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On the signal-to-noise problem in atmospheric response studies

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SUMMARY

The problem of identifying the mean atmospheric response to external forcing in the presence of the natural variability of the atmosphere is treated as a pattern-detection problem. It is shown that without application of filtering techniques to reduce the number of degrees of freedom of the response pattern the atmospheric response inferred from data or model experiments will normally fail a multi-variate significance test. A step-wise pattern construction method is proposed which avoids these difficulties. Starting from a given set of anticipated response patterns, a transformed set of patterns is derived which, used as a truncated basis set to represent the observed response, maximizes the statistical significance of the response. The patterns are ordered *a priori* in a sequence reflecting their anticipated contribution to the total response, the sequence being terminated when the net response falls below a prescribed significance level. In effect the method filters out the statistically significant components of the atmospheric response. For application to model experiments a multi-variate analysis of the low-frequency model variability is required.

1. INTRODUCTION

The long-term behaviour of the atmosphere is believed to be closely related to slow variations of the external boundary conditions of the atmospheric system. The prediction of modifications of the mean atmospheric circulation induced by such external changes therefore represents a central problem for extended-range weather forecasting and climate studies, and numerous investigations have accordingly been concerned with the atmospheric response to variations in sea surface temperature (s.s.t.), sea-ice extent, surface albedo, incident radiation, CO_2 concentration, and many other factors.

In most applications the interest lies in time scales large compared with the typical response time scale of the atmosphere, so that only the asymptotic equilibrium response of the atmosphere needs to be considered. This can be studied either empirically, for example by cross-correlating observed atmospheric anomaly fields with observed variations in the external forcing, or with the aid of models. Usually the models are sufficiently realistic (such as a general circulation model – GCM) to be able to simulate the natural variability of the atmosphere. The determination of the mean atmospheric response in the presence of this variability then presents, both for model experiments and real data, a basic signal detection problem. Often the length of data record available for filtering out the mean signal from the noise is rather limited, either because of the short time history of hemispheric or global recorded meteorological data, or because of the expense of running high resolution GCMs. The signal-to-noise problem can then become one of the severest limitations in the study of the atmospheric response to external forcing.

Since the signal represents a multi-dimensional vector, or field, the problem is essentially one of pattern recognition. The statistical concepts involved are in many respects very similar to those occurring in the inverse problem of constructing prediction models from data (Lorenz 1956; Davis 1976, 1977, 1978; Hasselmann 1978; Barnett and Hasselmann 1979). Basically, the techniques are well known from standard statistical literature, but apparently they have not yet been applied to the atmospheric response problem. Indeed, the signal-to-noise analysis has largely been formulated in previous studies in terms of individual gridpoint statistics, rather than the pattern response. Typically, a significance analysis based on individual gridpoint values may yield 95% significant response regions covering approximately 5% of the total area investigated; the question whether or not the response pattern, as a whole or in part, is statistically significant clearly cannot be resolved by such an approach. For a consistent pattern recognition analysis it is necessary to regard the signal and noise fields as multi-dimensional vector quantities, characterized by appropriate multi-dimensional joint probability distributions, and the significance analysis should accordingly be carried out with respect to this multi-variate statistical field, rather than in terms of individual gridpoint statistics.

Once the signal-to-noise problem has been appropriately formulated, the natural next step is to consider methods of signal processing to enhance the signal-to-noise ratio. If no signal processing is applied, the unmodified response field, as inferred from model experiments or real data, will in most cases fail the relevant multi-variate significance test. However, if *a priori* hypotheses regarding the general structure of the expected response can be formulated, pattern filtering can be applied to improve greatly the signal-to-noise ratio. In the atmospheric response problem, such hypotheses can normally be rather easily generated using simple, linearized models. The consistent application of pattern filtering techniques may therefore be expected to enhance considerably the effectiveness of GCM experiments and real data time series analysis in studies of the atmospheric response to external forcing.

