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HIGHER ORDER RECTIFIABILITY IN EUCLIDEAN SPACE

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Abstract

The first main goal of this thesis is to develop a concept of approximate differentiability of higher order for subsets of the Euclidean space that allows to characterize higher order rectifiable sets, extending somehow well known facts for functions. We emphasize that for every subset A of the Euclidean space and for every integer $k \geq 2$ we introduce the approximate differential of order k of A and we prove it is a Borel map whose domain is a (possibly empty) Borel set. This concept could be helpful to deal with higher order rectifiable sets in applications.

The other goal is to extend to general closed sets a well known theorem of Alberti on the second order rectifiability properties of the boundary of convex bodies. The Alberti theorem provides a stratification of second order rectifiable subsets of the boundary of a convex body based on the dimension of the (convex) normal cone. Considering a suitable generalization of this normal cone for general closed subsets of the Euclidean space and employing some results from the first part we can prove that the same stratification exists for every closed set.

The content of Chapters 2 and 3 has been published on ArXiv, see [San17].

Zusammenfassung

Das erste Ziel dieser Arbeit ist die Entwicklung eines Konzepts zur Beschreibung von Differenzierbarkeit höherer Ordnung für Teilmengen des euklidischen Raumes, welche es erlaubt von höherer Ordnung rektifizierbare Mengen zu charakterisieren. Wir betonen, dass wir für jede Teilmenge A des euklidischen Raumes und jede ganze Zahl $k \geq 2$ ein approximatives Differenzial der Ordnung k einführen und beweisen, dass es sich dabei um eine Borelfunktion handelt deren Definitionsbereich eine (möglicherweise leere) Borelmenge ist. Unser Konzept könnte hilfreich für die Behandlung von höherer Ordnung rektifizierbarer Mengen in Anwendungen sein.

Das andere Ziel ist die Verallgemeinerung auf beliebige abgeschlossene Mengen eines bekannten Satzes von Alberti über Rektifizierbarkeit zweiter Ordnung des Randes konvexer Körper. Für den Rand eines solchen konvexen Körper liefert Albertis Resultat eine Stratifikation durch von zweiter Ordnung rektifizierbare Teilmengen des Randes basierend auf der Dimension des (konvexen) Normalenkegels. Für eine geeignete Verallgemeinerung dieses Normalenkegels auf allgemeine abgeschlossene Teilmengen des euklidischen Raumes und unter Verwendung einiger Resultate aus dem ersten Teil können wir zeigen dass eine solche Stratifiaktion für alle abgeschlossenen Mengen existiert.

Der Inhalt der Abschnitte 2 und 3 wurde bereits auf ArXiv veröffentlicht, siehe [San17].

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Contents

Chapter 1

Introduction

After we have provided the basic notation and the terminology, we introduce the subject of the thesis and we give a detailed overview about its content.

Notation and basic definitions

The notation and the terminology used without comments agree with [Fed69, pp. 669-676]. However, for the reader's convenience, sometimes we use footnotes to point out the references in [Fed69]. Moreover we add the following classical definitions.

Fibers. Suppose X and Y are sets, $Q \subseteq X \times Y$.

In addition to the standard notation for the domain, the image and the restriction (over some subset of X) of Q (see [Fed69, p. 669]) we define the fiber of Q at a point $x \in X$ by

$$Q(x) = \operatorname{im}(Q|\{x\}) = Y \cap \{y : (x, y) \in Q\}.$$

In order to make our notation consistent with the universally accepted notation for functions we additionaly use the following convention. In case Q is a function and $x \in \operatorname{dmn} Q$ we identify the singleton $Q(x) \subseteq Y$ with the point $y \in Y$ such that $Q(x) = \{y\}$. Therefore $Q(x) = \{Q(x)\}$ whenever $x \in \operatorname{dmn} Q$ and $Q(x) = \varnothing$ whenever $x \notin \operatorname{dmn} Q$.

Distance function and nearest point projection[1]. Suppose $\varnothing \neq A \subseteq \mathbf{R}^n$.

We define $\boldsymbol{\delta}_A : \mathbf{R}^n \to \mathbf{R}$ to be the function such that

$$\boldsymbol{\delta}_A(x) = \inf\{|x - a| : a \in A\} \quad \text{for } x \in \mathbf{R}^n.$$

We define $\operatorname{Unp}(A)$ to be the set of $x \in \mathbf{R}^n$ such that there exists a unique $y \in A$ such that $\boldsymbol{\delta}_A(x) = |y - x|$ and $\boldsymbol{\xi}_A : \operatorname{Unp}(A) \to \mathbf{R}^n$ to be the function such that

$$\boldsymbol{\delta}_A(x) = |\boldsymbol{\xi}_A(x) - x| \quad \text{whenever } x \in \operatorname{Unp}(A).$$

Affine hulls, cones and duals. A subset $C \subseteq \mathbf{R}^n$ is called cone if and only if $\lambda x \in C$ whenever $x \in C$ and $\lambda > 0$. For each subset $S \subseteq \mathbf{R}^n$,

$$\operatorname{Dual} S = \mathbf{R}^n \cap \{v : v \bullet u \leq 0 \text{ whenever } u \in S\};$$

[1] This notation agrees with [Fed59, 4.1].

see [Fed59, 4.5] for further comments. Finally for each subset $S \subseteq \mathbf{R}^n$ we define

$$\operatorname{aff} S$$

as the smallest affine set of $\mathbf{R}^n$ containing S (a set $M \subseteq \mathbf{R}^n$ is called affine set if $\lambda x + (1-\lambda) \in M$ whenever $x \in M$, $y \in M$ and $\lambda \in \mathbf{R}$); see [Roc70, Section 1] for further comments.

Orthogonal projections. If $1 \leq m \leq n$ are integers we define $\mathbf{G}(n,m)$ to be the set of all m dimensional subspaces of $\mathbf{R}^n$. If $T \in \mathbf{G}(n,m)$ we define $T_\natural : \mathbf{R}^n \to \mathbf{R}^n$ to be the linear map such that

$$T_\natural^* = T_\natural, \quad T_\natural \circ T_\natural = T_\natural, \quad \operatorname{im} T_\natural = T,$$

and we define $T^\perp = \ker T_\natural$.

Pointwise differentiability for functions. Suppose X and Y are normed vector spaces, $k \geq 0$ is an integer, $0 \leq \alpha \leq 1$, g maps a subset of X into Y and $a \in X$. We say that g is pointwise differentiable of order (k,α) at a if and only if there exists an open set $U \subseteq X$ and a polynomial function $P : X \to Y$ of degree at most k such that $a \in U \subseteq \operatorname{dmn} g$, $g(a) = P(a)$,

$$\lim_{x \to a} \frac{|g(x) - P(x)|}{|x-a|^k} = 0 \ \text{ if } \alpha = 0, \quad \limsup_{x \to a} \frac{|g(x) - P(x)|}{|x-a|^{k+\alpha}} < \infty \ \text{ if } \alpha > 0.$$

In this case P is unique and the pointwise differentials of order i of f at a are defined by $\operatorname{pt} \mathrm{D}^i g(a) = \mathrm{D}^i P(a)$ for $i = 0, \ldots, k$.

Functions and submanifolds of class (k,α). Suppose X and Y are normed vector spaces, $k \geq 0$ is an integer, $0 \leq \alpha \leq 1$, g maps some open subset of X into Y and $a \in X$. We say that g is of class (k,α) if and only if g is of class k and each point of $\operatorname{dmn} g$ has an open neighbourhood U such that $(\mathrm{D}^k f)|U$ satisfies a Hölder condition with exponent α.

Suppose $k \geq 0$ is an integer and $0 \leq \alpha \leq 1$. The notion of diffeomorphism of class (k,α) is made by replacing "class k" with "class (k,α)" in [Fed69, 3.1.18]. Analogously the notion of μ dimensional submanifold of class (k,α) of $\mathbf{R}^n$ is made by replacing "class k" with "class (k,α)" in [Fed69, 3.1.19].

Second fundamental form. If $1 \leq m \leq n$ are integers, M is an m dimensional submanifold of class 2 of $\mathbf{R}^n$ and $a \in M$ then we call second fundamental form of M at a the unique symmetric 2 linear function

$$\mathbf{b}_M(a) : \operatorname{Tan}(M,a) \times \operatorname{Tan}(M,a) \to \operatorname{Nor}(M,a)$$

such that $\mathbf{b}_M(a)(u,v) \bullet \nu(a) = -\mathrm{D}\,\nu(a)(u) \bullet v$ for each $u, v \in \operatorname{Tan}(M,a)$, whenever $\nu : M \to \mathbf{R}^n$ is of class 1 relative to M with $\nu(x) \in \operatorname{Nor}(M,x)$ for every $x \in M$.

Higher order rectifiability[2]. Suppose $1 \leq m \leq n$ are integers and ϕ is a measure over $\mathbf{R}^n$. A subset $A \subseteq \mathbf{R}^n$ is called *countably (ϕ,m) rectifiable of class (k,α)* if and only if there exist countably many m dimensional submanifolds M_j of class (k,α) such that

$$\phi\big(A \sim \textstyle\bigcup_{j=1}^\infty M_j\big) = 0.$$

A subset $A \subseteq \mathbf{R}^n$ is called *(ϕ,m) rectifiable of class (k,α)* if it is countably (ϕ,m) rectifiable of class (k,α) and $\phi(A) < \infty$.

[2] When $\phi = \mathscr{H}^m$ this notion has been introduced in [AS94, 3.1].

Some further notation. Suppose $1 \le m \le n$ and $k \ge 1$ are integers, $0 \le \alpha \le 1$, $a \in \mathbf{R}^n$, $T \in \mathbf{G}(n,m)$, $0 \le \kappa < \infty$ and suppose $f : T \to T^\perp$ is a function such that $f(T_\natural(a)) = T_\natural^\perp(a)$.

Then we define[3]

$$\mathbf{X}_{k,\alpha}(a,T,f,\kappa) = \mathbf{R}^n \cap \{z : |f(T_\natural(z)) - T_\natural^\perp(z)| \le \kappa |T_\natural(z-a)|^{k+\alpha}\};$$

alternatively $\mathbf{X}_k(a,T,f,\kappa) = \mathbf{X}_{k,0}(a,T,f,\kappa)$. If $f(\chi) = T_\natural^\perp(a)$ for every $\chi \in T$ then we abbreviate $\mathbf{X}(a,T,\kappa) = \mathbf{X}_1(a,T,f,\kappa)$.

If $0 < s < \infty$ and $0 < t < \infty$ we define

$$\mathbf{C}(T,a,s,t) = \mathbf{R}^n \cap \{x : |T_\natural(x-a)| < s,\ |T_\natural^\perp(x-a)| < t\}.$$

Finally let $\operatorname{gr}(f) = \{\chi + f(\chi) : \chi \in T\}$.

Motivation

The concept of higher order rectifiability is a very natural (weak) notion of regularity that can be considered in the setting of Geometric Measure Theory.

First of all this notion turns out to be an interesting concept in order to study the regularity properties of solutions of variational problems involving elliptic functionals (in particular the area functional). These solutions can be modelled using the notion of varifold, originally introduced by Almgren in the 60's. Roughly speaking a varifold is a model for non smooth surfaces having multiple sheets (integral varifolds) and, more generally, non integer valued density (rectifiable varifolds). If we consider the class of varifolds V in $\mathbf{R}^n$ whose first variation with respect to the area functional is represented by integrating a function $\mathbf{h}(V,\cdot) \in \mathbf{L}_1^{\mathrm{loc}}(\|V\|, \mathbf{R}^n)$ (usually called mean curvature), it is well known that classical regularity almost everywhere fails to hold in many cases of interest. For example, there are integral varifolds with bounded mean curvature such that the set where the support does not locally correspond to a graph of a function of class 1 has positive measure (see [All72, 8.1(2)]). Therefore the need arises to look for weaker notions of regularity that allow to further investigate such classes of objects. Recently, in [Men13], it has been proved that the support of every m dimensional integral varifold V in $\mathbf{R}^n$ with mean curvature in $\mathbf{L}_1^{\mathrm{loc}}(\|V\|, \mathbf{R}^n)$ is countably $(\|V\|, m)$ rectifiable of class 2. By [All72, 8.3] we can deduce that if the mean curvature is in $\mathbf{L}_m^{\mathrm{loc}}(\|V\|, \mathbf{R}^n)$ then the support of $\|V\|$ is countably $(\mathscr{H}^m, m)$ rectifiable of class 2. Whether or not the support of an m dimensional stationary varifold is countably $(\mathscr{H}^m, m)$ rectifiable of class k, for every integer $k \ge 1$, is an open problem.

Rectifiability of class 2 is a natural concept of regularity in convex geometry. Moreover rectifiable sets of class k arise naturally as level sets of maps between Euclidean spaces with distributional derivatives up to order k representable by integration, see [BHS05, 1.6]. A sufficient condition for rectifiability of class $(1,\alpha)$ is obtained in [Kol16] using discrete curvatures. Higher order rectifiability of graphs is investigated in [Del14, Theorem 3.1].

[3] Compare this definition with similar ones introduced in [Fed69, 3.3.1] and [Mat95, 15.12].

Overview

The first main goal of this thesis is to develop a concept of approximate differentiability of higher order for subsets of the Euclidean space that allows to characterize higher order rectifiable sets. For functions whose domain is a subset of the Euclidean space this is a well known fact that it has been established in [Whi51], [Fed69, § 3.1] and [Isa87]. More specifically these results can be combined in order to get the following result, see 2.10 and 2.11.

1.1 Theorem (Federer, Isakov, Whitney)**.** *If $1 \le m < n$ and $k \ge 1$ are integers, $0 \le \alpha \le 1$, $A \subseteq \mathbf{R}^m$ is $\mathscr{L}^m$ measurable and $f : A \to \mathbf{R}^{n-m}$ is $\mathscr{L}^m \llcorner A$ measurable, then f is approximately differentiable*[4] *of order (k, α) at $\mathscr{L}^m$ a.e. $a \in A$ if and only if there exist countably many functions $g_j : \mathbf{R}^m \to \mathbf{R}^{n-m}$ of class (k, α) such that*

$$\mathscr{L}^m \left(A \sim \textstyle\bigcup_{j=1}^{\infty} \{x : g_j(x) = f(x)\} \right) = 0.$$

We establish this result for subsets of the Euclidean space. In fact, employing the notion of approximate differentiability of higher order for sets introduced in 3.8 we can prove, in 3.23 and 3.41, the following result.

1.2 Theorem. *If $1 \le m \le n$ and $k \ge 1$ are integers, $0 \le \alpha \le 1$, $A \subseteq \mathbf{R}^n$ is $\mathscr{H}^m$ measurable and $\mathscr{H}^m(A) < \infty$, then A is approximately differentiable of order (k, α) at $\mathscr{H}^m$ a.e. $a \in A$ if and only if A is $(\mathscr{H}^m, m)$ rectifiable of class (k, α).*

It is worth to compare this result with other results in the literature. First of all this result can be seen as a generalization to the case of higher order differentiability of the well known fact in Geometric Measure Theory that $(\mathscr{H}^m, m)$ rectifiable sets[5] of class 1 can be characterized among all the $\mathscr{H}^m$ measurable subsets of $\mathbf{R}^n$ with finite m dimensional Hausdorff measure through the existence of an m dimensional "measure theoretic tangent" plane at $\mathscr{H}^m$ a.e. points of the set. There are essentially two natural ways to define this notion of measure theoretic tangency. One uses a blow up procedure and the other one uses densities of Hausdorff measures.

1.3 Definition (Simon[6])**.** Suppose $1 \le m \le n$ are integers, $A \subseteq \mathbf{R}^n$ and $a \in \mathbf{R}^n$. An m dimensional plane $T \in \mathbf{G}(n, m)$ is the m dimensional approximate tangent plane of A at a if and only if there exists $0 < \theta < \infty$ such that

$$\lim_{r \to 0+} r^{-m} \int_A f((x-a)/r) d\mathscr{H}^m x = \theta \int_T f d\mathscr{H}^m \quad \text{whenever } f \in \mathscr{K}(\mathbf{R}^n).$$

1.4 Definition (Federer[7])**.** Suppose $1 \le m \le n$ are integers, $A \subseteq \mathbf{R}^n$ and $a \in \mathbf{R}^n$. A vector $v \in \mathbf{R}^n$ is an $(\mathscr{H}^m \llcorner A, m)$ approximate tangent vector at a if and only if $\mathbf{\Theta}^{*m}(\mathscr{H}^m \llcorner A \cap \mathbf{E}(a, v, \epsilon), a) > 0$ for every $\epsilon > 0$. An m dimensional plane $T \in \mathbf{G}(n, m)$ is the m dimensional approximate tangent plane of A at a if and only if T equals the set of all $(\mathscr{H}^m \llcorner A, m)$ approximate tangent vectors at a.

[4] For the definition of approximate differentiability for functions, see 2.2.

[5] By [Fed69, 3.2.29] the notion of rectifiability of class 1 coincides with the classical notion of rectifiability phrased in terms of images of Lipschitzian maps, see [Fed69, 3.2.14].

[6] See [Sim83, 11.2, 11.4] and [FM99, 2.2].

[7] See [Fed69, 3.2.16].

In 1.3 and in 1.4 it is not difficult to see that m and T are uniquely determined by A and a. Either employing the notion in 1.3 or the one in 1.4 the following well known classical result holds.

1.5 Theorem (Federer[8], Simon[9])**.** *Suppose $1 \leq m \leq n$ are integers and $A \subseteq \mathbf{R}^n$ is $\mathscr{H}^m$ measurable with $\mathscr{H}^m(A) < \infty$. Then A is $(\mathscr{H}^m, m)$ rectifiable of class 1 if and only if A admits the m dimensional approximate tangent plane at $\mathscr{H}^m$ a.e. $a \in A$.*

Suppose now $A \subseteq \mathbf{R}^n$ and $a \in \mathbf{R}^n$. For the case of order 1 the definition of approximate differentiability that we have introduced in 3.8 (see also 3.19) is equivalent to require the existence of a measure theoretic tangent space, $\operatorname{ap}\operatorname{Tan}(A, a)$, at a. In particular if $T \in \mathbf{G}(n, m)$ for some integer $1 \leq m \leq n$ then, as it is proved in 3.21 and 3.14,

$$T \text{ satisfies 1.3} \implies T = \operatorname{ap}\operatorname{Tan}(A, a) \implies T \text{ satisfies 1.4}.$$

The examples in 3.4 and 3.17 show that there are sets with finite m dimensional Hausdorff measure for which the reverse implications do not hold at every point. However if A is $\mathscr{H}^m$ measurable and $\mathscr{H}^m(A) < \infty$ the reverse implications hold at $\mathscr{H}^m$ a.e. points of A.

The problem of generalizing 1.5 to the case of higher order rectifiability has been already addressed in [AS94]. In that paper a notion of differentiability of order 2 and order $(1, \alpha)$, for every $0 < \alpha \leq 1$, are introduced by means of a blow up procedure naturally generalizing 1.3. However, as it is pointed out in [AS94, pp. 7–8], examples show that $(\mathscr{H}^m, m)$ rectifiable sets of class 2 may fail to be differentiable of order 2 in the sense of [AS94] at $\mathscr{H}^m$ a.e. points. Therefore, in order to generalize 1.5 they need, in [AS94, 3.5, 3.12], additional technical hypotheses on the structure of the sets. This pathological behaviour suggests that a different notion of differentiability for the case of order greater than 1 has to be considered. The definition introduced in this chapter rules out the pathologies of [AS94] and allows to get the generalization of 1.5 in the most natural setting.

For every integer $k \geq 2$ the notion of approximate differentiability of order k for a subset $A \subseteq \mathbf{R}^n$ naturally induces a notion of approximate differential of order k, $\operatorname{ap}\mathrm{D}^k A$, of A; see 3.20. For every $A \subseteq \mathbf{R}^n$ this is always a Borel map with values in $\bigodot^k(\mathbf{R}^n, \mathbf{R}^n)$ whose domain is a (possibly empty) Borel subset of $\mathbf{R}^n$, see 3.40. Moreover the approximate differential of order 2 naturally induces a notion of "approximate second fundamental form". In fact, for every $a \in \operatorname{dmn}\operatorname{ap}\mathrm{D}^2 A$ this can be defined as the symmetric bilinear form

$$\operatorname{ap}\mathrm{D}^2 A(a) | \operatorname{ap}\operatorname{Tan}(A, a) \times \operatorname{ap}\operatorname{Tan}(A, a).$$

In 3.24 and 3.36 two classical properties of the second fundamental form of submanifolds of class 2 are extended to our setting.

A notion of pointwise differentiability for subsets of the Euclidean space has been recently developed in [Men16] to study higher order differentiability properties of stationary varifolds. In 3.35 we establish the connection between the notion of approximate differentiability and pointwise differentiability.

[8] See [Fed69, 3.2.19, 3.3.17].
[9] See [Sim83, 11.6, 11.8].

The other goal of this thesis is to generalize a well known theorem of Alberti in convex geometry. In order to state this result we first introduce the following normal cone for convex sets.

1.6 Definition (Alberti[10])**.** Suppose C is a closed convex subset of $\mathbf{R}^n$ and $a \in C$. We define

$$\mathscr{N}(C,a) = \mathbf{R}^n \cap \{v : v \bullet (x-a) \leq 0 \text{ whenever } x \in C\}.$$

1.7 Theorem (Alberti[11])**.** *Suppose C is a closed convex subset of $\mathbf{R}^n$ with non empty interior. Then the set*

$$\mathbf{\Sigma}^m(C) = C \cap \{a : \mathscr{H}^{n-m}(\mathscr{N}(C,a)) > 0\}$$

is countably $(\mathscr{H}^m, m)$ rectifiable of class 2 whenever $m = 0, \ldots, n$.

