charges and thereby reveal how their degeneracies grow within one module. This, in particular, allows us to approximate the characters in the limit of small chemical potential, which source the respective charges. The latter give us insights into possible transformation properties of the characters. We also comment on the full generalized Gibbs ensemble and approximations to pure states.

---

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>From CFT to quantum Sinh-Gordon theory</b>	<b>5</b>
<b>3</b>	<b>KdV characters</b>	<b>8</b>
3.1	Warm-up: Q-character at large central charge	9
3.1.1	Vacuum Q character	9
3.1.2	Primary Q character	10
3.2	Higher charge characters	11
3.2.1	$\hat{K}_k$ vacuum character	12
3.2.2	$\hat{K}_k$ primary characters	14
3.3	Comment on the full GGE	14
3.3.1	Constraints on $\mu_l$	14
<b>4</b>	<b>Connection between high and low chemical potentials</b>	<b>15</b>
4.1	Connection to very high energies	16
<b>5</b>	<b>Comment on approximations of pure states</b>	<b>17</b>
<b>6</b>	<b>Conclusions</b>	<b>19</b>
<b>A</b>	<b>Classical integrals of motion</b>	<b>22</b>
A.1	Integral of motions for the modified KdV equation	22
A.2	Integral of motions for the KdV equation	22
<b>B</b>	<b>Quantum KdV charge of <math>2d</math> CFTs</b>	<b>22</b>
B.1	Composite operators	22
B.2	The charges	23
B.3	Some explicit checks on large $c$ eigenvalues of Q	23
B.4	Global Q-character	25
B.4.1	Closed forms for Global Q-characters	26
<b>C</b>	<b>Saddle</b>	<b>27</b>

---

# 1 Introduction

At the heart of the solvability of two-dimensional conformal field theories (CFTs) lies the infinite dimensional Virasoro algebra. This permits construction of an infinite set of mutually commuting charges, first exhibited in [1] and also [2]. A deep connection between CFTs and integrable theories arises when quantising soliton equations (specifically, the Sine-Gordon equation). The charges themselves are constructed as integrals over (finite) polynomials of the energy momentum tensor and as such are polynomial functions of the Virasoro generators. Often one calls them the Korteweg-de-Vries (KdV) charges of the conformal field theory,  $\hat{K}_n$ . Even though the infinite set of these charges in [1] are obtained for  $c \leq 1$ , there exists a proper analytic continuation to large  $c$  CFTs, as we show below.

All these charges commute with the Hamiltonian of the CFT,  $[H_{\text{CFT}}, \hat{K}_n] = 0$ . Thus they generate a rigid infinite dimensional abelian symmetry within the larger symmetry generated by the full enveloping Virasoro algebra. Therefore, the KdV charges are conserved quantities of the two-dimensional CFT. Due to their connection to the theory of integrable systems one can apply techniques thereof in the context of CFT. This was done in the series of remarkable papers [3–5], where they map the problem of finding the exact eigenvalues of all these charges in the  $c \leq 1$  CFTs to that of solving the pure quantum mechanical problem of a single particle in a specific potential. Moreover, under specific relevant perturbations, the integrability structure is retained. This has been utilized to find the exact S-matrix off-criticality [6, 7] and thereby provides additional motivation to understand the charges in CFTs.

The KdV charges can also be used to define a generalized partition function, or the Generalized Gibbs Ensembles (GGE). When a critical system is in equilibrium with some environment such partition functions naturally arise. If we do not specify the allowed interactions between the system and the environment then one needs to take all conserved quantities into account. Such a generic equilibrium state of the system with chemical potentials for every possible conserved quantity can be written

$$\rho(\{q, y_i\}) = q^{H_{\text{CFT}}} \prod_{i \geq 2} y_i^{\hat{K}_i}, \quad (1.1)$$

where  $q = e^{-\beta}$  with the inverse temperature  $\beta$ , and  $y_i \equiv e^{-\mu_i}$  with the chemical potential  $\mu_i$ . The GGE itself is then given by

$$Z(\{q, y_i\}) = \text{Tr} [\rho(\{q, y_i\})] = \text{Tr}_{[0]} [\rho(\{q, y_i\})] + \sum_p \text{Tr}_{[p]} [\rho(\{q, y_i\})], \quad (1.2)$$

where the first trace is over the full spectrum of the CFT, and  $\text{Tr}_{[p]}$  stands for the trace over a primary module  $[p]$ , with  $[0]$  denoting the vacuum module. The subtraces over specific modules will naturally be called the GGE characters.

Purely thermal states are obviously given by  $\rho(\{q, y_i = 1\})$ , i.e. all chemical potentials have to vanish. In two dimensional CFT, the thermal ensemble – or partition function – has a nice geometric description. If the theory is defined on a circle of length  $L$  then the thermal ensemble is the partition function of the theory on a rectangular torus with modular parameter  $\tau = i\beta/L$ . Furthermore, if one demands that the CFT is well defined on a torus then its partition function must be invariant under modular transformations

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}). \quad (1.3)$$

One of its generators is the so-called  $S$ -transformation  $\tau \mapsto -1/\tau$ . On the level of temperature, this means that the partition function must be invariant under  $\beta \mapsto \frac{L^2}{\beta}$  which directly relates different temperature regimes, in particular the partition function at very low temperature, where it is basically given by the vacuum result, to the very high temperature partition function, where high energy states dominate. This allows one to bootstrap the theory and compute universal results on the growth of its high energy spectrum [8, 9]. One can also bootstrap the CFT by considering correlation functions on the torus, which are not invariant but transform covariantly under modular transformations. This allows to obtain universal information about asymptotic matrix elements/OPE coefficients in 2d CFT [10–12].

A natural question while considering GGEs is whether or not similar features arise for the KdV charges. Namely, *is there a transformation  $y_i \mapsto \tilde{y}_i$  which relates the different regimes of the GGE and correlation functions in the corresponding state?* If such a transformation exists, can it be used to bootstrap the CFT? A related second order question is whether there is a geometrical description for GGEs such that  $y_i$  plays the role of a moduli. A motivation for this expectation comes from considering classical integrable systems. Riemann surfaces play an ubiquitous role in the analytic study of integrable systems. Solutions of the integrable hierarchy are related to finding sections of an eigenvector bundle over a Riemann surface [13]. The hope is that this classical connection also reflects itself in the GGE of the quantum theory.

Considerations of the GGE as a possible generalization of thermalization of pure states and in particular the Eigenstate Thermalization Hypothesis (ETH) also provides yet another strong motivation. It is widely believed that in many non-integrable quantum theories generic pure states look approximately thermal after waiting long enough. This, in particular, implies that static states, i.e. energy eigenstates, can be approximated by thermal states. The question arising here is whether generic pure states in integrable systems might be approximated by GGE states, rather than thermal states, when waiting long enough. This then implies that generic energy eigenstates would always look like states in equilibrium with respect to all the conserved charges of the system.

In this article we take small steps into the direction of addressing the above questions. In particular, we compute the GGE characters in the large central charge limit. Recently in [14–16], aspects of GGE was studied in the regime where the chemical potentials are parametrically smaller than the inverse temperature, which itself is small enough to probe energies much larger than the central charge. Our analysis is in a complimentary regime,  $\mu_l \gg \beta \gg 1/(c^{1-\frac{1}{2l}})$ . Large  $c$  is also the *classical* limit of the corresponding soliton theory, and the integrable charges can be computed by a relatively easy recursive formula. In this limit it is possible to diagonalise all charges simultaneously and give their eigenvalues in a closed form. This allows us to give good estimates on the large and small chemical potential regimes, such that possible transformations are tractable.

The article is organized as follows. In §2 we review the connection between quantum Sinh-Gordon theory and conformal field theory. Following [1] we explicitly demonstrate the analytical continuation to central charges  $c > 1$ . Our analysis reveals that the Feign-Fuchs form of the energy momentum tensor, which is needed to draw the connection to quantum integrable theory, is closely related to the bound state sectors of quantum Liouville theory that describes two-dimensional non-perturbative gravity. We will hopefully deepen the understanding of this connection in future work. In §3 we compute the GGE characters in mixed states with a single non-zero chemical potential at parametrically large central charge. We first concentrate on the second integrable charge and then generalize to higher ones. We again want to point out that our calculation is valid in regimes, where the chemical potentials are such that the eigenvalues that contribute dominantly to the GGE character are small compared to the central charge. In the following §4 we comment on possible transformations that relate regimes of high and low chemical potentials. For this we analyze how the charge degeneracy grows within a module and perform the sum over the vacuum module for small charges by a integral evaluated via the saddle point approximation. We also compare the result to the full GGE at parametrically small chemical potentials as considered in [14, 15]. In both cases it shows qualitatively the same behaviour. In particular this also matches the well known results of thermal states and might indicate that there indeed is some transformation that relates high and low chemical potentials. However, the results show that transformation of the chemical potentials of higher integrable charges cannot simply be the standard modular transformations and it is not clear to us if there might exist a geometric interpretation. The following §5 we shortly comment on the possible generalization of ETH in the presence of integrable charges. In §6 we conclude on our findings. In the appendices we give specific formulæ to compute the classical and quantum KdV charges, give some explicit checks on the large  $c$  eigenvalues of the second KdV charge, and compute the global character of the second charge which appears as a factor of the GGE characters in a primary module.

## 2 From CFT to quantum Sinh-Gordon theory

The connection between conformal and integrable field theories has a long history. We want to go back to some of the earliest works and want to briefly review how one can connect two-dimensional conformal field theories with models like Sinh-Gordon theory [1]. Although the integrable charges themselves can be constructed for any 2d CFT the latter connection is not always present. However, only with this connection it becomes clear why large  $c$  indeed is a *classical* limit and how the quantum integrable structure is related to the classical integrable structure. This connection is what gives the idea to search for a geometric meaning of the chemical potentials. In what follows we do, however, not concentrate on the  $c < 1$  case as in [1], but do a analytic continuation to  $c > 1$ . In fact it turns out that the theories we have to consider have  $c \geq 25$ .

