

# With Great Speed Come Small Buffers: Space-Bandwidth Tradeoffs for Routing

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## Abstract

We consider the Adversarial Queuing Theory (AQT) model, where packet arrivals are subject to a maximum average rate  $0 \leq \rho \leq 1$  and burstiness  $\sigma \geq 0$ . In this model, we analyze the size of buffers required to avoid overflows in the basic case of a path. Our main results characterize the space required by the average rate and the number of distinct destinations: we show that  $O(kd^{1/k})$  space suffice, where  $d$  is the number of distinct destinations and  $k = \lceil 1/\rho \rceil$ ; and we show that  $\Omega(\frac{1}{k}d^{1/k})$  space is necessary. For directed trees, we describe an algorithm whose buffer space requirement is at most  $1 + d' + \sigma$  where  $d'$  is the maximum number of destinations on any root-leaf path.

## 1 Introduction

**Background.** Routing, be it physical (cars, trains, air traffic) or virtual (data packets, voice call circuits, multicasts) has long been recognized as a critical component of any system that allows for interaction and communication. The introduction of packet networks and store-and-forward routing as the basic technique of the Internet (when it was still called DARPANET [11]) brought to the forefront an interesting parameter, namely the buffer space at nodes.

Buffers are, in some sense, the universal glue that connects different components of a system smoothly. Intuitively, buffers allow each part of a system to run at its own pace. This is necessary even in tightly synchronized systems, because local processing speed depends on local resources and local load, which are rarely uniform across the system. Obviously, buffers cannot make up for prolonged gaps between the rates of demand and capacity, but they help in overcoming short bursts by smoothing the demand over time. This partial decoupling of input and output makes buffer control and analysis a tricky issue. Andrews et al. [3] give strong results regarding the required buffer space and maximal latency, but assuming that all packets are available at start, and ignoring initial storage.

Typically, buffers are simple and easy to use so long as the load is light. However, problems arise once loads increase and buffers are filled: packet discards due to full buffers are usually viewed as a failure.<sup>1</sup> From the theoretical viewpoint (possibly because packet drops greatly complicate stochastic analysis à la queuing-theory), the issue of dropped packets was largely ignored by analytic studies in the Theory of Networks. The introduction of competitive analysis allowed for investigating the issue from the throughput viewpoint [13], gaining new insights into the problem

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<sup>1</sup>TCP, the Internet's prevalent transport protocol, is an exception in this respect. TCP in fact forces periodic packet drops, as a part of its ongoing quest to grab free bandwidth. Since no explicit information about utilization is assumed, the protocol keeps increasing demand until an indication of a packet drop is received.

(see, e.g., the survey [12]). But by and large, the positive results concerned a single device (a switch or a router), and results for even slightly more complex systems were mostly negative [14], in the sense that in the examined cases, the competitive ratio of any algorithm could be unacceptably large for practical applications.

A different approach was advocated by Adversarial Queuing Theory (AQT), introduced in [6]. AQT considers the scenario in which each packet has a predetermined route, and forwarding is *greedy*. That is, if there is a packet buffered for a link  $e$  at time  $t$ , then some packet will be forwarded over  $e$  at time  $t$ . A *scheduling policy* is a rule which selects which packet each buffer forwards each round. Classical AQT addresses the following qualitative question: Which scheduling policies ensure that all buffer occupancies are bounded irrespective of the duration of execution?

To have a meaningful answer for the above question, one must somehow restrict the demand presented by new packets. The criterion used by AQT (adopting ideas of Cruz [8]) is, roughly, that the set of routes associated with packets that appear together (“injected by the adversary”) at any step is edge-disjoint. This ensures that the raw bandwidth required by packets does not exceed the available bandwidth. (In fact, overlap is allowed, but not more than some  $\sigma \geq 0$ ; a second parameter,  $0 \leq \rho \leq 1$  bounds the maximum rate in which any edge is requested. See Def. 2.1.)

AQT was fruitful in identifying strengths and weaknesses of different (greedy) scheduling policies (e.g., [5]). However, the theory was limited in two important respects. First, restricting forwarding policies to be greedy turned out to be a serious handicap in terms of efficiency [2]. Secondly, the qualitative-only view of the required buffer space (either bounded or infinite) is quite coarse. For example, some protocols may be stable, but with space requirement which is exponential in the number of nodes. Notable exceptions to this state of affairs are the work of Adler and Rosén on DAGs [1] and the work by Rosén and Scalosub on lines [19]. The relative simplicity of the topologies in these studies is not coincidental. The difficulty of understanding maximum buffer size *quantitatively* drove research to examine simple cases first: Even the case of a line turned out to be fairly non-trivial (see [10] for a recent result).

**Recent progress: Single-destination trees.** Recently, several works have addressed the problem of quantifying the required buffer size under a  $(\rho, \sigma)$ -restricted packet injections, but without the requirement that the forwarding protocol must be greedy, and with an eye on *locality*. In [16], it was shown that in the case of a single-destination tree, a centralized algorithm can route any instance using  $O(\rho + \sigma)$  buffer space at every node. In [9] and [17] it is shown that  $\Theta(\rho \log n + \sigma)$  buffer space are necessary and sufficient for protocols with constant locality. That result was later extended in [18] to show that  $\Theta(\rho \lceil \frac{\log n}{r} \rceil + \sigma)$  is necessary and sufficient for protocols with locality  $r$ . An interesting consequence of this result is that  $O(\log n)$  locality is sufficient to perform as well as the best (offline) strategy.

**Our results: Multiple destinations.** In [17] it was shown that on a line,  $\Omega(d)$  space is necessary to avoid overflows if there are  $d$  distinct destinations, and if the average injection rate satisfies  $\rho > 1/2$ . In this paper we show that the latter condition was no accident: On one hand, we show that if the average rate satisfies  $\frac{1}{\ell+1} < \rho \leq \frac{1}{\ell}$  for some integer  $\ell$ , then there is a centralized online algorithm which routes all packets using buffer space  $O(\ell d^{1/\ell} + \sigma)$ ; and on the other hand, we prove a lower bound of  $\Omega(\frac{1}{\ell} d^{1/\ell} + \sigma)$  buffer space required by any (offline) algorithm, for any integer  $\ell > 1/\rho$ . In particular, this result means that if  $\rho \leq \frac{1}{\log d}$ ,  $O(\log d)$  buffer space suffices. In addition, for the directed tree topology, we prove that buffers of size  $1 + d' + \sigma$  are sufficient, where  $d'$  is an upper bound on the number of destinations along any leaf-root path.

**Implications and open problems.** One way to interpret our positive result is the following: Suppose that in a given line system, some buffer space suffices to avoid packet drops. If the system is modified so that the number of destinations is increased by a factor  $\alpha > 1$  without changing

the overall offered load per link, overflows can be avoided by either increasing the buffer sizes by factor  $\alpha$ , or by increasing both buffer space and link bandwidths by an  $O(\log \alpha)$  factor. This result corresponds nicely to previous results such as online virtual circuit admission control [4, 7] exhibiting  $O(\log n)$  competitiveness when demands are at most  $O(1/\log n)$  of the capacity.

Great challenges remain open on our way to fully understand the general case. These include finding a decentralized (local) algorithm for the multi-destination case, and extending the restricted topologies we consider to general topologies. Regarding the former, we expect the OED algorithm [9, 17] to be useful as it was for algorithms of parameterized locality [18]. Regarding the latter, we note that the case of a union of trees is also important, due to the fact that this topology is the output of many routing algorithms.

**Paper organization.** We present the algorithm gradually, from the simplest to the most general case: In Sec. 3 we present algorithms for any utilization parameter  $\rho$ , and also extend them to directed trees. The algorithm is then generalized in Sec. 4 to be hierarchical, with  $\lfloor \frac{1}{\rho} \rfloor$  hierarchy levels. The lower bound is presented in Sec. 5. Sec. 2 formalizes the model and defines some concepts and notation. Due to lack of space, many proofs appear in an appendix.

## 2 Preliminaries

Given a natural number  $n$ , we use the notation  $\langle n \rangle = \{0, \dots, n-1\}$ . We model a network as a directed graph  $G = (V, E)$ , where packets flow along the direction of edges. For the most part, in this paper we restrict attention to the case where  $G$  is a path, and take  $V = \langle n \rangle$ ,  $E = \{(i, i+1) : 0 \leq i < n-1\}$ .

An execution of an algorithm proceeds in synchronous rounds. Each round consists of two steps: an *injection step*, during which new packets arrive, and a *forwarding step*, during which the algorithm performs computations and forwards packets. In this paper we assume that in each round, at most one packet can be forwarded over each link. When we refer to a state at round  $t$ , we compute the state *after* the injection step, but *before* the forwarding step. We denote the time in round  $t$  immediately after the forwarding step by  $t+$ . E.g.,  $L^t(v)$  denotes the state of the buffer at  $v$  after packets are injected and before forwarding; and  $L^{t+}(v)$  denotes the state after forwarding, but before round  $t+1$  packets arrive.

A node may partition its buffer into several parts, which we refer to as *pseudo-buffers*. We use the notation  $L_k^t(v)$  and  $L_{j,k}^t(v)$  to denote pseudo-buffers, where  $j$  and  $k$  are parameters described below. We make no assumptions about the priority assigned to packets within a single (pseudo)-buffer, but for concreteness it is convenient to assume that all (pseudo)-buffers use Last-In, First-Out (LIFO) priority.

Throughout an execution, packets arrive in the network, controlled by an adversary. An *adversary* or *injection pattern* is a set  $A$  of packets. A packet  $P$  is represented by a triple  $P = (t, i_P, w_P)$  where  $t \in \mathbf{N}$  is injection round of  $P$ ,  $i_P \in V$  is  $P$ 's injection site, and  $w_P \in V$  is  $P$ 's destination. We denote the unique path from  $i_P$  to  $w_P$  by  $\text{Path}(i_P, w_P)$ .

We say that  $P$ 's path *contains* a buffer  $v$  if  $v \in \text{Path}(i_P, w_P)$ . For an adversary  $A$ , a set of rounds  $T$ , and a buffer  $v$ , we denote the number of packets injected during  $T$  whose trajectories contain  $v$  by  $N_T(v)$ . That is,  $N_T(v) = |\{(t, i_P, w_P) \in A : v \in \text{Path}(i_P, w_P) \text{ and } t \in T\}|$ .

**Definition 2.1.** For any  $\rho \in \mathbf{R}^+$  and  $\sigma \in \mathbf{N}$  and adversary  $A$ , we say that  $A$  is  $(\rho, \sigma)$ -bounded if for all buffers  $v$  and intervals of time  $I$ , we have  $N_T(v) \leq \rho|I| + \sigma$ .

Intuitively, an adversary  $A$  is  $(\rho, \sigma)$ -bounded if the average rate of packets needing to cross any particular edge  $i$  is at most  $\rho$ . This average may not be exceeded by more than  $\sigma$  packets when

taken over any contiguous interval of time. The term  $\sigma$  thus bounds the “burstiness” of  $A$ .