Evidently $\mathbf{\Sigma}^0(C) \subseteq \ldots \subseteq \mathbf{\Sigma}^{n-1}(C) \subseteq \mathbf{\Sigma}^n(C) = C$ and $\mathbf{\Sigma}^{n-1}(C) = \operatorname{Bdry} C$. A classical result due to Anderson and Klee, see [Sch14, 2.2.5], states that $\mathbf{\Sigma}^m(C)$ is a countable union of compact sets with finite m dimensional Hausdorff measure and the theorem of Alberti strenghtens, up to a set of $\mathscr{H}^m$ measure zero, their result. It could be thought that the second order rectifiability property of the sets $\mathbf{\Sigma}^m(C)$ is a prerogative to work in the context of convex sets. We prove that this is actually not the case. In fact 1.7 can be seen as a special instance of a general fact that holds for every closed subset of the Euclidean space. In order to describe our result we introduce the following definitions.

If $A \subseteq \mathbf{R}^n$ is closed and $a \in A$ we define the closed convex cone

$$\operatorname{nor}(A,a) = \operatorname{Clos}\{\lambda u : \lambda > 0,\ u \in \mathbf{R}^n,\ \boldsymbol{\delta}_A(a+u) = |u|\},$$

and for each integer $m = 0, \ldots, n$ we introduce the sets

$$\mathbf{\Sigma}^m(A) = A \cap \{a : \mathscr{H}^{n-m}(\operatorname{nor}(A,a)) > 0\}.$$

Evidently $\mathbf{\Sigma}^0(A) \subseteq \mathbf{\Sigma}^1(A) \subseteq \ldots \subseteq \mathbf{\Sigma}^{n-1}(A) \subseteq \mathbf{\Sigma}^n(A) = A$. We prove in 4.3 that $\mathbf{\Sigma}^m(A)$ is countably $(\mathscr{H}^m, m)$ rectifiable of class 2 whenever $m = 0, \ldots, n$. As it is pointed out in 4.1 and 4.2, if $A \subseteq \mathbf{R}^n$ is a closed convex set then $\operatorname{nor}(A,a) = \mathscr{N}(A,a)$ whenever $a \in A$. Therefore 1.7 is a special case of our theorem. Our result is new for sets of positive reach. In fact, if $A \subseteq \mathbf{R}^n$ is a set of positive reach then the stratification given by the sets $\mathbf{\Sigma}^m(A)$ has been already considered in [Fed59, 4.15(3)] (see 4.2 for further details), where it is proved that $\mathbf{\Sigma}^m(A)$ is countably m rectifiable for each $m = 0, \ldots, n$. Our result strenghtens this conclusion up to a set of $\mathscr{H}^m$ measure zero. It is worth to recall here that if $A \subseteq \mathbf{R}^n$ is a set of positive reach with non empty interior then it is not difficult to prove that $\mathbf{\Sigma}^{n-1}(A) = \operatorname{Bdry} A$.

For each integer $m = 1, \ldots, n-1$ there are examples of m (Hausdorff) dimensional closed sets $A \subset \mathbf{R}^n$ such that $\mathscr{H}^m(\mathbf{\Sigma}^m(A)) = 0$. For example if $A \subseteq \mathbf{R}^2$ is the graph of a function $f : \mathbf{R} \to \mathbf{R}$ of class 1 such that $\mathscr{L}^1\{x : f(x) = g(x)\} = 0$ whenever $g : \mathbf{R} \to \mathbf{R}$ is of class 2 (see, for instance, [Koh77]) then $\mathscr{H}^1(\mathbf{\Sigma}^1(A)) = 0$. This simple example raises the problem to

[10] See [Alb94, Definition 1.7].
[11] See [Alb94, Theorem 3].

find geometric characterizations for (some subclasses of) the class of all m dimensional closed subsets A of $\mathbf{R}^n$ for which $\mathscr{H}^m(A \sim \mathbf{\Sigma}^m(A)) = 0$. Observe that, by [Fed59, 4.15(4)], if $A \subseteq \mathbf{R}^n$ is an m dimensional set of positive reach then $\mathbf{\Sigma}^m(A) = A$. The result in 4.3 links the aforementioned problem with the reseach of sufficient criteria for second order rectifiability.

Finally we mention that the result in 4.3 raises the question about how the second order rectifiability property of the sets $\mathbf{\Sigma}^m(A)$ is related to the principal curvatures (support measures) on the normal bundle N_A (see 4.19 for the definition of N_A) introduced in [HLW04, §2]. An answer to this question is unknown even in the special case of sets of positive reach, for which case the principal curvatures on the normal bundle were introduced in [Zäh86].

A solution for the aforementioned problems will be part of the forthcoming work of the author.

Chapter 2

Approximate differentiability for functions

Here we present the classical theory of approximate differentiability for functions developed by the work of Federer, Whitney and Isakov. This theory will be applied in the subsequent chapters and it provides a scheme for the novel theory of approximate differentiability for sets introduced in chapter 3.

2.1. Suppose $1 \le m < n$ are integers, $A \subset \mathbf{R}^m$, $a \in \mathbf{R}^m$ and $P : \mathbf{R}^m \to \mathbf{R}^{n-m}$ is a polynomial function of degree at most k such that[1]

$$\operatorname{ap}\lim_{x \to a} \frac{(P|A)(x)}{|x-a|^k} = 0.$$

In particular $\mathbf{\Theta}^m(\mathscr{L}^m \llcorner \mathbf{R}^m \sim A, a) = 0$. Therefore we can use a lemma due to De Giorgi, see [Cam64, Lemma 2.I], to conclude that $P = 0$.

2.2 Definition. Let $1 \le m < n$ and $k \ge 0$ be integers, $0 \le \alpha \le 1$, $A \subset \mathbf{R}^m$, $f : A \to \mathbf{R}^{n-m}$ and $a \in \mathbf{R}^m$.

We say that f is *approximately differentiable of order* (k, α) *at* a (f *is approximately differentiable of order* k *at* a if $\alpha = 0$) if

$$\mathbf{\Theta}^m(\mathscr{L}^m \llcorner \mathbf{R}^m \sim A, a) = 0$$

and there exists a polynomial function $P : \mathbf{R}^m \to \mathbf{R}^{n-m}$ of degree at most k such that $P(a) = f(a)$ if $a \in A$,

$$\operatorname{ap}\lim_{x \to a} \frac{|f(x) - P(x)|}{|x-a|^k} = 0 \ \text{ if } \alpha = 0, \quad \operatorname{ap}\limsup_{x \to a} \frac{|f(x) - P(x)|}{|x-a|^{k+\alpha}} < \infty \ \text{ if } \alpha > 0.$$

[1] Let f be a function mapping a subset of $\mathbf{R}^m$ into some set Y and let $a \in \mathbf{R}^m$. If Y is a normed vector space, a point $y \in Y$ is the approximate limit of f at a if and only if

$$\mathbf{\Theta}^m(\mathscr{L}^m \llcorner \mathbf{R}^m \sim \{x : |f(x) - y| \le \epsilon\}, a) = 0 \quad \text{for every } \epsilon > 0$$

and we denote it by $\operatorname{ap}\lim_{x \to a} f(x)$. If $Y = \overline{\mathbf{R}}$, a point $t \in \overline{\mathbf{R}}$ is the approximate upper limit of f at a if and only if

$$t = \inf\{s : \mathbf{\Theta}^m(\mathscr{L}^m \llcorner \{x : f(x) > s\}, a) = 0\}$$

and we denote it by $\operatorname{ap}\limsup_{x \to a} f(x)$. This concept is a special case of [Fed69, 2.9.12].

2.3 Remark. The condition $\mathbf{\Theta}^m(\mathscr{L}^m \llcorner \mathbf{R}^m \sim A, a) = 0$ in 2.2 is redundant if $\alpha = 0$. By 2.1 we deduce that the polynomial function P in 2.2 is uniquely determined by f and a.

2.4 Definition. Let $A \subset \mathbf{R}^m$ and let $f : A \to \mathbf{R}^{n-m}$. For every non negative integer k the function $\operatorname{ap} \mathrm{D}^k f$ is defined to be the function whose domain consists of all $a \in \mathbf{R}^m$ such that f is approximately differentiable of order k at a and whose value at a equals $\mathrm{D}^k P(a)$, where P satisfies 2.2.

2.5 Remark. If $a \in A \subset \mathbf{R}^m$ and $f : A \to \mathbf{R}^{n-m}$ then f is approximately differentiable of order 0 at a if and only if f is $(\mathscr{L}^m, V)$ approximately continuous[2] at a. In this case $\operatorname{ap} \mathrm{D}^0 f(a) = f(a)$. Here V is the standard $\mathscr{L}^m$ Vitali relation, $V = \{(a, \mathbf{B}(a, r)) : a \in \mathbf{R}^m,\ 0 < r < \infty\}$.

In case $a \in A$ the notion of approximate differentiability of order 1 has been introduced in [Fed69, 3.1.2].

2.6 Lemma. *Suppose $1 \leq m < n$ are integers, $A \subseteq \mathbf{R}^m$, $a \in \mathbf{R}^m$, $f : A \to \mathbf{R}^{n-m}$, $\gamma \geq 1$, $0 < M < \infty$ and $0 \leq \lambda < \infty$ such that*

$$\limsup_{r \to 0+} \frac{\mathscr{L}^m(\mathbf{B}(a,r) \cap \{x : |f(x)| > \lambda\, r^\gamma\})}{\boldsymbol{\alpha}(m)\, r^m} < M.$$

Then $\mathbf{\Theta}^{*m}(\mathscr{L}^m \llcorner \{x : |f(x)| > 2^\gamma \lambda\, |x-a|^\gamma\}, a) < M(1 - 2^{-m})^{-1}$.

Proof. Let $\delta > 0$ such that

$$\mathscr{L}^m(\mathbf{B}(a,r) \cap \{x : |f(x)| > \lambda\, r^\gamma\}) < M\, \boldsymbol{\alpha}(m)\, r^m \quad \text{for } 0 < r \leq \delta.$$

Therefore for $0 < r \leq \delta$ we observe

$$\begin{aligned}
&\mathbf{B}(a,r) \cap \{x : |f(x)| > 2^\gamma \lambda\, |x-a|^\gamma\} \\
&\quad = \{a\} \cup \bigcup_{i=0}^{\infty} (\mathbf{B}(a, r/2^i) \sim \mathbf{B}(a, r/2^{i+1})) \cap \{x : |f(x)| > 2^\gamma \lambda\, |x-a|^\gamma\} \\
&\quad \subseteq \{a\} \cup \bigcup_{i=0}^{\infty} \mathbf{B}(a, r/2^i) \cap \{x : |f(x)| > \lambda\, (r/2^i)^\gamma\},
\end{aligned}$$

$$\mathscr{L}^m(\mathbf{B}(a,r) \cap \{x : |f(x)| > 2^\gamma \lambda\, |x-a|^\gamma\}) < M\, \boldsymbol{\alpha}(m) r^m (1 - 2^{-m})^{-1}$$

and the conclusion follows.

2.7 Theorem. *Let $1 \leq m < n$ and $k \geq 1$ be integers, $0 \leq \alpha \leq 1$, $A \subset \mathbf{R}^m$, $a \in \mathbf{R}^m$ and $f : A \to \mathbf{R}^{n-m}$.*

Then f is approximately differentiable of order (k, α) at a if and only if there exists a function $g : \mathbf{R}^m \to \mathbf{R}^{n-m}$ pointwise differentiable of order (k, α) at a such that $f(a) = g(a)$ if $a \in A$ and

$$\mathbf{\Theta}^m(\mathscr{L}^m \llcorner \mathbf{R}^m \sim \{x : g(x) = f(x)\}, a) = 0.$$

In this case $\operatorname{pt} \mathrm{D}^i g(a) = \operatorname{ap} \mathrm{D}^i f(a)$ for $i = 0, \ldots, k$.

[2] See [Fed69, 2.9.12].

Proof. Since one implication is elementary we suppose f is approximately differentiable of order (k, α) at a.

First we consider the case $\alpha = 0$. There exists a polynomial function $P : \mathbf{R}^m \to \mathbf{R}^{n-m}$ of degree at most k such that, if for every integer $i \geq 1$ we define

$$S_i = \{x : |f(x) - P(x)| < i^{-1}|x - a|^k\},$$

then there exists $\delta_i > 0$ such that $\mathscr{L}^m(\mathbf{B}(a, r) \sim S_i) < 2^{-i} r^m$ for $0 < r \leq \delta_i$. We can assume $\delta_{i+1} < \delta_i$ for each $i \geq 1$ and $\delta_i \to 0$ as $i \to \infty$. Let

$$T = \textstyle\bigcup_{i=1}^{\infty} \left[S_i \cap \mathbf{B}(a, \delta_i) \sim \mathbf{B}(a, \delta_{i+1})\right].$$

If $r > 0$ and $j \geq 1$ is an integer such that $\delta_{j+1} < r \leq \delta_j$ we compute

$$\mathscr{L}^m(\mathbf{B}(a, r) \sim T) \leq \mathscr{L}^m(\mathbf{B}(a, r) \sim S_j) + \sum_{l=j+1}^{\infty} \mathscr{L}^m(\mathbf{B}(a, \delta_l) \sim S_l) < r^m \sum_{l=j}^{\infty} 2^{-l}$$

and we conclude $\boldsymbol{\Theta}^m(\mathscr{L}^m \llcorner \mathbf{R}^m \sim T, a) = 0$. Moreover

$$\lim_{T \ni x \to a} \frac{|f(x) - P(x)|}{|x - a|^k} = 0.$$

If we define $g : \mathbf{R}^m \to \mathbf{R}^{n-m}$ as $g(x) = f(x)$ if $x \in T$ and $g(x) = P(x)$ if $x \in \mathbf{R}^m \sim T$, then we have $\boldsymbol{\Theta}^m(\mathscr{L}^m \llcorner \mathbf{R}^m \sim \{x : g(x) = f(x)\}, a) = 0$,

$$\lim_{x \to a} \frac{|g(x) - P(x)|}{|x - a|^k} = 0 \quad \text{and} \quad g(a) = P(a) \ \ (\text{since } a \notin T).$$

For the case $\alpha > 0$, once we have chosen $0 \leq \lambda < \infty$ such that

$$\operatorname{ap} \limsup_{x \to a} \frac{|f(x) - P(x)|}{|x - a|^{k+\alpha}} < \lambda,$$

we can use the same argument above replacing the sets S_i with the set

$$S = \{x : |f(x) - P(x)| < \lambda \, |x - a|^{k+\alpha}\}.$$

2.8 Remark. The proof of 2.7 has been adapted from [Fed69, 3.2.16] and [Fed69, 3.1.22].

2.9 Remark. Let $1 \leq m < n$ and $k \geq 1$ be integers, $0 \leq \alpha \leq 1$, let $A \subset \mathbf{R}^m$ be $\mathscr{L}^m$ measurable and let $f : A \to \mathbf{R}^{n-m}$ be $\mathscr{L}^m \llcorner A$ measurable. If there exist countably many functions $g_j : \mathbf{R}^m \to \mathbf{R}^{n-m}$ of class (k, α) such that

$$\mathscr{L}^m\left(A \sim \textstyle\bigcup_{j=1}^{\infty} A_j\right) = 0$$

where $A_j = A \cap \{z : g_j(z) = f(z)\}$ for $j \geq 1$, then using [Fed69, 2.10.19(4)] and 2.7 we can easily prove that f is approximately differentiable of order (k, α) at $\mathscr{L}^m$ a.e. $a \in A$ and, for each $j \geq 1$,

$$\mathrm{D}^i \, g_j(z) = \operatorname{ap} \mathrm{D}^i \, f(z) \quad \text{for } \mathscr{L}^m \text{ a.e. } z \in A_j \text{ and } i = 0, \ldots, k.$$

2.10 Theorem. *Let $1 \leq m < n$ and $k \geq 0$ be integers, $A \subset \mathbf{R}^m$ and let $f : A \to \mathbf{R}^{n-m}$ be approximately differentiable of order $(k, 1)$ at $\mathscr{L}^m$ a.e. $a \in A$. Then the following statements hold.*

(1) *f is approximately differentiable of order $k+1$ at $\mathscr{L}^m$ a.e. $x \in A$.*

(2) *A is $\mathscr{L}^m$ measurable and the functions $\operatorname{ap} \mathrm{D}^i f$ are $\mathscr{L}^m \llcorner A$ measurable for $i = 0, \ldots, k+1$.*

(3) *There exist countably many functions $g_j : \mathbf{R}^m \to \mathbf{R}^{n-m}$ of class $k+1$ such that*
$$\mathscr{L}^m \left(A \sim \textstyle\bigcup_{j=1}^{\infty} \{x : g_j(x) = f(x)\} \right) = 0.$$

Proof. First we observe that A is $\mathscr{L}^m$ measurable, f is $(\mathscr{L}^m, V)$ approximately continuous[3] at $\mathscr{L}^m$ a.e. $a \in A$ and f is $\mathscr{L}^m \llcorner A$ measurable by [Fed69, 2.9.11, 2.9.13].

If $k = 0$ the conclusions are consequences of [Fed69, 3.1.8, 3.1.4, 3.1.16] respectively. We use induction over k. Since f is approximately differentiable of order $(k-1, 1)$ at $\mathscr{L}^m$ a.e. point of A we inductively assume that $\operatorname{ap} \mathrm{D}^i f$ are $\mathscr{L}^m \llcorner A$ measurable for $i = 0, \ldots, k$. We use now [Isa87, Theorem 2] and [Fed69, 3.1.15] to deduce the existence of countably many functions $g_j : \mathbf{R}^m \to \mathbf{R}^{n-m}$ of class $k+1$ satisfying (3). Now (1) and (2) follow from 2.9.

2.11 Theorem. *Suppose $1 \leq m < n$ and $k \geq 1$ are integers, $0 \leq \alpha \leq 1$, $A \subset \mathbf{R}^m$ and $f : A \to \mathbf{R}^{n-m}$ is approximately differentiable of order (k, α) at $\mathscr{L}^m$ a.e. $a \in A$.*

Then the following statements hold.

(1) *A is $\mathscr{L}^m$ measurable and the functions $\operatorname{ap} \mathrm{D}^i f$ are $\mathscr{L}^m \llcorner A$ measurable for $i = 0, \ldots, k$.*

(2) *There exist countably many functions $g_j : \mathbf{R}^m \to \mathbf{R}^{n-m}$ of class (k, α) such that*
$$\mathscr{L}^m \left(A \sim \textstyle\bigcup_{j=1}^{\infty} \{x : g_j(x) = f(x)\} \right) = 0.$$

Proof. Since f is approximately differentiable of order $(k-1, 1)$ at every $x \in A$ then (1) follows from 2.10(2). Now we can apply [Isa87, Theorem 1] if $\alpha = 0$ or [Isa87, Theorem 2] if $\alpha > 0$ to get (2).

[3] As usual, $V = \{(a, \mathbf{B}(a, r)) : a \in \mathbf{R}^m,\ 0 < r < \infty\}$.

Chapter 3

Approximate differentiability for sets

In this chapter we introduce a novel theory of approximate differentiability for subsets of Euclidean space.

Basic properties and characterization

Here we provide the definition of approximate differentiability and approximate differentials of higher order for sets, a technical characterization in 3.14 and some examples. Moreover we study the approximate differentiability properties of higher order rectifiable sets in 3.23 and 3.24.

3.1 Definition (Lower and upper tangent cones to a measure)**.** Suppose X is a normed vector space, ϕ is a measure over X, m is a positive integer and $a \in X$.

We define *the m dimensional approximate upper tangent cone of ϕ at a* by[1]

$$\operatorname{Tan}^{*m}(\phi, a) = X \cap \{v : \boldsymbol{\Theta}^{*m}(\phi \llcorner \mathbf{E}(a, v, \epsilon), a) > 0 \text{ for every } \epsilon > 0\}$$

and *the m dimensional approximate lower tangent cone of ϕ at a* as the set $\operatorname{Tan}_*^m(\phi, a)$ of $v \in X$ such that for every $\epsilon > 0$ there exists $\eta > 0$ such that

$$\phi(\mathbf{U}(a + rv, \epsilon r)) \geq \eta r^m \quad \text{whenever } 0 < r \leq \eta.$$

In case $\operatorname{Tan}^{*m}(\phi, a) = \operatorname{Tan}_*^m(\phi, a)$, this set is denoted by $\operatorname{Tan}^m(\phi, a)$ and we call it *the m dimensional approximate tangent cone of ϕ at a*.

3.2 Remark. Evidently $\operatorname{Tan}_*^m(\phi, a) \subseteq \operatorname{Tan}^{*m}(\phi, a)$. Moreover one may easily verify that $\operatorname{Tan}_*^m(\phi, a)$ and $\operatorname{Tan}^{*m}(\phi, a)$ are closed cones. Finally

$$\boldsymbol{\Theta}^{*m}(\phi, a) > 0 \quad [\boldsymbol{\Theta}_*^m(\phi, a) > 0] \iff 0 \in \operatorname{Tan}^{*m}(\phi, a) \quad [0 \in \operatorname{Tan}_*^m(\phi, a)].$$

3.3 Remark. Observe that, in this case, our notation does not agree with [Fed69, 3.2.16]. In fact, $\operatorname{Tan}^{*m}(\phi, a)$ is denoted by $\operatorname{Tan}^m(\phi, a)$ in [Fed69, 3.2.16].

It is often useful to recall that if C is a compact subset of $X \sim \operatorname{Tan}^{*m}(\phi, a)$ and $T = \{a + rv : r \geq 0, v \in C\}$ then $\boldsymbol{\Theta}^m(\phi \llcorner T, a) = 0$. This is proved in [Fed69, 3.2.16].

[1] As in [Fed69, 3.2.16], $\mathbf{E}(a, v, \epsilon) = X \cap \{x : |r(x-a) - v| < \epsilon \text{ for some } 0 < r < \infty\}$.