2. Signal significance

Let $\bar{\phi} = (\bar{\phi}_i) = (\bar{\phi}_1, \bar{\phi}_2, \ldots, \bar{\phi}_n)$ denote the asymptotic mean change induced in an atmospheric field in response to some given constant change in the boundary conditions (for example a given s.s.t. anomaly field). The index i indicates here the dependence of the atmospheric field on discrete gridpoints, wavenumbers, or some other finite dimensional representation of the field. The mean value may be assumed to denote either the time mean, in the limit of an infinite averaging period, or an ensemble average of the response (for large time) over an infinite set of experiments with slightly perturbed initial states.

In practice, $\bar{\phi}$ cannot be determined exactly but must be estimated from averages over a finite time period or over a finite number of experiments, or a combination of both. The error $\delta\phi = \phi - \bar{\phi}$, between the estimated mean ϕ and the true mean $\bar{\phi}$ is then characterized by an n-dimensional probability distribution $p(\delta\phi)$. Normally this is approximately Gaussian:

$$p(\delta\phi) = (2\pi)^{-\frac{1}{2}n}|C|^{-\frac{1}{2}}\exp(-\rho^2/2), \tag{1}$$

where

$$\rho^2(\delta\phi) = \sum_{i,j} C_{ij}^{-1}\, \delta\phi_i \delta\phi_j, \tag{2}$$

and $C_{ij}, = \langle \delta\phi_i \delta\phi_j \rangle$, is the error covariance matrix.

An approximately normal distribution follows from the central limit theorem, independent of the detailed short-time statistics of the natural variability of the atmospheric field, if the averaging period used for estimating ϕ is large compared with the integral correlation time scale of the natural variability. This condition is required generally for an acceptable signal-to-noise ratio, and is therefore usually reasonably well satisfied.

The covariance matrix C_{ij} can be estimated theoretically from the structure of the time-lagged covariance functions of the natural or model variability (Leith 1973; Jenkins and Watts 1968). Since these can only be estimated, C_{ij} should be treated correctly as an estimated rather than a given statistical quantity, just as ϕ. However, for the following the estimation errors in C_{ij} can be neglected to first order. Since the generalizations needed to include the statistical indeterminacy of C_{ij}, although conceptually straightforward, rather complicate the algebra, C_{ij} will accordingly be regarded as a known quantity.

Using Eq. (1), a γ-probability region R in the n-dimensional space of the estimated response ϕ may be defined by

$$\int_R p(\phi)\, d\phi = \gamma \quad . \quad . \quad . \quad . \quad (3)$$

where R is bounded by a constant probability surface, $p =$ const. or $\rho^2(\delta\phi) = \rho_\gamma^2 =$ const. The radius ρ_γ of the n-dimensional ellipsoid R can be obtained from the one-dimensional probability distribution for ρ^2, which is a χ^2-distribution with n degrees of freedom:

$$p(\rho^2)d\rho^2 = \chi^2(\rho^2)d\rho^2 = (\Gamma(\tfrac{1}{2}n)\,2^{\frac{1}{2}n})^{-1}(\rho^2)^{\frac{1}{2}n-1}\exp(-\tfrac{1}{2}\rho^2)d\rho^2 \quad . \quad (4)$$

The estimated mean response ϕ is then termed statistically significant at the γ-significance level (sometimes, the $(1-\gamma)$-significance level) if the vector ϕ falls outside the γ-probability region associated with the null hypothesis of a zero true mean response $\bar{\phi} = 0$, i.e. if

$$\rho^2(\phi) > \rho_\gamma^2 \quad . \quad . \quad . \quad . \quad (5)$$

For large n,

$$\rho_\gamma^2 \approx n + b_\gamma\sqrt{n} \quad . \quad . \quad . \quad . \quad (6)$$

where, for example, $b_\gamma = 2{\cdot}33$ for $\gamma = 95\%$ (cf. Abramowitz and Stegun 1965. For intermediate n in the range $10 < n < 100$, $b_\gamma \approx 2{\cdot}5$ yields a somewhat better approximation.)