The term “excess” gives a measure of how much of the adversary’s burst budget has been expended.

**Definition 2.2.** Fix an adversary  $A$ , rate  $\rho$ , round  $t$ , and buffer  $v \in V$ . The *excess* of  $v$  at time  $t$ , denoted  $\xi^t(v)$ , is given by

$$\xi^t(v) = \max_{s \leq t} (\{N_{[s,t]}(v) - \rho(t - s + 1)\} \cup \{0\}). \quad (1)$$

The following lemma, which applies to graphs of any topology, gives basic properties of excess. The proof appears in Appendix A.

**Lemma 2.3.** Suppose  $A$  is  $(\rho, \sigma)$ -bounded. Then for all rounds  $t$  and buffers  $v \in V$ , we have (1)  $\xi^t(v) \leq \sigma$ , and (2)  $N_{\{t\}}(v) \leq \xi^t(v) - \xi^{t-1}(v) + \rho$ .

In what follows, we sometimes consider algorithms that only accept packets every few rounds. Specifically, for some fixed  $\ell$  and every  $k \in \mathbf{N}$ , the algorithm treats all packets injected at times  $t = (k - 1)\ell + 1, (k - 1)\ell + 2, \dots, k\ell$  as being injected at time  $k\ell$ . The algorithm then performs a single forwarding step at time  $k\ell$ , waits until  $(k + 1)\ell$  to perform its next forwarding step, etc.

**Definition 2.4.** Let  $A$  be an adversary and  $\ell$  a positive integer. The  $\ell$ -*reduction* of  $A$ , denoted  $A_\ell$  is the injection pattern defined by  $A_\ell = \{(\lfloor \frac{t-1}{\ell} \rfloor + 1, i_P, w_P) : (t, i_P, w_P) \in A\}$ .

The following lemma (proven in the appendix) characterizes  $A_\ell$  using the parameters of  $A$ .

**Lemma 2.5.** Suppose  $A$  is  $(\rho, \sigma)$  bounded and  $\ell$  a positive integer. Then  $A_\ell$  is  $(\ell \cdot \rho, \sigma)$  bounded.

### 3 The Basic Algorithms

In this section, we describe a simple algorithm that requires buffers of size  $O(d + \sigma)$  for all rates  $\rho \leq 1$  and  $d$  destinations. By Thm. 5.1, this space requirement is optimal for rates  $\rho$  satisfying  $1/2 < \rho \leq 1$ . We break the description into two parts. We first consider the case where all packets have a common destination: Algorithm “Peak-to-Sink” (PTS) solves it using the best possible buffer space of  $O(1 + \sigma)$  packets. We then reduce the case of multiple destinations to single destinations in Algorithm “Parallel Peak-to-Sink” (PPTS). Finally we extend the result to directed trees.

*Implementation convention.* In order to determine which buffers forward packets in round  $t$ , a buffer  $L^t(v)$  can be *active* or *inactive*. In each round, all buffers are initially inactive. During the forwarding step, our algorithms select buffers to activate, then all active buffers simultaneously forward. We denote the family of activated buffers by  $\mathcal{A}$ .

#### 3.1 Peak to Sink (PTS) forwarding

Assume first that all packets  $P \in A$  have a common destination  $w$ , which we take to be  $w = n - 1$  without loss of generality. Consider a configuration  $L^t$ . We say that a buffer  $i \in \langle n \rangle$  is *bad* if  $|L^t(i)| \geq 2$ . PTS (Alg. 1) is simple: if  $i$  is the left-most bad buffer, then all non-empty buffers  $i' \geq i$  forward. PTS requires the minimal possible buffer space, as we claim next (see Appendix B for a proof).

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#### Algorithm 1 PTS( $w$ )

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- 1: **for all** rounds  $t$  **do**
  - 2:    $i \leftarrow$  left-most buffer with  $|L^t(i)| \geq 2$
  - 3:    $\mathcal{A} \leftarrow [i, w - 1]$
  - 4:   forward all buffers in  $\mathcal{A}$
  - 5: **end for**
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**Proposition 3.1.** Suppose  $A$  is a  $(\rho, \sigma)$ -bounded adversary with  $\rho \leq 1$  and all packets  $P \in A$  have destination  $w$ . Then the maximum buffer size is at most  $2 + \sigma$ .

### 3.2 Parallel Peak to Sink (PPTS) forwarding

Assume now that there are  $d$  destinations. Let  $W = \{w_0, w_1, \dots, w_{d-1}\}$  denote the set of destinations, where  $w_0 < w_1 < \dots < w_{d-1}$ . The idea of the algorithm is for each  $w \in W$  to treat all packets with destination  $w$  as an instance of the single destination algorithm. Capacity constraints preclude us from performing all  $d$  instances in parallel every round, so PPTS chooses an appropriate maximal set of (pseudo)-buffers that can simultaneously forward.

To describe PPTS formally, we view each buffer  $i$  as a collection of  $d$  pseudo-buffers, where the  $k$ -th pseudo-buffer in  $i$  stores packets destined for  $w_k$ .<sup>2</sup> We denote the contents of  $i$ 's  $k$ -th pseudo-buffer by

$$L_k^t(i) = \{P \in L^t(i) \text{ and } P \text{ has destination } w_k\}.$$

For any destination  $w_k$ , we refer to pseudo-buffers  $L_k(i)$  as  **$k$ -pseudo-buffers**. We say that a pseudo-buffer  $L_k^t(i)$  is **bad** at time  $t$  if  $|L_k^t(i)| \geq 2$ . In this case we call  $L^t(i)$  **bad for  $k$** . We call a packet  $P$  in a  $k$ -pseudo-buffer  **$k$ -bad** if it is stored in a position  $p \geq 2$  in  $L_k^t(i)$ .

PPTS (Alg. 2) works as follows. Starting from the right-most destination,  $w_{d-1}$ , the algorithm searches for the left-most bad  $(d-1)$ -pseudo-buffer, if any. If a bad  $(d-1)$ -pseudo-buffer  $L_{d-1}^t(i_{d-1})$  is found, all  $(d-1)$ -pseudo-buffers in the range  $[i_{d-1}, w_{d-1} - 1]$  are activated. The algorithm then searches for bad  $d-2$  pseudo-buffers. Suppose  $L_{d-2}^t(i_{d-2})$  is the left-most  $(d-2)$ -pseudo-buffer. If  $i_{d-2} < i_{d-1}$ , then all  $(d-2)$ -pseudo-buffers in the interval  $[i_{d-2}, i_{d-1} - 1]$  are activated; otherwise, no  $(d-2)$ -pseudo-buffers are activated. This process is continued for destinations  $w_{d-3}, w_{d-4}, \dots, w_0$ , where an interval of  $k$ -pseudo-buffers is activated only if a  $k$ -bad pseudo-buffer is encountered to the left of all pseudo-buffers activated thus far. By construction, the intervals activated for each destination are disjoint, so forwarding can be implemented without violating capacity constraints.

We note that algorithm PPTS need not be told the set of destinations  $W$  to which an adversary  $A$  injects in advance. Instead, we can assume that all nodes  $i \in [1, n-1]$  are potential destinations, giving rise to  $n$  pseudo-buffers per node. However, if an adversary only injects packets with  $d$  distinct destinations, then at most  $d$  pseudo-buffers in any given buffer will be non-empty.

Our analysis shows that for any  $(\rho, \sigma)$ -bounded adversary  $A$ , PPTS maintains the following invariant: For each buffer  $i$ , the number of bad packets that must cross  $i$ —i.e., the number of bad packets in buffers  $i' \leq i$  with destinations  $w_k > i$ —is at most one plus the excess  $\xi^t(i) + 1 \leq \sigma + 1$ . In particular, every buffer  $L(i)$  contains at most  $1 + \sigma + d$  packets, as it can contain at most  $d$  packets that are not bad. The following proposition states the key property of PPTS.

**Proposition 3.2.** *Let  $A$  be any  $(\rho, \sigma)$ -bounded adversary such that all packets have destinations in  $W$  with  $d = |W|$ . Then the maximum buffer usage of PPTS is at most  $1 + d + \sigma$ .*

To facilitate the proof, we introduce the concept of “badness,” along with some new notation.

**Definition 3.3.** *Let  $i \in \langle n \rangle$  be a buffer, and  $w_k$  a destination such that  $w_k > i$ . Given  $t$ , we denote the number of  $k$ -bad packets in  $i$  at time  $t$  by  $\beta_k^t(i)$ . Formally,  $\beta_k^t(i) = \max\{|L_k^t(i)| - 1, 0\}$ . The*

<sup>2</sup>This is known in practice as “virtual output queuing” (e.g., [15]).

**$k$ -badness** of  $i$ , denoted  $B_k^t(i)$ , is the total number of bad packets in buffers  $i' \leq i$  with destination  $w_k$ . That is,  $B_k^t(i) = \sum_{i' \leq i} \beta_k^t(i')$ . The **badness** of  $i$ , denoted  $B^t(i)$ , is the total number of bad packets in buffers  $i' \leq i$  with destinations  $w > i$ . Thus,  $B^t(i) = \sum_{k:w_k > i} B_k^t(i)$ .

(We stress that  $B_k^t(i)$  and  $B^t(i)$  count also badness of packets upstream from  $i$ .) In the proof of Prop. 3.2, we use the following key lemma (proof in Appendix B).

**Lemma 3.4.** *Let  $L$  be a configuration of  $\langle n \rangle$  with destinations  $W = \{w_0 < w_1 < \dots < w_{d-1}\}$ . Suppose that for some  $k \in \langle d \rangle$ ,  $I_k = [a_k, b_k] \subseteq \langle n \rangle$  is an interval such that  $|L_k(a_k)| \geq 2$  and  $b_k < w_k$ . Let  $L^+$  be the configuration obtained by forwarding only from nonempty  $k$ -pseudo-buffers in  $I_k$ . For  $i \in \langle n \rangle$  let  $B_k(i)$  and  $B_k^+(i)$  be the  $k$ -badness of  $i$  before and after forwarding, respectively. Similarly,  $B(i)$  and  $B^+(i)$  denote the badness of  $i$  before and after forwarding. Then*

$$B_k^+(i) \leq \begin{cases} B_k(i) - 1 & \text{if } i \in I_k \\ B_k(i) & \text{if } i \notin I_k \end{cases}, \quad \text{and similarly,} \quad B^+(i) \leq \begin{cases} B(i) - 1 & \text{if } i \in I_k \\ B(i) & \text{if } i \notin I_k \end{cases}.$$

### 3.3 Extension to Trees

In Appendix B.2, we explain how to extend both PTS and PPTS to the case of directed trees. We assume that all edges are directed toward the root, and that packets follow directed paths. In this setting we prove the following.