3.4 Remark. It is natural to consider the following cone

$$T = X \cap \{v : \boldsymbol{\Theta}^m_*(\phi \llcorner \mathbf{E}(a,v,\epsilon), a) > 0 \text{ for every } \epsilon > 0\}.$$

Evidently $\operatorname{Tan}^m_*(\phi, a) \subseteq T$, but simple examples show that the opposite inclusion does not hold. In fact, if we consider $X = \mathbf{R}$, $\phi = \mathscr{L}^1 \llcorner A$, $m = 1$ and $a = 0$, where

$$A = \bigcup_{i=0}^{\infty} \mathbf{R} \cap \{t : 2^{-2i-1} < |t| < 2^{-2i}\},$$

then $\operatorname{Tan}^1_*(\phi, 0) = \{0\}$ and $T = \mathbf{R}$.

3.5 Remark. Suppose $1 \leq m \leq n$ are integers, $A \subseteq \mathbf{R}^n$, $B \subseteq \mathbf{R}^n$ and $a \in \mathbf{R}^n$.

If $\boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner A \sim B, a) = 0$ then it is not difficult to see that

$$\operatorname{Tan}^m_*(\mathscr{H}^m \llcorner A, a) \subseteq \operatorname{Tan}^m_*(\mathscr{H}^m \llcorner B, a),$$

$$\operatorname{Tan}^{*m}(\mathscr{H}^m \llcorner A, a) \subseteq \operatorname{Tan}^{*m}(\mathscr{H}^m \llcorner B, a).$$

3.6 Lemma. *Suppose $1 \leq m \leq n$ are integers, $A \subseteq \mathbf{R}^n$, $a \in \mathbf{R}^n$ and $T \in \mathbf{G}(n,m)$.*

Then the following three conditions are equivalent:

(1) $\operatorname{Tan}^{*m}(\mathscr{H}^m \llcorner A, a) \subseteq T$,

(2) $\boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner A \sim \mathbf{X}(a,T,\epsilon), a) = 0$ *whenever* $\epsilon > 0$,

(3) whenever $\epsilon > 0$

$$\lim_{r \to 0} \frac{\mathscr{H}^m(A \cap \mathbf{B}(a,r) \cap \{z : |T^\perp_\natural(z-a)| > \epsilon r\})}{\boldsymbol{\alpha}(m)\, r^m} = 0.$$

Proof. The fact that (1) implies (2) is a consequence of 3.3 and the fact that (3) follows from (2) is evident. If the condition in (3) holds for some $\epsilon > 0$ then we can argue as in 2.6 to show that

$$\lim_{r \to 0} \frac{\mathscr{H}^m(A \cap \mathbf{B}(a,r) \cap \{z : |T^\perp_\natural(z-a)| > 2\epsilon\, |z-a|\})}{\boldsymbol{\alpha}(m)\, r^m} = 0.$$

Therefore (3) implies (1).

3.7 Lemma. *Let $1 \leq m \leq n$ and $k \geq 1$ be integers, $0 \leq \alpha \leq 1$, $0 \leq \lambda < \infty$, $0 < M < \infty$, $A \subseteq \mathbf{R}^n$, $a \in \mathbf{R}^n$, $T \in \mathbf{G}(n,m)$ and let $f : T \to T^\perp$ be a function of class 1 such that $f(T_\natural(a)) = T^\perp_\natural(a)$ and $\mathrm{D}\, f(T_\natural(a)) = 0$. Suppose*

$$\lim_{r \to 0} \frac{\mathscr{H}^m(A \cap \mathbf{B}(a,r) \cap \{z : |T^\perp_\natural(z) - f(T_\natural(z))| > \epsilon r\})}{\boldsymbol{\alpha}(m)\, r^m} = 0 \text{ for every } \epsilon > 0,$$

$$\limsup_{r \to 0} \frac{\mathscr{H}^m(A \cap \mathbf{B}(a,r) \cap \{z : |T^\perp_\natural(z) - f(T_\natural(z))| > \lambda\, r^{k+\alpha}\})}{\boldsymbol{\alpha}(m)\, r^m} < M.$$

Then

$$\boldsymbol{\Theta}^{*m}(\mathscr{H}^m \llcorner A \sim \mathbf{X}_{k,\alpha}(a,T,f,\kappa), a) < M(1-2^{-m})^{-1}$$

for every $\kappa > 2^{k+\alpha}\lambda$.

Proof. Arguing as in the proof of 2.6 we conclude that

$$\mathbf{\Theta}^{*m}(\mathscr{H}^m \llcorner A \cap \{z : |f(T_\natural(z)) - T_\natural^\perp(z)| > 2^{k+\alpha}\lambda|z-a|^{k+\alpha}\}, a) < M(1-2^{-m})^{-1}.$$

Since $\mathrm{D}\, f(T_\natural(a)) = 0$ we can easily get that

$$\lim_{r\to 0} \frac{\mathscr{H}^m(A \cap \mathbf{B}(a,r) \cap \{z : |T_\natural^\perp(z-a)| > \epsilon\, r\})}{\boldsymbol{\alpha}(m)\, r^m} = 0 \text{ for every } \epsilon > 0$$

and applying 3.6 we conclude that $\mathbf{\Theta}^m(\mathscr{H}^m \llcorner A \sim \mathbf{X}(a,T,\epsilon), a) = 0$. Since

$$\begin{aligned} &\mathbf{X}(a,T,\epsilon) \cap \{z : |f(T_\natural(z)) - T_\natural^\perp(z)| \le 2^{k+\alpha}\lambda|z-a|^{k+\alpha}\} \\ &\quad \subseteq \mathbf{X}_{k,\alpha}(a,T,f,2^{k+\alpha}\lambda(1+\epsilon^2)^{(k+\alpha)/2}) \quad \text{for every } \epsilon > 0, \end{aligned}$$

the conclusion follows.

3.8 Definition (Approximate differentiability for sets)**.** Let $n \ge 1$ and $k \ge 1$ be integers, $0 \le \alpha \le 1$, $A \subseteq \mathbf{R}^n$, $a \in \mathbf{R}^n$ and $A_1 = \{x - a : x \in A\}$. We say that A is *approximately differentiable of order* (k,α) *at* a if there exist an integer $1 \le m \le n$, $T \in \mathbf{G}(n,m)$ and a polynomial function $P : T \to T^\perp$ of degree at most k such that $P(0) = 0$, $\mathrm{D}\, P(0) = 0$ and the following two conditions hold.

(1) For every $\epsilon > 0$ there exists $\rho > 0$ and $\eta > 0$ such that

$$\mathscr{H}^m(\mathbf{C}(T,z,\epsilon r,\epsilon r) \cap A_1) \ge \eta r^m$$

for every $z \in T \cap \mathbf{B}(0,r)$ and $0 \le r \le \rho$.

(2) For every $\epsilon > 0$

$$\lim_{r\to 0} \frac{\mathscr{H}^m\left(A_1 \cap \mathbf{B}(0,r) \cap \{z : \boldsymbol{\delta}_{\mathrm{gr}(P)}(z) > \epsilon\, r^k\}\right)}{\boldsymbol{\alpha}(m) r^m} = 0$$

and, if $\alpha > 0$, there exists $0 \le \lambda < \infty$ such that

$$\lim_{r\to 0} \frac{\mathscr{H}^m\left(A_1 \cap \mathbf{B}(0,r) \cap \{z : \boldsymbol{\delta}_{\mathrm{gr}(P)}(z) > \lambda\, r^{k+\alpha}\}\right)}{\boldsymbol{\alpha}(m) r^m} = 0.$$

3.9 Remark. If $k = 1$ and $\alpha = 0$ the conditions in 3.8 are equivalent to [Mat95, 15.7].

3.10 Remark. We prove that the condition

$$T \subseteq \mathrm{Tan}_*^m(\mathscr{H}^m \llcorner A, a)$$

is necessary and sufficient to have 3.8(1). The condition is clearly necessary. To prove the sufficiency assume $a = 0$, suppose $0 < \epsilon < 1$ and observe there exist an integer $l \ge 1$, $v_1, \ldots, v_l \in \mathbf{S}^{n-1} \cap T$ and a positive number η such that

$$T \cap \mathbf{S}^{n-1} \subseteq \textstyle\bigcup_{i=1}^l \mathbf{U}(v_i,\epsilon) \cap T,$$

$$\mathscr{H}^m(A \cap \mathbf{U}(rv_i,\epsilon r)) \ge \eta\epsilon^{-m} r^m \quad \text{whenever } 0 < r \le \eta \text{ and } i = 1, \ldots, l.$$

Since $\mathbf{\Theta}^{m}_{*}(\mathscr{H}^m \llcorner A, 0) > 0$ by 3.2, we can choose $\eta > 0$ smaller, if necessary, in order to have

$$\mathscr{H}^m(A \cap \mathbf{U}(0, \epsilon r)) \geq \eta r^m \quad \text{whenever } 0 < r \leq \eta.$$

We fix $0 < r \leq \eta$ and $z \in \mathbf{B}(0, r)$. If $|z| \leq \epsilon r$ then $\mathbf{U}(0, \epsilon r) \subseteq \mathbf{U}(z, 2\epsilon r)$ and

$$\mathscr{H}^m(A \cap \mathbf{U}(z, 2\epsilon r)) \geq \eta r^m.$$

If $|z| \geq \epsilon r$ then we choose $1 \leq i \leq l$ such that $|(z/|z|) - v_i| < \epsilon$ and we observe

$$\mathbf{U}(|z| v_i, |z|\epsilon) \subseteq \mathbf{U}(z, 2\epsilon|z|) \subseteq \mathbf{U}(z, 2\epsilon r),$$

$$\mathscr{H}^m(A \cap \mathbf{U}(z, 2\epsilon r)) \geq \eta \epsilon^{-m} |z|^m \geq \eta r^m.$$

3.11 Lemma. *Suppose $1 \leq m \leq n$ are integers, $0 < r < \infty$, $w \in \mathbf{R}^n \cap \mathbf{B}(0, r)$, $T \in \mathbf{G}(n, m)$ and $f : T \to T^{\perp}$ is a locally Lipschitzian function such that $f(0) = 0$.*

Then $\boldsymbol{\delta}_{\operatorname{gr} f}(w) \leq |T^{\perp}_{\natural}(w) - f(T_{\natural}(w))| \leq \big(2 + \operatorname{Lip}(f|\mathbf{B}(0, 2r))\big)\boldsymbol{\delta}_{\operatorname{gr} f}(w)$.

Proof. If we choose $\chi \in T$ so that $\boldsymbol{\delta}_{\operatorname{gr} f}(w) = |w - \chi - f(\chi)|$ then $\chi \in \mathbf{B}(0, 2r)$ and we get

$$\begin{aligned}\boldsymbol{\delta}_{\operatorname{gr} f}(w) &\leq |T^{\perp}_{\natural}(w) - f(T_{\natural}(w))| \\ &\leq |w - \chi - f(\chi)| + |\chi + f(\chi) - T_{\natural}(w) - f(T_{\natural}(w))| \\ &\leq \big(2 + \operatorname{Lip}(f|\mathbf{B}(0, 2r))\big)\boldsymbol{\delta}_{\operatorname{gr} f}(w).\end{aligned}$$

3.12 Lemma. *Suppose $1 \leq m \leq n$ are integers, $\gamma > 0$, $A \subseteq \mathbf{R}^n$, $B \subseteq \mathbf{R}^n$ such that $0 \in \operatorname{Clos} B$, $f : \mathbf{R}^n \to \mathbf{R}^n$ is an univalent map onto $\mathbf{R}^n$ such that $f(0) = 0$ and f and f^{-1} are locally Lipschitzian maps.*

Then the following two conditions are equivalent.

(1) For every $\epsilon > 0$ [for some $0 \leq \epsilon < \infty$]

$$\lim_{r \to 0} \frac{\mathscr{H}^m(A \cap \mathbf{B}(0, r) \cap \{z : \boldsymbol{\delta}_B(z) > \epsilon\, r^{\gamma}\})}{\boldsymbol{\alpha}(m) r^m} = 0.$$

(2) For every $\epsilon > 0$ [for some $0 \leq \epsilon < \infty$]

$$\lim_{r \to 0} \frac{\mathscr{H}^m(f[A] \cap \mathbf{B}(0, r) \cap \{z : \boldsymbol{\delta}_{f[B]}(z) > \epsilon r^{\gamma}\})}{\boldsymbol{\alpha}(m) r^m} = 0.$$

Proof. Since $f[\operatorname{Clos} B] = \operatorname{Clos} f[B]$ we assume B to be closed. Moreover if we prove one implication we immediately get the other one. Therefore we prove that (1) implies (2). Suppose $1 < \Gamma < \infty$ is such that

$$\Gamma^{-1}|z - w| \leq |f(z) - f(w)| \leq \Gamma|z - w|,$$
$$\Gamma^{-1}|z - w| \leq |f^{-1}(z) - f^{-1}(w)| \leq \Gamma|z - w|$$

whenever $z, w \in \mathbf{B}(0, 2)$. Evidently it is enough to show that

$$\begin{aligned}&f^{-1}[f[A] \cap \mathbf{B}(0, r/\Gamma^2) \cap \{w : \boldsymbol{\delta}_{f[B]}(w) > \epsilon r^{\gamma}\}] \\ &\quad \subseteq A \cap \mathbf{B}(0, r/\Gamma) \cap \{w : \boldsymbol{\delta}_B(w) > \Gamma^{-1}\epsilon r^{\gamma}\}\end{aligned}$$

for $\epsilon > 0$ and $0 < r \le 1$. Suppose $z \in f[A] \cap \mathbf{B}(0, r/\Gamma^2)$ such that $\boldsymbol{\delta}_{f[B]}(z) > \epsilon r^\gamma$. Let $w \in f[B]$ such that $|f^{-1}(z) - f^{-1}(w)| = \boldsymbol{\delta}_B(f^{-1}(z))$ and observe

$$\boldsymbol{\delta}_B(f^{-1}(z)) \le |f^{-1}(z)|, \quad |f^{-1}(w)| \le 2|f^{-1}(z)| \le 2\Gamma|z| \le 2\Gamma^{-1} r \le 2,$$
$$|w| \le \Gamma|f^{-1}(w)| \le 2r \le 2,$$
$$\boldsymbol{\delta}_B(f^{-1}(z)) \ge \Gamma^{-1}|z - w| \ge \Gamma^{-1}\boldsymbol{\delta}_{f[B]}(z) > \Gamma^{-1}\epsilon r^\gamma.$$

3.13 Lemma. *Let $1 \le m \le n$ and $k \ge 1$ be integers, $T \in \mathbf{G}(n, m)$ and let $P : T \to T^\perp$ and $Q : T \to T^\perp$ be polynomial functions of degree at most k such that $P(0) = 0$ and $\mathrm{D}^i Q(0) = 0$ for $i = 0, \dots, k-1$. Suppose for every $\epsilon > 0$ there exists $\rho > 0$ such that*

$$\mathbf{C}(T, z, \epsilon r, \epsilon r^k) \cap \{w : \boldsymbol{\delta}_{\mathrm{gr}(Q)}(w) \le \epsilon r^k\} \ne \varnothing$$

whenever $z \in \mathrm{gr}(P) \cap \mathbf{B}(0, r)$ and $0 < r \le \rho$.

Then $P = Q$.

Proof. Let $0 \le c < \infty$ such that $|P(\chi)| \le c|\chi|$ whenever $\chi \in T \cap \mathbf{B}(0, 1)$. If $0 < \epsilon \le 1$ and $0 < \rho \le 1$ are as in the hypothesis, $\chi \in \mathbf{B}(0, (1+c)^{-1}\rho) \cap T$ and $z = \chi + P(\chi)$ then $|z| \le (1+c)|\chi| \le \rho$. Therefore there exists

$$w \in \mathbf{C}(T, z, \epsilon(1+c)|\chi|, \epsilon(1+c)^k|\chi|^k)$$

such that $\boldsymbol{\delta}_{\mathrm{gr}(Q)}(w) \le \epsilon(1+c)^k|\chi|^k$. If $y \in \mathrm{gr}(Q)$ is such that $|w - y| = \boldsymbol{\delta}_{\mathrm{gr}(Q)}(w)$ then

$$|T_\natural(y) - \chi| \le |T_\natural(y - w)| + |T_\natural(w) - \chi| \le 2\epsilon(1+c)^k|\chi|,$$
$$|P(\chi) - Q(\chi)| \le |T_\natural^\perp(z) - T_\natural^\perp(w)| + |T_\natural^\perp(w) - T_\natural^\perp(y)| + |T_\natural^\perp(y) - Q(\chi)|,$$

and the Taylor's formula (see [Fed69, p. 46]) implies

$$Q(T_\natural(y)) - Q(\chi) = \sum_{i=1}^{k} \langle (T_\natural(y) - \chi)^i / i! \odot \chi^{k-i}/(k-i)!, \mathrm{D}^k Q(0) \rangle,$$

$$|Q(T_\natural(y)) - Q(\chi)| \le c_1 \epsilon |\chi|^k,$$

where $c_1 = \| \mathrm{D}^k Q(0) \| \sum_{i=1}^k 2^i (1+c)^{ki}/(i!(k-i)!)$. Therefore $|Q(\chi) - P(\chi)| \le (2(1+c)^k + c_1)\epsilon|\chi|^k$ and the conclusion follows.

3.14 Theorem. *Suppose $1 \le m \le n$ and $k \ge 1$ are integers, $0 \le \alpha \le 1$, $A \subseteq \mathbf{R}^n$, $a \in \mathbf{R}^n$, $A_1 = \{x - a : x \in A\}$, $T \in \mathbf{G}(n, m)$ and $P : T \to T^\perp$ is a polynomial function of degree at most k such that $P(0) = 0$, $\mathrm{D}\, P(0) = 0$.*

Then the following two conditions are equivalent.

(1) T and P satisfy 3.8(1) and 3.8(2).

(2) If $P_i(\chi) = \langle \chi^i / i!, \mathrm{D}^i P(0) \rangle$ for $\chi \in T$ and $i = 1, \dots, k$ and

$$A_i = \{x - P_{i-1}(T_\natural(x)) : x \in A_{i-1}\} \quad \text{for } i = 2, \dots, k,$$

then the following two conditions hold:

(a) for every $i = 1, \dots, k$ and for every $\epsilon > 0$ there exist $\rho > 0$ and $\eta > 0$ such that

$$\mathscr{H}^m(\mathbf{C}(T, z, \epsilon r, \epsilon r^i) \cap A_i) \ge \eta \boldsymbol{\alpha}(m) r^m$$

for every $z \in \mathrm{gr}(P_i) \cap \mathbf{B}(0, r)$ and $0 \le r \le \rho$, *(b) for every $i = 1, \dots, k$ and for every $\epsilon > 0$*

$$\lim_{r \to 0} \frac{\mathscr{H}^m \big(A_i \cap \mathbf{B}(0,r) \cap \{z : \boldsymbol{\delta}_{\mathrm{gr}(P_i)}(z) > \epsilon\, r^i\}\big)}{\boldsymbol{\alpha}(m) r^m} = 0$$

and, if $\alpha > 0$, there exists $0 \le \lambda < \infty$ such that

$$\lim_{r \to 0} \frac{\mathscr{H}^m \big(A_k \cap \mathbf{B}(0,r) \cap \{z : \boldsymbol{\delta}_{\mathrm{gr}(P_k)}(z) > \lambda\, r^{k+\alpha}\}\big)}{\boldsymbol{\alpha}(m) r^m} = 0.$$

In this case P is uniquely determined by a, A and k,

$$\boldsymbol{\Theta}^m_*(\mathscr{H}^m \llcorner A, a) > 0, \ \operatorname{Tan}^m_*(\mathscr{H}^m \llcorner A, a) = \operatorname{Tan}^{*m}(\mathscr{H}^m \llcorner A, a) = T$$

and A is approximately differentiable of order (l, β) whenever either $l < k$ and $0 \le \beta \le 1$ or $l = k$ and $0 \le \beta \le \alpha$.

Proof. Assume $a = 0$ and suppose $\sup\big\{1, \sum_{j=1}^k 2^j \|\operatorname{D}^j P(0)\|\big\} < \Gamma < \infty$.