The recipe for deciding whether or not a given response pattern ϕ, estimated from a numerical experiment or from the analysis of real data, is statistically significant is therefore very simple: first one evaluates the error covariance matrix C_{ij} associated with the particular finite averaging procedure used, applying the relations given, for example, in Leith (1973) or Jenkins and Watts (1968); then one forms the quadratic expression $\rho^2(\phi)$ and tests this against the appropriate confidence limit ρ_γ^2, given in standard statistical tables, for the specified number of degrees of freedom, n.

3. An example

The difference between the joint multi-variate χ^2 significance test and independent single-point tests is best demonstrated by a simple example. This will also serve to illustrate in familiar terms the considerable enhancement in significance which can be achieved by appropriate signal filtering, which will be discussed in more detail in the following section.

Let ϕ depend on only a single space coordinate x, at grid points $x_i = i\Delta x$, and let the true mean response of the system (real atmosphere or model) to some given external forcing be a simple sinusoid

$$\bar{\phi}_i = A_p\cos(2\pi ip/n + \alpha_p) \quad . \quad . \quad . \quad (7)$$

representing the pth harmonic of the fundamental analysis interval $n\Delta x$. Assume further that the estimation errors of the response are statistically homogeneous and can therefore be characterized by a covariance matrix

$$\langle\delta\phi_i\delta\phi_j\rangle = R_{i-j} \quad . \quad . \quad . \quad . \quad (8)$$

depending only on the difference of the indices.

The outcome of a particular numerical experiment (or real data analysis) will now yield a response consisting of the form (7) and a superimposed particular noise realization $\delta\phi_i$. Under what conditions will this response be recognized as statistically significant?

It is convenient to diagonalize the noise covariance matrix by transforming to a spectral representation (in a symmetrized complex notation with frequencies running from 1 to n), $\phi_i = \sum_{j=1}^{n} S_{ij}\hat{\phi}_j$ where $S_{ij}, = n^{-\frac{1}{2}}\exp\{2\pi\mathrm{i}ij/n\}$, is a unitary matrix. The inverse transformation is then given by $\hat{\phi}_j = \sum_{i=1}^{n} S_{ij}^* \phi_i$, the Fourier components $\hat{\phi}_j$ with $j > n/2$ being related to components with $j \leqslant n/2$ by the reality condition $\hat{\phi}_j = \hat{\phi}_{n-j}^*$ (* denotes the complex conjugate).

The relation (8) then transforms to

$$\langle \delta\hat{\phi}_i \delta\hat{\phi}_j^* \rangle = \delta_{ij} F_j \quad \text{where} \quad F_j = \sqrt{n} \sum_{i=1}^{n} S_{ij}^* R_i \qquad (9)$$

(R_n is defined as R_0 through the periodicity condition). With the observed response in spectral coordinates given by

$$\hat{\phi}_i = \delta_{ip} A_p \exp(i\alpha_p) + \delta\hat{\phi}_i, \qquad (10)$$

the ρ^2-statistic then takes the diagonal sum form

$$\rho^2 = |\delta\hat{\phi}_p + A_p \exp(i\alpha_p)|^2 / F_p + \sum_{i(i \neq p)} |\delta\hat{\phi}_i|^2 / F_i \qquad (11)$$

The signal (10) is accepted as significant at the γ-confidence level if ρ^2 satisfies condition (5), where for large n, ρ_γ^2 is given by (6). Now the average contribution of the noise to ρ^2 in (11) is equal to n, since $\langle |\delta\hat{\phi}_i|^2 \rangle = F_i$. Furthermore, for any particular noise realization, the noise contribution to ρ^2 will, in 95% of all cases, deviate from n by an amount which is smaller than the second term $b_\gamma \sqrt{n}$ in the right hand side of (6) (by definition of the confidence limit). Thus the signal will generally be recognized as noise only if the first term in the right hand side of (11) is larger than $b_\gamma \sqrt{n}$, or, approximately,

$$A_p^2 \gtrsim b_\gamma \sqrt{n}.F_p \qquad (12)$$

Equation (12) indicates that the critical signal amplitude required for statistical significance generally increases with the number of degrees of freedom, n. However, this conclusion requires a somewhat more careful examination, since the noise spectrum will generally depend on the discretization interval Δx and length of record $L = n\Delta x$, and is therefore not independent of n.