Let us consider some conformal field theory and assume that we can write its energy momentum tensor in Feign-Fuchs (or Coulomb-Gas) form

$$T^+(z) \equiv \frac{1}{\hbar} \left( \frac{1}{2} : (D\phi)^2 : - \lambda D^2 \phi + \frac{\lambda^2}{2} \right), \quad (2.1)$$

where  $\lambda$  is some real constant,  $D = iz\partial_z$  with  $z$  the compactified coordinate on the plane,  $z = e^{\tau+i\sigma}$ , and

$$\phi(z) = iq - i\alpha_0 \log(z) + i \sum_{n \neq 0} \frac{\alpha_n}{n} z^{-n} \quad (2.2)$$

is a generalized free field whose modes obey the standard commutation relations

$$[\alpha_n, \alpha_m] = \hbar n \delta_{n+m}, \quad [q, \alpha_n] = i\hbar \delta_n. \quad (2.3)$$

One can show that this energy momentum tensor corresponds to a Virasoro algebra with central charge  $c = 1 + \frac{12\lambda^2}{\hbar}$ . Its generators can be written as

$$\hbar L_n^+ = \frac{1}{2} \sum_p : \alpha_p \alpha_{n-p} : + i\lambda n \alpha_n + \frac{\lambda^2}{2} \delta_n. \quad (2.4)$$

We want to follow the logic of [1] and define a second energy momentum tensor as

$$T^-(z) \equiv \frac{1}{\hbar} \left( \frac{1}{2} : (D\phi)^2 : + \lambda D^2 \phi + \frac{\lambda^2}{2} \right). \quad (2.5)$$

However, in contrast to them, we define hermitian conjugation as  $(L_n^+)^{\dagger} = L_n^-$ , for which we need  $\alpha_n^{\dagger} = \alpha_n$ . Note, that by this we loose the direct connection between asymptotic in and out states.

We now shortly want to comment on the similarities of the present construction to Liouville theory. Its energy momentum tensor in a strip can actually be written

exactly as in (2.1) where the generalized field  $\phi$  can be expressed in terms of the Liouville field  $\varphi$  (see e.g. [18]). When considering bound states in the Liouville theory a simple connection between in and out states by hermitian conjugation, as in the case of scattering states, is now longer possible [18]. It seems reasonable that the integrable structure that we want to present here, is also visible in the transition amplitudes of bound states in Liouville theory on a strip.

Let us now consider the dressed Vertex operators

$$V^\pm = z^{-\beta\lambda}:e^{\pm\beta\phi}:, \quad \beta \in \mathbb{R}, \quad (2.6)$$

with conformal weight  $h = -\frac{\hbar}{2}\beta^2 + \beta\lambda$  w.r.t.  $T^\pm$  respectively. Now, to draw the connection between the CFT and the quantum Sinh-Gordon one needs  $h = 1$  as we will see later. The secondary field  $z^h V^\pm$  then naturally converges to  $e^{\pm\beta\phi}$  in the classical limit  $\hbar \rightarrow 0$ . Therefore, the operator

$$\hat{H} = \frac{1}{2\pi i\beta^2} \oint \frac{dz}{z} \left( 1 - \frac{1}{2} (z^{1-\beta\lambda}:e^{\beta\phi}: + z^{1-\beta\lambda}:e^{-\beta\phi}:) \right) \quad (2.7)$$

converges for  $\hbar \rightarrow 0$  to the classical Hamiltonian of Sinh-Gordon theory

$$H = \frac{1}{2\pi\beta^2} \int_0^{2\pi} d\sigma 1 - \cosh \beta\phi, \quad (2.8)$$

which, with the identification  $u = \frac{\beta}{2}\partial_\sigma\phi$ , canonically generates the Sinh-Gordon equation<sup>1</sup>

$$\partial_t\partial_\sigma\phi = \frac{2}{\beta}\partial_t u = \{u, H\} = -\frac{1}{\beta}\sinh \beta\phi. \quad (2.9)$$

An important point now is that there exist an infinite set of polynomial conserved quantities  $I_n$ , defined in (A.1), which is the same as for the modified Korteweg-de-Vries (mKdV) equation

$$u_t = u_{\sigma\sigma\sigma} - 6u^2u_\sigma. \quad (2.10)$$

Being a conserved quantity in particular means that  $\{I_n, H\} = 0$ .

Another observation is that the analytic continuation of the classical Miura transform

$$v = u^2 \pm u_\sigma \quad (2.11)$$

maps any solution of the above mKdV equation to a solution of the KdV equation

$$v_t = v_{\sigma\sigma\sigma} - 6vv_\sigma, \quad (2.12)$$

which also gives rise to an infinite set of polynomial conserved quantities  $K_n$  as defined in (A.3). They are connected to the conserved quantities of the mKdV by

$$K_n(v = u^2 \pm u_\sigma) = I_{n+1}(u). \quad (2.13)$$

---

<sup>1</sup>The canonical Poisson bracket needs to be  $\{\phi(\sigma), u(\sigma')\} = -\beta\pi\delta(\sigma - \sigma')$ .

It is clear that

$$\{K_n(v), H\} = 0 \quad (2.14)$$

must be true too.

Let us now come back to quantum theory and CFT. With the identification  $u = \frac{\beta}{2}D\phi$  as in the classical case, the energy momentum tensor of the CFT can be regarded as a quantum Miura transform of that field,

$$\mathcal{L}^+(z) = \hbar' \left( T^+(z) + \frac{1-c}{24} \right) = \frac{1}{2} (:u^2: - \lambda\beta Du) , \quad \hbar' = \left( \frac{\beta}{2} \right)^2 \hbar. \quad (2.15)$$

To see the mKdV hierarchy at quantum level we therefore demand that any hermitian operator/functional  $\hat{F}(\mathcal{L}^+)$  ( $\hat{F}(T^+)$ ) commutes with the quantum Sinh-Gordon hamiltonian  $[\hat{F}(\mathcal{L}^+), \hat{H}] = 0$ . We consider functionals of the form

$$F(T^+) = \frac{1}{2\pi i} \oint \frac{dz}{z} P^{(n)}(\mathcal{L}^+, D\mathcal{L}^+, \dots) \quad (2.16)$$

where  $P^{(n)}$  are polynomials of  $\mathcal{L}^+$  and its derivatives. All these functionals in particular commute with  $V^+$  if all the Virasoro generators commute with it,

$$[L_n^+, z^h V^+] = 0 \quad \forall n, \quad (2.17)$$

which is only possible if the previously defined dressed Vertex operator has unit conformal weight,

$$h = -\frac{\hbar}{2}\beta^2 + \beta\lambda = 1 \quad \Rightarrow \quad \lambda = \frac{1}{\beta} \left( 1 + \frac{\hbar}{2}\beta^2 \right). \quad (2.18)$$

The additional constraint

$$[F(\mathcal{L}^+), z^h V^-] = [F(\mathcal{L}^+)^{\dagger}, z^h V^-] = [F(\mathcal{L}^-), z^h V^-] = 0 \quad (2.19)$$

is respectively solved if for all  $n$  we demand  $[L_n^-, z^h V^-] = 0$  which also gives  $h = 1$ .

With these constraints we, in particular, obtain  $c = 13 + \frac{3}{\hbar'} + 12\hbar'$ , which has a minimal value of  $c = 25$  at  $\hbar' = \frac{1}{2}$  and in the classical limit  $\hbar' \rightarrow 0$  goes as  $c \approx \frac{3}{\hbar'}$ . It also becomes evident that in the classical limit the quantum Miura transformation (2.15) becomes the classical Miura transformation (2.11). Note that the precise shift of the energy momentum tensor in (2.15) is in fact needed to preserve the same kind of connection between the KdV and the mKdV structure on quantum level as in the classical case.

Following the logic in constructing the classical charges, one constructs the quantum KdV charges  $\hat{K}_n$  as functionals (2.16), i.e. as integrals of polynomials of  $\mathcal{L}^+(z)$  and its derivatives, that are related to the mKdV charges of the quantum Sinh-Gordon theory by the quantum Miura transform (2.15), and are constrained as follows:



- The charges are hermitian,  $\hat{K}_n = \hat{K}_n^\dagger$ .
- The charges commute among each other,  $[\hat{K}_n, \hat{K}_m] = 0$ .
- If we assign grade 2 to the field  $T$  and grade 1 to the derivative then  $P^{(n)}$  have total grade  $2n$ .
- In the classical limit  $\hbar' \rightarrow 0$  ( $c \rightarrow \frac{3}{\hbar'}$ ) the charges reproduce the classical integrals of motion (A.3).

There is no closed formula known for all charges. However, one can construct them step by step with increasing grade. The first two KdV charges in a CFT are given by

$$H \equiv \hat{K}_1 = \frac{1}{2\pi\hbar'i} \oint \frac{dz}{z} \mathcal{L}(z) = L_0 - \frac{c-1}{24} \quad (2.20)$$

$$\begin{aligned} Q \equiv \hat{K}_2 &= \frac{1}{2\pi\hbar'i} \oint \frac{dz}{z} 2\mathcal{L}^2(z) = 2\hbar' \left( 2 \sum_{n=1}^{\infty} L_{-n} L_n + L_0^2 - \frac{c-1}{12} L_0 + \frac{(c-1)^2}{576} \right) \\ &= Q^{(1)} + 2\hbar' \left( L_0^2 - \frac{c-1}{12} L_0 + \frac{(c-1)^2}{576} \right). \end{aligned} \quad (2.21)$$

More are given in (B.5). Note, that the charges we give here slightly differ from the the charges considered in [3], and also as they were used in [14, 15]. This is because we strictly follow the conventions of [1].

### 3 KdV characters

As a first step to understand the full GGE, we here want to investigate GGE characters which we define as the trace of some GGE state over a specific highest weight module,

$$Z_{\{q, y_i\}}^p = \text{Tr}_{[p]} \rho(\{q, y_i\}). \quad (3.1)$$

We will analyse this quantity in the *classical* limit, i.e. when  $c \rightarrow \frac{3}{\hbar'} \rightarrow \infty$ . We also assume that the chemical potentials are not too small which in our case means that  $\mu_i \gg 1/c^{1-\frac{1}{2i}}$ .<sup>2</sup> This probes regimes of the GGE characters for which the corresponding charge is small compared to  $c$ . In this limit we are able to diagonalise all KdV charges simultaneously in a simple basis. To do so let us consider the renormalized Virasoro generators  $L_n \rightarrow L'_n = \frac{L_n}{\sqrt{c}}$ , with

$$[L'_m, L'_n] = \frac{m^3 - m}{12} \delta_{m, -n} + \frac{m - n}{\sqrt{c}} L'_{m+n} \approx \frac{m^3 - m}{12} \delta_{m, -n}, \quad (3.2)$$

---

<sup>2</sup>This constraint comes from a saddle point analysis and will become clear in (3.30).

where approximations from now on only keep the leading order results at large  $c$ .<sup>3</sup> Now consider states in a primary  $p$ -module of the form

$$|\{n_i, k_i\}\rangle = \prod_{i=1}^M L'_{-n_i}{}^{k_i} |p\rangle. \quad (3.3)$$

From the above commutation relation it follows that they are orthogonal up to large  $c$  corrections. A possible basis is for example the set of all states with  $L'_{-n_i}$  ordered such that bigger  $n_i$  always appear on the left of smaller  $n_i$ .