For $i = 1, \dots, k$ we define $Q_i = \sum_{j=1}^i P_j$ and $f_i : \mathbf{R}^n \to \mathbf{R}^n$ by

$$f_i(x) = x - Q_i(T_\natural(x)) + P_i(T_\natural(x)) \quad \text{for } x \in \mathbf{R}^n.$$

We observe that for every $i = 1, \dots, k$ the map f_i is a diffeomorphism of class ∞ onto $\mathbf{R}^n$ and, by induction over i, one may easily prove that

$$f_i[A] = A_i, \quad f_i[\operatorname{gr} Q_i] = \operatorname{gr} P_i.$$

Now, using 3.12, it is easy to see that (2) implies (1). Henceforth, we assume (1). First we prove that

$$\boldsymbol{\delta}_{\operatorname{gr} P}(z) \ge \boldsymbol{\delta}_{\operatorname{gr} Q_i}(z) - \Gamma |z|^{i+1} \quad \text{for every } i = 1, \dots, k-1 \text{ and } z \in \mathbf{B}(0,1).$$

In fact if $w \in \operatorname{gr}(P)$ such that $|z - w| = \boldsymbol{\delta}_{\operatorname{gr}(P)}(z)$ then

$$\begin{aligned} &|w| \le 2|z|, \quad |P(T_\natural(w)) - Q_i(T_\natural(w))| \le \Gamma |z|^{i+1}, \\ |z - w| &\ge |z - T_\natural(w) - Q_i(T_\natural(w))| - |Q_i(T_\natural(w)) - P(T_\natural(w))| \\ &\ge \boldsymbol{\delta}_{\operatorname{gr} Q_i}(z) - \Gamma |z|^{i+1}. \end{aligned}$$

It follows, for $i = 1, \dots, k$, that

$$\lim_{r \to 0} \frac{\mathscr{H}^m \big(A_1 \cap \mathbf{B}(0,r) \cap \{z : \boldsymbol{\delta}_{\mathrm{gr}(Q_i)}(z) > \epsilon\, r^i\}\big)}{\boldsymbol{\alpha}(m) r^m} = 0 \quad \text{for every } \epsilon > 0.$$

and, for $i = 1, \dots, k-1$ that

$$\lim_{r \to 0} \frac{\mathscr{H}^m \big(A_1 \cap \mathbf{B}(0,r) \cap \{z : \boldsymbol{\delta}_{\mathrm{gr}(Q_i)}(z) > 2\Gamma r^{i+1}\}\big)}{\boldsymbol{\alpha}(m) r^m} = 0.$$

Therefore (2b) is a consequence of 3.12. Moreover, using 3.12, it follows that A is approximately differentiable of order (l, β) whenever either $l < k$ and $0 \le \beta \le 1$ or $l = k$ and $0 \le \beta \le \alpha$. We prove now (2a), whose proof is slightly more involved. We fix $2 \le i \le k$ and we replace Γ by a larger number, if necessary, in order to have

$$|f_i^{-1}(w) - f_i^{-1}(z)| \le \Gamma |w - z|, \quad |f_i(w) - f_i(z)| \le \Gamma |w - z|$$

for $w, z \in \mathbf{B}(0,4)$. Therefore we have

$$f_i[A \cap \mathbf{C}(T, z, \epsilon r/\Gamma, \epsilon r/\Gamma)] \subseteq A_i \cap \mathbf{C}(T, z, \epsilon r/\Gamma, 3r),$$

$$\mathscr{H}^m(A_i \cap \mathbf{C}(T, z, \epsilon r/\Gamma, 3r)) \ge \Gamma^{-m} \mathscr{H}^m(A \cap \mathbf{C}(T, z, \epsilon r/\Gamma, \epsilon r/\Gamma)),$$

whenever $0 < r \le 1$, $0 < \epsilon \le 1$ and $z \in \mathbf{B}(0, r/\Gamma)$. We fix $0 < \epsilon \le 1$ and, using 3.11, we can find $0 < \rho \le 1$ and $\eta > 0$ such that

$$\mathscr{H}^m(A_i \cap \mathbf{B}(0,5r) \cap \{w : |T_\natural^\perp(w) - P_i(T_\natural(w))| > \epsilon r^i\}) < \eta \Gamma^{-m} \boldsymbol{\alpha}(m) r^m,$$

$$\mathscr{H}^m(\mathbf{C}(T, z, \epsilon r/\Gamma, \epsilon r/\Gamma) \cap A) \ge 2\eta \boldsymbol{\alpha}(m) r^m \quad \text{for every } z \in T \cap \mathbf{B}(0,r),$$

whenever $0 < r \le \rho$. Let $0 < r \le \rho$ and $z \in \operatorname{gr}(P_i) \cap \mathbf{B}(0, r/\Gamma)$. Then

$$\begin{aligned}\mathbf{C}(T, z, \epsilon r/\Gamma, 3r) \subseteq \big(&\mathbf{B}(0,5r) \cap \{w : |P_i(T_\natural(w)) - T_\natural^\perp(w)| > \epsilon r^i\}\big) \\ &\cup \mathbf{C}(T, z, \epsilon r/\Gamma, 2\epsilon r^i).\end{aligned}$$

In fact if $w \in \mathbf{C}(T, z, \epsilon r/\Gamma, 3r)$ and $|P_i(T_\natural(w)) - T_\natural^\perp(w)| \le \epsilon r^i$ then

$$\begin{aligned}&|P_i(T_\natural(w)) - P_i(T_\natural(z))| \\ &\quad = \big| \textstyle\sum_{j=1}^i \langle (T_\natural(w-z)^j / j!) \odot (T_\natural(z)^{i-j}/(i-j)!), \mathrm{D}^i P(0) \rangle \big| \\ &\quad \le i \| \mathrm{D}^i P(0) \| \Gamma^{-i} \epsilon r^i \le \epsilon r^i\end{aligned}$$

and we infer

$$|T_\natural^\perp(z) - T_\natural^\perp(w)| \le 2\epsilon r^i, \quad w \in \mathbf{C}(T, z, \epsilon r/\Gamma, 2\epsilon r^i).$$

We can now conclude that

$$\mathscr{H}^m(A_i \cap \mathbf{C}(T, z, \epsilon r/\Gamma, 2\epsilon r^i)) \ge \Gamma^{-m} \eta \boldsymbol{\alpha}(m) r^m$$

and (2a) is proved.

By 3.8(1) we immediately conclude that $\boldsymbol{\Theta}_*^m(\mathscr{H}^m \llcorner A, 0) > 0$ and $T \subseteq \operatorname{Tan}_*^m(\mathscr{H}^m \llcorner A, 0)$. By (2b) and 3.6 we conclude that $\operatorname{Tan}^{*m}(\mathscr{H}^m \llcorner A, 0) \subseteq T$. Finally let $R : T \to T^\perp$ be a polynomial function of degree at most k such that $R(0) = 0$ and $\mathrm{D}\, R(0) = 0$ and satisfying 3.8(1) and 3.8(2). Let

$$R_i(\chi) = \langle \chi^i / i!, \mathrm{D}^i R(0) \rangle \quad \text{for } \chi \in T \text{ and } i = 1, \ldots, k,$$

$$B_1 = A_1, \quad B_i = \{x - R_{i-1}(T_\natural(x)) : x \in B_{i-1}\} \text{ for } i = 2, \ldots, k.$$

We prove by induction that $P_i = R_i$ for $i = 1, \ldots, k$. Assume, for $j = 1, \ldots, i$ and $i < k$, that $P_j = R_j$ and observe that $A_{i+1} = B_{i+1}$. Let $\epsilon > 0$, $0 < \rho \le 1$ and $\eta > 0$ such that

$$\mathscr{H}^m(\mathbf{C}(T, z, \epsilon r, \epsilon r^{i+1}) \cap B_{i+1}) \ge \eta \boldsymbol{\alpha}(m) r^m \text{ for every } z \in \mathbf{B}(0,r) \cap \operatorname{gr}(R_{i+1}),$$

$$\mathscr{H}^m\big(A_{i+1} \cap \mathbf{B}(0,2r) \cap \{z : \boldsymbol{\delta}_{\operatorname{gr}(P_{i+1})}(z) > \epsilon\, r^{i+1}\}\big) \le (\eta/2) \boldsymbol{\alpha}(m) r^m$$

whenever $0 < r \le \rho$. Therefore for every $z \in \mathbf{B}(0,r) \cap \operatorname{gr}(R_{i+1})$ and for every $0 < r \le \rho$ we conclude that

$$\mathscr{H}^m(B_{i+1} \cap \mathbf{C}(T, z, \epsilon r, \epsilon r^{i+1}) \cap \{z : \boldsymbol{\delta}_{\operatorname{gr}(P_{i+1})}(z) \le \epsilon\, r^{i+1}\}) \ge (\eta/2) \boldsymbol{\alpha}(m) r^m$$

and $P_{i+1} = R_{i+1}$ by 3.13.

3.15 Remark. A conceptually similar characterization has been proved for the notion of pointwise differentiability in [Men16, 3.22]. Moreover the reader may find useful to compare 3.14(2) and [AS94, 3.4], where a concept of approximate tangent paraboloid is introduced by means of inhomogeneous dilations and weak convergence of Radon measures.

3.16 Remark. Suppose $A \subseteq \mathbf{R}^n$ and $a \in \mathbf{R}^n$. It is not difficult to see that the condition

$$\operatorname{Tan}^m(\mathscr{H}^m \llcorner A, a) \in \mathbf{G}(n,m) \text{ for some integer } 1 \leq m \leq n$$

is necessary and sufficient to conclude that A is approximately differentiable of order 1 at a. In fact the necessity is asserted in 3.14, while the sufficiency follows from 3.10 and 3.6.

3.17 Remark. We describe now a simple example which illustrates some features of the notion of approximate differentiability of order 1.

With each $\gamma > 1$ and $\gamma^{-1} < \alpha < (\gamma-1)^{-1}$ we associate the family $F_{\alpha,\gamma}$ consisting of the subsets

$$\mathbf{R}^2 \cap \big(\{(n^{-\alpha}, t) : 0 \leq t \leq n^{-\alpha\gamma}\} \cup \{(-n^{-\alpha}, t) : 0 \leq t \leq n^{-\alpha\gamma}\}\big)$$

correspoding to the integers $n \geq 1$. We define

$$A_{\alpha,\gamma} = \big(\mathbf{R}^2 \cap \{(s,0) : -1 \leq s \leq 1\}\big) \cup \bigcup F_{\alpha,\gamma}.$$

Since $\alpha\gamma > 1$ then $\mathscr{H}^1(A_{\alpha,\gamma}) < \infty$. Moreover, for each $n \geq 1$,

$$(n-1)^\alpha \sum_{i=n}^{\infty} i^{-\alpha\gamma} \geq (n-1)^\alpha \int_n^\infty x^{-\alpha\gamma} d\mathscr{L}^1 x = (n-1)^\alpha(\alpha\gamma-1)^{-1} n^{1-\alpha\gamma} \to \infty$$

as $n \to \infty$. Therefore $\mathbf{\Theta}^1_*(\mathscr{H}^1 \llcorner A_{\alpha,\gamma}, 0) = \infty$. Finally $A_{\alpha,\gamma}$ is approximately differentiable of order 1 at 0 by 3.16, since

$$\operatorname{Tan}^1(\mathscr{H}^1 \llcorner A_{\alpha,\gamma}, 0) = \mathbf{R} \times \{0\}.$$

3.18 Remark. Let $A \subseteq \mathbf{R}^n$, $a \in \mathbf{R}^n$ and let $0 \leq \mu \leq \nu$ be integers. Since

$$\operatorname{Tan}^{*\nu}(\mathscr{H}^\nu \llcorner A, a) \subseteq \operatorname{Tan}^{*\mu}(\mathscr{H}^\mu \llcorner A, a),$$

we deduce by 3.14 that the integer m in 3.8 is uniquely determined by A and a.

3.19 Definition (Approximate tangent space)**.** Let $A \subseteq \mathbf{R}^n$ and $a \in \mathbf{R}^n$. Suppose A is approximately differentiable of order 1 at a and m and T are as in 3.8. We define *the approximate tangent space of A at a* to be the m dimensional subspace T and we denote it by $\operatorname{ap}\operatorname{Tan}(A,a)$. Moreover we define *the approximate normal space of A at a* to be

$$\operatorname{ap}\operatorname{Nor}(A,a) = \operatorname{Dual}\operatorname{ap}\operatorname{Tan}(A,a).$$

3.20 Definition (Approximate differentials of higher order)**.** Let $A \subseteq \mathbf{R}^n$, let $k \geq 2$ be an integer and $a \in \mathbf{R}^n$. If A is approximately differentiable of order k at a then we define *the approximate differential of order k of A at a* to be the symmetric k linear map

$$\operatorname{ap}\mathrm{D}^k A(a) = \mathrm{D}^k(P \circ T_\natural)(0) \in \bigodot^k(\mathbf{R}^n, \mathbf{R}^n),$$

where $T = \operatorname{ap}\operatorname{Tan}(A,a)$ and $P : T \to T^\perp$ is as in 3.8.

3.21 Remark. Suppose $A \subseteq \mathbf{R}^n$.

Following [Sim83, 11.2, 11.4] (see also [FM99, 2.2]) we consider the map P_A whose domain is given by the set of $a \in \mathbf{R}^n$ such that there exist an integer $1 \leq m \leq n$, $T \in \mathbf{G}(n,m)$ and $0 < \theta < \infty$ such that

$$\lim_{r \to 0+} r^{-m} \int_A f((x-a)/r) d\mathscr{H}^m x = \theta \int_T f d\mathscr{H}^m \quad \text{whenever } f \in \mathscr{K}(\mathbf{R}^n),$$

and whose value $P_A(a)$ at a equals T. In fact, one may readily verify that m, T and θ are uniquely determined by A and a.

Then it is not difficult to check that if $a \in \operatorname{dmn} P_A$ and $m = \dim P_A(a)$ then

$$P_A(a) \subseteq \operatorname{Tan}^m_*(\mathscr{H}^m \llcorner A, a), \quad \operatorname{Tan}^{*m}(\mathscr{H}^m \llcorner A, a) \subseteq P_A(a),$$

$$\mathbf{\Theta}^m(\mathscr{H}^m \llcorner A, a) = \theta.$$

Using 3.16 and 3.14 we deduce

$$\operatorname{dmn} P_A \subseteq \operatorname{dmn} \operatorname{ap} \operatorname{Tan}(A, \cdot), \quad P_A(a) = \operatorname{ap} \operatorname{Tan}(A,a) \text{ whenever } a \in \operatorname{dmn} P_A.$$

If $A_{\alpha,\gamma}$ is defined as in 3.17, then $0 \in (\operatorname{dmn} \operatorname{ap} \operatorname{Tan}(A_{\alpha,\gamma}, \cdot)) \sim (\operatorname{dmn} P_{A_{\alpha,\gamma}})$.

3.22 Remark. Let $1 \leq m \leq n$ and $k \geq 1$ be integers, $0 \leq \alpha \leq 1$, $A \subseteq \mathbf{R}^n$, $B \subseteq \mathbf{R}^n$ and $a \in \mathbf{R}^n$. Suppose A is approximately differentiable of order (k,α) at a, $m = \dim \operatorname{ap} \operatorname{Tan}(A,a)$ and

$$\mathbf{\Theta}^m(\mathscr{H}^m \llcorner A \sim B, a) = 0, \quad \mathbf{\Theta}^m(\mathscr{H}^m \llcorner B \sim A, a) = 0.$$

Then B is approximately differentiable of order (k,α) at a with

$$\operatorname{ap} \operatorname{Tan}(A,a) = \operatorname{ap} \operatorname{Tan}(B,a),$$
$$\operatorname{ap} \mathrm{D}^i A(a) = \operatorname{ap} \mathrm{D}^i B(a) \quad \text{for } i = 2, \ldots, k.$$

3.23 Theorem. *Let $1 \leq m \leq n$ and $k \geq 1$ be integers, $0 \leq \alpha \leq 1$ and let $A \subseteq \mathbf{R}^n$ be $\mathscr{H}^m$ measurable and $(\mathscr{H}^m, m)$ rectifiable of class (k,α).*

Then for $\mathscr{H}^m$ a.e. $a \in A$ the set A is approximately differentiable of order (k,α) at a with

$$\operatorname{ap} \operatorname{Tan}(A,a) \in \mathbf{G}(n,m).$$

Proof. Since an m dimensional submanifold M of class (k,α) of $\mathbf{R}^n$ locally corresponds at each $a \in M$ to a graph of function $f : \operatorname{Tan}(M,a) \to \operatorname{Nor}(M,a)$ of class (k,α) with $\mathrm{D} f(\operatorname{Tan}(M,a)_\natural(a)) = 0$, one readily checks that M is approximately differentiable of order (k,α) at each of its points. Then the conclusion follows from [Fed69, 2.10.19(4)] and 3.22.

3.24 Theorem. *Let $1 \leq m \leq n$ be integers, let $A \subseteq \mathbf{R}^n$ be $\mathscr{H}^m$ measurable and $(\mathscr{H}^m, m)$ rectifiable of class 1 and let $\nu : A \to \mathbf{R}^n$ be a map such that for $\mathscr{H}^m$ a.e. $x \in A$ there exists $0 \leq \lambda < \infty$ such that*

$$\mathbf{\Theta}^m(\mathscr{H}^m \llcorner A \cap \{z : |\nu(z) - \nu(x)| > \lambda |z-x|\}, x) = 0.$$

Then ν is $\mathscr{H}^m \llcorner A$ measurable and $(\mathscr{H}^m \llcorner A, m)$ approximately differentiable[2] at $\mathscr{H}^m$ a.e. $x \in A$.

[2]See [Fed69, 3.2.16].

If additionally $\nu(x) \in \operatorname{ap}\operatorname{Nor}(A,x)$ for $\mathscr{H}^m$ a.e. $x \in A$ and A is $(\mathscr{H}^m, m)$ rectifiable of class 2 then

$$(\mathscr{H}^m \llcorner A, m) \operatorname{ap} \mathrm{D}\, \nu(x)(u) \bullet v = - \operatorname{ap} \mathrm{D}^2 A(x)(u,v) \bullet \nu(x)$$

for every $u, v \in \operatorname{ap}\operatorname{Tan}(A,x)$ and for $\mathscr{H}^m$ a.e. $x \in A$.

Proof. By [Fed69, 3.2.29, 3.1.19(4), 2.10.19(4), 3.2.16] it is enough to prove the statement in the following special case: let $U \subseteq \mathbf{R}^n$, $V \subseteq \mathbf{R}^m$ be bounded open sets and let $\phi : U \to \mathbf{R}^m$, $\psi : V \to \mathbf{R}^n$ be maps of class 1 (of class 2 if A is $(\mathscr{H}^m, m)$ rectifiable of class 2) such that $A \subseteq \operatorname{im}\psi$ and $\phi \circ \psi = \mathbf{1}_V$. Let $M = \operatorname{im}\psi$ and observe that $\phi | M = \psi^{-1}$, $\phi[A]$ is an $\mathscr{H}^m$ measurable subset of $\mathbf{R}^m$. Moreover we can prove that ν is $\mathscr{H}^m \llcorner A$ measurable by [Fed69, 2.9.13]. In fact one verifies that $V = \{(a, \mathbf{B}(a,r)) : a \in \mathbf{R}^n,\ 0 < r < \infty\}$ is an $\mathscr{H}^m \llcorner A$ Vitali relation by [Fed69, 2.8.18] and, since $\boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner A, x) = 1$ for $\mathscr{H}^m$ a.e. $x \in A$ by [Fed69, 3.2.19], we conclude that ν is $(\mathscr{H}^m \llcorner A, V)$ approximately continuous[3] at $\mathscr{H}^m$ a.e. $x \in A$.

Let $\eta = \nu \circ (\psi | \phi[A])$ and, by [Fed69, 2.9.11], we deduce that η is approximately differentiable of order $(0,1)$ at $\mathscr{L}^m$ a.e. $\chi \in \phi[A]$. Therefore by 2.10(3) there exist countably many maps $\eta_j : \mathbf{R}^m \to \mathbf{R}^n$ of class 1 such that

$$\mathscr{L}^m \Big(\phi[A] \sim \textstyle\bigcup_{j=1}^{\infty} \{\chi : \eta_j(\chi) = \eta(\chi)\} \Big) = 0.$$

We deduce, by [Fed69, 2.10.19(4)], that ν is $(\mathscr{H}^m \llcorner A, m)$ approximately differentiable at $\mathscr{H}^m$ a.e. $x \in A$ because

$$\mathscr{H}^m \Big(A \sim \textstyle\bigcup_{j=1}^{\infty} \{x : (\eta_j \circ \phi)(x) = \nu(x)\} \Big) = 0.$$

If we further assume $\nu(x) \in \operatorname{ap}\operatorname{Nor}(A,x)$ for $\mathscr{H}^m$ a.e. $x \in A$ and A is $(\mathscr{H}^m, m)$ rectifiable of class 2 then, for every $j \geq 1$, we define

$$\nu_j(x) = \big(\operatorname{Nor}(M,x)_\natural \circ \eta_j \circ \phi \big)(x) \quad \text{for } x \in M,$$

we observe that ν_j is of class 1 relative to M and, by [Fed69, 2.10.19(4)] and 3.22,

$$\mathscr{H}^m \Big(A \sim \textstyle\bigcup_{j=1}^{\infty} \{x : \nu_j(x) = \nu(x)\} \Big) = 0.$$

Since, by [Fed69, 2.10.19(4), 3.2.16] and 3.22,

$$\mathrm{D}\, \nu_j(x)(u) \bullet v = - \operatorname{ap} \mathrm{D}^2 A(x)(u,v) \bullet \nu_j(x) \quad \text{for every } u, v \in \operatorname{ap}\operatorname{Tan}(A,x)$$

and $(\mathscr{H}^m \llcorner A, m) \operatorname{ap} \mathrm{D}\, \nu(x) = \mathrm{D}\, \nu_j(x)$ for $\mathscr{H}^m$ a.e. $x \in A$, the conclusion follows.

3.25 Remark. The conclusion of the second part of 3.24 may fail to hold at $\mathscr{H}^m$ a.e. $a \in A$ if we omit the hypothesis "A is $(\mathscr{H}^m, m)$ rectifiable of class 2", even if we assume that A is an m dimensional submanifold of class 1. This fact can be easily deduced from [Koh77] and 3.41.

Moreover the same conclusion may fail to hold at $\mathscr{H}^m$ a.e. $a \in A$ if we omit the hypothesis "A is $(\mathscr{H}^m, m)$ rectifiable of class 2" but we assume $\nu(x) = \zeta$ for $\mathscr{H}^m$ a.e. $x \in A$ for some $\zeta \in \mathbf{S}^{n-1}$. In fact it is proved in [AS94, Appendix]

[3] See [Fed69, 2.9.12].

that for every $0 < \alpha < 1$ there exists a function $f : \mathbf{R} \to \mathbf{R}$ of class $(1, \alpha)$ and a Cantor-type set $E \subseteq \mathbf{R}$ such that

$$\mathscr{L}^1(E) > 0, \quad \mathrm{D} f(x) = 0 \quad \text{for every } x \in E,$$
$$\mathscr{L}^1\big(E \cap \{x : f(x) = g(x)\}\big) = 0 \quad \text{whenever } g : \mathbf{R} \to \mathbf{R} \text{ is of class } 2.$$

If $A = \operatorname{gr}(f|E)$ then, by 3.41, $\mathscr{H}^1(A \cap \operatorname{dmn} \operatorname{ap} \mathrm{D}^2 A) = 0$.

Relation with pointwise differentiability

The concept of pointwise differentiability of higher order for sets has been recently introduced in [Men16]. In 3.35 we study its relation with the concept of approximate differentiability introduced in the previous section. As a corollary we derive in 3.36(3) a one-sided estimate for the approximate differential of second order at the set of points where the set can be touched by a full ball of the ambient space.