An exception is white noise, $C_{ij} = \delta_{ij} W$, for which the discrete spectrum is $F_i = W =$ const., and, as defined here, is independent of Δx, L and n. In this case equations (12) and (6) state that a sinusoidal signal will be accepted as a statistically significant response at the 95% confidence level if

$$A_p^2 > 2{\cdot}3\sqrt{n}.W \qquad (13)$$

Thus if $n = 100$, say, the sinusoidal amplitude would need to exceed the r.m.s. noise level by a factor $(2{\cdot}3\sqrt{100})^{\frac{1}{2}} = 4{\cdot}8$, and for higher resolution the factor would increase accordingly.

In the case of a normal spectrum corresponding to an autocovariance function of finite lag width the discrete spectrum increases proportionally to n as the discretization interval Δx approaches zero ($n \to \infty$ for fixed $L = n\Delta x$). Thus the significance criterion takes the form

$$A_p^2 > \text{const.} \times n^{\frac{3}{2}} . \qquad (14)$$

which increases still more rapidly with n than in the case of white noise.

The relations (13) and (14) generally represent considerably more stringent significance conditions than may have been anticipated from the application of independent significance tests for each point separately. Thus in the most favourable white-noise case, condition (13) implies that for $n = 100$ approximately 73% of the response values must exceed their individual 95% confidence limits in order for the sinusoidal pattern as a whole (plus noise realization) to be accepted as statistically significant.

The stringency of the multi-variate significance test, and the fact that increasing the resolution increases the critical signal level, is a typical property of multi-dimensional signal detection: the inclusion of additional information which adds to the noise without significantly improving the definition of the signal deteriorates the signal-to-noise ratio and makes the signal more difficult to detect. The same problem arises in the construction of statistical prediction models from data (Lorenz 1956; Davis 1976, 1977, 1978; Hasselmann 1978; Barnett and Hasselmann 1979), where the inclusion of too many noisy predictors in the model can mask the true predictors. The solution is to filter out as much of the noise as possible while still retaining the essential features of the signal by making use of known characteristics of the noise and the anticipated signal.

Thus in the present example, the signal-to-noise ratio can clearly be greatly enhanced if it is known *a priori* that the signal is a sinusoid of known frequency and phase. Rather than attempting to describe the complete response vector $\hat{\phi}$, one can then limit oneself to the determination of the single Fourier component $\hat{\phi}_p$, and the significance test for this component reduces to the much less stringent one-dimensional condition (for 95% significance) $|\hat{\phi}_p|^2 > (1{\cdot}96)^2 F_p$. More generally, one can consider the case that the signal consists of a superposition of a number of spectral lines of known frequencies, amplitude ratios and phase relationships (i.e. a given pulse shape). If the pulse shape is known *a priori*, the signal can in this case also be characterized by a single number, the amplitude of the pulse, and the significance test reduces again to the less stringent one-dimensional case.

This suggests a general strategy for the step-wise construction of the multi-dimensional response of the system. First one considers only a fixed response pattern characterized by a single degree of freedom: the pattern amplitude. If this one-component response passes the significance test, a second degree of freedom is added, and the resultant two-dimensional response is tested, and so forth. The procedure is terminated when the response pattern at some order $m \leqslant n$ fails to pass the significance test. In general, the maximum significant resolution order will be considerably smaller than n. The technique is conceptually very similar to the construction of optimum prediction models using nested model sequences (Hasselmann 1978; Barnett and Hasselmann 1979). The method has the advantage of automatically yielding the maximum degree of resolution which can be extracted from a given data set. However, as in the prediction model problem, it is dependent on the suitable *a priori* choice of the ordered set of basis vectors used to build up the response pattern.