All the above states are in fact approximate eigenstate of all KdV charges. Let us first show this for the second KdV charges  $Q$  as given in (2.21).

### 3.1 Warm-up: Q-character at large central charge

The above states (3.3) are clearly eigenstates of  $L_0$  with eigenvalue  $\sum_{i=1}^M k_i n_i$ . One can show that with the commutation relation (3.2) it follows that they are also eigenstates of  $Q^{(1)}$ . This is because only terms from the central extension contribute to order  $c^0$ . The approximate  $Q^{(1)}$  eigenvalue of a state  $|\{n_i, k_i\}\rangle$  is given by

$$\lambda_{\{n_i, k_i\}} \approx \sum_{i=1}^M k_i (n_i^3 - n_i). \quad (3.4)$$

In appendix B.3 we provide checks of the above equation by explicitly diagonalizing the  $Q^{(1)}$  matrix for different levels in the large  $c$  limit. Note that since the central term vanishes for  $L'_{-1}$ , contributions from possible  $L'_{-1}{}^m$  to the eigenvalue are of order  $1/c$ . They originate from the  $sl_2$  (global) part of  $Q^{(1)}$ , and are given by

$$4\hbar' L_{-1} L_1 L'_{-1}{}^m |\psi\rangle = 12L'_{-1} L_1 L'_{-1}{}^m |\psi\rangle \approx \frac{12}{c} (m^2 + (2h_\psi - 1)m) L'_{-1}{}^m |\psi\rangle, \quad (3.5)$$

where  $|\psi\rangle$  is some state of the form (3.3) without any  $L_{-1}$ , and  $h_\psi$  is its  $L_0$  eigenvalue.

Note, that all possible off-diagonal elements in a basis built from (3.3) are  $o(1/c)$ .

#### 3.1.1 Vacuum Q character

**First let us only consider  $Q^{(1)}$ :** A suitable basis in the vacuum module  $[0]$  is the set of all ordered states (3.3) without any  $L'_{-1}$  appearing. The vacuum  $Q^{(1)}$  character

---

<sup>3</sup>This unusual normalization of the Virasoro generators with  $\sqrt{c}$  also appears in [19] where the authors studied the CFT dual of graviton exchanges in the heavy-light conformal blocks.

is then given by

$$\begin{aligned}
\mathrm{Tr}_{[0]} y_1^{Q^{(1)}} &\approx \sum_{\{n_i, k_i\} \in [\Omega]} y_1^{\lambda_{\{n_i, k_i\}}} \\
&= \prod_{n=2}^{\infty} \sum_{k=0}^{\infty} y_1^{k(n^3-n)} = \prod_{n=2}^{\infty} \frac{1}{1 - y_1^{n^3-n}} \\
&= \prod_{n=1}^{\infty} \frac{1}{1 - y_1^{\frac{n(n+1)(n+2)}{6}}} \tag{3.6}
\end{aligned}$$

$$\equiv 1 + \sum_{m=1}^{\infty} P_{\{T_n\}}(m) x^m, \quad \text{with } x = y_1^6. \tag{3.7}$$

The numbers  $T_n = \frac{n(n+1)(n+2)}{6}$  are the tetrahedral numbers. They are defined as the amount of balls needed to construct a tetrahedral cluster with the close-packing of equal spheres. From (3.6) one can hence show that the vacuum  $Q^{(1)}$ -character is a generating function for the number of partitions of an integer  $n$  into tetrahedral numbers  $P_{\{T_n\}}(m)$ . We will see similar partitions for higher charges.

**Other relevant term in  $Q$ :** The other non-constant term relevant at large  $c$  is  $-\frac{h'(c-1)}{6} L_0 \approx -\frac{L_0}{2}$ . Taking it into account one obtains

$$\mathrm{Tr}_{[0]} y_1^Q \approx \prod_{n=2}^{\infty} \sum_{k=0}^{\infty} y_1^{k(n^3-\frac{3}{2}n)} = \prod_{n=2}^{\infty} \frac{1}{1 - y_1^{n^3-\frac{3}{2}n}} \tag{3.8}$$

The  $Q^{(1)}$  eigenvalues get shifted by the (scaled) energy of the corresponding states and at least for the low lying eigenvalues a lot of the degeneracy is broken. However for large  $n$ , the partitions still grow exponentially in the same way as the partition into tetrahedral numbers.

### 3.1.2 Primary $Q$ character

**Again first let us consider  $Q^{(1)}$ :** As mentioned earlier the possible  $L'_{-1}$  in the states contribute sub-leading to the eigenvalues with (3.5). For states in a primary module only containing  $L'_{-1}$  the  $Q^{(1)}$  eigenvalue is even sub-dominant. Taking this into account we can write the  $Q^{(1)}$ -character in the primary  $p$  module,  $[p]$ , as

$$\begin{aligned}
\mathrm{Tr}_{[p]} y_1^{Q^{(1)}} &\approx \prod_{n=2}^{\infty} \sum_{k=0}^{\infty} y_1^{k(n^3-n)} + \left( \sum_{k=1}^{\infty} y_1^{\frac{12}{c}(k(2p-1)+k^2)} \right) \prod_{n=2}^{\infty} \sum_{k=0}^{\infty} y_1^{k(n^3-(1-\frac{24}{c})n)} \\
&\approx \left( \sum_{k=0}^{\infty} y_1^{\frac{12}{c}(k(2p-1)+k^2)} \right) \prod_{n=2}^{\infty} \frac{1}{1 - y_1^{(n^3-n)}} \tag{3.9}
\end{aligned}$$

This result is the vacuum result (3.6) multiplied with the global character result, as given in (B.20).

**Full Q:** Again we can write the result as the vacuum character multiplied with the global character as given in (B.21). It is given by

$$\begin{aligned} \mathrm{Tr}_p y_1^Q &\approx \left( \sum_{k=0}^{\infty} y_1^{\frac{6(p+k)^2 + 12(k^2 + 2pk - k)}{c} - \frac{1}{2}(p+k)} \right) \prod_{n=2}^{\infty} \sum_{k=0}^{\infty} y_1^{\frac{1}{2}k(n^3 - \frac{3}{2}n)} \\ &= \left( \sum_{k=0}^{\infty} y_1^{\frac{6(p+k)^2 + 12(k^2 + 2pk - k)}{c} - \frac{p+k}{2}} \right) \prod_{n=2}^{\infty} \frac{1}{1 - y_1^{\frac{1}{2}(n^3 - \frac{3}{2}n)}}. \end{aligned} \quad (3.10)$$

One may estimate the term in the parentheses by extremizing the exponent with respect to  $k$  and approximate the sum by the corresponding  $k$  saddle value, see §B.4.1. Such an analysis however, is expected to be quite different from the case when  $c$  is fixed, since in that case the dropped terms  $\propto 1/c$  will contribute significantly to the character.

### 3.2 Higher charge characters

We now want to analyze all the higher charges in the large  $c$  limit. As mentioned before, this is the classical limit which in particular means that all the quantum corrections to the classical KdV charges/currents can be neglected. This is nice since while a closed form doesn't exist for the quantum charges, the classical charges are given explicitly in (A.3). To compute the quantum charges in the large  $c$  limit we simply have to replace  $v$  in (A.3) by  $2\mathcal{L}$  and consider the respective symmetric composite field. However, even that can be reduced further. In the classical limit we can write  $\mathcal{L} = \hbar'T - \frac{1}{8}$  (since  $c \rightarrow 3/\hbar'$  for  $\hbar' \rightarrow 0$ ) and the  $k$ th current ( $\equiv \mathcal{I}_k$ ) can be written in a power series in  $\hbar'$ ,

$$\hbar'\mathcal{I}_k \approx \sum_{n=0}^k P_k^{(n)}(T, D^2T, \dots) \hbar'^n, \quad (3.11)$$

where  $P_k^{(n)}(T, D^2T, \dots)$  is some polynomial of grade  $2n$  of  $T$  and its derivatives.  $T$  contributes two and any derivative one to the grade. It follows that  $P_k^{(n)}$  and  $P_k^{(n+1)}$  can at most give contributions to the eigenvalues of order  $c^{n/2}$ , where the eigenvectors are still given by (3.3). We can therefore neglect all terms in (3.11) with  $n > 2$  since they contribute at least of order  $c^{-1/2}$  to  $\mathcal{I}_k$ . The  $k$ th current can therefore be approximated by

$$\hbar'\mathcal{I}_k \approx (-1)^k \frac{Ct(k-1)}{2^{2k+1}} + P_k^{(1)}(T) \hbar' + P_k^{(2)}(T, DT, \dots) \hbar'^2. \quad (3.12)$$

$Ct(k)$  is the  $k$ th Catalan number and follows from the recursive definition of the classical charge.  $P_k^{(1)}(T)$  has the form

$$P_k^{(1)} = \frac{\sqrt{\pi}}{(k-1)!\Gamma(\frac{3}{2}-k)} T(z) + \text{total derivatives}, \quad (3.13)$$

and therefore contributes

$$\frac{\sqrt{\pi}}{(k-1)!\Gamma(\frac{3}{2}-k)}L_0 \quad (3.14)$$

to the charge.