3.26 Definition. Suppose X is a normed vector space, $B \subseteq X$ and $a \in X$.

We define *the upper tangent cone of B at a* by

$$\operatorname{Tan}^*(B, a) = X \cap \{v : \liminf_{r \to 0+} r^{-1} \boldsymbol{\delta}_B(a + rv) = 0\}$$

and *the lower tangent cone of B at a* by

$$\operatorname{Tan}_*(B, a) = X \cap \{v : \lim_{r \to 0+} r^{-1} \boldsymbol{\delta}_B(a + rv) = 0\}$$

In case $\operatorname{Tan}_*(B, a) = \operatorname{Tan}^*(B, a)$, this set is denoted by $\operatorname{Tan}(B, a)$ and we call it *the tangent cone of B at a*. Finally *the (lower, upper) normal cone of B at a* is defined by

$$\operatorname{Nor}_*(A, a) = \operatorname{Dual} \operatorname{Tan}_*(A, a), \quad \operatorname{Nor}^*(A, a) = \operatorname{Dual} \operatorname{Tan}^*(A, a)$$
$$\operatorname{Nor}(A, a) = \operatorname{Dual} \operatorname{Tan}(A, a).$$

3.27 Remark. If $1 \le m \le n$ are integers and $B \subseteq \mathbf{R}^n$ then one may verify that

$$\begin{array}{ccc} \operatorname{Tan}^{*m}(\mathscr{H}^m \llcorner B, a) & \subseteq & \operatorname{Tan}^*(B, a) \\ \cup\!\shortmid & & \cup\!\shortmid \\ \operatorname{Tan}^m_*(\mathscr{H}^m \llcorner B, a) & \subseteq & \operatorname{Tan}_*(B, a). \end{array}$$

Moreover one may readily verify that $\operatorname{Tan}_*(B, a)$ and $\operatorname{Tan}^*(B, a)$ are closed cones.

3.28 Remark. This notation does not agree with [Fed59, 4.3], [Fed69, 3.1.21] and [Men16]. In fact $\operatorname{Tan}^*(B, a)$ is denoted by $\operatorname{Tan}(B, a)$ therein.

3.29 Definition. Let k and n be positive integers, $0 \le \alpha \le 1$ and $B \subseteq \mathbf{R}^n$. We say that *B is pointwise differentiable of order (k, α) at a* if there exists a submanifold $M \subseteq \mathbf{R}^n$ of class (k, α) such that $a \in M$,

$$\lim_{r \to 0} r^{-1} \sup\{|\boldsymbol{\delta}_M(x) - \boldsymbol{\delta}_B(x)| : x \in \mathbf{B}(a, r)\} = 0,$$
$$\lim_{r \to 0} r^{-k} \sup\{\boldsymbol{\delta}_M(x) : x \in \mathbf{B}(a, r) \cap B\} = 0 \quad \text{if } \alpha = 0,$$
$$\limsup_{r \to 0} r^{-k-\alpha} \sup\{\boldsymbol{\delta}_M(x) : x \in \mathbf{B}(a, r) \cap B\} < \infty \quad \text{if } \alpha > 0.$$

3.30 Remark. This concept has been introduced in [Men16, 3.3]. In 3.35 and 3.36 we employ the concept of pointwise differential of order i for sets, introduced in [Men16, 3.12].

3.31 Remark. It is worth to mention that, for sets, pointwise differentiability does not imply approximate differentiability. In fact, suppose $n \geq 1$ is an integer and B is a countable dense subset of $\mathbf{R}^n$. Then for every integer $k \geq 1$ the set B is pointwise differentiable of order k at every $x \in \mathbf{R}^n$. But B is not approximately differentiable of order 1 at every $x \in \mathbf{R}^n$.

3.32 Lemma. *Let $B \subseteq \mathbf{R}^n$ and $a \in \operatorname{Clos} B$.*

Then the following statements hold.

(1) *If $M = \{a + v : v \in \operatorname{Tan}^*(B, a)\}$ then*

$$\lim_{r \to 0} r^{-1} \sup\{\boldsymbol{\delta}_M(x) : x \in \mathbf{B}(a, r) \cap B\} = 0.$$

(2) *If $M = \{a + v : v \in \operatorname{Tan}_*(B, a)\}$ then*

$$\lim_{r \to 0} r^{-1} \sup\{\boldsymbol{\delta}_B(x) : x \in \mathbf{B}(a, r) \cap M\} = 0.$$

(3) *The condition*

$$\operatorname{Tan}(B, a) \in \mathbf{G}(n, m) \quad \textit{for some integer } 0 \leq m \leq n$$

is necessary and sufficient to conclude that A is pointwise differentiable of order 1 at a.

Proof. Proof of (1). If there existed $\epsilon > 0$, $r_i > 0$, $r_i \to 0$ as $i \to \infty$ and $x_i \in B \cap \mathbf{B}(a, r_i)$ such that $\boldsymbol{\delta}_M(x_i) \geq \epsilon r_i$ then, possibly passing to a subsequence, we could assume there would exist $v \in \mathbf{S}^{n-1}$ such that $(x_i - a)/|x_i - a| \to v$ as $i \to \infty$. Then $v \in \operatorname{Tan}^*(B, a)$,

$$\epsilon \leq r_i^{-1} \big| x_i - a - |x_i - a| v \big| \leq |x_i - a|^{-1} \big| x_i - a - |x_i - a| v \big| \quad \text{for } i \geq 1$$

and we would get a contradiction.

Proof of (2). Suppose $\epsilon > 0$ and observe there exist an integer $l \geq 1$, $v_1, \ldots, v_l \in \operatorname{Tan}_*(B, a) \cap \mathbf{S}^{n-1}$ and $\eta > 0$ such that $r^{-1} \boldsymbol{\delta}_B(a + r v_i) < \epsilon$ whenever $i = 1, \ldots, l$ and $0 < r \leq \eta$ and

$$\operatorname{Tan}_*(B, a) \cap \mathbf{S}^{n-1} \subseteq \textstyle\bigcup_{i=1}^{l} \mathbf{B}(v_i, \epsilon).$$

If $0 < r \leq \eta$ and $v \in \mathbf{B}(0, r) \cap \operatorname{Tan}_*(B, a) \sim \{0\}$ then we choose $i = 1, \ldots, l$ such that $|(v/|v|) - v_i| \leq \epsilon$ and, since $\operatorname{Lip} \boldsymbol{\delta}_B \leq 1$, we conclude that $\boldsymbol{\delta}_B(a + v) \leq 2\epsilon |v|$.

Proof of (3). For the necessity, suppose M is as in 3.29 when $k = 1$ and $\alpha = 0$, observe that $\operatorname{Tan}(M, a) = \operatorname{Tan}^*(B, a)$ by [Men16, 3.4] and $\operatorname{Tan}(M, a) \subseteq \operatorname{Tan}_*(B, a)$ because

$$\lim_{\operatorname{Tan}(M, a) \ni v \to 0} |v|^{-1} \boldsymbol{\delta}_M(a + v) = 0.$$

For the sufficiency let $M = \{a + v : v \in \operatorname{Tan}(B, a)\}$ and, since $a \in \operatorname{Clos} B$, one verifies that

$$\begin{aligned} &\sup\{|\boldsymbol{\delta}_B(x) - \boldsymbol{\delta}_M(x)| : x \in \mathbf{B}(a, r)\} \leq \\ &\leq \sup(\{\boldsymbol{\delta}_B(x) : x \in \mathbf{B}(a, 2r) \cap M\} \cup \{\boldsymbol{\delta}_M(x) : x \in \mathbf{B}(a, 2r) \cap B\}), \end{aligned}$$

Therefore the conclusion comes from (1) and (2).

3.33 Remark. Compare 3.32(3) with the analogous result for approximate differentiability in 3.16. Moreover 3.32(3) is a restatement of [Men16, 3.19].

3.34 Remark. If M is an m dimensional submanifold of class 1 of $\mathbf{R}^n$ then, by 3.32(3), 3.16 and 3.27, one may readily infer that

$$\operatorname{Tan}(M, a) = \operatorname{Tan}^m(\mathscr{H}^m \llcorner M, a) \quad \text{for every } a \in M.$$

3.35 Theorem. *Let $1 \le m \le n$ and $k \ge 1$ be integers, $0 \le \alpha \le 1$, $A \subseteq \mathbf{R}^n$ and $a \in \mathbf{R}^n$. Suppose A is approximately differentiable of order (k, α) at a and $m = \dim \operatorname{ap} \operatorname{Tan}(A, a)$.*

Then there exists $B \subseteq A$ pointwise differentiable of order (k, α) at a such that

$$\begin{gathered} \boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner A \sim B, a) = 0, \\ \operatorname{ap} \operatorname{Tan}(A, a) = \operatorname{Tan}(B, a) = \operatorname{Tan}^m(\mathscr{H}^m \llcorner B, a), \\ \operatorname{pt} \mathrm{D}^i B(a, \operatorname{Tan}(B, a)) = \operatorname{ap} \mathrm{D}^i A(a) \quad \textit{for } i = 2, \ldots, k. \end{gathered}$$

Proof. Assume $a = 0$ and suppose $T = \operatorname{ap} \operatorname{Tan}(A, 0)$, $P : T \to T^\perp$ is defined by

$$P(\chi) = \textstyle\sum_{j=2}^k \langle \chi^j / j!, \operatorname{ap} \mathrm{D}^j A(0) \rangle \quad \text{for } \chi \in T$$

and $\Gamma = \sup\{1, \sum_{j=2}^k \| \operatorname{ap} \mathrm{D}^j A(0) \| / j!\}$. By 3.11 and 3.7 we infer that

$$\begin{gathered} \boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner A \sim \mathbf{X}_k(0, T, P, \epsilon), a) = 0 \quad \text{for every } \epsilon > 0 \text{ if } \alpha = 0, \\ \boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner A \sim \mathbf{X}_{k,\alpha}(0, T, P, \lambda), a) = 0 \quad \text{for some } 0 \le \lambda < \infty \text{ if } \alpha > 0. \end{gathered}$$

We fix $0 \le \lambda < \infty$ as above if $\alpha > 0$. We define, for every integer $i \ge 1$,

$$A_i = A \cap \mathbf{X}_k(0, T, P, (2i)^{-1}) \text{ if } \alpha = 0, \quad A_i = A \cap \mathbf{X}_{k,\alpha}(0, T, P, \lambda) \text{ if } \alpha > 0.$$

Let $Q_r = \mathbf{R}^n \cap \{z : |T_\natural^\perp(z)| \le r,\ |T_\natural(z)| \le r\}$ for $0 < r < \infty$. For every integer $i \ge 1$ let $\delta_i > 0$ be such that

$$\mathscr{H}^m(A \cap Q_r \sim A_i) \le 2^{-i} \boldsymbol{\alpha}(m)\, r^m \quad \text{whenever } 0 < r \le \delta_i$$

and we assume $\delta_{i+1} < \delta_i$, $\delta_i \to 0$ as $i \to \infty$,

$$\delta_1 \le (2\Gamma)^{-1} \ \text{ if } \alpha = 0, \quad \delta_1 \le (\lambda + \Gamma)^{-1/\alpha} \ \text{ if } \alpha > 0.$$

We define, for every integer $i \ge 1$,

$$C_i = T_\natural^{-1}[\mathbf{B}(0, \delta_i) \sim \mathbf{B}(0, \delta_{i+1})], \quad B = \textstyle\bigcup_{j=1}^\infty A_j \cap C_j.$$

Observe that $B \subseteq \mathbf{X}(0, T, 1)$ and

$$(Q_{\delta_j} \sim Q_{\delta_{j+1}}) \cap \mathbf{X}(0, T, 1) \sim C_j = \varnothing \quad \text{whenever } j \ge 1.$$

We can prove now that $\boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner A \sim B, 0) = 0$. In fact, by 3.6 and 3.14 we infer $\boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner A \sim \mathbf{X}(0, T, 1), 0) = 0$. Moreover, if $0 < r \le \delta_1$ and $i \ge 1$ are such that $\delta_{i+1} < r \le \delta_i$ then

$$\begin{gathered} Q_r \cap A \cap \mathbf{X}(0, T, 1) \sim B \subseteq (Q_r \cap A \sim A_i) \cup \textstyle\bigcup_{j=i+1}^\infty Q_{\delta_j} \cap A \sim A_j, \\ \mathscr{H}^m(Q_r \cap A \cap \mathbf{X}(0, T, 1) \sim B) \le \boldsymbol{\alpha}(m) r^m \textstyle\sum_{j=i}^\infty 2^{-j}. \end{gathered}$$

Since this implies $0 \in \operatorname{Clos} B$ by 3.14, it follows that

$$\lim_{r \to 0} r^{-k} \sup\{|P(T_\natural(z)) - T_\natural^\perp(z)| : z \in B \cap T_\natural^{-1}[\mathbf{B}(0,r)]\} = 0 \quad \text{if } \alpha = 0,$$

$$\limsup_{r \to 0} r^{-k-\alpha} \sup\{|P(T_\natural(z)) - T_\natural^\perp(z)| : z \in B \cap T_\natural^{-1}[\mathbf{B}(0,r)]\} \le \lambda \quad \text{if } \alpha > 0.$$

In particular, $\operatorname{Tan}^*(B, 0) \subseteq T$. By 3.5 and 3.14 we get that

$$T = \operatorname{Tan}_*^m(\mathscr{H}^m \llcorner A, 0) \subseteq \operatorname{Tan}_*^m(\mathscr{H}^m \llcorner B, 0).$$

Therefore B is pointwise differentiable of order 1 at a with $T = \operatorname{Tan}(B, 0) = \operatorname{Tan}^m(\mathscr{H}^m \llcorner B, 0)$ by 3.27 and 3.32(3). Moreover, since $\operatorname{Tan}(\operatorname{gr} P, 0) = T$, we can use 3.11 to check that the conditions in 3.29 hold with M replaced by $\operatorname{gr} P$. Therefore, by [Men16, 3.12] and 3.20, we conclude that

$$\operatorname{pt} \mathrm{D}^i B(0, T) = \operatorname{ap} \mathrm{D}^i A(0) \quad \text{for } i = 2, \ldots, k.$$

3.36 Theorem. *Let $A \subseteq \mathbf{R}^n$, $a \in \mathbf{R}^n$, $\nu \in \mathbf{S}^{n-1}$, $0 < r < \infty$ and suppose*

$$\mathbf{U}(a + r\nu, r) \cap A = \varnothing.$$

Then the following three statements hold.

(1) If A is a submanifold of class 2 and $a \in A$ then

$$\mathbf{b}_A(a)(v, v) \bullet \nu \le r^{-1}|v|^2 \quad \textit{whenever } v \in \operatorname{Tan}(A, a).$$

(2) If A is pointwise differentiable of order 2 at a then

$$\operatorname{pt} \mathrm{D}^2 A(a, \operatorname{Tan}(A, a))(v, v) \bullet \nu \le r^{-1}|v|^2 \quad \textit{whenever } v \in \operatorname{Tan}(A, a).$$

(3) If A is approximately differentiable of order 2 at a then

$$\operatorname{ap} \mathrm{D}^2 A(a)(v, v) \bullet \nu \le r^{-1}|v|^2 \quad \textit{whenever } v \in \operatorname{ap} \operatorname{Tan}(A, a).$$

Proof. Assume $a = 0$. Observe that $\nu \in \operatorname{Tan}^*(A, 0)^\perp$ in case the hypothesis of 1 or 2 are satisfied.

The statement in (1) is classical. We give a proof here for completeness. If $T = \operatorname{Tan}(A, 0)$ then there exist a function $f : T \to T^\perp$ of class 2 and an open neighbourhood U of $0 \in \mathbf{R}^n$ such that $\mathrm{D} f(0) = 0$, $T_\natural[U] = T_\natural[U \cap A]$ and $A \cap U = \{\chi + f(\chi) : \chi \in T_\natural[U]\}$. Since for every $\chi \in T_\natural[U]$

$$|\chi + f(\chi) - r\nu| \ge r, \quad 2r\, f(\chi) \bullet \nu \le |\chi|^2 + |f(\chi)|^2,$$

we conclude that $\mathrm{D}^2 f(0)(v, v) \bullet \nu \le r^{-1}|v|^2$ for every $v \in \operatorname{Tan}(A, 0)$ and, since $\mathbf{b}_A(0) = \mathrm{D}^2 f(0)$, the statement in (1) follows.

The statement in (2) is an immediate consequence of [Men16, 3.18]. In fact suppose $T = \operatorname{Tan}(A, 0)$, $P : T \to T^\perp$ is the homogeneous polynomial function of degree 2 such that $\operatorname{pt} \mathrm{D}^2 A(0, T) = \mathrm{D}^2(P \circ T_\natural)(0)$ (whose existence can be asserted, from instance, by [Men16, 3.22]) and $B = \{\chi + P(\chi) : \chi \in T\}$. If we prove that $\mathbf{U}(r\nu, r) \cap B = \varnothing$ then (2) is a consequence of (1). By contradiction let $x \in B \cap \mathbf{U}(r\nu, r)$ and, by [Men16, 3.18], for every positive integer i we can select $x_i \in A$ such that

$$|iT_\natural(x_i) + i^2 T_\natural^\perp(x_i) - x| \to 0 \quad \text{as } i \to \infty.$$

Since $|x_i - r\nu| \geq r$ for every $i \geq 1$, we get

$$\begin{aligned}
&|iT_\natural(x_i) + i^2 T_\natural^\perp(x_i) - r\nu|^2 \\
&\quad = i^2|x_i - r\nu|^2 + (i^4 - i^2)|T_\natural^\perp(x_i)|^2 + r^2 - i^2 r^2 \\
&\quad \geq (i^4 - i^2)|T_\natural^\perp(x_i)|^2 + r^2 \quad \text{for } i \geq 1;
\end{aligned}$$

yet $|iT_\natural(x_i) + i^2 T_\natural^\perp(x_i) - r\nu| < r$ for i large. This is a contradiction.

Finally (3) is a consequence of (2) and 3.35.

Rectifiability and Borel measurability

In 3.37, 3.38 and 3.40 we study the measurability properties of the approximate differentials. Then in 3.39 we prove a novel criterion for higher order rectifiability. This result together with 3.23 provides a full characterization of higher order rectifiable sets in terms of approximate differentiability.

3.37 Lemma. *Let $1 \leq m \leq n$ and $k \geq 1$ be integers, $0 \leq \alpha \leq 1$, $\gamma = k + \alpha$ and $A \subseteq \mathbf{R}^n$. Let Y be the set of*

$$(a, T, \phi_0, \ldots, \phi_k) \in \mathbf{R}^n \times \mathbf{G}(n,m) \times \prod_{i=0}^{k} \bigodot^i (\mathbf{R}^n, \mathbf{R}^n)$$

such that $\phi_0 = T_\natural^\perp(a)$ and

$$\lim_{r \to 0} \frac{\mathscr{H}^m\left(A \cap \mathbf{U}(a,r) \cap \{z : |T_\natural^\perp(z) - \sum_{j=0}^{k} \langle T_\natural(z-a)^j/j!, \phi_j\rangle| > \epsilon\, r^\gamma\}\right)}{\boldsymbol{\alpha}(m)\, r^m} = 0$$

for every $\epsilon > 0$ [for some $0 \leq \epsilon < \infty$].

Then Y is a Borel subset of $\mathbf{R}^n \times \mathbf{G}(n,m) \times \prod_{i=0}^{k} \bigodot^i (\mathbf{R}^n, \mathbf{R}^n)$.

Proof. Let $Z = \mathbf{R}^n \times \mathbf{G}(n,m) \times \prod_{j=0}^{k} \bigodot^j (\mathbf{R}^n, \mathbf{R}^n)$. If $0 < \epsilon < \infty$, $i \geq 1$ is an integer and $0 < r < \infty$ we define $W_{\epsilon,i,r}$ to be the set of $(a, T, \phi_0, \ldots, \phi_k) \in Z$ such that $\phi_0 = T_\natural^\perp(a)$ and

$$\mathscr{H}^m\left(A \cap \mathbf{U}(a,r) \cap \left\{z : |T_\natural^\perp(z) - \textstyle\sum_{l=0}^{k} \langle T_\natural(z-a)^l/l!, \phi_l\rangle| > \epsilon\, r^\gamma\right\}\right) \leq i^{-1}\, r^m.$$

Then $W_{\epsilon,i,r}$ is a closed subset of Z. In fact if $(a_j, T_j, \phi_{0,j}, \ldots, \phi_{k,j}) \in W_{\epsilon,i,r}$, $j \geq 1$, is a sequence converging to $(a, T, \phi_0, \ldots, \phi_k) \in Z$ as $j \to \infty$, we define

$$P_j(\chi) = \textstyle\sum_{l=0}^{k} \langle (\chi - T_{j\,\natural}(a_j))^l/l!, \phi_{l,j}\rangle \quad \text{for } \chi \in T_j \text{ and } j \geq 1,$$
$$P(\chi) = \textstyle\sum_{l=0}^{k} \langle (\chi - T_\natural(a))^l/l!, \phi_l\rangle \quad \text{for } \chi \in T,$$

and we observe that $P_j(T_{j\,\natural}(z)) \to P(T_\natural(z))$ as $j \to \infty$, whenever $z \in \mathbf{R}^n$. Let

$$B_j = A \cap \mathbf{U}(a_j, r) \cap \{z : |T_{j\,\natural}^\perp(z) - P_j(T_{j\,\natural}(z))| > \epsilon\, r^\gamma\},$$
$$B = A \cap \mathbf{U}(a, r) \cap \{z : |T_\natural^\perp(z) - P(T_\natural(z))| > \epsilon\, r^\gamma\}$$

and observe that

$$B \subseteq \bigcup_{j=1}^{\infty} \bigcap_{h=j}^{\infty} B_h.$$

Therefore, by [Fed69, 2.1.5(1)], we conclude that

$$\mathscr{H}^m(B) \leq \lim_{j\to\infty} \mathscr{H}^m(\bigcap_{h=j}^{\infty} B_h) \leq \liminf_{j\to\infty} \mathscr{H}^m(B_j) \leq i^{-1} r^m,$$

$(a, T, \phi_0, \dots, \phi_k) \in W_{\epsilon,i,r}$ and $W_{\epsilon,i,r}$ is closed.