**Proposition 3.5.** *Let  $A$  be any  $(\rho, \sigma)$ -bounded adversary such that all packets have destinations in  $W$ . Then the maximum buffer usage of PPTS is at most  $1 + d' + \sigma$ , where  $d'$  is an upper bound on the number of destinations in any leaf-root path.*

## 4 Hierarchical Algorithm

In this section, we describe Algorithm HPTS (Hierarchical Peak-to-Sink), that achieves significantly better performance than PPTS in the case where the rate of the adversary satisfies  $\rho \leq 1/2$ . The algorithm description is parametrized by the number of **levels** in the construction, denoted  $\ell \in \mathbb{N}$ . In the case  $\ell = 1$ , HPTS reduces to PPTS. In general case, we obtain the following result. For the remainder of the section, we assume that  $\rho$  and  $\ell$  satisfy  $\rho \cdot \ell \leq 1$ .

**Theorem 4.1.** *There exists an algorithm, HPTS, such that for every positive integer  $\ell$  and  $(\rho, \sigma)$ -bounded adversary  $A$  such that  $\rho \cdot \ell \leq 1$ , the space requirement of HPTS is at most  $\ell n^{1/\ell} + \sigma + 1$ .*

### 4.1 Partitioning the network

Our algorithm is based on a hierarchical partition of the line. For simplicity, we assume that  $n = m^\ell$  for some integers  $m$  and  $\ell$ . We consider the base- $m$  representation of buffer indices. That is, given  $i \in \langle n \rangle$  we write the string  $i = i_{\ell-1}i_{\ell-2} \dots i_0$ , where  $i_j \in \langle m \rangle$  for  $j = 0, \dots, \ell - 1$  and  $i = \sum_{j=0}^{\ell-1} i_j m^j$ . We refer to  $i_j$  as the  $j$ -th *digit*, or the digit in the  $j$ -th *position*, of  $i$ .

Using this convention, for  $j \in \langle \ell \rangle$  and  $r \in \langle m^{\ell-j-1} \rangle$ , define the intervals  $I_{j,r}$  by

$$I_{j,r} = \{rm^{j+1} + k : 0 \leq k < m^{j+1}\}.$$

Intuitively, the interval  $I_{j,r}$  consists of all  $m^j$  nodes whose most significant  $\ell - j - 1$  digits have value  $r$ . For each fixed  $j$ , the set of intervals  $\mathcal{I}_j = \{I_{j,r} : r \in \langle m^j \rangle\}$  forms a partition of  $\langle n \rangle$ . We call  $\mathcal{I}_j$  the **level- $j$**  partition, and refer to each  $I_{j,r} \in \mathcal{I}_j$  as a **level- $j$**  interval. For each  $j > 0$ , each

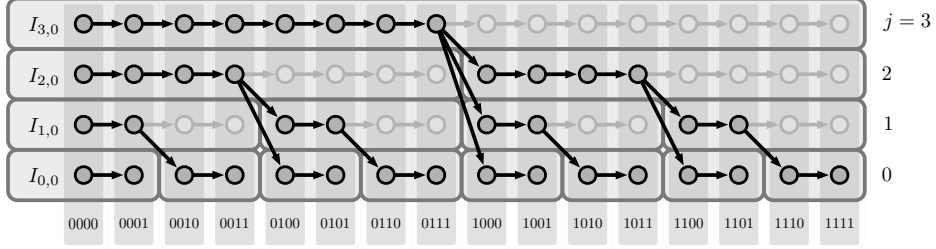


Figure 1: The network with  $n = 16$ ,  $m = 2$ , and  $\ell = 4$ . Each column represents a single buffer, and each row represents a level. The horizontal boxes indicate divisions between intervals in  $\mathcal{I}$ . Given a packet  $P$  injected into  $i$  with destination  $w$ , the virtual trajectory of  $P$  is given by the unique path from a pseudo-buffer in  $i$  to a level-0 pseudo-buffer in  $w - 1$ .

level- $j$  interval contains exactly  $m$  level- $(j - 1)$  subintervals. We denote the union of the partitions by  $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1 \cup \dots \cup \mathcal{I}_{\ell-1}$ .

The idea of HPTS is to run an independent instance of PPTS in each interval of  $\mathcal{I}$ , allowing each interval of level  $j$  to have exactly  $m$  (intermediate) destinations: one for each left endpoint of a subinterval of level  $j - 1$ . Intermediate destinations are chosen so as to “correct” the index of a packet’s location to the index of its final destination, digit by digit: position  $j$  of the index is corrected at level  $j$ . Formally, we have the following.

**Definition 4.2.** Let  $i, w \in \langle n \rangle$ . Assume that  $i < w$  and that the largest position (in base- $m$  notation) in which  $i$  and  $w$  differ is  $j$ . Then the **intermediate destination of  $i$  to  $w$**  is  $x(i, w) = \lfloor w/m^j \rfloor m^j$ . The route  $[i, x(i, w)]$  is called a **segment**, and its **level**, denoted  $\text{lv}(i, w)$ , is  $j$ .

A packet  $P$  with destination  $w$  residing in a buffer  $L(i)$ , **crosses  $i$  at level**  $\text{lv}(i, P) = \text{lv}(i, w)$ . If  $j = \text{lv}(i, P)$ , we define  $P$ ’s **level- $j$  intermediate destination** to be  $w_{P,j} = x(i, w)$ .

If the largest position in which  $i$  and  $w$  differ is  $j$ , then the segment from  $i$  to  $x(i, w)$  is contained in a level- $j$  interval  $I_j$ . If  $j \geq 1$ , then for all  $j' \leq j$ ,  $x(i, w)$  is the left endpoint of some level- $j'$  interval. If a packet is injected into buffer  $i$  with destination  $w$ , we can think of its trajectory  $[i, w]$  as being partitioned into segments  $[i, x(i, w) - 1]$ ,  $[x(i, w), x(x(i, w), w)]$ ,  $\dots$ , where the level of these segments is strictly decreasing. Further, with the possible exception of the initial segment, the left endpoints of all segments are left endpoints of intervals  $I \in \mathcal{I}$ .

When  $P$  is stored in a buffer  $i$ , we think of  $P$  as virtually residing in the subinterval  $I_j$  at level  $j = \text{lv}(i, P)$  containing  $i$ . When  $P$  is forwarded to its next intermediate destination,  $w_{P,j} = x(i, w)$ , its virtual location moves the corresponding interval at level  $j' = \text{lv}(w_{P,j}, P)$ . We emphasize that the *virtual* location of each packet  $P$  in a buffer  $i$  (i.e., level and intermediate destination) is a function of  $i$  and  $P$ ’s destination. Thus,  $i$  can compute the virtual location of each packet it stores locally. Figure 1 illustrates the virtual motion packets through the network. Formally, the HPTS algorithm implements the virtual packet movement by partitioning each buffer into pseudo-buffers, as defined below.

**Definition 4.3.** Let  $t \in \mathbf{N}$ ,  $i \in \langle n \rangle$ ,  $j \in \langle \ell \rangle$  and  $k \in \langle m \rangle$ . Let  $I_j(i)$  be the level- $j$  interval containing  $i$ , and let  $W_j(i) = \{w_0 < w_1 < \dots < w_{m-1}\}$  denote the left endpoints of the level- $(j - 1)$  intervals contained in  $I_j(i)$ . We define the  **$(j, k)$ -pseudo-buffer** of  $i$  at time  $t$ , denoted  $L_{j,k}^t(i)$ , to contain all packets in  $I_j(i)$  whose current segment level is  $j$  and whose next intermediate destination is  $w_k$ .

Since  $j \in \langle \ell \rangle$  and  $k \in \langle m \rangle$ , each buffer  $L^t(i)$  is partitioned into  $\ell m = \ell n^{1/\ell}$  pseudo-buffers. As

in our description of the PPTS algorithm, we define a notion of badness for packets, pseudo-buffers, and buffers.

**Definition 4.4.** Let  $t \in \mathbf{N}$ ,  $i \in \langle n \rangle$ ,  $j \in \langle \ell \rangle$ , and  $k \in \langle m \rangle$ . We say that the  $(j, k)$ -pseudo-buffer  $L_{j,k}^t(i)$  is **bad** at time  $t$  if  $|L_{j,k}^t(i)| \geq 2$ . A packet in  $L_{j,k}^t(i)$  is called **bad** if its position in  $L_{j,k}^t(i)$  is at least 2. The number of bad packets in  $L_{j,k}^t(i)$  is  $\beta_{j,k}^t(i) = \max \left\{ |L_{j,k}^t(i)| - 1, 0 \right\}$ .

In the definition below, note that the badness of a node accounts also for bad packets in other (upstream) nodes.

**Definition 4.5.** Let  $i \in \langle n \rangle$ ,  $j \in \langle \ell \rangle$ ,  $k \in \langle m \rangle$  and let  $I = [a, b]$  be a level- $j$  interval containing  $i$ . The  **$(j, k)$ -badness** of a node  $i$  at time  $t$ , denoted  $B_{j,k}^t(i)$ , is defined by  $B_{j,k}^t(i) = \sum_{i'=a}^i \beta_{j,k}^t(i')$ . That is,  $B_{j,k}^t(i)$  is the number of bad packets in buffers  $i' \in I$  with  $i' \leq i$  whose segment level at  $i$  is  $j$  and whose level  $j$  intermediate destination is  $w_k$ . The **level- $j$  badness** of  $i$  is  $B_j^t(i) = \sum_{k \in \langle m \rangle} B_{j,k}^t(i)$ , and the **badness** of  $i$  is  $B^t(i) = \sum_{j \in \langle \ell \rangle} B_j^t(i)$ .

## 4.2 Algorithm description

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### Algorithm 3 HPTS( $n, \ell, t$ ) step

```

1:  $\mathcal{A} \leftarrow \emptyset$            {active pseudo-buffers}
2:  $\lambda \leftarrow t \bmod \ell$ 
3: if  $\lambda = 0$  then
4:   accept round  $(t-\ell), \dots, (t-1)$ -injections
5: end if
6: for all  $r \in \langle m^{\ell-\lambda} \rangle$  do
7:    $\mathcal{A} \leftarrow \text{FormPaths}(I_{\lambda,r}, \lambda)$ 
8: end for
9: for  $j = \lambda - 1$  downto 0 do
10:   $\mathcal{A} \leftarrow \text{ActivatePreBad}(\mathcal{A}, j)$ 
11: end for
12: each nonempty  $L_{j,k}(i) \in \mathcal{A}$  forwards a
    packet

```

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### Algorithm 4 FormPaths( $I, j$ )

```

1: Let  $W = \{w_0 < w_1 < \dots < w_{m-1}\}$  be the intermedi-
   ate destinations in  $I$ 
2:  $\mathcal{A} \leftarrow \emptyset$ ;  $i' \leftarrow w_{m-1}$ 
3: for  $k \leftarrow m - 1$  downto 0 do
4:   if there exists  $i < i'$  such that  $i$  is bad for  $w_k$  then
5:      $i_k \leftarrow \min \left\{ i'' : |L_{j,k}^t(i'')| \geq 2 \right\}$ 
6:      $\mathcal{A} \leftarrow \mathcal{A} \cup \left\{ L_{j,k}^t(i) \mid i \in [i_k, \min \{i' - 1, w_k - 1\}] \right\}$ 
7:      $i' \leftarrow i_k$ 
8:   end if
9: end for
10: return  $\mathcal{A}$ 

```

---

The idea of Algorithm HPTS (Alg. 3) is as follows. Let us refer to a sequence of rounds  $t, t+1, \dots, t+\ell-1$  with  $t \equiv 0 \pmod{\ell}$  as a **phase**. That is, the  $\varphi$ -th phase consists of rounds  $(\varphi-1)\ell, \dots, \varphi\ell-1$ . We assume that all packets injected during a phase  $\varphi$  are only accepted during the first round of phase  $\varphi+1$  (Lines 3–5). This results in  $A_\ell$ , i.e., the  $\ell$ -reduction of  $A$  (cf. Def. 2.4).