The remaining polynomial can be written as

$$P_k^{(2)} = \sum_{m=0}^{k-2} (-1)^{k+m} b_m^k T(z) D^{2m} T(z) \quad \text{for } k \geq 2, \quad (3.15)$$

with  $b_m^k$  positive constants, where always  $b_{k-2}^k = 2$ . The other coefficients can e.g. be computed by the help of a computer algebra program. Integrating over this part of the current then gives

$$\frac{1}{2\pi i \hbar'} \oint \frac{dz}{z} P_k^{(2)} \hbar'^2 = 2\hbar' (-1)^k \sum_{n>0} \left( \sum_{m=0}^{k-2} b_m^k n^{2m} \right) L_{-n} L_n, \quad (3.16)$$

where for now we omit the term proportional to  $L_0^2$  which is sub-leading in large  $c$ . The relevant part of the  $k$ th charge in the large  $c$  limit can therefore be written as

$$\begin{aligned} \hat{K}_k &= 2\hbar' (-1)^k \sum_{n>0} \left( \sum_{m=0}^{k-2} b_m^k n^{2m} \right) L_{-n} L_n + \frac{\sqrt{\pi}}{(k-1)!\Gamma(\frac{3}{2}-k)} L_0 + (-1)^k \frac{C\mathfrak{t}(k-1)}{2^{2k+1}\hbar'}, \\ &\equiv K_k^{(1)} + \frac{\sqrt{\pi}}{(k-1)!\Gamma(\frac{3}{2}-k)} L_0 + (-1)^k \frac{C\mathfrak{t}(k-1)}{2^{2k+1}\hbar'}. \end{aligned} \quad (3.17)$$

With this it is clear that the previously chosen orthogonal basis in large  $c$  diagonalises all the higher charges as well. Firstly, note that we have only kept the leading order terms in large  $c$ . At  $\mathcal{O}(1/c)$ , the only contributions come from the  $L_0^2$ , and the  $L_{-1}L_1$  term in  $K_k^{(1)}$ . Secondly, note that the most suppressed term is

$$\hbar^{k-1} 2^{k-1} C\mathfrak{t}(k-1) L_0^k \approx \frac{6^{k-1}}{c^{k-1}} C\mathfrak{t}(k-1) L_0^k. \quad (3.18)$$

### 3.2.1 $\hat{K}_k$ vacuum character

Now we focus on the vacuum module. Using (3.17) we obtain,

$$\hat{K}_k |\{n_i, l_i\}\rangle = \mathfrak{K}_{k, \{n_i, l_i\}} |\{n_i, l_i\}\rangle \quad (3.19)$$

with

$$\begin{aligned} \mathfrak{K}_{k, \{n_i, l_i\}} &= \sum_{i=1}^M l_i \left[ (-1)^k (n_i^3 - n_i) \left( \sum_{m=0}^{k-2} \frac{b_m^k}{2} n_i^{2m} \right) + \frac{\sqrt{\pi} n_i}{(k-1)!\Gamma(\frac{3}{2}-k)} \right] + (-1)^k \frac{C\mathfrak{t}(k-1)}{2^{2k+1}\hbar'} \\ &\equiv \sum_{i=1}^M l_i \left[ (-1)^k \mathfrak{p}_{k, n_i}^{(1)} + \frac{\sqrt{\pi} n_i}{(k-1)!\Gamma(\frac{3}{2}-k)} \right] + (-1)^k \frac{C\mathfrak{t}(k-1)}{2^{2k+1}\hbar'}. \end{aligned} \quad (3.20)$$

Focussing on the  $K^{(1)}$  part of the  $l$ -th KdV charge, we find the character analogous to (3.6) as,

$$\begin{aligned} \mathrm{Tr}_{[0]} y_l^{K_l^{(1)}} &\approx \prod_{n=2}^{\infty} \sum_{k=0}^{\infty} y_l^{k \cdot \mathfrak{p}_{l,n}^{(1)}} \\ &= \prod_{n=2}^{\infty} \frac{1}{1 - y_l^{\mathfrak{p}_{l,n}^{(1)}}} \quad \text{for } l \geq 2, \text{ and even.} \end{aligned} \quad (3.21)$$

For  $\mu_l \ll 1$  ( $\mu_l = -\log y_l$ ) we can assume that the above is dominated by large  $n \sim n_* \gg 1$ . However since in our approximation  $c \gg n_*$ , we will also obtain a lower bound on  $\mu_l$ . For large  $n$ ,  $\mathfrak{p}_{l,n}^{(1)} \approx n^{2l-1}$ , hence

$$\mathrm{Tr}_{[0]} y_l^{K_l^{(1)}} \approx \prod_{n=2}^{\infty} \left(1 - y_l^{n^{2l-1}}\right)^{-1}. \quad (3.22)$$

Remarkably, (3.22) can now be expressed in an expansion in  $y_l$  weighted by power partitions,  $p_l(q_m)$ ,

$$\mathrm{Tr}_{[0]} y_l^{K_l^{(1)}} \approx \sum_{q_m=0, q_m \in \mathbb{Z}}^{\infty} p_l(q_m) y_l^{q_m}. \quad (3.23)$$

$p_l(q_m)$  is the number of ways  $q_m$  can be expressed as sum of integers which are of the form  $i^{2l-1}$  where  $i$  is also an integer (this is the quantum number of the  $l$ th KdV charge). Note that for  $l = 2$ , this is partition of  $n$  into cubes, which is the large  $n$  asymptotics of tetrahedral numbers. It turns out that there is a formula due to Hardy and Ramanujan [20, p.111] which gives an asymptotic analytic form for the power partitions,

$$p_l(q_m) \approx \exp\left(A(l) q_m^{\frac{1}{2l}}\right), \quad (3.24)$$

where,  $A(l) = 2l \left(\frac{1}{2l-1} \Gamma\left(\frac{2l}{2l-1}\right) \zeta\left(\frac{2l}{2l-1}\right)\right)^{(2l-1)/(2l)}$ . This then gives us the density of states in the vacuum module with  $K_l^{(1)}$  charge  $q_m$ , where  $q_m \gg 1$ . Adding the  $L_0$  contribution does not change the above logic which allows us to approximate the vacuum character by

$$\mathrm{Tr}_{[0]} y_l^{K_l} \approx \sum_{q_m=0, q_m \in \mathbb{Z}}^{\infty} \exp\left(A(l) q_m^{\frac{1}{2l}} - \mu_l (q_m - cB_l)\right) \quad (3.25)$$

where  $B_l = (-1)^{l+1} \frac{C_l(l-1)}{3^{2l+1}}$  comes from the constant part in the charge. We can further approximate the sum by an integral and obtain

$$\mathrm{Tr}_{[0]} y_l^{K_l} \approx e^{\mu_l c B} \int_0^{\infty} dq_n \exp\left(A(l) q_n^{\frac{1}{2l}} - \mu_l q_n\right) \quad (3.26)$$

$$\approx e^{\mu_l c B} \exp\left\{\frac{2l-1}{2l} A(l) \frac{2l}{2l-1} (2l\mu_l)^{\frac{1}{(1-2l)}}\right\}, \quad (3.27)$$

where the integral has been done by saddle (C.3).

### 3.2.2 $\hat{K}_k$ primary characters

**Only  $K_l^{(1)}$ :** As in the case of  $Q^{(1)}$  we only have to consider the additional contributions from  $L_{-1}L_1$  in the sum. It is sub-leading and adds

$$\frac{24}{c} (k(2(p+N) - 1) + k^2) \mathfrak{A}$$

to the dominant contribution, where  $N$  denotes the contribution to the conformal dimension from all excitations not including  $L_{-1}$  and  $\mathfrak{A} = \sum_{m=0}^{l-2} b_m^l$ . Analogously to  $Q^{(1)}$  we can therefore write

$$\begin{aligned} \mathrm{Tr}_{[p]} y_l^{K_l^{(1)}} &\approx \prod_{n=2}^{\infty} \sum_{k=0}^{\infty} y_l^{k p_{l,n}^{(1)}} + \left( \sum_{k=1}^{\infty} y_1^{\frac{12}{c} (k(2p-1)+k^2) \mathfrak{A}} \right) \prod_{n=2}^{\infty} \sum_{k=0}^{\infty} y_l^{k (p_{l,n}^{(1)} + \frac{24}{c} \mathfrak{A} n)} \\ &\approx \left( \sum_{k=0}^{\infty} y_1^{\frac{12}{c} \mathfrak{A} (k(2p-1)+k^2)} \right) \prod_{n=2}^{\infty} \frac{1}{1 - y_l^{p_{l,n}^{(1)}}} \quad \text{for } l \geq 2 \text{ even.} \end{aligned} \quad (3.28)$$

Again, this is the global answer multiplied with the vacuum result.

**Full  $K_l$ :** Again, analogously to the case of  $Q$ , the dominant contributions from  $L_{-1}$  to states in the  $p$ -module come from the part in  $\hat{K}_l$  proportional to  $L_0$ . Formally, we can write

$$\mathrm{Tr}_{[p]} y_l^{K_l} \approx \left( \sum_{k=0}^{\infty} y_1^{\frac{\sqrt{\pi}}{(k-1)! \Gamma(\frac{3}{2}-k)} (p+k) + \frac{12}{c} (k(2p-1)+k^2) \mathfrak{A} + \frac{b_0^l}{c} (p+k)^2} \right) \prod_{n=2}^{\infty} \frac{1}{1 - y_l^{\frac{p_{k,n}^{(1)} + \frac{\sqrt{\pi} n}{(k-1)! \Gamma(\frac{3}{2}-k)}}}}. \quad (3.29)$$

### 3.3 Comment on the full GGE

The full Generalized Gibbs Ensemble is obtained by taking the sum over all GGE characters that appear in the CFT. This sum, however, depends on the specifics of the theory. Note that we cannot approximate the result by e.g. taking the asymptotic formula for the growth of primary states [8, 9], because this formula is valid in energy regimes ( $\Delta_p \gg c$ ) that do not fit the regimes of validity that we consider ( $\Delta_p \ll c$ ). We would need the specifics of the spectrum at intermediate energies which is not universal [21].

#### 3.3.1 Constraints on $\mu_l$

- (a) **Lower bound on  $\mu_l$ :** We already mentioned it earlier and now want to show that one gets a lower bound on  $\mu_l$  by demanding that the dominant contribution from the saddle point of the above integral is smaller than  $c$ . The saddle turns out to be,

$$\begin{aligned} q_n^* &= 2^{\frac{2l}{1-2l}} \left( \frac{l \mu_l}{A(l)} \right)^{\frac{2l}{1-2l}} \\ \Rightarrow \mu_l &\gg \frac{A(l)}{2l} \frac{1}{c^{1-\frac{1}{2l}}} \approx \frac{1}{c^{1-\frac{1}{2l}}}. \end{aligned} \quad (3.30)$$

This implies that our result (3.27) is valid in the regime

$$c^{\frac{1}{2l}-1} \ll \mu_l \lll 1.$$

- (b) **Positivity of  $\mu_l$**  : Note, that while the leading terms are negative for the *odd* KdV charges (3.17), the  $L_0^k$  term is always positive (3.18). This is the term that controls the asymptotically high energy regimes arising in the trace formula for the character. Hence, for convergence  $\mu_l > 0$ , for both even as well as odd  $l$ . Since the leading terms in  $1/c$  expansion are also positive for even  $l$ , this is the case we have focussed on.