Henceforth Y is a Borel set because

$$Y = \bigcap_{l=1}^{\infty} \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap \{W_{l^{-1},i,r} : 0 < r \leq j^{-1}\},$$

$$\bigl[Y = \bigcup_{l=1}^{\infty} \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap \{W_{l,i,r} : 0 < r \leq j^{-1}\}\bigr].$$

3.38 Lemma. *Suppose $1 \leq m \leq n$ are integers, $A \subseteq \mathbf{R}^n$ and $\tau_a(x) = x - a$ whenever $a, x \in \mathbf{R}^n$. Let Y be the set of $(a, T) \in \mathbf{R}^n \times \mathbf{G}(n, m)$ such that for every $\epsilon > 0$ there exist $\eta > 0$ and $\rho > 0$ such that*

$$\mathscr{H}^m(\mathbf{C}(T, z, \epsilon r, \epsilon r) \cap \tau_a[A]) \geq \eta \boldsymbol{\alpha}(m) r^m$$

for every $0 < r \leq \rho$ and for every $z \in T \cap \mathbf{B}(0, r)$.

Then Y is a Borel subset of $\mathbf{R}^n \times \mathbf{G}(n, m)$.

Proof. We prove that $(\mathbf{R}^n \times \mathbf{G}(n, m)) \sim Y$ is a Borel subset of $\mathbf{R}^n \times \mathbf{G}(n, m)$. For every $\epsilon > 0$, $\eta > 0$ and $0 < \rho_2 < \rho_1$ suppose $W_{\epsilon,\eta,\rho_1,\rho_2}$ is the set of $(a, T) \in \mathbf{R}^n \times \mathbf{G}(n, m)$ such that

$$\mathscr{H}^m(\mathbf{C}(T, z, \epsilon r, \epsilon r) \cap \tau_a[A]) \leq \eta \boldsymbol{\alpha}(m) r^m$$

for some $z \in \mathbf{B}(0, r) \cap T$ and some $\rho_2 \leq r \leq \rho_1$. We prove that $W_{\epsilon,\eta,\rho_1,\rho_2}$ is a closed subset of $\mathbf{R}^n \times \mathbf{G}(n, m)$. Suppose $(a_j, T_j) \in W_{\epsilon,\eta,\rho_1,\rho_2}$, $j \geq 1$, is a sequence converging to $(a, T) \in \mathbf{R}^n \times \mathbf{G}(n, m)$ as $j \to \infty$. Therefore there exist sequences $\rho_2 \leq r_j \leq \rho_1$ and $z_j \in T_j \cap \mathbf{B}(0, r_j)$, for $j \geq 1$, such that

$$\mathscr{H}^m(\mathbf{C}(T_j, z_j, \epsilon r_j, \epsilon r_j) \cap \tau_{a_j}[A]) \leq \eta \boldsymbol{\alpha}(m) r_j^m \quad \text{for every } j \geq 1.$$

Then there exist $z \in \mathbf{R}^n$ and $r \in \mathbf{R}$ such that, possibly passing to a subsequence, $z_j \to z$ and $r_j \to r$ as $j \to \infty$. Observe that $z \in \mathbf{B}(0, r) \cap T$ and $\rho_2 \leq r \leq \rho_1$. For each $j \geq 1$ we define

$$B_j = \mathbf{C}(T_j, z_j, \epsilon r_j, \epsilon r_j) \cap \tau_{a_j}[A], \quad B = \mathbf{C}(T, z, \epsilon r, \epsilon r) \cap \tau_a[A],$$

and one may easily verify that

$$B \subseteq \bigcup_{h=1} \bigcap_{k=h}^{\infty} \tau_{a-a_k}[B_k].$$

Now we can use [Fed69, 2.1.5(1)] to conclude that

$$\mathscr{H}^m(B) \leq \liminf_{h\to\infty} \mathscr{H}^m\bigl(\tau_{a-a_h}[B_h]\bigr) \leq \boldsymbol{\alpha}(m)\eta r^m.$$

Therefore $(a,T) \in W_{\epsilon,\eta,\rho_1,\rho_2}$ and $W_{\epsilon,\eta,\rho_1,\rho_2}$ is a closed subset of $\mathbf{R}^n \times \mathbf{G}(n,m)$. If $E \subseteq \mathbf{R}$ is a countable set such that $\inf E = 0 \notin E$ then it is not difficult to see that

$$\big(\mathbf{R}^n \times \mathbf{G}(n,m)\big) \sim Y = \bigcup_{\epsilon \in E} \bigcap_{\eta \in E} \bigcap_{\rho_1 \in E} \bigcup_{\rho_2 \in E} W_{\epsilon,\eta,\rho_1,\rho_2}.$$

3.39 Theorem. *Suppose $1 \le m \le n$ and $k \ge 1$ are integers, $0 \le \alpha \le 1$, $A \subseteq \mathbf{R}^n$ such that $\mathscr{H}^m(A) < \infty$ and for every $a \in A$ there exists an m dimensional submanifold $B \subseteq \mathbf{R}^n$ of class (k,α) such that $a \in B$ and the following condition $(*)$ is satisfied. For every $\epsilon > 0$*

$$\lim_{r \to 0} \frac{\mathscr{H}^m\left(A \cap \mathbf{B}(a,r) \cap \{z : \boldsymbol{\delta}_B(z) > \epsilon\, r^k\}\right)}{\boldsymbol{\alpha}(m) r^m} = 0$$

and, if $\alpha > 0$, there exists $0 \le \lambda < \infty$ such that

$$\lim_{r \to 0} \frac{\mathscr{H}^m\left(A \cap \mathbf{B}(a,r) \cap \{z : \boldsymbol{\delta}_B(z) > \lambda\, r^{k+\alpha}\}\right)}{\boldsymbol{\alpha}(m) r^m} = 0.$$

Then A is $(\mathscr{H}^m, m)$ rectifiable of class (k,α).

Proof. If $a \in A$ and B is an m dimensional submanifold of class 1 such that $a \in B$ and $(*)$ is satisfied then, by 3.11 and 3.6, we get that $\operatorname{Tan}^{*m}(\mathscr{H}^m \llcorner A, a) \subseteq \operatorname{Tan}(B,a)$ and

$$\boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner A \sim \mathbf{X}(a, \operatorname{Tan}(B,a), \epsilon), a) = 0 \quad \text{for every } \epsilon > 0.$$

Therefore by [Fed69, 2.10.19(2), 3.3.17, 3.2.29] we conclude that A is $(\mathscr{H}^m, m)$ rectifiable of class 1.

Let $S \in \mathbf{G}(n,m)$, let $U \subseteq S$ be relatively open, let $f : U \to S^\perp$ be a function of class 1, $M = \{\chi + f(\chi) : \chi \in U\}$, $\operatorname{Lip} f < \infty$ and $\mathscr{H}^m(M) < \infty$. We prove that $M \cap A$ is $(\mathscr{H}^m, m)$ rectifiable of class (k,α). This evidently implies that A is $(\mathscr{H}^m, m)$ rectifiable of class (k,α).

Let X be the set of points $a \in \mathbf{R}^n$ such that there exists an m dimensional submanifold B of class (k,α) such that $a \in B$ and $(*)$ is satisfied. Then X is an $\mathscr{H}^m$ measurable subset of $\mathbf{R}^n$ by 3.11, 3.37 and [Fed69, 2.2.13]. Let $E \subseteq M$ be an $\mathscr{H}^m$ hull[4] of $M \cap A$ and we prove that $E \cap X$ is $(\mathscr{H}^m, m)$ rectifiable of class (k,α). Observe that $E \cap X$ satifies the same hypothesis A does. Let Y be the set of points $a \in E \cap X$ such that $\boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner M \sim (E \cap X), a) = 0$. We use 3.34, 3.5 and 3.27 to conclude that

$$\operatorname{Tan}^{*m}(\mathscr{H}^m \llcorner E \cap X, a) = \operatorname{Tan}(M,a) \quad \text{for every } a \in Y.$$

By [Fed69, 2.10.19(4)] we have $\mathscr{H}^m(E \cap X \sim Y) = 0$. Let

$$C = S \cap \{\chi : \chi + f(\chi) \in E \cap X\}, \quad D = S \cap \{\chi : \chi + f(\chi) \in Y\},$$

we observe that $\mathscr{H}^m(C \sim D) = 0$ and

$$\boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner S \sim C, \chi) = 0 \quad \text{for every } \chi \in D.$$

Let $\chi \in D$, $a = \chi + f(\chi)$ and suppose B is an m dimensional submanifold of class (k,α) such that $a \in B$ and $(*)$ is satisfied with A replaced by $E \cap X$.

[4] See [Fed69, 2.1.4].

Since $\operatorname{Tan}(B,a) \cap S^\perp = \{0\}$ there exist a function $g : S \to S^\perp$ of class (k,α) and an open neighbourhood V of a such that $B \cap V = \{\zeta + g(\zeta) : \zeta \in S\} \cap V$. Therefore, by 3.11,

$$\lim_{r\to 0} \frac{\mathscr{H}^m \Big(E \cap X \cap \mathbf{B}(a,r) \cap \{z : |g(S_\natural(z)) - S^\perp_\natural(z)| > \epsilon\, r^k\} \Big)}{\boldsymbol{\alpha}(m) r^m} = 0$$

for every $\epsilon > 0$ and, if $\alpha > 0$, there exists $0 \le \lambda < \infty$ such that

$$\lim_{r\to 0} \frac{\mathscr{H}^m \Big(E \cap X \cap \mathbf{B}(a,r) \cap \{z : |g(S_\natural(z)) - S^\perp_\natural(z)| > \lambda\, r^{k+\alpha}\} \Big)}{\boldsymbol{\alpha}(m) r^m} = 0.$$

Let $P : S \to S^\perp$ be the k jet of g at χ. If $\epsilon > 0$ then, possibly replacing λ by a larger number if $\alpha > 0$, we can choose $\rho > 0$ such that

$$|g(\zeta) - P(\zeta)| \le \lambda\, r^{k+\alpha} \ \text{ if } \alpha > 0, \quad |g(\zeta) - P(\zeta)| \le \epsilon\, r^k \ \text{ if } \alpha = 0,$$

for every $\zeta \in \mathbf{B}(\chi, r)$ and $0 < r \le \rho$. Let $\Gamma = (1 + (\operatorname{Lip} f)^2)^{1/2}$, $\gamma = \lambda + \Gamma^{k+\alpha}\,\lambda$ if $\alpha > 0$ and observe that whenever $0 < r \le \rho$

$$\begin{aligned}
&C \cap \mathbf{B}(\chi,r) \cap \{\zeta : |f(\zeta) - P(\zeta)| > \gamma\, r^{k+\alpha}\} \\
&\quad \subseteq S_\natural[E \cap X \cap \mathbf{B}(a, \Gamma\, r) \cap \{z : |S^\perp_\natural(z) - g(S_\natural(z))| > \lambda\, \Gamma^{k+\alpha}\, r^{k+\alpha}\}] \quad \text{if } \alpha > 0,\\
&C \cap \mathbf{B}(\chi,r) \cap \{\zeta : |f(\zeta) - P(\zeta)| > 2\epsilon\, r^{k}\} \\
&\quad \subseteq S_\natural[E \cap X \cap \mathbf{B}(a, \Gamma\, r) \cap \{z : |S^\perp_\natural(z) - g(S_\natural(z))| > \epsilon\ r^{k}\}] \quad \text{if } \alpha = 0.
\end{aligned}$$

Since χ is arbitrarily chosen in D we infer by 2.6 and 2.11 that there exist countably many functions $g_j : S \to S^\perp$ of class (k,α) such that

$$\mathscr{H}^m \Big(C \sim \textstyle\bigcup_{j=1}^\infty \{\zeta : f(\zeta) = g_j(\zeta)\} \Big) = 0,$$

implying that $E \cap X$ is $(\mathscr{H}^m, m)$ rectifiable of class (k,α).

3.40 Theorem. *Suppose $n \ge 1$ and $A \subseteq \mathbf{R}^n$.*

Then $\operatorname{ap}\operatorname{Tan}(A,\cdot)_\natural$ is a Borel map whose domain is a Borel subset of $\mathbf{R}^n$. The same conclusion is true for $\operatorname{ap}\mathrm{D}^k A$ for every $k \ge 2$.

Proof. This is a consequence of 3.37, 3.38 and [Men16, 4.1].

3.41 Theorem. *Suppose $1 \le m \le n$ and $k \ge 1$ are integers, $0 \le \alpha \le 1$, $A \subseteq \mathbf{R}^n$ such that $\mathscr{H}^m(A) < \infty$ and X is the set of $a \in \mathbf{R}^n$ such that A is approximately differentiable of order (k,α) at a with $\dim \operatorname{ap}\operatorname{Tan}(A,a) = m$.*

Then X is a Borel subset of $\mathbf{R}^n$ and $A \cap X$ is $(\mathscr{H}^m, m)$ rectifiable of class (k,α).

Proof. This is a consequence of 3.37, 3.38, [Men16, 4.1] and 3.39.

Chapter 4

A second order rectifiable stratification for sets

Statement

After we have introduced the necessary definitions in 4.2, we provide the statement of the main result of this chapter in 4.3.

4.1. First of all it is convenient to recall some basic facts about sets of positive reach and convex sets.

Suppose $0 \leq m \leq n$ are integers and $A \subseteq \mathbf{R}^n$ is a non empty closed set. Following [Fed59, 4.1] we define $\operatorname{reach}(A, a) = \sup\{r : \mathbf{U}(a, r) \subseteq \operatorname{Unp} A\}$ whenever $a \in A$ and $\operatorname{reach} A = \inf\{\operatorname{reach}(A, a) : a \in A\}$.

(1) The following fact is proved in [Fed59, 4.8(12)] and we state it here using the terminology we have introduced above. If $a \in A$ and $\operatorname{reach}(A, a) > 0$ then

$$\operatorname{Tan}_*(A, a) = \operatorname{Tan}^*(A, a), \quad \operatorname{Tan}(A, a) = \operatorname{Dual}\operatorname{Nor}(A, a).$$

Moreover if $\operatorname{reach}(A, a) > r > 0$ and

$$S = \{\lambda v : \lambda \geq 0,\ |v| = r,\ \boldsymbol{\xi}_A(a + v) = a\}$$

then either $S = \varnothing$ and $\operatorname{Nor}(A, a) = \{0\}$ or $S = \operatorname{Nor}(A, a)$.

(2) If A is convex then, by [Fed69, 4.1.16], we infer that $\operatorname{reach} A = \infty$. Moreover if $a \in A$ one may readily verify that

$$\operatorname{Tan}(A, a) = \operatorname{Clos}\{\lambda(x - a) : x \in A,\ 0 < \lambda < \infty\}$$

and, using [Roc70, 2.6.3, 6.2], one may infer that $\operatorname{Tan}(A, a)$ is the smallest closed convex cone containing A. Finally we observe that

$$\operatorname{Nor}(A, a) = \operatorname{Dual}\{x - a : x \in A\} \quad \text{whenever } a \in A.$$

In particular, $\mathscr{N}(A, a) = \operatorname{Nor}(A, a)$ whenever $a \in A$ (see 1.6).

4.2. Suppose $A \subseteq \mathbf{R}^n$ is closed and $a \in A$. Since by [Fed59, 4.8(2)]

$$\mathbf{R}^n \cap \{u : \boldsymbol{\delta}_A(a+u) = |u|\}$$

is a closed convex subset of $\operatorname{Nor}^*(A,a)$ containing 0 then, by 4.1 we define

$$\operatorname{nor}(A,a) = \operatorname{Tan}(\mathbf{R}^n \cap \{u : \boldsymbol{\delta}_A(a+u) = |u|\}, 0)$$

and we observe that $\operatorname{nor}(A,a) \subseteq \operatorname{Nor}^*(A,a)$. If $\operatorname{reach}(A,a) > 0$ we can use 4.1 to infer that

$$\operatorname{Nor}(A,a) = \operatorname{nor}(A,a).$$

For each integer $0 \leq m \leq n$ we define

$$\boldsymbol{\Sigma}^m(A) = A \cap \{a : \mathscr{H}^{n-m}(\operatorname{nor}(A,a)) > 0\}.$$

Evidently $\boldsymbol{\Sigma}^0(A) \subseteq \boldsymbol{\Sigma}^1(A) \subseteq \ldots \subseteq \boldsymbol{\Sigma}^{n-1}(A) \subseteq \boldsymbol{\Sigma}^n(A) = A$ and, by [Roc70, 6.2],

$$\boldsymbol{\Sigma}^m(A) = A \cap \{a : \mathscr{H}^{n-m}(\mathbf{R}^n \cap \{u : \boldsymbol{\delta}_A(a+u) = |u|\}) > 0\}$$

whenever $m = 0, \ldots, n$.

We can now state the main result of this chapter whose proof is postponed to 4.16.

4.3 Theorem. *Suppose $0 \leq m \leq n$ are integers and $A \subseteq \mathbf{R}^n$ is closed.*

Then $\boldsymbol{\Sigma}^m(A)$ is a Borel subset of $\mathbf{R}^n$ and it is countably $(\mathscr{H}^m, m)$ rectifiable of class 2.

Preparation

Here we collect several technical lemmas that we use in the proof of 4.3.

4.4. Suppose A is a non empty closed subset of $\mathbf{R}^n$ and $0 < r < s < \infty$. We define

$$S_r(A) = \mathbf{R}^n \cap \{z : \boldsymbol{\delta}_A(z) = r\},$$

$$S_r^s(A) = \{a + (r/s)u : a \in A,\ \boldsymbol{\delta}_A(a+u) = |u| = s\}.$$

By [Fed69, 3.2.12, 3.2.15] one may easily deduce, for $\mathscr{L}^1$ a.e. $r > 0$, that

$$\mathscr{H}^{n-1}(S_r(A) \cap K) < \infty \quad \text{whenever } K \subseteq \mathbf{R}^n \text{ is compact},$$

$$S_r(A) \text{ is countably } (\mathscr{H}^{n-1}, n-1) \text{ rectifiable of class 1.}$$

Evidently $S_r^s(A)$ is a Borel subset of $\mathbf{R}^n$ and $S_r^s(A) \subseteq \operatorname{Unp}(A) \cap S_r(A)$, whenever $0 < r < s < \infty$.

4.5 Lemma. *Suppose A is a non empty closed subset of $\mathbf{R}^n$ and $0 < r < s < \infty$.*

Then $\operatorname{Lip}(\boldsymbol{\xi}_A | S_r^s(A)) \leq (s-r)^{-1}s$.

Proof. The argument in this proof resembles the proof of [Fed59, 4.8(7), 4.8(8)].

Suppose $\xi = \boldsymbol{\xi}_A | S_r^s(A)$. Let $x \in S_r^s(A)$, $b \in A$,

$$u = \frac{x - \xi(x)}{|x - \xi(x)|}, \quad J = \{t : \boldsymbol{\delta}_A(\xi(x) + tu) = t\}.$$

Then $\sup J \geq s$. If $t \in J$ we have

$$|\xi(x) + tu - b| \geq \boldsymbol{\delta}_A(\xi(x) + tu) = t,$$

$$t^2 \leq |\xi(x) - b|^2 + t^2 + 2tu \bullet (\xi(x) - b),$$

$$(x - \xi(x)) \bullet (\xi(x) - b) \geq -(2t)^{-1} |\xi(x) - b|^2 |x - \xi(x)|$$

and, letting $t \to s-$, we conclude

$$(x - \xi(x)) \bullet (\xi(x) - b) \geq -(2s)^{-1} |\xi(x) - b|^2 r.$$

If $x, y \in S_r^s(A)$ then

$$(x - \xi(x)) \bullet (\xi(x) - \xi(y)) \geq -(2s)^{-1} |\xi(x) - \xi(y)|^2 r,$$

$$(y - \xi(y)) \bullet (\xi(y) - \xi(x)) \geq -(2s)^{-1} |\xi(x) - \xi(y)|^2 r,$$

$$\begin{aligned}
&|x - y||\xi(x) - \xi(y)| \geq (x - y) \bullet (\xi(x) - \xi(y)) \\
&\quad = (x - \xi(x)) \bullet (\xi(x) - \xi(y)) + |\xi(x) - \xi(y)|^2 + (\xi(y) - y) \bullet (\xi(x) - \xi(y)) \\
&\quad \geq s^{-1}(s - r)|\xi(x) - \xi(y)|^2.
\end{aligned}$$

4.6 Lemma. *Suppose A is a non empty closed subset of $\mathbf{R}^n$ and $0 < r < s < \infty$. If $\nu : S_r^s(A) \to \mathbf{R}^n$ is defined by $\nu(x) = r^{-1}(x - \boldsymbol{\xi}_A(x))$ whenever $x \in S_r^s(A)$ then the following three conditions hold.*

(1) Whenever $x \in S_r^s(A)$

$$\mathbf{U}\left(x - \tfrac{r}{2}\nu(x), \tfrac{r}{2}\right) \cap S_r(A) = \varnothing, \quad \mathbf{U}\left(x + \tfrac{s-r}{2}\nu(x), \tfrac{s-r}{2}\right) \cap S_r(A) = \varnothing.$$

(2) Whenever $x \in S_r^s(A)$

$$\limsup_{\delta \to 0+} \delta^{-2} \sup\{|\nu(x) \bullet (z - x)| : z \in S_r(A) \cap \mathbf{U}(x, \delta)\} < \infty.$$

(3) Whenever $x \in S_r^s(A)$

$$\limsup_{\delta \to 0+} \delta^{-2} \sup\{|\nu(x) \bullet (\boldsymbol{\xi}_A(z) - \boldsymbol{\xi}_A(x))| : z \in S_r^s(A) \cap \mathbf{U}(x, \delta)\} < \infty.$$

In particular $S_r^s(A)$ is countably $(\mathscr{H}^{n-1}, n-1)$ rectifiable of class 2.

Proof. Suppose $\xi = \boldsymbol{\xi}_A | S_r^s(A)$.