Due to capacity constraints, not all pseudo-buffers can simultaneously forward packets, so HPTS employs *time-division multiplexing*: At any time  $t$ , only the level- $\lambda$  intervals are activated, where  $\lambda \equiv t \pmod{\ell}$ . Since same-level intervals are edge-disjoint, all intervals in  $\mathcal{I}_\lambda$  can be active in parallel (Lines 6–8).

We note that the family of buffers activated by Algorithm 4 is identical to the activation pattern in an iteration of PPTS in which (1) only level- $j$  packets are considered, and (2) the destination of each packet is taken to be its level- $j$  intermediate destination. The difference between HPTS and PPTS is in Lines 9–11 of Alg. 3: Activating buffers in one level may cause the activation of buffers in lower levels. This is done to anticipate when a packet reaches an intermediate destination and switches level—and interval instance—where it may become bad in the new instance. Activating



the new instance just before a new packet arrives to a bad position is sufficient to avoid an increase of badness for any buffer in a given round. Formally, we define the notion of pre-badness as follows.

**Definition 4.6.** Suppose  $L_{j,k}^t(i)$  is activated and let  $P \in L_{j,k}^t(i)$  be the packet sent by  $L_{j,k}^t(i)$ . Suppose that  $P$ 's level- $j$  intermediate destination is  $i+1$  (i.e.,  $P$  is about to make the last hop in its level- $j$  segment), where it will be associated with pseudo-buffer  $L_{j',k}(i')$  with  $j' < j$ . We say that the packet  $P$  is **pre-bad** if  $|L_{j',k}(i+1)| \geq 1$ . In this case, we also call the buffer  $i+1$  **pre-bad for  $P$  at level  $j'$** .

### 4.3 Algorithm analysis

Our strategy in the analysis of HPTS is to bound the badness of each buffer  $i$  by  $\sigma$  at the end of each phase (just as our analysis of PPTS bounds the badness of each buffer at the end of each round). Since each buffer contains  $\ell \cdot m = \ell \cdot n^{1/\ell}$  pseudo-buffers, each buffer can contain at most  $\ell \cdot n^{1/\ell}$  non-bad packets. Thus Theorem 4.1 follows from giving an appropriate bound on the badness of each buffer.

The basic argument is the same as our analysis of PPTS: apply Lemma 2.3 to

show that the increase in badness of a buffer is at most one more than the increase in excess of the buffer, then show that forwarding reduces the badness of the buffer by at least one (if it is positive). First, we show that the forwarding in Line 12 of HPTS is feasible (proof in Appendix C).

**Lemma 4.7.** Consider the set  $\mathcal{A}$  of active pseudo-buffers immediately before Line 12 of HPTS. Then  $\mathcal{A}$  is feasible, i.e., for each  $i \in \langle n \rangle$ , there is at most one active pseudo-buffer  $L_{j,k}^t(i) \in \mathcal{A}$ .

Lemma 4.8 asserts that if a node has positive badness at the beginning of a phase, its badness strictly decreases during the phase. Thus, it is analogous to the analysis of PTS and PPTS (see App. B). Thm. 4.1 follows from an argument analogous to the proofs of Prop. 3.1 and Prop. 3.2.

**Lemma 4.8.** Let  $t = (\varphi - 1)\ell$  so that the  $\varphi$ -th phase consists of rounds  $t+1, t+2, \dots, t+\ell$ . Assume that no new packets arrive during rounds  $t+2, t+3, \dots, t+\ell$ . Then the HPTS algorithm guarantees that for all  $i \in \langle n \rangle$ ,  $B^{(t+\ell)^+}(i) \leq \max\{B^{t+1}(i) - 1, 0\}$ .

The proof of Lemma 4.8 (in appendix C) follows from two claims. First, in each round of a phase, the badness of each node is non-increasing. Second, if  $B^{t+1}(i) > 0$ , then  $B(i)$  strictly decreases during some round of the phase. Essentially we show that if  $B_\lambda(i) > 0$  during the  $\lambda$ -th round of a phase, then either  $B(i)$  decreases, or a bad packet (for  $i$ ) at level  $\lambda$  is replaced with a bad packet at some level  $j < \lambda$  (that was pre-bad before forwarding). Since levels are activated in decreasing order over the course of a phase,  $B(i)$  strictly decreases in some phase.

*Proof of Theorem 4.1.* Let  $A$  be a  $(\rho, \sigma)$ -bounded adversary such that  $\rho \cdot \ell \leq 1$ , and let  $A_\ell$  be the  $\ell$ -reduction of  $A$  (Definition 2.4). For each  $\varphi$ , let  $t(\varphi) = (\varphi - 1) \cdot \ell + 1$  denote the beginning of the  $\varphi$ -th phase, and let  $t(\varphi+)$  denote the time at the end of phase  $\varphi$ —that is, after the forwarding step of round  $t(\varphi) + \ell - 1$ .

We claim that for each  $\varphi$  and buffer  $i \in \langle n \rangle$ , we have  $B^{t(\varphi)}(i) \leq \xi^{t(\varphi)}(i) + 1$  and  $B^{t(\varphi)^+}(i) \leq \max\{B^{t(\varphi)}(i) - 1, 0\}$ . The argument follows by induction. For the base case, both inequalities hold

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#### Algorithm 5 ActivatePreBad( $\mathcal{A}, j$ )

---

```

1: for all  $r \in \langle m^{\ell-j-1} \rangle$  do
2:    $I = [a, b] \leftarrow I_{j,r}$ 
3:   if  $\exists P$  pre-bad for  $a$  at level  $j$  and  $a$  is inactive then
4:      $k \leftarrow$  index s.t.  $w_k = w_{P,j}$ 
5:      $w \leftarrow \max\{i \in I : i \leq w_k \text{ and } [a, i] \text{ is inactive}\}$ 
6:      $\mathcal{A} \leftarrow \mathcal{A} \cup \{L_{j,k}^t(i) : i \in [a, w]\}$ 
7:   end if
8: end for
9: return  $\mathcal{A}$ 

```

---

trivially, as no packets have been accepted. For the inductive step, suppose the equations above hold for phase  $\varphi - 1$ . By Lemmas 2.3 and 2.5, the number of new packets arriving whose paths contain  $i$  during phase  $\varphi - 1$  satisfies

$$N_{[t(\varphi-1)+1, t(\varphi)]}(i) \leq \xi^{t(\varphi)}(i) - \xi^{t(\varphi-1)+1}(i) + \ell \cdot \rho.$$

Therefore, by the inductive hypothesis, we have

$$B^{t(\varphi)}(i) \leq B^{t(\varphi-1)+1}(i) + N_{[t(\varphi-1)+1, t(\varphi)]}(i) \leq \xi^{t(\varphi)}(i) + \ell \cdot \rho \leq \xi^{t(\varphi)}(i) + 1.$$

The inequality  $B^{t(\varphi+)}(i) \leq \max\{B^{t(\varphi)}(i) - 1, 0\}$  follows from Lemma 4.8.  $\square$

## 5 Lower Bound

We show now that the upper bound achieved by HPTS is optimal, up to a factor of  $O(\rho^{-2})$ .

**Theorem 5.1.** *For any  $\rho > 1/(\ell + 1)$ , there exists a  $(\rho, 1)$ -bounded adversary such that for all forwarding protocols, the buffer requirement is  $\Omega(\frac{(\ell+1)\rho-1}{2\ell}n^{1/\ell})$ .*

Assume that  $n$  is of the form  $n = (\ell + 1)m^\ell$  for some  $m, \ell \in \mathbf{N}$ ,  $\ell \geq 2$ . We define an injection pattern  $A$  consisting of  $m^\ell$  phases each of length  $m$ . During each phase,  $A$  injects packets with  $\ell + 1$  non-overlapping routes, for a total of  $\rho m(\ell + 1)$  packets. The right-most injection site, denoted  $F(t)$ , decreases by some predetermined amount at the end of each phase, and routes for the next phase are adjusted correspondingly. The routes are constructed such that by the time *any* packet reaches its destination, it must be to the right of the current value of  $F(t)$ . Thm. 5.1 follows by establishing the following dichotomy: when  $F(t)$  is decreased at the end of each phase, either the average load of the interval  $[F(t + 1), F(t)]$  is  $\Omega(m)$ , or the average load behind  $F(t + 1)$  increased during the round. Thus, we show that at the end of the execution, either some interval  $[F(t + 1), F(t)]$  had a large average load at the end of some phase, or the average load in  $[0, F(t)]$  is sufficiently large.

We express each round number  $t \geq 0$  in as its base- $m$  representation  $t_\ell t_{\ell-1} \dots t_0$  where  $t = \sum_{i=0}^{\ell} t_i m^i$  with each  $t_i \in \langle m \rangle$ . For each round  $t \geq 0$ , the phase containing  $t$  is called the  $t_\ell \dots t_1$ -**phase**. Equivalently, the  $t_\ell \dots t_1$ -phase consists of the  $m$  rounds starting with  $\sum_{k=1}^{\ell} t_k m^k$ .

In each  $t_\ell \dots t_1$ -phase, the packet injection sites are defined based on the values of  $t_\ell, \dots, t_1$  as follows. For each  $t_\ell \dots t_1$ -phase, and  $i \in [\ell]$  we define the  $i$ -th site

$$v_i(t_\ell \dots t_1) = \sum_{k=i}^{\ell} \left( (k+1)m^k - (t_k+1)km^{k-1} \right).$$

**The Injection Pattern** During the  $t_\ell \dots t_1$ -phase,  $A$  injects packets as follows:

- inject  $\rho m$  packets into buffer  $v_1(t_\ell \dots t_1)$  with destination  $n$ ;
- for each  $k = 2, \dots, \ell$ , inject  $\rho m$  packets into buffer  $v_k(t_\ell \dots t_1)$  with destination  $v_{k-1}(t_\ell \dots t_1)$ ;
- inject  $\rho m$  packets into buffer 0 with destination  $v_\ell(t_\ell \dots t_1)$ .

We define the **type** of a packet by its injection site: for each  $k \in \{1, \dots, \ell\}$ , packets injected at buffer  $v_k(t_\ell \dots t_1)$  are **type- $k$**  packets and those injected at buffer 0 are **type- $(\ell + 1)$**  packets.