## 4 Connection between high and low chemical potentials

As mentioned in our introduction one of our motivations to study integrable charges and the GGE is to reveal some possible transformation property of the chemical potential under which the full GGE or GGE characters behaves in a certain way. In the section we use our results allow to discuss some aspects of the transformation.

First, let us collect the results at large and small chemical potential and large  $c$  with a single integrable charge  $\hat{K}_l$ . At high chemical potential the corresponding GGE is dominated by the constant part of the charge because it has the highest power in  $c$ . However, the contribution is not necessarily given by the vacuum contribution because the charges are not necessarily positive. The state with lowest charge eigenvalue dominates. Nevertheless we can only restrict ourselves to the constant part, because all other contributions in this regime are hugely sub-dominant. So at large chemical potential  $1 \ll \mu_l \ll c$ , and with all other chemical potentials turned off, we can approximate the GGE by

$$Z(\mu_l) \approx e^{\mu_l c B_l}, \quad (4.1)$$

where  $B_l = (-1)^{l+1} \frac{C_l(l-1)}{2^{2l+1}}$  is the constant part computed in the last section.

In the regime of small chemical potential,  $1/c^{1-1/2l} \ll \mu_l \ll 1$ , we in particular computed the  $\hat{K}_l$  vacuum character in (3.27),

$$\text{Tr}_{[0]} y_l^{\hat{K}_l} \approx e^{\mu_l c B_l} \exp \left\{ \frac{\mathcal{C}_l}{\mu_l^{\frac{1}{2l-1}}} \right\}, \quad (4.2)$$

where  $\mathcal{C}_l = \Gamma\left(\frac{2l}{2l-1}\right) \zeta\left(\frac{2l}{2l-1}\right)$ . The factor in the front,  $e^{\mu_l c B_l}$ , again comes from the constant part. However in the case of small chemical potentials we can show that this factor is negligible for large  $l$ . The competition is between the factors,  $\mu_l c B_l$  and  $\mathcal{C}_l \mu_l^{\frac{1}{1-2l}}$ . When  $\mu_l$  is taken the smallest it can be (to be in the valid regime of our analysis, i.e.,  $\mu_l \sim c^{-1+\frac{1}{2l}}$ ), the two terms are of the order,  $\sqrt{c^l} B_l$  and  $\sqrt{c^l} \mathcal{C}_l$ .



Therefore we need to compare  $\mathcal{C}_l$  and  $B_l$ . For large  $l$ , using the asymptotics of Gamma and Zeta functions,  $\mathcal{C}_l \sim 2l - 1$  whereas, using the asymptotics of Catalan numbers,  $|B_l| \sim l^{-3/2}/(8\sqrt{\pi})$ . Therefore clearly, for small  $\mu_l$ , as  $l$  gets bigger,

$$\mathrm{Tr}_{[0]} y_l^{\hat{K}_l} \approx \exp \left\{ \frac{\mathcal{C}_l}{\mu_l^{\frac{1}{2l-1}}} \right\}. \quad (4.3)$$

If we now assume that there exist any nice transformation property that relates the low and high chemical potentials relating the respective GGEs, it is reflected in the second factor  $\exp \left\{ \mathcal{C}_l / \mu_l^{\frac{1}{(2l-1)}} \right\}$ . Since the vacuum character appears in all primary character we can assume that this factor multiplies the full GGE.

It follows immediately that, if  $\mu_i \neq \beta$ , it is not possible to connect the low and high potential regime by an 'S transformation',  $\mu_i \rightsquigarrow \frac{1}{\mu_i}$ . Rather, the transformation that connects the high and low chemical potential regime seems to have the form

$$\mu_i \rightarrow \mathcal{D}_i \mu_i^{\frac{1}{1-2i}}, \quad (4.4)$$

where  $\mathcal{D}_i$  is independent of  $\mu_i$ , just depending on the charge and  $c$ . It is clear that the GGE can definitely not be invariant under this transformation. This is because any function of  $\mu_i$  that is invariant under such a transformation would be constant. This follows easily by performing  $\mu_i \rightarrow \mathcal{D}_i \mu_i^{\frac{1}{1-2i}}$  several times.

$$f(\mu_i) = f \left( \mathcal{D}_i^{\sum_{j=0}^{k-1} \frac{1}{(1-2i)^j}} \mu_i^{\frac{1}{(1-2i)^k}} \right) \quad \forall k \in \mathbb{N}, \mu_i \in \mathbb{R}^+. \quad (4.5)$$

Taking the limit  $k \rightarrow \infty$  then tells that  $f(\mu_i) = f(\mathcal{D}_i^{\frac{2i-1}{2i}})$ .

#### 4.1 Connection to very high energies

Recent study of KdV in 2D CFTs have explored various aspects of the GGE in the regimes of asymptotically large energies and KdV charges [14–16]. In particular, the chemical potentials are outside the regime that we have studied, viz.,  $\mu_i \ll \frac{1}{c^{1-1/2i}}$ . Therefore in this regime, the GGE partition function is dominated by  $h_*, n_* \gg c$ . Additionally, the investigations also focus on the limit,  $\{\mu_l\} \ll \beta \ll 1$ , therefore the KdV charges are perturbatively treated on the torus. It was shown that when high energies dominate, the main contribution to the GGE comes from primaries, and hence the GGE can be approximated by (using (3.18))<sup>4</sup>,

$$Z(\{q, y_i\}) = \mathrm{Tr} q^H \prod_i y_i^{\hat{K}_i} \approx \int_0^\infty dh \rho(h) \exp \left( -\beta h - \sum_i \mu_i \frac{6^{i-1} \mathrm{Ct}(i-1)}{c^{i-1}} h^i \right). \quad (4.6)$$

<sup>4</sup> Our coefficients in the exponent differ from those in [14–16], since the expressions for KdV charges (for  $c > 1$ ) are different.

In this high energy regime unlike [14–16], we shall focus on  $1 \gg \{\mu_i\} \gg \beta$ . The integral can in principle be solved by a saddle point approximation. This, however, becomes very complicated for arbitrary  $\mu_i \ll 1/c^{1-\frac{1}{2i}}$ . The situation simplifies when we take all potentials parametrically small with respect to one particular potential. In that case, the saddle is always dominated by the contribution of that particular potential with corrections given by a power series in all the other ones. The dominant saddle  $h^*$  for  $\mu_n \gg \{\mu_i\}, \beta$  is given by

$$h^* = \left( \frac{\pi}{n6^{i-1/2} Ct(i-1)\mu_n} \right)^{\frac{2}{2n-1}} c + \sum_{i \neq n} C_i \mu_i + \mathcal{O}(\mu_i \mu_j), \quad (4.7)$$

where the coefficients  $C_i \equiv C_i(\mu_n, c)$  are such that the corrections are small. Plugging back this saddle, the partition function is approximately given by

$$Z(1 \gg \mu_n \gg \mu_i) \approx \exp \left[ \frac{(2n-1)c}{6} \left( \frac{\pi^{2n}}{Ct(n-1)n^{2n}\mu_n} \right)^{\frac{1}{2n-1}} + \mathcal{O}(\mu_i) \right]. \quad (4.8)$$

Thus a bit surprisingly, we see that the GGE for very low chemical potential  $\mu_i \ll \frac{1}{c^{1-1/2i}}$  shows the same  $\mu_i$  dependence that we saw when we analyzed the universal vacuum character in the range  $\frac{1}{c^{1-1/2i}} \ll \mu_i \ll 1$ , (4.3).

This is also compatible with the known results from the thermal partition function. The Virasoro characters and the full partition function at high temperature show qualitatively the same behavior. The sum over all characters must lead to a quantitative agreement and, hence, gives tight constraints on the spectrum of the theory. The latter e.g. allows to compute the asymptotic growth of the density of primary states. In the GGE case, using (4.8) we can derive the growth of the asymptotic density of states carrying the  $n$ -th KdV charge ( $q_n$ ) as,  $\exp\left(Aq_n^{\frac{1}{2n}}\right)$  (same  $q_n$  dependence as in (3.24)).

## 5 Comment on approximations of pure states

Studying the GGE can also shed more light on how the eigenstate thermalization hypothesis (ETH) [22–24] might be realized in two-dimensional conformal field theory. It states that for a particular class of simple operators generic energy eigenstates can be well approximated by thermal states. It is, however, only expected to be true in theories exhibiting chaotic dynamics and known to fail in integrable theories, where integrable here not only means that there exists an infinite set of conserved charges but also that these trivialize the dynamics. In case of 2d CFTs with  $c > 1$  we saw that there exist an infinite set of conserved charges. However, the dynamics is not necessarily trivial. For example in [25] it has been shown that 2d CFTs exhibit chaotic dynamics and, hence, a generalization of ETH might hold. The generalization takes not only the Hamiltonian/energy into account but considers all conserved

quantities. One might expect that a generic static state can for some class of simple operators be approximated by some equilibrium state. The latter defines a GGE by taking the trace over the spectrum of the theory.