Let $x \in S_r^s(A)$, $c = x + ((s-r)/2)\nu(x)$ and $b = x - (r/2)\nu(x)$. First observe that $\mathbf{U}(b, r/2) \subseteq \mathbf{U}(\xi(x), r)$ and $\mathbf{U}(\xi(x), r) \cap S_r(A) = \varnothing$. Suppose $w \in S_r(A)$ and $a \in A$ so that

$$\boldsymbol{\delta}_{S_r(A)}(c) = |w - c|, \quad \boldsymbol{\delta}_A(w) = |w - a|.$$

We observe that $\xi(c) = \xi(x)$, $|c-\xi(c)| = |c-x|+|x-\xi(x)|$ and $|c-\xi(c)| \le |c-a|$. Therefore

$$\begin{aligned}|c-x| &= |c-\xi(c)| - |x-\xi(x)| \le |c-a| - |x-\xi(x)| \\ &\le |c-w| + |w-a| - |x-\xi(x)| = |c-w|.\end{aligned}$$

This implies that $\boldsymbol{\delta}_{S_r(A)}(c) = |c-x| = (s-r)/2$ and (1) is proved. Observe that (2) is an immediate consequence of (1). Finally in order to prove (3) we observe

$$\begin{aligned}&(\nu(z)-\nu(x)) \bullet \nu(x) = -(1/2)|\nu(z)-\nu(x)|^2, \\ &(\xi(z)-\xi(x)) \bullet \nu(x) = r(\nu(x)-\nu(z)) \bullet \nu(x) + (z-x) \bullet \nu(x), \\ &|(\xi(z)-\xi(x)) \bullet \nu(x)| \le (r/2)\,\mathrm{Lip}(\nu)^2\,|z-x|^2 + |(z-x) \bullet \nu(x)|\end{aligned}$$

whenever $z, x \in S_r^s(A)$. Therefore (3) follows from (2).

The postscript is a consequence of 3.39 and [Fed69, 3.1.15].

4.7. The following proposition is classical and it is based on [Fed69, 2.10.26].

Suppose X and Y are metric spaces, X is compact, $f : X \to Y$ is a continuous map, F is a family of open subsets of X, ζ is a function such that

$$0 \le \zeta(S) \le \infty \quad \text{whenever } S \in F,$$

ψ is the result of the Caratheodory's construction from ζ on F and ϕ_δ is the size δ approximating measure for every $0 < \delta \le \infty$ (see [Fed69, 2.10.1]). Then the function mapping $y \in Y$ onto $\psi(f^{-1}\{y\})$ is Borel. In fact, if $t > 0$, letting

$$V = Y \cap \{y : \psi(f^{-1}\{y\}) \le t\},$$

$$V_i = Y \cap \{y : \phi_{1/i}(f^{-1}\{y\}) < t + (1/i)\} \quad \text{for every integer } i \ge 1,$$

we observe that

$$V = \bigcap_{i=1}^{\infty} V_i$$

and V_i is open whenever $i \ge 1$. The latter can be proved observing that if $y \in V_i$ then there exists a countable open covering G of $f^{-1}\{y\}$ such that

$$\operatorname{diameter} S \le 1/i \ \text{ whenever } S \in G, \quad \sum_{S \in G} \zeta(S) < t + (1/i);$$

since X is compact, there exists $\epsilon > 0$ such that

$$f^{-1}[\mathbf{U}(y,\epsilon)] \subseteq \bigcup G.$$

4.8 Lemma. *Suppose $n \ge m \ge \mu$, $\nu \ge \mu$,*

$W \subseteq \mathbf{R}^n$ is countably $(\mathscr{H}^m, m)$ rectifiable, $\mathscr{H}^m$ measurable and $\mathscr{H}^m(K \cap W) < \infty$ whenever K is a compact subset of $\mathbf{R}^n$,

$Z \subseteq \mathbf{R}^\nu$ is countably $(\mathscr{H}^\mu, \mu)$ rectifiable, $\mathscr{H}^\mu$ measurable and $\mathscr{H}^\mu(Z \cap K) < \infty$ whenever K is a compact subset of $\mathbf{R}^\nu$,

$$f : W \to Z \text{ is a Lipschitzian map,}$$

$$V = W \cap \{w : \operatorname{ap} J_\mu f(w) = 0\},$$

where the $(\mathscr{H}^m \llcorner W, m)$ *approximate* μ *dimensional Jacobian of* f *at* w *is defined by the formula*

$$\operatorname{ap} J_\mu f(w) = \| \textstyle\bigwedge_\mu \big((\mathscr{H}^m \llcorner W, m) \operatorname{ap} \mathrm{D} f(w) \big) \|$$

whenever $\operatorname{Tan}^m(\mathscr{H}^m \llcorner W, w) \in \mathbf{G}(n, m)$ *and* f *is* $(\mathscr{H}^m \llcorner W, m)$ *approximately differentiable*[1] *at* w.

Then the following three statements hold:

(1) if $X \subseteq Z$ *is* $\mathscr{H}^\mu$ *measurable then* $f^{-1}[X] \sim V$ *is* $\mathscr{H}^m$ *measurable;*

(2) if $n = m$, $\mathscr{L}^n(V) = 0$ *and* $\mathscr{H}^{n-\mu}(f^{-1}(z)) > 0$ *for* $\mathscr{H}^\mu$ *a.e.* $z \in Z$ *then there exists a countable collection* G *of* $\mathscr{H}^\mu$ *measurable subsets of* W *such that* $\mathscr{H}^\mu\left(Z \sim \bigcup\{f[P] : P \in G\}\right) = 0$ *and, if* $P \in G$, *then*

$$f|P \text{ is univalent,} \quad \operatorname{Lip}\big((f|P)^{-1}\big) < \infty,$$
$$P \text{ is contained in some affine set of } \mathbf{R}^n \text{ of dimension } \mu;$$

(3) if $\mathscr{H}^m(V) = 0$ *and* $\mathscr{H}^{m-\mu}(f^{-1}(z)) > 0$ *for* $\mathscr{H}^\mu$ *a.e.* $z \in Z$ *then there exists a countable collection* G *of* $\mathscr{H}^\mu$ *measurable subsets of* W *such that* $\mathscr{H}^\mu\left(Z \sim \bigcup\{f[P] : P \in G\}\right) = 0$ *and, if* $P \in G$, *then*

$$f|P \text{ is univalent,} \quad \operatorname{Lip}\big((f|P)^{-1}\big) < \infty,$$
$$P \text{ is } \mu \text{ rectifiable.}$$

Proof. In order to prove (1) we observe, by [Fed69, 2.2.3], that there exists a Borel set $B \subseteq X$ such that $\mathscr{H}^\mu(X \sim B) = 0$ and another Borel set $A \subseteq Z$ such that $X \sim B \subseteq A$ and $\mathscr{H}^\mu(A) = 0$. Noting [Fed69, 2.10.19(4)], by [Fed69, 3.2.22(3)] with W, Z and f replaced by $f^{-1}[A]$, A and $f|f^{-1}[A]$,

$$\int_{f^{-1}[A] \sim V} \operatorname{ap} J_\mu f(w) d\mathscr{H}^m w = \int_A \mathscr{H}^{m-\mu}\left(f^{-1}(z)\right) d\mathscr{H}^\mu z = 0,$$

$$\mathscr{H}^m\left(f^{-1}[A] \sim V\right) = 0.$$

Since $f^{-1}[X] \sim V = \big(f^{-1}[B] \sim V\big) \cup \big(f^{-1}[X \sim B] \sim V\big)$ we conclude that $f^{-1}[X] \sim V$ is $\mathscr{H}^m$ measurable.

In order to prove (2) we first notice, by [Fed69, 2.10.19(4), 3.2.16] that (see 2.2)

$$(\mathscr{L}^n \llcorner W, n) \operatorname{ap} \mathrm{D} f(w) = \operatorname{ap} \mathrm{D} f(w) \quad \text{for } \mathscr{L}^n \text{ a.e. } w \in W.$$

We consider the class Ω of all families G of $\mathscr{H}^\mu$ measurable subsets P of W such that $f[P] \cap f[Q] \neq \varnothing$ if and only if $P = Q$ and such that if $P \in G$ then

$$\mathscr{H}^\mu(P) > 0, \quad f|P \text{ is univalent,} \quad \operatorname{Lip}\big((f|P)^{-1}\big) < \infty,$$
$$P \text{ is contained in some } \mu \text{ dimensional affine set of } \mathbf{R}^n.$$

[1] For the notion of (ϕ, m) approximate differentiability for functions, where ϕ is a measure over some normed vector space X and m is a positive integer, see [Fed69, 3.2.16].

Let G be a maximal element of Ω with respect to inclusion and note that G is countable. If the $\mathscr{H}^\mu$ measurable set

$$Z_1 = Z \sim \bigcup \{ f[P] : P \in G \}$$

had positive $\mathscr{H}^\mu$ measure, then $f^{-1}[Z_1]$ would be $\mathscr{L}^n$ measurable by (1) and $\mathscr{L}^n(f^{-1}[Z_1]) > 0$ by [Fed69, 3.2.22(3)]. Then one could choose $T \in \mathbf{G}(n, \mu)$ such that $\mathscr{L}^n(W_1) > 0$, where

$$W_1 = f^{-1}[Z_1] \cap \{ w : \| \textstyle\bigwedge_\mu (\operatorname{ap} \mathrm{D} f(w) | T) \| > 0 \},$$

and apply [Fed69, 2.6.2(3)] to infer that there exists $\eta \in \mathbf{R}^n$ such that

$$\mathscr{H}^\mu(R) > 0$$

where $R = W_1 \cap \{ \zeta : \zeta - \eta \in T \}$. Furthermore there exists a Lipschitzian function $F : \mathbf{R}^n \to \mathbf{R}^\nu$ such that $F | W = f$ by [Fed69, 2.10.43]. Since F is pointwise differentiable at w with $\mathrm{D} F(w) = \operatorname{ap} \mathrm{D} f(w)$ whenever $w \in W_1$ by [Fed69, 3.1.5], we can use [Fed69, 3.2.2] to infer the existence of a Borel set $P \subseteq R$ such that the family $G \cup \{P\}$ belongs to Ω, in contradiction with the maximality of G.

Finally we prove (3). Suppose K_i is a sequence of compact subsets of $\mathbf{R}^m$ and $\psi_i : \mathbf{R}^m \to \mathbf{R}^n$ is a sequence of Lipschitzian maps satisfying the conclusion of [Fed69, 3.2.18] with respect to W, for some $1 < \lambda < \infty$. For every integer $i \geq 1$ let

$$Z_i = \mathbf{R}^\nu \cap \{ z : \mathscr{H}^{m-\mu}(f^{-1}(z) \cap \psi_i[K_i]) > 0 \},$$

and observe, by [Fed69, 3.2.22(3)], that $\mathscr{H}^\mu(Z \sim \bigcup_{i=1}^\infty Z_i) = 0$. By 4.7 the set Z_i is a Borel subset of $\mathbf{R}^\nu$. We define

$$W_i = f^{-1}[Z_i] \cap \psi_i[K_i] \quad \text{for every integer } i \geq 1$$

and we observe they are Borel subsets of $\mathbf{R}^n$. For every integer $i \geq 1$ let $f_i : \psi_i^{-1}[W_i] \to Z_i$ be given by

$$f_i = f \circ (\psi_i | \psi_i^{-1}[W_i])$$

and $V_i = \psi_i^{-1}[W_i] \cap \{ x : \| \bigwedge_\mu (\operatorname{ap} \mathrm{D} f_i(x)) \| = 0 \}$. We use [Fed69, 2.10.19(4), 3.2.22(3)] to infer that

$$\mathscr{H}^m(\psi_i[K_i] \sim f^{-1}[Z_i]) = 0 \quad \text{for every } i \geq 1.$$

Therefore

$$\mathscr{H}^m \left(W \sim \textstyle\bigcup_{i=1}^\infty W_i \right) = 0.$$

Moreover $\mathscr{H}^{m-\mu}(f_i^{-1}(z)) > 0$ for every $z \in Z_i$. Finally combining [Fed69, 2.10.43, 3.1.5, 3.2.17] we infer that

$$\psi_i[V_i \cap \{ x : \boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner W \sim W_i, \psi_i(x)) = 0 \}] \subseteq V \cap W_i$$

and, by [Fed69, 2.10.19(4)],

$$\mathscr{L}^m(V_i \sim \{ x : \boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner W \sim W_i, \psi_i(x)) = 0 \}) = 0$$

whenever $i \geq 1$. Therefore we can apply (2) to conclude that for every $i \geq 1$ there exists a countable family G_i of $\mathscr{H}^\mu$ measurable and μ rectifiable subsets of W_i such that

$$\mathscr{H}^\mu\left(f[W_i] \sim \bigcup\{f[P] : P \in G_i\}\right) = 0,$$
$$f|P \text{ is univalent}, \quad \operatorname{Lip}\big((f|P)^{-1}\big) < \infty.$$

Applying [Fed69, 2.10.25] with A replaced by $f^{-1}\big[f[W] \sim \bigcup_{i=1}^\infty f[W_i]\big]$ and [Fed69, 2.10.26] we infer that $\mathscr{H}^\mu\left(f[W] \sim \bigcup_{i=1}^\infty f[W_i]\right) = 0$; therefore

$$\mathscr{H}^\mu\left(f[W] \sim \bigcup_{i=1}^\infty \bigcup\{f[P] : P \in G_i\}\right) = 0.$$

4.9. Suppose A is a non empty closed subset of $\mathbf{R}^n$. We define

$$Q = (A \times \mathbf{R}^n) \cap \{(a,u) : \boldsymbol{\delta}_A(a+u) = |u| \leq 1\},$$
$$N = (A \times \mathbf{R}^n) \cap \{(a,u) : u \in \operatorname{aff} Q(a)\}.$$

Observe that $\operatorname{aff} Q(a)$ is a linear subspace of $\mathbf{R}^n$ for every $a \in A$. Evidently Q is a closed subset of $\mathbf{R}^n \times \mathbf{R}^n$. Moreover it is not difficult to see that N is a countable union of compact sets. In fact, if

$$Q_n = \{(a, u_1, \dots, u_n) : a \in A,\ u_i \in Q(a) \ \text{ for } i = 1, \dots, n\}$$

and $L : (\mathbf{R}^n)^{n+1} \times \mathbf{R}^n \to \mathbf{R}^n \times \mathbf{R}^n$ is defined by

$$L(a, u_1, \dots, u_n, \lambda_1, \dots, \lambda_n) = (a, \textstyle\sum_{i=1}^n \lambda_i\, u_i),$$

we observe that Q_n is a closed subset of $(\mathbf{R}^n)^{n+1}$ and $N = L[Q_n \times \mathbf{R}^n]$.

Since $Q(a)$ is convex (by [Fed59, 4.8(2)]) and

$$\operatorname{nor}(A,a) = \operatorname{Clos}\{\lambda u : \lambda > 0,\ u \in Q(a)\}$$

for each $a \in A$, we can use [Roc70, 2.6.3, 6.2] to deduce that

$$\operatorname{aff} \operatorname{nor}(A,a) = N(a) \quad \text{whenever } a \in A.$$

4.10 Lemma. *Suppose $A \subseteq \mathbf{R}^n$ is closed.*

Then for each integer $0 \leq m \leq n$ the set $\boldsymbol{\Sigma}^m(A)$ is a Borel subset of $\mathbf{R}^n$.

Proof. We assume $m < n$ since $\boldsymbol{\Sigma}^n(A) = A$.

Since N is a countable union of compact sets then

$$\{a : N(a) \cap W \neq \varnothing\}$$

is a countable union of compact sets, whenever $W \subseteq \mathbf{R}^n$ is open. Therefore we can apply [CV77, III.7] to infer the existence of countably many Borel functions $u_j : A \to \mathbf{R}^n$ such that

$$\operatorname{Clos}\{u_j(a) : j \geq 1\} = N(a) \quad \text{whenever } a \in A.$$

Since $\boldsymbol{\delta}_{N(a)}(v) = \inf\{|v - u_j(a)| : j \geq 1\}$ whenever $(a,v) \in A \times \mathbf{R}^n$, we conclude that $\boldsymbol{\delta}_{N(a)}(v)$ is a Borel function with respect to $(a,v) \in A \times \mathbf{R}^n$. Let D be a countable dense subset of $\mathbf{G}(n, m+1)$. Then it is not difficult to show that

$$A \cap \{a : \dim N(a) \leq m\} = \bigcap_{S \in D} \big\{a : \sup\{\boldsymbol{\delta}_{N(a)}(v) : v \in S \cap \mathbf{S}^{n-1}\} \geq 1\big\}.$$

Therefore $\dim N(a)$ is a Borel function with respect to $a \in A$. We use now 4.9 to conclude that $A \cap \{a : \dim N(a) \geq n - m\} = \boldsymbol{\Sigma}^m(A)$.

4.11. The following facts will be used in the proof of 4.12 and 4.15.

(1) If X is a metric space then every Borel subset of X is a Suslin subset of X. This is proved in [Fed69, 2.2.10, p. 66].

(2) If X is a topological space, S is a Suslin subset of X and T is a Suslin subset of S then T is a Suslin subset of X. In fact, using the notation of [Fed69, 2.2.10], if $C \subseteq X \times \mathscr{N}$ is a closed set such that $T = p[C \cap (S \times \mathscr{N})]$ then $T = p[C] \cap S$, implying that T is a Suslin subset of X by [Fed69, 2.2.10, p. 66].

(3) If X is a complete separable metric space and B is a Borel subset of X then the following conditions are equivalent for $S \subseteq B$.

   (a) S is a Suslin subset of B;

   (b) S is a Suslin subset of X;

   (c) there exists a complete separable metric space Y and $f : Y \to X$ continuous such that $\operatorname{im} f = S$ (this is the definition of Suslin subset used in [CV77], see [CV77, III.17]).

   By (1) and (2) we deduce that (3a) implies (3b), by [Fed69, 2.2.6] we may infer that (3b) implies (3c) and by [Fed69, 2.2.10, p. 65] we get that (3c) implies (3a).

(4) If X and Y are topological spaces, $f : X \to Y$ is continuous and F is the Borel family generated by the Suslin subsets of X then $f^{-1}[S] \in F$ whenever S is an element of the Borel family generated by the Suslin subsets of Y. In fact, by [Fed69, 2.2.10, p. 66],

$$\mathbf{2}^Y \cap \{T : f^{-1}[T] \in F\}$$

   is a Borel family with respect to Y containing the Suslin subsets of Y.

4.12 Lemma. *Suppose $A \subseteq \mathbf{R}^n$ is closed.*
Then there exist $\rho : \boldsymbol{\Sigma}^{n-1}(A) \to \mathbf{R}$ and $\zeta : \boldsymbol{\Sigma}^{n-1}(A) \to \mathbf{R}^n$ such that

$$\rho(a) > 0, \quad \zeta(a) \in N(a), \quad \mathbf{U}(\zeta(a), \rho(a)) \cap N(a) \subseteq Q(a),$$

whenever $a \in \boldsymbol{\Sigma}^{n-1}(A)$. Moroever ρ is a Borel function and ζ is so that for each open $U \subseteq \mathbf{R}^n$ the set $\zeta^{-1}[U]$ belongs to the Borel family generated by the Suslin subsets of $\boldsymbol{\Sigma}^{n-1}(A)$.

Proof. For each integer $i \geq 1$ let

$$R_i = N \cap \{(a, u) : \text{either } \boldsymbol{\delta}_A(a+u) \leq |u| - i^{-1} \text{ or } \boldsymbol{\delta}_A(a+u) = |u| \geq 1\}$$

and observe that $R_i \subseteq R_{i+1}$ whenever $i \geq 1$, $R_i(a) = \varnothing$ whenever $i \geq 1$ and $a \in A \sim \boldsymbol{\Sigma}^{n-1}(A)$,

$$\textstyle\bigcup_{i=1}^{\infty} \operatorname{dmn} R_i = \boldsymbol{\Sigma}^{n-1}(A).$$

Fix $i \geq 1$. Since N is a countable union of compact sets by 4.9, so is R_i. Therefore $\{a : R_i(a) \cap W \neq \varnothing\}$ is a countable union of compact sets whenever

$W \subseteq \mathbf{R}^n$ is open and, applying [CV77, III.7], we conclude there exist countably many Borel functions $w_{i,j} : \operatorname{dmn} R_i \to \mathbf{R}^n$ such that

$$\operatorname{Clos}\{w_{i,j}(a) : j \geq 1\} = R_i(a) \quad \text{whenever } a \in \operatorname{dmn} R_i.$$

Thereofore $\boldsymbol{\delta}_{R_i(a)}(v) = \inf\{|w_{i,j}(a) - v| : j \geq 1\}$ is a Borel function with respect to $(a, v) \in (\operatorname{dmn} R_i) \times \mathbf{R}^n$. Since[2]

$$\boldsymbol{\delta}_{N(a)\sim Q(a)}(v) = \inf\{\boldsymbol{\delta}_{R_i(a)}(v) : i \geq 1\}$$

whenever $(a, v) \in \boldsymbol{\Sigma}^{n-1}(A) \times \mathbf{R}^n$, we conclude that

$$\boldsymbol{\delta}_{N(a)\sim Q(a)}(v) \text{ is a Borel function with respect to } (a, v) \in \boldsymbol{\Sigma}^{n-1}(A) \times \mathbf{R}^n.$$

Therefore we define $\rho : \boldsymbol{\Sigma}^{n-1}(A) \to \mathbf{R}$ by

$$\rho(a) = \sup\{\boldsymbol{\delta}_{N(a)\sim Q(a)}(v) : v \in N(a)\} \quad \text{whenever } a \in \boldsymbol{\Sigma}^{n-1}(A)$$

and observe[3] it is a Borel function such that $\rho(a) > 0$ whenever $a \in \boldsymbol{\Sigma}^{n-1}(A)$. Since $N \cap \{(a, v) : \boldsymbol{\delta}_{N(a)\sim Q(a)}(v) = \rho(a)\}$ is a Borel subset of $\boldsymbol{\Sigma}^{n-1}(A) \times \mathbf{R}^n$, we can apply [CV77, Theorem III.18] in combination with 4.11(3) to deduce the existence of a function

$$\zeta : \boldsymbol{\Sigma}^{n-1}(A) \to \mathbf{R}^n$$

such that for every open subset U of $\mathbf{R}^n$ the subset $\zeta^{-1}[U]$ belongs to the Borel family generated by the Suslin subsets of $\boldsymbol{\Sigma}^{n-1}(A)$ and

$$\zeta(a) \in N(a), \quad \boldsymbol{\delta}_{N(a)\sim Q(a)}(\zeta(a)) = \rho(a),$$

whenever $a \in \boldsymbol{\Sigma}^{n-1}(A)$.