4. Construction of the response pattern

In order to increase the statistical significance of the response analysis, the number of degrees of freedom used to represent the response must be reduced. This requires some prior knowledge regarding the expected structure of the response. Assume therefore that, with the aid of some simpler model or from general theoretical considerations, one has arrived at a first guess $\mathbf{g}_1 = (g_{11}, g_{21}, \ldots, g_{n1})$ of the expected response. Assume furthermore that one can estimate the likely structure of the deviations of the first guess from the true response, so that an improved guess of the true response may be represented by a linear combination of two vectors $\mathbf{g}_1$ and $\mathbf{g}_2$. In general, let $\mathbf{g}_1 \ldots, \mathbf{g}_p$ represent p vectors which span the space of the expected response to order p.

It can readily be seen that the use of the vectors $\mathbf{g}_\alpha$ directly as a truncated set of basis vectors to represent the pth-order response would not optimize the signal-to-noise ratio, even when the true response lies in the p-dimensional space spanned by the set $\mathbf{g}_\alpha$, since the set $\mathbf{g}_\alpha$ was introduced solely to represent the anticipated response, without reference to the properties of the noise field. It may therefore be expected that higher significance values can be achieved by introducing a new set of basis vectors $\mathbf{b}_\alpha$ which eliminate as much of the noise as possible while still retaining a high proportion of the expected signal. Although the vector set $\mathbf{b}_\alpha$ is no longer optimum with respect to the representation of the total variance of the expected response, it represents an optimum filter for identifying those features of the response which can be most readily distinguished from the noise.

Consider then the truncated representation

$$\tilde{\phi}_i = \sum_{\alpha=1}^{p} b_{i\alpha}\psi_\alpha, \quad \text{or} \quad \tilde{\phi} = B\psi, \qquad (15)$$

of the true n-dimensional response ϕ in terms of the p components ψ_α with respect to the space of basis vectors $\mathbf{b}_\alpha$. The components ψ_α, chosen to minimize the error $|\tilde{\phi}-\phi|^2$, are determined by the set of equations resulting from scalar multiplication of (15) by the basis set $\mathbf{b}_\alpha$,

$$\sum_{\alpha=1}^{p} G_{\beta\alpha}\psi_\alpha = h_\beta, \quad \beta = 1\ldots, p \qquad (16)$$

with

$$G_{\beta\alpha} = \sum_{i=1}^{n} b_{i\beta}b_{i\alpha}, \quad h_\beta = \sum_i b_{i\beta}\phi_i \qquad (17)$$

It is convenient to express the solution of (16) in terms of the set of adjoint basis vectors

$$b_\alpha^i = \sum_\beta G_{\alpha\beta}^{-1}\, b_{i\beta} \qquad (18)$$

in the form

$$\psi_\alpha = \sum_i b_\alpha^i \phi_i \qquad (19)$$

For formal reasons related to the nonorthogonal linear transformations considered below, the adjoint basis vectors have been defined here as upper-index contravariant vectors, as opposed to the lower-index covariant basis vectors.

The two vector sets are orthogonal,

$$\sum_i b_\alpha^i b_{i\beta} = \delta_{\alpha\beta} \qquad (20)$$

We wish now to determine the set of basis vectors $\mathbf{b}_\alpha$ such that the statistic

$$\rho^2 = \sum_{\alpha,\beta=1}^{p} \psi_\alpha\psi_\beta\langle\delta\psi_\alpha\delta\psi_\beta\rangle^{-1} \qquad (21)$$

characterizing the statistical significance of the response in the truncated $\mathbf{b}_\alpha$ space is a maximum for any response vector ϕ lying in the anticipated response space spanned by the vectors $\mathbf{g}_\alpha$.

The problem is most easily solved by transforming to ortho-normal coordinates

$$\phi_i' = \sum_{j=1}^{n} T_{ij}\phi_j, \quad \text{or} \quad \phi' = T\phi, \qquad (22)$$

with $\langle\delta\phi_i'\delta\phi_j'\rangle = \delta_{ij}$.