We, here, want to check how well a GGE dominated by a single chemical potential,  $\mu_l$ , can approximate a typical eigenstate of the corresponding KdV charge,  $\hat{K}_l$ , in the regime that we probe. For now let us choose such a state  $|q_m\rangle$ , whose  $\hat{K}_l$  eigenvalue is  $q_m - cB_l$ , where  $1 \ll q_m \ll c$ . Restricting to the vacuum part of the GGE, in the regime  $\{\mu_i, \beta\} \ll \mu_l \ll 1$ , then we can compute  $\langle \hat{K}_l \rangle$  in the mixed state  $e^{-\mu_l \hat{K}_l}$  reduced to the vacuum module with the help of (3.27) as

$$\begin{aligned} \langle \hat{K}_l \rangle_{q_l} &= -\partial_{\mu_l} \log Z_{GGE}^{0,l} \\ &\approx -\partial_{\mu_l} \left\{ \frac{2l-1}{2l} A(l)^{\frac{2l}{2l-1}} (2l\mu_l)^{\frac{1}{1-2l}} + \mu_l c B_l \right\} \\ &= (2lA(l)\mu_l)^{\frac{2l}{1-2l}} - cB_l. \end{aligned} \tag{5.1}$$

Hence, a necessary condition for  $e^{-\mu_l \hat{K}_l}$  being a good approximation to  $|q_m\rangle\langle q_m|$  is when we equalize the eigenvalue,  $q_m - cB_l$  to the above ensemble answer. This gives,

$$\mu_l = \frac{1}{2lA(l)} q_m^{\frac{1-2l}{2l}}. \tag{5.2}$$

Note, that since the GGE should be a positive quantity,  $\mu_l$  should be a real quantity, implying positivity of  $q_m$ . Thus in the cases where  $q_m < 0$  it appears that we cannot have equilibration. In the leading order in  $c$  are the terms of the type  $L_{-n}L_n$  ( $n > 1$ ) and  $L_0$ , see (3.17). For the even KdV charges that we are considering the former gives a positive or zero contribution to  $q_m$  always. It is however is sensitive only to the number of excitations on top of primary modules. It is zero acting on highest weight states. The coefficient of the  $L_0$  term, on the other hand, is always negative for even  $l$  (due to the  $\Gamma(\frac{3}{2} - l)$  factor). This can therefore make  $q_m < 0$ , when there are too many light excitations, e.g., many  $L_{-1}^k$  in  $|q_m\rangle$  (the  $c^0$  term from  $L_{-n}L_n$  is zero, while  $L_0$  gives a negative contribution at this order). This also is an artifact of the large  $c$  regime we are working in, since higher powers of  $L_0$  (which are subleading in  $\mathcal{O}(1/c)$ ) can make  $q_m > 0$ .

It is interesting to note, that exactly these states can be well approximated by a purely thermal state in the large  $c$  limit [26]. However, in [26] they also conjecture – and show it for particular states explicitly – that descendants far from their primary state cannot be approximated well by a purely thermal state, even at large  $c$ ! For such states,  $q_m \gg 0$  and GGE states might be a necessity for a good approximation of those.

## 6 Conclusions

With this note we shed some light on the implications that follow from the integrable structure of 2D CFT. We have first shown under which conditions one can not only construct an infinite set of commuting charges in a CFT with  $c > 1$ , but also connect it to the quantum Sinh-Gordon theory. Our analysis implies that if one can write the energy momentum tensor in the Feigin-Fuchs form, then the quantum KdV charges of CFT with  $c = 13 + \frac{3}{\hbar} + 12\hbar \geq 25$  are related to the quantum mKdV charges of sinh-Gordon (sG) theory with the Hamiltonian

$$\hat{H} = \frac{1}{2\pi i} \oint \frac{dz}{z} \left( 1 - \frac{1}{2z^{\frac{\hbar}{2}}} (:e^{\beta\phi}: + :e^{-\beta\phi}:) \right). \quad (6.1)$$

The relation between the quantum KdV and quantum mKdV charges are given by the Miura transform (2.15).

Next we have concentrated on the KdV *characters* at large central charge which on the sG side corresponds to taking the classical limit ( $\hbar \sim 3/c \rightarrow 0$ ). They are the building blocks of generalized Gibbs ensembles and can help in understanding its general features. The KdV characters can be computed for any integrable charge  $\hat{K}_l$ , as long as the corresponding chemical potential  $\mu_l$  is bounded from below by  $1/c^{1-\frac{1}{2l}}$ . This probes regimes where the corresponding eigenvalues of the charge are small compared to the central charge, which is always kept parametrically large in our analysis. The key idea is that to leading order in large  $c$  all states of the form (3.3) diagonalize the charges with eigenvalues given by (3.20).

A key motivation behind this work, has been whether there exists a connection between the high and low chemical potential GGE. By using the Hardy-Ramanujan result (3.24) on the number of partitions into monomials we can find the behaviour of the KdV vacuum character at small chemical potential (3.27). This factor appears in the other characters as well, and is universal. The self-consistency of our analysis bounds the chemical potential  $\mu_l$  from below by,  $1/c^{1-\frac{1}{2l}}$ . When considering much lower chemical potentials, s.t. eigenvalues much larger than  $c$  dominate the GGE, one can also extract the result for the full GGE (4.8) from the results in [14]. Both methods show the same  $\mu_l$  dependence in the GGE answer, which suggests that this behavior is valid for a rather large range of  $\mu_l \ll 1$ . On the other end, for large  $\mu_l$ , the GGE is dominated by the vacuum. In case of the first KdV charge, namely the Hamiltonian, the two regimes are directly related by the standard  $\mathcal{S}$ -transformation, which takes  $\beta \rightarrow 1/\beta$ . It turns out, that for the higher chemical potentials a transformation of the form,  $\mu_l \rightarrow \mu_l^{\frac{1}{1-2l}}$ , (4.4) relates the low and the high  $\mu_l$  GGEs .

We also briefly commented on a generalization of the eigenstate thermalization hypothesis. If a thermal state fails to approximate an energy eigenstate it might be cured by considering GGE states. Once again in the restricted regime of a single

chemical potential dominated GGE, we show the conditions when such an approximation is well defined. States that are already known to be approximated well by a thermal state, namely high energy primary states and descendant states with only light excitations on top of the primary [26], can - at least in the large  $c$  regime of validity of our analysis - not be approximated by a GGE state with only a single charge other than the Hamiltonian turned on, since the corresponding chemical potential is then no longer *real*. On the other hand for pure states made out of large excitations above the highest weight state, one can solve for the GGE chemical potential that needs to be turned on (5.2).

A number of interesting directions are left for future work. Here we list some of the more pressing ones,

- going beyond the large  $c$  approximation : the large  $c$  analysis is marred with issues of convergence, which is very explicit for the *odd* KdV charges. The leading order term in  $K_k^{(1)}$  for  $k$  odd, is in this case negative. The naive way to get by this, is to work with a negative  $\mu_k$ . However this is *wrong* as it will lead to non-convergence (see §3.3.1(b)), the correct choice being  $\mu_k > 0$ . Now to proceed in the  $1/c$  calculation of the GGE character, one needs to know more sub-leading terms in  $K_k$  beyond the regime of the present analysis. In future work, we shall try to see if the issue of finite  $c$  can be addressed by extending the technologies developed in [3–5] to  $c > 1$ . Since the problem of finding the eigenvalues of all KdV charges in the CFT can be mapped to the problem of solving a particle in a specific potential it might also be possible that symmetries of the GGE are directly related to symmetries in the spectrum of the quantum particle.
- investigate the GGE in specific models : until now, the GGE with only the second charge turned on, has not even been computed for the free theories. We in particular hope, that a closed form of the GGE might reveal more general transformation properties of the chemical potentials and show that the connection between high and low chemical potential that we see in this paper is not an artifact of taking large  $c$ .
- issues of equilibration : the pure states in 2D CFT are of the form,  $|\psi\rangle = \prod_i L_{-n_i}^{k_i} |p\rangle$  where,  $|p\rangle$  is a highest weight state, with  $L_0$  eigenvalue  $E = p + \sum_i n_i k_i$ . Hence the thermal ensemble is only sensitive to the number  $E$ . On the other hand, to fix a KdV GGE has to use more information regarding the construction of the state. There are clear indications of this even in large  $c$ . For example, the large  $c$ , eigenvalue  $\lambda_{\{n_i, k_i\}}$  of the second KdV charge, (3.4), is not just the combination  $E$ , and thus probes a finer microstructure of  $|\psi\rangle$ . Higher KdV charges probe even finer microstructures, however to understand

this completely, one will need to go beyond large  $c$ , since in leading order in  $1/c$ , the  $\lambda_{\{n_i, k_i\}}$  eigenvalues continue to dominate.

- holographic interpretation : adding polynomials of the stress tensor and its derivatives with respective chemical potentials to the action of a CFT corresponds to a multitrace deformation in the CFT. In a potential dual description of the CFT these deformations also appear as multitrace deformation of the action of gravitational action (or its higher spin extension). As discussed in [27] these can for example be studied in the Chern-Simons formulation. The deformation appears as a boundary term and which does not change the bulk field equations and hence a classical BTZ black hole solution stays unchanged. However, the free energy/partition function is different from the that of a usual BTZ black hole because of the additional boundary terms. In fact, the change looks like a generalized Legendre transform and which changes the partition function to the so-called tau-function of the classical KdV hierarchy. When quantizing the Chern-Simons theory with matter this leads to more structure in the gravitational solution and could lead to explicit constructions of black holes with quantum hair.

Apart from the above, it will also be interesting to understand the geometric interpretation of the KdV charges, in the same way that the Hamiltonian translates states along the Euclidean time. A generalization of the mapping of GGE to a quantum mechanics problem [3–5] for  $c > 1$  and for arbitrary chemical potentials is another fascinating future direction.

## Acknowledgements

It is a pleasure to thank George Jorjadze, and Stefan Theisen for encouragement and useful discussions at various stages of the project. We would also like to thank Shouvik Datta who was part of the initial collaboration. DD is grateful to the MPI Partner group grant.

## Attachments

We add two Mathematica<sup>®</sup> notebooks. The file LARGE\_C.NB is concerned with explicit checks of the large  $c$  eigenvalues of the second charge. This notebook uses VIRASORO.NB by Matthew Headrick, which can be downloaded from <http://people.brandeis.edu/~headrick/Mathematica/Virasoro.nb>. Finally in KDV\_CHARGES.NB we explicitly construct the leading order parts of the first  $n$  charges in large  $c$ . It in particular confirms the coefficients we give in §3.2.