For any round  $t$ , we denote  $F(t) = v_1(t_\ell \dots t_1)$ . That is,  $F(t)$  is the site of all injections of type-1 packets (which coincides with destination of all injected type-2 packets) during the  $t_\ell \dots t_1$ -phase. For any packet  $P$ , let  $P(t)$  denote the buffer in which packet  $P$  is stored in round  $t$ . We assign each packet a status depending on the relative values of  $P(t)$  and  $F(t)$ : we say that  $P$  is **fresh** in round  $t$  if  $P(t) \leq F(t)$ . Otherwise, if  $P(t) > F(t)$ , then  $P$  is **stale**. Note that a packet is fresh when it is

injected in a round  $t$ , since  $P(t)$  is either 0 or  $v_1(t_\ell \cdots t_1) = F(t)$ . We say that  $P$  **becomes stale at the end of round  $t$**  if  $P$  is fresh in round  $t$  and is stale in round  $t+1$ . For any  $t_\ell \in \{0, \dots, m-1\}$ , let  $f(t_\ell)$  denote the number of fresh packets in the network at the beginning of the  $t_\ell 0 \cdots 0$ -phase. The following lemma gives conditions under which a packet can become stale.

**Lemma 5.2.** *A packet  $P$  can only become stale at the end of a round  $t$  if either (1)  $P(t) = F(t)$  and  $P$  is forwarded in round  $t$ ; or (2)  $F(t+1) < F(t)$  and  $P(t+1) \in [F(t+1) + 1, F(t)]$ .*

As packets can become stale for different reasons, we say that a packet is  **$\alpha$ -stale** if it became stale due to condition 1 of Lemma 5.2, and  $P$  is  **$\beta$ -stale** if it became stale due to condition 2. The following lemma shows that no packet is delivered while it is fresh. The idea is that, in the time it takes for a packet to be delivered,  $F(t)$  moves to the left of the packet's destination.

**Lemma 5.3.** *If  $P$  has reached its destination in round  $t'$ , then  $P$  is stale in round  $t'$ .*

We now give a lower bound on the number of fresh packets in the network at any given time. To do so, we give an upper bound on the number of packets that go stale in each  $(t_\ell \cdots t_1)$ -phase.

**Lemma 5.4.** *Over any interval of  $\tau \geq 0$  rounds, the number of packets that become  $\alpha$ -stale is at most  $\tau$ . Let  $t \in \langle m^\ell \rangle$ . If  $t$  is not the last round of the  $t_\ell \cdots t_1$ -phase, then no packets become  $\beta$ -stale at the end of round  $t$ . If  $t$  is the last round of the  $t_\ell \cdots t_1$ -phase, and  $k$  is the smallest integer in  $\langle \ell \rangle$  such that  $t_{k+1} < m-1$ , then the number of packets that become  $\beta$ -stale at the end of round  $t$  is  $L^{t+1} \left( \left[ v_1(t_\ell \cdots t_1) - \frac{m(m^k-1)}{m-1}, v_1(t_\ell \cdots t_1) \right] \right)$ .*

**Lemma 5.5.** *Consider any fixed  $t'_\ell \in \{0, \dots, m-2\}$ . At least one of the following scenarios occurs:*

1. *There exists a  $k \in \{0, \dots, \ell-1\}$  such that  $t_{k+1} < m-1$  and at least  $((\ell+1)\rho-1)m^{k+1}/2\ell$  packets become  $\beta$ -stale at the end of the  $t'_\ell t_{\ell-1} \cdots t_{k+1} (m-1) \cdots (m-1)$ -phase.*
2.  *$f(t'_\ell + 1) \geq f(t'_\ell) + ((\ell+1)\rho-1)m^\ell/2$ .*

*Proof of Thm. 5.1.* First, suppose that the first scenario of Lemma 5.5 is satisfied, i.e., there exists a  $t_\ell \in \{0, \dots, m-2\}$  and an  $k \in \{0, \dots, \ell-1\}$  such that  $t_{k+1} < m-1$  and at least  $((\ell+1)\rho-1)m^{k+1}/2\ell$  packets become  $\beta$ -stale at the end of the  $t_\ell \cdots t_{k+1} (m-1) \cdots (m-1)$ -phase. Then, by Lemma 5.4,  $L^{t+1}([v_1(t_\ell \cdots t_1) - m(m^k-1)/(m-1), v_1(t_\ell \cdots t_1)]) \geq ((\ell+1)\rho-1)m^{k+1}/2k$ . As  $[v_1(t_\ell \cdots t_1) - m(m^k-1)/(m-1), v_1(t_\ell \cdots t_1)]$  consists of  $O(m^k)$  buffers, the average load of a buffer in this range is at least  $\Omega(((\ell+1)\rho-1)m/2\ell)$ . Hence, there is at least one buffer with load at least  $\Omega(((\ell+1)\rho-1)m/2\ell)$ , giving the desired conclusion.

Otherwise, suppose that the first scenario from Lemma 5.5 is never satisfied in any  $t_\ell \cdots t_1$ -phase with  $t_\ell \in \{0, \dots, m-2\}$ . Then, by Lemma 5.5, the second scenario occurs for every  $t_\ell \in \{0, \dots, m-2\}$ . Therefore,  $f(m-1) \geq (m-1)((\ell+1)\rho-1)m^\ell/2$ . By Lemma 5.3, the total load over all buffers in the first round  $t$  of the  $(m-1)0 \cdots 0$ -phase is bounded below by the number of fresh packets, i.e.,  $L^t([n]) \geq f(m-1) \geq (m-1)((\ell+1)\rho-1)m^\ell/2$ . Therefore, the average load in the network in round  $t$  is at least  $(m-1)((\ell+1)\rho-1)m^\ell/2n = (m-1)((\ell+1)\rho-1)m^\ell/2(\ell+1)m^\ell = (m-1)((\ell+1)\rho-1)/2(\ell+1) \in \Omega(\frac{(\ell+1)\rho-1}{2\ell}n^{1/\ell})$ , and the conclusion of the theorem follows.  $\square$

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# APPENDIX

## A Proofs for Section 2

*Proof of Lemma 2.3.* The first claim follows from the definition of  $(\rho, \sigma)$  bound: For all  $T = [s, t]$  we have  $N_T(v) \leq \rho |T| + \sigma$ , hence  $N_T(v) - \rho |T| \leq \sigma$ . For the second claim, suppose  $\nu$  packets are injected in round  $t$  whose trajectories contain  $v$ , and let  $T' = [s, t-1]$  be an interval achieving the maximum in the definition of  $\xi^{t-1}(v)$ . Then we have

$$\begin{aligned} \xi^t(v) &\geq N_{[s,t]}(v) - \rho(t-s+1) \\ &= N_{[s,t-1]}(v) + \nu - \rho((t-1)-s+1) - \rho \\ &\geq \xi^{t-1}(v) + \nu - \rho \end{aligned}$$

Rearranging this expression gives  $\nu \leq \xi^t(v) - \xi^{t-1}(v) + \rho$ , as desired.  $\square$

*Proof of Lemma 2.5.* Fix an interval of time  $T = [s, t]$  and buffer  $v \in V$ . From the definition of  $A_\ell$ , we have

$$\begin{aligned} N_T(v) &= |\{(u, i_P, w_P) \in A : \ell \cdot s + 1 \leq u \leq \ell \cdot t\}| \\ &\leq \rho \cdot \ell(t-s+1) + \sigma. \end{aligned}$$

The inequality holds because  $A$  is  $(\rho, \sigma)$ -bounded. Therefore,  $A_\ell$  is  $(\ell \cdot \rho, \sigma)$ -bounded, as desired.  $\square$

## B Additional Material for Section 3

### B.1 Proofs

*Proof of Prop. 3.1.* For each time  $t$ , let  $B^t(w-1)$  denote the number of bad packets in the network after the  $t$ -th injection step. That is

$$B^t(w-1) = \sum_{i \in \langle n \rangle} \max \{L^t(i) - 1, 0\}.$$

Similarly, let  $B^{t+}(w-1)$  be the number of bad packets after the  $t$ -th forwarding step. We will argue by induction that for all  $t$

$$B^t(w-1) \leq \xi^t(w-1) + 1 \quad \text{and} \quad B^{t+}(w-1) \leq \xi^t(w-1). \quad (2)$$

Since the maximum load of any buffer can be at most  $B(t) + 1$ , the proposition follows.

For the base case,  $t = 0$ , (2) trivially holds as all quantities are zero. For the inductive step, suppose (2) holds for round  $t-1$ . Let  $\nu(t)$  denote the number of new packets injected during the  $t$ -th injection step. By part 2 of Lemma 2.3,  $\nu(t) \leq \xi_{w-1}(t) - \xi_{w-1}(t-1) + 1$ . Therefore

$$B^t(w-1) \leq B^{t-1}(w-1) + \nu(t) \leq \xi^t(w-1) + (\xi^t(w-1) - \xi^{t-1}(w-1) + 1) = \xi^t(w-1) + 1.$$

Thus the first expression in (2) holds, as desired. To show the second expression also holds, it suffices to show that if  $B^t(w-1) \geq 1$ , then  $B^{t+}(w-1) = B^t(w-1) - 1$ . To this end, let  $L^{t+}$  denote the configuration after forwarding. Let  $i$  be the left-most buffer for which  $|L^t(i)| \geq 2$ . Since

$i$  forwards, but  $i - 1$  does not, we have  $L^{t+}(i) = L^t(i) - 1 \geq 1$ . Since every non-empty  $i' > i$  forwards, and no  $i'$  receives more than one packet from  $i' - 1$ , we have for all  $i' > i$

$$|L^{t+}(i')| \leq \max 1, |L^t(i')|.$$

Therefore, we compute

$$\begin{aligned} B^{t+}(w-1) &= (|L^{t+}(i)| - 1) + \sum_{i' > i} \max \{|L^{t+}(i')| - 1, 0\} \\ &\leq L^t(i) - 2 + \sum_{i' > i} \max \{|L^t(i')| - 1, 0\} \\ &= B^t(w-1) - 1, \end{aligned}$$

as desired.  $\square$

To prove Prop. 3.2, we first show that the forwarding pattern prescribed by PPTS is feasible, in the sense that at most one pseudo-buffer in a buffer  $i$  forwards in each round.

**Lemma B.1.** *Consider a single step of PPTS, and  $I_k$  be a the (possibly empty) interval of  $k$ -pseudo-buffers activated during that step. Then for any distinct  $k, k' \in \langle d \rangle$  we have  $I_k \cap I_{k'} = \emptyset$ .*

*Proof of Lemma B.1.* Denote the endpoints of an interval  $I_k$  by  $[a_k, b_k]$ . Let  $i(k)$  be the value of  $i$  at the beginning of iteration  $k$  of the loop in Lines 3–9 PPTS. Note that  $I_k$  is empty, unless the condition of Line 4 is satisfied. In the former case, we have  $i(k+1) = i(k)$ , and in the latter case we have  $i(k+1) = i_k < i(k)$  and  $I_k = [i_k, i(k) - 1]$ . Therefore, the sequence  $i(d), i(d-1), \dots$  is weakly decreasing. Thus, for  $k < k'$ , if  $I_k, I_{k'} \neq \emptyset$ , there is some  $k''$  satisfying  $k < k'' \leq k'$  such that  $b_k = i(k'') - 1 < i_{k''} \leq i_{k'} = a_{k'}$  which gives the desired result.  $\square$

*Proof of Lemma 3.4.* First observe that since only  $k$ -packets are forwarded, so only  $k$ -loads—hence  $k$ -badness—changes after forwarding. Thus, the second claim follows immediately from the first.