The transformation can be represented as the product, $T = \Lambda^{-1}R$, of a rotation R to orthogonal coordinates, the matrix R consisting of the set of eigenvectors (principal components, or empirical orthogonal functions (EOF)) of the covariance matrix $C_{ij} = \langle\delta\phi_i\delta\phi_j\rangle$,

$$\sum_j (C_{ij}-\sigma_j^2\delta_{ij})R_{jk} = 0,$$

and a subsequent normalization by multiplication with the inverse of the diagonal matrix $\Lambda_{ij} = \delta_{ij}\sigma_i$, where σ_i^2 is the eigenvalue (variance) of the ith principal component.

The transformation leaves equation (19), and therefore also (21), invariant, provided a distinction is made between covariant vectors, which transform in accordance with (22), and contravariant vectors, whose transformation is given by

$$\phi^{i'} = \sum_{j=1}^{n} (T^{-1})^{+}_{ij}\phi^j = \sum_j (\Lambda R)_{ij}\phi^j \quad . \quad . \quad (23)$$

where the symbol + denotes the transposed matrix. (Note that the transformations apply only to Latin indices running from 1 to n, not to Greek indices running from 1 to p.)

Substituting (19) in the ortho-normal reference frame, the covariance matrix of the estimation errors $\delta\psi_\alpha$ may then be written

$$\langle \delta\psi_\alpha \delta\psi_\beta \rangle = \sum_{i,j} b^{i'}_\alpha b^{j'}_\beta \langle \delta\phi'_i \delta\phi'_j \rangle = \sum_i b^{i'}_\alpha b^{i'}_\beta \quad . \quad . \quad (24)$$

Without loss of generality we may now assume that the vector set $b^{i'}_\alpha$ is ortho-normal, $\sum_i b^{i'}_\alpha b^{i'}_\beta = \delta_{\alpha\beta}$. If they are not ortho-normal originally, they may be transformed to an ortho-normal set by a linear transformation (with respect to the α, β index space), which leaves the scalar form (21) invariant. The statistic ρ^2 then reduces to the Euclidean form

$$\rho^2 = \sum_\alpha (\psi_\alpha)^2 \quad . \quad . \quad . \quad . \quad (25)$$

Similarly, the vector set $\mathbf{g}_\alpha$ may be redefined such that the new vectors span the same p-dimensional space, but become ortho-normal after the transformation (22), $\sum_i g'_{i\alpha} g'_{i\beta} = \delta_{\alpha\beta}$.

After these transformations, the basis vector set b^i_α which yields the maximum value of $\rho^2 = \sum_\alpha \sum_i (b^{i'}_\alpha \phi'_i)^2$ for any vector $\boldsymbol{\phi}'$ lying in the space $\{\mathbf{g}'_\alpha\}$ can immediately be seen to be given by

$$b^{i'}_\alpha = g'_{i\alpha} \quad . \quad . \quad . \quad . \quad (26)$$

(or any equivalent ortho-normal set).

Having established the equality of the spaces spanned by the vector sets $g'_{\alpha i}$ and $b^{i'}_\alpha$, the intermediate step of transforming to ortho-normal vectors in ϕ' space can now be dropped, and equation (26) may be interpreted as applying directly to the original vectors in $\boldsymbol{\phi}'$ space.

Transforming Eq. (26) back to the vectors in $\boldsymbol{\phi}$ space, and noting that the left and right hand sides obey different transformation rules, we obtain for the conjugate basis vectors $\hat{\mathbf{b}}_\alpha = (b^i_\alpha)$

$$\hat{\mathbf{b}}_\alpha = T^+ T \mathbf{g}_\alpha = R^+(\Lambda^2)^{-1} R \mathbf{g}_\alpha \quad . \quad . \quad . \quad (27)$$

The original basis vectors $\mathbf{b}_\alpha$ may then be obtained by solving (18). However, since we are interested only in the space spanned by $\mathbf{b}_\alpha$, which is identical to the space spanned by the set $\hat{\mathbf{b}}_\alpha$, it is irrelevant which vector set is chosen as basis.