4.13 Lemma. *Suppose $A \subseteq \mathbf{R}^n$ is a closed set and Q is defined as in 4.9. If $0 < r < 1$, $a \in A$ and $C \subseteq \mathbf{R}^n$ is a convex cone such that*

$$\varnothing \neq C \cap \{u : |u| = r\} \subseteq Q(a),$$

then $\boldsymbol{\delta}_{\operatorname{Dual} C}(b - a) \leq (2r)^{-1}|b - a|^2$ whenever $b \in A$.

Proof. The proof resembles [Fed59, 4.18]. We fix $b \in A$. If $v \in C$, $v \neq 0$, $u = (r/|v|)v$ and

$$J = \{t : \boldsymbol{\delta}_A(a + tu) = t\,r\},$$

then $u \in Q(a)$ and $\sup J \geq 1$. Observe that if $t \in J$ then $|a + tu - b| \geq tr$. Therefore we can compute

$$\begin{aligned}
&(a - b) \bullet v \geq -(2r)^{-1}|a - b|^2|v|,\\
&|(b - a) - v|^2 \geq |b - a|^2 + |v|^2 - r^{-1}|a - b|^2|v|\\
&\quad = |b - a|^2 + |v|^2 - r^{-1}|a - b|^2|v| + (4r^2)^{-1}|b - a|^4 - (4r^2)^{-1}|b - a|^4\\
&\quad \geq |b - a|^2 - (4r^2)^{-1}|b - a|^4.
\end{aligned}$$

[2]More precisely for each integer $i \geq 1$ we define $\psi_i : \boldsymbol{\Sigma}^{n-1}(A) \times \mathbf{R}^n \to \overline{\mathbf{R}}$ as $\psi_i(a, v) = \boldsymbol{\delta}_{R_i(a)}(v)$ if $(a, v) \in (\operatorname{dmn} R_i) \times \mathbf{R}^n$ and $\psi_i(a, v) = \infty$ if $(a, v) \in (\boldsymbol{\Sigma}^{n-1}(A) \sim \operatorname{dmn} R_i) \times \mathbf{R}^n$ and we observe that it is a Borel function such that

$$\boldsymbol{\delta}_{N(a)\sim Q(a)}(v) = \inf\{\psi_i(a, v) : i \geq 1\} \quad \text{whenever } (a, v) \in \boldsymbol{\Sigma}^{n-1}(A) \times \mathbf{R}^n.$$

[3]If the functions $u_j : A \to \mathbf{R}^n$ are defined as in the proof of 4.10 for each $j \geq 1$, then

$$\rho(a) = \sup\{\boldsymbol{\delta}_{N(a)\sim Q(a)}(u_j(a)) : j \geq 1\} \quad \text{whenever } a \in \boldsymbol{\Sigma}^{n-1}(A).$$

Therefore $[\boldsymbol{\delta}_C(b-a)]^2 \geq |b-a|^2 - (4r^2)^{-1}|b-a|^4$ and, by [Fed59, 4.16], we infer that
$$[\boldsymbol{\delta}_{\operatorname{Dual} C}(b-a)]^2 \leq \frac{|b-a|^4}{4r^2}.$$

4.14. In 4.15 we use the concept of universally measurable set as it is given in [CV77, Definition 21]. For reader's convenience we give an equivalent formulation of this concept using the terminology of [Fed69].

Suppose F is a Borel family[4] with respect to X. A subset of X is called F *universally measurable subset of* X if and only if it is ϕ measurable for every measure ϕ over X such that every element of F is ϕ measurable.

The family of all F universally measurable subsets of X is a Borel family with respect to X containing F.

In case X is a topological space and F is the Borel family of the Borel subsets of X then the term "F universally measurable" is replaced by "universally measurable".

Finally one may readily verify the following two statements.

(1) If F and G are Borel families with respect to X, H is the Borel family of all F universally measurable subsets of X and $F \subseteq G \subseteq H$ then the Borel family of all G universally measurable subsets of X equals H.

(2) If X is a topological space, A is a Borel subset of X and S is a universally measurable subset of A then S is a universally measurable subset of X

4.15. Suppose A is a non empty closed subset of $\mathbf{R}^n$, Q and N are defined as in 4.9 and ρ and ζ are as in 4.12. For every $a \in \boldsymbol{\Sigma}^{n-1}(A)$ we define
$$C(a) = \{\lambda(\zeta(a)+v) : v \in N(a),\ \lambda > 0,\ |v| < \rho(a),\ \zeta(a) \bullet v > 0\} \quad \text{if } \zeta(a) \neq 0,$$
$$C(a) = N(a) \quad \text{if } \zeta(a) = 0.$$
For each $a \in \boldsymbol{\Sigma}^{n-1}(A)$ we observe that $C(a)$ is a (non empty) convex cone by [Roc70, 2.6.3]; by 4.12, 4.2 and 4.9,
$$C(a) \subseteq \operatorname{nor}(A,a), \quad \operatorname{aff} C(a) = \operatorname{aff}\operatorname{nor}(A,a) = N(a),$$
and $C(a)$ is relatively open in $N(a)$ whenever $a \in \boldsymbol{\Sigma}^{n-1}(A)$. Moreover it is not difficult to see that if $a \in \boldsymbol{\Sigma}^{n-1}(A)$, $0 < r < \infty$ and $\rho(a) \geq 2r$ then
$$\varnothing \neq C(a) \cap \{u : |u| = r\} \subseteq Q(a).$$
In fact, recalling that $\mathbf{U}(\zeta(a), \rho(a)) \cap N(a) \subseteq Q(a)$ by 4.12, we argue as follows. The conclusion is evidently true if $\zeta(a) = 0$. In case $0 < |\zeta(a)| < r$ then, for each $u \in C(a)$ such that $|u| = r$ we get $|\zeta(a) - u| < 2r \leq \rho(a)$ and $u \in Q(a)$. Finally if $|\zeta(a)| \geq r$, $u \in C(a)$, $\lambda > 0$ and $v \in N(a)$ are such that $u = \lambda(\zeta(a)+v)$, $|v| < \rho(a)$, $|u| = r$ and $v \bullet \zeta(a) > 0$ then
$$\zeta(a) + v \in Q(a), \quad r = |u| = \lambda|\zeta(a)+v| \geq \lambda|\zeta(a)|,$$
$$\lambda \leq 1, \quad u \in Q(a),$$
since $Q(a)$ is a convex set containing 0.

[4]Borel families are termed "tribes" or "σ fields" in [CV77, p. 60].

Suppose $0 < r < s < \infty$, $\xi = \boldsymbol{\xi}_A | S_r^s(A)$, $\nu(x) = r^{-1}(x - \xi(x))$ for $x \in S_r^s(A)$. We prove that (notice 4.4)

$$B_r^s = S_r^s(A) \cap \{x : \nu(x) \in C(\xi(x))\}$$

is a universally measurable subset of $S_r^s(A)$. In fact let G be the set of all $(x, v) \in S_r^s(A) \times \mathbf{R}^n$ such that

$$\nu(x) = \frac{\zeta(\xi(x)) + v}{|\zeta(\xi(x)) + v|}, \quad v \in N(\xi(x)), \quad |v| < \rho(\xi(x)), \quad \zeta(\xi(x)) \bullet v > 0.$$

Employing 4.9, 4.12 and 4.11(4) it is not difficult to see that the set G belongs to the smallest Borel family with respect to $S_r^s(A) \times \mathbf{R}^n$ containing all the subsets $\alpha \times \beta$ where α belongs to the Borel family generated by the Suslin subsets of $S_r^s(A)$ and β is a Borel subset of $\mathbf{R}^n$. Therefore one may deduce that $\operatorname{dmn} G$ is a universally measurable subset of $S_r^s(A)$ by suitably combining [CV77, Theorem III.23], 4.11(1), [Fed69, 2.2.12] and 4.14(1). Moreover

$$B_r^s \cap \{x : \zeta(\xi(x)) = 0\} = S_r^s(A) \cap \{x : \nu(x) \in N(\xi(x)),\ \zeta(\xi(x)) = 0\}$$

is a universally measurable subset of $S_r^s(A)$ by 4.9, 4.12, 4.11(4) and [Fed69, 2.2.12]. Since $\operatorname{dmn} G = B_r^s \cap \{x : \zeta(\xi(x)) \neq 0\}$ we get the conclusion.

In particular, employing 4.4, 4.14(2) and 4.6 we conclude that B_r^s is $\mathscr{H}^{n-1}$ measurable and countably $(\mathscr{H}^{n-1}, n-1)$ rectifiable of class 2 whenever $0 < r < s < \infty$.

Proof and discussion

Eventually the proof of 4.3 is given in 4.16. We provide few additional comments and remarks at the end of the section.

4.16. The case $m = n$ is trivial.

The case $m = 0$ is an elementary consequence of the fact that $\mathscr{H}^n$ is finite on bounded subsets of $\mathbf{R}^n$. In fact for $0 < r < \infty$ and $a \in A$ let

$$X_r(a) = \{a + \lambda\, u : \boldsymbol{\delta}_A(a + ru) = r,\ |u| = 1,\ 0 \leq \lambda < r/2\}.$$

These are Borel subsets of $\mathbf{R}^n$ contained in $\operatorname{Unp}(A)$ and $X_r(a) \cap X_r(b) = \varnothing$ if $a \neq b$ since $X_r(a) \subseteq \boldsymbol{\xi}_A^{-1}(a)$ whenever $a \in A$. Therefore if $K \subseteq \mathbf{R}^n$ is compact, $\epsilon > 0$ and $0 < r < \infty$ then

$$A \cap K \cap \{a : \mathscr{H}^n(X_r(a)) \geq \epsilon\}$$

is a set of finite cardinality. Since by 4.2

$$\boldsymbol{\Sigma}^0(A) = A \cap \{a : \mathscr{H}^n(X_r(a)) > 0 \text{ for some } r > 0\},$$

we conclude that $\boldsymbol{\Sigma}^0(A)$ has to be at most countable.

From now on we assume $1 \leq m < n$. Suppose $\rho : \boldsymbol{\Sigma}^{n-1}(A) \to \mathbf{R}$ is given by 4.12 and

$$\Sigma_r = \boldsymbol{\Sigma}^m(A) \cap \{x : \rho(x) \geq 2r\} \quad \text{whenever } 0 < r < \infty.$$

Observe Σ_r is a Borel subset of $\mathbf{R}^n$ whenever $0 < r < \infty$ by 4.12 and 4.10, $\Sigma_r \subseteq \Sigma_s$ if $s < r$, $\Sigma_r = \varnothing$ if $r > 1/2$ (since $\rho(x) \leq 1$ whenever $x \in \mathbf{\Sigma}^{n-1}(A)$) and

$$\mathbf{\Sigma}^m(A) = \bigcup_{r>0} \Sigma_r.$$

Therefore we fix $0 < r < 1/2$ and we prove that Σ_r is countably $(\mathscr{H}^m, m)$ rectifiable of class 2. Suppose $C(a)$ is chosen as in 4.15 whenever $a \in \mathbf{\Sigma}^{n-1}(A)$,

$$S = S^r_{r/2}(A), \quad B = B^r_{r/2}, \quad \xi = \boldsymbol{\xi}_A | S,$$

and $\nu : S \to \mathbf{R}^n$ is defined by $\nu(x) = (2/r)^{-1}(x - \xi(x))$ whenever $x \in S$. We observe that

$$\Sigma_r \subseteq \{a : \mathscr{H}^{n-m-1}(\xi^{-1}(a) \cap B) > 0\},$$

because if $a \in \Sigma_r$ then $\{a + u : u \in C(a),\, |u| = r/2\} \subseteq \xi^{-1}(a) \cap B$ by 4.15 and $\mathscr{H}^{n-m-1}(\{a + u : u \in C(a),\, |u| = r/2\}) > 0$ by [Fed69, 3.2.22(3)] (recall that $C(a)$ is a convex cone with $\operatorname{aff} C(a) = \operatorname{aff} \operatorname{nor}(A, a)$ according to 4.15). By 4.15 and [Fed69, 2.10.26, 3.2.31],

$$\mathbf{R}^n \cap \{a : \mathscr{H}^{n-m-1}\left(\xi^{-1}(a) \cap \mathbf{B}(0, i) \cap B\right) \geq j^{-1}\}$$

is $\mathscr{H}^m$ measurable and $(\mathscr{H}^m, m)$ rectifiable whenever $i \geq 1$ and $j \geq 1$ are integers. Therefore we fix $i \geq 1$ and $j \geq 1$ integers, we use [Fed69, 2.2.3] and [Fed69, 3.2.19] to select a Borel subset Y of $\mathbf{R}^n$ such that

$$Y \subseteq \Sigma_r \cap \{a : \mathscr{H}^{n-m-1}\left(\xi^{-1}(a) \cap \mathbf{B}(0, i) \cap B\right) \geq j^{-1}\},$$

$$\mathscr{H}^m\big(\Sigma_r \cap \{a : \mathscr{H}^{n-m-1}\left(\xi^{-1}(a) \cap \mathbf{B}(0, i) \cap B\right) \geq j^{-1}\} \sim Y\big) = 0,$$

$$\operatorname{Tan}^m(\mathscr{H}^m \llcorner Y, a) \in \mathbf{G}(n, m) \quad \text{whenever } a \in Y,$$

and we prove that Y is $(\mathscr{H}^m, m)$ rectifiable of class 2.

Suppose

$$V = \xi^{-1}[Y] \cap \{z : \operatorname{ap} J_m \xi(z) = 0\},$$

$$X = Y \cap \{a : \mathscr{H}^{n-m-1}\left(\xi^{-1}(a) \cap V\right) = 0\},$$

observe that X is $\mathscr{H}^m$ measurable by [Fed69, 2.10.26] and $\xi^{-1}[X] \sim V$ is $\mathscr{H}^{n-1}$ measurable by 4.8(1). Moreover $\mathscr{H}^m(Y \sim X) = 0$ by [Fed69, 3.2.22(3)] and

$$\mathscr{H}^{n-m-1}(\xi^{-1}(a) \cap B \sim V) > 0 \quad \text{whenever } a \in X.$$

We prove now that

$$\limsup_{\delta \to 0+} \delta^{-2} \sup\{|T_\natural^\perp(\xi(z) - \xi(w))| : z \in \mathbf{U}(w, \delta) \cap S\} < \infty$$

whenever $w \in \xi^{-1}[X] \cap B$ and $T = \operatorname{Tan}^m(\mathscr{H}^m \llcorner Y, \xi(w))$. Arguing by contradiction, we assume there exist $w \in \xi^{-1}[X] \cap B$ and sequences $\delta_i > 0$ and $z_i \in \mathbf{U}(w, \delta_i) \cap S$ such that, if $T = \operatorname{Tan}^m(\mathscr{H}^m \llcorner Y, \xi(w))$, then

$$\delta_i \to 0, \quad \delta_i^{-2}|T_\natural^\perp(\xi(z_i) - \xi(w))| \to \infty,$$

as $i \to \infty$. Let $P = \operatorname{Dual} C(\xi(w))$. Since $\operatorname{Lip}\xi \le 2$ by 4.5, we use 4.15 and 4.13 to infer the existence of a sequence $c_i \in P$ such that

$$\begin{aligned}|c_i - (\xi(z_i) - \xi(w))| &= \boldsymbol{\delta}_P(\xi(z_i) - \xi(w)) \\ &\le (2r)^{-1}|\xi(z_i) - \xi(w)|^2 \le (2/r)|z_i - w|^2 \le (2/r)\delta_i^2 \quad \text{whenever } i \ge 1.\end{aligned}$$

Therefore $\delta_i^{-2}|T_\natural^\perp(c_i)| \to \infty$ as $i \to \infty$. On the other hand by 4.6(3) there exists $M < \infty$ such that $|\,(\xi(z_i) - \xi(w)) \bullet \nu(w)| \le M\delta_i^2$ whenever $i \ge 1$. Therefore

$$\begin{aligned}|c_i \bullet \nu(w)| &\le |\,(c_i - (\xi(z_i) - \xi(w))) \bullet \nu(w)| + |\,(\xi(z_i) - \xi(w)) \bullet \nu(w)| \\ &\le (2/r)\delta_i^2 + M\delta_i^2 \quad \text{whenever } i \ge 1\end{aligned}$$

and, since $\nu(w) \in C(\xi(w))$ and $c_i \in P$, we conclude that

$$0 \ge c_i \bullet \nu(w) \ge -(2/r)\delta_i^2 - M\delta_i^2 \quad \text{whenever } i \ge 1.$$

Since P is a convex cone and $T \subseteq P$ (since $C(\xi(w)) \subseteq T^\perp$) we have

$$T_\natural^\perp(c_i) = c_i - T_\natural(c_i) \in P \quad \text{whenever } i \ge 1.$$

Let $\gamma_i = |T_\natural^\perp(c_i)|^{-1}T_\natural^\perp(c_i) \in T^\perp \cap P \cap \mathbf{S}^{n-1}$ and assume $\gamma_i \to \gamma$ as $i \to \infty$ for some $\gamma \in T^\perp \cap P \cap \mathbf{S}^{n-1}$. Noting that $T_\natural(c_i) \bullet \nu(w) = 0$ whenever $i \ge 1$, we conclude that

$$\begin{gathered}0 \ge \gamma_i \bullet \nu(w) = |T_\natural^\perp(c_i)|^{-1}\, c_i \bullet \nu(w) \ge -|T_\natural^\perp(c_i)|^{-1}((2/r) + M)\delta_i^2, \\ \gamma_i \bullet \nu(w) \to 0, \qquad \gamma \bullet \nu(w) = 0.\end{gathered}$$

Since by 4.15 we have that $\operatorname{aff} C(\xi(w)) = T^\perp$ and $C(\xi(w))$ is relatively open in $T^\perp$ we finally get a contradiction.

We can now apply 4.8(3), with $W = \xi^{-1}[X] \cap B \sim V$ and $Z = X$, and 3.39 to infer that X is $(\mathscr{H}^m, m)$ rectifiable of class $(1,1)$. Therefore by [Fed69, 3.1.15] we conclude that X is $(\mathscr{H}^m, m)$ rectifiable of class 2 and the same conclusion holds for Y.

4.17 Remark. In the last paragraph, in order to conclude the second order rectifiability of X, it is important to notice that we use in a crucial way the fact that X can be covered, up to a set of $\mathscr{H}^m$ measure zero, by countably many sets of the form $\boldsymbol{\xi}_A[P]$, where P is an m rectifiable subset of $\xi^{-1}[X] \cap B \sim V$ such that $\boldsymbol{\xi}_A|P$ is a bi-Lipschitzian homeomorphism (by 4.8(3)). This idea originates from unplublished notes of Ulrich Menne, where a similar approach has been used to prove 4.3 in case A is assumed to be convex (henceforth re-proving 1.7).

The extension of this approach to general closed sets is crucially based, first, on the replacement of $\operatorname{Nor}^*(A,a)$ with $\operatorname{nor}(A,a)$ in the the definition of $\boldsymbol{\Sigma}^m(A)$ and, second, on the fact that the Lipschitzian constant of $\boldsymbol{\xi}_A$ remains finite if we consider its restriction on suitable second order rectifiable subsets of the level sets of $\boldsymbol{\delta}_A$, see 4.5 and 4.6.

4.18 Remark. The proof of 1.7 is achieved in [Alb94] by a method that is completely different from the one employed here.

4.19. Finally we want to link our work with the results of [HLW04]. In this important paper the authors, generalizing the results of [Sta79], introduce principal curvatures on the (generalized) normal bundle N_A of a general closed set $A \subseteq \mathbf{R}^n$ and they use them to prove a Steiner type formula for N_A.

Following [HLW04, §2.1] if $A \subseteq \mathbf{R}^n$ is closed then we define

$$\boldsymbol{\nu}_A : (\operatorname{Unp} A) \sim A \to \mathbf{S}^{n-1}$$

by $\boldsymbol{\nu}_A(x) = |x - \boldsymbol{\xi}_A(x)|^{-1}(x - \boldsymbol{\xi}_A(x))$ whenever $x \in (\operatorname{Unp} A) \sim A$ and the (generalized) normal bundle of A by

$$N_A = \{(\boldsymbol{\xi}_A(x), \boldsymbol{\nu}_A(x)) : x \in (\operatorname{Unp} A) \sim A\}.$$

Since $N_A(a) = \mathbf{S}^{n-1} \cap \{\lambda u : \lambda > 0,\ u \in \mathbf{R}^n,\ \boldsymbol{\delta}_A(a+u) = |u|\}$ whenever $a \in A$, we can use 4.2, [Roc70, 2.6.3, 6.2] and [Fed69, 3.2.22(3)] to conclude that

$$\mathscr{H}^{n-m}(\operatorname{nor}(A,a)) > 0 \iff \mathscr{H}^{n-m-1}(N_A(a)) > 0$$

whenever $a \in A$ and $m = 0, \ldots, n-1$.

It is proved in [HLW04, Lemma 2.3] that if $A \subseteq \mathbf{R}^n$ is closed then there exists a sequence $A_i \subseteq \mathbf{R}^n$ of sets of positive reach such that

$$N_A \subseteq \bigcup_{i=1}^{\infty} N_{A_i}.$$

Therefore, instead of directly proving 4.3 for a general closed set, we could have proved it for sets of positive reach and then use [HLW04, Lemma 2.3] to get the same conclusion for every closed set. However this alternative approach, apart from simplifying the measurability questions, would be essentially the same of the one presented here.

On the other hand, the proof we give here is independent from the concept of set of positive reach.

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