For a buffer  $i$ , let  $\beta_k(i)$  and  $\beta_k^+(i)$  denote the number of  $k$ -bad packets stored in  $i$  before and after forwarding, respectively. We consider 4 cases separately:

- $i < a_k$  or  $i > b_k + 1$ . In this case,  $i$  neither sends nor receives a packet so that  $\beta_k^+(i) = \beta_k(i)$ .
- $i = a_k$ . In this case,  $L_k(a_k) \geq 2$ . Since  $a_k$  sends but does not receive a packet, we have  $\beta_k^+(a_k) = \beta_k(a_k) - 1$ .
- $a_k + 1 \leq i \leq b_k$ . Observe that  $i$  receives at most one packet, and if  $L_k(i) \geq 1$ ,  $i$  forwards one packet. Thus  $L_k^+(i) \leq \max \{L_k(i), 1\}$ , hence  $\beta_k^+(i) \leq \beta_k(i)$ .
- $i = b_k + 1$ . In this case,  $i$  receives at most one packet, hence  $\beta_k^+(i) \leq \beta_k(i) + 1$ .

The lemma follows from the four cases above, using the definitions of  $B_k(i)$  and  $B_k^+(i)$ .  $\square$

*Proof of Proposition 3.2.* For each round  $t > 0$  we, use  $t+$  to denote the time after the forwarding step of round  $t$ . For example,  $B^{t+}(i)$  is the badness of  $i$  after the  $t$ -th forwarding step (whereas  $B^t(i)$  is the badness after the injection step, before forwarding). We will show that for all rounds  $t$  and buffers  $i \in \langle n \rangle$ , we have

$$B^t(i) \leq \xi^t(i) + 1 \quad \text{and} \quad B^{t+}(i) \leq \xi^t(i). \quad (3)$$

To see that the proposition follows from (3), we compute

$$|L^t(i)| = \sum_{k \in \langle d \rangle} |L_k^t(i)| \leq \sum_{k \in \langle d \rangle} (1 + \beta_k^t(i)) \leq d + \sum_{k \in \langle d \rangle} \beta_k^t(i) \leq d + \sum_{k \in \langle d \rangle} B_k^t(i) = d + B^t(i).$$

Therefore, from (3), we obtain  $|L^t(i)| \leq d + 1 + \xi^t(i)$ . By Claim 1 of Lemma 2.3,  $\xi^t(i) \leq \sigma$ , so that  $L^t(i) \leq d + 1 + \sigma$ , as desired.

In order to prove (3), we argue by induction on  $t$ . The proof for  $t = 0$  is trivial, as no packets have yet been injected. Suppose that (3) holds for every  $i \in \langle n \rangle$  at time  $t - 1$ . Let  $N_{\{t\}}(i)$  denote the number of packets injected in round  $t$  whose paths contain  $i$ . We compute

$$\begin{aligned} B^t(i) &\leq B^{(t-1)+}(i) + N_{\{t\}}(i) \\ &\leq \xi^{t-1}(i) + N_{\{t\}}(i) && \text{(inductive hypothesis)} \\ &\leq \xi^{t-1}(i) + (\xi^t(i) - \xi^{t-1}(i) + 1) && \text{(Lemma 2.3)} \\ &= \xi^t(i) + 1. \end{aligned}$$

Therefore, the first equation of (3) holds.

To prove the second equation in (3), it suffices to show that if  $B^t(i) > 0$ , then after the  $t$ -th forwarding step we have  $B^{t+}(i) = B^t(i) - 1$ . For each  $k \in \langle d \rangle$ , let  $I_k$  be the (possibly empty) interval of  $k$ -pseudo-buffers activated in Line 6 of PPTS. We have the following.

**Claim 1.** *Let  $\iota \in \langle n \rangle$ . If  $B^t(\iota) > 0$ , then there exists  $\kappa \in \langle d \rangle$  such that  $\iota \in I_\kappa$ .*

*Proof of Claim 1.* Let  $\kappa$  be the maximum index such that  $B_\kappa^t(\iota) > 0$ . Thus, for all  $k > \kappa$ , there is no  $k$ -bad packet in any buffer  $i \leq \iota$ . Therefore, in the corresponding iteration  $k = \kappa$  of PPTS, we have  $i_k \leq \iota$  in Line 5. Since  $i$  is only ever assigned to a buffer containing a bad packet (Line 7 of PPTS), during the  $k$ -th iteration we must have had  $\iota < i$  in Line 6 when  $I_k = [i_k, i - 1]$  is activated. Thus,  $\iota \in I_\kappa$ , as desired.  $\square$

Finally, suppose  $i$  satisfies  $B^t(i) > 0$ . Applying the claim, let  $I_k = [a_k, b_k]$  be the activated interval containing  $i$ . Further, by the definition of  $i_k (= a_k)$  in Line 5 of PPTS, we have  $\beta_k^t(a_k) > 0$ . Therefore, by Lemma 3.4, we have  $B_k^{t+}(i) \leq B_k^t(i) - 1$ . For all  $k' \neq k$ , Lemma 3.4 gives  $B_{k'}^{t+}(i) \leq B_{k'}^t(i)$ . Summing over all destinations gives

$$B^{t+}(i) = \sum_{k' \in \langle d \rangle} B_{k'}^{t+}(i) = B_k^{t+}(i) + \sum_{k' \neq k} B_{k'}^{t+}(i) \leq B_k^t(i) - 1 + \sum_{k' \neq k} B_{k'}^t(i) \leq B_k^t(i) - 1,$$

as desired.  $\square$

## B.2 Generalization to directed trees

Both PTS and PPTS algorithms generalize to directed trees topologies. In this section we explain how.

A *directed tree*  $G = (V, E)$  is a rooted tree, where all edges are directed toward the root  $r \in V$ . We assume that the trajectories of all packets are directed paths in  $G$ .

The orientation of edges  $e \in E$  toward  $r$  induces a partial order  $\prec$  on  $V$ : for  $u, v \in V$ , we have  $u \prec v$  if and only if  $v$  is on the unique path from  $u$  to  $r$ . Thus, leaves are minimal with respect to  $\prec$ , and the root  $r$  is maximal. By extension, given a configuration  $L$  of  $G$ ,  $\prec$  induces a partial order on packets, where  $P \prec P'$  if  $P \in L(u)$ ,  $P' \in L(v)$ , and  $u \prec v$ .

**Single destination case.** Suppose first that all packets have destination  $r$ , the root of  $G$ . Again, we say that a packet  $P \in L^t(v)$  is bad if it stored at position  $p \geq 2$ , and  $L^t(v)$  is bad if  $|L^t(v)| \geq 2$ . The generalized PTS in this case selects a maximal set of “farthest” bad buffers from the root, and activates the union of paths from these buffers to the root. Formally, we have the following.

**Definition B.2.** Let  $\mathcal{B}^t \subseteq V$  be the set of bad buffers at time  $t$ .  $\mathcal{P} \subseteq \mathcal{B}^t$  is a **low-antichain** if (1) For all  $u, v \in \mathcal{P}$ ,  $u \not\preceq v$  and  $v \not\preceq u$ ; and (2) For every  $v \in \mathcal{B}^t$  there exists  $u \in \mathcal{P}$  such that  $u \preceq v$ .

Given  $\mathcal{B}^t$ , there is a unique low-antichain  $\min(\mathcal{B}^t) = \{u \in \mathcal{B}^t : \forall v \in \mathcal{B}^t, v \not\preceq u\}$ . Our generalization of PTS activates buffers as follows: Let  $\mathcal{P} = \min(\mathcal{B}^t)$ . Then define  $\mathcal{A}$  to be  $\mathcal{A} = \bigcup_{u \in \mathcal{P}} \text{Path}(u, r)$ , where  $\text{Path}(u, v)$  returns the set of nodes on the unique path from  $u$  to  $v$  in  $G$ . More concisely, PTS activates every buffer  $v \in V$  such that there exists a bad buffer  $u$  with  $u \preceq v$ . We state the performance of this variant of PTS in the following.

**Proposition B.3.** Suppose  $G$  is a directed tree,  $A$  is a  $(\rho, \sigma)$ -bounded adversary with  $\rho \leq 1$ , and all packets  $P \in A$  have destination  $r$ . Then the maximum buffers size is at most  $2 + \sigma$ .

Proposition B.3 is proven analogously to Proposition 3.1. Specifically, we bound the number of bad packets “behind” any node  $v \in V$  by  $\xi^t(v) + 1$  during any step of the algorithm. To this end, we generalize the notion of badness of a node to directed trees.

**Definition B.4.** Let  $v \in V$ . Let  $G_v = (U_v, E_v)$  denote the (directed) subtree of  $G$  rooted at  $v$ , so that  $U_v = \{u \in V : u \preceq v\}$ . For each  $v \in V$  we define  $\beta^t(v) = \max\{|L^t(v)| - 1, 0\}$  to be the number of bad packets in  $v$ . The **badness** of  $v$  is  $B^t(v) = \sum_{u \in U_v} \beta^t(u)$ . That is,  $B^t(v)$  is the number of bad packets whose paths contain  $v$ .

**Lemma B.5.** Let  $L^t$  be a configuration and  $\mathcal{A}$  the set of activated nodes described above. For any  $v \in V$ , let  $B^{t+}(v)$  denote the badness of  $v$  immediately after forwarding. Then  $B^{t+}(v) \leq \max\{B^t(v) - 1, 0\}$ .

Given Lemma B.5, Prop. B.3 follows from Lemma 2.3 as in the proof of Prop. 3.1.

Next, we generalize Algorithm PPTS to directed trees. As before let  $W = \{w_1, w_2, \dots, w_d\}$  be a set of possible destinations of packets. We assume that the sequence  $w_1, w_2, \dots, w_d$  is a topological sorting of  $W$  so that  $w_i \prec w_j$  implies that  $i < j$ . As in Section 3.2, we partition in  $L^t(v)$  into pseudo-buffers  $\{L_k^t(v) : k \in \langle d \rangle\}$ , where  $L_k^t(v)$  stores packets destined for  $w_k$ . Badness of pseudo-buffers is defined analogously to before:

$$\beta_k^t(v) = \max\{|L_k^t(v)| - 1, 0\}, \quad B_k^t(v) = \sum_{u \in U_v} \beta_k^t(u), \quad B^t(v) = \sum_{k \in \langle d \rangle} B_k^t(v).$$

We denote  $\mathcal{B}_k^t = \{v \in V : \beta_k^t(v) \geq 1\}$  to be the set of nodes containing  $k$ -bad pseudo-buffers.