## A Classical integrals of motion

### A.1 Integral of motions for the modified KdV equation

The classical integrals of motion defining the mKdV hierarchy are given by

$$I_n = -\frac{1}{2\pi} \int_0^{2\pi} d\sigma \frac{1}{2} u(\sigma) Y_{2n-1}(u) \quad (\text{A.1})$$

with

$$Y_{n+1} = \partial_\sigma Y_n + u \sum_{k=1}^{n-1} Y_k Y_{n-k}, \quad Y_1 = -u. \quad (\text{A.2})$$

### A.2 Integral of motions for the KdV equation

The classical integrals of motion defining the KdV hierarchy are given by

$$K_n(v) = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \frac{1}{2} v(\sigma) Z_{2n+1}(v) \quad (\text{A.3})$$

with

$$Z_{n+1} = \partial_\sigma Z_n + v \sum_{k=1}^{n-1} Z_k Z_{n-k}, \quad Z_1 = 1. \quad (\text{A.4})$$

## B Quantum KdV charge of 2d CFTs

### B.1 Composite operators

A composite operator  $AB(z)$  of two quantum fields  $A(z) = \sum_n A_n z^{-n}$  and  $B(z) = \sum_n B_n z^{-n}$  is defined as

$$AB(z) \equiv \frac{1}{2\pi i} \oint_{c_z} \frac{d\zeta}{\zeta} \frac{z}{\zeta - z} \mathcal{R}(A(\zeta)B(z)) \quad (\text{B.1})$$

with  $\mathcal{R}$  denotes radial ordering, i.e.

$$\mathcal{R}(A(\zeta)B(z)) = \begin{cases} A(\zeta)B(z) & \text{for } |\zeta| \geq |z| \\ B(z)A(\zeta) & \text{for } |\zeta| < |z|. \end{cases} \quad (\text{B.2})$$

The modes of the composite field are then given by

$$AB_n = \sum_{l=1}^{\infty} A_{-l} B_{l+n} + \sum_{l=0}^{\infty} B_{n-l} A_l. \quad (\text{B.3})$$

This definition gives you some sort of normal ordering with respect to the modes  $A_n$  and  $B_n$ . However, note that there is no additional normal ordering part. The

composition of several operators is always defined by first composing the most left operators and successively composing the operators that follow to the right, e.g.  $ABC(z)$  is defined by composing  $AB(z)$  with  $C(z)$ .

Furthermore, any composite operator from now on is understood as a symmetric composition, i.e.

$$\begin{aligned}\langle AB \rangle &= \frac{1}{2} (AB(z) + BA(z)) \\ \langle A_1 A_2 \dots A_n \rangle &= \frac{1}{n!} \sum_p A_{p_1} A_{p_2} \dots A_{p_n}\end{aligned}\tag{B.4}$$

where the sum is taken over all permutations and we omit the brackets in what follows.

## B.2 The charges

The first few charges are given by

$$\begin{aligned}H \equiv \hat{K}_1 &= \frac{1}{2\pi\hbar'i} \oint \frac{dz}{z} \mathcal{L}(z) = L_0 - \frac{c-1}{24} \\ Q \equiv \hat{K}_2 &= \frac{1}{2\pi\hbar'i} \oint \frac{dz}{z} 2\mathcal{L}^2(z) = 2\hbar' \left( 2 \sum_{n=1}^{\infty} L_{-n} L_n + L_0^2 - \frac{c-1}{12} L_0 + \frac{(1-c)^2}{576} \right) \\ \hat{K}_3 &= \frac{1}{2\pi\hbar'i} \oint \frac{dz}{z} 8\mathcal{L}^3 + \frac{2\hbar'}{3} (2-c)(D\mathcal{L})^2 \\ &= 8\hbar'^2 \left( \sum_{n,m=1}^{\infty} 2L_{-m} L_{-n+m} L_n + L_{-n-m} L_m L_n + L_{-n} L_{-m} L_{n+m} + \right. \\ &\quad \left. + \sum_{n=0}^{\infty} (n+1) L_{-n} L_0 L_n - \frac{2(c-2)n^2 + 3(c-1)}{12} L_{-n} L_n \right. \\ &\quad \left. + \frac{1-c}{8} L_0^2 + \frac{(1-c)^2}{192} L_0 + \frac{(1-c)^3}{13824} \right) \\ K_4 &= \frac{40}{2\pi\hbar'i} \oint \frac{dz}{z} \left[ \mathcal{L}^4 - \frac{\hbar'}{3} (c+2) \mathcal{L} (D\mathcal{L})^2 + \right. \\ &\quad \left. + \frac{\hbar'^2}{180} (c+2) \left( c - \frac{1}{2} \right) (D^2 \mathcal{L})^2 - \frac{\hbar'^2}{24} (c+2) (D\mathcal{L})^2 \right].\end{aligned}\tag{B.5}$$

In the limit  $\hbar' \rightarrow \frac{3}{c} \rightarrow 0$  they converge to the classical KdV charges (A.3) when we replace  $\mathcal{L} = \frac{v}{2}$  up to an overall factor of  $\frac{1}{\hbar'}$ .

## B.3 Some explicit checks on large $c$ eigenvalues of $\mathbf{Q}$

We can in principle compute the eigenvalues of  $\tilde{Q}^{(1)} = 2 \sum_{n=1}^{\infty} L_{-n} L_n$  with the help of Mathematica up to arbitrary level. However, the calculation time grows rapidly



with the level. We here give the results up to level 6 in the vacuum module and show that the large  $c$  eigenvalues are given by  $k_i \frac{n_i^3 - n_i}{6} c$ .

**Level 0 & 1:**  $\tilde{Q}^{(1)}$  annihilates the vacuum and there is no level 1 in the vacuum module. So no contribution here.

**Level 2:** A (not normalized) basis is  $\{L_{-2}|0\rangle\}$ . Its only basis element's eigenvalue is simply  $c$  and matches  $c \frac{n^3 - n}{6} \stackrel{n=2}{=} c$ .

**Level 3:** A (not normalized) basis is  $\{L_{-3}|0\rangle\}$ . Its only basis element's eigenvalue is  $4c + 4$ . In the large  $c$  limit this gives  $4c = \frac{3^3 - 3}{6} c$ .

**Level 4:** A (not orthonormal) basis is  $\{L_{-4}|0\rangle, L_{-2}^2|0\rangle\}$ . The matrix representation of  $\tilde{Q}^{(1)}$  in this basis is

$$\begin{pmatrix} 10c + 20 & 3c + 6 \\ 6 & c + 8 \end{pmatrix} \quad (\text{B.6})$$

with eigenvalues  $\lambda_{1/2} = 9 + 3c \pm \sqrt{37 + 22c + 4c^2}$ . At large  $c$  this gives

$$\lambda_1 = 10c = \frac{4^3 - 4}{6} c \quad (\text{B.7})$$

$$\lambda_2 = 2c = 2 \frac{2^3 - 2}{6} c \quad (\text{B.8})$$

**Level 5:** A (not orthonormal) basis is  $\{L_{-5}|0\rangle, L_{-3}L_{-2}|0\rangle\}$ . The matrix representation of  $\tilde{Q}^{(1)}$  in this basis is

$$\begin{pmatrix} 20c + 50 & 9c + 18 \\ 30 & 5c + 50 \end{pmatrix} \quad (\text{B.9})$$

with eigenvalues  $\lambda_{1/2} = \frac{100 + 25c \pm \sqrt{45(48 + 24c + 5c^2)}}{2}$ . At large  $c$  this gives

$$\lambda_1 = 20c = \frac{5^3 - 5}{6} c \quad (\text{B.10})$$

$$\lambda_2 = 5c = \left( \frac{3^3 - 3}{6} + \frac{2^3 - 2}{6} \right) c \quad (\text{B.11})$$

**Level 6:** A (not orthonormal) basis is  $\{L_{-6}|0\rangle, L_{-4}L_{-2}|0\rangle, L_{-3}L_{-3}|0\rangle, L_{-2}L_{-2}L_{-2}|0\rangle\}$ . The matrix representation of  $\tilde{Q}^{(1)}$  in this basis is

$$\begin{pmatrix} 35c + 88 & 12c & 36c + 72 & 48c + 48 \\ 36 & 11c + 42 & 108 & 18c + 132 \\ 18 & 24 & 8c + 52 & 48 \\ 0 & 11 & 0 & 3c + 48 \end{pmatrix}. \quad (\text{B.12})$$

At large  $c$  the four eigenvalues are

$$\lambda_1 = 35c = \frac{6^3 - 6}{6}c \quad (\text{B.13})$$

$$\lambda_2 = 11c = \left( \frac{4^3 - 4}{6} + \frac{2^3 - 2}{6} \right) c \quad (\text{B.14})$$

$$\lambda_3 = 8c = 2 \frac{3^3 - 3}{6}c \quad (\text{B.15})$$

$$\lambda_4 = 3c = 3 \frac{2^3 - 2}{6}c \quad (\text{B.16})$$

#### B.4 Global Q-character

By the global character we denote the object that follows from taking the trace over states in the sub-module corresponding to the global sub-algebra of the full Virasoro algebra, i.e.  $sl_2 \subset \text{Vir}_c$ . The  $sl_2$  sub-module  $[p]_{sl_2}$  of the primary module  $[p]$  is built upon states of the form  $L_{-1}^k |p\rangle$ ,  $k \in \mathbb{N}$ .

We also have to restrict  $Q$  to the parts that act within this global sub-module. In the present case this means we only keep the terms that include  $sl_2$  generators,

$$Q_{sl_2} \equiv 2\hbar' \left( 2c L'_{-1} L'_1 + L_0^2 - \frac{c-1}{12} L_0 \right) \equiv Q_{sl_2}^{(1)} + 2\hbar' L_0^2 - 2\hbar' \frac{c-1}{12} L_0. \quad (\text{B.17})$$

All the terms that we omitted act like

$$\begin{aligned} 2\hbar' c L'_{-m} L'_m L'^k_{-1} |p\rangle &= 2\hbar' \frac{(m+1)!}{c^{\frac{m-1}{2}}} \binom{k}{k-m} \left( \frac{k-m}{m+1} + p \right) L'_{-m} L'^{k-m}_{-1} |p\rangle \\ &\equiv Q_{k,m,p} L'_{-m} L'^{k-m}_{-1} |p\rangle \notin [p]_{sl_2} \quad \text{for } m > 1. \end{aligned} \quad (\text{B.18})$$

They do not act within the global sub-representation and hence have to be omitted.

The  $Q_{sl_2}^{(1)}$  eigenvalues of a global state  $L_{-1}^k |p\rangle$  are exactly the one in (3.5) with  $h_\psi = p$ , so

$$Q_{sl_2}^{(1)} L_{-1}^k |p\rangle = 4\hbar' (k^2 + (2p-1)k) L_{-1}^k |p\rangle. \quad (\text{B.19})$$

It follows immediately that the  $Q_{sl_2}^{(1)}$  character in the global module  $[p]_{sl_2}$  is given by

$$\text{Tr}_{[p]_{sl_2}} y_1^{Q_{sl_2}^{(1)}} = \left( \sum_{k=0}^{\infty} y_1^{4\hbar' (k^2 + (2p-1)k)} \right). \quad (\text{B.20})$$

The  $L_0$  eigenvalues of the previous states are simply  $(p+k)$ . The full  $Q_{sl_2}$  character is, hence, given by

$$\text{Tr}_{[p]_{sl_2}} q^{Q_{sl_2}} = \sum_{k=0}^{\infty} y_1^{2\hbar' (2k^2 + (2p-1)2k + (p+k)^2 - \frac{c-1}{12}(p+k))}. \quad (\text{B.21})$$

We see here that we cannot simply interchange the limit and the summation. This is because the eigenvalues switch sign at  $k \sim c$  which completely changes the behavior of the sum. In particular, when  $c$  is very large, then  $k \lesssim c$  gives huge contributions to the character!