The physical interpretation of (27) is most easily seen in terms of a principal component (EOF) representation using statistically orthogonal, but non-normalized variables. Assuming that the EOF representation coincides with the original choice of variables ϕ_i, so that $R \equiv 1$, Eq. (27) reduces, in component representation, to

$$\hat{b}_{\alpha i} \equiv b^i_\alpha = g_{\alpha i}/\sigma_i^2 \quad . \quad . \quad . \quad (28)$$

Thus the individual components of the vector $\hat{\mathbf{b}}_\alpha$ are reduced relative to $\mathbf{g}_\alpha$ by a factor inversely proportional to the noise variance. The vector $\hat{\mathbf{b}}_\alpha$ is thereby skewed away from the

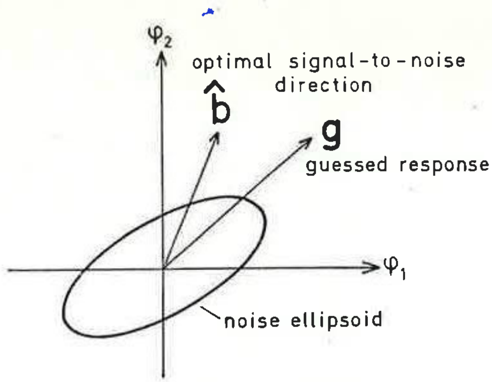

Figure 1. Relation between guessed response pattern vector **g** and maximal significance direction $\hat{\mathbf{b}}$ relative to error ellipsoid, see Eq. (27).

high noise components towards the low noise components, see Fig. 1. It is of interest that a similar modification of the signal response function by a transfer function inversely proportional to the noise variance occurs in the signal processing theory of continuous time series (Wainstein and Zubakov 1962).

5. Application to the atmosphere

The example discussed in section 3 suggests that in most applications it will prove virtually impossible to demonstrate the statistical significance of atmospheric response relations derived from numerical experiments with GCMs or from the analysis of real data without the application of filtering techniques. This is required primarily to reduce the number of degrees of freedom employed in the unmodified description of the complete field (typically of order 10^3–10^4) to the much smaller number of leading components which are able to satisfy the χ^2 significance criterion. Experience with the analogous problem of constructing prediction models from data, with comparable statistics, suggests that this number will generally lie in the range 5 to 20 (Barnett and Hasselmann 1979). In order to achieve an optimum separation of the signal from the noise through pattern filtering, it is clearly necessary to have some prior knowledge regarding the expected structure of the atmospheric response and noise statistics.

In most cases, a reasonable first-order guess at the atmospheric response can be obtained with the aid of rather simple, linear, quasi-geostrophic models (Egger 1977), or from more general considerations. For example, if the response is expected to be of large scale and relatively strong, so that a reasonable number of degrees of freedom can be incorporated in the pattern description, it may be adequate to build up the response using low-order spherical harmonics, without particular regard to the ordering of the first few components of the sequence.

More effort will generally be required to determine the covariance matrix of the estimation errors. The techniques are basically well known (Leith 1973; Jenkins and Watts 1968) and involve estimating the integrals of the covariance functions of the atmospheric variability (or, expressed differently, determining the covariance spectral density of the atmospheric variability at zero frequency). The resulting error covariance matrix C_{ij} must then be rotated to orthogonal principal components in order to transform the anticipated response patterns $\mathbf{g}_\alpha$ into maximum significance patterns $\mathbf{b}_\alpha$. (In practice, the orthogonalization of

C_{ij} is usually also the most efficient way of inverting the covariance matrix in forming the statistic ρ^2, independent of the question of constructing optimum pattern filters.)

For real data, the derivation of the required error statistics will generally present no basic difficulty. However, the analogous error statistics for models are more difficult to reconstruct, since continuous model time series of comparable length to analysed global or hemispheric real data are not available. Nonetheless, estimates of the variability statistics of models have been published (Chervin and Schneider 1976), although in terms of individual gridpoint statistics rather than EOFs. It would be valuable to reanalyse the available model variability data with respect to the multi-variate error statistics relevant for pattern detection, thereby providing the basic statistical calibration needed for the application of the models in atmospheric response studies.

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