The PPTS algorithm for directed trees is described in Algorithm 6. In order to quantify the performance of Algorithm 6, we define the **destination depth**  $d' = d'(G, W)$  of  $G$  with destinations  $W$  to be the maximal number of destinations on a leaf-root path, or, equivalently, the length of the longest chain (w.r.t.  $\prec$ ) of destinations. Note that  $d' \leq \min\{d, D\}$  where  $D$  is the depth of  $G$ .

The proof of Proposition 3.5 is analogous to the proofs of Propositions 3.2 and B.3. Specifically, an argument analogous to Lemma B.5 shows that forwarding  $\mathcal{A}$  decreases the badness of each buffer. Combined with Lemma 2.3, this implies that the badness of each node  $v$  is bounded by  $\xi^t(v) + 1 \leq \sigma + 1$ . Since each  $v$  can contain at most  $d'$  non-bad packets, Proposition 3.5 follows from this bound on the badness of each node.

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**Algorithm 6** PPTS( $G, W$ ) (tree variant)

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1:  $\mathcal{A} \leftarrow \emptyset$ 
2: for  $k \leftarrow d - 1$  downto 0 do
3:    $\mathcal{P} \leftarrow \min(\mathcal{B}_k^t)$ 
4:    $\mathcal{A}_k \leftarrow (\bigcup_{u \in \mathcal{P}} \text{Path}(u, w_k)) \setminus \mathcal{A}$ 
5:    $\mathcal{A} \leftarrow \mathcal{A} \cup \mathcal{A}_k$ 
6: end for
7: every active (nonempty) pseudo-buffer in  $\mathcal{A}$ 
   forwards a packet

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## C Proofs for Section 4

*Proof of Lemma 4.7.* Consider the state of  $\mathcal{A}$  immediately after the calls `FormPaths` in Lines 6–8. The intervals  $I_{\lambda,r}$  are disjoint, and on each interval,  $I_{\lambda,r}$  the level- $\lambda$  pseudo-buffers are activated in the same pattern as a call to `PPTS`. Thus the feasibility of  $\mathcal{A}$  after Line 8 follows from the observation that the activation pattern for `PPTS` is feasible.  $\mathcal{A}$  remains feasible after each call to `ActivatePreBad`, because only previously inactive buffers are activated in Line 5.  $\square$

In order to prove Lemma 4.8, we will apply the following result.

**Claim 2.** *Let  $t = (\varphi - 1)\ell$  and  $s = t + \lambda$  for some  $\lambda \in \langle \ell \rangle$ . Then for all  $i \in \langle n \rangle$  the following hold:*

1. For all  $j < \lambda$ ,  $B_j^{s+}(i) = B_j^s(i)$ .
2.  $B_\lambda^{s+}(i) \leq \max\{B_\lambda^s(i) - 1, 0\}$ .
3.  $B^{s+}(i) \leq B^s(i)$ .

*Proof.* Item 1 holds because `HPTS` can only activate buffers at levels  $j' \leq \lambda$  in round  $t + \lambda$  and no packet ever moves from level  $j'$  to  $j$  with  $j > j'$ . Thus states of level- $j$  pseudo buffers are unchanged after forwarding for  $j > \lambda$ . For Item 2, observe that for each interval  $I \in \mathcal{I}_\lambda$  the paths activated in  $I$  are precisely the same pseudo-buffers as are activated by `PPTS` (where the intermediate destinations play the role of destinations in the `PPTS` algorithm). Thus, Item 2 follows from Lemma 3.4.

For Item 3, consider a level- $j$  segment  $A$  added to  $\mathcal{A}$  in a call to `ActivatePreBad`, and let  $I_j = [a_j, b_j] \in \mathcal{I}$  be the level- $j$  interval containing  $A$ . Thus  $A = [a_j, i_j]$  for some  $i_j < b_j$ . Let  $P$  be the unique pre-bad packet for  $a_j$ , and let  $k$  be the index such that  $w_{P,j}$  is  $I_j$ 's  $k$ -th destination. Observe that one of the following events occurred:

- (a)  $i_j = w_{P,j} - 1$  and activating  $A$  creates a new pre-bad packet  $P'$  in buffer  $L_{j,k}^s(i_j)$ .
- (b)  $i_j = w_{P,j} - 1$  but activating  $A$  does not create another pre-bad packet.
- (c)  $i_j < w_{P,j} - 1$  and every  $i \in I_j$  with  $i > i_j$  is active at level  $\lambda$ .<sup>3</sup>

From items (a)–(c) above, each pre-bad  $P$  created in the call to `FormPaths` induces a (possibly empty) sequence  $A_1, A_2, \dots$  of consecutive segments at decreasing levels, where each segment  $A_{i+1}$  is activated because event (a) occurred in the activation of  $A_i$ . Conversely, every pre-bad packet  $P'$  at a level  $j < \lambda$  is induced from a unique pre-bad packet  $P$  at level  $\lambda$ . To prove part 3 of the claim, let  $i \in I_j$  for some  $j < \lambda$ . We consider three (exhaustive) cases separately.

**Case 1:**  $i \in A_\alpha$  where  $A_\alpha$  is the unique activated containing  $i$ .

**Case 2:**  $i$  is active at level  $\lambda$ , and an active segment  $A_\alpha \subseteq I_j$  terminates as in case (c) above (necessarily with  $i_j < i$ ).

**Case 3:**  $i$  is not active.

In Case 1, suppose the segment  $A_\alpha$  is at level  $j$ . First observe that for  $j' \neq j$ , we have  $B_{j'}^{s+}(i) = B_{j'}^s(i)$  because no buffer  $i' \leq i$  at level  $j'$  containing packets with intermediate destination  $w \geq i$  is active. As for the level- $j$  badness of  $i$ , we have  $B_j^{s+}(i) \leq B_j^s(i)$ . To see this, let  $I_j = [a_j, b_j]$  be the level- $j$  interval containing  $i$ , so that  $a_j$  receives a pre-bad packet  $P$  during forwarding. Let  $k$  be the index such that  $w_{P,j}$  is  $I_j$ 's  $k$ -th intermediate destination. If  $|L_{j,k}^s(a_j)| \geq 2$ , then forwarding  $A_\alpha$  strictly decreases  $B_{j,k}(i)$  by Lemma 3.4, while  $B_j(i)$  increases by one as a result of  $P$  being forwarded. Similarly, if  $|L_{j,k}^s(a_j)| = 1$ , forwarding  $A_\alpha$  does not increase  $B_{j,k}(i)$ . In this case,  $P$  being forwarded to  $a_j$  does not increase  $B_j(i)$ , because  $L_{j,k}^s(a_j)$ 's sole occupant is forwarded.

<sup>3</sup>The second conclusion holds because intervals activated in `FormPaths` always terminate at intermediate destinations for level  $\lambda$ . Since  $j < \lambda$ ,  $I_j$  is contained in the interval between consecutive intermediate level  $\lambda$  intermediate destinations, if some  $i \in I_j$  is active at level  $\lambda$ , all buffers  $i' \in I_j$  with  $i' \geq i$  are active at level  $\lambda$ .

In Case 2, for all  $j' \neq \lambda, j$ , we have  $B_{j'}^{s+}(i) = B_{j'}^s(i)$ .  $B_\lambda^{s+}(i) \leq B_\lambda^s(i) - 1$  by Lemma 3.4. We argue that  $B_j^{s+}(i) \leq B_j^s(i) + 1$ . To see this first consider the case where  $|L_{j,k}(a_j)| \geq 2$ . In this case,  $\beta_{j,k}^{s+}(a_j) = \beta_{j,k}^s(a_j)$  because  $a_j$  forwards one bad packet, and receives one bad packet. For  $i' \in [a_j + 1, i_j]$ ,  $\beta_{j,k}(i')$  does not increase because each non-empty  $L_{j,k}(i')$  forwards. Finally  $\beta_{j,k}(i_j + 1)$  increases by at most one as a result of  $i_j + 1$  receiving a level- $j$  packet, but sending none. Since no other level- $j$  pseudo-buffers forward, in this case,  $B_j^{s+}(i) \leq B_j^s(i) + 1$ . If  $|L_{j,k}(i)| = 1$ , then forwarding  $P$  does not create a bad packet, but forwarding  $A_\alpha$  could create a single new bad packet (in buffer  $i_j + 1$ ), so the same conclusion holds.

Finally for Case 3, no packet that is pre-bad for any level  $j$  has a path crossing  $i$  at level  $j$  (otherwise Cases 1 or 2 would have occurred). Therefore, forwarding pre-bad packets will not increase  $B_j(i)$  for any  $j < \lambda$ . Further  $B_j^{s+}(i) = B_j^s(i)$  for all  $j \geq \lambda$ , as no level- $j$  buffers are active that forward packets whose paths cross  $i$  at level  $j$ .  $\square$

*Proof of Lemma 4.8.* Since we assume no injections are made in rounds  $t + 2, t + 3, \dots, t + \ell$ , we have  $B^{s+1}(i) = B^{s+}(i)$  for all  $s = t + 1, t + 2, \dots, t + \ell$ . Applying Item 3 of Claim 2 iteratively for  $t + 1, t + 2, \dots, t + \ell$ , we find that  $B^{(t+\ell)+}(i) \leq B^t(i)$ . Thus, it suffices to show that if  $B^{t+1}(i) \geq 1$ , there is some round  $s = t + \lambda$  for  $\lambda \in \langle \ell \rangle$  such that  $B^{s+}(i) < B^s(i)$ . By Claim 2, in each round  $s$  such that  $B^s(i) > 0$ , either  $B^{s+}(i) \leq B^s(i) - 1$ , or

$$B_{\lambda+1}^{s+}(i) + B_{\lambda+2}^{s+}(i) + \dots + B_\ell^{s+}(i) = B_{\lambda+1}^s(i) + B_{\lambda+2}^s(i) + \dots + B_\ell^s(i) + 1.$$

That is, either the badness of  $i$  decreases, or the level- $j$  badness of  $i$  for some  $j < \lambda$  increases. Consider the largest  $s = t + \lambda$  such that  $B_\lambda^s(i) > 0$ . Then the level- $j$  badness of  $i$  did not increase for any  $j < \lambda$ , hence we have  $B^{s+}(i) = B^s(i)$ , as desired.  $\square$

## D Proofs for Section 5

*Proof of Lemma 5.2.* As  $F(t)$  is non-increasing, there are two possible cases:

- Suppose that  $F(t + 1) = F(t)$ . If  $P$  becomes stale at the end of round  $t$ , it means that  $P$  is fresh in round  $t$  and stale in round  $t + 1$ . In particular,  $P(t) \leq F(t) = F(t + 1) < P(t + 1)$ , which implies that  $P$  is forwarded in round  $t$ . As  $P$  can only be forwarded once in round  $t$ , it follows that  $P(t + 1) = F(t) + 1$ , so  $P(t) = F(t)$ .
- Suppose that  $F(t + 1) < F(t)$ . As  $P$  is stale in round  $t + 1$ , we have  $P(t + 1) > F(t + 1)$ , so  $P(t + 1) \geq F(t + 1) + 1$ .