### B.4.1 Closed forms for Global Q-characters

The eigenvalues

$$\lambda(k) = 2\hbar' \left( 2k^2 + (2p-1)2k + (p+k)^2 - \frac{c-1}{12}(p+k) \right)$$

are negative between

$$k_{\pm} = \frac{c + 23 - 72p \pm \sqrt{529 + 46c + c^2 - 3456p + 3456p^2}}{72}.$$

Hence, for large  $c \gg p$ ,  $\lambda$  is negative between  $k_- < k_0 = 0$  and  $k_+ \approx \frac{c}{36}$ . The minimum lies at  $k_{\min} = \frac{23+c-72p}{72} \approx \frac{c}{72}$ , with a minimal value of

$$\lambda(k_{\min}) = -\hbar \frac{c^2 + 46c + 3456p^2 - 3456p + 529}{864} \approx -\frac{\hbar c^2}{864}.$$

The sum (3.10) for  $k < k_+$  can be approximated by an integral for  $\mu \gg 1/c$  and  $c \gg p$ . It gives

$$\sum_{k=0}^{k_+} y_1^{\lambda(k)} \approx \sqrt{\frac{\pi c}{18\mu_1}} \exp \left[ \mu_1 \left( \frac{c}{288} + \frac{23}{144} + \frac{12p(p-1) + \frac{529}{288}}{c} \right) \right]. \quad (\text{B.22})$$

It is obvious that the contribution to the global character by negative eigenvalues is huge.

Note, that in case of higher charges one also needs to consider the first sub-leading contribution from the charge. Higher order corrections in  $1/c$  are not needed to render the sum finite. The global character looks like (B.21) but dressed with coefficients that follow from (3.17).

Note further, that it seems reasonable that the global character behave somehow nice under modular transformation, since their form looks much like the well known  $\theta$  functions. If we for example take the primary to be  $p = \frac{1}{2}$ , then (B.20) is given by  $\frac{1}{2}(\theta_3(\tau) + 1)$  with  $\tau = i\frac{24\mu_1}{c\pi}$  and

$$\theta_3(\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{n^2}{2}}.$$

Or for  $p = 1$  it can be written as  $\frac{q^{1/4}}{2}\theta_2(\tau)$  with

$$\theta_2(\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{(n+\frac{1}{2})^2}{2}}.$$

It is possible to write the general global character in any primary module as the generalized  $\theta$ -function given by

$$\begin{aligned} \theta[\alpha, \beta](\tau, z) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\alpha)^2} e^{2\pi i(n+\alpha)(z+\beta)} \\ &= \frac{\eta(\tau) e^{2\pi i\alpha(z+\beta)}}{q^{\frac{\frac{1}{24}-\alpha^2}{2}}} \prod_{n=1}^{\infty} \left( 1 + q^{n+\alpha-\frac{1}{2}} e^{2\pi i(z+\beta)} \right) \left( 1 + q^{n-\alpha-\frac{1}{2}} e^{-2\pi i(z+\beta)} \right), \end{aligned} \quad (\text{B.23})$$

which behave under modular transformations as

$$\begin{aligned}\theta[\alpha, \beta](\tau + 1, z) &= e^{-i\pi\alpha(\alpha-1)}\theta\left[\alpha, \alpha + \beta - \frac{1}{2}\right](\tau, z), \\ \theta[\alpha, \beta]\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) &= \sqrt{-i\tau}e^{2\pi i\alpha\beta + i\pi\frac{z^2}{\tau}}\theta[\beta, -\alpha](\tau, z).\end{aligned}\tag{B.24}$$

## C Saddle

We want to evaluate by saddle the following integral :

$$\int dn \exp (A(l)n^{1/2l} - \mu_l n) =$$

The saddle point equation yields,

$$\begin{aligned}\frac{A(l)}{2l}n_s^{\frac{1-2l}{2l}} - \mu_l &= 0. \\ \text{or, } n_s &= \left(\frac{2l\mu_l}{A(l)}\right)^{\frac{2l}{1-2l}}.\end{aligned}\tag{C.1}$$

Plugging back the saddle point into the integral we obtain,

$$\begin{aligned}\exp (A(l)n_s^{1/2l}) y_l^{n_s} &= \exp \left( A(l) \left( \frac{2l\mu_l}{A(l)} \right)^{\frac{1}{1-2l}} \right) y_l^{\frac{(2l\mu_l)^{2l}}{A(l)^{1-2l}}}, \\ &= \exp \left( A(l)^{\frac{2l}{2l-1}} (2l\mu_l)^{\frac{1}{1-2l}} (1 - (2l)^{-1}) \right)\end{aligned}\tag{C.2}$$

Thus,

$$\int dn \exp (A(l)n^{1/2l}) y_l^n \approx \exp \left\{ \left( 1 - \frac{1}{2l} \right) A(l)^{\frac{2l}{2l-1}} (2l\mu_l)^{\frac{1}{1-2l}} \right\}.\tag{C.3}$$

## References

- [1] R. Sasaki and I. Yamanaka, *Virasoro Algebra, Vertex Operators, Quantum Sine-Gordon and Solvable Quantum Field Theories*, *Adv. Stud. Pure Math.* **16** (1988) 271–296.
- [2] T. Eguchi and S.-K. Yang, *Deformations of Conformal Field Theories and Soliton Equations*, *Phys. Lett.* **B224** (1989) 373–378.
- [3] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, *Integrable structure of conformal field theory, quantum KdV theory and thermodynamic Bethe ansatz*, *Commun. Math. Phys.* **177** (1996) 381–398, [[hep-th/9412229](#)].
- [4] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, *Spectral determinants for Schrodinger equation and Q operators of conformal field theory*, *J. Statist. Phys.* **102** (2001) 567–576, [[hep-th/9812247](#)].

- [5] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, *Higher level eigenvalues of  $Q$  operators and Schroedinger equation*, *Adv. Theor. Math. Phys.* **7** (2003) 711–725, [[hep-th/0307108](#)].
- [6] S. Negro, *Integrable structures in quantum field theory*, *Journal of Physics A Mathematical General* **49** (Aug., 2016) 323006, [[1606.02952](#)].
- [7] A. B. ZAMOLODCHIKOV, *Integrals of motion and  $s$ -matrix of the (scaled)  $t = tc$  ising model with magnetic field*, *International Journal of Modern Physics A* **04** (1989) 4235–4248, [<https://doi.org/10.1142/S0217751X8900176X>].
- [8] W. Nahm, *A semiclassical calculation of the mass spectrum of relativistic strings*, *Nuclear Physics B* **81** (Oct., 1974) 164–178.
- [9] J. L. Cardy, *Operator Content of Two-Dimensional Conformally Invariant Theories*, *Nucl. Phys.* **B270** (1986) 186–204.
- [10] P. Kraus and A. Maloney, *A cardy formula for three-point coefficients or how the black hole got its spots*, *JHEP* **05** (2017) 160, [[1608.03284](#)].
- [11] D. Das, S. Datta and S. Pal, *Charged structure constants from modularity*, *JHEP* **11** (2017) 183, [[1706.04612](#)].
- [12] E. M. Brehm, D. Das and S. Datta, *Probing thermality beyond the diagonal*, [[1804.07924](#)].
- [13] B. Eynard, *Lectures notes on compact Riemann surfaces*, *arXiv e-prints* (May, 2018) arXiv:1805.06405, [[1805.06405](#)].
- [14] A. Maloney, S. G. Ng, S. F. Ross and I. Tsiaras, *Generalized Gibbs Ensemble and the Statistics of KdV Charges in 2D CFT*, [[1810.11054](#)].
- [15] A. Dymarsky and K. Pavlenko, *Generalized Gibbs Ensemble of 2d CFTs at large central charge in the thermodynamic limit*, [[1810.11025](#)].
- [16] A. Maloney, S. Ng, S. F. Ross and I. Tsiaras, *Thermal Correlation Functions of KdV Charges in 2D CFT*, [[1810.11053](#)].
- [17] R. Sasaki and I. Yamanaka, *Virasoro Algebra, Vertex Operators, Quantum Sine-Gordon and Solvable Quantum Field Theories*, *Adv. Stud. Pure Math.* **16** (1988) 271–296.
- [18] H. Dorn and G. Jorjadze, *Operator Approach to Boundary Liouville Theory*, *Annals Phys.* **323** (2008) 2799–2839, [[0801.3206](#)].
- [19] A. L. Fitzpatrick, J. Kaplan, M. T. Walters and J. Wang, *Hawking from Catalan*, *JHEP* **05** (2016) 069, [[1510.00014](#)].
- [20] G. H. Hardy and S. Ramanujan, *Asymptotic formulae in combinatory analysis*, *Proceedings of the London Mathematical Society* **s2-17** (1918) 75–115, [<https://londmathsoc.onlinelibrary.wiley.com/doi/pdf/10.1112/plms/s2-17.1.75>].
- [21] T. Hartman, C. A. Keller and B. Stoica, *Universal Spectrum of 2d Conformal Field Theory in the Large  $c$  Limit*, *JHEP* **09** (2014) 118, [[1405.5137](#)].

- [22] M. Srednicki, *Thermal fluctuations in quantized chaotic systems*, *Journal of Physics A: Mathematical and General* **29** (1996) L75.
- [23] J. M. Deutsch, *Quantum statistical mechanics in a closed system*, *Phys. Rev. A* **43** (Feb, 1991) 2046–2049.
- [24] M. Rigol, V. Dunjko and M. Olshanii, *Thermalization and its mechanism for generic isolated quantum systems*, *Nature* **452** (2008) 854–858.
- [25] D. A. Roberts and D. Stanford, *Two-dimensional conformal field theory and the butterfly effect*, *Phys. Rev. Lett.* **115** (2015) 131603, [[1412.5123](#)].
- [26] W.-Z. Guo, F.-L. Lin and J. Zhang, *Note on ETH of descendant states in 2D CFT*, [1810.01258](#).
- [27] J. de Boer and D. Engelhardt, *Remarks on thermalization in 2D CFT*, *Phys. Rev.* **D94** (2016) 126019, [[1604.05327](#)].