Next, as  $P$  is fresh in round  $t$ , we have  $P(t) \leq F(t)$ . Assuming condition (1) doesn't hold, there are two cases to consider. In the case where  $P(t) < F(t)$ , since  $P$  can only be forwarded once in round  $t$ , it follows that  $P(t + 1) \leq F(t)$ . In the case where  $P(t) = F(t)$  and  $P$  isn't forwarded in round  $t$ , then  $P(t + 1) = F(t)$ .  $\square$

*Proof of Lemma 5.3.* We first prove the following claim that relates the values of  $F(t)$  and  $F(t')$  depending on the value of  $t' - t$ .

**Claim 3.** *Consider any round  $t$  and any round  $t' > t$ . If there exists  $k \in \{1, \dots, \ell\}$  such that  $t'_k > t_k$  and  $t'_i = t_i$  for all  $i \in \{k + 1, \dots, \ell\}$ , then  $F(t') < v_k(t_\ell \dots t_1)$ .*

By definition,

$$\begin{aligned} F(t') &= v_1(t'_\ell \cdots t'_1) = v_1(t_\ell \cdots t_{k+1} t'_k \cdots t'_1) \\ &= \sum_{j=1}^k [(j+1)m^j - (t'_j+1)jm^{j-1}] + \sum_{j=k+1}^{\ell} [(j+1)m^j - (t_j+1)jm^{j-1}] . \end{aligned}$$

By definition,  $v_k(t_\ell \cdots t_1) = \sum_{j=k}^{\ell} [(j+1)m^j - (t_j+1)jm^{j-1}]$ . Therefore,

$$\begin{aligned} F(t') - v_k(t_\ell \cdots t_1) &= \sum_{j=1}^k [(j+1)m^j - (t'_j+1)jm^{j-1}] - [(k+1)m^k - (t_k+1)km^{k-1}] \\ &= \sum_{j=1}^{k-1} [(j+1)m^j - (t'_j+1)jm^{j-1}] - [(t'_k+1)km^{k-1} - (t_k+1)km^{k-1}] \\ &< \sum_{j=1}^{k-1} [(j+1)m^j - (t'_j+1)jm^{j-1}] - [km^{k-1}] \\ &\leq \sum_{j=1}^{k-1} [(j+1)m^j - jm^{j-1}] - [km^{k-1}] \\ &= [km^{k-1} - 1] - [km^{k-1}] \\ &= -1 . \end{aligned}$$

It follows that  $F(t') < v_k(t_\ell \cdots t_1)$ , which concludes the proof of the claim.

To prove Lemma 5.3, suppose a packet  $P$  is injected in round  $t$ . There are several cases to consider, based on the packet type.

**$P$  is a type-1 packet:** In this case  $P$  is injected into buffer  $v_1(t_\ell \cdots t_1)$  with destination  $n$ . Since  $t' \geq t$ , we have

$$\begin{aligned} F(t') &\leq F(t) = v_1(t_\ell \cdots t_1) \\ &= \sum_{j=1}^{\ell} [(j+1)m^j - (t_j+1)jm^{j-1}] \\ &\leq \sum_{j=1}^{\ell} [(j+1)m^j - jm^{j-1}] \\ &= (\ell+1)m^\ell - 1 = n - 1 < P(t') . \end{aligned}$$

**$P$  is a type- $(\ell+1)$  packet:** In this case  $P$  is injected into buffer 0 with destination  $v_\ell(t_\ell \cdots t_1)$ .

By the definition of  $v_\ell(t_\ell \cdots t_1)$ , and since  $t_\ell \in \{0, \dots, m-1\}$ , packet  $P$ 's destination is  $(\ell+1)m^\ell - (t_\ell+1)\ell m^{\ell-1} \geq (\ell+1)m^\ell - \ell m^\ell = m^\ell$ . As packet  $P$  moves at most one hop per round, at least  $m^\ell$  rounds elapse before  $P$  reaches its destination, i.e.,  $t' - t \geq m^\ell$ . This means that, in the base- $m$  representations of  $t$  and  $t'$ , we have  $t'_\ell > t_\ell$ , so by Claim 3,  $F(t') < v_k(t_\ell \cdots t_1) = P(t')$ .

**$P$  is a type- $k$  packet with  $k \in \{2, \dots, \ell\}$ :** In this case  $P$  is injected into buffer  $v_k(t_\ell \cdots t_1)$  with

destination  $v_{k-1}(t_\ell \cdots t_1)$ . The distance between these two buffers is

$$\begin{aligned} \sum_{j=k-1}^{\ell} [(j+1)m^j - (t_j+1)jm^{j-1}] - \sum_{j=k}^{\ell} [(j+1)m^j - (t_j+1)jm^{j-1}] \\ = km^{k-1} - (t_{k-1}+1)(k-1)m^{k-2} \\ \geq km^{k-1} - m(k-1)m^{k-2} \\ = m^{k-1}. \end{aligned}$$

As packet  $P$  moves at most one hop per round, at least  $m^{k-1}$  rounds elapse before  $P$  reaches its destination, i.e.,  $t' - t \geq m^{k-1}$ . This means that, in the base- $m$  representations of  $t$  and  $t'$ , there exists a  $k' \geq k-1$  such that  $t'_{k'} > t_{k'}$  and  $t'_i = t_i$  for all  $i \in \{k'+1, \dots, \ell\}$ . So by Claim 3,  $F(t') < v_{k'}(t_\ell \cdots t_1) \leq v_{k-1}(t_\ell \cdots t_1) = P(t')$ .

In all cases, we showed that  $P(t') > F(t')$ , which means that  $P$  is stale in round  $t'$ , as desired.  $\square$

*Proof of Lemma 5.4.* As the buffer  $F(t)$  can forward at most one packet per round, the number of packets that can go  $\alpha$ -stale in any interval of  $\tau \geq 0$  rounds is at most  $\tau$ .

Next, note that if  $t$  is not the last round of the  $t_\ell \cdots t_1$ -phase, then rounds  $t$  and  $t+1$  belong to the  $t_\ell \cdots t_1$ -phase, which implies that  $F(t+1) = v_1(t_\ell \cdots t_1) = F(t)$ . Therefore, condition 2 of Lemma 5.2 cannot be satisfied, so no packets become  $\beta$ -stale at the end of round  $t$ .

Next, suppose that  $t$  is the last round of the  $t_\ell \cdots t_1$ -phase, and let  $k$  be the smallest integer in  $\{0, \dots, \ell-1\}$  such that  $t_{k+1} < m-1$ . Namely, round  $t$  belongs to the  $t_\ell \cdots t_{k+1}(m-1) \cdots (m-1)$ -phase, and round  $t+1$  belongs to the  $t_\ell \cdots (t_{k+1}+1)0 \cdots 0$ -phase.

By definition, the value of  $F(t)$  is

$$\begin{aligned} v_1(t_\ell \cdots t_{k+1}(m-1) \cdots (m-1)) &= \sum_{j=1}^k [(j+1)m^j - jm^{j-1}] + \sum_{j=k+1}^{\ell} [(j+1)m^j - (t_j+1)jm^{j-1}] \\ &= \sum_{j=1}^k m^j + \sum_{j=k+1}^{\ell} [(j+1)m^j - (t_j+1)jm^{j-1}]. \end{aligned}$$

Further, the value of  $F(t+1)$  is

$$\begin{aligned} v_1(t_\ell \cdots (t_{k+1}+1)0 \cdots 0) &= \sum_{j=1}^k [(j+1)m^j - jm^{j-1}] + \sum_{j=k+1}^{\ell} [(j+1)m^j - (t_j+1)jm^{j-1}] - (k+1)m^k \\ &= ((k+1)m^k - 1) + \sum_{j=k+1}^{\ell} [(j+1)m^j - (t_j+1)jm^{j-1}] - (k+1)m^k \\ &= -1 + \sum_{j=k+1}^{\ell} [(j+1)m^j - (t_j+1)jm^{j-1}]. \end{aligned}$$

Thus,  $F(t+1) + 1 = F(t) - \sum_{j=1}^k m^j = F(t) - m(m^k - 1)/(m-1)$ . By Lemma 5.2, all packets that become  $\beta$ -stale at the end of round  $t$  are located in buffer interval  $[F(t+1) + 1, F(t)]$  in round  $t+1$ , which means that the number of such packets is  $L^{t+1}([v_1(t_\ell \cdots t_1) - m(m^k - 1)/(m-1), v_1(t_\ell \cdots t_1)])$ .  $\square$

*Proof of Lemma 5.5.* We assume that scenario 1 does not occur, and show that this implies the inequality in scenario 2.

For any fixed  $t'_\ell \in \{0, \dots, m-2\}$  and any fixed  $k \in \{0, \dots, \ell-1\}$ , the number of distinct  $t'_\ell t_{\ell-1} \dots t_{k+1} (m-1) \dots (m-1)$ -phases is at most  $m^{\ell-1-k}$ , as  $t_{\ell-1}, \dots, t_{k+1}$  can take on at most  $m$  values each. So, if we assume that scenario 1 does not occur, i.e., at most  $((\ell+1)\rho-1)m^{k+1}/2\ell$  packets become  $\beta$ -stale at the end of each such phase, then the total number packets that become  $\beta$ -stale over all  $t'_\ell t_{\ell-1} \dots t_1$ -phases is at most

$$\sum_{k=0}^{\ell-1} (m^{\ell-1-k}) \frac{((\ell+1)\rho-1)m^{k+1}}{2\ell} = \sum_{k=0}^{\ell-1} \frac{((\ell+1)\rho-1)m^\ell}{2\ell} = \frac{((\ell+1)\rho-1)m^\ell}{2}.$$

Next, for fixed  $t'_\ell$ , there are  $m^{\ell-1}$   $t'_\ell t_{\ell-1} \dots t_1$ -phases, and each consists of  $m$  rounds, so the number of rounds that elapse over all such phases is  $m^\ell$ . By Lemma 5.4, the total number of packets that become  $\alpha$ -stale over all  $t'_\ell t_{\ell-1} \dots t_1$ -phases is at most  $m^\ell$ . Therefore, the total number of packets that become stale over all  $t'_\ell t_{\ell-1} \dots t_1$ -phases is at most  $m^\ell + \frac{((\ell+1)\rho-1)m^\ell}{2}$ . However, our injection pattern injects  $(\ell+1)\rho m$  fresh packets in each  $t'_\ell t_{\ell-1} \dots t_1$ -phase, so the total number of fresh packets injected over all  $t'_\ell t_{\ell-1} \dots t_1$ -phases is  $m^{\ell-1}((\ell+1)\rho m) = (\ell+1)\rho m^\ell = m^\ell + ((\ell+1)\rho-1)m^\ell$ . So, of all of the packets injected from the start of the  $t'_\ell 0 \dots 0$ -phase until the end of the  $t'_\ell (m-1) \dots (m-1)$ -phase, at least  $\frac{1}{2}((\ell+1)\rho-1)m^\ell$  of them do not go stale, which proves that  $f(t'_\ell+1) \geq f(t'_\ell) + \frac{1}{2}((\ell+1)\rho-1)m^\ell$ .  $\square$